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Gaiotto's Lagrangian Subvarieties via Derived Symplectic Geometry

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Abstract Let Bun_G be the moduli space of G-bundles on a smooth complex projective curve. Motivated by a study of boundary conditions in mirror symmetry, Gaiotto (2016) associated to any symplectic representation of G a Lagrangian subvariety of T^*Bun_G. We give a simple interpretation of (a generalization of) Gaiotto's construction in terms of derived symplectic geometry. This allows to consider a more general setting where symplectic G-representations are replaced by arbitrary symplectic manifolds equipped with a Hamiltonian G-action and with an action of the multiplicative group that rescales the symplectic form with positive weight.

Keywords Lagrangian subvariety · G-bundles · Derived symplectic stack

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1 Statement of the Result

We will use the language of derived stacks. Throughout, a 'stack' means a 'derived Artin stack over $k = \mathbb{C}$' in the sense of [5] and [11]. We write $B\mathbb{G} = \mathrm{pt}/\mathbb{G}$ for the classifying stack of a group $\mathbb{G}$. We fix a smooth complex projective variety X and let K_X denote the canonical bundle. We write G for an algebraic group and $Bun_G(X)$, resp. $Higgs_G(X)$, for the stack

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To Alexander Alexandrovich Kirillov on his 80th Birthday.

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of G-bundles, resp. Higgs bundles, on X. One has a canonical isomorphism $Bun_G(X) \cong \mathrm{Map}(X, BG)$, where $\mathrm{Map}(X, Z)$ denotes a mapping stack that classifies morphisms $X \to Z$.

Given a $\mathbb{G}_m$-stack $\mathcal{Y}$ and a $\mathbb{G}_m$-bundle $L \to X$, there is an associated bundle $\mathcal{Y}_L := \mathcal{Y} \times_{\mathbb{G}_m} L$. Let $\mathcal{S}ect_X(\mathcal{Y}_L)$ be the stack of sections of the projection $\mathcal{Y}_L \to X$. By definition, we have $\mathcal{S}ect_X(\mathcal{Y}_L) = \{\mathrm{Id}_X\} \times_{\mathrm{Map}(X,X)} \mathrm{Map}(X, \mathcal{Y}_L)$. The $\mathbb{G}_m$-action on the first factor of $\mathcal{Y} \times L$ descends to a $\mathbb{G}_m$-action along the fibers of $\mathcal{Y}_L \to X$. This induces a natural $\mathbb{G}_m$-action on $\mathcal{S}ect_X(\mathcal{Y}_L)$.

Remarks 1.1 Let $L \to X$ be a $\mathbb{G}_m$-bundle and $\mathcal{L}$ an associated line bundle on X.

(i) We will abuse the notation and write $Y_{\mathcal{L}}$ for Y_L.

(ii) For a $\mathbb{G}_m$-stack $\mathcal{Y}$, there is a canonical isomorphism $\mathcal{Y}_L \cong \mathcal{Y}/\mathbb{G}_m \times_{B\mathbb{G}_m} X$, where we have used the map $X = L/\mathbb{G}_m \to B\mathbb{G}_m = \mathrm{pt}/\mathbb{G}_m$ that classifies L.

(iii) For a $(G \times \mathbb{G}_m)$-stack $\mathcal{Y}$, we will often use natural identifications $(\mathcal{Y}/G)_L = (\mathcal{Y} \times L)/(G \times \mathbb{G}_m) = (\mathcal{Y}_L)/G$.

Let M be a smooth symplectic algebraic manifold equipped with a $G \times \mathbb{G}_m$-action such that the action of the group $G = G \times \{1\}$ on M is Hamiltonian and the symplectic 2-form has weight $\ell \geq 1$ with respect to the action of $\mathbb{G}_m = \{1\} \times \mathbb{G}_m$. Assume that there exists a line bundle $K_X^{1/\ell}$, an ℓ-th root of K_X, and fix a choice of $K_X^{1/\ell}$.

Following Gaiotto, [6], we consider the stack $\mathcal{S}ect_X(M_{K_X^{1/\ell}}/G)$. This stack classifies pairs (P, s), where P is a $(G \times \mathbb{G}_m)$-bundle on X and $s : P \to M \times \overset{\circ}{K}{}_X^{1/\ell}$ is a $(G \times \mathbb{G}_m)$-equivariant morphism that intertwines the natural projections $P \to X$ and $M \times \overset{\circ}{K}{}_X^{1/\ell} \to X$. Here $\overset{\circ}{K}{}_X^{1/\ell}$ denotes the $\mathbb{G}_m$-bundle obtained from $K_X^{1/\ell}$ by removing the zero section. The group G acts on $M \times \overset{\circ}{K}{}_X^{1/\ell}$ through its action on the first factor and $\mathbb{G}_m$ acts diagonally.

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^*$ the dual of $\mathfrak{g}$. The group $G \times \mathbb{G}_m$ acts on $\mathfrak{g}^*$, where G acts by the coadjoint action and $\mathbb{G}_m$ acts by dilations. The symplectic 2-form on M being of weight ℓ, the moment map $\mu : M \to \mathfrak{g}^*$ intertwines, for any $t \in \mathbb{G}_m$, the t-action on M with dilation by t^ℓ on $\mathfrak{g}^*$. It follows that μ gives a well defined morphism $M_{K_X^{1/\ell}} \to \mathfrak{g}^*_{K_X}$, of stacks over X. Therefore, there is an induced morphism

$$\mu_{\mathcal{S}ect} : \mathcal{S}ect_X(M_{K_X^{1/\ell}}/G) \longrightarrow \mathcal{S}ect_X(\mathfrak{g}^*_{K_X}/G). \tag{1.1}$$

We now specialize to the case where $X = \Sigma$ is a smooth projective curve and G is reductive. In such a case, we have $\mathcal{S}ect_\Sigma(\mathfrak{g}^*_{K_\Sigma}/G) \cong Higgs_G(X) \cong T^*Bun_G(\Sigma)$. Let $T^*Bun_G(\Sigma)^{reg}$ be an open substack of $T^*Bun_G(\Sigma)$ that corresponds to the Higgs bundles whose only automorphisms lie in the center. It is known that $T^*Bun_G(\Sigma)^{reg}$ is a smooth variety that comes equipped with a natural symplectic 2-form ω.

Theorem 1.2 *The map $\mu_{\mathcal{S}ect}$ is Lagrangian, specifically, the 2-form $\mu^*_{\mathcal{S}ect}(\omega)$ vanishes on the preimage of $T^*Bun_G(\Sigma)^{reg}$.*

The above result was discovered by Gaiotto [6] in the linear case, i.e. in the special case where M is a symplectic representation of G. In this case, $\mathbb{G}_m$ acts on M, a symplectic vector space, by dilations and the symplectic form on M has weight 2.

One of the goals of this paper is to show that Theorem 1.2 is a simple consequence of some very general results of derived symplectic geometry.

2 Derived Symplectic Geometry

Let n be an integer and Y a stack equipped with an n-shifted symplectic structure in the sense of [11]. There is a notion of "Lagrangian structure" on a morphism $Z \to Y$, see [11, §2.2] and [2]. One has the following result, where part (i) is [11, Theorem 0.4], resp. part (ii) is [2, Therorem 2.10].

Theorem 2.1 *Let X be a smooth projective Calabi-Yau variety of dimension d. Then, one has:*

(i) *An n-shifted symplectic structure on a stack Y gives rise to a natural $(n-d)$-shifted symplectic structure on* $\mathrm{Map}(X, Y)$.

(ii) *A Lagrangian structure $f : Z \to Y$ gives rise to a natural Lagrangian structure on* $\mathrm{Map}(X, Z) \to \mathrm{Map}(X, Y)$*, the morphism of mapping stacks induced by f.*

It was shown, see [11, Corollary 2.6(2)], that part (i) of the theorem implies the following

Corollary 2.2 *For any smooth projective Calabi-Yau variety X of dimension d the stack $Higgs_G(X)$ has a canonical $2(1-d)$-shifted symplectic structure.*

In the case where X is a Fano variety suitable analogs of the statements of Theorem 2.1 were proved by Spaide [14], Theorems 3.3 and 3.5.

Below, we propose a modification of the above results that holds for more general, not necessarily Calabi-Yau, varieties X.

To this end, we recall some notions from derived algebraic geometry. For a (derived) stack $\mathcal{X}$, we will denote by $\mathrm{QCoh}(\mathcal{X})$ the (unbouded) derived ∞-category of quasi-coherent sheaves on $\mathcal{X}$ (see, e.g. [5] Vol. 1, Chapter 3, for a detailed account of this ∞-category). We will refer to objects of $\mathrm{QCoh}(\mathcal{X})$ as "sheaves on $\mathcal{X}$". Given $\mathcal{M} \in \mathrm{QCoh}(\mathcal{X})$, we will denote by $\Gamma(\mathcal{X}, \mathcal{M}) = Hom(\mathcal{O}_{\mathcal{X}}, \mathcal{M})$, the (*derived*) functor of global sections.

Let $f : Y \to \mathcal{X}$ be a map of stacks and $\mathbb{L}_{Y/\mathcal{X}} \in \mathrm{QCoh}(Y)$ the relative cotangent complex of f. One has a sheaf

$$\tilde{\mathcal{A}}^p_{\mathcal{X}}(Y) := f_*(\wedge^p \mathbb{L}_{Y/\mathcal{X}}) \in \mathrm{QCoh}(\mathcal{X}),$$

of relative p-forms. There is also a sheaf $\tilde{\mathcal{A}}^{p,cl}_{\mathcal{X}}(Y) \in \mathrm{QCoh}(\mathcal{X})$, of relative closed p-forms. The sheaf $\tilde{\mathcal{A}}^{p,cl}_{\mathcal{X}}(Y)$ comes equipped with a forgetful map $\tilde{\mathcal{A}}^{p,cl}_{\mathcal{X}}(Y) \to \tilde{\mathcal{A}}^p_{\mathcal{X}}(Y)$ which assigns to a closed p-form its underlying p-form (see [4, Sect. 1] or [5, Vol. II, Chapter 9] for a discussion of relative differential forms). Note that in the derived setting, a closed p-form is a p-form equipped with additional closure data (as opposed to satisfying a condition).

We will use the following basic result about relative differential forms:

Lemma 2.3 *Let*

$$\begin{array}{ccc} Y_2 & \longrightarrow & Y_1 \\ \downarrow & & \downarrow \\ \mathcal{X}_2 & \xrightarrow{g} & \mathcal{X}_1 \end{array}$$

be a commutative square of stacks. Then, for each $i \geq 0$, there is a natural map

$$\phi_{i,cl} : g^*(\tilde{\mathcal{A}}^{i,cl}_{\mathcal{X}_1}(Y_1)) \to \tilde{\mathcal{A}}^{i,cl}_{\mathcal{X}_2}(Y_2).$$

Moreover, if the square is Cartesian and $\mathbb{L}_{Y_1/\mathcal{X}_1}$ is perfect (more generally, it is sufficient to require $\mathbb{L}_{Y_1/\mathcal{X}_1}$ be bounded below) then the map $\phi_{p,cl}$ is an isomorphism.

Definition Let $p : Y \to \mathcal{X}$ be a map of stacks and $\mathcal{L}$ a line bundle on $\mathcal{X}$. We put

$$\mathcal{A}^i(Y/\mathcal{X};\mathcal{L}) := \Gamma(\mathcal{X}, \tilde{\mathcal{A}}^i_{\mathcal{X}}(Y)\otimes\mathcal{L}), \text{ and } \mathcal{A}^{i,cl}(Y/\mathcal{X};\mathcal{L}) := \Gamma(\mathcal{X}, \tilde{\mathcal{A}}^{i,cl}_{\mathcal{X}}(Y)\otimes\mathcal{L}).$$

(i) Assume the relative cotangent complex of $p : Y \to \mathcal{X}$ is perfect. *An $\mathcal{L}$-twisted n-shifted relative symplectic structure on Y* is a twisted relative closed 2-form $\omega \in Hom(k, \mathcal{A}^{2,cl}(Y/\mathcal{X};\mathcal{L})[n])$ such that the underlying 2-form is nondegenerate, i.e. it induces an isomorphism

$$\mathbb{L}^\vee_{Y/\mathcal{X}} \xrightarrow{\sim} \mathbb{L}_{Y/\mathcal{X}}[n]\otimes p^*(\mathcal{L}).$$

(ii) Assume that $p : Y \to \mathcal{X}$ is equipped with an $\mathcal{L}$-twisted n-shifted relative symplectic structure and let $f : Z \to Y$ be a map of stacks with perfect relative cotangent complex. *An ($\mathcal{L}$-twisted n-shifted) Lagrangian structure on f* is a nullhomotopy of $f^*(\omega) \in Hom(k, \mathcal{A}^{2,cl}_{\mathcal{X}}(Z;\mathcal{L})[n])$ such that the map

$$\mathbb{L}^\vee_{Z/\mathcal{X}} \to \mathbb{L}_{Z/Y}[n-1]\otimes(f\circ p)^*(\mathcal{L}),$$

induced by the nullhomotopy of the underlying 2-form, is an isomorphism.

The proposition below gives a preliminary version of our main construction. In Section 3, we will describe how to obtain relative twisted symplectic, resp. Lagrangian, structures from symplectic, resp. Lagrangian, sturctures of a fixed weight on a $\mathbb{G}_m$-stack.

Proposition 2.4 *Let X be a smooth projective variety of dimension d and Y, Z a pair of stacks.*

(i) *A K_X-twisted relative n-shifted symplectic structure on a morphism $Y \to X$ induces an $(n-d)$-shifted symplectic structure on $\mathcal{S}ect_X(Y)$.*
(ii) *A K_X-twisted relative Langrangian structure on $Z \to Y$ induces a Lagrangian structure on*

$$\mathcal{S}ect_X(Z) \to \mathcal{S}ect_X(Y).$$

Proof Following [11], we consider the evaluation map

$$\mathcal{S}ect_X(Y)\times X \xrightarrow{\text{ev}} Y,$$

a map of stacks over X. By Lemma 2.3, there is a pull-back morphism in QCoh(X):

$$ev^* : \tilde{\mathcal{A}}^{2,cl}_X(Y)\otimes_{\mathcal{O}_X} K_X \to \mathcal{A}^{2,cl}(\mathcal{S}ect_X(Y))\otimes_k K_X.$$

Using an integration map $\int_X : \Gamma(X, K_X) \to k[-d]$ provided by Serre duality, one obtains a map

$$\left(\text{Id}\times\int_X\right)\circ ev^* : \mathcal{A}^{2,cl}(Y/X;K_X) \to \mathcal{A}^{2,cl}(\mathcal{S}ect_X(Y)).$$

Now, the same argument as in [11] shows that if the twisted 2-form ω on Y is nondegenerate then so is the 2-form

$$\omega_{\mathcal{S}ect} := \left(\text{Id}\times\int_X\right)\circ ev^*)(\omega).$$

This proves part (i) of Proposition 2.4. The proof of part (ii) is obtained by similarly tweaking the proof of [2, Therorem 2.10]. □

Remarks 2.5 (i) The same proof works in a more general setting where X is any strictly $\mathcal{O}$-compact stack in the sense of [11, Definition 2.1] equipped with a line bundle K_X and a map $\int_X : \Gamma(X, K_X) \to k[-d]$ that induces a perfect pairing as in [11, Definition 2.4]. For instance, one can take X be any proper Gorenstein (derived) scheme.

(ii) It is tempting to try to develop a formalism of 'derived hyper-Kähler geometry', at least a notion of 'derived twistor space'. One could then consider an analog of Proposition 2.4, as well as analogs of various results below, with a hyper-Kähler target Y and hyper-Lagrangian structures $Z \to Y$.

3 Equivariance and Twistings

Let Y be a $\mathbb{G}_m$-stack. Given an integer m, let $Y^{(m)}$ denote the $\mathbb{G}_m$-stack with the same underlying stack as Y and the $\mathbb{G}_m$-action given by precomposition with the homomorphism $\mathbb{G}_m \to \mathbb{G}_m$, $t \mapsto t^m$. The space of (closed) p-forms on the $\mathbb{G}_m$-stack Y carries a natural $\mathbb{Z}$-grading, to be referred to as 'weight'. Thus, one can consider n-shifted symplectic structures on Y of weight m.

Given a $\mathbb{G}_m$-stack Z, we say that f is a map from Z to Y of weight m if f is a $\mathbb{G}_m$-equivariant map $Z \to Y^{(m)}$. Heuristically, a map $f : Z \to Y$ has weight m if $f(tz) = t^m f(z)$ for all $t \in \mathbb{G}_m$.

Definition Fix an n-shifted symplectic structure on Y of weight m. This gives, for each $\ell \geq 1$, an n-shifted symplectic structure on $Y^{(\ell)}$ of weight $m\ell$.

(i) An equivariant Lagrangian structure is an equivariant map $f : Z \to Y$, of $\mathbb{G}_m$-stacks, equipped with a nullhomotopy, *in the space of closed 2-forms on Z of weight m*, of the pullback of the n-shifted symplectic form, satisfying a non-degeneracy condition.

(ii) An equivariant Lagrangian structure $f : Z \to Y^{(\ell)}$ will be called a Lagrangian structure of weight ℓ.

Let X be a smooth projective variety of dimension d (or, more generally, a derived stack with a twisted orientation of degree d as in Remark 2.5). Fix $m \in \mathbb{Z}$ and a choice, $K^{1/m}$, of an m-th root of the line bundle K_X on X.

Lemma 3.1 *Let Y be a $\mathbb{G}_m$-stack equipped with an n-shifted symplectic form of weight $m \geq 1$ with respect to the $\mathbb{G}_m$-action. Let $\mathcal{L}$ be a line bundle on X and L the corresponding $\mathbb{G}_m$-torsor. Then the stack $Y_L \to X$ carries an $\mathcal{L}^{\otimes m}$-twisted relative n-shifted symplectic structure of weight m.*

Proof Let $\lambda : X \times B\mathbb{G}_m \to B\mathbb{G}_m$ be the map classifying the line bundle $\mathcal{L} \boxtimes \mathcal{O}(-1)$. We have a diagram with cartesian squares:

$$\begin{array}{ccccc} Y_L & \longrightarrow & Y_L/\mathbb{G}_m & \longrightarrow & X \times Y/\mathbb{G}_m \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & X \times B\mathbb{G}_m & \xrightarrow{p_X \times \lambda} & X \times B\mathbb{G}_m \end{array}$$

By Lemma 2.3, we get an isomorphism

$$\tilde{\mathcal{A}}^{2,cl}_{X \times B\mathbb{G}_m}(Y_L/\mathbb{G}_m) \simeq (p_X \times \lambda)^*(\tilde{\mathcal{A}}^{2,cl}_{X \times B\mathbb{G}_m}(X \times Y/\mathbb{G}_m)).$$

In particular, the sheaf of weight m relative closed 2-forms on Y_L is given by

$$\tilde{A}_X^{2,cl}(Y_L)(m) \simeq \mathcal{L}^{\otimes(-m)} \otimes \mathcal{A}^{2,cl}(Y)(m).$$

By adjunction, we obtain a map

$$\mathrm{twist}_L : \mathcal{A}^{2,cl}(Y)(m) \to \Gamma(X, \tilde{\mathcal{A}}_X^{2,cl}(Y_L)(m) \otimes \mathcal{L}^{\otimes m}). \tag{3.1}$$

Thus, an n-shifted symplectic form of weight m on Y gives an $\mathcal{L}^{\otimes m}$-twisted relative closed 2-form of weight m on Y_L. Moreover for a $\mathbb{G}_m$-equivariant Lagrangian map $f : Z \to Y$, functoriality of twist_L induces a relative isotropic structure on $f_L : Z_L \to Y_L$. Now, to see that the twisted relative closed 2-form on Y_L is nondegenerate (resp. that f_L is Lagrangian), it suffices to check this locally on X. Thus, we can assume that L is the trivial line bundle in which case the statement is manifest. □

The following is one of the main results of the paper.

Theorem 3.2 *Let Y be a $\mathbb{G}_m$-stack equipped with an n-shifted symplectic form of weight $m \geq 1$. Then, one has:*

(i) *The stack $\mathcal{S}ect_X(Y_{K_X^{1/m}})$ has a natural $(n-d)$-shifted symplectic structure of weight m.*

(ii) *For any Lagrangian structure $f : Z \to Y$, of weight ℓ, the map $\mathcal{S}ect_X(Z_{K_X^{1/\ell m}}) \to \mathcal{S}ect_X(Y_{K_X^{1/m}})$, induced by f, has a natural Lagrangian structure of weight ℓ.*

Proof Put $\mathcal{L} = K_X^{1/m}$, and let $L \to X$ be the corresponding $\mathbb{G}_m$-torsor. By Lemma 3.1, we have that $Y_L \to X$ has a K_X-twisted relative n-shifted symplectic structure of weight m. By Proposition 2.4 we obtain an $(n-d)$-shifted symplectic structure on $\mathcal{S}ect_X(Y_L)$, resp. Lagrangian structure, on $\mathcal{S}ect_X(Z_L) \to \mathcal{S}ect_X(Y_L)$. Moreover, since the maps

$$\mathcal{S}ect_X(Y_L) \leftarrow \mathcal{S}ect_X(Y_L) \times X \to Y_L$$

are $\mathbb{G}_m$-equivariant, the corresponding symplectic structure has weight m. The required statements now follow from an observation that, for any $\mathbb{G}_m$-stack and a $\mathbb{G}_m$-bundle $L \to X$, one has natural isomorphisms of $\mathbb{G}_m$-stacks $\mathcal{S}ect_X(Y_{L^{\otimes m}})^{(m)} \simeq \mathcal{S}ect_X(Y_L^{(m)})$. □

We apply the above result to get a description of the symplectic structure on cotangent stacks to mapping stacks.

Proposition 3.3 *Let $Y = T^*[n]Z$ be the shifted cotangent stack with its n-shifted symplectic structure of weight 1. In this case, there is a natural isomorphism of $(n-d)$-shifted symplectic stacks*

$$\mathcal{S}ect_X(Y_{K_X}) \simeq T^*[n-d]\,\mathrm{Map}(X, Z).$$

Proof The symplectic form on $T^*[n]Z$ is given by the deRham differential of the canonical n-shifted 1-form on $T^*[n]Z$. Therefore, it will suffice to construct an isomorphism of derived stacks $\mathcal{S}ect_X(Y_{K_X}) \simeq T^*[n-d]\,\mathrm{Map}(X, Z)$ such that the transgression of the canonical 1-form is the canonical 1-form.

Recall that given a stack W together with a quasi-coherent sheaf $\mathcal{E} \in \mathrm{QCoh}(W)$, we can form the "total space of $\mathcal{E}$" as the stack $T(\mathcal{E})$ defined as follows. A map from a test scheme S to $T(\mathcal{E})$ is a map $f : S \to W$ together with a section of $f^*(\mathcal{E})$. For instance, the stack $T^*[n]Z$ is the total space of the sheaf $\mathbb{L}_Z[n]$ on Z and the canonical 1-form on $T^*[n]Z$ is given by the image of the section obtained from the identity map on $T^*[n]Z$ along

$$p^*\mathbb{L}_Z[n] \to \mathbb{L}_{T^*[n]Z}[n],$$

where $p : T^*[n]Z \to Z$ is the projection map.

The projection map $p : T^*[n]Z \to Z$ gives a map $f : Y_{K_X} \to Z \times X$. In fact, by construction, Y_{K_X} is the total space of the sheaf $\mathbb{L}_Z[n] \boxtimes K_X$ on $Z \times X$. In particular, we have a section of $\mathbb{L}_{Y_{K_X}/X}[n] \otimes K_X$ given by the image of the canonical section of $f^*(\mathbb{L}_Z[n] \boxtimes K_X)$ along the natural map

$$f^*(\mathbb{L}_Z[n] \boxtimes K_X) \to \mathbb{L}_{Y_{K_X}/X}[n] \otimes K_X.$$

Moreover, the map f induces the map

$$g : \mathcal{S}ect_X(Y_{K_X}) \to \mathrm{Map}(X, Z),$$

together with a section of $ev^*(\mathbb{L}_{Y_{K_X}/X}[n] \otimes K_X)$, where

$$ev : \mathcal{S}ect_X(Y_{K_X}) \times X \to \mathcal{S}ect_X(Y_{K_X})$$

is the evaluation map. Integrating along X, we obtain a section of $\pi_*(ev^*(\mathbb{L}_{Y_{K_X}/X}[n] \otimes K_X)) \simeq g^*(\mathbb{L}_{\mathrm{Map}(X,Z)}[n-d])$. This gives the desired map of derived stacks

$$h : \mathcal{S}ect_X(Y_{K_X}) \to T^*[n-d]\,\mathrm{Map}(X, Z),$$

which is easily seen to be an isomorphism. Moreover, by construction, the pullback of the canonical 1-form on $T^*[n-d]\,\mathrm{Map}(X, Z)$ along h is identified with the transgression of the canonical 1-form on $T^*[n]Z$, as desired. □

In addition to equivariant symplectic structures, we will also need to consider equivariant Calabi-Yau structures.

Definition Let S be a stack with a $\mathbb{G}_m$-action. A d-Calabi-Yau structure of weight m on S is a map

$$\Gamma(S, \mathcal{O}_X) \to \mathbb{C}[-d]$$

of weight m satisfying the nondegeneracy condition of [11, Definition 2.4]. Equivalently, such a structure is given by a map of quasi-coherent sheaves on $B\mathbb{G}_m$

$$\pi_*(\mathcal{O}_{S/\mathbb{G}_m}) \to \mathbb{C}(m)[-d],$$

where $\pi : S/\mathbb{G}_m \to B\mathbb{G}_m$ is the projection map.

Theorem 3.4 *Let S be a $\mathbb{G}_m$-stack with a d'-Calabi-Yau structure of weight m. Let X be a smooth projective variety of dimension d (or more generally, a derived stack with a twised orientation K_X of degree d as above) together with a choice of $K_X^{1/m}$. Then:*

(i) *The stack $\tilde{X} := X \underset{B\mathbb{G}_m}{\times} S/\mathbb{G}_m$ has a natural $(d + d')$ Calabi-Yau structure of weight m, where the map $X \to B\mathbb{G}_m$ classifies the line bundle $K_X^{1/m}$.*

(ii) *Given an n-shifted symplectic stack Y, there is a natural $\mathbb{G}_m$-equivaraint equivalence of $(n - d - d')$-shifted symplectic stacks of weight m*

$$\mathrm{Map}(\tilde{X}, Y) \simeq \mathcal{S}ect_X(\mathrm{Map}(S, Y)_{K_X^{1/m}}).$$

Proof We have the Cartesian square of stacks

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & S/\mathbb{G}_m \\ \downarrow & & \downarrow \pi \\ X & \xrightarrow{l} & B\mathbb{G}_m \end{array}$$

Therefore, by base change, we have

$$\Gamma(\tilde{X}, \mathcal{O}_X) \simeq \Gamma(X, l^*\pi_*(\mathcal{O}_{S/\mathbb{G}_m})).$$

The desired Calabi-Yau structure on $\tilde{X}$ is then given as the composition of Calabi-Yau structures on S and X:

$$\Gamma(X, l^*\pi_*(\mathcal{O}_{S/\mathbb{G}_m})) \to \Gamma(X, l^*(\mathbb{C}(m)[d'])) \to \Gamma(X, K_X[d']) \to \mathbb{C}[d + d'].$$

Now, we have isomorphisms

$$\mathrm{Map}(\tilde{X}, Y) \simeq \mathcal{S}ect_X(\mathrm{Map}_{/X}(\tilde{X}, Y \times X)) \simeq \mathcal{S}ect_X(\mathrm{Map}(S, Y)_{K^{1/m}}),$$

which by construction of the Calabi-Yau structure on $\tilde{X}$ are compatible with the $(n - d - d')$-shifted symplectic structures of weight m. □

4 The Case of G-Bundles

For any stack $\mathcal{Y}$ and an integer n, the n-shifted cotangent stack $T^*[n]\mathcal{Y}$ comes equipped with a natural n-shifted symplectic form, see [11, Proposition 1.21] and also [3]. This 2-form has weight 1 with respect to the $\mathbb{G}_m$-action on $T^*[n]\mathcal{Y}$ by dilations along the fibers of the cotangent bundle. The zero section $\mathcal{Y} \hookrightarrow T^*[n]\mathcal{Y}$ has a natural Lagrangian structure.

One has a canonical isomorphism $\mathfrak{g}^*/G = T^*[1]BG$, which provides the stack $\mathfrak{g}^*/G$ with a natural 1-shifted symplectic structure of weight 1.

In what follows, it will be convenient to have another description of this 1-shifted symplectic stack as a mapping stack. Recall that an Ad-invaraint nondegenerate symmetric bilinear form κ on $\mathfrak{g}$ gives a 2-shifted symplectic structure on the stack BG. Now, let $S = \widehat{B\mathbb{G}_a}$, the formal completion of $B\mathbb{G}_a$ at a point, with its natural $\mathbb{G}_m$ action. We have that $\Gamma(S, \mathcal{O}_S) \simeq \mathbb{C}[\epsilon]$, where $|\epsilon| = 1$ and the map $\mathbb{C}[\epsilon] \to \mathbb{C}[-1]$, given by $\epsilon \mapsto 1$ gives S a 1-Calabi-Yau structure of weight 1. We then have:

Lemma 4.1 *There is a canonical isomorphism of 1-shifted symplectic stacks of weight 1*

$$\mathrm{Map}(S, BG) \simeq T^*[1]BG.$$

Proof We have a $\mathbb{G}_m$-equivariant isomorphism of derived stacks $\mathrm{Map}(S, BG) \simeq T[-1]BG \simeq \mathfrak{g}/G$. Recall that the 2-shifted symplectic structure on BG is given by the image of an Ad-invariant symmetric bilinear form κ on $\mathfrak{g}$ under the natural map

$$\left(\oplus_{i\geq 0} Sym^{2+i}(\mathfrak{g}^*)[-2-2i]\right)^G \to \mathcal{A}^{2,cl}(BG).$$

Unraveling the definitions, we have that the composite map

$$\left(\oplus_{i\geq 0} Sym^{2+i}(\mathfrak{g}^*)[-2-2i]\right)^G \to \mathcal{A}^{2,cl}(BG) \to \mathcal{A}^{2,cl}(\mathfrak{g}/G)[-1]$$

factors through the map

$$\left(\oplus_{p+q\geq l}\Omega^p(\mathfrak{g})\otimes_{\mathbb{C}} Sym^q(\mathfrak{g}^*)[2-p-2q]\right)^G \to \mathcal{A}^{2,cl}(\mathfrak{g}/G),$$

where the differential in the complex on the left is given by the sum of the internal differential and the deRham differential on $\mathfrak{g}$. Thus, we obtain that the only nonzero component of the 1-shifted symplectic structure on $\mathfrak{g}/G$ is given by the image of κ along the map

$$Sym^2(\mathfrak{g}^*) \to \Omega^1(\mathfrak{g})\otimes\mathfrak{g}^* \simeq \mathfrak{g}^*\otimes\mathfrak{g}^*\otimes\mathcal{O}_{\mathfrak{g}}.$$

It follows that the $\mathbb{G}_m$ equivariant identification $\mathfrak{g}/G \simeq \mathfrak{g}^*/G$ induced by κ upgrades to an isomorphism of 1-shifted symplectic stacks of weight 1. □

The map $0/G \to \mathfrak{g}^*/G$, induced by the imbedding $\{0\} \hookrightarrow \mathfrak{g}^*$, may be identified with the zero section $\iota : BG \to T^*[1]BG$.

Let M be a smooth symplectic variety equipped with a Hamiltonian G-action. It was observed by Calaque [2], that the map $M/G \to \mathfrak{g}^*/G$, induced by the moment map $\mu : M \to \mathfrak{g}^*$, has a natural Lagrangian structure. Hence, from Theorem 3.2 in the special case where $Y = \mathfrak{g}^*/G$ and $n = 1$ we deduce the following result.

Corollary 4.2 (i) *For any $m \geq 1$, the stack $\mathcal{S}ect_X(\mathfrak{g}^*_{K_X^{1/m}}/G)$ has a canonical $(1-d)$-shifted symplectic structure structure of weight m.*

(ii) *For a smooth symplectic $G \times \mathbb{G}_m$-variety M such that the action of the group G is hamiltonian and the symplectic 2-form has weight $\ell \geq 1$ with respect to the $\mathbb{G}_m$-action, the map $\mathcal{S}ect_X(M_{K_X^{1/m\ell}}) \to \mathcal{S}ect_X(\mathfrak{g}^*_{K_X^{1/m}}/G)$, induced by the moment map $M \to \mathfrak{g}^*$, has a natural Lagrangian structure of weight ℓ.*

We now specialize to the case where $\Sigma = X$ is a smooth projective curve. The stack of Higgs bundles on Σ is defined as $Higgs_G(\Sigma) := \mathrm{Map}(\Sigma_{Dol}, BG)$, where Σ_{Dol} is the Dolbeault stack, see [11]. Since $d = \dim\Sigma = 1$, the stack $Higgs_G(\Sigma)$ is equipped with a 0-shifted symplectic structure, by [11, Corollary 2.6(2)].

Lemma 4.3 *There are natural isomorphisms of 0-shifted symplectic stacks*

$$\mathrm{Map}(\Sigma_{Dol}, BG) \simeq \mathcal{S}ect_\Sigma(\mathfrak{g}^*_K/G) \simeq T^*Bun_G(\Sigma).$$

Proof By definition, Σ_{Dol} is identified with $X \underset{B\mathbb{G}_m}{\times} S$, where the map $X \to B\mathbb{G}_m$ classifies K_X. Moreover, by construction of the Calabi-Yau structure in Theorem 3.4(i), this isomorphism gives an isomorphism of 1-CY stacks. The first, resp. second, isomorphism of the lemma then follows from Theorem 3.4(ii), resp. Proposition 3.3. □

Using the above lemma, from Corollary 4.2 we deduce

Theorem 4.4 *Let M be a smooth symplectic $G \times \mathbb{G}_m$-variety such that the action of the group G is hamiltonian and the symplectic 2-form has weight $\ell \geq 1$ with respect to the $\mathbb{G}_m$-action. Then, the map*

$$\mathcal{S}ect_\Sigma(M_{K_\Sigma^{1/\ell}}/G) \longrightarrow \mathcal{S}ect_\Sigma(\mathfrak{g}^*_{K_\Sigma}/G) = T^*Bun_G(\Sigma), \quad (4.1)$$

induced by the moment map $M \to \mathfrak{g}^*$, has a natural Lagrangian structure of weight ℓ.

To complete the proof of Theorem 1.2 one observes that, on the locus $T^*Bun_G(\Sigma)^{reg}$ where $T^*Bun_G(\Sigma)$ is a smooth variety, the 0-shifted symplectic 2-form is nothing but the standard symplectic 2-form ω on $T^*Bun_G(\Sigma)^{reg}$ in the ordinary sense. Similarly, if Λ is a smooth variety and a map $f : \Lambda \to T^*Bun_G(X)^{reg}$ has a Lagrangian structure then one has $f^*\omega = 0$. Thus, Theorem 1.2 follows from Theorem 4.4.

5 Additional Comments and Speculations

5.1 A Generalization of Gaiotto's Argument

In the linear case, an 'infinite dimensional' approach to Theorem 1.2 is explained in [6]. Gaiotto's approach is based on a standard differential geometric interpretation of $Bun_G(\Sigma)$ as a quotient of an infinite dimensional space of $\bar\partial$-connections by a gauge group. It was suggested to us by Gaiotto that the argument in [6] can be adapted to the more general, nonlinear setting of Theorem 1.2 as follows. Below, we assume that $\ell = 2$, for simplicity.

Fix a principal C^∞-bundle $P \xrightarrow{G} \Sigma$ and let $Conn_{\bar\partial}(P)$ be (an infinite dimensional) space of $\bar\partial$-connections on P. Further, let $\mathcal{S}ect_{\Sigma,C^\infty}(M_{K_\Sigma^{1/2}} \times_G P)$ be (an infinite dimensional) space of C^∞-sections of an associated bundle $M_{K_\Sigma^{1/2}} \times_G P \to \Sigma$. Let $z \in \mathcal{S}ect_{\Sigma,C^\infty}(M_{K_\Sigma^{1/2}} \times_G P)$ be such a section and $A \in Conn_{\bar\partial}(P)$ a $\bar\partial$-connection. Then $\nabla_A z$, a covariant derivative of z with respect to A, is a C^∞-section of $z^*T_M \otimes K_\Sigma^{1/2} \otimes \Omega_\Sigma^{0,1}$, where T_M stands for the holomorphic tangent sheaf on M and $\Omega_\Sigma^{p,q}$ is the sheaf of C^∞ differential forms on Σ of type (p,q). Further, let $\lambda_M = i_{\mathrm{eu}_M}\omega_M$, where ω_M is the (holomorphic) symplectic form on M and eu_M is the Euler field that generates the $\mathbb{G}_m$-action on M. Thus, $z^*\lambda_M$ is a C^∞-section of $z^*T_M^* \otimes K_\Sigma^{1/2}$. Using the canonical pairing $\langle -, - \rangle$ of holomorphic vector fields and holomorphic 1-forms on M, we obtain a C^∞-section $\langle \nabla_A z, z^*\lambda_M\rangle$ of the sheaf $K_\Sigma^{1/2} \otimes \Omega_\Sigma^{0,1} \otimes K_\Sigma^{1/2} = K_\Sigma \otimes \Omega_\Sigma^{0,1} = \Omega_\Sigma^{1,1}$.

In the above setting, the role of the potential from [6, formula (2.3)] is played by a function on $\mathcal{S}ect_{\Sigma,C^\infty}(M_{K_\Sigma^{1/2}} \times_G P) \times Conn_{\bar\partial}(P)$ defined by the formula

$$W(z, A) = \int_\Sigma \langle \nabla_A z, z^*\lambda_M\rangle. \tag{5.1}$$

To prove that the map $\mu_{\mathcal{S}ect}$ in Theorem 1.2 is Lagrangian we show, by a calculation similar to the one in [6, Appendix A], that Eq. 5.1 is a generating function (aka 'Lagrange multiplier') for $\mathcal{S}ect_\Sigma(M_{K_\Sigma^{1/2}}/G)$.

To this end, observe that an infinitesimal variation of z is given by a section $\dot z$ of $z^*T_M \otimes K_\Sigma^{1/2}$. The corresponding variation of the (1, 1)-form $\langle \nabla_A z, z^*\lambda_M\rangle$ reads

$$\frac{\delta\langle \nabla_A z, z^*\lambda_M\rangle}{\delta z}(\dot z) = \bar\partial\langle \nabla_A \dot z, z^*\lambda_M\rangle + (d\lambda_M)(\nabla_A z, \dot z),$$

where the operator $\bar\partial$ that appears in the first summand on the right is the Dolbeault differential $\bar\partial : \Omega_\Sigma^{1,0} \to \Omega_\Sigma^{1,1}$. Using that $d\lambda_M = \omega_M$ and that, on $\Omega_\Sigma^{1,0}$, one has $\bar\partial = d$, we find that the variation of Eq. 5.1 equals

$$\frac{\delta W}{\delta z}(\dot z) = \int_\Sigma d\langle \nabla_A \dot z, z^*\lambda_M\rangle + \int_\Sigma \omega_M(\nabla_A z, \dot z).$$

The first summand on the right vanishes by Stokes' theorem. Hence, the form ω_M being nondegerate, we deduce that the equation $\frac{\delta W}{\delta z}(\dot{z}) = 0$ holds for all $\dot{z}$ if and only if $\nabla_A z = 0$, that is, if and only if the section z is holomorphic with respect to the complex structure on $M_{K_\Sigma^{1/2}} \times_G P$ determined by the $\bar{\partial}$-connection A.

Next, let $\dot{A}$ be an infinitesimal variation of A. Then, it is easy to check that $\frac{\delta W}{\delta A}(\dot{A}) = \mu_{\mathcal{S}ect}(z, A)(\dot{A})$, proving that W is a generating function for $\mathcal{S}ect_\Sigma(M_{K_\Sigma^{1/2}}/G)$.

Remark 5.1 Let eu, resp. $\text{eu}_{\mathcal{S}ect}$, be the Euler vector field on $T^*Bun_G(\Sigma)$, resp. $\mathcal{S}ect_\Sigma(M_{K_\Sigma^{1/2}}/G)$, that generates the $\mathbb{G}_m$-action. Recall that $\omega = d\lambda$ where $\lambda = i_{\text{eu}}\omega$ is the Liouville 1-form on $T^*Bun_G(\Sigma)$. The map $\mu_{\mathcal{S}ect}$ in Eq. 1.1 being of weight ℓ, one finds:

$$\mu^*_{\mathcal{S}ect}(\lambda) = \mu^*_{\mathcal{S}ect}(i_{\text{eu}}\omega) = \frac{1}{\ell} \cdot i_{\text{eu}_{\mathcal{S}ect}}\mu^*(\omega).$$

It follows, as has been observed by Hitchin [8], that Theorem 4.4 is equivalent to the equation $\mu^*_{\mathcal{S}ect}(\lambda) = 0$.

5.2 Relation to the Global Nilpotent Cone

Let B be a Borel subgroup of G, so G/B is the flag variety. The symplectic form on $T^*(G/B)$ has weight 1 and the moment map $\mu : T^*(G/B) \to \mathfrak{g}^*$ is the Springer resolution $T^*(G/B) \to \mathcal{N}$, where $\mathcal{N} \subset \mathfrak{g}^*$ is the nilpotent cone. The stack $\mathcal{S}ect_\Sigma((\mathcal{N}/G)_{K_\Sigma})$ can be identified with $\mathcal{N}_\Sigma$, the *global nilpotent cone* in $T^*Bun_G(\Sigma)$. Further, the stack $\mathcal{S}ect_\Sigma((T^*(G/B)_{K_\Sigma}/G)$ can be identified with $T^*Bun_B(\Sigma)$. Explicitly, writing $\mathfrak{n}$ for the nilradical of Lie B, the stack $T^*Bun_B(\Sigma)$ classifies triples (P, σ, ϕ), where P is a G-bundle on Σ, $\sigma : \Sigma \to P/B$ is a section, i.e. a reduction of P to a B-bundle, and $\phi : P \times_B \mathfrak{n} \to (P \times_B \mathfrak{n}) \otimes K_\Sigma$ is a Higgs field. Assume that the genus of the curve Σ is greater than 1. Then, the derived stacks $T^*Bun_G(\Sigma)$ and $T^*Bun_B(\Sigma)$ are concentrated in homological degree 0, i.e. they are actually non-derived stacks. The stack $\mathcal{N}_\Sigma$ is not concentrated in homological degree 0, and one can consider $\mathcal{N}_\Sigma^{\text{classical}}$, its non-derived counterpart, which is an ordinary substack of $T^*Bun_G(\Sigma)$.

The map $(P, \sigma, \phi) \mapsto (P, \phi)$, that forgets reduction of the structure group, may be identified with the composition

$$\mu_{\mathcal{S}ect} : T^*Bun_B(\Sigma) \xrightarrow{\pi_1} \mathcal{N}_\Sigma \xrightarrow{\pi_2} T^*Bun_G(\Sigma). \tag{5.2}$$

The map $\mu_{\mathcal{S}ect}$ has a Lagrangian structure by Corollary 4.2. One can show that the map π_2 has a natural coisotropic structure in the sense of [9]. However, this coisotropic structure is easily seen to be *not* Lagrangian.

On the other hand, it was shown in [7] that, for any field extension K/k, the map $\pi_1^{\text{classical}} : T^*Bun_B(\Sigma)(\text{Spec}\, K) \to \mathcal{N}_\Sigma^{\text{classical}}(\text{Spec}\, K)$, of K-points of the corresponding *non-derived* stacks, is surjective. This result was used in [7] to prove that $\mathcal{N}_\Sigma^{\text{classical}}$ is (as opposed to its derived analog) a Lagrangian substack of $T^*Bun_G(\Sigma)$ in the sense explained in *loc cit*.

More generally, let $\widetilde{Y} \to Y$ be a $(G \times \mathbb{G}_m)$-equivariant symplectic resolution such that Y is affine, the $\mathbb{G}_m$-action on Y is a contraction to a unique $\mathbb{G}_m$-fixed point and, moreover, the symplectic form on $\widetilde{Y}$ has weight $m \geq 1$. Then, we have $k[\widetilde{Y}] = k[Y]$, so the Poisson bracket on the algebra $k[\widetilde{Y}]$ provides Y with a $(G \times \mathbb{G}_m)$-equivariant Poisson structure. Also, the moment map $\widetilde{Y}/G \to \mathfrak{g}^*/G$ factors through Y/G. Therefore, there is a chain of induced maps $\mathcal{S}ect_\Sigma((\widetilde{Y}/G)_{K_\Sigma^{1/m}}) \xrightarrow{\pi_1} \mathcal{S}ect_\Sigma(Y_{K_\Sigma^{1/m}}/G) \xrightarrow{\pi_2} T^*Bun_G(\Sigma)$ such that $\pi_2 \circ \pi_1 = \mu_{\mathcal{S}ect}$. The map $\mu_{\mathcal{S}ect}$ has a Lagrangian structure, by Theorem 4.4. Again, one can show that the map $\pi_2 : \mathcal{S}ect_\Sigma(Y_{K_\Sigma^{1/m}}/G) \xrightarrow{\pi_2} T^*Bun_G(\Sigma)$ has a natural coisotropic structure.

Question 5.2 Is $\mathcal{S}ect_\Sigma(Y_{K_\Sigma^{1/m}}/G)^{\text{classical}}$, a non-derived counterpart of $\mathcal{S}ect_\Sigma(Y_{K_\Sigma^{1/m}}/G)$, isotropic in the sense of [7], specifically, is it possible to partition $\mathcal{S}ect_\Sigma(Y_{K_\Sigma^{1/m}}/G)^{\text{classical}}$ as a disjoint union of substacks such that the pull-back of the symplectic 2-form on $T^*Bun_G(\Sigma)$ to each of these substacks vanishes?

5.3 Hamiltonian Reduction

Let M be a stack equipped with a 0-shifted symplectic structure and with a Hamiltonian G-action with moment map μ. The stack $\mu^{-1}(0)/G$, a stacky Hamiltonian reduction of M, comes equipped with a canonical 0-shifted symplectic structure. On the other hand, let $\Lambda_1 = 0/G \to \mathfrak{g}^*/G$ be the map induced by the imbedding $\{0\} \hookrightarrow \mathfrak{g}^*$ and $\Lambda_2 = M/G \to \mathfrak{g}^*/G$ be the map induced by μ. One has a natural isomorphism, see [13],

$$\Lambda_1 \times_{\mathfrak{g}^*/G} \Lambda_2 = 0/G \times_{\mathfrak{g}^*/G} M/G \cong \mu^{-1}(0)/G. \tag{5.3}$$

Recall that the stack $\mathfrak{g}^*/G$ has the canonical 1-shifted symplectic structure and each of the two maps $\Lambda_i \to \mathfrak{g}^*/G,\ i = 1, 2$, has a Lagrangian structure, cf. §4. Further, according to [11, Theorem 0.5], for any stack $\mathcal{Y}$ equipped with an n-shifted symplectic structure and a pair $\Lambda_i \to \mathcal{Y},\ i = 1, 2$, of Lagrangian structures, the stack $\Lambda_1 \times_{\mathcal{Y}} \Lambda_2$ has a natural $(n-1)$-shifted symplectic structure. Therefore, the stack $0/G \times_{\mathfrak{g}^*/G} M/G$ comes equipped with a 0-shifted symplectic structure. It was shown by Calaque [2] that the isomorphism in Eq. 5.3 respects the 0-shifted symplectic structures.

Next, we fix a smooth projective curve Σ and let $K = K_\Sigma$. The stack $\mathfrak{g}^*_K/G = T^*Bun_G(\Sigma)$, a global counterpart of $\mathfrak{g}^*/G$, has the 0-shifted symplectic structure of weight 1. Also, the Lagrangian structure on the map $0/G \to \mathfrak{g}^*/G$ induces, for any ℓ, a weight ℓ Lagrangian structure $\mathcal{S}ect_\Sigma((0/G)_{K^{1/\ell}}) \to \mathcal{S}ect_\Sigma((\mathfrak{g}^*/G)_K)$. The latter Lagrangian structure corresponds, via the isomorphisms $T^*Bun_G(\Sigma) \cong \mathfrak{g}^*_K/G$ and $Bun_G(\Sigma) \cong \mathcal{S}ect_\Sigma((0/G)_K)$, to an obvious Lagrangian structure on the zero section $Bun_G(\Sigma) \to T^*Bun_G(\Sigma)$. (We have used here that for any variety $\mathcal{Y}$ equipped with a trivial $\mathbb{G}_m$-action and any $\mathbb{G}_m$-bundle L on Σ, one has $\mathcal{S}ect_X(\mathcal{Y}_L) = \mathrm{Map}(\Sigma, \mathcal{Y})$, in particular, we have $\mathcal{S}ect_\Sigma((0/G)_K) = \mathrm{Map}(\Sigma, BG) = Bun_G(\Sigma)$.)

Now, let M be a symplectic manifold equiped with a $(G \times \mathbb{G}_m)$-action such that the symplectic 2-form has weight $\ell \geq 1$ and the G-action is Hamiltonian. One has canonical isomorphisms

$$\begin{aligned}\mathcal{S}ect_\Sigma((0/G)_{K^{1/\ell}}) \times_{T^*Bun_G(\Sigma)} \mathcal{S}ect_\Sigma((M/G)_{K^{1/\ell}}) &\cong \mathcal{S}ect_\Sigma\big((0/G)_{K^{1/\ell}} \times_{\mathfrak{g}^*_K/G} (M/G)_{K^{1/\ell}}\big)\\ &\cong \mathcal{S}ect_\Sigma((\mu^{-1}(0)/G)_{K^{1/\ell}}).\end{aligned} \tag{5.4}$$

Here, the fiber product on the left involves the map (4.1), which has a weight ℓ Lagrangian structure, by Theorem 4.4. Thus, according to [11, Theorem 0.5], the fiber product of Lagrangians on the left of Eq. 5.4 has a (-1)-shifted symplectic structure. On the other hand, the 0-shifted symplectic structure on $\mu^{-1}(0)/G$ induces, by Theorem 3.2(i), a (-1)-shifted symplectic structure of weight ℓ on $\mathcal{S}ect_\Sigma((\mu^{-1}(0)/G)_{K^{1/\ell}})$, the stack on the right of Eq. 5.4. One can check that the composite isomorphism in Eq. 5.4 respects the (-1)-shifted symplectic structures described above.

Let $\mathcal{X}$ be a stack and assume there is a line bundle $K_{\mathcal{X}}^{1/2}$, a square root of the dualizing complex of $\mathcal{X}$. In [12], Pridham shows that an (-1)-shifted symplectic structure on $\mathcal{X}$ gives rise to a canonical self-dual quantization of $K_{\mathcal{X}}^{1/2}$. Moreover, associated with that quantization, there is a constructible complex on $\mathcal{X}$, of vanishing cycles. Therefore, one might expect that, in the setting of the previous paragraph, the stack $\mathcal{S}ect_{\Sigma}((\mu^{-1}(0)/G)_{K^{1/\ell}})$ comes equipped (perhaps, under some additional assumptions) with a natural constructible complex of vanishing cycles.

The linear case, where $\ell = 2$ and M is a linear symplectic representation of G, has been considered in the physics literature in the framework of Coulomb branches for 3-dimensional gage theories, cf. [6] and references therein. The special case where $M = E \oplus E^*$ is a direct sum of a pair of dual representations of G is simpler than the general case. In that case, the geometry of $\mathcal{S}ect_{\Sigma}((\mu^{-1}(0)/G)_{K^{1/2}})$ can be reduced, in a sense, to the geometry of $\mathcal{S}ect_{\Sigma}(E_{K^{1/2}})$. Such a reduction allows to avoid the use of vanishing cycles. A mathematical theory of Coulomb branches in the case $M = E \oplus E^*$ was developed by H. Nakajima [10], cf. also [1].

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