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Symmetries and Connected Components of the AR-quiver of a Gorenstein Local Ring

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Abstract

Let $(A, \mathfrak{m})$ be a commutative complete equi-characteristic Gorenstein isolated singularity of dimension d with $k = A/\mathfrak{m}$ algebraically closed. Let $\Gamma(A)$ be the AR (Auslander-Reiten) quiver of A. Let $\mathcal{P}$ be a property of maximal Cohen-Macaulay A-modules. We show that some naturally defined properties $\mathcal{P}$ define a union of connected components of $\Gamma(A)$. So in this case if there is a maximal Cohen-Macaulay module satisfying $\mathcal{P}$ and if A is not of finite representation type then there exists a family $\{M_n\}_{n\geq 1}$ of maximal Cohen-Macaulay indecomposable modules satisfying $\mathcal{P}$ with multiplicity $e(M_n) > n$. Let $\underline{\Gamma}(A)$ be the stable quiver. We show that there are many symmetries in $\underline{\Gamma}(A)$. As an application we show that if $(A, \mathfrak{m})$ is a two dimensional Gorenstein isolated singularity with multiplicity $e(A) \geq 3$ then for all $n \geq 1$ there exists an indecomposable self-dual maximal Cohen-Macaulay A-module of rank n.

Keywords Artin-reiten quiver · Hensel rings · Indecomposable modules · Ulrich modules · Periodic modules · Non-periodic modules with bounded betti numbers

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1 Introduction

Let us recall that a commutative Noetherian local ring $(A, \mathfrak{m})$ is called an isolated singularity if A_P is a regular local ring for all prime ideals $P \neq \mathfrak{m}$. We note that with this definition if A is Artinian local then it is an isolated singularity. This is not a usual practice, nevertheless in this paper Artin rings will be considered as isolated singularities. Also recall that if a local Noetherian ring $(B, \mathfrak{n})$ is Henselian then it satisfies Krull-Schmidt property, i.e., every finitely generated B-module is uniquely a direct sum of indecomposable B-modules. Now

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assume that B is Cohen-Macaulay with a canonical module K_B. Then we say B is of finite (Cohen-Macaulay) representation type if B has only finitely many indecomposable maximal Cohen-Macaulay B-modules. In a remarkable paper, [1], Auslander proved that in this case B is an isolated singularity (also see [25, Theorem 4.22]). Note that if A is not necessarily Henselian but of finite CM representation type then also by a result of Huneke and Leuschke we get that A has isolated singularities [19, Corollary 2].

To study (not necessarily commutative) Artin algebra's Auslander and Reiten introduced the theory of almost-split sequences. These are now called AR-sequences. The AR-sequences are organized to form the AR-quiver. Later Auslander and Reiten extended the theory of AR-sequences to the case of commutative Henselian isolated singularities.

If $(A, \mathfrak{m})$ is a Henselian Cohen-Macaulay isolated singularity then we denote its AR-quiver by $\Gamma(A)$. A good reference for this topic is [25]. The motivation for this paper comes from the following crucial fact about AR quivers (under some conditions on A), see [25, 6.2]:

If $\mathcal{C}$ is a *non-empty* connected component of $\Gamma(A)$ and if A is not of finite representation type then there exist a family $\{M_n\}_{n\geq 1}$ of maximal Cohen-Macaulay indecomposable modules in $\mathcal{C}$ with multiplicity $e(M_n) > n$.

Let $\mathcal{P}$ be a property of maximal Cohen-Macaulay A-modules. We show that some naturally defined properties $\mathcal{P}$ define a union of connected components of the AR quiver of A. Thus the above mentioned observation still holds. Therefore if there is a maximal Cohen-Macaulay module satisfying $\mathcal{P}$ then there exists a family $\{M_n\}_{n\geq 1}$ of maximal Cohen-Macaulay indecomposable modules satisfying $\mathcal{P}$ with multiplicity $e(M_n) > n$.

1.1 Our Assumptions on the Ring

For the rest of the paper let us assume that $(A, \mathfrak{m})$ is a complete equi-characteristic Gorenstein isolated singularity of dimension d. Assume $k = A/\mathfrak{m}$ is algebraically closed. Some of our results are applicable more generally. However for simplicity we will make this hypothesis throughout this paper. We will also assume that A does not have finite representation type. This is automatic if A is not a hypersurface, see [25, 8.15]. Furthermore if A is a hypersurface ring with $\dim A \geq 2$ and $e(A) \geq 3$ then also A is not of finite representation type, [25, 8.1 and 8.10].

Now we describe our results. We first describe our results on
Connected Components of the AR-quiver:

I *Modules with periodic resolution:*
Let M be a maximal Cohen-Macaulay *non-free* A-module. Let $\operatorname{Syz}_n(M)$ be the n^{th}-syzygy module of M. We say M has periodic resolution if there exists a non-negative integer n and a positive integer p with $\operatorname{Syz}_{n+p}(M) \cong \operatorname{Syz}_n(M)$. The smallest p for which this holds is called the *period* of the resolution. We say M has property $\mathcal{H}$ if it has a periodic resolution.

If A is a hypersurface ring then any non-free maximal Cohen-Macaulay A-module has periodic resolution with period ≤ 2 and in fact $\operatorname{Syz}_3(M) \cong \operatorname{Syz}_1(M)$. There exists maximal Cohen-Macaulay modules with periodic resolutions if A is a complete intersection of any codimension $c \geq 1$. Again it can be shown that in this case the period is ≤ 2 and in fact $\operatorname{Syz}_3(M) \cong \operatorname{Syz}_1(M)$.

For general Gorenstein local rings there is no convenient criterion to determine when A has a module with periodic resolution (however see [12, 5.8] for a criterion). It was conjectured by Eisenbud that if a module M has a periodic resolution then the period is ≤ 2, see [14, p. 37]. This was disproved by Gasharov and Peeva, see [15, Theorem 1.3]. Our first result is

Theorem 1.2 *(with hypotheses as in 1.1.)* $\mathcal{H}$ *defines a union of connected components of* $\Gamma(A)$.

Remark 1.3 In the theory of (not necessarily commutative) Artin Algebra's a module M is said to be DTr periodic if $(DTr)^n(M) = M$ for some $n \geq 1$. For symmetric algebras (in particular for Artinian Gorenstein local rings) we have $DTr(M) = \mathrm{Syz}_2(M)$. By a result of Happel, Preiser and Ringel (cf. [8, 4.16.2]) it is known that if $\mathcal{C}$ is a connected component of the stable quiver containing a DTr periodic module then every module in $\mathcal{C}$ is DTr periodic. Furthermore if $\mathcal{C}$ is infinite then it is a tube. Note that this result *doe not* imply our result even in the case of Gorenstein Artin local rings, because of the following reason:

Suppose $(A, \mathfrak{m})$ is a Gorenstein Artin local ring.

Our results imply that periodic modules define union of connected components of the whole AR-quiver and not just the stable quiver. We need this added information as we need to show that existence of a single periodic module implies existence of indecomposable periodic modules $\{M_n\}_{n\geq 1}$ such that $e(M_n) > n$.

We now give more refined versions of Theorem 1.2:

Assume $A = Q/(f)$ where $(Q, \mathfrak{n})$ is a Gorenstein local ring and $f \in \mathfrak{n}^2$ is a Q-regular element. Let M be a maximal Cohen-Macaulay *non-free* A-module. We say M has property $\mathcal{P}_Q$ if $\operatorname{projdim}_Q M$ finite. In this case it is easy to prove that M has a periodic-resolution over A with period ≤ 2. There is essentially a unique method to construct non-free maximal Cohen-Macaulay modules over A having finite projective dimension over Q. This is essentially due to Buchweitz *et al,* see [11, 2.3]. Also see the paper [18, 1.2] by Herzog *et al.* Our next result is:

Theorem 1.4 *[with hypotheses as in 1.4.]* $\mathcal{P}_Q$ *defines a union of connected components of* $\Gamma(A)$.

Again our results implies existence of indecomposable maximal Cohen-Macaulay A-modules with arbitrarily high multiplicity and satisfying property $\mathcal{P}_Q$. However our method does not give a way to construct these modules.

Eisenbud's conjecture (as stated above) is valid if M has the so-called finite CI-dimension [6, 7.2]. We say M has property $\mathcal{P}_O$ if M has finite CI-dimension over A and has a periodic resolution over A. We say M has property $\mathcal{P}_E$ if M has periodic resolution over A but it has infinite CI dimension over A. Our next result is

Theorem 1.5 *[with hypotheses as in 1.1.]*

(1) $\mathcal{P}_O$ *defines a union of connected components of* $\Gamma(A)$.
(2) $\mathcal{P}_E$ *defines a union of connected components of* $\Gamma(A)$.

We note that in [15, 3.1] a family A_α of an Artininian Gorenstein local ring is constructed with each having a *single* module M_α having periodic resolution of period > 2 is given. As the period of M_α is greater than two it cannot have finite CI-dimension over A_α. Thus our result implies existence of indecomposable modules with arbitrary length, having a periodic resolution and having infinite CI dimension over A_α.

Note that till now our results does not give any information regarding period's. In dimension two we can say something, see Theorem 1.13.

Now assume that A is a complete intersection of codimension $c \geq 2$. There is a theory of support varieties for modules over A. Essentially for every finitely generated module E over A an algebraic set $V(E)$ in the projective space $\mathbb{P}^{c-1}$ is attached, see [4, 6.2]. Conversely it is known that if V is an algebraic set in $\mathbb{P}^{c-1}$ then there exists a finitely generated module E with $V(E) = V$, see [9, 2.3]. It is known that $V(\mathrm{Syz}_n(E)) = V(E)$ for any $n \geq 0$. Thus we can assume E is maximal Cohen-Macaulay. If E has periodic resolution over A then $V(E)$ is a finite set of points. The converse is also true, see [7, Theorem II]. If further E is indecomposable then $V(E)$ is a singleton set, see [9, 3.2]. Let $a \in \mathbb{P}^{c-1}$. We say a maximal Cohen-Macaulay A-module M satisfies property $\mathcal{P}_a$ if $V(M) = \{a\}$. We prove:

Theorem 1.6 *[with hypotheses as in 1.8.] Let $a \in \mathbb{P}^{c-1}$. Then $\mathcal{P}_a$ defines a union of connected components of $\Gamma(A)$. Conversely if $\mathcal{C}$ is a non-empty connected component of $\Gamma(A)$ containing a periodic module M then for any $[N] \in \mathcal{C}$ we have $V(N) = V(M)(= \{p\})$. In particular $\Gamma(A)$ has at least $|k|$ connected components.*

II *Modules with bounded betti-numbers but not having a periodic resolution:*
For a long time it was believed that if a module M has a bounded resolution (i.e., there exists c with $\beta_i(M) \leq c$ for all $i \geq 0$) then it is periodic. If A is a complete intersection then modules having bounded resolutions are periodic [14, 4.1]. In [15, 3.2] there are examples of modules M having a bounded resolution but M is not periodic.

If M is a maximal Cohen-Macaulay A-module having a bounded resolution but M is not periodic then we say that M has property $\mathcal{B}_{NP}$. We prove

Theorem 1.7 *[with hypotheses as in 1.1.] $\mathcal{B}_{NP}$ defines a union of connected components of $\Gamma(A)$.*

We note that if M has bounded resolution but not periodic then there exists c with $e(\mathrm{Syz}_n(M)) \leq c$ for all $n \geq 0$. Our result implies the existence of modules with bounded but not periodic resolution of arbitrary multiplicity.

III: *Ulrich modules:*
Let M be a maximal Cohen-Macaulay A-module. It is well-known that $e(M) \geq \mu(M)$ (here $\mu(M)$ denotes the cardinality of a minimal generating set of M). A maximal Cohen-Macaulay module M is said to be an Ulrich module if its multiplicity $e(M) = \mu(M)$. In this case we say M has property $\mathcal{U}$.

If $\dim A = 1$ then A has a Ulrich module. It is known that if A is a strict complete intersection (i.e, the associated graded ring of A is a complete intersection) of any dimension d then it has a Ulrich module, see [18, 2.5]. In particular if A is a hypersurface ring then it has a Ulrich module. There are some broad class of examples of Gorenstein normal domain (but not complete intersections) of dimension two that admit an Ulrich module see [10, 4.8]. However there are no examples of Gorenstein local rings R (but not complete intersections) with $\dim R \geq 3$ such that R admits an Ulrich module (note we are not even insisting that R is reduced).

Even if A is a hypersurface there is essentially a unique way to construct an Ulrich modules. We show

Theorem 1.8 *[with hypotheses as in 1.1.] Further assume that either A is a hypersurface ring of even dimension $d \geq 2$ and multiplicity $e(A) \geq 3$ OR A is Gorenstein of dimension two. Then $\mathcal{U}$ defines a union of connected components of $\Gamma(A)$.*

The reason we cannot say anything about Ulrich modules over hypersurface rings of *odd* dimension is due to a peculiar nature of AR-sequences, see remark 8.1. Also note that if $e(A) = 2$ then any non-free MCM A-module is an Ulrich module.

We now describe our result on:

IV: *Symmetries of the AR-quiver:*
Let $\Gamma_0(A)$ be the connected component of $\Gamma(A)$ containing the vertex $[A]$. Set $\widetilde{\Gamma}(A) = \Gamma(A) \setminus \Gamma_0(A)$. Let $\underline{\Gamma}(A)$ denote the stable AR-quiver of A, i.e., we delete the vertex $[A]$ from $\Gamma(A)$ and all arrows connecting to $[A]$. Also set $\underline{\Gamma}_0(A)$ to be the stable part of $\Gamma_0(A)$.

Our starting point is the observation that for simple singularities $\underline{\Gamma}(A)$ is trivially isomorphic to its reverse graph (see [25, p. 95]). Recall if G is a directed graph then it's reverse graph G^{rev} is a graph with the same vertices as G and there is an arrow from vertex u to v in G^{rev} if and only if there is an arrow from vertex v to u in G. In fact we construct

Theorem 1.9 *[with hypotheses as in 1.1.] There exists isomorphisms $D, \lambda \colon \underline{\Gamma}(A) \to \underline{\Gamma}(A)^{rev}$ as graphs. If A is not a hypersurface ring then*

(1) $D \neq \lambda$.
(2) *There exists indecomposable maximal Cohen-Macaulay modules M, N with $\lambda(M) \neq M$ and $D(N) \neq N$.*

The first isomorphism D is just the dual functor i.e., $D(M) = \operatorname{Hom}_A(M, A)$. The next map λ arises in the theory of *horizontal linkage* defined by Martsinkovsky and Strooker, see [20, p. 592]. See 9.1 for the definition of λ. We note that the assumption A not a hypersurface is essential for the later part of Theorem 1.9, for in the case of simple singularities it is known that $\lambda(M) = M$ for each non-free indecomposable M, see [20, Theorem 3].

For $n \geq 0$ let Syz_n be the n^{th} syzygy functor. As A is Gorenstein we can also define for integers $n \leq -1$ the n^{th} cosyzygy functor (for maximal Cohen-Macaulaymodules) which we again denote with Syz_n. By the definition of horizontal linkage we have $\operatorname{Syz}_{-1} \circ D = \lambda$. Thus $\operatorname{Syz}_{-1} = \lambda \circ D^{-1}$ and $\operatorname{Syz}_1 = D \circ \lambda^{-1}$. So under the assumptions as in 1.1 we get that $\operatorname{Syz}_n \colon \underline{\Gamma}(A) \to \underline{\Gamma}(A)$ is an isomorphism of *graphs* for all $n \in \mathbb{Z}$.

Remark 1.10 In 9.1 we have shown that using D, Syz_1 we can define action of the infinite Dihedral group on $\underline{\Gamma}(A)$ and its connected components.

We prove:

Theorem 1.11 *[with hypotheses as in 1.1] Let $\mathcal{C}$ be a connected component of $\underline{\Gamma}(A)$. For $[M] \in \mathcal{C}$, set $I(M) = \{n \mid [\operatorname{Syz}_n(M)] \in \mathcal{C}\}$. Then*

(1) *$I(M)$ is an ideal in $\mathbb{Z}$ (possibly zero).*
(2) *$I(N) = I(M)$ for all $[N] \in \mathcal{C}$.*

If A is not of finite representation type then there is practically no information on connected components of $\underline{\Gamma}(A)$. The only case known is when A is a hypersurface there is information on connected components of $\widetilde{\Gamma}(A)$, see [13, Theorem I]. It is easy to show that $\underline{\Gamma}_0(A)$ has only finitely many components. As an application of Theorem 1.11 we show:

Corollary 1.12 *[with hypotheses as in 1.1.] Assume further that A is not a hypersurface ring. Let $\mathcal{D}$ be a connected component of $\underline{\Gamma}_0(A)$. Then*

(1) *$\mathcal{D}$ has infinitely many vertices.*
(2) *The function $[M] \to e(M)$ is unbounded on $\mathcal{D}$.*

V: *Structure of the AR-quiver:*
If A is of finite representation type then the structure of the AR-quiver is known, see [25]. For hypersurface rings which are not of finite representation type there is some information regarding connected components of A not containing the vertex $[A]$. For two dimensional Gorenstein rings we show:

Theorem 1.13 *[with hypothesis as in 1.1.] Assume $\dim A = 2$ and $e(A) \geq 3$. Let $\mathcal{C}$ be a non-empty component of $\Gamma(A)$. Then $\mathcal{C}$ is of the form*

$$M_1 \leftrightarrows M_2 \leftrightarrows M_3 \leftrightarrows M_4 \leftrightarrows \cdots \leftrightarrows M_n \leftrightarrows \cdots$$

where $e(M_n) = ne(M_1)$ for all $n \geq 1$. Furthermore

(1) *If $C = \Gamma_0(A)$ then $M_1 = A$. Furthermore $M_n^* \cong M_n$ for all $n \geq 1$.*
(2) *Assume now that A is not a hypersurface ring. Then*
   (a) *If M_j is periodic with period c for some j then M_n is periodic with period c for all $n \geq 1$.*
   (b) *Let $\underline{\mathcal{C}}$ denote the stable part of $\mathcal{C}$. Let $[M_i] \in \underline{\mathcal{C}}$. If the Poincare series of M_i is rational then the Poincare series of M is rational for all $[M] \in \underline{\mathcal{C}}$. Furthermore all of them share a common denominator.*

In the Theorem above the Poincare series $P_M(z)$ of a module M is $\sum_{n\geq 0} \dim_k \operatorname{Tor}_n^A(M, k)z^n$. We also note that the structure of all components of $\Gamma(A) \setminus \Gamma_0(A)$ is already known, see [13, Theorem 17].

We have several interesting consequences of Theorem 1.13. A direct consequence of this theorem is that if $\dim A = 2$ and $e(A) \geq 3$ then for all $n \geq 1$ there exist's an indecomposable maximal Cohen-Macaulay A-module M_n of rank n with $M_n^* \cong M_n$. I do not know whether such a result holds for higher dimensional rings.

A simple consequence of Theorem's 1.17 and 1.15 is the following:

Corollary 1.14 *[with hypotheses as in 1.1.] Assume A is not a hypersurface ring. Also assume $\dim A = 2$. Then $\operatorname{Syz}_n(\underline{\Gamma}_0(A))$ are distinct components of $\underline{\Gamma}(A)$ for all $n \in \mathbb{Z}$.*

We now describe in brief the contents of this paper. In section two we discuss some preliminary results that we need. In section three we discuss lifts of irreducible maps. In the next section we discuss non-free indecomposable summands of maximal Cohen-Macaulay approximation of the maximal ideal. In section five we give proof's of Theorem's 1.2, 1.4 and 1.6. In the next section we give a proof of Theorem 1.8. In section seven we discuss our notion of quasi AR-sequences and in the next section we give a proof of Theorem 1.7. In section nine we prove Theorem 1.8. In the next section we give a few obstructions to existence of quasi-AR sequences. In section 11 we describe the describe $\Gamma_0(A)$ when A is a two dimensional with $e(A) \geq 3$. In section twelve we give a proof of Theorem 1.11 and Corollary 1.12. In the last section we discuss curvature and complexity of MCM modules and as an application give a proof of Theorem 1.6.

Remark 1.15 Srikanth Iyengar informed me about the excellent paper [16] where the authors considered AR-quiver of self-injective Artin algebra's. Note that commutative Artin Gorenstein rings is an extremely special case of self-injective Artin Algebra's. So our results in this case is sharper than that of [16]. I do not believe that the results of this paper when A is commutative Artin Gorenstein ring will hold for the more general case of self-injective Artin algebra's.

2 Some Preliminaries

In this paper all rings will be Noetherian local. All modules considered are *finitely* generated. Let $(A, \mathfrak{m})$ be a local ring and let $k = A/\mathfrak{m}$ be its residue field. Let $\dim A = d$. If M is an A-module then $\mu(M) = \dim_k M/\mathfrak{m}M$ is the number of a minimal generating set of M. Also let $\ell(M)$ denote its length. In this section we discuss a few preliminary results that we need.

Let M be an A-module. For $i \geq 0$ let $\beta_i(M) = \dim_k \operatorname{Tor}_i^A(M, k)$ be its i^{th} *betti*-number. Let $P_M(z) = \sum_{n\geq 0} \beta_n(M) z^n$, the *Poincare series* of M. Set

$$\operatorname{cx}(M) = \inf\{d \mid \limsup \frac{\beta_n(M)}{n^{d-1}} < \infty\} \quad \text{and}$$

$$\operatorname{curv} M = \limsup(\beta_n(M))^{\frac{1}{n}}.$$

It is possible that $\operatorname{cx}(M) = \infty$, see [5, 4.2.2]. However $\operatorname{curv}(M)$ is finite for any module M [5, 4.1.5]. It can be shown that if $\operatorname{cx}(M) < \infty$ then $\operatorname{curv}(M) \leq 1$.

It can be shown that for any A-module M we have

$$\operatorname{cx}(M) \leq \operatorname{cx}(k) \quad \text{and} \quad \operatorname{curv}(M) \leq \operatorname{curv}(k).$$

see [5, 4.2.4].

If A is a complete intersection of co-dimension c then for any A-module M we have $\operatorname{cx}(M) \leq c$. Furthermore for each $i = 0, \ldots, c$ there exists an A-module M_i with $\operatorname{cx}(M_i) = i$. Also note that $\operatorname{cx}(k) = c$. [5, 8.1.1(2)]. If A is a complete intersection and M is a module with $\operatorname{cx}(M) = \operatorname{cx}(k)$ then we say M is *extremal*.

If A is *not* a complete intersection then $\operatorname{curv}(k) > 1$. [5, 8.2.2]. In this case we say a module M is *extremal* if $\operatorname{curv}(M) = \operatorname{curv}(k)$.

Let M be an A-module and let $\mathbb{F}_M \colon \cdots \to F_n \xrightarrow{d_n} F_{n-1} \to \cdots$ be a minimal resolution of M. Set $\operatorname{Syz}_n(M) = \ker d_n$, the n^{th}-syzygy of M. It is well-defined upto (non-unique) isomorphism.

Let M, N be A-modules and let $f \colon M \to N$ be A-linear. Let $\mathbb{F}_M$ be a minimal resolution of M and let $\mathbb{F}_N$ be a minimal resolution of N. Then f induces a lift $\widetilde{f} \colon \mathbb{F}_M \to \mathbb{F}_N$. This map $\widetilde{f}$ is unique up to homotopy. The chain map $\widetilde{f}$ induces an A-linear map $f_n \colon \operatorname{Syz}_n(M) \to \operatorname{Syz}_n(N)$ for all $n \geq 1$.

Denote by $\beta(M, N)$ the set of A-homomorphisms of M to N which pass through a free module. That is, an A-linear map $f : M \to N$ lies in $\beta(M, N)$ if and only if it factors as

$$\begin{array}{ccc} M & & \\ {\scriptstyle u}\downarrow & \searrow^{f} & \\ F & \xrightarrow[v]{} & N \end{array}$$

where $F \cong A^n$ for some $n \geq 1$.

Remark 2.1 If $f : M \to N$ is A-linear and if $f_1, f_1' : \operatorname{Syz}_1(M) \to \operatorname{Syz}_1(N)$ are two lifts of f then it is well known and easily verified that $f_1 - f_1' \in \beta(\operatorname{Syz}_1(M), \operatorname{Syz}_1(N))$.

Recall that an A-module M is called *stable* if M has no free summands. We need the following:

Proposition 2.2 *Let $(A, \mathfrak{m})$ be a Noetherian local ring and let M, N be A-modules. Set $\Lambda = \operatorname{End}_A(M)$ and $\mathfrak{r} = \operatorname{rad}\Lambda$. Let $f \in \Lambda$. Also suppose there exists A-linear maps $u : M \to N$ and $v : N \to M$. Set $g = v \circ u$.*

(1) *If $f(M) \subseteq \mathfrak{m}M$ then $f \in \mathfrak{r}$.*
(2) *If M is stable and $f \in \beta(M, M)$ then $f \in \mathfrak{r}$.*
(3) *If $1 - g \in \mathfrak{r}$ then u is a split mono and v is a split epi.*

Proof (1) This is well known.
(2) Assume $f = \beta \circ \alpha$ where $\alpha : M \to F$ and $\beta : F \to M$ and $F \cong A^n$. As M is stable it follows that $\alpha(M) \subseteq \mathfrak{m}F$ and so $f(M) \subseteq \mathfrak{m}M$. The result follows from (1).
(3) Let $1 - g = h$ where $h \in \mathfrak{r}$. Then $g = 1 - h$ is invertible in Λ. So there exists $\tau \in \Lambda$ with $\tau \circ g = g \circ \tau = 1_M$. The result follows. □

Let $G(A) = \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ be the associated graded ring of A (with respect to $\mathfrak{m}$). Furthermore if M is an A-module then let $G(M) = \bigoplus_{n \geq 0} \mathfrak{m}^n M/\mathfrak{m}^{n+1} M$ be the associated graded module of M. The Hilbert series of M is the Hilbert series of $G(M)$. Set $H_M(z) = \sum_{n \geq 0} \ell(\mathfrak{m}^n M/\mathfrak{m}^{n+1} M) z^n$. It is well-known that

$$H_M(z) = \frac{h_M(z)}{(1-z)^r};$$

where $r = \dim M$ and $h_M(z) \in \mathbb{Z}[z]$ and $h_M(1) \neq 0$. We call $h_M(z)$ as the *h-polynomial* of M. If f is a polynomial then we use $f^{(i)}$ to denote the i^{th} formal derivative of f. Set

$$e_i(M) = \frac{h_M^{(i)}(1)}{i!}.$$

Then $e_i(M)$ is called the i^{th}-Hilbert coefficient of M (with respect to $\mathfrak{m}$). We denote $e(M) = e_0(M) = h_M(1)$; the multiplicity of M.

An element $x \in \mathfrak{m}$ is said to be *superficial* for M if there exists an integer $c > 0$ such that

$$(\mathfrak{m}^n M :_M x) \cap \mathfrak{m}^c M = \mathfrak{m}^{n-1} M \quad \text{for all} \quad n > c.$$

Superficial elements always exist if k is infinite [24, p. 7]. A sequence $x_1, x_2, \ldots, x_r$ in a local ring $(A, \mathfrak{m})$ is said to be a *superficial sequence* for M if x_1 is superficial for M and x_i is superficial for $M/(x_1, \ldots, x_{i-1})M$ for $2 \leq i \leq r$.

If depth $M > 0$ and x is M-superficial then x is also M-regular (see [24, 2.1] for the case $M = A$; the proof works in general). Set $N = M/xM$. We have an equality of Hilbert coefficients $e_i(N) = e_i(M)$ for $0 \leq i \leq \dim N$; see [21, Corollary 10]).

Let us recall the definition of transpose of a module. Let $F \xrightarrow{f} G \to M \to 0$ be a presentation of M (not necessarily minimal). Then we set $\operatorname{Tr}(M) = \operatorname{coker}(\operatorname{Hom}_A(f, A))$. Note that $\operatorname{Tr}(M)$ depends on the presentation of M. However if $F' \xrightarrow{f'} G' \to M \to 0$ is another presentation of M then it can be shown that there exists free modules U, V such that $U \oplus \operatorname{coker}(\operatorname{Hom}_A(f, A)) \cong V \oplus \operatorname{coker}(\operatorname{Hom}_A(f', A))$. Thus $\operatorname{Tr}(M)$ is well defined upto free summands.

Now assume A is Cohen-Macaulay. Let M, N be maximal Cohen-Macaulay A-modules and let $f : M \to N$ be A-linear. Recall f is said to be *irreducible* if

(1) f is not a split epimorphism and not a split monomorphism.
(2) If X is a maximal Cohen-Macaulay A-module and if there is a commutative diagram

$$\begin{array}{ccc} M & & \\ {\scriptstyle u}\downarrow & \searrow^{f} & \\ X & \xrightarrow[v]{} & N \end{array}$$

then either u is a split monomorphism or v is a split epimorphism.

Remark 2.3 Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring. If M is a maximal Cohen-Macaulay A-module then it is easy to verify that $\operatorname{Syz}_n(M)$ is stable for each $n \geq 1$.

A maximal Cohen-Macaulay module M is said to be *Ulrich* if $e(M) = \mu(M)$. If $x_1, \ldots, x_d$ is a maximal $A \oplus M$-superficial sequence then note that by 2.12 we get $e_0(M) = \ell(M/\mathbf{x}M)$. Thus M is Ulrich if and only if $\mathbf{x}M = \mathfrak{m}M$. It follows that $M/(\mathbf{x})M \cong k^{\mu(M)}$ if M is Ulrich (and conversely).

The following result is well-known. We give a proof for the convenience of the reader.

Proposition 2.4 *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d and with a canonical module ω_A. Assume that the residue field of A is infinite. Let M be a maximal Cohen-Macaulay A-module. Set $M^\dagger = \operatorname{Hom}_A(M, \omega_A)$. If M is Ulrich then so is $M^\dagger$.*

Proof Let $x_1, \ldots, x_d$ be a $A \oplus \omega_A \oplus M \oplus M^\dagger$-superficial sequence. If E is an A-module then set $E_0 = E$ and $E_i = E/(x_1, \ldots, x_i)E$ for $i = 1, \ldots, d$. Note ω_i is a canonical module of A_i. The exact sequence

$$0 \to \omega_0 \xrightarrow{x_1} \omega_0 \to \omega_1 \to 0$$

induces an exact sequence

$$0 \to M_0^\dagger \xrightarrow{x_1} M_0^\dagger \to \operatorname{Hom}_{A_1}(M_1, \omega_1) \to 0,$$

(as $\operatorname{Ext}^1_{A_0}(M_0, \omega_0) = 0$). So in particular we have that $M_0^\dagger/x_1 M_0^\dagger \cong (M_1)^\dagger$. Iterating we get

$$M_0^\dagger/(x_1, \ldots, x_d)M_0^\dagger \cong (M_d)^\dagger.$$

As M is Ulrich we get that $M_d^\dagger \cong k^{\mu(M)}$. As $\operatorname{Hom}_{A_d}(k, \omega_d) \cong k$ it follows that $M_0^\dagger/(x_1, \ldots, x_d)M_0^\dagger$ is a k-vector space. Thus $M^\dagger$ is also Ulrich. □

Now assume that A is a Henselian Cohen-Macaulay isolated singularity of dimension d. Also assume that A has a canonical module ω_A. Then the category of maximal Cohen-Macaulay$A$-modules admits Auslander-Reiten(AR) sequences; see [25, Chapter 3]. Recall an exact sequence $s\colon 0 \to N \to E \xrightarrow{p} M \to 0$ of maximal Cohen-Macaulay A-modules is an AR-sequence if

(1) s is not split.
(2) M, N are indecomposable maximal Cohen-Macaulay A-modules.
(3) If L is a maximal Cohen-Macaulay A-module and if $q\colon L \to M$ is a *not* a split epimorphism then there exists $f\colon L \to E$ such that $q = p \circ f$.

We call s the AR-sequence ending at M (equivalently starting at N). Also note that $N \cong \operatorname{Hom}_A(\operatorname{Syz}_d(\operatorname{Tr}(M), \omega_A)$; see [25, 3.11]. We set $\tau(M) = N$ and call it the Auslander-Reiten translate of M.

Suppose M is a maximal Cohen-Macaulay over a local Gorenstein ring A. Then $M^* = \operatorname{Hom}_A(M, A)$ is also a maximal Cohen-Macaulay module. Furthermore $\operatorname{Ext}^i_A(M, A) = 0$ for $i > 0$. We also have $(M^*)^* \cong M$. Notice if

$$0 \to M_1 \xrightarrow{\alpha_1} M_2 \xrightarrow{\alpha_2} M_3 \to 0,$$

is a short exact sequence of maximal Cohen-Macaulay A-modules then we have the following short exact sequence

$$0 \to M_3^* \xrightarrow{\alpha_2^*} M_2^* \xrightarrow{\alpha_1^*} M_1^* \to 0,$$

of maximal Cohen-Macaulay A-modules.

Remark 2.5 Let $(A, \mathfrak{m})$ be a Gorenstein local ring and let M be a maximal Cohen-Macaulay A-module. Let $0 \to N \to G \to M^* \to 0$ be a minimal presentation of M^*. As $M \cong M^{**}$ we have a short exact sequence $0 \to M \to G^* \to N^* \to 0$. Set $\operatorname{Syz}_{-1}(M) = N^*$. Iteratively define $\operatorname{Syz}_{-n}(M) = \operatorname{Syz}_{-1}(\operatorname{Syz}_{-n+1}(M))$ for all $n \geq 1$. For $m \leq -1$ we call $\operatorname{Syz}_m(M)$ as the m^{th} co-syzygy of M.

Remark 2.6 Due to 2.19 the following assertions hold:
Let M_1, M_2 and N_1, N_2 are maximal Cohen-Macaulay A-modules and let F, G be free A-modules. Suppose we have exact sequences:

$$0 \to M_1 \to F \to M_2 \to 0 \quad \text{and} \quad 0 \to N_1 \to G \to N_2 \to 0.$$

If there exists an A-linear map $\psi_1\colon M_1 \to N_1$ then:

(1) there exists A-linear maps $\psi_2\colon M_2 \to N_2$ and $\phi\colon F \to G$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_1 & \longrightarrow & F & \longrightarrow & M_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi_2} & & \\
0 & \longrightarrow & N_1 & \longrightarrow & G & \longrightarrow & N_2 & \longrightarrow & 0
\end{array}$$

(2) If $\psi'\colon M_2 \to N_2$ and $\phi'\colon F \to G$ are another pair of maps such that the above commutative diagram holds then $\psi_2 - \psi_2' \in \beta(M_2, N_2)$.

Definition 2.7 (with hypotheses as in 2.6.) We call ψ_2 to be a *co-lift* of ψ_1.

The following is an easy consequence of 2.19.

Proposition 2.8 *Let* $(A, \mathfrak{m})$ *be a local Gorenstein ring and let* M, N *be maximal Cohen-Macaulay* A*-modules. Let* $f: M \to N$ *be* A*-linear and let* $f^*: N^* \to M^*$ *be the induced map. We have:*

(1) *f is a split mono if and only if f^* is a split epi.*
(2) *f is a split epi if and only if f^* is a split mono.*
(3) *f is irreducible if and only if f^* is irreducible.*

Let $(A, \mathfrak{m})$ be a Gorenstein local ring. Let us recall the definition of Cohen-Macaulay approximation from [2]. A Cohen-Macaulay approximation of an A-module M is an exact sequence

$$0 \longrightarrow Y \longrightarrow X \longrightarrow M \longrightarrow 0,$$

where X is a maximal Cohen-Macaulay A-module and Y has finite projective dimension. Such a sequence is not unique but X is known to unique up to a free summand and so is well defined in the stable category $\underline{\mathrm{CM}}(A)$. We denote by $X(M)$ the maximal Cohen-Macaulay approximation of M.

If M is Cohen-Macaulay then maximal Cohen-Macaulay approximation of M are very easy to construct. We recall this construction from [2]. Let $n = \operatorname{codim} M = \dim A - \dim M$. Let $M^\vee = \operatorname{Ext}^n_A(M, A)$. It is well-known that $M^\vee$ is Cohen-Macaulay module of codim n and $M^{\vee\vee} \cong M$. Let $\mathbb{F}$ be any free resolution of $M^\vee$ with each $\mathbb{F}_i$ a finitely generated free module. Note $\mathbb{F}$ need not be minimal free resolution of M. Set $S_n(\mathbb{F}) = \operatorname{image}(\mathbb{F}_n \xrightarrow{\partial_n} \mathbb{F}_{n-1})$. Then note $S_n(\mathbb{F})$ is maximal Cohen-Macaulay A-module. It can be easily proved that $X(M) \cong S_n(\mathbb{F})^*$ in $\underline{\mathrm{CM}}(A)$. The following result is well-known. We give a proof for the convenience of the reader.

Proposition 2.9 *Let* $(A, \mathfrak{m})$ *be a Gorenstein local ring and let* M *be an* A*-module. Then*

$$X(\operatorname{Syz}_1(M)) \cong \operatorname{Syz}_1(X(M)) \quad \textit{in } \underline{\mathrm{CM}}(A).$$

Proof Let $0 \to Y \to X(M) \to M \to 0$ be a MCM approximation of M. Let $\alpha: G \to Y$ and $\gamma: F \to M$ be projective covers of Y and M respectively. Then we have a following diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G & \longrightarrow & U & \longrightarrow & F & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & Y & \longrightarrow & X(M) & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

Note U is free and β is surjective. So we have an exact sequence

$$0 \to \operatorname{Syz}_1(Y) \to \operatorname{Syz}_1(X(M)) \oplus H \to \operatorname{Syz}_1(M) \to 0;$$

where H is free (possibly zero). We note that $\operatorname{Syz}_1(Y)$ has finite projective dimension and $\operatorname{Syz}_1(X(M))$ is maximal Cohen-Macaulay. The result follows. □

Now assume $(A, \mathfrak{m})$ is a Henselian Gorenstein isolated singularity. Let M be a maximal Cohen-Macaulay, indecomposable, non-free A-module. Let $\tau(M) = \operatorname{Hom}(\operatorname{Syz}_d(\operatorname{Tr}(M), A)$ be the Auslander-Reiten translate of M. Then it is well known that $\tau(M) \cong \operatorname{Syz}_{2-d}(M)$. We give a short proof for the convenience of the reader. Let $F \xrightarrow{f} G \to M \to 0$ be a minimal presentation of M. Then note that we can choose $\operatorname{Tr}(M) = (\operatorname{Syz}_2(M))^*$. Notice that

$$\operatorname{Syz}_d(\operatorname{Tr} M) = \operatorname{Syz}_d((\operatorname{Syz}_2(M))^*) \cong (\operatorname{Syz}_{2-d}(M))^*.$$

It follows that $\tau(M) = \operatorname{Syz}_{2-d}(M)$.

3 Lifts of Irreducible Maps

In this section $(A, \mathfrak{m})$ is a Gorenstein local ring, not necessarily an isolated singularity. Also A need not be Henselian. The following is the main result of this section.

Theorem 3.1 *(with hypotheses as above.) Let M, N be stable maximal Cohen-Macaulay A-modules and let $f: M \to N$ be A-linear. Let $f_1: \operatorname{Syz}_1(M) \to \operatorname{Syz}_1(N)$ be any lift of f. If f is irreducible then f_1 is also an irreducible map.*

We need a few preliminaries to prove Theorem 3.1.
We first prove:

Lemma 3.2 *Let $(A, \mathfrak{m})$ be a local Gorenstein ring and let M, N be stable maximal Cohen-Macaulay A-modules. Let $f: M \to N$ be A-linear and let $\delta \in \beta(M, N)$. If f is irreducible then so is $f + \delta$.*

Proof Assume $\delta = v \circ u$ where $u: M \to F$ and $v: F \to N$ and $F \cong A^n$. As M is stable it follows that $u(M) \subseteq \mathfrak{m}F$ and so $\delta(M) \subseteq \mathfrak{m}N$.

Claim-1: $f + \delta$ is not a split mono.
Suppose it is so. Then there exists $\sigma: N \to M$ with $\sigma \circ (f + \delta) = 1_M$. So $\sigma \circ f + \sigma \circ \delta = 1_M$. As $\delta(M) \subseteq \mathfrak{m}N$ we get $\sigma \circ \delta(M) \subseteq \mathfrak{m}M$. It follows that $\sigma \circ \delta \in \operatorname{rad}\operatorname{End}_A(M)$. By 2.2(3) it follows that f is a split mono. This is a contradiction as f is irreducible.

Claim-2: $f + \delta$ is not a split epi.
Suppose it is so. Then there exists $\sigma: N \to M$ with $(f + \delta) \circ \sigma = 1_N$. So $f \circ \sigma + \delta \circ \sigma = 1_N$. Notice

$$\delta \circ \sigma(N) \subset \delta(M) \subseteq \mathfrak{m}N.$$

It follows that $\delta \circ \sigma \in \operatorname{rad}\operatorname{End}_A(N)$. By 2.2(3) it follows that f is a split epimorphism. This is a contradiction as f is irreducible.

Claim-3: Suppose X is maximal Cohen-Macaulay and we have a commutative diagram

$$\begin{array}{ccc} M & & \\ {\scriptstyle g}\downarrow & \searrow^{f+\delta} & \\ X & \xrightarrow[h]{} & N \end{array}$$

then either g is a split monomorphism or h is a split epimorphism.

Proof of Claim 3: Notice we have a commutative diagram

$$\begin{array}{ccc} M & & \\ {\scriptstyle (g,-u)}\downarrow & \searrow^{f} & \\ X \oplus F & \xrightarrow[h+v]{} & N \end{array}$$

As f is irreducible either $(g, -u)$ is a split mono or $h + v$ is a split epi. We assert:

Subclaim-1: If $(g, -u)$ is a split mono then g is a split mono.

Subclaim-2: If $h + v$ is a split epi then h is a split epi.
Notice that Subclaim 1 and 2 will finish the proof of Claim 3. Also Claims 1,2,3 implies the assertion of the Lemma. We now give:

Proof of Subclaim-1: As $(g, -u)$ is a split mono there exits $\sigma: X \oplus F \to M$ such that $\sigma \circ (g, -u) = 1_M$. Write $\sigma = \sigma_1 + \sigma_2$ where $\sigma_1: X \to M$ and $\sigma_2: F \to M$. Thus we have $\sigma_1 \circ g - \sigma_2 \circ u = 1_M$. As M is stable $u(M) \subseteq \mathfrak{m}F$. So $\sigma_2 \circ u(M) \subseteq \mathfrak{m}M$. Thus $\sigma_2 \circ u \in \operatorname{rad}\operatorname{End}_A(M)$. It follows from 2.2(3) that g is a split mono.
We now give:

Proof of Subclaim-2: As $h+v$ is a split epi there exists $\sigma: N \to X \oplus F$ with $(h+v)\circ\sigma = 1_N$. Write $\sigma = (\sigma_1, \sigma_2)$ where $\sigma_1: N \to X$ and $\sigma_2: N \to F$. It follows that $h \circ \sigma_1 + v \circ \sigma_2 = 1_N$.

As N is stable $\sigma_2(N) \subseteq \mathfrak{m}F$. So $v \circ \sigma_2(N) \subseteq \mathfrak{m}N$. Thus $v \circ \sigma_2 \in \operatorname{rad}\operatorname{End}_A(N)$. It follows from 2.2(3) that h is a split epi. □

We also need

Lemma 3.3 *Let* $(A, \mathfrak{m})$ *be a Gorenstein local ring and let* M *be a stable maximal Cohen-Macaulay* A*-module. Let* $F = A^{\mu(M)}$ *and let* $\epsilon: F \to M$ *be a minimal map. Set* $M_1 = \ker\epsilon \cong \operatorname{Syz}_1(M)$. *Let* X *be another maximal Cohen-Macaulay* A*-module (not necessarily stable) and let* $\eta: G \to X$ *be a surjective map with* G*-free (not necessarily minimal). Let* $X_1 = \ker\eta$. *Let* $\alpha: M \to X$ *be* A*-linear and let* $\alpha_1: M_1 \to X_1$ *be any lift of* α. *If* α *is a split mono then* α_1 *is a split mono.*

Proof We note that M_1 is also stable. Let $\phi: X \to M$ be such that $\phi \circ \alpha = 1_M$. Let $\phi_1: X_1 \to M_1$ be a lift of ϕ. Then note that $\phi_1 \circ \alpha_1$ is a lift of 1_M. Thus $\phi_1 \circ \alpha_1 - 1 \in \beta(M_1, M_1)$. The result now follows from 2.2. □

The following is a dual version of Lemma 3.3 and can be proved similarly.

Lemma 3.4 *Let* $(A, \mathfrak{m})$ *be a Gorenstein local ring and let* N *be a stable maximal Cohen-Macaulay* A*-module. Let* $F = A^{\mu(N)}$ *and let* $\epsilon: F \to N$ *be a minimal map. Set* $N_1 =$

$\ker \epsilon \cong \mathrm{Syz}_1(N)$. *Let X be another maximal Cohen-Macaulay A-module (not necessarily stable) and let $\eta\colon G \to X$ be a surjective map with G-free (not necessarily minimal). Let $X_1 = \ker \eta$. Let $\beta\colon X \to N$ be A-linear and let $\beta_1\colon X_1 \to N_1$ be any lift of α. If β is a split epi then β_1 is a split epi.* □

We now give:

Proof of Theorem 3.1. Set $M_1 = \mathrm{Syz}_1(M)$ and $N_1 = \mathrm{Syz}_1(N)$.

Claim-1: f_1 is not a split mono.

Suppose if possible f_1 is a split mono. Then there exists $\sigma_1\colon N_1 \to M_1$ with $\sigma_1 \circ f_1 = 1_{M_1}$. Let $\sigma\colon N \to M$ be a co-lift of σ (see 2.7 for this notion). Then $\sigma \circ f$ is a co-lift of 1_{M_1}. Notice 1_M is a co-lift of of 1_{M_1}. Then by 2.6 we get that $1_M - \sigma \circ f \in \beta(M, M)$. As M is stable we get by 2.2 that f is a split mono. This is a contradiction as f is irreducible.

Claim-2: f_1 is not a split epi.

Suppose if possible f_1 is a split epi. Then there exists $\sigma_1\colon M_1 \to N_1$ with $f_1 \circ \sigma_1 = 1_{N_1}$. Let $\sigma\colon N \to M$ be a co-lift of σ. Then $f \circ \sigma$ is a co-lift of 1_{N_1}. Notice 1_N is a co-lift of of 1_{N_1}. Then by 2.6 we get that $1_N - f \circ \sigma \in \beta(N, N)$. As N is stable we get by 2.2 that f is a split epi. This is a contradiction as f is irreducible.

Claim-3: If X_1 is a maximal Cohen-Macaulay A-module and if there is a commutative diagram

$$\begin{array}{ccc} M_1 & & \\ {\scriptstyle u_1}\downarrow & \searrow^{f_1} & \\ X_1 & \xrightarrow[v_1]{} & N_1 \end{array}$$

then either u_1 is a split monomorphism or v_1 is a split epimorphism.

Proof of Claim-3: By 2.5 there exists an exact sequence

$$0 \to X_1 \to L \to X \to 0,$$

with L_1 free and X maximal Cohen-Macaulay.

Let $u\colon M \to X$ be a co-lift of u_1 and let $v\colon X \to N$ be a co-lift of v_1. Then notice $v \circ u$ is a co-lift of $f_1 = v_1 \circ u_1$. As f by definition is a co-lift of f_1 we get that $v \circ u = f + \delta$ for some $\delta \in \beta(M, N)$.

By 3.2 we get that $f + \delta$ is irreducible. So u is a split mono or v is a split epi. By Lemma's 3.3 and 3.4 we get that u_1 is a split mono or v_1 is a split epi.

By Claims 1, 2 and 3 the result follows. □

4 Indecomposable Non-free Summands of Maximal Cohen-Macaulay Approximation of the Maximal Ideal

In this section $(A, \mathfrak{m})$ is a Henselian Gorenstein local ring. Let $X(\mathfrak{m})$ be a maximal Cohen-Macaulay approximation of the maximal ideal. In this section we are concerned with non-free indecomposable summands of $X(\mathfrak{m})$. Our results are:

Theorem 4.1 *Let $(A, \mathfrak{m})$ be a Henselian Gorenstein local ring of dimension d and let $X(\mathfrak{m})$ be a maximal approximation of* $\mathfrak{m}$. *Let M be an indecomposable non-free summand of $X(\mathfrak{m})$. Then M is extremal, i.e.,*

(1) *If A is a complete intersection of codimension c then* $\operatorname{cx}(M) = \operatorname{cx}(k) = c$.
(2) *If A is not a complete intersection then* $\operatorname{curv}(M) = \operatorname{curv}(k) > 1$.

We also prove:

Theorem 4.2 *Let $(A, \mathfrak{m})$ be a Henselian Gorenstein local ring of dimension $d \geq 1$ and infinite residue field k. Let $e(A) \geq 3$. Assume either* $\dim A = 2$ *or A is a hypersurface ring (with no restriction on dimension) with multiplicity $e(A) \geq 3$. Let M be an indecomposable Ulrich A-module. Then neither M or* $\operatorname{Syz}_1(M)$ *is a summand of $X(\mathfrak{m})$.*

Remark 4.3 Note that Theorem 4.2 is false if $e(A) = 2$. Note that if $e(A) = 2$ then A is a hypersurface with minimal multiplicity. In this case it is known that any stable maximal Cohen-Macaulay module is Ulrich. For a proof note as M is stable we get that $M = \operatorname{Syz}_1(L)$ for a MCM A-module. Let $0 \to M \to F \to L \to 0$. We go mod $\mathbf{x}$; a maximal $M \oplus A \oplus L$-superficial sequence. Set $\overline{(-)} = - \otimes A/(\mathbf{x})$. We note that $\overline{M} \subseteq \mathfrak{m}\overline{F}$. As A has minimal multiplcity we get that $\overline{\mathfrak{m}}^2 = 0$; see [21, Theorem 16]. So $\overline{\mathfrak{m}}\overline{M} = 0$. It follows that M is Ulrich; see 2.16. So any non-free indecomposable summand of $X(\mathfrak{m})$ is Ulrich.

Motivation: Our motivation to prove the above results is the following: Assume A is a Gorenstein Henselian isolated singularity. If M is a maximal Cohen-Macaulay *non-free* indecomposable module then there exists an irreducible morphism from $M \to A$ only if M is a summand of $X(\mathfrak{m})$, see [25, 4.2.1]. In our Theorems we have to show that the vertex $[A]$ does not belong to certain components of $\Gamma(A)$.

We first give:

Proof of Theorem 4.1 By Proposition 2.9 we get that $X(\mathfrak{m}) \oplus F = \operatorname{Syz}_1(X(k)) \oplus G$ for some free modules F, G. Thus it suffices to prove that if M is a direct summand of $X(k)$ then it is extremal. By 2.25 it suffices to prove that if M is a summand of $\operatorname{Syz}_d(k)^*$ then M is extremal. We prove by induction on d that if M is a summand of $\operatorname{Syz}_n(k)^*$ for some $n \geq d$ then M is extremal.

We first consider the case $d = 0$. We note that as k is indecomposable $\operatorname{Syz}_n(k)$ is indecomposable for all $n \geq 0$ (this is a result of Herzog, cf. [25, 8.17]). So $\operatorname{Syz}_n(k)^*$ is indecomposable. Therefore $M = \operatorname{Syz}_n(k)^*$. Notice $\operatorname{Syz}_n(\operatorname{Syz}_n(k)^*) = k$. It follows that M is extremal.

We now assume that $d \geq 1$ and the result has been proved for Gorenstein Henselian rings of dimension $d-1$. Let $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ be a non-zero divisor on A. Set $B = A/(x)$ and for any A-module N set $\overline{N} = N/xN$. We note that for $n \geq d$

$$\overline{\operatorname{Syz}_n^A(k)} \cong \operatorname{Syz}_n^B(k) \oplus \operatorname{Syz}_{n-1}^B(k).$$

It follows that

$$\overline{\operatorname{Syz}_n^A(k)^*} \cong \operatorname{Syz}_n^B(k)^* \oplus \operatorname{Syz}_{n-1}^B(k)^*.$$

If M is a summand of $\operatorname{Syz}_n^A(k)^*$ then $\overline{M}$ is a summand of $\overline{\operatorname{Syz}_n^A k^*}$. Let E be an irreducible summand of $\overline{M}$. Then by Krull-Schmidt it is an irreducible summand of either $\operatorname{Syz}_n^B(k)^*$ or of $\operatorname{Syz}_{n-1}^B(B)^*$. By induction hypothesis we get that E is extremal. It follows that M is extremal. □

We now give:

Proof of Theorem 4.2

Case 1 We first consider the case when A is a hypersurface of dimension $d \geq 1$ and multiplicity $e(A) \geq 3$.

Let M be an indecomposable Ulrich A-module. Suppose if possible M is a summand of $X(\mathfrak{m})$. By Proposition 2.9 we get that $X(\mathfrak{m}) \oplus F = \mathrm{Syz}_1(X(k)) \oplus G$ for some free modules F, G. It follows that $\mathrm{Syz}_{-1}(M)$ is a summand of $X(k)$. As M has no free summands we get $\mathrm{Syz}_1(M) = \mathrm{Syz}_{-1}(M)$. By 2.25 we get $\mathrm{Syz}_1(M)$ is a summand of $\mathrm{Syz}_d(k)^*$. It follows that $\mathrm{Syz}_1(M)^*$ is a summand of $\mathrm{Syz}_d(k)$. But

$$\mathrm{Syz}_1(M)^* \cong \mathrm{Syz}_{-1}(M^*) \cong \mathrm{Syz}_1(M^*).$$

Notice if M is Ulrich then M^* is also Ulrich; see 2.4. Similarly if we set $N = \mathrm{Syz}_1(M)$ is a summand of $\mathbf{X}(\mathfrak{m})$ then M^* is a summand of $\mathrm{Syz}_d(k)$.

By the arguments in the previous paragraph it suffices to prove that if E is an Ulrich A-module then neither E nor $\mathrm{Syz}_1(E)$ is a summand of $\mathrm{Syz}_d(k)$. This we prove by induction on d.

We first consider the case $d = 1$. Then as $e(A) \geq 3$ we have that $\mathfrak{m} = \mathrm{Syz}_1(k)$ is indecomposable [23, Theorem A].

If E is a summand of $\mathfrak{m}$ then $\mathfrak{m} = E$. Let x be E-superficial. Then as $E = \mathfrak{m}$ is Ulrich we get that $\mathfrak{m}\mathfrak{m} = x\mathfrak{m}$. So $\mathfrak{m}^2 = x\mathfrak{m}$. So A has minimal multiplicity. It follows that $e(A) = 2$. This a contradiction.

If $\mathrm{Syz}_1(E)$ is a summand of $\mathfrak{m}$ then $\mathfrak{m} = \mathrm{Syz}_1(E)$. Using [22, Theorem 2] we get the h-polynomial of $\mathfrak{m}$ is

$$h_{\mathfrak{m}}(z) = \mu(\mathfrak{m})(1 + z + z^2 + \cdots + z^{e-2}) \quad \text{where } e = e(A) \geq 3.$$

Here $\mu(\mathfrak{m})$ denotes the number of minimal generators of $\mathfrak{m}$. Note $\mu(\mathfrak{m}) \geq 2$ as A is not regular. It follows that the h-polynomial of A is

$$h_A(z) = 1 + z(h_{\mathfrak{m}}(z) - 1) = \mu(\mathfrak{m})z^{e-1} + \text{lower terms in } z.$$

This is a contradiction as A is a hypersurface.

Now assume that $d \geq 2$ and the result has been proved for hypersurface rings of dimension $d - 1$. Let $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ be sufficiently general. Then x is A-regular and $A \oplus E \oplus \mathrm{Syz}^A_1(E)$-superficial. Set $B = A/(x)$ and if V is an A-module set $\overline{V} = V/xV$. Then notice

$$\overline{\mathrm{Syz}^A_d(k)} = \mathrm{Syz}^B_d(k) \oplus \mathrm{Syz}^B_{d-1}(k).$$

Note $\overline{E}$ is an Ulrich B-module. Let $\overline{E} = U_1 \oplus U_2 \oplus \cdots \oplus U_s$ where U_i are indecomposable B-modules. Then each U_i is an Ulrich B-module. We also have

$$\overline{\mathrm{Syz}^A_1(E)} \cong \mathrm{Syz}^B_1(\overline{E}) \cong \mathrm{Syz}^B_1(U_1) \oplus \cdots \oplus \mathrm{Syz}^B_1(U_s).$$

By [25, 8.17], $\mathrm{Syz}^B_1(U_i)$ is an indecomposable B-module for $i = 1, \ldots, s$.

If E is a summand of $\mathrm{Syz}^A_d(k)$ then $\overline{E}$ is a summand of $\overline{\mathrm{Syz}^A_d(k)}$. So U_1 is a summand of $\mathrm{Syz}^B_d(k)$ or $\mathrm{Syz}^B_{d-1}(k)$. By our induction hypothesis U_1 is not a summand of $\mathrm{Syz}^B_{d-1}(k)$. It follows that U_1 is a summand of $\mathrm{Syz}^B_d(k)$. Therefore $\mathrm{Syz}^B_1(U_1)$ is a summand of $\mathrm{Syz}^B_{d+1}(k) \cong \mathrm{Syz}^B_{d-1}(k)$, a contradiction. A similar argument will show that $\mathrm{Syz}^A_1(E)$ is not a summand of $\mathrm{Syz}^A_d(k)$.

Case 2 We now consider the case when A is a Gorenstein local ring of dimension 2 but not a hypersurface.
Notice $e(A) \geq 4$. Let M be an indecomposable non-free, maximal Cohen-Macaulay A-module. If M is a summand of $X(\mathfrak{m})$ then $\operatorname{Syz}^A_{-1}(M)$ is a summand of $X(k)$. Notice $\operatorname{Syz}^A_{-1}(M)$ is an indecomposable non-free A-module. Therefore $\operatorname{Syz}^A_{-1}(M)$ is a summand of $\operatorname{Syz}^A_2(k)^*$. By [23, Theorem B], $\operatorname{Syz}^A_2(k)$ is an indecomposable maximal Cohen-Macaulay A-module. It follows that $\operatorname{Syz}^A_{-1}(M) \cong \operatorname{Syz}^A_2(k)^*$. Notice

$$\operatorname{Syz}_1(M^*) \cong \operatorname{Syz}^A_{-1}(M)^* \cong \operatorname{Syz}^A_2(k). \tag{4.4.1}$$

Let x_1, x_2 be a $A \oplus M^* \oplus \operatorname{Syz}_1(M^*)$-superficial sequence. Set $C = A/(x_1, x_2)$ and if E is an A-module, set $\overline{E} = E/(x_1, x_2)E$. We note that

$$\overline{\operatorname{Syz}^A_2(k)} = \operatorname{Syz}^C_2(k) \oplus \operatorname{Syz}^C_1(k)^2 \oplus \operatorname{Syz}^C_0(k). \tag{4.4.2}$$

Set $N = \overline{M^*}$. Then $\overline{\operatorname{Syz}^A_1(M^*)} \cong \operatorname{Syz}^C_1(N)$. We now consider two cases.

Case 1 M is Ulrich.
Then notice M^* is also Ulrich; see 2.4. Then $N = k^a$ where $a = \mu(N)$. Let $\mathfrak{n}$ be the maximal ideal of C. Note it is indecomposable as a C-module. By 4.4.2 and 4.4.1 we get

$$\mathfrak{n}^a = \operatorname{Syz}^C_2(k) \oplus \operatorname{Syz}^C_1(k)^2 \oplus k.$$

By Krull-Schmidt we get that $k = \mathfrak{n}$. So $\mathfrak{n}^2 = 0$. It follows that $e(A) = e(C) = 2$, a contradiction (here the first equality holds by 2.12).

Case 2 $M = \operatorname{Syz}^A_1(D)$ where D is Ulrich.
Then $D = \operatorname{Syz}^A_{-1}(M)$. It follows that $\operatorname{Syz}^A_2(k)^*$ is Ulrich. Therefore $\operatorname{Syz}^A_2(k)$ is also Ulrich. Therefore $\overline{\operatorname{Syz}^A_2(k)} \cong k^l$ for some l. So by 4.4.2 it follows that $\mathfrak{n}$ the maximal ideal of C is isomorphic to k. Therefore $\mathfrak{n}^2 = 0$ and so $e(A) = e(C) = 2$ a contradiction. □

5 Proofs of Theorem's 1.2, 1.5 and 1.7

We first give

Proof of Theorem 1.2 We have nothing to prove if A is a hypersurface ring. So we can consider the case when A is *not* a hypersurface ring. Let M be an indecomposable periodic maximal Cohen-Macaulay A-module. Let N, L be indecomposable non-free maximal Cohen-Macaulay A-modules and assume there exists irreducible maps $u : N \to M$ and $v : M \to L$. Let

$$0 \to \tau(M) \to E_M \to M \to 0, \quad 0 \to M \to V_M \to \tau^{-1}(M) \to 0$$

be the AR-sequences starting and ending at M.

We first note that $\operatorname{Syz}_i(u) : \operatorname{Syz}_i(N) \to \operatorname{Syz}_i(M)$ and $\operatorname{Syz}_i(v) : \operatorname{Syz}_i(M) \to \operatorname{Syz}_i(L)$ are irreducible, see 3.1. Let p be the period of M. Note $\operatorname{Syz}_p(M) \cong M$.

We have irreducible maps $\operatorname{Syz}_{ip} : \operatorname{Syz}_{ip}(N) \to M$ for all $i \geq 1$. As $\operatorname{Syz}_{ip}(N)$ is indecomposable maximal Cohen-Macaulay A-module we get that $\operatorname{Syz}_{ip}(N)$ are factors of E_M for all $i \geq 1$. By Krull-Schmidt theorem we get that N is periodic. A dual argument gives that L is periodic.

If there exists an irreducible map from $M \to A$ then M is factor of $X(\mathfrak{m})$ the maximal Cohen-Macaulay approximation of $\mathfrak{m}$. So by 4.1 we get that M is extremal. As A is not a hypersurface this is a contradiction.

Notice $\tau(M) = \operatorname{Syz}_{-d+2}(M)$ as A is Gorenstein. If there is an irreducible map from $A \to M$ then A is a factor of E_M and so there exit's an irreducible map from $\tau(M) \to A$. By previous argument we get that $\tau(M)$ is extremal. So M is extremal. As A is not a hypersurface this is a contradiction,

Thus $\mathcal{H}$ defines a union of connected components of $\Gamma(A)$. □

Remark 5.1 In Proposition 5.2 of [3] it is shown that if A is a self-injective Artin algebra and M, N are non-projective indecomposable A-modules such that there is an irreducible map $f\colon M \to N$ then M is a periodic module if and only if N is. To the best of my knowledge this proof does not generalize to higher dimensions.

We now give

Proof of Theorem 1.4 There is nothing to prove if Q is regular local. So we can consider the case when Q is not regular. In particular A is not regular. Let M be an indecomposable maximal Cohen-Macaulay A-module such that $\operatorname{projdim}_Q M$ is finite. Then M is a periodic A-module with period ≤ 2. As A is not a hypersurface ring then by proof of previous Theorem there is no irreducible map from $A \to M$ or $M \to A$.

Let N, L be indecomposable non-free maximal Cohen-Macaulay A-modules and assume there exists irreducible maps $u\colon N \to M$ and $v\colon M \to L$. Let

$$0 \to \tau(M) \to E_M \to M \to 0, \quad 0 \to M \to V_M \to \tau^{-1}(M) \to 0$$

be the AR-sequences starting and ending at M. Notice $\tau(M) = \operatorname{Syz}^A_{-d+2}(M)$ and $\tau^{-1}(M) = \operatorname{Syz}^A_{d-2}(M)$. It follows that $\operatorname{projdim}_Q \tau(M)$ and $\operatorname{projdim}_Q \tau^{-1}(M)$ is finite. Therefore $\operatorname{projdim}_Q E_M$ and $\operatorname{projdim}_Q V_M$ is finite.

By [25, 5.5, 5.6], we get that N, L are direct summands of E_M and V_M respectively. It follows that $\operatorname{projdim}_Q N$ and $\operatorname{projdim}_Q L$ are finite. Thus $\mathcal{P}_Q$ determines a union of connected components of $\Gamma(A)$. □

Let us recall that a quasi-deformation $A \to B \longleftarrow Q$ of A is a flat local map $A \to B$ and a deformation $Q \xrightarrow{\eta} B$ (i.e., $\ker \eta$ is generated by a Q-regular sequence). We say CI-dimension of an A-module M is finite if there is a quasi-deformation $A \to B \longleftarrow Q$ with $\operatorname{projdim}_Q M \otimes_A B$ is finite.

We now give:

Proof of Theorem 1.5 If A is a hypersurface ring then we have nothing to show. So we can assume that A is not a hypersurface. Let M be an indecomposable maximal Cohen-Macaulay A-module which is periodic and has a finite CI-dimension over A. Let $A \to B \longleftarrow Q$ be a quasi-deformation of A with $\operatorname{projdim}_Q M \otimes_A B$ finite. As A is not a hypersurface by proof of Theorem 1.2 there is no irreducible map from $A \to M$ or $M \to A$.

Let N, L be indecomposable non-free maximal Cohen-Macaulay A-modules and assume there exists irreducible maps $u\colon N \to M$ and $v\colon M \to L$. Let

$$0 \to \tau(M) \to E_M \to M \to 0, \quad 0 \to M \to V_M \to \tau^{-1}(M) \to 0$$

be the AR-sequences starting and ending at M. Notice $\tau(M) = \operatorname{Syz}^A_{-d+2}(M)$ and $\tau^{-1}(M) = \operatorname{Syz}^A_{d-2}(M)$.

Notice $\tau(M) \otimes_A B = \mathrm{Syz}^B_{-d+2}(M \otimes_A B)$. Thus CI-dimension of $\tau(M)$ is finite. As the quasi-deformations involved are the same we get that E_M also has finite CI-dimension over A. A similar argument yields that V_M has finite CI dimension over A. As N and L are summands of E_M and V_M respectively we get that CI dimension of N and L are finite. It follows that $\mathcal{P}_O$ defines a union of connected components of $\Gamma(A)$.

Let $\mathcal{C}$ be the union of connected components of $\Gamma(A)$ consisting of periodic indecomposable maximal Cohen-Macaulay A-modules and let $\mathcal{C}_O$ be the union of connected components of $\Gamma(A)$ consisting of periodic indecomposable maximal Cohen-Macaulay A-modules having finite CI-dimension over A. Then as $\mathcal{C}_O \subseteq \mathcal{C}$ we get that $\mathcal{C} \setminus \mathcal{C}_O$ is a union of connected components of $\Gamma(A)$. Notice $\mathcal{C} \setminus \mathcal{C}_O$ consists of precisely those periodic maximal Cohen-Macaulay A-modules which has infinite CI-dimension over A. □

6 Proof of Theorem 1.6

We need to recall some preliminaries regarding support varieties. This is relatively simple in our case since A is complete with algebraically closed residue field.

Let $A = Q/(\mathbf{u})$ where $(Q, \mathfrak{n})$ is regular local and $\mathbf{u} = u_1, \ldots, u_c \in \mathfrak{n}^2$ is a regular sequence. We need the notion of cohomological operators over a complete intersection ring.

The *Eisenbud operators*, [14] are constructed as follows:

Let $\mathbb{F}\colon \cdots \to F_{i+2} \xrightarrow{\partial} F_{i+1} \xrightarrow{\partial} F_i \to \cdots$ be a complex of free A-modules.

Step 1: Choose a sequence of free Q-modules $\widetilde{F}_i$ and maps $\widetilde{\partial}$ between them:

$$\widetilde{\mathbb{F}}\colon \cdots \to \widetilde{F}_{i+2} \xrightarrow{\widetilde{\partial}} \widetilde{F}_{i+1} \xrightarrow{\widetilde{\partial}} \widetilde{F}_i \to \cdots$$

so that $\mathbb{F} = A \otimes \widetilde{\mathbb{F}}$

Step 2: Since $\widetilde{\partial}^2 \equiv 0$ modulo $(\mathbf{u})$, we may write $\widetilde{\partial}^2 = \sum_{j=1}^{c} u_j \widetilde{t}_j$ where $\widetilde{t}_j\colon \widetilde{F}_i \to \widetilde{F}_{i-2}$ are linear maps for every i.

Step 3: Define, for $j = 1, \ldots, c$ the map $t_j = t_j(Q, \mathbf{f}, \mathbb{F})\colon \mathbb{F} \to \mathbb{F}(-2)$ by $t_j = A \otimes \widetilde{t}_j$.

The operators $t_1, \ldots, t_c$ are called *Eisenbud's operator's* (associated to $\mathbf{u}$) . It can be shown that

(1) t_i are uniquely determined up to homotopy.
(2) t_i, t_j commute up to homotopy.

Let $R = A[t_1, \ldots, t_c]$ be a polynomial ring over A with variables $t_1, \ldots, t_c$ of degree 2. Let M, N be finitely generated A-modules. By considering a free resolution $\mathbb{F}$ of M we get well defined maps

$$t_j\colon \mathrm{Ext}^n_A(M, N) \to \mathrm{Ext}^{n+2}_R(M, N) \quad \text{for } 1 \le j \le c \text{ and all } n,$$

which turn $\mathrm{Ext}^*_A(M, N) = \bigoplus_{i \ge 0} \mathrm{Ext}^i_A(M, N)$ into a module over R. Furthermore these structure depend on $\mathbf{u}$, are natural in both module arguments and commute with the connecting maps induced by short exact sequences.

Gulliksen, [17, 3.1], proved that $\mathrm{Ext}^*_A(M, N)$ is a finitely generated R-module. We note that $\mathrm{Ext}^*(M, k)$ is a finitely generated graded module over $T = k[t_1, \ldots, t_c]$. Define $V^*(M) = Var(\mathrm{ann}_T(\mathrm{Ext}^*(M, k))$ in the projective space $\mathbb{P}^{c-1}$. We call $V^*(M)$ the support variety of a module M. *Note that in* [7] *support varities are defined as* $Var(\mathrm{ann}_T(\mathrm{Ext}^*(M, k))$ *in* k^c. *However as the ideal involved is homogeneous we get a similar notion.* We need the following

Lemma 6.1 *Let $(Q, \mathfrak{n})$ be a complete regular local ring with algebraically closed residue field k. Let $\mathbf{f} = f_1, \ldots, f_c \in \mathfrak{n}^2$ be a regular sequence. Assume $c \geq 2$. Set $A = Q/(\mathbf{f})$ and let $d = \dim A$. Let W be an irreducible non-empty sub-variety of $\mathbb{P}^{c-1}$. Then there exists an indecomposable non-free maximal Cohen-Macaulay A-module M with $V^*(M) = W$.*

Proof By [9, 2.3], there exists an A-module E with $V^*(E) = W$. Then $N = \operatorname{Syz}_{d+1}(E)$ is a maximal Cohen-Macaulay A-module and $V^*(N) = V^*(E) = W$ (see [7, 2.4(3)]). Notice N has no free summands. If N is indecomposable then we are done. Otherwise $N = N_1 \oplus N_2$ where N_1 and N_2 are maximal Cohen-Macaulay A-modules with no free-summands. Let $T = k[t_1, \ldots, t_c]$ and let $\operatorname{Ext}^*_A(N, k)$, $\operatorname{Ext}^*_A(N_1, k)$ and $\operatorname{Ext}^*_A(N_2, k)$ be given T-module structure as above.

As $\operatorname{Ext}^*(N, k) = \operatorname{Ext}^*_A(N_1, k) \oplus \operatorname{Ext}^*_A(N_2, k)$ we get that

$$\operatorname{ann}_T \operatorname{Ext}^*(N, k) = \operatorname{ann}_T \operatorname{Ext}^*(N_1, k) \cap \operatorname{ann}_T \operatorname{Ext}^*(N_2, k).$$

It follows that

$$W = W_1 \cap W_2 \quad \text{where } W_i = V^*(N_i) \text{ for } i = 1, 2.$$

As W is irreducible we get that $W = W_1$ or $W = W_2$. Iterating this procedure we get our result. □

The following result yields Theorem 1.6 as an easy corollary.

Theorem 6.2 *Let $Q = k[[x_1, \ldots, x_n]]$ be the formal power series over an algebraically closed field k. Let $\mathbf{u} = u_1, \ldots, u_c \in \mathfrak{n}^2$ be a regular sequence. Assume $c \geq 2$. Set $A = Q/(\mathbf{u})$ and let $d = \dim A$. Assume A is an isolated singularity. Let W be a proper sub-variety of $\mathbb{P}^{c-1}$. Let*

$$\mathcal{C}_W = \{M \mid M \text{ is indecomposable MCM } A\text{-module with } V^*(M) = W\}.$$

Then $\mathcal{C}_W$ defines an union of connected components of $\Gamma(A)$. If W is irreducible then $\mathcal{C}_W$ is non-empty.

Proof Let $M \in \mathcal{C}_W$. Notice M is not free. As W is a proper subset of $\mathbb{P}^{c-1}$ we get that $\dim W \leq c - 2$. So $\operatorname{cx} M = \dim W + 1 \leq c - 1$. In particular M is not extremal. So there is no irreducible map $M \to A$. As $\tau(M) = \operatorname{Syz}_{-d+2}(M)$ is also not extremal there is no irreducible map $\tau(M) \to A$. So there is no irreducible map $A \to M$.

Let N, L be indecomposable non-free maximal Cohen-Macaulay A-modules and suppose there exists an irreducible map $u \colon N \to M$ and an irreducible map $v \colon M \to L$.

Claim: $V^*(N) = V^*(M) = W$ and $V^*(L) = V^*(M) = W$.

Suppose there exists $a \in V^*(N) \setminus V^*(M)$. Let D be a maximal Cohen-Macaulay A-module with $V^*(D) = \{a\}$. As $\{a\} \cap W = \emptyset$ we get that $\operatorname{Ext}^A_i(D, M) = 0$ for all $i \gg 0$, see [7, Theorem I]). As $V^*(\tau(M)) = V^*(M) = W$ (see [7, 2.4(3)]) we also get $\operatorname{Ext}^A_i(D, \tau(M)) = 0$ for $i \gg 0$. Thus $\operatorname{Ext}^A_i(D, E_M) = 0$ for $i \gg 0$. As N is a summand of E_M we get that $\operatorname{Ext}^A_i(D, N) = 0$ for all $i \gg 0$. This implies $a \notin V^*(N)$, a contradiction. Thus $V^*(N) \subseteq V^*(M)$. We now notice that there exists an irreducible map $M \to \tau^{-1}(N)$. By the previous argument we get that $V^*(M) \subseteq V^*(\tau^{-1}(N)) = V^*(N)$. Thus $V^*(N) = V^*(M) = W$.

As there exist's an irreducible map $v\colon M \to L$. Then there exists an irreducible map from $v'\colon L \to \tau^{-1}(M)$. By the previous argument we get $V^*(L) = V^*(\tau^{-1}(M))$. As $V^*(\tau^{-1}(M)) = V^*(M) = W$ we get $V^*(L) = W$. Thus we have proved our Claim.

By our claim and as there are no irreducible maps from $M \to A$ and $A \to M$ we get that $\mathcal{C}_W$ is a union of connected components of $\Gamma(A)$. If W is irreducible then by 6.1 we get that $\mathcal{C}_W$ is non-empty. □

We now give

Proof of Theorem 1.6 By 6.2 the result follows. □

7 Quasi AR-sequences

Setup: In this section $(A, \mathfrak{m})$ is a Henselian Gorenstein local ring with algebraically closed residue field k. We also assume A is an isolated singularity. For the notion of AR -sequences see [25, Chapter 2]. In this section we introduce the notion of quasi AR-sequences.

Definition 7.1 Let M be an indecomposable non-free maximal Cohen-Macaulay A-module. By a *quasi-AR sequence* **ending** at M we mean an exact sequence $s\colon 0 \to K \to E \xrightarrow{\phi} M$ such that

(1) E is a stable maximal Cohen-Macaulay A-module.
(2) ϕ is irreducible.
(3) If L is a stable maximal Cohen-Macaulay A-module and if there is an A-linear map $\sigma : L \to M$ which is not a split epi then there exist's a map $\xi\colon L \to E$ such that $\phi \circ \xi - \sigma \in \beta(L, M)$.

Remark 7.2 Unlike AR-sequences the module K *need not* be maximal Cohen-Macaulay. Also the map ϕ need not be surjective.

A consequence of the definition of quasi AR-sequence is the following:

Proposition 7.3 *[with hypothesis as in 7.1.] Suppose σ is irreducible. Then ξ is a split monomorphism.*

Proof By 3.2, $\phi \circ \xi$ is irreducible. Now ϕ is irreducible. So in particular it is not a split epi. It follows that ξ is a split mono. □

We need the following analogue to Corollary 2.12 from [25].

Lemma 7.4 *Let $(A, \mathfrak{m})$ be a Henselian Gorenstein local ring with algebraically closed residue field k. Let M, L be indecomposable non-free maximal Cohen-Macaulay A-modules and let $s\colon 0 \to K \to E \xrightarrow{\phi} M$ be a quasi AR-sequence ending at M. Then the following two conditions are equivalent:*

(i) *There is an irreducible morphism from L to M.*
(ii) *L is isomorphic to a direct summand of E.*

Proof $(i) \implies (ii)$. This follows from 7.3.

$(ii) \implies (i)$. Assume the decomposition of E is given by $E = L \oplus Q$. Denote $\phi = (f, g)$ along this decomposition. We claim that f is irreducible.

Clearly f is not a split epi as ϕ is not a split epi. If f is a split mono then as L and M are indecomposable we get that f is an isomorphism and so a split epi, a contradiction.

The rest of the proof is similar to the proof of $(ii) \implies (i)$ of Corollary 2.12 from [25]. □

We give two constructions of quasi AR-sequences. The first one comes from AR sequences.

Proposition 7.5 *Let $(A, \mathfrak{m})$ be a Henselian Gorenstein local ring with algebraically closed residue field k. Let M be an indecomposable non-free maximal Cohen-Macaulay A-modules and let $l\colon 0 \to N \to E_M \xrightarrow{p} M \to 0$ be an AR-sequence ending at M. Suppose $E_M = E \oplus F$ where F is a free A-module and E has no free summands. Denote $p = (\phi, \psi)$ along this decomposition. Assume $E \neq 0$. Let $K = \ker \phi$. Then $s\colon 0 \to K \to E \xrightarrow{\phi} M$ is a quasi AR-sequence ending at M.*

Proof By Corollary 2.12 from [25] we get that ϕ is an irreducible map. Now let L be a stable maximal Cohen-Macaulay A-module and let $f\colon L \to M$ be A-linear which is not a split epi. Then as l is an AR sequence ending at M there exist's an A-linear map $g\colon L \to E_M$ with $f = p \circ g$. Suppose

$$g = \begin{pmatrix} \sigma \\ \delta \end{pmatrix} \quad \text{where } \sigma\colon L \to E \text{ and } \delta\colon L \to F.$$

So we get $f = \phi \circ \sigma + \psi \circ \delta$. Notice $\psi \circ \delta \in \beta(L, M)$. It follows that s is a quasi AR sequence ending at M. □

Let $l\colon 0 \to N \to E_M \xrightarrow{p} M \to 0$ be an AR-sequence ending at M then it is not true that a lift of p; $q\colon \operatorname{Syz}_1(E) \to \operatorname{Syz}_1(M)$ is surjective and defines a AR sequence ending at $\operatorname{Syz}_1(M)$. The great advantage of quasi AR sequences is that it behaves well under lifting (and also co-lifting).

Theorem 7.6 *Let $(A, \mathfrak{m})$ be a Henselian Gorenstein local ring with algebraically closed residue field k. Let M be an indecomposable non-free maximal Cohen-Macaulay A-modules and let $s\colon 0 \to K \to E \xrightarrow{\phi} M$ be a quasi AR-sequence ending at M. Let ψ be any lift of ϕ. Set $K' = \ker \psi$. Then $s'\colon 0 \to K' \to \operatorname{Syz}_1(E) \xrightarrow{\psi} \operatorname{Syz}_1(M)$ is a quasi AR sequence ending at $\operatorname{Syz}_1(M)$. Similarly if θ is any co-lift of ϕ. Then $\widetilde{s}\colon 0 \to \widetilde{K} \to \operatorname{Syz}_{-1}(E) \xrightarrow{\theta} \operatorname{Syz}_{-1}(M)$ is a quasi AR sequence ending at $\operatorname{Syz}_{-1}(M)$.*

Proof As E is a stable maximal Cohen-Macaulay A-module we get that $\operatorname{Syz}_1(E)$ is also a stable maximal Cohen-Macaulay A-module. By Theorem 3.1 we get that ψ is an irreducible map.

Let L be a stable maximal Cohen-Macaulay A-module and let $f\colon L \to \operatorname{Syz}_1(M)$ be an A-linear map which is not a split epi. Let $g\colon \operatorname{Syz}_{-1}(L) \to M$ be any co-lift of f. Then by 3.3 we get that g is not a split epi. It follows that there exists $\xi\colon \operatorname{Syz}_{-1}(L) \to M$ such that $\phi \circ \xi - g = \delta$ where $\delta \in \beta(\operatorname{Syz}_{-1}(L), M)$. Let $\xi'\colon L \to \operatorname{Syz}_1(M)$ be a lift of ξ. Then notice by construction $\psi \circ \xi' - g$ is a lift of δ. It follows that $\psi \circ \xi' - g \in \beta(L, \mathrm{Syz}_1(M))$. Thus s' is a quasi AR-sequence ending at $\mathrm{Syz}_1(M)$.

The assertion regarding $\widetilde{s}$ can be proved similarly. $\square$

Till now we have not used the fact that k, the residue field of A is algebraically closed. We will now use this fact. Let us recall the following notion from [25, Chapter 5].

Let M, N be maximal Cohen-Macaulay A-modules. Set $(M, N) = \mathrm{Hom}_A(M, N)$.

Decompose $M = \bigoplus_{i=1}^m M_i$ and $N = \bigoplus_{j=1}^n N_j$ where M_i, N_j are indecomposable A-modules for all i, j For $g \in (M, N)$ decompose $g = (g_{ij})$ where $g_{ij}: M_i \to N_j$.

Definition 7.7 We say $g \in (M, N)_*$ if no g_{ij} is an isomorphism.

We define the following descending chain $\{(M, N)_n\}_{n \geq 1}$ of A-submodules of (M, N) as follows:

$(M, N)_n$ consists of those $f \in (M, N)$ such that there is a sequence $X_0, \ldots, X_n$ of maximal Cohen-Macaulay A-modules with $X_0 = M$ and $X_n = N$ and $f_i \in (X_{i-1}, X_i)_*$ such that $f = f_n \circ f_{n-1} \circ \cdots \circ f_1$.

It is easy to see that $(M, N)_n$ are A-submodules of (M, N) and that

$$(M, N) \supseteq (M, N)_1 \supseteq \cdots \supseteq (M, N)_n \supseteq (M, N)_{n+1} \supseteq \cdots .$$

It is not difficult to see that $(M, N)_1/(M, N)_2$ is a $k = A/\mathfrak{m}$ vector space. It is finite dimensional since it is finitely generated as an A-module. Set

$$\mathrm{irr}(M, N) = \dim_k \frac{(M, N)_1}{(M, N)_2}.$$

Let us restate the following basic result from [25, 5.5].

Lemma 7.8 *[with hypothesis as in 7.1.] Let M, N be indecomposable maximal Cohen-Macaulay A-modules. Assume there is an AR-sequence ending at M*

$$0 \to \tau(M) \to E_M \to M \to 0.$$

Let n be the number of copies of N in direct summands of E_M (note that $n = 0$ is possible). Then the following equality holds:

$$\mathrm{irr}(N, M) = n.$$

We note that the assumption k is algebraically closed is used in the proof of Lemma 7.8. The following is a basic result in our theory of quasi AR-sequences.

Theorem 7.9 *[with hypothesis as in 7.1.] Let M, N be indecomposable non-free maximal Cohen-Macaulay A-modules. Let $0 \to K \to E \xrightarrow{\phi} M$ be a quasi AR sequence ending at M. Let n be the number of copies of N in direct summands of E (note that $n = 0$ is possible). Then the following equality holds:*

$$\mathrm{irr}(N, M) = n.$$

Proof Set $S(N, E) = (N, E)/(N, E)_1$. Then by proof of Lemma 5.5. in [25] it follows that $S(N, E) \cong k^n$. Define

$$\theta\colon S(N, E) \to \frac{(N, M)_1}{(N, M)_2},$$
$$[f] \to [\phi \circ f].$$

By 7.4 we get that θ is a well-defined k-linear map.

We first show that θ is surjective. Let $\sigma : N \to M$ be an irreducible map. Denote by $[\sigma]$ it's class in $(N, M)_1/(N, M)_2$. By our definition of quasi AR sequence there exists $\xi : N \to E$ such that $\phi \circ \xi - \sigma = g \in \beta(N, M)$. By 7.3 we get that ξ is a split monomorphism. As N, M are both non-free indecomposable A-modules we get that $g \in (N, M)_2$. Therefore we get $\theta([\xi]) = [\sigma]$.

Next we show that θ is injective . Let $[h] \in S(N, E)$ be non-zero. Thus $h\colon N \to E$ is a split mono. Then by 7.4 we get that $\phi \circ h\colon N \to M$ is an irreducible map. Thus $\theta([h]) = [\phi \circ h] \neq 0$. Therefore θ is injective. □

A consequence of the previous two results is the following:

Corollary 7.10 *[with hypothesis as in 7.1.] Let M be indecomposable non-free maximal Cohen-Macaulay A-module. Suppose the following is an AR-sequence ending at M:*

$$t :\ 0 \to \tau(M) \to E_M \xrightarrow{p} M \to 0.$$

Further assume E_M has no free summnads. Let

$$s : 0 \to K \to E \xrightarrow{\phi} M$$

be a quasi AR sequence ending at M. Then $E \cong E_M$ and ϕ is surjective. Furthermore s is also an AR-sequence ending at M (and so $K \cong \tau(M)$).

Proof By 7.8 and 7.9 it follows that $E \cong E_M$. As $p\colon E_M \to M$ is an indecomosable map, by defining property of quasi AR-sequences there exists a map $\xi : E_M \to E$ such that $\phi \circ \xi - p = \delta \in \beta(E_M, M)$. As p is indecomposable, by 7.3 we get ξ is a split mono. As $E \cong E_M$, by Krull-Schmidt we get ξ is an isomorphism. It follows that

$$\phi - p \circ \xi^{-1} = \delta \circ \xi^{-1} := \eta \in \beta(E, M).$$

Set $\psi = p \circ \xi^{-1}\colon E \to M$. Notice ψ is surjective. As E, M has no free summands we get that $\eta(E) \subseteq \mathfrak{m}M$. It follows that the maps $\overline{\phi}, \overline{\psi}\colon E/\mathfrak{m}E \to M/\mathfrak{m}M$ are equal. As $\overline{\psi}$ is surjective, it follows that $\overline{\phi}$ is surjective. So by Nakayama's Lemma we get that ϕ is surjective.

We now use that t is an AR-sequence. As ϕ is irreducible, it is not a split epi. Therefore there exists $\theta\colon E \to E_M$ such that $p \circ \theta = \phi$. As ϕ is irreducible we get that θ is a split mono. Since $E \cong E_M$, by Krull-Schmidt we get θ is an isomorphism. Note there exists

$f\colon K \to \tau(M)$ which makes the following diagram commute:

$$\begin{array}{ccccccccc} s\colon 0 & \longrightarrow & K & \longrightarrow & E & \xrightarrow{\phi} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle 1_M} & & \\ t\colon 0 & \longrightarrow & \tau(M) & \longrightarrow & E_M & \xrightarrow{p} & M & \longrightarrow & 0 \end{array}$$

By Snake Lemma we get that f is also an isomorphism. So $K \cong \tau(M)$. In the terminology of [25, 2.3] we get $s \sim t$. So s is an AR-sequence ending at M. □

The following consequence of Corollary 7.10 is significant:

Lemma 7.11 *[with hypothesis as in 7.1.] Let M be an indecomposable maximal Cohen-Macaulay non-free A-module. Let $t\colon 0 \to \tau(M) \to E_M \xrightarrow{p} M \to 0$ be an AR-sequence ending at M. If there is no irreducible maps $A \to M$ and $A \to \mathrm{Syz}_1(M)$ then we have*

$$\mu(E_M) = \mu(M) + \mu(\tau(M)).$$

Proof Set $N = \tau(M)$, $M_1 = \mathrm{Syz}_1(M)$ and $E_1 = \mathrm{Syz}_1(E)$. As there is no irreducible maps from $A \to M$ we get that E_M is stable. In particular t is a quasi AR-sequence, see 7.5. Let $\phi\colon E_1 \to M_1$ be any lift of p. Then we have a quasi AR-sequence

$$s\colon 0 \to K_1 \to E_1 \xrightarrow{\phi} M_1.$$

As there are no irreducible maps from $A \to M_1$ we get that E_{M_1} is stable. Therefore by Corollary 7.10 we get that ϕ is surjective and s is an AR-sequence ending at M_1. Furthermore $E_1 \cong E_{M_1}$ and $K_1 \cong \tau(M_1)$.

Let $F \to E_M$ and $G \to M$ be projective covers. Then we have an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_1 & \longrightarrow & F & \longrightarrow & E_M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle p} & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & G & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Set $N = \tau(M)$. As ϕ, p are surjective we get θ is surjective. As G is free it is in fact a split epi. Set $H = \ker\theta$. Then H is free and $\mu(H) = \mu(E_M) - \mu(M)$.

As ϕ is surjective we get by Snake Lemma that the induced map $H \to N$ is surjective. It follows that $\mu(N) \le \mu(E_M) - \mu(M)$. As there is an exact sequence $0 \to N \to E_M \to M \to 0$ it follows (after tensoring with $A/\mathfrak{m}$) that $\mu(N) \ge \mu(E_M) - \mu(M)$. The result follows. □

8 Proof of Theorem 1.8

In this section we give a proof of Theorem 1.8.

[with hypotheses as in 1.1.] Recall if M is an indecomposable non-free A-module then it's AR-translate is $\tau(M) = \mathrm{Syz}_{-d+2}(M)$. So if $d = 2$ or if A is a hypersurface of even dimension then $\tau(M) = M$.

We now give:

Proof of Theorem 1.8. Let M be an idecomposable Ulrich A-module. By 8.1 we get that $\tau(M) = M$. Then by 4.2 there is no irreducible map from $M \to A$ and $\mathrm{Syz}_1(M) \to A$. By our assumptions on the ring there is no irreducible map from $A \to M$ and $A \to \mathrm{Syz}_1(M)$. Let $s\colon 0 \to M \to E_M \to M \to 0$ be the AR-sequence ending at M. Then by 7.11 we get $\mu(E_M) = 2\mu(M)$. Also note that $e(E_M) = 2e(M)$. So $e(E_M) = \mu(E_M)$. Therefore E_M is Ulrich A-module.

Let N be a non-free indecomposable maximal Cohen-Macaulay A-module. If there is an irreducible morphism $N \to M$ then N is a summand of E_M. As E_M is Ulrich we also get N is Ulrich. If there is an irreducible morphism from $M \to N$ then by our assumptions on the ring there is also an irreducible morphism from $N \to M$. By our earlier argument we get N is Ulrich.

As there is no irreducible map from $A \to M$ or from $M \to A$ it follows that $\mathcal{U}$ defines a union of connected components of $\Gamma(A)$. □

Remark 8.1 If $A = Q/(f)$ is a hypersurace ring with dim A is odd then note that Auslander-Reiten translate $\tau(M) = \mathrm{Syz}_1^A(M)$ which in general is not an Ulrich module (even if M is Ulrich), see [22, Theorem 2]. So for odd dimensions our technique to produce an infinite family of indecomposable Ulrich modules with unbounded multiplicities fails.

9 Proof of Theorem 1.9

In this section we give a proof of Theorem 1.9.

We recall the definition of linkage of modules as given in [20]. Throughout $(A, \mathfrak{m})$ is a Gorenstein local ring of dimension d.

Let us recall the definition of transpose of a module. Let $F_1 \xrightarrow{\phi} F_0 \to M \to 0$ be a minimal presentation of M. Let $(-)^* = \mathrm{Hom}(-, A)$. The *transpose* $\mathrm{Tr}(M)$ is defined by the exact sequence

$$0 \to M^* \to F_0^* \xrightarrow{\phi^*} F_1^* \to \mathrm{Tr}(M) \to 0.$$

Set $\lambda(M) = \mathrm{Syz}_1(\mathrm{Tr}(M))$. We note that if M is a stable maximal Cohen-Macaulay A-module then $\mathrm{Tr}(M) = (\mathrm{Syz}_2(M))^*$.

Definition 9.1 Two A-modules M and N are said to be *horizontally linked* if $M \cong \lambda(N)$ and $N \cong \lambda(M)$.

If E is a stable maximal Cohen-Macaulay A-module then it is known that E is linked to $\lambda(E)$, i.e., $\lambda^2(E) = E$ see [20, Corollary 7]. Note if M is an indecomposable non-free maximal Cohen-Macaulay A-module then so is $\lambda(M)$.

Proof of Theorem 1.9 We prove the result only for λ. The proof for D is in fact simpler.

Let M, N be indecomposable non-free maximal Cohen-Macaulay A-modules. Using terminology from 7.11 it suffices to prove that there exists an isomorphism

$$\phi\colon \frac{(M, N)_1}{(M, N)_2} \to \frac{(\lambda(N), \lambda(M))_1}{(\lambda(N), \lambda(M))_2}.$$

Let $f \in (M, N)_1$. Then as M, N are indecomposable we get that f is not an isomorphism. In particular it is not a split mono. Let $f_2\colon \mathrm{Syz}_2(M) \to \mathrm{Syz}_2(N)$ be a lift of f. By 3.3 and

2.8 it follows that f_2 is not a split mono. Let $f_2^*\colon \operatorname{Tr}(N) \to \operatorname{Tr}(M)$ be the dual of f_2. Then f_2^* is not a split epi. Let $g_f\colon \lambda(N) \to \lambda(M)$ be any lift of f_2^*. Define

$$\begin{aligned}\widetilde{\phi}\colon (M,N)_1 &\to \frac{(\lambda(N),\lambda(M))_1}{(\lambda(N),\lambda(M))_2},\\ f &\mapsto g_f + (\lambda(N),\lambda(M))_2.\end{aligned}$$

We first show that this map is independent of the choices we made. If f_2' is another lift of f then $f_2 - f_2' \in \beta(\operatorname{Syz}_2(M), \operatorname{Syz}_2(N))$. So $f_2^* - (f_2')^* \in \beta(\operatorname{Tr}(N), \operatorname{Tr}(M))$. We know that if $\sigma \in \beta(\operatorname{Tr}(N), \operatorname{Tr}(M))$ then any lift of σ is in $\beta(\lambda(N), \lambda(M))$. Thus we have $g_f - g'_f = \delta \in \beta(\lambda(N), \lambda(M))$. As $\lambda(N), \lambda(M)$ are indecomposable and non-free we get that $\delta \in (\lambda(N), \lambda(M))_2$. Thus $\widetilde{\phi}$ is well-defined. It is elementary to show that $\widetilde{\phi}$ is A-linear.

Now let $f \in (M,N)_2$. Then there exists a maximal Cohen-Macaulay A-module X and a commutative diagram

$$\begin{array}{ccc} M & & \\ {\scriptstyle u}\downarrow & \searrow^{f} & \\ X & \xrightarrow[v]{} & N \end{array}$$

such that u is not a split mono and v is not a split epi. Let $u_2\colon \operatorname{Syz}_2(M) \to \operatorname{Syz}_2(X)$ be a lift of u and $v_2\colon \operatorname{Syz}_2(X) \to \operatorname{Syz}_2(N)$ be a lift of v. By 3.3 and 3.4 we get that u_2 is not a split mono and v_2 is not a split epi. Then $f_2 = v_2 \circ u_2$ is a lift of f. Then $f_2^* = u_2^* \circ v_2^*$. Also u_2^* is not a split epi and v_2^* is not a split mono. Let $\operatorname{Syz}_1(u_2^*)$ be a lift of u_2^* and $\operatorname{Syz}_1(v_2^*)$ be a lift of v_2^*. Then $g_f = \operatorname{Syz}_1(u_2^*) \circ \operatorname{Syz}_1(v_2^*)$ is a lift of f_2^*. By 3.3 and 3.4 we get that $\operatorname{Syz}_1(u_2^*)$ is not a split epi and $\operatorname{Syz}_1(v_2^*)$ is not a split mono. So $g_f \in (\lambda(N), \lambda(M))_2$. Thus we have a well-defined A-linear map

$$\phi\colon \frac{(M,N)_1}{(M,N)_2} \to \frac{(\lambda(N),\lambda(M))_1}{(\lambda(N),\lambda(M))_2}.$$

As $\lambda^2(M) = M$ and $\lambda^2(N) = N$ we have a well defined A-linear map

$$\psi\colon \frac{(\lambda(N),\lambda(M))_1}{(\lambda(N),\lambda(M))_2} \to \frac{(M,N)_1}{(M,N)_2}.$$

Finally it is tautological that ϕ and ψ are inverses of each other. Thus $\lambda\colon \underline{\Gamma}(A) \to \underline{\Gamma}(A)^{rev}$ is an isomorphism.

Now assume that A is not a hypersurface ring. We first note that $\operatorname{Syz}_1(\lambda(M)) = M^*$ when M is stable maximal Cohen-Macaulay A-module. If $\lambda(M) = D(M)$ for all indecomposable maximal non-free A-modules then M^* (and so M) has a periodic resolution with period 1. It follows that A is a hypersurface ring, a contradiction.

Next we show that there exists E with $D(E) \neq E$. As A is not a hypersurface there exists an MCM module M which is not periodic. Let $M_1 = \operatorname{Syz}_1(M)$. As M is not periodic either $M \neq M^*$ or $M_1 \neq M_1^*$.

If $\lambda(M) = M$ for all indecomposable maximal Cohen-Macaulay non-free M then note that $\operatorname{Syz}_1(M) = M^*$ for all such M. We now note that

$$\operatorname{Syz}_{-2}(M^*) \cong (\operatorname{Syz}_2(M))^* \cong \operatorname{Syz}_1(\operatorname{Syz}_2(M)) = \operatorname{Syz}_3(M)$$

We now note that

$$\operatorname{Syz}_{-2}(M^*) = \operatorname{Syz}_{-2}(\operatorname{Syz}_1(M)) = \operatorname{Syz}_{-1}(M)$$

It follows that M is periodic for all indecomposable maximal Cohen-Macaulay non-free M. Thus A is a hypersurface, a contradiction.

Thus $\lambda, D \colon \underline{\Gamma}(A) \to \underline{\Gamma}(A)^{rev}$ are distinct isomorphism's if A is not a hypersurface ring. Furthermore $\lambda \neq 1$ and $D \neq 1$. □

The following result is immediate:

Corollary 9.2 *(with hypotheses as in 1.1). For all $n \in \mathbb{Z}$ the map $\operatorname{Syz}_n \colon \underline{\Gamma}(A) \to \underline{\Gamma}(A)$ is an isomorphism of graphs*

Proof We have $\operatorname{Syz}_1 \circ \lambda = D$. So $\operatorname{Syz}_1 = D \circ \lambda$ and $\operatorname{Syz}_{-1} = \lambda \circ D$. The result follows. □

Remark 9.3 Let $G = \{g, h \mid h^2 = 1; g^n \neq 1 \text{ for all } n \neq 1; gh = hg^{-1}\}$ be the infinite dihedral group. Then G acts on $\underline{\Gamma}(A)$ via $g[M] = [\operatorname{Syz}_1(M)]$ and $h[M] = [M^*]$. Our results show that we also have an action on the connected components of $\underline{\Gamma}(A)$.

We now give

Proof of Theorem 1.11. Let $[M]$ in $\mathcal{C}$. Recall

$$I(M) = \{n \mid [\operatorname{Syz}_n(M)] \in \mathcal{C}\}.$$

We first show that $I(M)$ is an ideal in $\mathbb{Z}$. As $M = \operatorname{Syz}_0(M)$ we get $0 \in I(M)$. Now let $n \in I(M)$. The isomorphism $\operatorname{Syz}_{-n} \colon \underline{\Gamma}(A) \to \underline{\Gamma}(A)$ maps $\mathcal{C}$ to itself since $\operatorname{Syz}_{-n}(\operatorname{Syz}_n(M)) = M$. In particular we have $[\operatorname{Syz}_{-n}(M)] = \operatorname{Syz}_{-n}([M]) \in \mathcal{C}$. If $m, n \in \mathcal{C}$ then note that the isomorphism $\operatorname{Syz}_n \colon \underline{\Gamma}(A) \to \underline{\Gamma}(A)$ maps $\mathcal{C}$ to itself as $\operatorname{Syz}_n([M]) = [\operatorname{Syz}_n(M)] \in \mathcal{C}$. Therefore $[\operatorname{Syz}_{n+m}(M)] = \operatorname{Syz}_n([\operatorname{Syz}_m(M)]) \in \mathcal{C}$. Thus $I(M)$ is an ideal in $\mathbb{Z}$. In particular there exists a unique non-negative integer $i(M)$ such that $I(M) = i(M)\mathbb{Z}$.

To prove rest of the assertion of the theorem we first make a convention: if $[X], [Y] \in \mathbb{C}$ then write $[X] \longleftrightarrow [Y]$ if there is an irreducible map from X to Y OR there is an irreducible map from $[Y]$ to $[X]$.

As $[M], [N]$ are in $\mathcal{C}$ there is a sequence

$$[M = X_0] \longleftrightarrow [X_1] \longleftrightarrow \cdots \longleftrightarrow [X_{n-1}] \longleftrightarrow [X_n = N],$$

in $\mathcal{C}$. Set $a = i(M)$ and $b = i(N)$. By 9.2 we have the following sequence in $\underline{\Gamma}(A)$:

$$[\operatorname{Syz}_a(M)] \longleftrightarrow [\operatorname{Syz}_a(X_1)] \longleftrightarrow \cdots \longleftrightarrow [\operatorname{Syz}_a(N)].$$

As $[\operatorname{Syz}_a(M)] \in \mathcal{C}$ we get $[\operatorname{Syz}_a(N)] \in \mathcal{C}$. So $a \in I(N)$ and therefore $I(M) \subseteq I(N)$. Similarly we get $I(N) \subseteq I(M)$. Thus $I(M) = I(N)$. □

We now give

Proof of Corollary 1.12 (1) Suppose if possible $\mathcal{D}$ has only finitely many vertices. Then $\operatorname{Syz}_n(\mathcal{D})$ cannot be a component of $\widetilde{\Gamma}(A) = \Gamma(A) \setminus \Gamma_0(A)$. As $\underline{\Gamma}_0(A)$ has only finitely many components we get $\operatorname{Syz}_n(\mathcal{D}) = \operatorname{Syz}_m(\mathcal{D})$ for some $n > m$. Set $c = n - m$. Then $\operatorname{Syz}_c(\mathcal{D}) = \mathcal{D}$. Therefore $\operatorname{Syz}_{lc}(\mathcal{D}) = \mathcal{D}$ for all $l \in \mathbb{Z}$.

We note that the function Syz_{lc} permutes vertices of $\mathcal{D}$ among itself. As $\mathcal{D}$ is finite it follows that all modules in $\mathcal{D}$ is periodic.

As $\mathcal{D}$ is a connected component of $\underline{\Gamma}_0(A)$ it follows that there exists $[M] \in \mathcal{D}$ such that there is an irreducible map either from M to A or an irreducible map from A to M. In the first case M is a component of $X(\mathfrak{m})$ the maximal Cohen-Macaulay approximation of $\mathfrak{m}$. By 4.1 we get that M is extremal. As M is periodic we get that A is a hypersurface ring, a contradiction. In the second case there is an irreducible map from $\operatorname{Syz}_{-d+2}(M) \to A$. Note as M is periodic then so is $\mathrm{Syz}_{-d+2}(M)$. An argument similar to the earlier case yields that A is a hypersurface ring, a contradiction.

(2) Suppose if possible the function $f\colon Vert(\mathcal{D}) \to \mathbb{Z}$ given by $f([M]) = e(M)$ is bounded. As $e(M) \geq \mu(M)$ and $e(\mathrm{Syz}_1(M)) = e(A)\mu(M) - e(M)$, it follows that the multiplicity function on $Vert(\mathrm{Syz}_1(\mathcal{D}))$ is bounded. Iterating we get that the multiplcity function on $Vert(\mathrm{Syz}_n(\mathcal{D}))$ is bounded for each $n \geq 1$. Then $\mathrm{Syz}_n(\mathcal{D})$ cannot be a component of $\widetilde{\Gamma}(A) = \Gamma(A) \setminus \Gamma_0(A)$. As $\underline{\Gamma}_0(A)$ has only finitely many components we get $\mathrm{Syz}_n(\mathcal{D}) = \mathrm{Syz}_m(\mathcal{D})$ for some $n > m$. Set $c = n - m$. Then $\mathrm{Syz}_c(\mathcal{D}) = \mathcal{D}$. Therefore $\mathrm{Syz}_{lc}(\mathcal{D}) = \mathcal{D}$ for all $l \in \mathbb{Z}$. In particular there exists c such that

$$(**) \qquad \beta_{il}(M) \leq c \qquad \text{for all } \ l \geq 0 \ \text{ and all } [M] \text{ in } \mathcal{D}.$$

As $\mathcal{D}$ is a connected component of $\underline{\Gamma}_0(A)$ it follows that there exists $[M] \in \mathcal{D}$ such that there is an irreducible map either from M to A or an irreducible map from A to M. In the first case M is a component of $X(\mathfrak{m})$ the maximal Cohen-Macaulay approximation of $\mathfrak{m}$. By 4.1 we get that M is extremal. As A is not an hypersurface we get the following

(1) If A is a complete intersection of codimension $c \geq 2$ then $\mathrm{cx}(M) = c$. Furthermore $\lim \beta_i(M) = \infty$. In particular the sequence $\{\beta_{il}(M)\}$ is unbounded. Thus (**) is not possible in this case.
(2) If A is Gorenstein but not a complete intersection then $\mathrm{curv}(M) = \mathrm{curv}(k) > 1$. So there exists $r > 1$ such that $\beta_i(M) > r^i$ for all $i \gg 0$. Thus (**) is not possible in this case too.

In the second case note that there is an irreducible map from $N = \mathrm{Syz}_{-d+2}(M)$ to A. We then have that for all $i \geq 0$

$$\beta_{il+d-2}(N) \leq c$$

Then an argument similar to above gives a contradiction. □

10 Obstruction to quasi AR-sequences

Let the setup be as in 1.1. Let M be a non-free maximal Cohen-Macaulay indecomosable A-module.

Let $s\colon 0 \to \tau(M) \to E_M \to M \to 0$ be the AR-sequence ending at M. Then using Proposition 7.5 we get the following:
There is no quasi-AR sequence ending at $M \iff E_M$ is free.

The next result gives an essential obstruction to non-existence of quasi AR-sequences.

Lemma 10.1 *[with hypothesis as in 1.1] Further assume $d \neq 1$. Suppose there is a non-free indecomposable maximal Cohen-Macaulay module M such that there is no quasi AR-sequence ending at M. Then A is a hypersurface ring.*

Proof By 10.1 it follows that $\tau(M) = \mathrm{Syz}_1(M)$. By construction $\tau(M) = \mathrm{Syz}_{-d+2}(M)$. Therefore we get that $M \cong \mathrm{Syz}_{-d+1}(M)$. As $d \neq 1$ we get that M (and so $\tau(M)$) is periodic.

E_M is non-zero and free. In particular it has A as a summand. So there is an irreducible map from $\tau(M) \to A$. It follows that $\tau(M)$ is a summand of $X(\mathfrak{m})$, the maximal Cohen-Macaulay approximation of $\mathfrak{m}$. By 4.1 we get that $\tau(M)$ is extremal A-module. It is also periodic. So A is a hypersurface ring. □

We now analyze hypersuface rings having perhaps modules M such that there is no quasi AR-sequence ending at M. Let $< M >$ denote the isomorphism class of a module M. Set

$$Q^c(A) = \{< M > \mid [M] \in \underline{\Gamma}(A) \text{ with no quasi AR-sequence ending at } M\}.$$

We show

Proposition 10.2 *Let $(A, \mathfrak{m})$ be a complete equicharacteristic hypersurface isolated singularity. Assume $d = \dim A$ is even and non-zero. Also assume that $k = A/\mathfrak{m}$ is algebraically closed. Then*

(1) *$Q^c(A)$ is a finite set (possibly empty).*
(2) *If A is not of finite representation type and $\operatorname{Syz}_d(k)$ is indecomposable then $Q^c(A)$ is empty.*
(3) *If $Q^c(A)$ is non-empty and if $< M > \in Q_c(A)$ then*

  (a) *$\operatorname{Syz}_n(M) = M$ for all $n \in \mathbb{Z}$.*
  (b) *$[M]$ is an isolated component of $\underline{\Gamma}(A)$.*

Proof We note that as A is a hypersurface and d is even we get that $\tau(M) = M$ for any non-free maximal Cohen-Macaulay indecomposable A-module M.

(1) If $< M > \in Q^C(A)$ then there is an irreducible map from $M = \tau(M) \to A$. So M is a component of $X(\mathfrak{m})$. It follows that $Q^c(A)$ is a finite set.
(2) As $\operatorname{Syz}_d(k)$ is indecomposable there is a unique non-free component of $X(\mathfrak{m})$. It follows that $\sharp Q^c(A) \leq 1$. If $< M > \in Q^c(A)$ then note that $[M] \leftrightarrows [A]$ is a connected component of $\Gamma(A)$. It follows that A is of finite representation type, a contradiction.
(3)(a) As there is no quasi AR-sequence ending at M we get that E_M is free. So $\tau(M) = \operatorname{Syz}_1(M)$. As $\dim A$ is even we get $M = \tau(M)$. As A is a hypersurface we get $\operatorname{Syz}_n(M) = M$ for all $n \in \mathbb{Z}$.
(3)(b) Notice $[M]$ is only connected to $[A]$. So we get that $[M]$ is an isolated component in $\underline{\Gamma}(A)$. □

11 Structure of $\Gamma_0(A)$

In this section we completely determine the structure of $\Gamma_0(A)$ when $\dim A = 2$ and its multiplicity $e(A) \geq 3$.

Theorem 11.1 *[with hypothesis as in 1.1.] Assume $\dim A = 2$ and $e(A) \geq 3$. Set $M_1 = A$. Then $\Gamma_0(A)$ is of the form*

$$M_1 \leftrightarrows M_2 \leftrightarrows M_3 \leftrightarrows M_4 \leftrightarrows \cdots \leftrightarrows M_n \leftrightarrows \cdots$$

where $e(M_n) = ne(M_1)$ for all $n \geq 1$. Furthermore

(1) *$X(\mathfrak{m}) = M_2 \oplus F$ where F is free.*
(2) *$M_n^* = M_n$ for all $n \geq 1$.*

Remark 11.2 We do not have any idea of the structure of $\Gamma_0(A)$ when $e(A) = 2$ (so necessarily A is a hypersurface) and A is of infinite representation type. The reason is that Proposition 11.3 given below breaks down in the case $e(A) = 2$.

The following result is essential in our proof of Theorem 11.1.

Proposition 11.3 *(with hypotheses as in Theorem 11.1). Let $X(\mathfrak{m})$ be a MCM approximation of $\mathfrak{m}$. Write $X(\mathfrak{m}) = M \oplus F$ where F is free and M has no free summands. Then*

(1) *M is indecomposable.*
(2) $\operatorname{rank} M = 2$.
(3) $M \cong M^*$.

Proof (1) By [23, Theorem B]; $\operatorname{Syz}_2^A(k)$ is indecomposable. So $X(k) = \operatorname{Syz}_2^A(k)^*$ is indecomposable. As $X(\mathfrak{m}) = \operatorname{Syz}_1^A(X(k)) \oplus G$ where G is free we get that $M = \operatorname{Syz}_1^A(\operatorname{Syz}_2^A(k)^*)$ is indecomposable by [25, 8.17].

(2) We get $M^* = \operatorname{Syz}_{-1}^A(\operatorname{Syz}_2^A(k))$. Let x, y be a $A \oplus M \oplus \operatorname{Syz}_2^A(k)$-superficial sequence. Set $C = A/(x, y)$. If E is an A-module then set $\overline{E} = E/(x, y)E$. Notice

$$\overline{M^*} \cong \operatorname{Syz}_{-1}^C(\overline{\operatorname{Syz}_2^A(k)}) \quad \text{and} \quad \overline{\operatorname{Syz}_2^A(k)} \cong \operatorname{Syz}_2^C(k) \oplus \operatorname{Syz}_1^C(k)^2 \oplus \operatorname{Syz}_0^C(k).$$

Therefore

$$\overline{M^*} \cong \operatorname{Syz}_1^C(k) \oplus \operatorname{Syz}_0^C(k)^2 \oplus \operatorname{Syz}_{-1}^C(k).$$

We note that as we have an exact sequence $0 \to k = \operatorname{soc}(C) \to C \to C/\operatorname{soc}(C) \to 0$. Thus $\operatorname{Syz}_{-1}^C(k) = C/\operatorname{soc}(C)$. Let $\mathfrak{n}$ be the maximal ideal of C. Thus we have

$$\overline{M^*} \cong \mathfrak{n} \oplus k^2 \oplus C/\operatorname{soc}(C).$$

So $\ell(\overline{M^*}) = 2\ell(C)$. Therefore

$$e(M) = e(M^*) = e(\overline{M^*}) = \ell(\overline{M^*}) = 2\ell(C) = 2e(C) = 2e(A);$$

(here the second and the last equality holds by 2.12). It follows that $\operatorname{rank} M = 2$.

(3) As there is an irreducible map $M \to A$ there exists an irreducible map $A \to M^*$. As $\dim A = 2$ we have $\tau(M^*) = M^*$. So there is an irreducible map from $M^* \to A$. Thus M^* is a non-free irreducible component of $X(\mathfrak{m})$. By (1) we have $M^* \cong M$. □

We now give

Proof of Theorem 11.1. Set $X(\mathfrak{m}) = M_2 \oplus F$ where F is free and M_2 has no free summands. By Proposition 11.3 we get that M_2 is indecomposable of rank 2. We have the AR-sequence

$$0 \to M_2 \to M_1^a \oplus X \to M_2 \to 0.$$

Thus $a + \operatorname{rank} X = 4$. By Lemma 10.1 and Proposition 10.2(2) we get that $X \neq 0$. Thus $1 \leq a \leq 3$. We assert $a = 1$. We prove this by showing that the cases $a = 2$ or 3 do not occur.

Claim 1: $a \neq 3$.
Suppose if possible $a = 3$ then rank X is one. So X is indecomposable. As $\dim A = 2$ and there is an irreducible map from X to M_2, there is an irreducible

map from $M_2 \to X$. By rank considerations we get that the AR-quiver ending at X is

$$0 \to X \to M_2 \to X \to 0.$$

It follows that M_1, M_2 and X constitute a connected component of $\Gamma_0(A)$ and so it is equal to $\Gamma_0(A)$. Therefore A has finite representation type, a contradiction.

Claim 2: $a \neq 2$.
If possible assume $a = 2$. It follows that rank $X = 2$. We assert:

Subclaim 3: X is indecomposable.
Suppose if possible $X = X_1 \oplus X_2$ where rank $X_i = 1$. As dim $A = 2$ and there is an irreducible map from X_i to M_2, there is an irreducible map from $M_2 \to X_i$. By rank considerations we get that the AR-quiver ending at X_i for $i = 1, 2$ is

$$0 \to X_i \to M_2 \to X_i \to 0.$$

It follows that M_1, M_2, X_1 and X_2 constitute a connected component of $\Gamma_0(A)$ and so it is equal to $\Gamma_0(A)$. It follows that A has finite representation type, a contradiction. Thus X is indecomposable.

The AR-sequence ending at X is

$$0 \to X \to M_2 \oplus X_1 \to X \to 0.$$

By an argument similar to Subclaim-3 we get that X_1 is indecomposable of rank 2. Set $X_0 = X$.

For $i \geq 1$, by an argument similar to Subclaim-3 we get that there exists indecomposable module X_{i+1} of rank 2 such that the AR-sequence ending at X_i is

$$0 \to X_i \to X_{i-1} \oplus X_{i+1} \to X_i \to 0.$$

Thus $\Gamma_0(A)$ consists of the modules $\{M_1, M_2, X_i \mid i \geq 0\}$, Also rank $X_i = 2$. This implies that A is of finite representation type (see [25, 6.2]), a contradiction.

By claims 1, 2 we get $a = 1$. Thus rank $X = 3$.

Claim 4: X is indecomposable.
Suppose if possible this is not so. Then either

Subcase 5: $X = X_1 \oplus X_2 \oplus X_3$ where rank $X_i = 1$ for $1 \leq i \leq 3$, OR

Subcase 6: $X = X_1 \oplus X_2$ where rank $X_i = i$ for $i = 1, 2$.
We show that subcase 5, 6 are not possible. If subcase 5 occurs then by rank considerations the AR-quiver ending at X_i is

$$0 \to X_i \to M_2 \to X_i \to 0 \quad \text{for } i = 1, 2, 3.$$

Thus the vertices of $\Gamma_0(A)$ will be

$$\{M_1, M_2, X_1, X_2, X_3\}.$$

This implies that A has finite representation type, a contradiction.

If subcase 6 occurs then by rank considerations the AR-quiver ending at X_1 is

$$0 \to X_1 \to M_2 \to X_1 \to 0.$$

Furthermore the AR-quiver ending at X_2 is

$$0 \to X_2 \to M_2 \oplus X_3 \to X_2 \to 0.$$

Note rank $X_3 = 2$. By an argument similar to that of subcase 5 we get that X_3 is indecomposable. Iterating we obtain rank two indecomposable modules X_i for $i \geq 4$ such that the AR-quiver ending at X_i is

$$0 \to X_i \to X_{i+1} \oplus X_{i-1} \to X_i \to 0.$$

It follows that the vertices of $\Gamma_0(A)$ is

$$\{M_1, M_2, X_i \mid i \geq 1\}.$$

As there is a bound on the ranks of vertices of $\Gamma_0(A)$ it follows that A is of finite representation type, a contradiction.

Set $M_3 = X$. We have rank $M_3 = 3$ and that M_3 is indecomposable. Inductively assume that we have indecomposable MCM A-modules $M_1, \ldots, M_n$ with $n \geq 3$ and rank $M_i = i$ such that the AR-sequence ending at M_j for $j \leq n-1$ is

$$0 \to M_j \to M_j \oplus M_{j+1} \to M_j \to 0.$$

Let the AR-sequence ending at M_n be

$$0 \to M_n \to M_{n-1} \oplus Y \to M_n \to 0.$$

Clearly rank $Y = n+1$. If we prove that Y is indecomposable then we can set $M_{n+1} = Y$ and we will be done by induction.

Let Z be an indecomposable summand of Y. Then the AR-sequence ending at Z is

$$0 \to Z \to M_n \oplus W \to Z \to 0,$$

where W is an MCM A-module (possibly zero). Nevertheless we get that rank $Z \geq n/2$.

As $n \geq 3$, rank $Y = n+1$ and an indecomposable summand Z of Y has rank atleast $n/2$ it follows that Y has at most two indecomposable summnads.

We want to prove that Y is indecomposable. Suppose it is not so. Then by our previous argument it has two indecomposable summands say Y_1 and Y_2. Suppose rank $Y_1 \leq$ rank Y_2. Then we have

$$\frac{n}{2} \leq \operatorname{rank} Y_1 \leq \frac{n+1}{2}.$$

We consider two cases:

Case 1: $n = 2m+1$ is odd.

We get rank $Y_1 = m+1$. So rank $Y_2 = m+1$ also. Let the AR-sequence ending at Y_1 be

$$0 \to Y_1 \to M_n \oplus T \to Y_1 \to 0.$$

Thus T has rank 1. The AR-sequence ending at T is

$$0 \to T \to Y_1 \oplus L \to T \to 0.$$

As $m+1 \leq 2$ we get $m \leq 1$. As $m \geq 1$ we get $m = 1$. Therefore $n = 2m+1 = 3$. Now consider the case $n = 3$. We get rank $Y_j = 2$ for $j = 1, 2$ and rank $T = 1$. Furthermore $L = 0$. Similarly the AR-sequence ending at Y_2 will be

$$0 \to Y_2 \to M_3 \oplus T' \to Y_2 \to 0,$$

where T' has rank 1. The AR-sequence ending at T' is

$$0 \to T' \to Y_2 \to T' \to 0.$$

It follows that the vertices of $\Gamma_0(A)$ will be

$$\{M_1, M_2, M_3, Y_1, Y_2, T, T'\}.$$

It follows that A has finite representation type, a contradiction.

Case 2: $n = 2m$ is even.

We get rank $Y_1 = m$ and rank $Y_2 = m + 1$. The AR sequence ending at Y_1 is

$$0 \to Y_1 \to M_n \to Y_1 \to 0.$$

The AR sequence ending at Y_2 is

$$0 \to Y_2 \to M_n \oplus T \to Y_2 \to 0.$$

It follows that rank $T = 2$. We have to consider two sub cases:

Subcase-1: T is decomposable. In this case $T = T_1 \oplus T_2$ where rank $T_i = 1$ for $i = 1, 2$. The AR-sequence ending at T_1 is

$$0 \to T_1 \to Y_2 \oplus L \to T_1 \to 0.$$

We have $2 = m + 1 + \operatorname{rank} L$. As $m \geq 1$ we get $m = 1$ and $L = 0$. So $n = 2$. We have already dealt with this case.

Subcase-2: T is indecomposable. The AR-sequence ending at T is

$$0 \to T \to Y_2 \oplus W \to T \to 0.$$

We have $4 = m + 1 + \operatorname{rank} W$. As $m \geq 1$ the possibilities for m is 1, 2, 3. If $m = 1$ then $n = 2$. This case has been discussed earlier. Next we consider the case $m = 3$. In this case $W = 0$. So the vertices of $\Gamma_0(A)$ will be

$$\{M_i, Y_1, Y_2, T \mid 1 \leq i \leq n\}.$$

It follows that A has finite representation type, a contradiction.

Finally we consider the case when $m = 2$. So $n = 4$. Thus rank $W = 1$. The AR-sequence ending at W is

$$0 \to W \to T \to W \to 0.$$

Thus the vertices of $\Gamma_0(A)$ will be

$$\{M_i, Y_1, Y_2, T, W \mid 1 \leq i \leq n\}.$$

It follows that A has finite representation type, a contradiction.

(2) We note that the dual map $D\colon \underline{\Gamma}(A) \to \underline{\Gamma}(A)^{rev}$ is an isomorphism of graphs. As $D(M_2) = M_2^* \cong M_2$ and as $\underline{\Gamma}_0(A)$ is connected we get that D maps $\underline{\Gamma}_0(A)$ to itself. Comparing ranks we get $M_n^* \cong M_n$ for all $n \geq 3$. □

12 Proof of Theorem 1.13 and Corollary 1.14

In this section we prove results as stated in the title of the section. Throughout $(A, \mathfrak{m})$ is an equi-characteristic Gorenstein isolated singularity of dimension two. We also assume that A is complete and the residue field k is algebraically closed. Furthermore we assume that $e(A) \geq 3$.

We first give

Proof of Corollary 1.14 It suffices to show that $\operatorname{Syz}_n(M) \notin \underline{\Gamma}_0(A)$ for all $M \in \underline{\Gamma}_0(A)$ and for all $n \neq 0$. Using the terminology of Theorem 1.11 we need to show $I(M) = 0$ for all M

in $\underline{\Gamma}_0(A)$. We also recall that $I(M) = I(N)$ for all $M, N \in \underline{\Gamma}_0(A)$. We denote this common value by c.

We want to show $c = 0$. If possible assume $c > 0$. Set

$$V = \{|i - j| \mid M_j = \mathrm{Syz}_n M_i \text{ for some } n \neq 0\} \quad \text{and } r = \min V.$$

Notice $c \neq 0$ if and only if $V \neq \emptyset$.

We first consider the case when $r = 0$. Say $M_i = \mathrm{Syz}_n M_i$ for some $n \neq 0$. We may assume $n > 0$. Then M_i is periodic. As A is not a hypersurface this is a contradiction by Theorems 1.2 and 4.1.

We now assume $r \geq 1$. Say $M_{i+r} = \mathrm{Syz}_n(M_i)$ for some $r > 0$ and for some $n \neq 0$. *Note we are not assuming* $n > 0$. As we have an irreducible morphism from $M_{i+r-1} \to M_{i+r}$ we have an irreducible map from

$$\mathrm{Syz}_{-n}(M_{i+r-1}) \to M_i.$$

So we have $M_{i+1} = \mathrm{Syz}_{-n}(M_{i+r-1})$ or $M_{i-1} = \mathrm{Syz}_{-n}(M_{i+r-1})$. The first case cannot occur as $r = \min V$. So $M_{i-1} = \mathrm{Syz}_{-n}(M_{i+r-1})$ and therefore $M_{i+r-1} = \mathrm{Syz}_n(M_{i-1})$. Iterating this procedure we get that $M_{2+r} = \mathrm{Syz}_n(M_2)$. We have irreducible maps from M_{2+r-1} and M_{2+r+1} to $M_{2+r} = \mathrm{Syz}_n(M_2)$. So we have an irreducible map from $\mathrm{Syz}_{-n}(M_{2+r-1})$ and $\mathrm{Syz}_{-n}(M_{2+r+1})$ to M_2. It follows that atleast one of $\mathrm{Syz}_{-n}(M_{2+r-1})$ and $\mathrm{Syz}_{-n}(M_{2+r+1})$ is A. This is a contradiction. □

Next we give

Proof of Theorem 1.13 The assertion on the structure of $\mathcal{C}$ follows from Theorem 11.1 and [8, 4.16.2].

(1) This follows from Theorem 11.1.

(2)(a) Let $\mathcal{C}$ be a connected component of $\Gamma(A)$ such that $[M] \in Vert(\mathcal{C})$ is a periodic module. Then by Theorem 1.2 all the modules N in $Vert(\mathcal{C})$ is periodic. We note that $\mathrm{Syz}_n(\mathcal{C})$ consists of periodic modules and so $[A] \notin Vert(\mathrm{Syz}_n(\mathcal{C})$ for all $n \in \mathbb{Z}$ (see Theorem 4.1). Using Theorem 7.6 and Corollary 7.10 we get that if $[M] \in Vert(\mathcal{C})$ and if $0 \to M \to E_M \to M \to 0$ is an AR sequence ending at M then for all $n \in \mathbb{Z}$ the AR-sequence ending at $\mathrm{Syz}_n(M)$ is of the form $0 \to \mathrm{Syz}_n(M) \to \mathrm{Syz}_n(E) \to \mathrm{Syz}_n(M) \to 0$.

Now consider the structure of $\mathcal{C}$ as given in (1). Let period of M_1 be c. We first show that $I(M_1) = c\mathbb{Z}$ (notation as in Theorem 1.11). Note $c \in I(M_1)$. If $I(M_1) \neq c\mathbb{Z}$ then there exists a with $1 < a < c$ such that $[\mathrm{Syz}_a(M_1)] \in Vert(\mathcal{C})$. We note that $0 \to \mathrm{Syz}_a(M_1) \to \mathrm{Syz}_a(M_2) \to \mathrm{Syz}_a(M_1) \to 0$ is the AR sequence ending at $\mathrm{Syz}_a(M_1)$. As M_1 is the unique vertex in $\mathcal{C}$ which is connected to only one other vertex we get that $\mathrm{Syz}_a(M_1) = M_1$. This contradicts the fact that period of M_1 is c.

We show by induction on $n \geq 2$ that the period of M_n is c. We first consider the case $n = 2$. As period of M_1 is c we get that $0 \to M_1 \to \mathrm{Syz}_c(M_2) \to M_1 \to 0$ is also an AR-sequence ending at M_1. By uniqueness of AR sequences we get $M_2 \cong \mathrm{Syz}_c(M_2)$. Suppose for some a with $1 \leq a < c$ we have $\mathrm{Syz}_a(M_2) = M_2$ then note that $a \in I(M_2) = I(M_1) = c\mathbb{Z}$, a contradiction. Thus period of M_2 is c.

Now assume that period of $M_1, \ldots, M_n$ is c. We prove that period of M_{n+1} is also c. As the period of M_{n-1} and M_n is c we get that $0 \to M_n \to M_{n-1} \oplus \mathrm{Syz}_c(M_{n+1}) \to M_n \to 0$ is another AR-sequence ending at M_n. By uniqueness of AR-sequences we get that $M_{n+1} \cong \mathrm{Syz}_c(M_{n+1})$. Suppose for some a with $1 \leq$

$a < c$ we have $M_{n+1} = \operatorname{Syz}_a(M_{n+1})$. Then $a \in I(M_{n+1}) = I(M_1) = c\mathbb{Z}$, a contradiction. Thus period of M_{n+1} is c. The result follows.

(2)(b) By 1.14 there exists *at-most* one $m_0 \geq 1$ such that $\operatorname{Syz}_{m_0}(\mathcal{C}) = \underline{\Gamma}_0(A)$. Thus for $n > m_0$ we have that $[A] \notin Vert(\operatorname{Syz}_n(\mathcal{C}))$. Set $M_0 = 0$. We have that for all $n > m_0$ the sequence $0 \to \operatorname{Syz}_n(M_i) \to \operatorname{Syz}_n(M_{i-1}) \oplus \operatorname{Syz}_n(M_{i+1}) \to \operatorname{Syz}_n(M_i) \to 0$ is the AR quiver ending at M_i for all $i \geq 1$. By Lemma 7.11 we get that for all $n > m$ and for all $i \geq 1$

$$2\beta_n(M_i) = \beta_n(M_{i-1}) + \beta_n(M_{i+1}).$$

As $M_0 = 0$ an easy recursion yields that $\beta_n(M_i) = i\beta_n(M_1)$. The result follows.

□

13 Curvature and Complexity

If $(A, \mathfrak{m})$ is a complete intersection of codimension c then it is known that for any non-zero module M we have $0 \leq \operatorname{cx} M \leq c$. Furthermore for any integer i with $0 \leq i \leq c$ there exists an A-module M with complexity i. If A is not a complete intersection then $\operatorname{cx} k = \infty$. To deal with this situation the notion of curvature was introduced. It can be shown that $1 < \operatorname{curv} k < \infty$ (see [5, 8.2.2]) and for any non-zero module with infinite projective dimension we have $1 \leq \operatorname{curv} M \leq \operatorname{curv} k$ [5, 4.1.9]). Furthermore if $\operatorname{cx} M < \infty$ then $\operatorname{curv} M = 1$. We first prove

Proposition 13.1 *Let* $(A, \mathfrak{m})$ *be an equi-charactersitic complete Gorenstein isolated singularity with algebraically closed residue field k. Assume A is not a complete intersection. Then*

(1) *For any* $i \geq 1$ *the modules M with complexity i form a union of connected components of* $\Gamma(A)$.
(2) *For any* $\alpha \in [1, \operatorname{curv} k)$ *the modules M with curvature* α *form a union of connected components of* $\Gamma(A)$.

We first show

Lemma 13.2 *[with hypotheses as in Proposition 13.1] Let* $1 \leq \alpha < \operatorname{curv} k$. *Let* $\mathcal{V}_\alpha$ *be the collection of all indecomposable modules M with* $\operatorname{curv} M \leq \alpha$. *Then* V_α *is a union of connected components of* $\Gamma(A)$. *Furthermore* $\Gamma_0(A) \nsubseteq V_\alpha$.

Proof Let $M \in V_\alpha$. Note that $\tau(M) = \operatorname{Syz}_{-d+2}(M) \in V_\alpha$. As $\alpha < \operatorname{curv} k$ it follows that there is no irreducible map from M to A or from A to M, see 4.1.

Clearly $\operatorname{Syz}_n(M) \in V_\alpha$ for all $n \in \mathbb{Z}$. By a similar argument as before there is no irreducible map from $\operatorname{Syz}_n(M)$ to A or from A to $\operatorname{Syz}_n(M)$ for all $n \in \mathbb{Z}$.

Let $0 \to \tau(M) \to E_M \to M \to 0$ be the AR-sequence ending at M. By 7.9 and 7.10 we get that

(1) $0 \to \operatorname{Syz}_n(\tau(M)) \to \operatorname{Syz}_n(E_M) \to \operatorname{Syz}_n(M) \to 0$ is the AR-sequence ending at $\operatorname{Syz}_n(M)$ for all $n \geq 0$.
(2) $\beta_n(E_M) = \beta_n(M) + \beta_n(\tau(M))$ for all $n \geq 0$.

Thus we have $\operatorname{curv}(E) \leq \alpha$. If there is an irreducible map from N to M then N is a factor of E_M and so $\operatorname{curv}(N) \leq \operatorname{curv}(E) \leq c$. Thus $N \in V_\alpha$. In a similar fashion if there is an irreducible map from M to N then also $N \in V_\alpha$. Thus V_α is a union of connected components of $\Gamma(A)$. Also clearly $\Gamma_0(A) \nsubseteq V_\alpha$. □

As an immediate consequence we get

Corollary 13.3 *[with hypotheses as in Proposition 13.1] Let* $1 < \beta < \operatorname{curv} k$. *Let* $\mathcal{U}_\beta$ *be the collection of all indecomposable modules* M *with* $\operatorname{curv} M < \beta$. *Then* U_β *is a union of connected components of* $\Gamma(A)$. *Furthermore* $\Gamma_0(A) \nsubseteq U_\beta$.

Proof Let $1 = \alpha_1 < \alpha_2 < \cdots < \alpha_n < \alpha_{n+1} < \cdots$ be any strictly monotonically increasing sequence converging to β. Notice

$$U_\beta = \bigcup_{n \geq 1} V_{\alpha_n}$$

The result now follows from Lemma 13.2. □

We now give

Proof of Proposition 13.1 We first prove (2). Let C_α = the collection of modules with complexity α. Notice (with notation as in Lemma 13.2 and Corollary 13.3

(a) $C_1 = V_1$.
(b) For $1 < \alpha < \operatorname{curv}(k)$ we have $C_\alpha = V_\alpha - U_\alpha$.

Thus (2) follows.

(1) This is similar to (2). We have to prove results analogous to Lemma 13.2 and Corollary 13.3 first. □

We now give

Proof of Theorem 1.7 Suppose A has a module M with bounded betti-numbers but not periodic. Then note that A is not a complete intersection. We note that a MCM A-module M will have bounded betti-numbers if and only if $\operatorname{cx}(M) \leq 1$. By Proposition 13.1, $\mathcal{D}$ the collection of all such modules defines a union of connected components of $\Gamma(A)$. We note that modules M having a periodic resolution will form a subset $\mathcal{C}$ of $\mathcal{D}$. By Theorem 1.2 we get that $\mathcal{C}$ is a union of connected components of $\Gamma(A)$. It follows that $\mathcal{D} \setminus \mathcal{C}$ is a union of connected components of $\Gamma(A)$. If M is not periodic but has a bounded resolution then $[M] \in \mathcal{D} \setminus \mathcal{C}$. The result follows. □

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