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n-Ary Generalized Lie-Type Color Algebras Admitting a Quasi-multiplicative Basis

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Abstract

The class of generalized Lie-type color algebras contains the ones of generalized Lie-type algebras, of n-Lie algebras and superalgebras, commutative Leibniz n-ary algebras and superalgebras, among others. We focus on the class of generalized Lie-type color algebras $\mathfrak{L}$ admitting a quasi-multiplicative basis, with restrictions neither on the dimensions nor on the base field $\mathbb{F}$ and study its structure. We state that if $\mathfrak{L}$ admits a quasi-multiplicative basis then it decomposes as $\mathfrak{L} = \mathcal{U} \oplus (\sum \mathfrak{J}_k)$ with any $\mathfrak{J}_k$ a well described color gLt-ideal of $\mathfrak{L}$ admitting also a quasi-multiplicative basis, and $\mathcal{U}$ a linear subspace of $\mathbb{V}$. Also the minimality of $\mathfrak{L}$ is characterized in terms of the connections and it is shown that the above direct sum is by means of the family of its minimal color gLt-ideals, admitting each one a μ-quasi-multiplicative basis inherited by the one of $\mathfrak{L}$.

Keywords Generalized Lie-type algebra · n-Lie algebra · n-Leibniz algebra · Superalgebra · Color algebra · Quasi-multiplicative basis · Structure theory

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1 Introduction

The concept of multiplicative bases appears in a natural way in the study of different physical problems. In fact, one may expect that there are problems which are naturally and more simply formulated exploiting multiplicative bases. There are many works about a study of algebras with a multiplicative basis [1–5, 8–11, 16–20, 23, 25, 30–32].

The interest in the study of associative algebras admitting a quasi-multiplicative basis also comes from the viewpoint of the theory of infinite dimensional Lie algebras. Since we can recover many classes of Lie algebras from an associative algebra with involution, (see for instance [29, Section 6]), it is interesting to know the structure of the initial associative algebra to understand the one of the Lie algebra. In the framework of infinite dimensional Lie algebras we find many classes of Lie algebras which admits, in a natural way, a quasi-multiplicative basis. For instance, we have the semisimple separable L^*-algebras, the semisimple locally finite split Lie algebras over a field of characteristic zero and the graded Lie algebras considered in [14, Section 3]. Taking into account these comments, it seems natural to study associative and non-associative algebras with a quasi-multiplicative basis to a better understanding of these classes of Lie algebras. The first attempt of a study of algebras with a quasi-multiplicative basis was given in [15].

This interest leads us in a natural way to study Lie algebras admitting quasi-multiplicative basis. But instead of study this class of algebras, we are going to extend our framework in two different ways, on the one hand we will consider the widest n-ary extension by dealing with generalized Lie-type algebras. On the other hand we will consider the colored version of this category of algebras, that is, the generalized Lie-type color algebras. Hence, our aim is to study generalized Lie-type color algebras admitting quasi-multiplicative bases.

The paper is organized as follows. In the Section 3 we introduce relations techniques on the set of indexes I of the quasi-multiplicative basis so as to become a powerful tool for the study of this class of algebras. By making use of these techniques we show that any generalized Lie-type color algebra $\mathfrak{L}$ admitting a quasi-multiplicative basis is of the form $\mathfrak{L} = \mathcal{U} \oplus (\sum \mathfrak{J}_k)$ with any $\mathfrak{J}_k$ a well described color gLt-ideal of $\mathfrak{L}$ admitting also a quasi-multiplicative basis, and $\mathcal{U}$ a linear subspace of $\mathbb{V}$. In the Section 4 the minimality of $\mathfrak{L}$ is characterized in terms of the quasi-multiplicative basis and it is shown that, under mild conditions, the above decomposition of $\mathfrak{L}$ is actually the direct sum of the family of its minimal color gLt-ideals.

2 Basic Definitions

2.1 Generalized Lie-Type Algebras

Let us introduce now the notion of generalized Lie-type algebras which extends some well-known classes of algebras.

Definition 1 An n-ary algebra $(\mathfrak{L}, \langle \cdot, \overset{n)}{\ldots}, \cdot \rangle)$ is called a *generalized Lie-type algebra* (or shortly *gLt-algebra*) if it satisfies the following n identities:

$$\langle y_1, \ldots, \underbrace{\langle x_1, \ldots, x_n \rangle}_{\text{pos } k}, \ldots, y_{n-1} \rangle =$$

$$\sum_{\substack{1 \le i, j \le n \\ \sigma_1 \in \mathbb{S}_n \\ \sigma_2 \in \mathbb{S}_{n-1}}} \alpha_{i,j,k}^{\sigma_1, \sigma_2} \langle x_{\sigma_1(1)}, \ldots, x_{\sigma_1(i-1)}, \langle y_{\sigma_2(1)}, \ldots, \underbrace{x_{\sigma_1(i)}}_{\text{pos } j}, \ldots, y_{\sigma_2(n-1)} \rangle, x_{\sigma_1(i+1)}, \ldots, x_{\sigma_1(n)} \rangle, \tag{1}$$

for $k = 1, \ldots, n$, $x_1, \ldots, x_n, y_1, \ldots, y_{n-1} \in \mathfrak{L}$, being $\alpha_{i,j,k}^{\sigma_1,\sigma_2} \in \mathbb{F}$, and where pos j means that the element $x_{\sigma_1(i)}$ is in the position j in the inside n-product.

Observe that depending of the values of $\alpha_{i,j,k}^{\sigma_1,\sigma_2}$ we obtain several binary algebras:

- Lie algebras, Leibniz algebras, Novikov algebras, associative algebras, alternative algebras, bicommutative algebras, commutative pre-Lie algebras, etc.;

and several n-ary algebras:

- n-Lie (Filippov) algebras, commutative Leibniz n-ary algebras, totally associative-commutative n-ary algebras, etc.

2.2 Color Ω-Algebras

In this subsection we discuss about color n-ary algebras and color Ω-algebras. In the end of the subsection we give some defintions of classical color algebras.

Definition 2 Let $\mathbb{G}$ be an abelian group. A *graded n-ary algebra* $(\mathfrak{L}, \langle \cdot, \overset{n)}{\ldots}, \cdot \rangle)$ is a $\mathbb{G}$-graded vector space $\mathfrak{L} = \oplus_{g \in \mathbb{G}} \mathfrak{L}_g$ provided with a graded n-linear map $\langle \cdot, \overset{n)}{\ldots}, \cdot \rangle : \mathfrak{L} \times \cdots \times \mathfrak{L} \to \mathfrak{L}$ satisfying

$$\langle \mathfrak{L}_{g_1}, \ldots, \mathfrak{L}_{g_n} \rangle \subset \mathfrak{L}_{g_1 + \cdots + g_n},$$

for $g_1, \ldots, g_n \in \mathbb{G}$.

Definition 3 Let $\mathbb{F}$ be a field and $\mathbb{G}$ an abelian group. A map $\epsilon : \mathbb{G} \times \mathbb{G} \to \mathbb{F} \setminus \{0\}$ is called a *bicharacter on* $\mathbb{G}$ if it satisfies:

1. $\epsilon(k, g + h) = \epsilon(k, g)\epsilon(k, h)$,
2. $\epsilon(g + h, k) = \epsilon(g, k)\epsilon(h, k)$,
3. $\epsilon(g, h)\epsilon(h, g) = 1$.

for all $g, h, k \in \mathbb{G}$.

Let $\mathfrak{L} = \oplus_{g \in \mathbb{G}} \mathfrak{L}_g$ be a graded n-ary algebra. An element x is called a *homogeneous element of degree* g if $x \in \mathfrak{L}_g$ and denoted by $\deg(x) = g$. From now on, unless stated otherwise, we assume that all elements are homogeneous. Let ϵ be a bicharacter of $\mathbb{G}$. Given two homogeneous elements $x, y \in \mathfrak{L}$ we set $\epsilon(x, y) := \epsilon(\deg(x), \deg(y))$. Now we recall the notion of color n-ary Ω-algebra for an arbitrary family of polynomial identities Ω (see [6, 28] for more details).

Definition 4 For a (possible n-ary) multilinear polynomial $f(x_1, \ldots, x_n)$ we fix the order of indexes $\{i_1, \ldots, i_n\}$ of one non-associative word $\langle x_{i_1}, \ldots, x_{i_n}\rangle_\beta$ from the polynomial f. Here,

$$f = \sum_{\beta,\sigma\in S_n} \alpha_{\sigma,\beta}\langle x_{\sigma(i_1)}, \ldots, x_{\sigma(i_n)}\rangle_\beta,$$

where $\mathbb{S}_n$ is the permutation group of n elements and β is an arrangement of brackets in the non-associative word. For the shift $\mu_i : \{j_1, \ldots, j_n\} \mapsto \{j_1, \ldots, j_{i+1}, j_i, \ldots, j_n\}$ we define the element $\epsilon(x_{j_i}, x_{j_{i+1}}) \in \mathbb{F} \setminus \{0\}$. Now, for arbitrary non-associative word $\langle x_{\sigma(i_1)}, \ldots, x_{\sigma(i_n)}\rangle_\beta$ its order of indexes is a composition of suitable shifts μ_i, and for this word we set ϵ_σ defined as the product of corresponding $\epsilon(x_{j_i}, x_{j_{i+1}})$. Now, for the multilinear polynomial f, we define the *color multilinear polynomial*

$$f_{co} = \sum_{\beta,\sigma\in S_n} \alpha_{\sigma,\beta}\epsilon_\sigma\langle x_{\sigma(i_1)}, \ldots, x_{\sigma(i_n)}\rangle_\beta.$$

Let $\Omega = \{f_i\}$ be a family of n-ary multilinear polynomials. We say that a *n-ary* algebra $\mathfrak{L}$ is a *Ω-algebra* if it satisfies the family of polynomial identities $\Omega = \{f_i\}$. Also an *n-ary color Ω-algebra* is an n-ary color algebra $\mathfrak{L}$ satisfying the family of color multilinear polynomials $\Omega_{co} = \{(f_i)_{co}\}$.

Some examples of n-ary color algebras are Lie and Jordan superalgebras [26, 27], Leibniz color algebras [24], Filippov (n-Lie) superalgebras [7, 22, 33, 34] and 3-Lie color algebras [35]. Let us give definitions of some color algebras.

Definition 5 A Leibniz color algebra $(\mathfrak{L}, [\cdot,\cdot], \epsilon)$ is a $\mathbb{G}$-graded vector space $\mathfrak{L} = \bigoplus_{g\in\mathbb{G}} \mathfrak{L}_g$ with a bicharacter ϵ, an even bilinear map $[\cdot,\cdot] : \mathfrak{L}\times\mathfrak{L} \to \mathfrak{L}$ satisfying

$$[x, [y, z]] = [[x, y], z] + \epsilon(x, y)[y, [x, z]].$$

Definition 6 An n-Lie color algebra $(\mathfrak{L}, [\cdot, \ldots, \cdot], \epsilon)$ is a $\mathbb{G}$-graded vector space $\mathfrak{L} = \bigoplus_{g\in\mathbb{G}} \mathfrak{L}_g$ with an n-linear map $[\cdot, \ldots, \cdot] : \mathfrak{L}\times\ldots\times\mathfrak{L} \to \mathfrak{L}$ satisfying

$$[x_1, \ldots, x_i, x_{i+1}, \ldots, x_n] = -\epsilon(x_i, x_{i+1})[x_1, \ldots, x_{i+1}, x_i, \ldots, x_n],$$

$$[x_1,\ldots, x_{n-1},[y_1,\ldots,y_n]] = \sum_{i=1}^{n} \epsilon(X_{n-1},Y_{i-1})[y_1,\ldots, y_{i-1},[x_1,\ldots, x_{n-1}, y_i], y_{i+1},\ldots,y_n],$$

where $X_{i-1} = \sum_{k=1}^{i-1} x_k,\ Y_{i-1} = \sum_{k=1}^{i-1} y_k$.

Definition 7 For any σ from the permutation group of n elements $\mathbb{S}_n$, we use the notation

$$\langle x_1, \ldots, x_j, \ldots, x_n\rangle_\sigma = \langle x_{\sigma(1)}, \ldots, x_{\sigma(j)}, \ldots, x_{\sigma(n)}\rangle.$$

2.3 Multiplicative and Quasi-multiplicative Basis

Firstly we establish the natural definition of multiplicative basis of graded n-ary algebras.

Definition 8 A basis of homogeneous elements $\mathfrak{B} = \{e_i\}_{i\in I}$ of a graded n-ary algebra $\mathfrak{L}$ is *multiplicative* if for any $i_1, \ldots, i_n \in I$ we have $\langle e_{i_1}, \ldots, e_{i_n}\rangle \in \mathbb{F}e_j$ for some $j \in I$.

In particular, this definition extends the one considered in [19, 20]. Also this definition is more general than the usual one in the literature [23, 25, 30]. In fact, in these references, a basis $\mathfrak{B} = \{e_i\}_{i \in I}$ is *multiplicative* if for any $i, j \in I$ we have either $e_i e_j = 0$ or $0 \neq e_i e_j = e_k$ for some $k \in I$.

We wish to go a step further by introducing a more general concept than the one of multiplicative basis as follows.

Definition 9 A graded n-ary algebra $\mathfrak{L}$ admits a *quasi-multiplicative basis* if $\mathfrak{L} = \mathbb{V} \oplus \mathbb{W}$ with $\mathbb{V}$ and $0 \neq \mathbb{W}$ graded linear subspaces in such a way that there exists a basis of homogeneous elements $\mathfrak{B} = \{e_i\}_{i \in I}$ of $\mathbb{W}$ satisfying:

1. For $i_1, \dots, i_n \in I$ we have either $\langle e_{i_1}, \dots, e_{i_n} \rangle \in \mathbb{F} e_j$ for some $j \in I$ or $\langle e_{i_1}, \dots, e_{i_n} \rangle \in \mathbb{V}$.
2. Given $0 < k < n$, for $i_1, \dots, i_k \in I$ and $\sigma \in \mathbb{S}_n$ we have $\langle e_{i_1}, \dots, e_{i_k}, \mathbb{V}, \dots, \mathbb{V} \rangle_\sigma \subset \mathbb{F} e_{j_\sigma}$ for some $j_\sigma \in I$.
3. We have either $\langle \mathbb{V}, \dots, \mathbb{V} \rangle \subset \mathbb{F} e_j$ for some $j \in I$ or $\langle \mathbb{V}, \dots, \mathbb{V} \rangle \subset \mathbb{V}$.

Observe that in item 2. we only consider $0 < k < n$ because $k = n$ is contemplated in item 1. and $k = 0$ in item 3. We also note that if the linear subspace $\mathbb{V}$ is trivial or 1-dimensional this definition agrees with the one of multiplicative basis.

We note that a different concept of quasi-multiplicative basis to the one given in Definition 9 can be found, in a context of category theory, in the reference [1].

Examples of gLt-algebras admitting a quasi-multiplicative basis are any n-ary algebras admitting a multiplicative basis (case $\mathbb{V} = \{0\}$). We also have that any finite-dimensional associative algebra A of finite representation type (that is, there are only finitely many isomorphism classes of indecomposable finite-dimensional A-modules) has also a multiplicative basis [5, 30]. Multiplicative bases are also well-related to Gröbner basis. In fact, it is well-known that an algebra with a multiplicative basis has a Gröbner basis if there is an admissible order on the basis [25].

3 Decomposition as Direct Sum of Ideals

In what follows $\mathfrak{L} = \mathbb{V} \oplus \mathbb{W}$ denotes a color gLt-algebra admitting a quasi-multiplicative basis of homogeneous elements $\mathfrak{B} = \{e_i\}_{i \in I}$ of $\mathbb{W} \neq 0$. We begin this section by developing connection techniques among the elements in the set of indexes I as a main tool in our study.

Consider v an external element to I and define the set

$$\mathfrak{I} := I \dot{\cup} \{v\}.$$

The element $\mathbb{V}$ gives us information about the behavior of the linear subspace $\mathbb{V}$ with respect to the elements in the basis $\mathfrak{B}$. For each $j \in \mathfrak{I}$, a new assistant variable $\overline{j} \notin \mathfrak{I}$ is introduced and we consider the set $\overline{I} := \{\overline{i} : i \in I\}$, so

$$\overline{\mathfrak{I}} := \{\overline{j} : j \in \mathfrak{I}\} = \overline{I} \dot{\cup} \{\overline{v}\}$$

consists of all these new symbols. We also denote by $\mathcal{P}(A)$ the power set of a given set A.

Next, we consider the following operations which recover, in a sense, certain multiplicative relations among the elements in $\mathfrak{I}$. Given $\sigma \in \mathbb{S}_n$ we define

- ⋆ $a_\sigma : \mathfrak{I} \times \overset{n)}{\cdots} \times \mathfrak{I} \to \mathcal{P}(\mathfrak{I})$ such as
  - $a_\sigma(i_1, \dots, i_n) := \begin{cases} \{r\}, & \text{if } 0 \neq \langle e_{i_{\sigma(1)}}, \dots, e_{i_{\sigma(n)}} \rangle \in \mathbb{F}e_r; \\ \{v\}, & \text{if } 0 \neq \langle e_{i_{\sigma(1)}}, \dots, e_{i_{\sigma(n)}} \rangle \in \mathbb{V}. \end{cases}$
  - For $0 < k < n$,
    $a_\sigma(i_1, \dots, i_k, v, \dots, v) := \{ \{r\}, \text{ if } 0 \neq \langle e_{i_1}, \dots, e_{i_k}, \mathbb{V}, \dots, \mathbb{V} \rangle_\sigma \subset \mathbb{F}e_r\}.$
  - $a_\sigma(v, \dots, v) := \begin{cases} \{r\}, & \text{if } 0 \neq \langle \mathbb{V}, \dots, \mathbb{V} \rangle \subset \mathbb{F}e_r; \\ \{v\}, & \text{if } 0 \neq \langle \mathbb{V}, \dots, \mathbb{V} \rangle \subset \mathbb{V}. \end{cases}$
  - $\emptyset$ in the remaining cases.
- ⋆ $b_\sigma : \mathfrak{I} \times \overline{\mathfrak{I}} \times \overset{n-1)}{\cdots} \times \overline{\mathfrak{I}} \to \mathcal{P}(\mathfrak{I})$ such as
  - For $1 \leq k \leq n$,
$$b_\sigma(i, \overline{i}_2, \dots, \overline{i}_k, \overline{v}, \dots, \overline{v}) := \Big\{ i' \in I : a_\sigma(i', i_2, \dots, i_k, v, \dots, v) = \{i\} \Big\}$$
$$\cup \Big\{ \{v\}, \text{ if } a_\sigma(i_2, \dots, i_k, v, \dots, v) = \{i\} \Big\}.$$
  - $b_\sigma(v, \overline{i}_2, \dots, \overline{i}_n) := \{i' \in I : a_\sigma(i', i_2, ..., i_n) = \{v\} \}.$
  - $b_\sigma(v, \overline{v}, \dots, \overline{v}) := \{ \{v\}, \text{ if } a_\sigma(v, \dots, v) = \{v\}\}.$
  - $\emptyset$ in the remaining cases.

Then, we consider the operation

$$\mu : (\mathfrak{I} \,\dot{\cup}\, \overline{\mathfrak{I}}) \times ((\mathfrak{I} \times \overset{n-1)}{\cdots} \times \mathfrak{I}) \dot{\cup} (\overline{\mathfrak{I}} \times \overset{n-1)}{\cdots} \times \overline{\mathfrak{I}})) \to \mathcal{P}(\mathfrak{I})$$

given by:

- For any $j, j_1, ..., j_{n-1} \in \mathfrak{I}$, $\mu(j, j_1, \dots, j_{n-2}, j_{n-1}) := \bigcup_{\sigma \in \mathbb{S}_n} a_\sigma(j, j_1, \dots, j_{n-2}, j_{n-1})$
- For any $j \in \mathfrak{I}$ and $\overline{j}_1, ..., \overline{j}_{n-1} \in \overline{\mathfrak{I}}$, $\mu(j, \overline{j}_1, \dots, \overline{j}_{n-2}, \overline{j}_{n-1}) := \bigcup_{\sigma \in \mathbb{S}_n} b_\sigma(j, \overline{j}_1, \dots, \overline{j}_{n-2}, \overline{j}_{n-1})$
- For any $j, j_1, ..., j_{n-1} \in \mathfrak{I}$, $\mu(\overline{j}, j_1, \dots, j_{n-1}) := \bigcup_{k \in \{1, \dots, n-1\}, \sigma \in \mathbb{S}_{n-1}} b_\sigma(j_k, \overline{j}, \overline{j}_1, ..., \overline{j}_{k-1}, \overline{j}_k, ..., \overline{j}_{n-1})$
- $\mu(\overline{\mathfrak{I}}, \overline{\mathfrak{I}}, \dots, \overline{\mathfrak{I}}) := \emptyset.$

From now on, given any $\overline{j} \in \overline{\mathfrak{I}}$ we denote $\overline{(\overline{j})} := j$. Given also any subset J of $\mathfrak{I} \dot{\cup} \overline{\mathfrak{I}}$, we write by $\overline{J} := \{\overline{j} : j \in J\}$ if $J \neq \emptyset$ and $\overline{\emptyset} := \emptyset$.

Lemma 10 *Let $i, j \in I$ and elements $a_2, \dots, a_n$ of $(\mathfrak{I} \times \overset{n-1)}{\cdots} \times \mathfrak{I}) \dot{\cup} (\overline{\mathfrak{I}} \times \overset{n-1)}{\cdots} \times \overline{\mathfrak{I}})$. It holds that $i \in \mu(j, a_2, \dots, a_n)$ if and only if $j \in \mu(i, \overline{a}_2, \dots, \overline{a}_n)$.*

Proof First let us suppose $i \in \mu(j, a_2, \dots, a_n)$. If $\{a_2, \dots, a_n\} \subset \mathfrak{I}$, then there exists $\sigma \in \mathbb{S}_n$ such that $\{i\} = a_\sigma(j, a_2, \dots, a_n)$. Then

$$j \in b_\sigma(i, \overline{a}_2, \dots, \overline{a}_n) \subset \mu(i, \overline{a}_2, \dots, \overline{a}_n).$$

In another case, if $\{a_2, \dots, a_n\} \subset \overline{\mathfrak{I}}$, then exists $\sigma \in \mathbb{S}_n$ such that $i \in b_\sigma(j, a_2, \dots, a_n)$ and so

$$\{j\} = a_\sigma(i, \overline{a}_2, \dots, \overline{a}_n) \subset \mu(i, \overline{a}_2, \dots, \overline{a}_n).$$

To prove the converse we can argue in a similar way. □

Lemma 11 *Let $i \in I$, $j \in \overline{I}$ and elements $a_1, \ldots, a_{n-1}$ of $(\mathfrak{I} \times \overset{n-1)}{\cdots} \times \mathfrak{I})\dot{\cup}(\overline{\mathfrak{I}} \times \overset{n-1)}{\cdots} \times \overline{\mathfrak{I}})$. It holds that $i \in \mu(j, a_1, \ldots, a_{n-1})$ if and only if $\overline{j} \in \mu(\overline{i}, a_1, \ldots, a_{n-1})$.*

Proof Suppose $i \in \mu(j, a_1, \ldots, a_{n-1})$. Since $\mu(\overline{\mathfrak{I}}, \overline{\mathfrak{I}}, \ldots, \overline{\mathfrak{I}}) := \emptyset$, we just have to consider the case in which $\{a_1, \ldots, a_{n-1}\} \subset \mathfrak{I}$. Then there exist $k \in \{1, ..., n-1\}$ and $\sigma \in \mathbb{S}_{n-1}$ such that $i \in b_\sigma(a_k, j, \overline{a}_1, .., \overline{a}_{k-1}, \overline{a}_{k+1}, \ldots, \overline{a}_{n-1})$. From here,

$$\{a_k\} = a_\sigma(i, \overline{j}, a_1, \ldots, a_{k-1}, a_k, \ldots, a_{n-1}) = a_\nu(\overline{j}, i, a_1, \ldots, a_{k-1}, a_k, \ldots, a_{n-1})$$

where $\nu = (1, 2)\sigma$. Hence $\overline{j} \in b_\nu(a_k, \overline{i}, \overline{a}_1, .., \overline{a}_{k-1}, \overline{a}_{k+1}, \ldots, \overline{a}_{n-1}) \subset \mu(\overline{i}, a_1, \ldots, a_{n-1})$.

The converse is proved similarly. □

The map μ is not adequate for our purposes in the sense that we require to send a subset of $I\dot{\cup}\overline{I}$ in a subset of the same union. So we need to introduce the following map:

$$\phi : \mathcal{P}(I\dot{\cup}\overline{I}) \times ((\mathfrak{I} \times \overset{n-1)}{\cdots} \times \mathfrak{I})\dot{\cup}(\overline{\mathfrak{I}} \times \overset{n-1)}{\cdots} \times \overline{\mathfrak{I}})) \to \mathcal{P}(I\dot{\cup}\overline{I}),$$

as

- $\phi(\emptyset, (\mathfrak{I} \times \overset{n-1)}{\cdots} \times \mathfrak{I})\dot{\cup}(\overline{\mathfrak{I}} \times \overset{n-1)}{\cdots} \times \overline{\mathfrak{I}})) := \emptyset$,
- For any $\emptyset \neq J \in \mathcal{P}(I\dot{\cup}\overline{I})$ and $a_2, \ldots, a_n \in (\mathfrak{I} \times \overset{n-1)}{\cdots} \times \mathfrak{I})\dot{\cup}(\overline{\mathfrak{I}} \times \overset{n-1)}{\cdots} \times \overline{\mathfrak{I}})$,

$$\phi(J, a_2, \ldots, a_n) := \Big(\big(\bigcup_{j \in J} \mu(j, a_2, \ldots, a_n)\big) \setminus \{v\}\Big) \cup \overline{\Big(\big(\bigcup_{j \in J} \mu(j, a_2, \ldots, a_n)\big) \setminus \{v\}\Big)}.$$

Note that for any $J \in \mathcal{P}(I\dot{\cup}\overline{I})$ and $a_2, \ldots, a_n \in (\mathfrak{I} \times \overset{n-1)}{\cdots} \times \mathfrak{I})\dot{\cup}(\overline{\mathfrak{I}} \times \overset{n-1)}{\cdots} \times \overline{\mathfrak{I}})$ we have that

$$\phi(J, a_2, \ldots, a_n) = \overline{\phi(J, a_2, \ldots, a_n)} \tag{2}$$

and

$$\phi(J, a_2, \ldots, a_n) \cap I = \Big(\bigcup_{j \in J} \mu(j, a_2, \ldots, a_n)\Big) \setminus \{v\}. \tag{3}$$

Lemma 12 *Consider $J \in \mathcal{P}(I\dot{\cup}\overline{I})$ such that $J = \overline{J}$, $a_2, \ldots, a_n \in \mathfrak{I}\dot{\cup}\overline{\mathfrak{I}}$ and $i \in I$. The following statements are equivalent:*

1. $i \in \phi(J, a_2, \ldots, a_n)$;
2. *either $\phi(\{i\}, \overline{a}_2, \ldots, \overline{a}_n) \cap J \cap I \neq \emptyset$ or $\phi(\{\overline{i}\}, a_2, \ldots, a_n) \cap J \cap I \neq \emptyset$.*

Proof It is straightforward to verify that for any $i \in I$ and $a_2, \ldots, a_n \in \mathfrak{I}\dot{\cup}\overline{\mathfrak{I}}$ we have that

$$i \in \mu(j, a_2, \ldots, a_n)$$

for some $j \in I$ if and only if $j \in \mu(i, \overline{a}_2, \ldots, \overline{a}_n)$ (see Lemma 10), while $i \in \mu(j, a_2, \ldots, a_n)$ for some $j \in \overline{I}$ if and only if $\overline{j} \in \mu(\overline{i}, a_2, \ldots, a_n)$. Since these facts together with Eqs. 2 and 3 we conclude the result. □

For an easier comprenhesion we firstly present a shorter notation. Given m a natural number, we denote $X_m := (a_{m,2}, \ldots, a_{m,n}) \in \mathfrak{I} \dot{\cup} \overline{\mathfrak{I}} \times \overset{n-1)}{\cdots} \times \mathfrak{I} \dot{\cup} \overline{\mathfrak{I}}$. Let us also denote

$$\overline{X}_m := (\overline{a}_{m,2}, \ldots, \overline{a}_{m,n}).$$

Additionally, for $t \geq 1$, by $\{X_1, X_2, \ldots, X_t\}$ we mean the set of elements

$$\{a_{1,2}, \ldots, a_{1,n}, a_{2,2}, \ldots, a_{2,n}, \ldots, a_{t,2} \ldots, a_{t,n}\}.$$

Finally, for $\mathfrak{A} \in \mathcal{P}(I\dot{\cup}\overline{I})$ we denote $\phi(\mathfrak{A}, X_m) := \phi(\mathfrak{A}, a_{m,2}, \ldots, a_{m,n})$.

Definition 13 Let i and j be distinct elements in I. We say that i is *connected* to j if there exists a subset $\{X_1, \ldots, X_t\} \subset \mathfrak{I} \dot{\cup} \overline{\mathfrak{I}}$, for certain $t \geq 1$, such that the following conditions hold:

1. $\widetilde{i} \in \{i, \overline{i}\}$,
2. $\phi(\{\widetilde{i}\}, X_1) \neq \emptyset$,
   $\phi(\phi(\{\widetilde{i}\}, X_1), X_2) \neq \emptyset$,
   $\vdots$
   $\phi(\phi(\ldots\phi(\{\widetilde{i}\}, X_1), \ldots), X_{t-1}) \neq \emptyset$.
3. $j \in \phi(\phi(\ldots\phi(\{\widetilde{i}\}, X_1), \ldots), X_t)$.

The subset $\{X_1, \ldots, X_t\}$ is a *connection* from i to j and by convention i is connected to itself.

Our aim is to show that the connection relation is of equivalence. Previously we check the symmetric property.

Proposition 14 *The relation $\sim$ in I, defined by $i \sim j$ if and only if i is connected to j, is an equivalence relation.*

Proof The reflexive character of $\sim$ is given by the Definition 13. Let us see the symmetric character of $\sim$: If $i \sim j$ then there exists a connection

$$\{X_1, \ldots, X_t\} \subset \mathfrak{I}\dot{\cup}\overline{\mathfrak{I}}$$

from i to j satisfying conditions in Definition 13. In case $t = 1$ we have $j \in \phi(\{\widetilde{i}\}, X_1)$. If $\widetilde{i} = i \in I$ then $i \in \phi(\{j\}, \overline{X}_1)$. If $\widetilde{i} = \overline{i} \in \overline{I}$ then $i \in \phi(\{\overline{j}\}, X_1)$. So $i \in \phi(\{\widetilde{j}\}, \widetilde{X}_1)$ with $(\widetilde{j}, \widetilde{X}_1) \in \{(j, \overline{X}_1), (\overline{j}, X_1)\}$, that is, $\{\widetilde{X}_1\}$ is a connection from j to i.

Suppose $t \geq 2$ and let us show that we can find a set

$$\{\widetilde{X}_t, \ldots, \widetilde{X}_1\} \subset \mathfrak{I}\dot{\cup}\mathfrak{I},$$

where $\widetilde{X}_m \in \{X_m, \overline{X}_m\}$, for $1 \leq m \leq t$, which gives rise to a connection from j to i. Indeed, Eq. 2 shows

$$\phi(\ldots(\phi(\{\widetilde{i}\}, X_1), \ldots), X_{t-1}) = \overline{\phi(\ldots(\phi(\{\widetilde{i}\}, X_1), \ldots), X_{t-1})}$$

and so by taking $J := \phi(\ldots(\phi(\{\widetilde{i}\}, X_1), \ldots), X_{t-1})$ we have $J \in \mathcal{P}(I\dot{\cup}\overline{I})$ and $J = \overline{J}$, so we can apply Lemma 12 to the expression

$$j \in \phi(\phi(\ldots(\phi(\{\widetilde{i}\}, X_1), \ldots), X_{t-1}), X_t)$$

to get that either

$$\phi(\{j\}, \overline{X}_t) \cap \phi(\ldots(\phi(\{\widetilde{i}\}, X_1), \ldots), X_{t-1}) \cap I \neq \emptyset$$

or

$$\phi(\{\overline{j}\}, X_t) \cap \phi(\ldots(\phi(\{\widetilde{i}\}, X_1), \ldots), X_{t-1}) \cap I \neq \emptyset$$

and so

$$\phi(\{\widetilde{j}\}, \widetilde{X}_t) \neq \emptyset$$

with $(\widetilde{j}, \widetilde{X}_t) \in \{(j, \overline{X}_t), (\overline{j}, X_t)\}$.

By taking

$$k \in \phi(\{\widetilde{j}\}, \widetilde{X}_t) \cap \phi(\ldots(\phi(\{\widetilde{i}\}, X_1), \ldots), X_{t-1}) \cap I,$$

Eq. 2 and Lemma 12, the fact $k \in \phi(\cdots(\phi(\{\widetilde{i}\}, X_1), \ldots), X_{t-1})$ and $k \in \phi(\{\widetilde{j}\}, \widetilde{X}_t)$ imply now either

$$\phi(\phi(\{\widetilde{j}\}, \widetilde{X}_t), \overline{X}_{t-1}) \cap \phi(\ldots\phi(\phi(\{i\}, X_1), \ldots), X_{t-2}) \cap I \neq \emptyset$$

or

$$\phi(\phi(\{\widetilde{j}\}, \widetilde{X}_t), X_{t-1}) \cap \phi(\ldots\phi(\phi(\{\widetilde{i}\}, X_1), \ldots), X_{t-2}) \cap I \neq \emptyset$$

and consequently

$$\phi(\phi(\{\widetilde{j}\}, \widetilde{X}_t), \widetilde{X}_{t-1}) \neq \emptyset$$

for some $\widetilde{X}_{t-1} \in \{X_{t-1}, \overline{X}_{t-1}\}$.

By iterating this process we get

$$\begin{aligned}&\phi(\phi(\ldots(\phi(\{\widetilde{j}\}, \widetilde{X}_t), \ldots), \widetilde{X}_{t-m+1}), \widetilde{X}_{t-m})\\ &\cap\phi(\phi(\ldots(\phi(\{\widetilde{i}\}, X_1), \ldots), X_{t-m-2}), X_{t-m-1}) \cap I \neq \emptyset\end{aligned}$$

for $0 \leq m \leq t-2$. In particular, we have for the case $m = t-2$ that

$$\phi(\phi(\cdots(\phi(\{\widetilde{j}\}, \widetilde{X}_t), \ldots), \widetilde{X}_3), \widetilde{X}_2) \cap \phi(\{\widetilde{i}\}, X_1) \cap I \neq \emptyset.$$

Since either $\widetilde{i} = i$ or $\widetilde{i} = \overline{i}$, if we write $J := \phi(\phi(\ldots(\phi(\{\widetilde{j}\}, \widetilde{X}_t), \ldots), \widetilde{X}_3), \widetilde{X}_2)$, the previous equation allows us to assert that either $\phi(\{i\}, X_1) \cap J \cap I \neq \emptyset$ or $\phi(\{\overline{i}\}, X_1) \cap J \cap I \neq \emptyset$ with $i \in I$. Hence Lemma 12 applies to get

$$i \in \phi(\phi(\ldots(\phi(\{\widetilde{j}\}, \widetilde{X}_t), \ldots), \widetilde{X}_2), \widetilde{X}_1)$$

for some $\widetilde{X}_1 \in \{X_1, \overline{X}_1\}$ and conclude $\sim$ is symmetric.

Finally, let us verify the transitive character of $\sim$. Suppose $i \sim j$ and $j \sim k$, and write $\{X_1, \ldots, X_t\}$ for a connection from i to j and $\{Y_1, \ldots, Y_s\}$ for a connection from j to k. If $i = j$ so $\{Y_1, \ldots, Y_s\}$ is a connection from i to k. If $k = j$ thus $\{X_1, \ldots, X_t\}$ is a connection from i to k. Finally, if $t \geq 1$ and $s \geq 1$, taking into account Eq. 2 we easily have that $\{X_1, \ldots, X_t, Y_1, \ldots, Y_s\}$ is a connection from i to k. We have shown that the connection relation is an equivalence relation. □

By the above proposition we can consider the quotient set

$$I/\sim = \{[i] : i \in I\},$$

becoming $[i]$ the set of elements in I which are connected to i.

Definition 15 A *color gLt-subalgebra* of $\mathfrak{L}$ is a $\mathbb{G}$-graded subspace $\mathfrak{S}$ of $\mathfrak{L}$ verifying $\langle\mathfrak{S}, \ldots, \mathfrak{S}\rangle \subset \mathfrak{S}$. A $\mathbb{G}$-graded subspace $\mathcal{I} \subset \mathfrak{L}$ is a *color gLt-ideal* of $\mathfrak{L}$ if $\langle\mathcal{I}, \mathfrak{L}, \ldots, \mathfrak{L}\rangle_\sigma \subset \mathcal{I}$, for any $\sigma \in \mathbb{S}_n$.

Our next goal in this section is to associate an n-ary ideal $\mathfrak{J}_{[i]}$ of $\mathfrak{L}$ to any $[i] \in I/\sim$. Fix $i \in I$, we start by defining the sets

$$\begin{aligned}\mathbb{V}_{[i]} &:= \Big(\sum_{i_1,\ldots,i_n\in[i]} \mathbb{F}\langle e_{i_1}, \ldots, e_{i_n}\rangle\Big) \cap \mathbb{V} \subset \mathbb{V},\\ \mathbb{W}_{[i]} &:= \bigoplus\nolimits_{j\in[i]} \mathbb{F}e_j \subset \mathbb{W}\end{aligned}$$

Finally, we denote by $\mathfrak{J}_{[i]}$ the direct sum of the two subspaces above, that is,

$$\mathfrak{J}_{[i]} := \mathbb{V}_{[i]} \oplus \mathbb{W}_{[i]}.$$

Definition 16 Let $\mathfrak{L} = \mathbb{V} \oplus \mathbb{W}$ be a graded n-ary algebra admitting a quasi-multiplicative basis $\mathfrak{B} = \{e_i\}_{i \in I}$ with $\mathbb{W} \neq 0$. It is said that a n-ary graded subalgebra $\mathfrak{S}$ of $\mathfrak{L}$ has a quasi-multiplicative basis *inherited* by the one of $\mathfrak{L}$ if $\mathfrak{S} = \mathbb{V}_{\mathfrak{S}} \oplus \mathbb{W}_{\mathfrak{S}}$ with $\mathbb{V}_{\mathfrak{S}}$ a graded linear subspace of $\mathbb{V}$, and $0 \neq \mathbb{W}_{\mathfrak{S}}$ a graded linear subspace of $\mathbb{W}$ admitting $\mathfrak{B}' \subset \mathfrak{B}$ as a basis.

Proposition 17 *For any $i \in I$, the linear subspace $\mathfrak{J}_{[i]}$ is a color gLt-ideal of $\mathfrak{L}$ admitting a quasi-multiplicative basis inherited by the one of $\mathfrak{L}$.*

Proof Given $\sigma \in \mathbb{S}_n$, we can write

$$\langle \mathfrak{J}_{[i]}, \mathfrak{L}, \dots, \mathfrak{L} \rangle_\sigma = \langle \mathbb{V}_{[i]} \oplus \mathbb{W}_{[i]}, \Big(\mathbb{V} \oplus (\bigoplus_{r \in I} \mathbb{F} e_r)\Big), \dots, \Big(\mathbb{V} \oplus (\bigoplus_{s \in I} \mathbb{F} e_s)\Big) \rangle_\sigma. \quad (4)$$

In case $\langle e_j, e_{i_2}, \dots, e_{i_n} \rangle_\sigma \neq 0$ for some $j \in [i]$ and $i_2, \dots, i_n \in I$, we have that either $0 \neq \langle e_j, e_{i_2}, \dots, e_{i_n} \rangle_\sigma \in \mathbb{F} e_l$ with $l \in I$ or $0 \neq \langle e_j, e_{i_2}, \dots, e_{i_n} \rangle_\sigma \in \mathbb{V}$. In the first case the connection $\{i_2, \dots, i_n\}$ gives us $j \sim l$, so $l \in [i]$ and then $\langle e_j, e_{i_2}, \dots, e_{i_n} \rangle_\sigma \in \mathbb{W}_{[i]}$. In the second case, for all $2 \leq k \leq n$ we get $i_k \in b_\tau(v, \overline{j}, \overline{i}_2, \dots, \overline{i}_{k-1}, \overline{i}_{k+1}, \dots, \overline{i}_n)$ for some $\tau \in \mathbb{S}_n$, and so $i_k \in \mu(\overline{j}, v, i_2, \dots, i_{k-1}, i_{k+1}, \dots, i_n)$. Hence the set $\{v, i_2, \dots, i_{k-1}, i_{k+1}, \dots, i_n\}$ is a connection from j to i_k and so $i_k \in [i]$ for $2 \leq k \leq n$. Therefore $\langle e_j, e_{i_2}, \dots, e_{i_n} \rangle_\sigma \in V_{[i]}$. Hence we get

$$\langle \mathbb{W}_{[i]}, (\bigoplus_{r \in I} \mathbb{F} e_r), \dots, (\bigoplus_{s \in I} \mathbb{F} e_s) \rangle_\sigma \subset \mathfrak{J}_{[i]}. \quad (5)$$

For some $j \in [i]$, if we have $0 \neq \langle e_j, \mathbb{V}, \dots, \mathbb{V}, e_{i_2}, \dots, e_{i_k} \rangle_\sigma \subset \mathbb{F} e_l$ for certain $l \in I$ and where $k \geq 2$, then $l \in \mu(j, v, \dots, v, \dots, i_2, \dots, i_k)$. So $\{v, \dots, v, \dots, i_2, \dots, i_k\}$ is a connection from j to $\mathfrak{L}$ then $l \in [i]$. From here we have (taking also into account Equation (5)) that:

$$\langle \mathbb{W}_{[i]}, \mathfrak{L}, \dots, \mathfrak{L} \rangle_\sigma \subset \mathbb{W}_{[i]}. \quad (6)$$

Suppose there exist $i_1, \dots, i_n \in [i]$ with $0 \neq \langle e_{i_1}, \dots, e_{i_n} \rangle_\sigma \in \mathbb{V}$, that is, $v \in \mu(i_1, \dots, i_n)$, in such a way that $0 \neq \langle \langle e_{i_1}, \dots, e_{i_n} \rangle_\sigma, e_{j_1}, \dots, e_{j_{n-1}} \rangle_\tau \in \mathbb{F} e_m$ for some $j_1, \dots, j_{n-1} \in I$. By Eq. 1 we get that

$$0 \neq \langle e_{i_{\sigma_1(1)}}, \dots, e_{i_{\sigma_1(k-1)}}, \langle e_{j_{\sigma_2(1)}}, \dots, \underbrace{e_{i_{\sigma_1(k)}}}_{\text{pos } h}, \dots, e_{j_{\sigma_2(n-1)}} \rangle_{\tau_1}, e_{i_{\sigma_1(k+1)}}, \dots, e_{i_{\sigma_1(n)}} \rangle_{\tau_2} \in \mathbb{F} e_m$$

for certain $1 \leq h, k \leq n$, $\sigma_1 \in \mathbb{S}_n, \sigma_2 \in \mathbb{S}_{n-1}$. The connection $\{i_{\sigma_1(2)}, \dots, i_{\sigma_1(k-1)}, i', i_{\sigma_1(k+1)}, \dots i_{\sigma_1(n)}\}$ where either $i' \in I$ if $0 \neq \langle e_{j_{\sigma_2(1)}}, \dots, e_{i_{\sigma_1(k)}}, \dots, e_{j_{\sigma_2(n-1)}} \rangle_{\tau_1} \in \mathbb{F} e_{i'}$ or $i' = v$ when $0 \neq \langle e_{j_{\sigma_2(1)}}, \dots, e_{i_{\sigma_1(k)}}, \dots, e_{j_{\sigma_2(n-1)}} \rangle_{\tau_2} \in \mathbb{V}$, gives us that $i_{\sigma_1(1)}$ is connected to m and so $m \in [i]$. From here

$$\langle \mathbb{V}_{[i]}, (\bigoplus_{r \in I} \mathbb{F} e_r), \dots, (\bigoplus_{s \in I} \mathbb{F} e_s) \rangle_\sigma \subset \mathbb{W}_{[i]}. \quad (7)$$

Now, suppose there exist $i_1, \dots, i_n \in [i]$ with $v \in \mu(i_1, \dots, i_n)$ satisfying $0 \neq \langle \langle e_{i_1}, \dots, e_{i_n} \rangle_\sigma, \mathbb{V}, \dots, \mathbb{V} \rangle_\tau$. We have two possibilities, in the first one $0 \neq$

$\langle\langle e_{i_1}, \ldots, e_{i_n}\rangle_\sigma, \mathbb{V}, \ldots, \mathbb{V}\rangle_\tau \subset \mathbb{F}e_k$ and we have as above that $k \in [i]$. In the second one $0 \neq \langle\langle e_{i_1}, \ldots, e_{i_n}\rangle_\sigma, \mathbb{V}, \ldots, \mathbb{V}\rangle_\tau \subset \mathbb{V}$ and we get by Eq. 1

$$0 \neq \langle e_{i_{\sigma_1(1)}}, \ldots, e_{i_{\sigma_1(k-1)}}, \langle \mathbb{V}, \ldots, e_{i_{\sigma_1(k)}}, \ldots, \mathbb{V}\rangle_{\tau_1}, e_{i_{\sigma_1(k+1)}}, \ldots, e_{i_{\sigma_1(n)}}\rangle_{\tau_2} \subset \mathbb{V}$$

being then $\mu(i_{\sigma_1(k)}, v, \ldots, v) = \{r\}$, for $r \in I$, with $\{v, \ldots, v\}$ a connection from $i_{\sigma_1(k)}$ to r. Hence $r \in [i]$ and

$$\begin{aligned} 0 \neq\ & \langle e_{i_{\sigma_1(1)}}, \ldots, e_{i_{\sigma_1(k-1)}}, \langle \mathbb{V}, \ldots, e_{i_{\sigma_1(k)}}, \ldots, \mathbb{V}\rangle_{\tau_1}, e_{i_{\sigma_1(k+1)}}, \ldots, e_{i_{\sigma_1(n)}}\rangle_{\tau_2} \\ & \subset \mathbb{F}\langle e_{i_{\sigma_1(1)}}, \ldots, e_{i_{\sigma_1(k-1)}}, e_r, e_{i_{\sigma_1(k+1)}}, \ldots, e_{i_{\sigma_1(n)}}\rangle_{\tau_2} \cap \mathbb{V} \subset \mathbb{V}_{[i]}. \end{aligned}$$

We have shown

$$\langle \mathbb{V}_{[i]}, \mathbb{V}, \ldots, \mathbb{V}\rangle_\tau \subset \mathfrak{I}_{[i]}. \tag{8}$$

Finally, in case there exist $i_1, \ldots, i_n \in [i]$ with $v \in \mu(i_1, \ldots, i_n)$ satisfying $0 \neq \langle\langle e_{i_1}, \ldots, e_{i_n}\rangle_\sigma, \mathbb{V}, \ldots, \mathbb{V}, e_{j_2}, ..., e_{j_k}\rangle_\tau$ with $k \geq 2$. We have that necessarily $0 \neq \langle\langle e_{i_1}, \ldots, e_{i_n}\rangle_\sigma, \mathbb{V}, \ldots, \mathbb{V}, e_{j_2}, ..., e_{j_k}\rangle_\tau \subset \mathbb{F}e_l$ for some $l \in I$, and we have as above that $l \in [i]$. Consequently

$$\langle \mathbb{V}_{[i]}, \mathbb{V}, \ldots, \mathbb{V}, \mathbb{W}_{[i]}, ..., \mathbb{W}_{[i]}\rangle_\tau \subset \mathfrak{I}_{[i]}. \tag{9}$$

From Eqs. 4–9 we conclude that $\mathfrak{I}_{[i]}$ is a color gLt-ideal of $\mathfrak{L}$.

Finally, observe that the decomposition $\mathfrak{I}_{[i]} = \mathbb{V}_{[i]} \oplus \mathbb{W}_{[i]}$ together with the basis

$$\{e_j : j \in [i]\}$$

of $\mathbb{W}_{[i]}$ allow us to assert that $\mathfrak{I}_{[i]}$ admits a quasi-multiplicative basis inherited by the one of $\mathfrak{L}$. □

Definition 18 A color gLt-algebra $\mathfrak{L}$ is *simple* if its unique non-zero color gLt-ideals are $\{0\}$ and $\mathfrak{L}$.

Corollary 19 *If $\mathfrak{L}$ is simple, then there exists a connection between any couple of elements in the index set I.*

Proof The simplicity of $\mathfrak{L}$ applies to get that $\mathfrak{I}_{[i_0]} = \mathfrak{L}$ for any $i_0 \in I$. Hence $[i_0] = I$ and so any couple of elements in I are connected. □

Lemma 20 *If $[i] \neq [h]$ for some $i, h \in I$ then $\langle \mathfrak{I}_{[i]}, \mathfrak{I}_{[h]}, \mathfrak{L}, \ldots, \mathfrak{L}\rangle_\sigma = 0$ for any $\sigma \in \mathbb{S}_n$.*

Proof We have to study the product $\langle \mathbb{V}_{[i]} \oplus \mathbb{W}_{[i]}, \mathbb{V}_{[h]} \oplus \mathbb{W}_{[h]}, \mathfrak{L}, \ldots, \mathfrak{L}\rangle_\sigma$. By Eqs. 6 and 7 we have the following subsets of $\mathbb{W}_{[i]} \cap \mathbb{W}_{[h]}$ satisfy

$$\langle \mathbb{V}_{[i]}, \mathbb{W}_{[h]}, \mathfrak{L}, \ldots, \mathfrak{L}\rangle_\sigma = \langle \mathbb{W}_{[i]}, \mathbb{V}_{[h]}, \mathfrak{L}, \ldots, \mathfrak{L}\rangle_\sigma = \{0\}. \tag{10}$$

We also have as consequence of the previous comments to Eq. 5 and to Eq. 8 that

$$\langle \mathbb{W}_{[i]}, \mathbb{W}_{[h]}, \mathfrak{L}, \ldots, \mathfrak{L}\rangle_\sigma \cap (\bigoplus_{k\in I} \mathbb{F}e_k) \subset \mathbb{W}_{[i]} \cap \mathbb{W}_{[h]} = \{0\}$$

and

$$\langle \mathbb{V}_{[i]}, \mathbb{V}_{[h]}, \mathfrak{L}, \ldots, \mathfrak{L}\rangle_\sigma \cap (\bigoplus_{k\in I} \mathbb{F}e_k) \subset \mathbb{W}_{[i]} \cap \mathbb{W}_{[h]} = \{0\}$$

respectively. From here, it just remains to consider the products $\langle e_{i'}, e_{h'}, e_{k_3}, \ldots, e_{k_n}\rangle_\sigma \in \mathbb{V}$ for $i' \in [i]$, $h' \in [h]$, $k_3, \ldots, k_n \in I$ and $\langle\langle e_{i'_1}, \ldots, e_{i'_n}\rangle_{\sigma_1}, \langle e_{h'_1}, e_{h'_2}, \ldots, e_{h'_n}\rangle_{\sigma_2}, \mathbb{V},$

$\mathbb{V}, \ldots, \mathbb{V}\rangle_{\sigma_3} \in \mathbb{V}$ for $i'_1, \ldots, i'_n \in [i]$, $h'_1, \ldots, h'_n \in [h]$ with $\phi(i'_1, \ldots, i'_n) = \phi(h'_1, \ldots, h'_n) = \{v\}$. In the first situation, if $\langle e_{i'}, e_{h'}, e_{k_3}, \ldots, e_{k_n}\rangle_\sigma \neq 0$, then $i' \in b_\sigma(v, \overline{h'}, \overline{k_3}, ..., \overline{k_n})$ and so $i' \in \mu(\overline{h'}, v, k_3, ..., k_n)$. From here the connection $\{v, k_3, \ldots, k_n\}$ gives us h' is connected to i', that is $[i] = [h]$ a contradiction, so $\langle e_{i'}, e_{h'}, e_{k_3}, \ldots, e_{k_n}\rangle_\sigma = 0$.

In the second situation we deal, by Eq. 1, with n-ary products of the form $\langle e_{h'_2}, \ldots, e_{h'_{k-1}}, e_{i'_{\sigma_1(1)}}, e_{h'_{k+1}}, \ldots e_{h'_n}\rangle_\sigma$ for certain $\sigma_1 \in \mathbb{S}_n$ and $2 \leq k \leq n$. In case some $\langle e_{h'_2}, \ldots, e_{h'_{k-1}}, e_{i'_{\sigma_1(1)}}, e_{h'_{k+1}}, \ldots e_{h'_n}\rangle_\sigma \neq 0$ we would have $\langle \mathbb{V}_{[i]}, \mathbb{W}_{[h]}, \ldots, \mathbb{W}_{[h]}\rangle_\sigma \neq 0$ what contradicts Eq. 10. From here any $\langle \mathbb{V}_{[i]}, \mathbb{W}_{[h]}, \ldots, \mathbb{W}_{[h]}\rangle_\sigma = 0$, then $\langle\langle e_{i'_1}, \ldots, e_{i'_n}\rangle_{\sigma_1}, \langle e_{h'_1}, e_{h'_2}, \ldots, e_{h'_n}\rangle_{\sigma_2}, \mathbb{V}, \ldots, \mathbb{V}\rangle_{\sigma_3} = 0$ and the proof is complete. □

Theorem 21 *A color gLt-algebra $\mathfrak{L} = \mathbb{V} \oplus \mathbb{W}$ admitting a quasi-multiplicative basis of $\mathbb{W} \neq 0$ decomposes as*

$$\mathfrak{L} = \mathcal{U} \oplus \Big(\sum_{[i]\in I/\sim} \mathfrak{J}_{[i]}\Big),$$

where $\mathcal{U}$ is a linear complement of $\sum_{[i]\in I/\sim} \mathbb{V}_{[i]}$ in $\mathbb{V}$ and any $\mathfrak{J}_{[i]}$ is one of the color gLt-ideals, admitting a quasi-multiplicative basis inherited by the one of $\mathfrak{L}$, given in Proposition 17. Furthermore

$$\langle \mathfrak{J}_{[i]}, \mathfrak{J}_{[h]}, \mathfrak{L}, \ldots, \mathfrak{L}\rangle_\sigma = 0$$

whenever $[i] \neq [h]$.

Proof Since we can write

$$\mathfrak{L} = \mathbb{V} \oplus \Big(\bigoplus_{i\in I} \mathbb{F}e_i\Big)$$

and

$$\mathbb{V} = \mathcal{U} \oplus \Big(\sum_{[i]\in I/\sim} \mathbb{V}_{[i]}\Big), \quad \bigoplus_{i\in I} \mathbb{F}e_i = \bigoplus_{[i]\in I/\sim} W_{[i]}$$

we clearly have

$$\mathfrak{L} = \mathcal{U} \oplus \Big(\sum_{[i]\in I/\sim} \mathfrak{J}_{[i]}\Big)$$

being each $\mathfrak{J}_{[i]}$ a color gLt-ideal of $\mathfrak{L}$, admitting a quasi-multiplicative basis inherited by the one of $\mathfrak{L}$, satisfying $\langle \mathfrak{J}_{[i]}, \mathfrak{J}_{[h]}, \mathfrak{L}, \ldots, \mathfrak{L}\rangle = 0$ when $[i] \neq [h]$ by Proposition 17 and Lemma 20. □

In case $\mathfrak{L}$ admits a multiplicative basis (see Definition 8) we have as an immediate consequence of Theorem 21 the next result.

Corollary 22 *If $\mathfrak{L}$ admits a multiplicative basis, then*

$$\mathfrak{L} = \bigoplus_{[i]\in I/\sim} \mathfrak{J}_{[i]},$$

where any $\mathfrak{J}_{[i]}$ is one of the color gLt-ideals given in Proposition 17, admitting each one a multiplicative basis inherited by the one of $\mathfrak{L}$.

Definition 23 Let $\mathfrak{L} = \mathbb{V} \oplus \mathbb{W}$ be a color gLt-algebra admitting a quasi-multiplicative basis. We call the *center* of $\mathfrak{L}$ the set

$$\mathcal{Z}(\mathfrak{L}) := \{x \in \mathfrak{L} : \langle x, \mathfrak{L}, \ldots, \mathfrak{L}\rangle_\sigma = 0 \ \text{ for any } \ \sigma \in \mathbb{S}_n\}.$$

We also say that $\mathbb{V}$ is *tight* whence $\mathbb{V} = \{0\}$ or $\mathbb{V} = \sum_{\substack{i_1,\ldots,i_n \in I \\ \mu(i_1,\ldots,i_n)=\{v\}}} \mathbb{F}\langle e_{i_1}, \ldots, e_{i_n}\rangle$.

Corollary 24 *Suppose $\mathfrak{L}$ is centerless and $\mathbb{V}$ is tight, then $\mathfrak{L}$ decomposes as the direct sum of the color gLt-ideals given in Proposition 17,*

$$\mathfrak{L} = \bigoplus_{[i]\in I/\sim} \mathfrak{I}_{[i]}.$$

Proof By Theorem 21, since $\mathcal{U} = 0$, we just have to show the direct character of the sum. Given

$$x \in \mathfrak{I}_{[i]} \cap \sum_{\substack{[j]\in I/\sim \\ j \not\sim i}} \mathfrak{I}_{[j]},$$

by using the fact $\langle \mathfrak{I}_{[i]}, \mathfrak{I}_{[h]}, \mathfrak{L}, \ldots, \mathfrak{L}\rangle_\sigma = 0$ for $[i] \neq [h]$ and any $\sigma \in \mathbb{S}_n$ we obtain

$$\langle x, \mathfrak{I}_{[i]}, \mathfrak{L}, \ldots, \mathfrak{L}\rangle_\sigma = \langle x, \sum_{\substack{[j]\in I/\sim \\ j \not\sim i}} \mathfrak{I}_{[h]}, \mathfrak{L}, \ldots, \mathfrak{L}\rangle_\sigma = 0.$$

It implies $\langle x, \mathfrak{L}, \ldots, \mathfrak{L}\rangle_\sigma = 0$, so $x \in \mathcal{Z}(\mathfrak{L}) = 0$, as desired. □

4 The Minimal Components

In this section we study the minimality of the components in the decompositions of color gLt-algebras given in Theorem 21, Corollary 22 and Corollary 24. So we introduce the next concept.

Definition 25 Let $\mathfrak{L} = \mathbb{V} \oplus \mathbb{W}$ be a color gLt-algebra admitting a quasi-multiplicative basis $\mathfrak{B} = \{e_i\}_{i\in I}$ with $\mathbb{W} \neq 0$. It is said that $\mathfrak{L}$ is *minimal* if its only non-zero color gLt-ideal admitting a basis inherited by the one of $\mathfrak{L}$ is itself.

Let us introduce the notion of μ-multiplicativity in the framework of color gLt-algebras with quasi-multiplicative bases in a similar way to the ones of closed-multiplicativity for associative quasi-multiplicative algebras, graded associative algebras, graded Lie algebras, split Leibniz algebras or split Lie triple systems (see [12–15, 21] for these notions and examples). From now on, for any $\overline{i} \in \overline{I}$ we denote $e_{\overline{i}} = 0$.

Definition 26 Let $\mathfrak{L} = \mathbb{V} \oplus \mathbb{W}$ be a color gLt-algebra admitting a quasi-multiplicative basis $\mathfrak{B} = \{e_i\}_{i\in I}$ of $\mathbb{W} \neq 0$. We say that $\mathfrak{L}$ admits a *μ-quasi-multiplicative basis* if given $i \in I$ and $k_1, \ldots, k_n \in (\mathfrak{I} \times \cdots \times \mathfrak{I})\dot{\cup}(\overline{\mathfrak{I}} \times \cdots \times \overline{\mathfrak{I}})$ such that

$$i \in \mu(k_1, \ldots, k_n) \text{ then } e_i \in \mathbb{F}\langle u_{k_1}, \ldots, u_{k_n}\rangle_\sigma$$

for some $\sigma \in S_n$, where $u_{k_r} = e_{k_r} + e_{\overline{k_r}}$ if $k_r \notin \{v, \overline{v}\}$ or $u_{k_r} = \mathbb{V}$ if $k_r \in \{v, \overline{v}\}$, for $1 \leq r \leq n$.

Theorem 27 *Suppose $\mathfrak{L}$ admits a μ-quasi-multiplicative basis and $\mathbb{V}$ is tight. It holds that $\mathfrak{L}$ is minimal if and only if I has all of its elements connected.*

Proof If $\mathfrak{L}$ is minimal, for the color gLt-ideals defined in Proposition 17 we have $\mathfrak{J}_{[i]} = \mathfrak{L}$ for any $[i]$. Hence, $[i] = I$. To prove the converse, consider $\mathfrak{J}$ a non-zero color gLt-ideal of $\mathfrak{L}$ admitting a basis inherited by the one of $\mathfrak{L}$. Since $\mathfrak{J} \neq 0$, we can take some $i_0 \in I$ such that

$$0 \neq e_{i_0} \in \mathfrak{J}. \tag{11}$$

Taking into account that I has all of its elements connected, we have that for any $i \in I$, we can consider a connection

$$\{a_{1,2}, \ldots, a_{1,n}, a_{2,2}, \ldots, a_{2,n}, \ldots, a_{t,2}, \ldots, a_{t,n}\} \tag{12}$$

from i_0 to i, being $t \geq 1$. We know by Eq. 2 that

$$\phi(\{\tilde{i_0}\}, a_{1,2}, \ldots, a_{1,n}) \cap I \neq \emptyset$$

and so for any $j_1 \in \phi(\{\tilde{i_0}\}, a_{1,2}, \ldots, a_{1,n}) \cap I$ we have, taking into account $\tilde{i_0} \in \{i_0, \overline{i_0}\}$ that either $j_1 \in \mu(i_0, a_{1,2}, ..., a_{1,n}) \setminus \{v\}$ or $j_1 \in \mu(\overline{i_0}, a_{1,2}, ..., a_{1,n}) \setminus \{v\}$, being necessarily any $a_{1,k} \in \mathfrak{J}$ in the second possibility. By Eq. 11 we get in the first possibility that $0 \neq e_{j_1} \in \mathbb{F}\langle e_{i_0}, u_{a_{1,2}}, \ldots, u_{a_{1,n}}\rangle_\sigma \subset \mathfrak{J}$ for some $\sigma \in \mathbb{S}_n$, and with $u_{a_{1,k}} = e_{a_{1,k}} + \overline{e_{a_{1,k}}}$ if $a_{1,k} \in I \cup \overline{I}$ or $u_{a_{1,k}} = \mathbb{V}$ if $a_{1,k} \in \{v, \overline{v}\}$. In the second possibility, we get by Eq. 11 and the μ-quasi-multiplicativity of $\mathfrak{B}$ that $e_{j_1} \in \mathbb{F}\langle e_{i_0}, u_{a_{1,2}}, \ldots, u_{a_{1,n}}\rangle_\sigma \subset \mathfrak{J}$ for some $\sigma \in \mathbb{S}_n$, where $u_{a_{1,k}} = e_{a_{1,k}}$ if $a_{1,k} \in I$ or $u_{a_{1,k}} = \mathbb{V}$ if $a_{1,k} = v$.

Hence we can assert

$$\bigoplus_{j \in \phi(\{i_0\}, a_{1,2}, \ldots, a_{1,n}) \cap I} \mathbb{F}e_j \subset \mathfrak{J}. \tag{13}$$

Since

$$\phi(\phi(\{i_0\}, a_{1,2}, \ldots, a_{1,n}), a_{2,2}, \ldots, a_{2,n}) \cap I \neq \emptyset,$$

we can argue as above, taking into account Eq. 13, to get

$$\bigoplus_{j \in \phi(\phi(\{i_0\}, a_{1,2}, \ldots, a_{1,n}), a_{2,2}, \ldots, a_{2,n}) \cap I} \mathbb{F}e_j \subset \mathfrak{J}.$$

By reiterating this process with the connection (12) we obtain

$$\bigoplus_{j \in \phi(\phi(\ldots\phi(\{i_0\}, a_{1,2}, \ldots, a_{1,n}), \ldots), a_{t,2}, \ldots, a_{t,n}) \cap I} \mathbb{F}e_j \subset \mathfrak{J}.$$

Taking now into account $i \in \phi(\phi(\ldots\phi(\{i_0\}, a_{1,2}, \ldots, a_{1,n}), \ldots), a_{t,2}, \ldots, a_{t,n}) \cap I$ we conclude $e_i \in \mathfrak{J}$ and so

$$\mathbb{W} = \bigoplus_{i \in I} \mathbb{F}e_i \subset \mathfrak{J}. \tag{14}$$

Taking now into account that $\mathbb{V}$ is tight, Eq. 14 allows us to assert

$$\mathbb{V} \subset \mathfrak{J}. \tag{15}$$

Finally, since $\mathfrak{L} = \mathbb{V} \oplus \mathbb{W}$, Eqs. 14 and 15 give us $\mathfrak{J} = \mathfrak{L}$. □

Theorem 28 *Suppose $\mathfrak{L}$ admits a μ-quasi-multiplicative basis. If $\mathfrak{L}$ is centerless and with $\mathbb{V}$ tight then*

$$\mathfrak{L} = \bigoplus_k \mathfrak{J}_k$$

is the direct sum of the family of its minimal color gLt-ideals, each one admitting a μ-quasi-multiplicative basis inherited by the one of $\mathfrak{L}$.

Proof By Corollary 24 we have that $\mathfrak{L} = \bigoplus_{[i]\in I/\sim} \mathfrak{J}_{[i]}$ is the direct sum of the color gLt-ideals $\mathfrak{J}_{[i]}$.

We wish to apply Theorem 27 to any $\mathfrak{J}_{[i]}$, so we have to verify that

$$\mathfrak{J}_{[i]} = \mathbb{V}_{[i]} \oplus \mathbb{W}_{[i]}$$

admits a μ-quasi-multiplicative basis, $\mathbb{V}_{[i]}$ is tight and the basis $\{e_i \ : \ i \ \in \ [i]\}$ of $\mathbb{W}_{[i]}$ satisfies that all of the elements in the index set $[i]$ are $[i]$-connected (connected through connections contained in $([i]\dot{\cup}v)\dot{\cup}(\overline{[i]}\dot{\cup}\overline{v})$).

We clearly have that $\mathfrak{J}_{[i]}$ admits a μ-quasi-multiplicative basis as consequence of having a basis inherited from the one of $\mathfrak{L}$ and that the linear space $\mathbb{V}_{[i]}$ is tight by construction.

Finally, since it is easy to verify that $[i]$ has all of its elements $[i]$-connected we can apply Theorem 27 to any $\mathfrak{J}_{[i]}$ so as to conclude $\mathfrak{J}_{[i]}$ is minimal. It is clear that the decomposition $\mathfrak{L} = \bigoplus_{[i]\in I/\sim} \mathfrak{J}_{[i]}$ satisfies the assertions of the theorem. □

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