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On the Cohomology of Certain Rank 2 Vector Bundles on G/B

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Abstract Let G be a semisimple, simply connected, linear algebraic group over an algebraically closed field k. Donkin (In J. Algebra, 307, 570–613 2007), Donkin gave a recursive description for the characters of the cohomology of line bundles on the three dimensional flag variety in prime characteristic. The recursion involves not only line bundles but also certain natural rank 2 bundles associated to two dimensional $B-$modules $N_\alpha(\lambda)$, where λ in an integral weight and α is a simple root. In this paper we compute the cohomology of these rank 2 bundles and simplify the recursion in Donkin (In J. Algebra, 307, 570–613 2007). We also compute the socle of $N_\alpha(\lambda)$ and give a rank 2 version of Kempf's vanishing theorem.

Keywords Algebraic groups · Flag varieties · Cohomology · Vector bundles

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1 Introduction

In [5], Donkin gave a recursive description for the characters of the cohomology of line bundles on the three dimensional flag variety in prime characteristic. The recursion involves not only line bundles but also certain natural rank 2 bundles associated to two dimensional

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$B-$modules $N_\alpha(\lambda)$, where λ in an integral weight and α is a simple root. These rank 2 bundles also appear in the similar recursive computations for the algebraic group of type G_2 [3]. For $G = SL_3(k)$ the results given in [5] are the only results which give a complete description for the characters of $H^i(G/B, k_\lambda)$ but due to their recursive nature it is very hard to use them. The purpose of this paper is to compute $H^i(G/B, N_\alpha(\lambda))$. Our results significantly reduce the recursion on the results given in [5]. In characteristic zero we give a complete result for the cohomology. In characteristic $p > 0$ we show that if p does not divide $\langle\lambda, \alpha^\vee\rangle$ then $H^i(G/B, N_\alpha(\lambda)) = H^i(G/B, k_\lambda) \oplus H^i(G/B, k_{\lambda-\alpha})$, for all $i \geq 0$. We also show that if p divides $\langle\lambda, \alpha^\vee\rangle$ then $H^1(G/B, N_\alpha(\lambda))$ is indecomposable as a $G-$module. We also compute its $G-$socle. We give a version of Kemp's vanishing theorem for these rank 2 bundles. In characteristic zero the modules $N_\alpha(\lambda)$ are a special case of the modules $V_{\lambda,\alpha}$ appearing in the Demazure's [4] famous proof of the Borel-Weil-Bott theorem. The modules $V_{lambda,\ alpha}$ also appear in [2].

Let k be an algebraically closed field and let G be a semisimple simply connected linear algebraic group over k. Let B be a Borel subgroup of G and $T \subset B$ be a maximal torus of G. For an algebraic group J, write mod(J) for the category of finite dimensional rational $J-$modules over k. Define $X(T)$ to be the group of multiplicative characters of T. For a $T-$module V and $\lambda \in X(T)$, write V^λ for corresponding weight space of V. Those λ's for which V^λ is non-zero are called weights of V. The Weyl group W acts on T and $X(T)$ in the usual way.

Choose a real, positive definite, $W-$invariant, symmetric, non-singular, bilinear form on $\mathbb{R} \otimes_{\mathbb{Z}} X(T)$. Let Φ be the set of non-zero weights for action of T on Lie algebra of T, then $(\mathbb{R} \otimes_{\mathbb{Z}} X(T), \Phi)$ is a root system. We choose the system of positive roots Φ^+ for which B is the negative Borel subgroup and let S denote the set of simple roots. For $\alpha \in \Phi$ the corresponding coroot $\alpha^\vee$ is given by $\frac{2\alpha}{(\alpha,\alpha)}$. The longest element of W will be denoted by w_0. Let $X^+(T)$ denotes the set of dominant weights. The element $\rho \in X^+(T)$ is defined by $2\rho = \Sigma_{\alpha\in\Phi^+} \alpha$. The dot action of W on $X(T)$ is given by $w \cdot \lambda = w(\lambda + \rho) - \rho$ where $\lambda \in X(T)$ and $w \in W$. For $\alpha \in S$, we denote by P_α the parabolic subgroup containing B which has α as its only positive root. Given an algebraic group J and a closed subgroup H we have the induction functor $\mathrm{Ind}_H^J : \mathrm{mod}(H) \to \mathrm{mod}(J)$. The category of rational $G-$modules has enough injectives and so we may define $\mathrm{Ext}^*(-, -)$ as usual by using injective resolutions see e.g. [8]. For $H \leq J \leq K$ and V an $H-$module we have a Grothendieck spectral sequence converging to $R^*\mathrm{Ind}_H^K\ V$, with E_2 page $R^i\mathrm{Ind}_J^K R^j\mathrm{Ind}_H^J\ V$. We will use the special case of Grothendieck spectral sequence for $B \leq P_\alpha \leq G$ with E_2 page $R^i\mathrm{Ind}_{P_\alpha}^G R^j\mathrm{Ind}_B^{P_\alpha}\ V$ and V a $B-$module. We will denote by V^* the dual of V.

For $\lambda \in X(T)$ we denote by k_λ the one dimensional (rational) $B-$module on which T acts via λ. In what follows we will also denote k_λ simply by λ. For $\lambda \in X^+(T)$ let $\nabla_\alpha(\lambda)$ denote the induced module $\mathrm{Ind}_B^{P_\alpha}\lambda$. We will write $\nabla(\lambda)$ for the induced module $\mathrm{Ind}_B^G\lambda$ and $\Delta(\lambda) = \nabla(-w_0\lambda)^*$. We will also write $H^i(M)$ for $R^i\mathrm{Ind}_B^G M$. We will denote by $P_\alpha/R_u(P_\alpha)$ the $P_\alpha-$module on which the unipotent radical $R_u(P_\alpha)$ acts trivially [6].

Let $F : G \to G$ denote the usual Frobenius morphism. By [8, II, 5.20 Proposition a] there is a unique (up to isomorphism) two dimensional indecomposable $B-$module with character $e(0)+e(-\alpha)$ denoted here by $N(\alpha)$ and let $N_\alpha(\lambda)$ denote the $B-$module $\lambda\otimes N(\alpha)$, $\lambda \in X(T)$. It is clear that $N_\alpha(\lambda) = \nabla_\alpha(\rho) \otimes (\lambda - \rho)$.

We will use the socle part of the result of H. H. Andersen given in [1] to give the socle of $H^1(N_\alpha(\lambda))$ in positive characteristic. The following proposition states the result for $H^1(\lambda)$ in the notation of [8, II, 5.15].

Proposition 1.1 *(Andersen [1])*
Suppose char(k)= $p \neq 0$. Let α be a simple root and $\lambda \in X(T)$ with $\langle s_\alpha \cdot \lambda, \alpha^\vee \rangle \geq 0$.

(1) *Suppose $\langle s_\alpha \cdot \lambda, \alpha^\vee \rangle = ap^n - 1$ for some non-negative integer n and for some positive integer a with $0 < a < p$. Then*

$$H^1(\lambda) \neq 0 \Longleftrightarrow s_\alpha \cdot \lambda \in X^+(T),$$

and if so, then $L(s_\alpha \cdot \lambda) = Soc_G(H^1(\lambda))$.

(2) *Let $\langle s_\alpha \cdot \lambda, \alpha^\vee \rangle = \sum_{j=0}^{n} a_j p^j$ with $0 \leq a_j < p$ and $a_n \neq 0$. Suppose there is some $j < n$ with $a_j < p - 1$. Then*

$$H^1(\lambda) \neq 0 \Longleftrightarrow \lambda + a_n p^n \alpha \in X^+(T),$$

and if so, then $Soc_G(H^1(\lambda)) = L(\lambda + a_n p^n \alpha)$.

We will also use the socle of the Weyl module for $G = SL_2(k)$ to compute the socle of $H^1(N_\alpha(\lambda))$. The following result is a special case of the above proposition

Proposition 1.2 [7, 6.9] *Let $G = SL_2(k)$ and $\mu \in X^+(T)$. Moreover $\mu + \rho = \sum_{j=0}^{n} a_j p^j \rho$ with $0 \leq a_j < p$ and $a_n \neq 0$ then*

$$Soc_G(\Delta((\sum_{j=0}^{n} a_j p^j - 1)\rho)) = L((a_n p^n - \sum_{j=0}^{n-1} a_j p^j - 1)\rho).$$

Let G_1 be the kernel of F and E be the natural module for $SL_2(k)$. From the above proposition we can easily deduce the following

Remark 1.3 For $G = SL_2(k)$ the module $E \otimes \Delta(p-1)$ has simple G_1-socle. This is clear that $E \otimes \Delta(p-1)$ is principal indecomposable as a G_1-module. Moreover $\mathrm{Soc}_{G_1}(E \otimes \Delta(p-1)) = L(p-2)$.

We will first look at the case when $\mathrm{char}(k) = 0$.

2 Characteristic Zero Case

Throughout this section k will be a field of characteristic zero.

Theorem 2.1 *Let $\lambda \in X(T)$, $\alpha \in S$ and $i > 0$. Then*

$$H^i(N_\alpha(\lambda)) = \begin{cases} H^i(\lambda) \oplus H^i(\lambda - \alpha) & \langle \lambda, \alpha^\vee \rangle \neq 0 \\ 0 & \langle \lambda, \alpha^\vee \rangle = 0. \end{cases}$$

Proof We give the proof in separate cases.

(1) Let $\langle \lambda, \alpha^\vee \rangle < -1$. Using the spectral sequence we have

$$H^i(N_\alpha(\lambda)) = R^{i-1}\mathrm{Ind}_{P_\alpha}^{G} R^1\mathrm{Ind}_B^{P_\alpha}(N_\alpha(\lambda)).$$

Since $\langle\lambda, \alpha^\vee\rangle < -1$ we have $\mathrm{Ind}_B^{P_\alpha}(\lambda-\alpha) = 0$. Moreover P_α/B is one dimensional so $R^i\mathrm{Ind}_B^{P_\alpha}(\lambda-\alpha) = 0$ for all $i \geq 2$. Hence from the exact sequence $0 \to \lambda-\alpha \to N_\alpha(\lambda) \to \lambda \to 0$ we get

$$0 \to R^1\mathrm{Ind}_B^{P_\alpha}(\lambda-\alpha) \to R^1\mathrm{Ind}_B^{P_\alpha}(N_\alpha(\lambda)) \to R^1\mathrm{Ind}_B^{P_\alpha}(\lambda) \to 0.$$

Since all modules for $P_\alpha/R_u(P_\alpha)$ are completely reducible so we get $R^1\mathrm{Ind}_B^{P_\alpha}(N_\alpha(\lambda)) \simeq R^1\mathrm{Ind}_B^{P_\alpha}(\lambda-\alpha) \oplus R^1\mathrm{Ind}_B^{P_\alpha}(\lambda)$. Therefore $H^i(N_\alpha(\lambda)) = R^{i-1}\mathrm{Ind}_{P_\alpha}^G(R^1\mathrm{Ind}_B^{P_\alpha}(\lambda-\alpha) \oplus R^1\mathrm{Ind}_B^{P_\alpha}(\lambda))$ and we get the result.

(2) For $\langle\lambda, \alpha^\vee\rangle = -1$ we get $H^i(\lambda) = 0$ for all i. Also $H^0(\lambda-\alpha) = 0$. Therefore by the long exact sequence of induction we get $H^i(N_\alpha(\lambda)) = H^i(\lambda-\alpha)$.

(3) Consider now the case $\langle\lambda, \alpha^\vee\rangle = 0$ we get $\langle\lambda-\rho, \alpha^\vee\rangle = -1$ and hence $R^j\mathrm{Ind}_B^{P_\alpha}(\lambda-\rho) = 0$. So we get

$$\begin{aligned} H^i(N_\alpha(\lambda)) &= R^i\mathrm{Ind}_{P_\alpha}^G R^j\mathrm{Ind}_B^{P_\alpha}(\nabla_\alpha(\rho) \otimes (\lambda-\rho)) \\ &= R^i\mathrm{Ind}_{P_\alpha}^G(\nabla_\alpha(\rho) \otimes R^j\mathrm{Ind}_B^{P_\alpha}(\lambda-\rho)) = 0. \end{aligned}$$

(4) For $\langle\lambda, \alpha^\vee\rangle = 1$ we consider the long exact sequence

$$\begin{aligned} 0 &\to H^0(\lambda-\alpha) \to H^0(N_\alpha(\lambda)) \to H^0(\lambda) \to H^1(\lambda-\alpha) \\ &\to H^1(N_\alpha(\lambda)) \to H^1(\lambda) \to H^2(\lambda-\alpha) \to \cdots \end{aligned}$$

Since $\langle\lambda-\alpha, \alpha^\vee\rangle = -1$ we get $H^i(\lambda-\alpha) = 0$. Hence $H^i(N_\alpha(\lambda)) = H^i(\lambda)$.

(5) Let $\langle\lambda, \alpha^\vee\rangle \geq 2$. Using the spectral sequence we have

$$\begin{aligned} &R^i\mathrm{Ind}_{P_\alpha}^G R^j\mathrm{Ind}_B^{P_\alpha}(\nabla_\alpha(\rho) \otimes (\lambda-\rho)) \\ &= R^i\mathrm{Ind}_{P_\alpha}^G(\nabla_\alpha(\rho) \otimes R^j\mathrm{Ind}_B^{P_\alpha}(\lambda-\rho)). \end{aligned}$$

Since $\langle\lambda, \alpha^\vee\rangle \geq 2$ we have that $R^j\mathrm{Ind}_B^{P_\alpha}(\lambda-\rho)$ is zero for all $j \neq 0$. Therefore $R^i\mathrm{Ind}_{P_\alpha}^G R^j\mathrm{Ind}_B^{P_\alpha}(N_\alpha(\lambda)) = R^i\mathrm{Ind}_{P_\alpha}^G(\nabla_\alpha(\rho) \otimes \nabla_\alpha(\lambda-\rho))$. Since the weights of $\nabla_\alpha(\rho)$ are ρ and $\rho-\alpha$ and in characteristic zero we get

$$H^i(N_\alpha(\lambda)) = H^i(\lambda) \oplus H^i(\lambda-\alpha).$$

This gives us the required result.

□

Remark 2.2 If $\langle\lambda, \alpha^\vee\rangle = 1$ then $H^i(N_\alpha(\lambda)) = H^i(\lambda)$. Moreover for $\langle\lambda, \alpha^\vee\rangle = -1$ we have $H^i(N_\alpha(\lambda)) = H^i(\lambda-\alpha)$.

3 Characteristic *p* Case

In this section we look at the case when $\mathrm{char}(k) = p > 0$. The following theorem gives the cohomology of $N_\alpha(\lambda)$ when λ is a dominant weight. This is essentially a version of Kempf's vanishing theorem for these rank 2 bundles.

Theorem 3.1 *Let* $\lambda \in X^+(T)$ *and* $\alpha \in S$.

(1) *If* $\langle\lambda, \alpha^\vee\rangle > 0$ *then* $H^i(N_\alpha(\lambda)) = 0$ *for* $i > 0$.

(2) *If* $\langle \lambda, \alpha^\vee \rangle = 0$ *then* $H^i(N_\alpha(\lambda)) = 0$ *for all* i.

Proof We have a short exact sequence

$$0 \to \lambda - \alpha \to N_\alpha(\lambda) \to \lambda \to 0.$$

This gives rise to a long exact sequence of induction given by

$$\begin{aligned} 0 \to H^0(\lambda - \alpha) \to H^0(N_\alpha(\lambda)) \to\ & H^0(\lambda) \to H^1(\lambda - \alpha) \\ \to\ & H^1(N_\alpha(\lambda)) \to H^1(\lambda) \to \cdots \end{aligned} \tag{1}$$

Now $\lambda \in X^+(T)$ so $H^i(\lambda) = 0$ for all $i > 0$ by Kempf's vanishing theorem. For $\langle \lambda, \alpha^\vee \rangle > 0$ we have $\langle \lambda - \alpha, \alpha^\vee \rangle \geq -1$ and $H^i(\lambda - \alpha) = 0$ because either $\langle \lambda - \alpha, \alpha^\vee \rangle = -1$ or $\lambda - \alpha \in X^+(T)$ and the result is true by Kemp's vanishing theorem. So $H^i(N_\alpha(\lambda)) = 0$.

For $\langle \lambda, \alpha^\vee \rangle = 0$ we use the spectral sequence given by $R^i\mathrm{Ind}_{P_\alpha}^G R^j\mathrm{Ind}_B^{P_\alpha}(\nabla_\alpha(\rho) \otimes (\lambda - \rho)) = R^i\mathrm{Ind}_{P_\alpha}^G(\nabla_\alpha(\rho) \otimes R^j\mathrm{Ind}_B^{P_\alpha}(\lambda - \rho))$. Since $\langle \lambda, \alpha^\vee \rangle = 0$ so $\langle \lambda - \rho, \alpha^\vee \rangle = -1$ and hence $R^j\mathrm{Ind}_B^{P_\alpha}(\lambda - \rho) = 0$. □

Remark 3.2 Let $\lambda \in X(T)$ and $\alpha \in S$ with $\langle \lambda, \alpha^\vee \rangle = -1$. We have $H^i(N_\alpha(\lambda)) = H^i(\lambda - \alpha)$. This follows directly from the long exact sequence given in (1).

The following theorem gives us a condition when $H^1(N_\alpha(\lambda))$ will have the same form as in characteristic zero.

Theorem 3.3 *Let* $\lambda \in X(T)$ *and* $\alpha \in S$ *with* $\langle \lambda, \alpha^\vee \rangle \leq -2$.

(1) *If* p *does not divide* $\langle \lambda, \alpha^\vee \rangle$ *then for all* $i \geq 0$ *we have*

$$H^i(N_\alpha(\lambda)) \simeq H^i(\lambda) \oplus H^i(\lambda - \alpha).$$

(2) *If* p *divides* $\langle \lambda, \alpha^\vee \rangle$ *then* $H^1(N_\alpha(\lambda))$ *is indecomposable as a* G*−module.*

Proof Since $\langle \lambda, \alpha^\vee \rangle \leq -2$ by spectral sequence we have $H^1(N_\alpha(\lambda)) = \mathrm{Ind}_{P_\alpha}^G R^1\mathrm{Ind}_B^{P_\alpha}(N_\alpha(\lambda))$. Since $\mathrm{Ind}_B^{P_\alpha}(\lambda - \alpha) = 0$ and $R^2\mathrm{Ind}_B^{P_\alpha}(\lambda - \alpha) = 0$ we get the short exact sequence

$$0 \to R^1\mathrm{Ind}_B^{P_\alpha}(\lambda - \alpha) \to R^1\mathrm{Ind}_B^{P_\alpha}(N_\alpha(\lambda)) \to R^1\mathrm{Ind}_B^{P_\alpha}(\lambda) \to 0. \tag{2}$$

Note that $R^1\mathrm{Ind}_B^{P_\alpha}(N_\alpha(\lambda)) = \nabla_\alpha(\rho) \otimes R^1\mathrm{Ind}_B^{P_\alpha}(\lambda - \rho)$. By applying Serre duality on $R^1\mathrm{Ind}_B^{P_\alpha}(\lambda - \rho)$ we get from (2)

$$0 \to \Delta_\alpha(-\lambda) \to \nabla_\alpha(\rho) \otimes \Delta_\alpha(-\lambda + \rho - \alpha) \to \Delta_\alpha(-\lambda - \alpha) \to 0. \tag{3}$$

We switch to the SL_2−notation here and take $\langle \lambda, \alpha^\vee \rangle = -m$, with $m \geq 2$ to get the exact sequence

$$0 \to \Delta(m) \to \nabla_\alpha(\rho) \otimes \Delta(m-1) \to \Delta(m-2) \to 0. \tag{4}$$

Using the linkage principle this sequence splits whenever m and $m-2$ are not linked. It is clear that m and $m-2$ are linked if p divides m. Therefore the sequence (4) splits if p does not divide m. Therefore the sequence (3) splits if p does not divide $\langle \lambda, \alpha^\vee \rangle$.

Now suppose p divides $\langle \lambda, \alpha^\vee \rangle$ and consider the sequence

$$0 \to \Delta_\alpha(-\lambda) \to \nabla_\alpha(\rho) \otimes \Delta_\alpha(-\lambda + \rho - \alpha) \to \Delta_\alpha(-\lambda - \alpha) \to 0.$$

Since $\nabla_\alpha(\rho) \otimes \Delta_\alpha(-\lambda + \rho - \alpha)$ is projective as a $(G_\alpha)_1-$module but $\Delta_\alpha(-\lambda)$ is not, therefore this sequence is never split. Hence the sequence (2) splits if and only if p does not divide $\langle \lambda, \alpha^\vee \rangle$.

Now if p does not divide $\langle \lambda, \alpha^\vee \rangle$ then $R^1\mathrm{Ind}_B^{P_\alpha}(N_\alpha(\lambda)) \simeq R^1\mathrm{Ind}_B^{P_\alpha}(\lambda) \oplus R^1\mathrm{Ind}_B^{P_\alpha}(\lambda - \alpha)$. Therefore $H^i(N_\alpha(\lambda)) \simeq R^{i-1}\mathrm{Ind}_{P_\alpha}^G(R^1\mathrm{Ind}_B^{P_\alpha}(\lambda) \oplus R^1\mathrm{Ind}_B^{P_\alpha}(\lambda - \alpha))$. Since $\langle \lambda, \alpha^\vee \rangle \le -2$ by [8, II, 5.4 proposition (c)] we have

$$R^{i-1}\mathrm{Ind}_{P_\alpha}^G(R^1\mathrm{Ind}_B^{P_\alpha}(\lambda - \alpha)) = H^i(\lambda - \alpha).$$

Combining this with the above statement we get the required result in part (1).

Suppose p divides $\langle \lambda, \alpha^\vee \rangle$. We will first now show that $H^1(N_\alpha(\lambda)) = \mathrm{Ind}_{P_\alpha}^G(\nabla_\alpha(\rho) \otimes \Delta_\alpha(-\lambda + \rho - \alpha))$ is indecomposable by proving that it has a simple socle. We have

$$\begin{aligned}
&\mathrm{Hom}_G(L(\mu), \mathrm{Ind}_{P_\alpha}^G(\nabla_\alpha(\rho) \otimes \Delta_\alpha(-\lambda + \rho - \alpha)))\\
&= H^0(G, L(-w_0\mu) \otimes \mathrm{Ind}_{P_\alpha}^G(\nabla_\alpha(\rho) \otimes \Delta_\alpha(-\lambda + \rho - \alpha)))\\
&= H^0(G, \mathrm{Ind}_{P_\alpha}^G(L(-w_0\mu) \otimes \nabla_\alpha(\rho) \otimes \Delta_\alpha(-\lambda + \rho - \alpha)))\\
&= H^0(P_\alpha, L(-w_0\mu) \otimes \nabla_\alpha(\rho) \otimes \Delta_\alpha(-\lambda + \rho - \alpha))\\
&= H^0(G_\alpha, H^0(R_u(P_\alpha)), L(-w_0\mu) \otimes \nabla_\alpha(\rho) \otimes \Delta_\alpha(-\lambda + \rho - \alpha))\\
&= [L_\alpha(-s_\alpha\mu) \otimes \nabla_\alpha(\rho) \otimes \Delta_\alpha(-\lambda + \rho - \alpha)]^{G_\alpha}\\
&= \mathrm{Hom}_{G_\alpha}(L_\alpha(\mu), \nabla_\alpha(\rho) \otimes \Delta_\alpha(-\lambda + \rho - \alpha))
\end{aligned}$$

Since p divides $\langle \lambda, \alpha^\vee \rangle$ and we are now in the case of G_α therefore $\langle -\lambda + \rho - \alpha, \alpha^\vee \rangle \equiv 1$ mod p. We get $E \otimes \Delta((p-1) + pm) = E \otimes \Delta(p-1) \otimes \Delta(m)^F$. Also

$$\mathrm{Soc}_G(E \otimes \Delta(p-1) \otimes \Delta(m)^F) = \mathrm{Soc}_G(\mathrm{Soc}_{G_1}(E \otimes \Delta(p-1) \otimes \Delta(m)^F)).$$

By remark 1.3 we get

$$\mathrm{Soc}_G(E \otimes \Delta(p-1) \otimes \Delta(m)^F) = \mathrm{Soc}_G(L(p-2) \otimes \Delta(m)^F).$$

Also by proposition 1.2, $\Delta(m)$ has simple socle therefore we compute

$$\mathrm{Hom}_G(L(p-2) \otimes L(r)^F, L(p-2) \otimes \Delta(m)^F) = \mathrm{Hom}_G(L(r), \Delta(m)).$$

It is clear from this argument that $\mathrm{Ind}_{P_\alpha}^G(\nabla_\alpha(\rho) \otimes \Delta_\alpha(-\lambda + \rho - \alpha))$ has simple socle and hence is indecomposable. □

The following proposition and the subsequent corollary and its remark describes the main difficulty in the case when p divides $\langle \lambda, \alpha^\vee \rangle$.

Proposition 3.4 *If $\langle \lambda, \alpha^\vee \rangle = -np$ for some $n \le p$, then*

$$H^{i+1}(N_\alpha(\lambda)) = H^i(N_\alpha(s_\alpha\lambda)),$$

for all i.

Proof Consider $R^1\mathrm{Ind}_B^{P_\alpha}(N_\alpha(\lambda)) = \nabla_\alpha(\rho) \otimes R^1\mathrm{Ind}_B^{P_\alpha}(\lambda - \rho)$. If $\langle \lambda, \alpha^\vee \rangle = -np$ we get $R^1\mathrm{Ind}_B^{P_\alpha}(N_\alpha(\lambda)) = \mathrm{Ind}_B^{P_\alpha}(N_\alpha(s_\alpha\lambda))$. This gives us the required result. □

Corollary 3.5 *If $\langle \lambda, \alpha^\vee \rangle = -np$ for some $n \le p$ and $s_\alpha\lambda$ belongs to the closure of the dominant Weyl chamber* [8] *then $H^i(N_\alpha(\lambda)) = 0$ for all $i \ne 1$.*

Proof The result is clear from the above proposition. □

Remark 3.6 It is worth mentioning here that the sequence

$$0 \to H^1(\lambda - \alpha) \to H^1(N_\alpha(\lambda)) \to H^1(\lambda) \to 0.$$

is not always exact. For example take $\langle \lambda, \alpha^\vee \rangle = -np$ and if $s_\alpha \lambda$ belongs to the closure of dominant Weyl chamber then whenever $H^2(\lambda - \alpha) \neq 0$ we have an exact sequence

$$0 \to H^1(\lambda - \alpha) \to H^1(N_\alpha(\lambda)) \to H^1(\lambda) \to H^1(\lambda - \alpha) \to 0$$

but the three term sequence

$$0 \to H^1(\lambda - \alpha) \to H^1(N_\alpha(\lambda)) \to H^1(\lambda) \to 0$$

is not exact. For a special case in type A_2 let $\lambda = -p w_\alpha + (p-1) w_\beta$, where w_α, w_β are fundamental dominant weights.

The following result gives the socle of $H^1(N_\alpha(\lambda))$.

Proposition 3.7 *Suppose $\lambda \in X(T)$ with $\langle \lambda, \alpha^\vee \rangle \leq -2$.*

(1) *If $p \nmid \langle \lambda, \alpha^\vee \rangle$ then*

$$Soc_G(H^1(N_\alpha(\lambda))) = Soc_G(H^1(\lambda)) \oplus Soc_G(H^1(\lambda - \alpha)).$$

(2) *If $p \mid \langle \lambda, \alpha^\vee \rangle$ write $\langle \lambda, \alpha^\vee \rangle = -pm,\ m > 0$ with $m + 1 = \sum_{j=0}^{n} a_j p^j$. We have*

$$Soc_G(H^1(N_\alpha(\lambda))) = L(a_n p^n - \sum_{j=0}^{n-1} a_j p^j - 1).$$

Proof If $p \nmid \langle \lambda, \alpha^\vee \rangle$ then by Theorem 3.3 we get $H^1(N_\alpha(\lambda)) = H^1(\lambda) \oplus H^1(\lambda - \alpha)$ and the result is clear. Moreover $\mathrm{Soc}_G(H^1(\lambda))$ and $\mathrm{Soc}_G(H^1(\lambda - \alpha))$ are given by Proposition 1.1.

When $p \mid \langle \lambda, \alpha^\vee \rangle$ we compute

$$\begin{aligned} \mathrm{Hom}_G(L(\mu), H^1(N_\alpha(\lambda)) &= H^0(G, L(-w_0\mu) \otimes H^1(N_\alpha(\lambda)) \\ &= \left[R^1 \mathrm{Ind}_B^G (L(-w_0\mu) \otimes H^1(N_\alpha(\lambda))) \right]^G \end{aligned}$$

By using Frobenius reciprocity we get

$$\left[R^1 \mathrm{Ind}_B^G (L(-w_0\mu) \otimes H^1(N_\alpha(\lambda))) \right]^G = R^1 \mathrm{Ind}_B^G (L(-w_0\mu) \otimes H^1(N_\alpha(\lambda)))^G$$

We know that $R^1 \mathrm{Ind}_B^G M = \mathrm{Ind}_{P_\alpha}^G R^1 \mathrm{Ind}_B^{P_\alpha} M$ and $N_\alpha(\lambda) = \nabla_\alpha(\rho) \otimes (\lambda - \rho)$ therefore

$$\begin{aligned} & R^1 \mathrm{Ind}_B^G (L(-w_0\mu) \otimes H^1(N_\alpha(\lambda)))^G \\ &= \mathrm{Ind}_{P_\alpha}^G (L(-w_0\mu) \otimes \nabla_\alpha(\rho) \otimes R^1 \mathrm{Ind}_B^{P_\alpha} (\lambda - \alpha))^G \\ &= \mathrm{Ind}_{P_\alpha}^G (L(-w_0\mu) \otimes \nabla_\alpha(\rho) \otimes \Delta_\alpha(-\lambda + \rho - \alpha))^G \\ &= (L(-w_0\mu) \otimes \nabla_\alpha(\rho) \otimes \Delta_\alpha(-\lambda + \rho - \alpha))^{P_\alpha} \end{aligned}$$

We have $G_\alpha = P_\alpha/R_u(P_\alpha)$ and $R_u(P_\alpha)$ acts trivially on $\nabla_\alpha(\rho) \otimes \Delta_\alpha(-\lambda + \rho - \alpha)$. Also $(L(-w_0\mu))^{R_u(P_\alpha)} = L_{G_\alpha}(-s_\alpha\mu)$ therefore we get

$$\begin{aligned}\mathrm{Hom}_G(L(\mu), H^1(N_\alpha(\lambda)) &= [L_\alpha(-s_\alpha\mu) \otimes \nabla_\alpha(\rho) \otimes \Delta_\alpha(-\lambda + \rho - \alpha)]^{G_\alpha} \\ &= \mathrm{Hom}_{G_\alpha}(L_\alpha(\mu), \nabla_\alpha(\rho) \otimes \Delta_\alpha(-\lambda + \rho - \alpha))\end{aligned}$$

We take $\langle \lambda, \alpha^\vee \rangle = -pm$. Following the argument of Theorem 3.3 we get

$$\mathrm{Soc}_{SL_2}(E \otimes \Delta((p-1) + pm)) = \mathrm{Soc}_{SL_2}(L(p-2) \otimes \Delta(m))^F.$$

Also

$$\mathrm{Hom}_{SL_2}(L(p-2) \otimes L(r)^F, L(p-2) \otimes \Delta(m)^F) = \mathrm{Hom}_{SL_2}(L(r), \Delta(m)).$$

We follow the notation of Proposition 1.2 and write $m + 1 = \sum_{j=0}^{n} a_j p^j$. We have

$$\mathrm{Soc}_{SL_2}\Delta(m) = L(a_n p^n - \sum_{j=0}^{n-1} a_j p^j - 1).$$ □

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