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Lie Algebras of Slow Growth and Klein-Gordon PDE

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Abstract We discuss the notion of characteristic Lie algebra of a hyperbolic PDE. The integrability of a hyperbolic PDE is closely related to the properties of the corresponding characteristic Lie algebra χ. We establish two explicit isomorphisms:

1) the first one is between the characteristic Lie algebra $\chi(\sinh u)$ of the sinh-Gordon equation $u_{xy} = \sinh u$ and the non-negative part $\mathcal{L}(\mathfrak{sl}(2,\mathbb{C}))^{\geq 0}$ of the loop algebra of $\mathfrak{sl}(2,\mathbb{C})$ that corresponds to the Kac-Moody algebra $A_1^{(1)}$
$$\chi(\sinh u) \cong \mathcal{L}(\mathfrak{sl}(2,\mathbb{C}))^{\geq 0} = \mathfrak{sl}(2,\mathbb{C}) \otimes \mathbb{C}[t].$$
2) the second isomorphism is for the Tzitzeica equation $u_{xy} = e^u + e^{-2u}$
$$\chi(e^u + e^{-2u}) \cong \mathcal{L}(\mathfrak{sl}(3,\mathbb{C}),\mu)^{\geq 0} = \bigoplus_{j=0}^{+\infty} \mathfrak{g}_{j \,(\mathrm{mod}\, 2)} \otimes t^j,$$
where $\mathcal{L}(\mathfrak{sl}(3,\mathbb{C}),\mu) = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_{j\,(\mathrm{mod}\,2)} \otimes t^j$ is the twisted loop algebra of the simple Lie algebra $\mathfrak{sl}(3,\mathbb{C})$ that corresponds to the Kac-Moody algebra $A_2^{(2)}$.

Hence the Lie algebras $\chi(\sinh u)$ and $\chi(e^u + e^{-2u})$ are slowly linearly growing Lie algebras with average growth rates $\frac{3}{2}$ and $\frac{4}{3}$ respectively.

Keywords Characteristic Lie algebra · Naturally graded Lie algebra · Loop algebra · Kac-Moody algebra · Hyperbolic PDE · Sine-Gordon equation · Tzitzeica equation · Bell polynomial · Gelfand-Kirillov dimension

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1 Introduction

The concept of characteristic Lie algebra $\chi(f)$ of a hyperbolic system of PDE

$$u^i_{xy} = f^i(u^1, \dots, u^n), i = 1, \dots, n, \tag{1}$$

was introduced by Leznov, Smirnov, Shabat and Yamilov [18, 24]. It is a natural generalization of the notion of characteristic vector field of a hyperbolic PDE that was first proposed by Goursat in 1899. In his classical paper [7] Goursat introduced a very effective algebraic approach to the problem of classifying Darboux-integrable equations.

In spite of the rather large number of papers where this algebraic object is studied [18, 22, 24, 26, 27], it can not be said that there exists any completely unambiguous definition of characteristic Lie algebra χ of a hyperbolic non-linear PDE. We use in the present article the definition of characteristic Lie algebra proposed in the original papers [18, 24].

An important step in the study of hyperbolic nonlinear Liouville-type systems was made in [15, 16, 18] where exponential hyperbolic systems were considered

$$u^j_{xy} = e^{\rho_j}, \;\; \rho_j = a_{j1}u^1 + \cdots + a_{jn}u^n, \;\; j = 1, \dots, n. \tag{2}$$

It pas proved in [15] that if $A = (a_{ij})$ is a non-degenerate Cartan matrix then the corresponding exponential hyperbolic system (2) is Darboux-integrable. The proof [15] consists in finding an explicit solution which depends on $2n$ arbitrary functions, i.e. it generalizes the one-dimensional case of the classical Liouville equation $u_{xy} = e^u$. Later it was claimed in the preprint [24] that the main result in [15] can be extended to an arbitrary generalized Cartan matrix A (possibly degenerate) by applying the inverse scattering problem method. The two-dimensional case $n = 2$ was studied explicitly in [18, 24].

$$\begin{cases} u^1_{xy} = e^{(a_{11}u^1 + a_{12}u^2)}, \\ u^2_{xy} = e^{(a_{21}u^1 + a_{22}u^2)}, \end{cases} \;, \;\; A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{3}$$

It was proved in [18, 24] that for the generalized Cartan matrices

$$A_1 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \;\; A_2 = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$$

the corresponding exponential systems (3) are integrable by the inverse scattering method. Moreover, the commutants $[\chi(A_1), \chi(A_1)]$, $[\chi(A_2), \chi(A_2)]$ of the corresponding characteristic Lie algebras $\chi(A_1)$, $\chi(A_2)$ are isomorphic to maximal pro-nilpotent subalgebras $N(A_1^{(1)})$, $N(A_2^{(2)})$ of the Kac-Moody algebras $A_1^{(1)}$, $A_2^{(2)}$ respectively (that correspond to the generalized Cartan matrices A_1 and A_2). Exponential systems (3) corresponding to nondegenerate Cartan 2×2-matrices

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

of the semisimple Lie algebras $A_1 \oplus A_1$, A_2, C_2, G_2 are Darboux-integrable. Their characteristic Lie algebras are finite-dimensional Borel subalgebras in semisimple Lie algebras listed above. These finite-dimensional solvable Lie algebras and infinite-dimensional characteristic Lie algebras $\chi(A_1)$, $\chi(A_2)$ can be unified into a class of slowly growing Lie algebras [18, 24].

Originally, when it was talked about the characteristic Lie algebra of slow growth [18, 24], it was in mind Kac's classification [10] of simple $\mathbb{Z}$-graded Lie algebras of finite growth. The condition of simple $\mathbb{Z}$-grading is very restrictive, meanwhile, the growth of a characteristic Lie algebra $\chi(f)$ must be understood from the point of view of the behavior of its growth function $F_{\mathfrak{g}}(n)$, i.e. the asymptotics of the dimension $F_{\mathfrak{g}}(n) = \dim V_n$ of the space V_n of commutators of order at most n of generators.

A finitely generated characteristic Lie algebra $\chi(f)$ of a hyperbolic Klein-Gordon system (1) is a pro-solvable Lie algebra whose commutant $[\chi(f), \chi(f)]$ is a pro-nilpotent naturally graded Lie algebra.

By **Lemma 2** we assert that the growth functions of $\chi(f)$ and its commutant $[\chi(f), \chi(f)]$ differ by a positive constant $C(\chi(f))$, which equals to the dimension of the maximal toral subalgebra of $\chi(f)$.

$$F_{\chi(f)}(n) = F_{[\chi(f),\chi(f)]}(n) + C(\chi(f)).$$

Thus, the study of the growth function $F_{\chi(f)}(n)$ of the entire characteristic Lie algebra $\chi(f)$ reduces to studying the growth of the commutant $[\chi(f), \chi(f)]$.

The problem of classification of $\mathbb{N}$-graded Lie algebras of slow growth is much more complicated problem than the classification of simple $\mathbb{Z}$-graded Lie algebras of finite growth. The Kac list [10] contains a countably many different Lie algebras, meanwhile in the case of naturally graded Lie algebras with two generators, an uncountable family of pairwise non-isomorphic Lie algebras of linear growth appears [20]. There are only three Klein-Gordon equations admitting non-trivial higher symmetries [28].

- Liouville equation $u_{xy} = e^u$;
- sinh-Gordon equation $u_{xy} = \sinh u$;
- Tzitzeica equation $u_{xy} = e^u + e^{-2u}$.

1) It's an elementary exercise to show that the characteristic Lie algebra $\chi(e^u)$ of the Liouville equation is the two-dimensional solvable Lie algebra. It can be defined by its basis X_0, X_1 and the unique relation $[X_0, X_1] = X_1$. Its commutant $[\chi(e^u), \chi(e^u)]$ is one-dimensional abelian Lie algebra spanned by X_1.

   We study two remaining cases and prove

2) **Theorem 2**. *The characteristic Lie algebra $\chi(\sinh u)$ of the sinh-Gordon equation $u_{xy} = \sinh u$ is isomorphic to the polynomial loop algebra $\mathcal{L}(\mathfrak{sl}(2,\mathbb{C}))^{\geq 0}$ (current Lie algebra)*

$$\chi(\sinh u) \cong \mathcal{L}(\mathfrak{sl}(2,\mathbb{C}))^{\geq 0} = \mathfrak{sl}(2,\mathbb{C}) \otimes \mathbb{C}[t],$$

   Its commutant $[\chi(\sinh u), \chi(\sinh u)]$ is isomorphic to the maximal pro-nilpotent Lie subalgebra $N(A_1^{(1)})$ of the Kac-Moody algebra $A_1^{(1)}$.

3) **Theorem 3**. The characteristic Lie algebra $\chi(e^u+e^{-2u})$ of the Tzitzeica equation $u_{xy} = e^u+e^{-2u}$ is isomorphic to the twisted polynomial loop algebra $\mathcal{L}(\mathfrak{sl}(3,\mathbb{C}),\mu)^{\geq 0}$

$$\chi(e^u+e^{-2u}) \cong \mathcal{L}(\mathfrak{sl}(3,\mathbb{C}),\mu)^{\geq 0} = \bigoplus_{j=0}^{+\infty} \mathfrak{g}_{j \,(\mathrm{mod}\, 2)} \otimes t^j, \ \mathfrak{sl}(3,\mathbb{C}) = \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

   where μ is a diagram automorphism of $\mathfrak{sl}(3,\mathbb{C})$, $\mu^2 = \mathrm{Id}$, and $\mathfrak{g}_0, \mathfrak{g}_1$ are eigen-spaces of μ corresponding to eigen-values $1, -1$ respectively, $[\mathfrak{g}_s, \mathfrak{g}_q] \subset \mathfrak{g}_{s+q \,(\mathrm{mod}\, 2)}$.

Its commutant $[\chi(e^u+e^{-2u}), \chi(e^u+e^{-2u})]$ is isomorphic to the maximal pro-nilpotent Lie subalgebra $N(A_2^{(2)})$ of the Kac-Moody algebra $A_2^{(2)}$.

At this point some very important observation need to be made.

It was discussed in [17, 18] that there is a reduction of two-dimensional systems (3) with matrices A_1 and A_2 to the sine-Gordon and Tzitzeika equations respectively. However, explicitly the characteristic Lie algebras $\chi(\sinh u)$ and $\chi(e^u+e^{-2u})$ have not been described there. The question of describing such algebras is very important, because, the characteristic Lie algebras of the one-dimensional and two-dimensional systems (3) are different by the definition. This circumstance, as well as some gaps in proofs of [17, 18] led to the appearance of [22, 26], where the problem of an explicit description of characteristic Lie algebras $\chi(\sinh u)$ and $\chi(e^u+e^{-2u})$ was posed and solved. It was solved from the point of view of constructing infinite bases and structure relations (different from the bases and relations proposed in this article). However the extremely important relationship between the characteristic Lie algebras of $\chi(\sinh u)$ and $\chi(e^u+e^{-2u})$ of sinh-Gordon and Tzitzeica equations and affine Kac-Moody algebras $A_1^{(1)}$ and $A_2^{(2)}$ escaped the attention of the authors in [22, 26]. In addition, we wrote the generators of these algebras in terms of Bell polynomials, which helped us to determine and relate various gradings of $\chi(\sinh u)$ and $\chi(e^u+e^{-2u})$. Also an interesting feature was the observation that the Lie algebras $\chi(\sinh u)$ and $\chi(\sin u)$ are non-isomorphic over $\mathbb{R}$ (but isomorphic over $\mathbb{C}$).

2 Characteristic Lie Algebra of Hyperbolic Non-linear PDE

Here and in the sequel, we define, unless otherwise stated, all Lie algebras over the field $\mathbb{K}$, which is either the field $\mathbb{R}$ of reals or the field $\mathbb{C}$ of complex numbers.

Consider a system of hyperbolic PDE

$$u^j_{xy} = f^j(u),\ \ j = 1, \dots, n,\ \ u = (u^1, \dots, u^n), \tag{4}$$

where each function $f^j(u)$, $j = 1, \dots, n$, belongs to a $\mathbb{K}$-algebra $C^\omega(\Omega)$ of analytic $\mathbb{K}$-valued functions of n real variables $u = (u^1, \dots, u^n)$ defined on some open domain $\Omega \subset \mathbb{R}^n$ (it is more convinient to consider germs instead of functions, but we will keep the definition from [18, 24]). By x, y we denote two coordinates on the real plane $\mathbb{R}^2$ and assume solutions of Eq. 4 to be analytic functions of x, y.

Take an algebra $C^\omega(\Omega)[u_1, u_2, \dots] = C^\omega(\Omega)[u^1_1, \dots, u^n_1, u^1_2, \dots, u^n_2, \dots]$ of polynomials in an infinite set of variables $\{u_i = (u^1_i, \dots, u^n_i), i \geq 1\}$ with coefficients in $C^\omega(\Omega)$. The multiplicative structure in $C^\omega(\Omega)[u_1, u_2, \dots]$ is defined as the standard product of polynomials.

Example 1 For $n = 2$ the following polynomial

$$P(u^1, u^2; u^1_1, u^2_1, u^1_2, u^2_2, \dots) = \sin(u^1+2u^2) \cdot (u^1_1)^2 + 2\cos u^1 \cdot (u^1_2)^3,$$

belongs to the algebra $C^\omega(\Omega)[u^1_1, u^2_1, u^1_2, u^2_2, \dots] = C^\omega(\Omega)[u_1, u_2, \dots]$.

Define a Lie algebra $\mathcal{L}$ of first order linear differential operators of the form

$$X = \sum_{k=1}^{+\infty} P^\alpha_k(u; u_1, u_2, \dots)\frac{\partial}{\partial u^\alpha_k}, \tag{5}$$

where all coefficients $P_i^\alpha(u; u_1, u_2, \dots), \alpha = 1, \dots, n, i \geq 1$, are polynomials from $C^\omega(\Omega)[u_1, u_2, \dots]$. We used tensor rules in Eq. 5 for summation

$$P_i^\alpha(u; u_1, u_2, \dots)\frac{\partial}{\partial u_k^\alpha} = \sum_{\alpha=1}^{n} P_i^\alpha(u; u_1, u_2, \dots)\frac{\partial}{\partial u_k^\alpha}.$$

Remark 1 We have already said in the Introduction that there does not seem to exist a canonical definition of the characteristic Lie algebra of a hyperbolic PDE. To all appearances, the characteristic Lie algebra $\chi(f)$ of a hyperbolic equation $u_{xy} = f(u)$ with additional structure of a $(\mathbb{K}, C^\omega(\Omega)$-Lie algebra with trivial $\chi(f)$-action on $C^\omega(\Omega)$ is called the characteristic Lie ring of $u_{xy} = f(u)$ in a series of papers [22, 26, 27] et al. Linear dependence or independence of vector fields, the choice of basis in the characteristic Lie algebra $\chi(f)$ is understood in [22, 26, 27] with respect to the left module structure over the localization of $C^\omega(\Omega)$.

In [24], one of the very first and key papers on the characteristic Lie algebras of hyperbolic systems of PDE, vector fields are considered for some fixed value u_M of the variables $u = (u^1, \dots, u^n)$.

More precisely, let $M = (u_M^1, \dots, u_M^n) = u_M$ be a fixed point in Ω. One can consider an evaluation map $ev : \mathcal{L} \to \mathcal{L}$ defined by

$$X = \sum_{k=1}^{+\infty} P_k^\alpha(u; u_1, u_2, \dots)\frac{\partial}{\partial u_k^\alpha} \xrightarrow{\text{ev}_M} X_M = \sum_{k=1}^{+\infty} P_k^\alpha(u_M; u_1, u_2, \dots)\frac{\partial}{\partial u_k^\alpha}.$$

Sometimes by characteristic Lie algebra $\chi(f)$ of a hyperbolic equation $u_{xy} = f(u)$ is called the image $ev_M(\chi(f))$ of the evaluation map ev_M for some choice of a point $M \in \Omega$ [24]. Thus, the Lie algebra $ev_M(\chi(f))$ consists of first order linear differential operators $\sum_{k=1}^{+\infty} P_k \frac{\partial}{\partial u_k}$ with coefficients P_k taken from the standard polynomial ring $\mathbb{C}[u_1, u_2, \dots]$ [24].

The following formulas are valid

$$\left[\frac{\partial}{\partial u^j}, X\right] = \left[\frac{\partial}{\partial u^j}, \sum_{k=1}^{+\infty} P_k^\alpha(u; u_1, u_2, \dots)\frac{\partial}{\partial u_k^\alpha}\right] = \sum_{k=1}^{+\infty} \frac{\partial P_k^\alpha(u; u_1, u_2, \dots)}{\partial u^j}\frac{\partial}{\partial u_k^\alpha}$$

Consider an operator $D : C^\omega(\Omega)[u_1, u_2, \dots] \to C^\omega(\Omega)(\Omega)[u_1, u_2, \dots]$.

$$D = u_1^\alpha\frac{\partial}{\partial u^\alpha} + u_2^\alpha\frac{\partial}{\partial u_1^\alpha} + u_3^\alpha\frac{\partial}{\partial u_2^\alpha} + \dots + u_{k+1}^\alpha\frac{\partial}{\partial u_k^\alpha} + \dots, \tag{6}$$

The operator D is called the operator of the full partial derivative $\frac{\partial}{\partial x}$. The definition of the operator D has a formal algebraic meaning, but the formula (6) defining it has a completely concrete analytic origin. Indeed, consider a solution $u(x, y) = (u^1(x, y), \dots, u^n(x, y))$ of the system (4). Let $g^j(u; u_1, u_2, \dots) \in C^\omega(\Omega)[u_1, u_2, \dots], j = 1, \dots, n$. Define with a help of $u(x, y)$ a composite function $g(x, y) = (g^1(x, y), \dots, g^n(x, y))$ of two arguments x, y:

$$g^j(x, y) = g^j(u(x, y); u_x^1(x, y), \dots, u_x^n(x, y), u_{xx}^1(x, y), \dots, u_{xx}^n(x, y), \dots)$$

In other words, we have a parametrization $u_j^\alpha = \frac{\partial^j u^\alpha}{\partial x^j}, \alpha = 1, \dots, n, j \geq 1$.

$$(u_1^1, \dots, u_1^n) = (u_x^1, \dots, u_x^n), (u_2^1, \dots, u_2^n) = (u_{xx}^1, \dots, u_{xx}^n), \dots,$$

In particular we have obvious formulas

$$\frac{\partial u^\alpha}{\partial x} = D(u^\alpha) = u_1^\alpha, \frac{\partial u_k^\alpha}{\partial x} = D(u_k^\alpha) = u_{k+1}^\alpha, \ \alpha = 1, \dots, n, k \geq 1.$$

Computing the partial derivative $\frac{\partial g^j}{\partial x}$ of the composite function $g^j(x, y)$, we obtain

$$\frac{\partial g^j}{\partial x} = \frac{\partial u^\alpha}{\partial x}\frac{\partial g^j}{\partial u^\alpha} + \frac{\partial u_1^\alpha}{\partial x}\frac{\partial g^j}{\partial u_1^\alpha} + \dots + \frac{\partial u_k^\alpha}{\partial x}\frac{\partial g^j}{\partial u_k^\alpha} + \dots = u_1^\alpha\frac{\partial g^j}{\partial u^\alpha} + u_2^\alpha\frac{\partial g^j}{\partial u_1^\alpha} + \dots + u_{k+1}^\alpha\frac{\partial g^j}{\partial u_k^\alpha} + \dots$$

Similar arguments lead us to the formula for $\frac{\partial}{\partial y}$ "on solutions of" Eq. 4

$$X(f) = \frac{\partial}{\partial y} = f^\alpha\frac{\partial}{\partial u_1^\alpha} + D(f^\alpha)\frac{\partial}{\partial u_2^\alpha} + D^2(f^\alpha)\frac{\partial}{\partial u_3^\alpha} + \dots + D^{k+1}(f^\alpha)\frac{\partial}{\partial u_k^\alpha} + \dots$$

Definition 1 ([18, 24]) A Lie algebra $\chi(f)$ generated by $n+1$ vector fields

$$X(f), \frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^n},$$

is called characteristic Lie algebra of the hyperbolic system (4).

A linear span of $\frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^n}$ determines an abelian subalgebra $\chi_0(f)$ of $\chi(f)$. One can easily verify the followng commutation relations $\frac{\partial}{\partial u^j}$ with $X(f)$

$$\left[\frac{\partial}{\partial u^j}, X(f)\right] = X\left(\frac{\partial f}{\partial u^j}\right) = \sum_{k=1}^{+\infty} D^{k-1}\left(\frac{\partial f^\alpha}{\partial u^j}\right)\frac{\partial}{\partial u_k^\alpha}, j = 1, \dots, n.$$

We denote by $\chi_1(f)$ the smallest invariant subspace of χ_0-action on $\chi(f)$ containing the operator $X(f)$. The subspace $\chi_1(f)$ coinsides with the linear span of all operators $X\left(\frac{\partial^s f}{\partial u^{j_1}\dots\partial u^{j_s}}\right)$, $s \geq 0$ and we have

$$[\chi_0(f), \chi_1(f)] = \chi_1(f).$$

In this article we are interested mainly in the one-dimensional case $n = 1$. The corresponding scalar PDE is well known and sometimes it is called the Klein-Gordon equation [27, 28].

Indeed, consider the classical Klein-Gordon equation

$$u_{tt} - u_{zz} = f(u).$$

Making a linear change of variables $x = \frac{z+t}{2}$, $y = \frac{z-t}{2}$ we'll get

$$u_{xy} = f(u), \tag{7}$$

where we assume that $f(u)$ is an analytic function on one variable u.

The operator D of the full derivative with respect to x is

$$D = u_1\frac{\partial}{\partial u} + u_2\frac{\partial}{\partial u_1} + u_3\frac{\partial}{\partial u_2} + \dots + u_{n+1}\frac{\partial}{\partial u_n} + \dots,$$

We recall that u_i are parametrized by some solution $u(x, y)$ of Eq. 7.

$$u_1 = u_x, u_2 = u_{xx}, \dots, u_i = \frac{\partial^i u}{\partial x^i}, \dots$$

In this case we have

$$\frac{\partial}{\partial x}(g(u(x, y), u_1(x, y), u_2(x, y), \dots) = D(g(u(x, y), u_1(x, y), u_2(x, y), \dots)).$$

Example 2 It's an elementary exercise to verify by recursion that

$$D^k(e^{\lambda u}) = e^{\lambda u} B_k(\lambda u_1, \dots, \lambda u_k),$$

where

$$B_k(\lambda u_1, \dots, \lambda u_k) = u_1^k \lambda^k + \dots + u_k \lambda, \quad k = 0, 1, 2, \dots,$$

are complete Bell polynomials of degree k. Complete Bell polynomials are well-known combinatorial object and they have a lot of properties and applications, see [2] for references. We want just to recall only a few basic facts about them.

Complete Bell polynomials can be defined recursively by the formula

$$B_{n+1}(u_1, u_2, \dots, u_{n+1}) = \sum_{i=0}^{n} \binom{n}{i} B_{n-i}(u_1, u_2, \dots, u_{n-i}) u_{i+1},$$

with the initial condition $B_0 = 1$.

The first few complete Bell polynomials are:

$$B_1(u_1) = u_1,\ B_2(u_1, u_2) = u_1^2 + u_2,\ B_3(u_1, u_2, u_3) = u_1^3 + 3u_1u_2 + u_3,$$
$$B_4(u_1, u_2, u_3, u_4) = u_1^4 + 6u_1^2u_2 + 4u_1u_3 + 3u_2^2 + u_4, \dots$$

The generating function for complete Bell polynomials is

$$\exp\left(\sum_{i=1}^{+\infty} u_i \frac{t^i}{i!}\right) = \sum_{n=0}^{+\infty} B_n(u_1, \dots, u_n) \frac{t^n}{n!}.$$

An one-dimensional version of the general Definition 1 is

Definition 2 The characterisitic Lie algebra $\chi(f)$ of Klein-Gordon equation (7) is a Lie algebra of vector fields generated by two vector fields X_0 and X_1

$$X_0 = \frac{\partial}{\partial u},\ X_1 = X(f) = f\frac{\partial}{\partial u_1} + D(f)\frac{\partial}{\partial u_2} + \dots + D^{n-1}(f)\frac{\partial}{\partial u_n} + \dots$$

It is an elementary exercise to express $D^k(f)$ for an arbitrary analytic f in terms of complete differential Bell polynomials $B_n(u_1\frac{d}{du}, \dots, u_n\frac{d}{du})$, i.e.

$$D^n(f) = B_n\left(u_1\frac{d}{du}, \dots, u_n\frac{d}{du}\right)(f),$$

where first four differential Bell polynomials are

$$B_0 = 1,\ B_1(u_1\tfrac{d}{du}) = u_1\tfrac{d}{du},\ B_2(u_1\tfrac{d}{du}, u_2\tfrac{d}{du}) = u_1^2\tfrac{d^2}{du^2} + u_2\tfrac{d}{du},$$
$$B_3(u_1\tfrac{d}{du}, u_2\tfrac{d}{du}, u_3\tfrac{d}{du}) = u_1^3\tfrac{d^3}{du^3} + 3u_1u_2\tfrac{d^2}{du^2} + u_3\tfrac{d}{du},$$
$$B_4(u_1\tfrac{d}{du}, u_2\tfrac{d}{du}, u_3\tfrac{d}{du}, u_4\tfrac{d}{du}) = u_1^4\tfrac{d^4}{du^4} + 6u_1^2u_2\tfrac{d^3}{du^3} + (4u_1u_3 + 3u_2^2)\tfrac{d^2}{du^2} + u_4\tfrac{d}{du}.$$

One can express $D^n(f)$ in terms of incomplete Bell polynomials $B_{n,k}(u_1, \dots, u_{n-k+1})$ (see [2] for references)

$$D^n(f) = \sum_{k=1}^{n} B_{n,k}(u_1, \dots, u_{n-k+1}) \frac{d^k f}{dx^k}.$$

We recall here only the generating function for incomplete Bell polynomials

$$\exp\left(z\sum_{i=1}^{+\infty} u_i \frac{t^i}{i!}\right) = \sum_{n,k \geq 0} B_{n,k}(u_1, \dots, u_{n-k+1}) z^k \frac{t^n}{n!}.$$

and the relationship between the complete and incomplete polynomials

$$B_n(u_1,\dots,u_n)=\sum_{k=1}^{n} B_{n,k}(u_1,\dots,u_{n-k+1})$$

We denote by Y_1 the commutator

$$Y_1=[X_0,X_1]=f_u\frac{\partial}{\partial u_1}+D(f_u)\frac{\partial}{\partial u_2}+\dots+D^{n-1}(f_u)\frac{\partial}{\partial u_n}+\dots$$

We have also

$$[X_0,Y_1]=[X_0,[X_0,X_1]]=f_{uu}\frac{\partial}{\partial u_1}+D(f_{uu})\frac{\partial}{\partial u_2}+\dots+D^{n-1}(f_{uu})\frac{\partial}{\partial u_n}+\dots$$

Example 3 Let $f(u)=e^u$. We have the Liouville equation $u_{xy}=e^u$. It follows that $[X_0,X_1]=X_1$ in this case. Hence the characteristic Lie algebra $\chi(e^u)$ of the Liouville equation is the non-abelian two-dimensional solvable Lie algebra. It can be defined by its basis X_0, X_1 and the unique commutation relation

$$[X_0,X_1]=X_1.$$

We have already noted that the implication of the canonical definition of the characteristic Lie algebra is related to its auxiliary, albeit very important, role in the search for integrals and higher symmetries of corresponding hyperbolic equations.

Definition 3 An analytic function $w(u;u_1,\dots,u_n)$ is called x-integral of PDE $u_{xy}=f(u)$ if

$$\frac{\partial}{\partial y}w\left(u,u_x,u_{xx},\dots,\frac{\partial^n u}{\partial x^n}\right)=0, \tag{8}$$

where $u(x,y)$ is a solution of $u_{xy}=f(u)$.

Respectively an analytic function $w(u_1,\dots,u_n)$ is called y-integral of PDE $u_{xy}=f(u)$ if

$$\frac{\partial}{\partial x}w\left(u,u_y,u_{yy},\dots,\frac{\partial^n u}{\partial y^n}\right)=0,\ \ u_{xy}=f(u).$$

Evidently in our symmetric case $u_{xy}=f(u)$ a x-integral w defines a y-integral and vise versa. The equation (8) can be written as

$$u_1\frac{\partial w}{\partial u}+X(f)w=0.$$

and it is equivalent to the system

$$\frac{\partial w}{\partial u}=0,\,X(f)w=0.$$

In other words, a x-integral w is annihilated by two generators $\frac{\partial}{\partial u}, X(f)$ of characteristic Lie algebra $\chi(f)$ and hence it is annihilated by the whole Lie algebra $\chi(f)$.

Example 4 A second order polynomial $w_2(u_1,u_2)=\frac{1}{2}(u_1)^2-u_2$ determines both x-, y-integrals of the Liouville equation $u_{xy}=e^u$.

$$X(e^u)w_2(u_1,u_2)=e^u\left(\frac{\partial}{\partial u_1}+u_1\frac{\partial}{\partial u_2}\right)\left(\frac{1}{2}(u_1)^2-u_2\right)=0.$$

Or one can verify directly that $w_2(u_x, u_{xx}) = \frac{1}{2}(u_x)^2 - u_{xx}$ is a x-integral

$$((u_x)^2 - 2u_{xx})_y = 2u_x u_{xy} - 2u_{xyx} = 2e^u(u_x - u_x) = 0.$$

Definition 4 A hyperbolic one-dimensional PDE $u_{xy} = f(u)$ is called Darboux-integrable if it admits both non-trivial x-, y-integrals.

Obviously the Liouville equation $u_{xy} = e^u$ is Darboux-integrable. Moreover there is a well-known classical formula found by Liouville himself for its general solution in terms of two arbitrary functions $\varphi(t)$, $\psi(t)$ of one variable t

$$u(x, y) = \log \frac{2\varphi'(x)\psi'(y)}{(1 - \varphi(x)\psi(y))^2}.$$

Technical details of a transition from the formulas for x, y-integrals of the Liouville equation to this explicit expression for $u(x, y)$ can be found in [4, 8].

Consider the sinh-Gordon equation $u_{xy} = \sinh u$. It is well-known that it is not Darboux-integrable but it is integrable by the inverse scattering problem method (see [28, 30] for references). In the framework of inverse scattering method one is looking for higher symmetries of the non-linear PDE under the study. We will not discuss details and remark only that we are looking now for non-trivial solutions of the so-called defining equation

$$DX(f)\phi = f'(u)\phi. \tag{9}$$

Example 5 A polynomial $\phi_3(u_1, u_2, u_3) = u_3 - \frac{1}{2}u_1^3$ is a solution of the defining equation (9) for the sinh-Gordon equation $u_{xy} = \sinh u$. It is not difficult to verify that for a function $u(x, y)$ satisfying the sinh-Gordon equation $u_{xy} = \sinh u$ we have

$$(u_{xxx} - \frac{1}{2}u_x^3)_{xy} = \cosh u(u_{xxx} - \frac{1}{2}u_x^3).$$

A method was developed in [27] such that, with the help of operators from the characteristic Lie algebra $\chi(\sinh)$, one can obtain all higher symmetries of the sinh-Gordon equation.

We finish this section with one simple technical lemma, which we will need in the sequel.

Lemma 1 ([27]) *Let X be a differential operator*

$$X = \sum_{i=1}^{+\infty} P_i \frac{\partial}{\partial u_i}, \quad P_i = P_i(u, u_1, \dots, u_n, \dots),$$

such that $[X, D] = 0$. *Then* $X = 0$.

Proof The proof from [27] is quite elementary and we present it here.

$$[D, X] = \sum_{i=1}^{+\infty} D(P_i)\frac{\partial}{\partial u_i} - P_1\frac{\partial}{\partial u} - \sum_{i=1}^{+\infty} P_{i+1}\frac{\partial}{\partial u_i}.$$

It follows that if $[X, D] = 0$ then

$$P_1 \equiv 0, D(P_i) = P_{i+1}, i = 1, 2, \dots, n, \dots$$

It means that all polynomials P_i have to vanish, i.e. $P_i \equiv 0, \forall i \geq 1$. □

Corollary 1

$$[D, X_1] = \sum_{i=1}^{+\infty} D(D^{i-1}(f))\frac{\partial}{\partial u_i} - f\frac{\partial}{\partial u} - \sum_{i=1}^{+\infty} D^i(f)\frac{\partial}{\partial u_i} = -fX_0. \tag{10}$$

3 Narrow Positively Graded Lie Algebras and Loop Algebras

Definition 5 A Lie algebra $\mathfrak{g}$ is called $\mathbb{N}$-graded (positively graded) if there is a decomposition of $\mathfrak{g}$ into a direct sum of linear subspaces

$$\mathfrak{g}=\bigoplus_{i=1}^{+\infty}\mathfrak{g}_i,\ \ [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j},\ \text{ for all } i, j \in \mathbb{N}.$$

Example 6 Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{K}$. Then the Lie algebra $L_+(\mathfrak{g}) = \oplus_{k=1}^{+\infty}\mathfrak{g} \otimes t^k$ with a bracket $[,]_L$ defined by

$$[g \otimes P(t), h \otimes Q(t)]_L = [g, h] \otimes P(t)Q(t),$$

where $[,]$ is the Lie bracket in $\mathfrak{g}$ is $\mathbb{N}$-graded and dimensions of all its homogeneous components are equal to $\dim \mathfrak{g}$. $L_+(\mathfrak{g})$ can be regarded as the positive part of the corresponding $\mathbb{Z}$-graded loop algebra $L(\mathfrak{g}) = \oplus_{k\in\mathbb{Z}}\mathfrak{g} \otimes t^k$.

Definition 6 ([23]) A $\mathbb{N}$-graded Lie algebra $\mathfrak{g}$ is called of width d if all its homogeneous components is uniformly bounded by $d \geq 1$.

$$\dim \mathfrak{g}_i \leq d, \forall i \in \mathbb{N}, \tag{11}$$

where the constant d is the smallest with the property (11).

Shalev and Zelmanov introduced a notion of *narrow* Lie algebra, i.e. a $\mathbb{N}$-graded Lie algebra $\mathfrak{g} = \oplus_{i\in\mathbb{N}}\mathfrak{g}_i$ of width $d = 1$ or $d = 2$.

Example 7 The Lie algebra $\mathfrak{m}_0$ is defined by its infinite basis $e_1, e_2, \ldots, e_n, \ldots$ with the commutation relations:

$$[e_1, e_i] = e_{i+1},\ \forall\ i \geq 2.$$

The remaining brackets among basis elements vanish: $[e_i, e_j] = 0$ if $i, j \neq 1$.

We will always omit the trivial commutator relations $[e_i, e_j] = 0$ in the definitions of Lie algebras.

Example 8 The Lie algebra $\mathfrak{m}_2$ is defined by its infinite basis $e_1, e_2, \ldots, e_n, \ldots$ and the commutating relations:

$$[e_1, e_i] = e_{i+1}, \quad \forall\ i \geq 2; \qquad [e_2, e_j] = e_{j+2}, \quad \forall\ j \geq 3.$$

Example 9 The positive part W^+ of the Witt algebra. It can be also defined by its infinite basis and the commutating relations

$$[e_i, e_j] = (j - i)e_{i+j},\ \forall\ i, j \in \mathbb{N}.$$

These three infinite-dimensional algebras $\mathfrak{m}_0, \mathfrak{m}_2, W^+$ are the narrowest possible $\mathbb{N}$-graded Lie algebras. They are all generated by two elements e_1, e_2 of degrees one and two respectively.

Example 10 The loop algebra $\mathcal{L}(\mathfrak{sl}(2,\mathbb{K}))$ and its positive part $\mathfrak{n}_1$.

Consider the loop algebra $\mathcal{L}(\mathfrak{sl}(2,\mathbb{K})) = \mathfrak{sl}(2,\mathbb{K}) \otimes \mathbb{K}[t,t^{-1}]$, where $\mathbb{K}[t,t^{-1}]$ is the ring of Laurent polynomials over $\mathbb{K}$. It has a Lie subalgebra of "polynomial loops"

$$\mathcal{L}(\mathfrak{sl}(2,\mathbb{K}))^{\geq 0} = \mathfrak{sl}(2,\mathbb{K}) \otimes \mathbb{K}[t]$$

that we will call in the sequel the non-negative part of the loop algebra $\mathcal{L}(\mathfrak{sl}(2,\mathbb{K}))$. Consider an infinite set of polynomial matrices defined for $k \in \mathbb{Z}$ by

$$e_{3k+1}=\frac{1}{2}\begin{pmatrix} 0 & t^k \\ 0 & 0 \end{pmatrix}, e_{3k-1}=\begin{pmatrix} 0 & 0 \\ t^k & 0 \end{pmatrix}, e_{3k}=\frac{1}{2}\begin{pmatrix} t^k & 0 \\ 0 & -t^k \end{pmatrix}. \qquad (12)$$

Evidently this set of matrices is an infinite basis of the loop algebra $\mathcal{L}(\mathfrak{sl}(2,\mathbb{K}))$.

The linear span of its half $\langle e_0, e_1, e_2, e_3, \dots, e_n, \dots \rangle$ is an infinite basis of the non-negative part $\mathcal{L}(\mathfrak{sl}(2,\mathbb{K}))^{\geq 0} = \mathfrak{sl}(2,\mathbb{K}) \otimes \mathbb{K}[t]$. It is $\mathbb{Z}_{\geq 0}$-graded with one-dimensional homogeneous components:

$$\mathcal{L}(\mathfrak{sl}(2,\mathbb{K}))^{\geq 0} = \oplus_{i=0}^{+\infty} \langle e_i \rangle \subset \mathfrak{sl}(2,\mathbb{K}) \otimes \mathbb{K}[t],$$

The structure relations for basic elements $e_i, e_j, i, j \geq 0$, are given by the rule

$$[e_i, e_j] = c_{i,j} e_{i+j},\ c_{i,j} = \begin{cases} 1, \text{if } j-i \equiv 1 \bmod 3; \\ 0, \text{if } j-i \equiv 0 \bmod 3; \\ -1, \text{if } j-i \equiv -1 \bmod 3. \end{cases} \qquad (13)$$

Now consider the positive part $\mathfrak{n}_1$ of the loop algebra $\mathcal{L}(\mathfrak{sl}(2,\mathbb{K}))$. It is defined as the linear span $\langle e_1, e_2, e_3, \dots, e_n, \dots \rangle$ and it is a $\mathbb{N}$-graded Lie algebra with one-dimensional homogeneous components:

$$\mathfrak{n}_1 = \oplus_{i=1}^{+\infty} \langle e_i \rangle \subset \mathfrak{sl}(2,\mathbb{K}) \otimes \mathbb{K}[t].$$

The Lie algebra $\mathfrak{n}_1$ is a codimension one ideal in $\mathcal{L}(\mathfrak{sl}(2,\mathbb{K}))^{\geq 0}$.

Example 11 The twisted loop algebra $\mathcal{L}(\mathfrak{sl}(3,\mathbb{K}), \mu)$ and its positive part $\mathfrak{n}_2$.

Consider a diagram automorphism μ of $\mathfrak{sl}(3,\mathbb{K})$ of the second order $\mu^2 = 1$ [9].

$$\mu : \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \to \begin{pmatrix} -a_{33} & a_{23} & -a_{13} \\ a_{32} & -a_{22} & a_{12} \\ -a_{31} & a_{21} & -a_{11} \end{pmatrix}$$

The simple Lie algebra $\mathfrak{sl}(3,\mathbb{K})$ is decomposed into the sum of eigensubspaces $\mathfrak{g}_0, \mathfrak{g}_1$ of μ coressponding to eigenvalues $1, -1$ respectively

$$\mathfrak{sl}(3,\mathbb{K}) = \mathfrak{g}_0 \oplus \mathfrak{g}_1, [\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{g}_0, [\mathfrak{g}_0, \mathfrak{g}_1] \subset \mathfrak{g}_1, [\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{g}_0.$$

One can choose a basis $f_{-1}, f_0, f_1, f_2, \dots, f_6$ of $\mathfrak{sl}(3,\mathbb{K})$, such that $\mathfrak{g}_0 = \langle f_{-1}, f_0, f_1 \rangle$ and $\mathfrak{g}_1 = \langle f_2, f_3, f_4, f_5, f_6 \rangle$

$$f_{-1}=\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, f_0=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, f_1=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, f_2=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$f_3=\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, f_4=\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, f_5=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, f_6=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

We recall (see [9]) that the twisted loop algebra $\mathcal{L}(\mathfrak{sl}(3,\mathbb{K}),\mu)$ is a Lie subalgebra of the loop algebra $\mathcal{L}(\mathfrak{sl}(3,\mathbb{K}))$ defined by

$$\mathcal{L}(\mathfrak{sl}(3,\mathbb{K}),\mu)=\bigoplus_{j\in\mathbb{Z}}\mathfrak{g}_{j\,(\mathrm{mod}\;2)}\otimes t^j.$$

There is an infinite basis of $\mathcal{L}(\mathfrak{sl}(3,\mathbb{K}),\mu)$ (see [9], Exercise 8.12).

$$\begin{gathered} f_{8k-1}{=}f_{-1}\otimes t^{2k},\; f_{8k}{=}f_0\otimes t^{2k},\; f_{8k+1}{=}f_1\otimes t^{2k},\\ f_{8k+2}{=}f_2\otimes t^{2k+1},\; f_{8k+3}{=}f_3\otimes t^{2k+1},\; f_{8k+4}{=}f_4\otimes t^{2k+1},\\ f_{8k+5}{=}f_5\otimes t^{2k+1},\; f_{8k+6}{=}f_6\otimes t^{2k+1},\;\; k\in\mathbb{Z}. \end{gathered}\tag{14}$$

It's easy to calculate the commutators $[f_q,f_l]$ of all these basic elements

$$[f_q,f_l]=d_{q,l}f_{q+l},\;\; q,l\in\mathbb{N}.\tag{15}$$

where the structure constants $d_{q,l}$ are presented in the Table 1.

The matrix $(d_{q,l})$ is skew-symmetric, its elements $d_{q,l}$ depend only on the residue generated by dividing positive integers q and l by 8. Moreover $(d_{q,l})$ satisfy the following relations (see [9] for references):

$$d_{i,j}+d_{q,l}=0,\;\text{ if } i{+}q\equiv 0 \bmod 8,\; j{+}l\equiv 0 \bmod 8.$$

We define the non-negative part $\mathcal{L}(\mathfrak{sl}(3,\mathbb{K}),\mu)^{\geq 0}$ of the twisted loop algebra $\mathcal{L}(\mathfrak{sl}(3,\mathbb{K}),\mu)$ as

$$\mathcal{L}(\mathfrak{sl}(3,\mathbb{K}),\mu)^{\geq 0}=\bigoplus_{j=0}^{+\infty}\mathfrak{g}_{j\,(\mathrm{mod}\;2)}\otimes t^j.$$

It coinsides with an infinite-dimensional linear span $\langle f_0,f_1,f_2,f_3,\dots,f_n,\dots\rangle$.

Now we introduce the positive part $\mathfrak{n}_2$ of the twisted loop algebra $\mathcal{L}(\mathfrak{sl}(3,\mathbb{K}),\mu)$ by setting

$$\mathfrak{n}_2=\oplus_{i=1}^{+\infty}\langle f_i\rangle=\bigoplus_{j=1}^{+\infty}\mathfrak{g}_{j\,(\mathrm{mod}\;2)}\otimes t^j,$$

Evidently $\mathfrak{n}_2$ is a $\mathbb{N}$-graded Lie algebra of width one.

Fialowski classified [5] the narrowest $\mathbb{N}$-graded Lie algebras, i.e. $\mathbb{N}$-graded Lie algebras $\mathfrak{g}=\oplus_{i\in\mathbb{N}}\mathfrak{g}_i$ with one-dimensional homogeneous components $\mathfrak{g}_i$ that are generated by two elements from $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively. Fialowski's classification list contains the Lie algebras

Table 1 Structure constants for $\mathfrak{n}_2$

	f_{8j}	f_{8j+1}	f_{8j+2}	f_{8j+3}	f_{8j+4}	f_{8j+5}	f_{8j+6}	f_{8j+7}
f_{8i}	0	1	−2	−1	0	1	2	−1
f_{8i+1}	−1	0	1	1	−3	−2	0	1
f_{8i+2}	2	−1	0	0	0	1	−1	0
f_{8i+3}	1	−1	0	0	3	−1	1	−2
f_{8i+4}	0	3	0	−3	0	3	0	−3
f_{8i+5}	−1	2	−1	1	−3	0	0	−1
f_{8i+6}	−2	0	1	−1	0	0	0	1
f_{8i+7}	1	−1	0	2	3	1	−1	0

$\mathfrak{m}_0, \mathfrak{m}_2, W^+, \mathfrak{n}_1, \mathfrak{n}_2$ considered above and a special multiparametric family of pairwise non-isomorphic Lie algebras. Later a part of Fialowski's theorem was rediscovered by Shalev and Zelmanov [23].

4 Naturally Graded Pro-nilpotent Lie Algebras

Definition 7 A Lie algebra $\mathfrak{g}$ is called pro-nilpotent if for the ideals $\mathfrak{g}^i$, $\mathfrak{g} = \mathfrak{g}^1$, $\mathfrak{g}^i = [\mathfrak{g}, \mathfrak{g}^{i-1}]$, $i \geq 2$, of its descending central sequence we have:

$$\cap_{i=1}^{+\infty} \mathfrak{g}^i = \{0\}, \ \dim \mathfrak{g}/\mathfrak{g}^i < +\infty.$$

It is clear that a finite-dimensional nilpotent Lie algebra $\mathfrak{g}$ is pro-nilpotent. Moreover, it follows from the Definition 7 that every quotient $\mathfrak{g}/\mathfrak{g}^i$ of a pro-nilpotent Lie algebra is finite-dimensional nilpotent Lie algebra and there is an inverse spectrum of finite-dimensional nilpotent Lie algebras

$$\dots \stackrel{p_{k+2,k+1}}{\longrightarrow} \mathfrak{g}/\mathfrak{g}^{k+1} \stackrel{p_{k+1,k}}{\longrightarrow} \mathfrak{g}/\mathfrak{g}^k \stackrel{p_{k,k-1}}{\longrightarrow} \dots \stackrel{p_{3,2}}{\longrightarrow} \mathfrak{g}/\mathfrak{g}^2 \stackrel{p_{2,1}}{\longrightarrow} \mathfrak{g}/\mathfrak{g}^1,$$

We denote by $\widehat{\mathfrak{g}}$ the projective (inverse) limit $\widehat{\mathfrak{g}} = \varprojlim_k \mathfrak{g}/\mathfrak{g}^k$. We call $\mathfrak{g}$ complete if $\widehat{\mathfrak{g}} = \mathfrak{g}$ ($\mathfrak{g} = \widehat{\mathfrak{g}}$ is an inverse limit of finite-dimensional nilpotent Lie algebras).

For a given pro-nilpotent Lie algebra $\mathfrak{g}$ one can consider a sequence of projections of a pro-nilpotent Lie algebra $\mathfrak{g}$ to its finite-dimensional quotients:

$$p_m : \mathfrak{g} \to \mathfrak{g}/\mathfrak{g}^m, \ m \in \mathbb{N}.$$

They determine the topology of the inverse limit of finite-dimensional spaces on $\mathfrak{g}$, i.e., smallest topology on $\mathfrak{g}$ for which all these maps p_m are continuous.

Example 12 We have considered three infinite-dimensional $\mathbb{N}$-graded Lie algebras

$$\mathfrak{m}_0, \ \mathfrak{m}_2, \ W^+.$$

All of them are pro-nilpotent and not complete. Their completions $\widehat{\mathfrak{m}}_0, \widehat{\mathfrak{m}}_2, \widehat{W}^+$ are the spaces of formal series $\sum_{k=1}^{+\infty} \alpha_k e_k$ of corresponding basic vectors e_k, $k \in \mathbb{N}$.

Definition 8 A Lie algebra $\mathfrak{g}$ is called pro-solvable if for the ideals $\mathfrak{g}^{(i)}$, $\mathfrak{g} = \mathfrak{g}^{(0)}$, $\mathfrak{g}^{(i)} = [\mathfrak{g}^{(i-1)}, \mathfrak{g}^{(i-1)}]$, $i \geq 1$, of its derived sequence of ideals we have:

$$\cap_{i=1}^{+\infty} \mathfrak{g}^{(i)} = \{0\}, \ \dim \mathfrak{g}/\mathfrak{g}^{(i)} < +\infty.$$

The descending central series $\{\mathfrak{g}^k\}$ of a pro-nilpotent Lie algebra $\mathfrak{g}$ determines a decreasing filtration

$$\mathfrak{g} = \mathfrak{g}^1 \supset \mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}] \cdots \supset \mathfrak{g}^m \supset \mathfrak{g}^{m+1} \supset \dots, \ [\mathfrak{g}^m, \mathfrak{g}^n] \subset \mathfrak{g}^{m+n}, \ m, n \in \mathbb{N}.$$

and one can consider the associated graded Lie algebra $\mathrm{gr}_C \mathfrak{g}$

$$\mathrm{gr}_C \mathfrak{g} = \oplus_{i=1}^{+\infty} \left(\mathrm{gr}_C \mathfrak{g}\right)_i = \oplus_{i=1}^{+\infty} \left(\mathfrak{g}^i/\mathfrak{g}^{i+1}\right)$$

with the bracket defined on its homogeneous components $\left(\mathrm{gr}_C \mathfrak{g}\right)_i, \left(\mathrm{gr}_C \mathfrak{g}\right)_j$ by

$$\left[x + \mathfrak{g}^{i+1}, y + \mathfrak{g}^{j+1}\right] = [x, y] + \mathfrak{g}^{i+j+1}, x \in \mathfrak{g}^i, y \in \mathfrak{g}^j.$$

Definition 9 A pro-nilpotent Lie algebra $\mathfrak{g}$ is called naturally graduable if it is isomorphic to its associated graded $\mathrm{gr}_C\mathfrak{g}$.

Definition 10 A $\mathbb{N}$-grading $\mathfrak{g} = \oplus_{i=1}^{+\infty}\mathfrak{g}_i$ of a naturally graduable pro-nilpotent Lie algebra $\mathfrak{g}$ is called natural grading if there exist a graded isomorphism

$$\varphi : \mathrm{gr}_C\mathfrak{g} \to \mathfrak{g},\ \varphi((\mathrm{gr}_C\mathfrak{g})_i) = \mathfrak{g}_i,\ i \in \mathbb{N}.$$

The Lie algebra $\mathfrak{m}_0$ considered above is naturally graduable. However its grading of width one considered above is not natural

$$\left(\mathrm{gr}_C m_0\right)_1 = \langle e_1, e_2\rangle, \left(\mathrm{gr}_C m_0\right)_i = \langle e_{i+1}\rangle, i \geq 2.$$

The positive part W^+ of the Witt algebra and $\mathfrak{m}_2$ are not naturally graded Lie algebras, one can easily verify the following isomorphisms:

$$\mathrm{gr}_C\mathfrak{m}_2 \cong \mathrm{gr}_C W^+ \cong \mathrm{gr}_C\mathfrak{m}_0 \cong \mathfrak{m}_0.$$

Definition 11 A $\mathbb{N}$-graded Lie algebra $\mathfrak{g} = \oplus_{i=1}^{+\infty}\mathfrak{g}_i$ is naturally graded if and only if $[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}, i \in \mathbb{N}$.

In particular it means that a naturally graded Lie algebra $\mathfrak{g} = \oplus_{i=1}^{+\infty}\mathfrak{g}_i$ is generated by its first homogeneous component $\mathfrak{g}_1$. The equivalence of two different definitions of a naturally graded Lie algebra follows from the basic properties of the descending central series of a Lie algebra.

The notion of naturally graded Lie algebra is the infinite-dimensional generalization of so-called Carnot algebra.

Definition 12 ([1]) A finite-dimensional Lie algebra $\mathfrak{g}$ is called Carnot algebra if it admits a $\mathbb{N}$-grading $\mathfrak{g} = \oplus_{i=1}^{n}\mathfrak{g}_i$ such that

$$[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}, i = 1, 2, \ldots, n-1,\ [\mathfrak{g}_1, \mathfrak{g}_n] = 0. \tag{16}$$

Proposition 1 *The Lie algebras* $\mathfrak{n}_1$ *and* $\mathfrak{n}_2$ *are naturally graded Lie algebras of width two.*

Proof For the proof we will introduce new bases for both algebras.

In the case $\mathfrak{n}_1$ we define new basic vectors $a_{2k+1}, b_{2k+1}, c_{2k}$ by the rule:

$$a_{2k+1}{=}e_{3k+1},\ b_{2k+1}{=}e_{3k+2},\ c_{2k}{=}e_{3k},\ \text{for all } k \in \mathbb{Z}_+.$$

The structure relations now look as follows

$$[a_{2k+1}, b_{2l+1}]{=}c_{2(k+l+1)},\ [c_{2k}, a_{2l+1}]{=}a_{2(k+l)+1},\ [c_{2k}, b_{2l+1}]{=}-b_{2(k+l)+1}, \tag{17}$$

One can easily verify by recursion that

$$C^{2m+1}\mathfrak{n}_1 = \mathrm{Span}(a_{2m+1}, b_{2m+1}, c_{2m+2}, \ldots,),\quad C^{2m}\mathfrak{n}_1 = \mathrm{Span}(c_{2m}, a_{2m+1}, b_{2m+1}, \ldots,).$$

Hence the natural grading is defined by

$$\mathfrak{n}_1 = \oplus_{i=1}^{+\infty}\mathfrak{n}_{1,i},\ \text{where}\ \mathfrak{n}_{1,2m+1} = \langle a_{2m+1}, b_{2m+1}\rangle, \mathfrak{n}_{1,2m} = \langle c_{2m}\rangle$$

i.e. with one-dimensional even and two-dimensional odd homogeneous components.

In the case $\mathfrak{n}_2$ we define new basic vectors a_i, b_{6q+1}, b_{6q+5} by

$$\begin{matrix} a_{6q+1}=e_{8k+1}, & a_{6q+4}=e_{8k+5}, & \\ a_{6q+2}=e_{8k+3}, & a_{6q+5}=e_{8k+6}, & b_{6q+1}=e_{8k+2}, \\ a_{6q+3}=e_{8k+4}, & a_{6q+6}=e_{8k+8}, & b_{6q+5}=e_{8k+7}, \end{matrix}$$

Then

$$\mathfrak{n}_2 = \oplus_{i=1}^{+\infty}\mathfrak{n}_{2,i}, \ \mathfrak{n}_{2,i} = \langle a_i, b_i\rangle, \text{ if } i = 6q+2 \text{ or } i = 6q+5, \ \mathfrak{n}_{2,i} = \langle a_i\rangle \text{ in other cases.}$$

The proof is completely analogous to the previous case and is reduced to the direct calculation of ideals $C^k\mathfrak{n}_2$. □

We define the set of polynomial matrices for positive integers $k = 1, 2, \dots$:

$$u_{2k-1}=\begin{pmatrix} 0 & t^{2k-1} & 0 \\ -t^{2k-1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, v^{\pm}_{2k-1}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & t^{2k-1} \\ 0 & \mp t^{2k-1} & 0 \end{pmatrix}, w^{\pm}_{2k}=\begin{pmatrix} 0 & 0 & t^{2k} \\ 0 & 0 & \\ \mp t^{2k} & 0 & 0 \end{pmatrix}.$$

One can easily verify the commutation relations between them

$$[u_{2k-1}, v^{\pm}_{2l-1}] = w^{\pm}_{2(k+l)-2}, \ [v^{\pm}_{2k-1}, w^{\pm}_{2l}] = \pm u_{2(k+l)-1}, \ [w^{\pm}_{2k}, u_{2l+1}] = v^{\pm}_{2(k+l)-1}.$$

The linear span $\mathfrak{n}_1^+=\langle u_1, v_1^+, w_2^+, \dots, u_{2k+1}, v^+_{2k+1}, w^+_{2k+2}, \dots\rangle$ is a naturally graded subalgebra in $\mathfrak{so}(3, \mathbb{K}) \otimes \mathbb{K}[t]$ and $\mathfrak{n}_1^-=\langle u_1, v_1^-, w_2^-, \dots, u_{2k+1}, v^-_{2k+1}, w^-_{2k+2}, \dots\rangle$ is a naturally graded subalgebra in $\mathfrak{so}(2, 1) \otimes \mathbb{K}[t]$ respectively.

Proposition 2 ([20]) *$\mathfrak{n}_1^{\pm}$ are isomorphic over $\mathbb{C}$ and non-isomorphic over $\mathbb{R}$.*

The latter fact is not surprising, given the fact that $\mathfrak{so}(3, \mathbb{R})$ and $\mathfrak{sl}(2, \mathbb{R})$ are the different real forms of $\mathfrak{sl}(2, \mathbb{C})$.

Let us introduce more examples of naturally graded Lie algebras.

1) Define a Lie algebra $\mathfrak{n}_2^3$ as an one-dimensional central extension of $\mathfrak{n}_2$:

$$\mathfrak{n}_2^3 = \mathfrak{n}_2 \oplus \langle c\rangle, \ [f_2, f_3]=c, \ [c, f_i]=0, i \in \mathbb{N}.$$

2) Let S be a subset (finite or infinite) of the set of positive odd integers

$$S = (3 \le 2s_1+1 \le 2s_2+1 \le 2s_3+1 \le \dots \le 2s_n+1 \le \dots,)$$

Define a central extension $\mathfrak{m}_0^S = \mathfrak{m}_0 \oplus \langle c_{2s_1+1}, c_{2s_2+1}, \dots, c_{2s_n+1}, \dots\rangle$ of $\mathfrak{m}_0$

$$\begin{gathered} [e_1, e_l] = e_{l+1}, l \ge 2, \ [e_i, e_j] = 0, i + j \ne 2s_j+1 \in S; \\ [e_k, e_{2s_j+1-k}] = (-1)^k c_{2s_j+1}, k = 2, \dots, s_j, \ 2s_j+1 \in S, \\ [c_{2s_j+1}, e_l] = 0, \forall l \in \mathbb{N}, \forall 2s_j+1 \in S. \end{gathered}$$

Theorem 1 ([20]) *Let $\mathfrak{g} = \bigoplus_{i=1}^{+\infty}\mathfrak{g}_i$ be an infinite-dimensional naturally graded Lie algebra over $\mathbb{R}$ such that*

$$\dim\mathfrak{g}_i + \dim\mathfrak{g}_{i+1} \le 3, \ i \in \mathbb{N}. \tag{18}$$

Then $\mathfrak{g}$ is isomorphic to the one and only one Lie algebra from the following list:

$$\mathfrak{n}_1^{\pm}, \mathfrak{n}_2, \mathfrak{n}_2^3, \mathfrak{m}_0, \{\mathfrak{m}_0^S, S \subset \{3, 5, 7, \dots, 2m+1, \dots\}.$$

5 Kac-Moody Algebras $A_1^{(1)}$ and $A_2^{(2)}$

Let A be a generalized Cartan $(n \times n)$-matrix and $\mathfrak{g}(A)$ be the corresponding Kac-Moody affine algebra (see [9] for necessary definitions and details). By the definition $\mathfrak{g}(A)$ is generated by $3n$ elements $e_i, h_i, f_i, i = 1, \dots, n$ satisfying the following relations

$$\begin{array}{l} [h_i, h_j] = 0, [e_i, f_j] = \delta_{ij} h_i, \\ [h_i, e_j] = a_{ij} e_j, [h_i, f_j] = -a_{ij} h_j, \\ ad^{1-a_{ij}} e_i(e_j) = 0, ad^{1-a_{ij}} f_i(f_j) = 0, \\ i, j = 1, \dots, n, \end{array} \tag{19}$$

where a_{ij} are entries of our generalized Cartan matrix A.

The Kac-Moody affine algebra $\mathfrak{g}(A)$ has the maximal nilpotent subalgebra $N(A) \subset \mathfrak{g}(A)$ and it can be defined by its generators $e_1, e_2, \dots, e_n$ and the relations

$$ade_i^{-a_{ij}+1}(e_j) = 0, 1 \le i \neq j \le n.$$

The Lie algebra $N(A)$ is $\mathbb{Z}_{\ge 0} \oplus \dots \oplus \mathbb{Z}_{\ge}$-graded

$$N(A) = \oplus_{k_1 \ge 0, k_2 \ge 0, \dots, k_n \ge 0}^{+\infty} N(A)_{(k_1, k_2, \dots, k_n)}, \tag{20}$$

where a homogeneous subspace $N(A)_{(k_1, k_2, \dots, k_n)}$ is spanned by all commutator monomials involving precisely k_i generators $e_i, i = 1, \dots, n$.

Proposition 3 *$N(A)$ is naturally graded. Natural grading is just the sum of the components of canonical grading (20)*

$$N(A) = \bigoplus_{N=1}^{+\infty} N(A)_{(K)}, \;\; N(A)_{(K)} = \bigoplus_{k_1 + \dots + k_n = K} N(A)_{(k_1, \dots, k_n)}$$

Proof It follows from the fact that all structure relations of $N(A)$ are defined by homogeneous monomials. □

Definition 13 We call the Lie algebra $N(A)$ the positive part of a Kac-Moody algebra $\mathfrak{g}(A)$, where A is the corresponding generalized Cartan matrix.

It is a classical fact that all Kac-Moody algebras can be realized as affine Lie algebras $\hat{\mathcal{L}}(\mathfrak{g}) = \mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}c \oplus \mathbb{C}d$ or twisted affine Lie algebras $\hat{\mathcal{L}}(\mathfrak{g}, \mu) = \mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}c \oplus \mathbb{C}d$, double extensions of loop $\mathcal{L}(\mathfrak{g})$ and twisted loop algebras $\mathcal{L}(\mathfrak{g}, \mu)$ respectively (see [9]) of complex simple Lie algebras.

We briefly recall the definitions of two affine algebras $A_1^{(1)}$ and $A_2^{(2)}$.

The affine algebra $A_1^{(1)}$ corresponds to 2 by 2 generalized Cartan matrix $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. It can be realized as a double extension of of the loop algebra of $\mathfrak{sl}(2, \mathbb{C})$.

$$A_1^{(1)} = \hat{\mathcal{L}}(\mathfrak{sl}(2, \mathbb{C})) = \mathcal{L}(\mathfrak{sl}(2, \mathbb{C})) \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

Its maximal nilpotent subalgebra $N(A_1^{(1)})$ is isomorphic to the positive part $\mathfrak{n}_1$ of the loop algebra $\mathcal{L}(\mathfrak{sl}(2, \mathbb{C}))$ and it is generated by two elements e_1, e_2 related by

$$ad^3 e_2(e_1) = [e_2, [e_2, [e_2, e_1]]] = 0, \;\; ad^3 e_1(e_2) = [e_1, [e_1, [e_1, e_2]]] = 0.$$

The twisted affine algebra $A_2^{(2)}$ in its turn corresponds to another generalized 2 by 2 Cartan matrix $\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$. Its maximal nilpotent subalgebra $N(A_2^{(2)})$ is isomorphic to the positive part $\mathfrak{n}_2$ of the twisted loop algebra $\mathcal{L}(\mathfrak{sl}(3,\mathbb{C}),\mu)$ and it is generated by two elements e_1, e_2 related by

$$ad^2 e_2(e_1) = [e_2,[e_2,e_1]]=0,\;\; ad^5 e_1(e_2)=[e_1,[e_1,[e_1,[e_1,[e_1,e_2]]]]] = 0.$$

Both Lie algebras $A_1^{(1)}$ and $A_2^{(2)}$ are canonically $\mathbb{Z}\oplus\mathbb{Z}$-graded as Kac-Moody algebras, their maximal nilpotent subalgebras $\mathfrak{n}_1 = N(A_1^{(1)})$ and $\mathfrak{n}_2 = N(A_2^{(2)})$ are $\mathbb{Z}_{\geq 0}\oplus\mathbb{Z}_{\geq}$-graded.

$$N(A_i^{(i)}) = \mathfrak{n}_i = \oplus_{p+q=1}^{+\infty}(N(A_i^{(i)}))_{(p,q)},\, i = 1, 2,$$

where $(N(A_i^{(i)}))_{p,q}, i{=}1, 2$, is the linear span of all commutator monomials involving precisely p generators e_1 and q generators e_2. Generators e_1, e_2 have degrees $(1, 0)$ and $(0, 1)$ respectively.

How are the gradings of the Lie algebras $\mathfrak{n}_1$ and $\mathfrak{n}_2$, as defined in previous sections, related to the gradings of $N(A_1^{(1)})$ and $N(A_2^{(2)})$ just considered? The connection between the natural grading of $\mathfrak{n}_i$ and the canonical one of $N(A_i^{(i)})$ is already established in the Proposition 3. What about the narrow $\mathbb{N}$-gradings of $\mathfrak{n}_1$ and $\mathfrak{n}_2$?

One can verify that the canonical degree $deg(f_{8m+s})$ of a basic element f_{8m+s} in $\mathfrak{n}_2$ is defined for $-1 \leq s \leq 6$, by

$$deg(f_{8m+s}) = \begin{cases} (4m{+}s, 2m), & \text{if } s \leq 1; \\ (4m{+}s{-}2, 2m{+}1), & \text{if } s \geq 2. \end{cases} \tag{21}$$

For instance f_{8m+7} has the canonical bidegree equal to $(4m{+}3, 2m{+}2)$. In its turn the bidegree of f_{8m+6} equals $(4m{+}4, 2m{+}1)$. Hence both of them has the natural degree $6m{+}5$.

For canonical bidegrees of basic elements e_i of $\mathfrak{n}_1$ we have

$$deg(e_{3k+1}) = (k+1, k),\;\; deg(e_{3k+2}) = (k, k+1),\;\; deg(e_{3k}) = (k, k).$$

6 Two-dimensional Integrable Hyperbolic Systems

Consider an exponential hyperbolic system

$$u^j_{xy} = e^{\rho_j},\;\; \rho_j = a_{j1}u^1 + \cdots + a_{jn}u^n,\;\; j = 1, \ldots, n. \tag{22}$$

where $u(x, y)^j$, $j = 1, \ldots, n$ are analytic functions on variables x, y. For an arbitrary n by n matrix A define vector fields

$$X_\alpha = e^{-\rho_\alpha}\sum_{k=1}^{+\infty} D^{k-1}(e^{\rho_\alpha})\frac{\partial}{\partial u_k^\alpha} = \sum_{k=1}^{+\infty} B_{k-1}(\rho_\alpha^1, \ldots, \rho_\alpha^{k-1})\frac{\partial}{\partial u_k^\alpha},\;\; \alpha = 1, \ldots, n, \tag{23}$$

where $\rho_\alpha = a_{\alpha 1}u^1 + \cdots + a_{\alpha n}u^n$ and we introduced linear functions ρ_α^i, $i \geq 1$, defined by

$$\rho_\alpha^i = a_{\alpha 1}u_i^1 + \cdots + a_{\alpha n}u_i^n,\, D(\rho_\alpha^i) = \rho_\alpha^{i+1},\;\; i \geq 1.$$

It pas proved in [15] that if A is the Cartan matrix of a semisimple Lie algebra $\mathfrak{g}$ of the rank n then the exponential hyperbolic system (2) is integrable. The proof consisted in the explicit construction of a complete solution of the equation which depends on $2n$ arbitrary functions, thus generalizing the one-dimensional case of the classical Liouville equation $u_{xy} = e^u$. An essential condition in the proof was the nondegeneracy of the Cartan matrix A.

Later it was claimed in the preprint [24] that the main result in [15] can be generalzed for an arbitrary generalized Cartan matrix A (possibly degenerate) if we apply the inverse scattering problem method. In the proof [24], however, there are unclear points.

We consider two-dimensional case $n = 2$ that was studied explicitly in [18, 24].

$$\begin{cases} u^1_{xy} = e^{(a_{11}u^1 + a_{12}u^2)}, \\ u^2_{xy} = e^{(a_{21}u^1 + a_{22}u^2)}, \end{cases}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Characteristic equation $\frac{\partial}{\partial x} w(u_1, u_2, \dots,) = 0$ is equivalent to the system

$$X_1 w = X_2 w = 0.$$

where for the basic fields $X_\alpha, \alpha = 1, 2$, we have the following expansions

$$X_\alpha = \frac{\partial}{\partial u^\alpha_1} + (a_{\alpha 1} u^1_1 + a_{\alpha 2} u^2_1) \frac{\partial}{\partial u^\alpha_2} + \left((a_{\alpha 1} u^1_1 + a_{\alpha 2} u^2_1)^2 + (a_{\alpha 1} u^1_2 + a_{\alpha 2} u^2_2) \right) \frac{\partial}{\partial u^\alpha_3} + \dots$$

For instance this system is consistent and it is easy to verify that it has an integral of the second order for arbitrary matrix A

$$w \equiv w^{(2)}(u_1, u_2) = 2a_{21}u^1_2 + 2a_{12}u^2_2 - a_{11}a_{21}(u^1_1)^2 - 2a_{12}a_{21}u^1_1 u^2_1 - a_{22}a_{12}(u^2_1)^2$$

In the text of papers [18, 24] we encounter another definition of the characteristic Lie algebra $\chi(A)$ of an exponential hyperbolic system.

Definition 14 ([18, 24]) A Lie algebra $\chi(A)$ of vector fields generated by n operators $X_\alpha, \alpha = 1, \dots, n$, that are defined by Eq. 23 is called characteristic Lie algebra of the hyperbolic exponential system (2) defined by a matrix A.

Remark 2 We do not see operators $\frac{\partial}{\partial u^j}$ among the generators of our algebra. And hence $\chi(A)$ is pro-nilpotent.

It was proved in [18, 24] that

1) for the generalized degenerate Cartan matrix $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ the corresponding characteristic Lie algebra $\chi(A) = \mathcal{L}ie_{\mathbb{C}}(X_1, X_2)$ is isomorphic to the positive part $\mathfrak{n}_1$ of the affine Kac-Moody algebra $A^{(1)}_1$. The corresponding exponential system is integrable in the framework of the inverse scattering method;
2) for the generalized degenerate Cartan matrix $A = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$ the corresponding characteristic Lie algebra $\chi(A) = \mathcal{L}ie_{\mathbb{C}}(X_1, X_2)$ is isomorphic to the positive part $\mathfrak{n}_2$ of the affine Kac-Moody algebra $A^{(2)}_2$. Like in the previous case the hyperbolic system is integrable if we apply the inverse scattering problem method.

Remark 3 Hyperbolic exponential systems corresponding to nondegenerate Cartan 2×2-matrices of semisimple Lie algebras ($A_1 \oplus A_1$, A_2, C_2, G_2) are Darboux-integrable.

7 Growth of Lie Algebras

In the late sixties Victor Kac studied simple $\mathbb{Z}$-graded Lie algebras $\mathfrak{g} = \oplus_{k\in\mathbb{Z}}\mathfrak{g}_k$ of finite growth in the following sense

$$\dim \mathfrak{g}_k \leq P(|k|),\ \ k \in \mathbb{Z},$$

for some polynomial $P(t)$. We recall that a $\mathbb{Z}$-graded Lie algebra $\mathfrak{g} = \oplus_{k\in\mathbb{Z}}\mathfrak{g}_k$ is called simple graded if it does not contain non-trivial homogeneous ideal $I = \oplus_{k\in\mathbb{Z}}I_k$ where $I_k = I\cap\mathfrak{g}_k$. Kac [10] proved that an infinite-dimensional simple $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}$ of finite growth that satisfies the following two technical conditions

$$\begin{array}{l} 1)\ \ \mathfrak{g}\ \text{is generated by its "local part"}\ \mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1;\\ 2)\ \ \text{the}\ \mathfrak{g}_0-\text{module}\ \mathfrak{g}_{-1}\ \text{is irreducible.}\end{array} \tag{24}$$

is isomorphic to one Lie algebra of the following types:

– loop algebras $\mathcal{L}(\mathfrak{g}) = \mathfrak{g}\otimes\mathbb{C}[t, t^{-1}]$, where $\mathfrak{g}$ is finite-dimensional simple Lie algebra and $\mathbb{C}[t, t^{-1}]$ is the ring of Laurent polynomials over complex numbers. Namely there are four infinite series and five exceptional so-called centerless affine Lie algebras [9]

$$A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}, F_4^{(1)}, G_2^{(1)}.$$

– twisted loop algebras

$$\mathcal{L}(\mathfrak{g}, \mu) = \bigoplus_{\substack{i\in\mathbb{Z},\, i\equiv j \text{ mod n},\\ j=0,1,\dots,n-1}} \mathfrak{g}_j\otimes t^i \subset \mathfrak{g}\otimes\mathbb{C}[t, t^{-1}],$$

where a simple finite-dimensional Lie algebra $\mathfrak{g} = \oplus_{i=0}^{n-1}\mathfrak{g}_i$ is graded by the cyclic group $\mathbb{Z}_n$ (eigensubspaces of an automorphism μ of $\mathfrak{g}$). Here we have two infinite series and two exceptional centerless twisted affine Lie algebras [9]

$$A_n^{(2)}, D_n^{(2)}, E_6^{(2)}, D_4^{(3)}.$$

– the Lie algebras W_n, S_n, K_n, H_n of Cartan type, for instance W_n is the Lie algebra of derivations of the ring of polynomials $\mathbb{C}[x_1,\dots,x_n]$;

Moreover, Kac conjectured that dropping the condition (24) would add only the Witt algebra W to the classification list.

Remark 4 The Witt algebra W and W_1 (with no grading) do not satisfy the first condition from Eq. 24.

Kac's conjecture was proved in 1990 by Mathieu [19].

Suppose that an infinite-dimensional Lie algebra $\mathfrak{g}$ is generated by its finite-dimensional subspace $V_1(\mathfrak{g})$. For $n > 1$ we denote by $V_n(\mathfrak{g})$ the $\mathbb{K}$-linear span of all products in elements of $V_1(\mathfrak{g})$ of length at most n with arbitrary arrangements of brackets. We have an ascending chain of finite-dimensional subspaces of $\mathfrak{g}$:

$$V_1(\mathfrak{g}) \subset V_2(\mathfrak{g}) \subset \cdots \subset V_n(\mathfrak{g}) \subset \dots,\ \ \cup_{i=1}^{+\infty}V_i(\mathfrak{g}) = \mathfrak{g}.$$

The Gelfand-Kirillov dimension of $\mathfrak{g}$ [6] is

$$GKdim\mathfrak{g} = \limsup_{n\to+\infty}\frac{\log\dim V_n(\mathfrak{g})}{\log n}.$$

A finite Gelfand-Kirillov dimension means that there exists a polynomial $P(x)$ such that $\dim V_n(\mathfrak{g}) < P(n)$ for all $n > 1$. In particular if $\mathfrak{g}$ is finite-dimensional then $GKdim\mathfrak{g} = 0$.

The growth function $F_{\mathfrak{g}}(n) = \dim V_n(\mathfrak{g})$ depends on the choice of the generating subspace $V_1(\mathfrak{g})$ (see [12] for details). For instance if we choose another generating subspace $\tilde{V}_1(\mathfrak{g})$ such that $\tilde{V}_1(\mathfrak{g}) \subset V_m(\mathfrak{g})$ for some positive integer m, then we have an obvious estimate

$$\tilde{F}_{\mathfrak{g}}(n) = \dim \tilde{V}_n(\mathfrak{g}) \leq \dim V_{mn}(\mathfrak{g}) = F_{\mathfrak{g}}(mn).$$

In fact, the growth of a Lie algebra $\mathfrak{g}$ is called the equivalence class of some of its growth functions $F_{\mathfrak{g}}(n)$ [12]. Two monotone increasing functions $f, g : \mathbb{N} \to \mathbb{N}$ are called equivalent if there exist $c, m, \tilde{c}, \tilde{m} \in \mathbb{N}$ such that

$$f(n) \leq cg(mn), g(n) \leq \tilde{c} f(\tilde{m}n),$$

for almost all $n \in \mathbb{N}$.

For a naturally graded pro-nilpotent Lie algebra $\mathfrak{g} = \oplus_{i=1}^{+\infty}\mathfrak{g}_i$ one can define its natural growth function $F_{\mathfrak{g}}^{gr}(n)$, choosing as the generating subspace its first homogeneous component $\mathfrak{g}_1$ of the Lie algebra $\mathfrak{g} = \oplus_{i=1}^{+\infty}\mathfrak{g}_i$. Obviously we have

$$F_{\mathfrak{g}}^{gr}(n) = \dim V_n(\mathfrak{g}) = \sum_{i=1}^{n} \dim \mathfrak{g}_i = \dim(\mathfrak{g}/\mathfrak{g}^{n+1}),$$

where $\mathfrak{g}^{n+1}$ denotes the $(n+1)$-th ideal of the lower central series of $\mathfrak{g}$.

For the Lie algebra $\mathfrak{m}_0$ considered above we have the slowest possible growth

$$F_{\mathfrak{m}_0}^{gr}(n) = n+1.$$

For an arbitrary naturally graded Lie algebra $\mathfrak{g} = \oplus_{i=1}^{+\infty}\mathfrak{g}_i$ of width d the function $F_{\mathfrak{g}}^{gr}(n)$ grows not faster than dn:

$$F_{\mathfrak{g}}^{gr}(n) \leq dn.$$

All Lie algebras of finite width have $GKdim\mathfrak{g} = 1$.

Consider natural growth functions of the Lie algebras $\mathfrak{n}_1$ and $\mathfrak{n}_2$.

$$\frac{3n}{2} \leq F_{\mathfrak{n}_1}^{gr}(n) \leq \frac{3n+1}{2}, \ \frac{4n}{3} \leq F_{\mathfrak{n}_2}^{gr}(n) \leq \frac{4n+2}{3}, \ \forall n \in \mathbb{N}.$$

Hence the piecewise linear functions $F_{\mathfrak{n}_1}^{gr}(n)$ and $F_{\mathfrak{n}_2}^{gr}(n)$ grow on average at rates of $\frac{3}{2}$ and $\frac{4}{3}$ respectively.

Remark 5 The next remark is that there is an continuum family of pairwise nonisomorphic linearly growing Lie algebras $\mathfrak{m}_0^S$ indexed by subsets $S \subset \{3, 5, 7, \dots\}$, while according to the Mathieu theorem [19] there is only a countable number of pairwise nonisomorphic simple $\mathbb{Z}$-graded Lie algebras of finite growth.

Lemma 2 *Suppose an infinite-dimensional Lie algebra $\tilde{\mathfrak{g}}$ is generated by its finite-dimensional subspace*

$$V_1(\tilde{\mathfrak{g}}) = \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

where $\mathfrak{g}_0$ is an abelian Lie subalgebra in $\tilde{\mathfrak{g}}$ and $\mathfrak{g}_1$ is an invariant subspace of $\mathfrak{g}_0$-action on $\tilde{\mathfrak{g}}$. Assume also that the $\mathfrak{g}_0$-module $\mathfrak{g}_1$ is diagonalizable and corresponding weights (roots) $\alpha_1, \dots, \alpha_q \in \mathfrak{g}_0^*$ are non-zero. Define a subalgebra $\mathfrak{g} \subset \tilde{\mathfrak{g}}$ generated by the subspace $V_1(\mathfrak{g}) = \mathfrak{g}_1$.

Then the corresponding growth functions $F_{\mathfrak{g}}(n)$, $F_{\tilde{\mathfrak{g}}}(n)$ are related

$$F_{\tilde{\mathfrak{g}}}(n) = F_{\mathfrak{g}}(n) + \dim \mathfrak{g}_0.$$

Hence $\mathfrak{g}$ *and* $\tilde{\mathfrak{g}}$ *have equal Gelfand-Kirillov dimensions*

$$GKdim\tilde{\mathfrak{g}} = GKdim\mathfrak{g}.$$

Proof For simplicity we consider the case $\dim \mathfrak{g}_0 = 1$. In addition to everything else, this is the case we will need for applications. However the general case is proved in a completely analogous way. First of all we fix a non-trivial X_0 in one-dimensional $\mathfrak{g}_0$. Choose a basis $X_1, \dots, X_q$ of $\mathfrak{g}_1$ consisting of eigen-vectors of adX_0 corresponding to eigenvalues $\lambda_1 = \alpha_1(X_0), \dots, \lambda_q = \alpha_q(X_0)$ respectively

$$adX_0(X_j) = [X_0, X_j] = \lambda_j X_j, \;\; j = 1, \dots, q.$$

Let $X_{i_1,\dots,i_m} = X_{i_1} \dots X_{i_m}$ be an element of $\mathfrak{g}$ represented by a m-word, where $X_{i_s} \in \{X_1, \dots, X_q\}$, $s = 1, \dots, m$, with an arbitrary (but fixed) arrangement of brackets. Then

$$adX_0(X_{i_1,\dots,i_m}) = (\lambda_{i_1} + \dots + \lambda_{i_m})X_{i_1,\dots,i_m}, \;\; q = 1, \dots, m, \tag{25}$$

We will prove (25) by recursion. We start by $m = 2$:

$$adX_0([X_{i_1}, X_{i_2}]) = [\lambda_{i_1} X_{i_1}, X_{i_2}] + [X_{i_1}, \lambda_{i_2} X_{i_2}] = (\lambda_{i_1} + \lambda_{i_2})[X_{i_1}, X_{i_2}].$$

Assume that Eq. 25 is valid for commutators of orders less than m. We take a m-word $X_{i_1,\dots,i_q,i_{q+1},\dots,i_m}$ that can be written as a bracket $[X_{i_1,\dots,i_q}, X_{i_{q+1},\dots,i_m}]$ of its subwords $X_{i_1,\dots,i_q}$ and $X_{i_{q+1},\dots,i_m}$.

$$\begin{gathered} adX_0(X_{i_1,\dots,i_q,i_{q+1},\dots,i_m}) = adX_0([X_{i_1,\dots,i_q}, X_{i_{q+1},\dots,i_m}]) = \\ = [(\lambda_{i_1}+\dots+\lambda_{i_q})X_{i_1,\dots,i_q}, X_{i_{q+1},\dots,i_m}]+[X_{i_1,\dots,i_q}, (\lambda_{i_{q+1}}+\dots+\lambda_{i_m})X_{i_{q+1},\dots,i_m}] = \\ = (\lambda_{i_1}+\dots+\lambda_{i_m})X_{i_1,\dots,i_q,i_{q+1},\dots,i_m}. \end{gathered}$$

Now we consider an arbitrary element of $\tilde{\mathfrak{g}}$ represented by a n-th order commutator $X_{i_1,\dots,i_n}$ where some lower indices can equal zero, in other words, $X_{i_1,\dots,i_n}$ may contain X_0 in a certain number among its own letters. Let s be a total number of occurences of the letter X_0 in the word $X_{i_1,\dots,i_n}$, then it follows from Eq. 25 that

$$X_{i_1,\dots,i_n} \in V_{n-s}(\mathfrak{g}).$$

Hence we have

$$V_n(\tilde{\mathfrak{g}}) = \langle X_0 \rangle \oplus V_n(\mathfrak{g}), \;\; F_{\tilde{\mathfrak{g}}}(n) = F_{\mathfrak{g}}(n) + 1.$$

□

8 The Bigraded Lie Subalgebra Diff($\mathcal{F}$)

We introduce a non-positive grading in the ring $\mathbb{K}[u_1, \dots, u_n, \dots]$ of polynomials over infinite number of variables $u_1, \dots, u_n, \dots$. We define it by recursion with respect to the power of polynomials.

1) We define the degrees (weights) $\mathrm{wt}(u_n)$ of generators u_n, $n \geq 1$, and unit 1 by the rule

$$\mathrm{wt}(1) = 0, \mathrm{wt}(u_n) = -n, n \in \mathbb{N}.$$

2) Let P_1, P_2 be two homogeneous polynomials of weights $\mathrm{wt}(P_1) = -p_1$ and $\mathrm{wt}(P_2) = -p_2$ respectively. Then their product $P_1 P_2$ is a homogeneous polynomial of weight $-p_1 - p_2$.
3) Let P_1, P_2 be two homogeneous polynomials of weight $\mathrm{wt}(P_1) = \mathrm{wt}(P_2) = -p$. Then their sum $P_1 + P_2$ is a homogeneous polynomial of weight $-p$.

For instance $wt(u_1^3u_3) = -6$ and a Bell polynomial $B_n(u_1,\dots,u_n)$ is a homogeneous polynomial of weight $-n$:

$$\mathrm{wt}(B_2(u_1,u_2)) = \mathrm{wt}(u_1^2+u_2) = -2,\ \mathrm{wt}(B_3(u_1,u_2,u_3)) = \mathrm{wt}(u_1^3+3u_1u_2+u_3) = -3.$$

Now we consider a subalgebra $\mathcal{F} \subset C^\omega(\Omega)[u_1,u_2,\dots]$ of quasipolynomials

$$Q(u,u_1,\dots,u_n,\dots) = \sum_{i=-m}^{M} e^{\alpha_i u} P_i(u_1,\dots,u_{n_i}),$$

where $\alpha_i \in \mathbb{Z}$ and $P_i(u_1,\dots,u_{n_i})$ stands for a polynomial of variables $u_1,\dots,u_{n_i}$ taken from the ring $\mathbb{K}[u_1,\dots,u_n,\dots]$.

The $\mathbb{K}$-algebra $\mathcal{F}$ admits a $\mathbb{Z}_{\le 0}\times\mathbb{Z}$-grading

$$\mathcal{F} = \bigoplus_{k\in\mathbb{Z}_{\le 0}, q\in\mathbb{Z}} \mathcal{F}_{k,q},\ \ \mathcal{F}_{k,q} = \{e^{qu}P(u_1,\dots,u_n), \mathrm{wt}(P)=k\}.$$

This bigrading is compatible with the product structure in the ring $\mathcal{F}$

$$\mathcal{F}_{k,q}\cdot\mathcal{F}_{l,r} \subset \mathcal{F}_{k+l,q+r}.$$

We consider the Lie algebra $\mathrm{Diff}(C^\omega(\Omega)[u_1,u_2,\dots])$ of all derivations of the algebra $C^\omega(\Omega)[u_1,u_2,\dots]$ and a Lie subalgebra $\mathrm{Diff}(\mathcal{F}) \subset \mathrm{Diff}(C^\omega(\Omega)[u_1,u_2,\dots])$ of first order differential operators

$$X = \sum_{j=1}^{+\infty} Q_j(u,u_1,\dots,u_n,\dots)\frac{\partial}{\partial u_j},$$

where $Q_j(u,u_1,\dots,u_n,\dots) \in \mathcal{F}$ are quasipolynomials.

The Lie subalgebra $\mathrm{Diff}(\mathcal{F})$ is $\mathbb{Z}\times\mathbb{Z}$-graded

$$\mathrm{Diff}(\mathcal{F}) = \bigoplus_{m\in\mathbb{Z}, r\in\mathbb{Z}} \mathrm{Diff}_{m,r}(\mathcal{F}),\ \ [\mathrm{Diff}_{m,r}(\mathcal{F}), \mathrm{Diff}_{n,q}(\mathcal{F})] \subset \mathrm{Diff}_{m+n,r+q}(\mathcal{F}),$$

where a homogeneous subspace $\mathrm{Diff}_{m,r}(\mathcal{F})$ is a linear subspace of first order differential operators

$$\mathrm{Diff}_{m,r}(\mathcal{F}) = \left\{ e^{ru}\sum_{j=1}^{+\infty} P_j(u_1,\dots,u_{s_j})\frac{\partial}{\partial u_j}, \mathrm{wt}(P_j)+j=m \right\}, (m,r)\in\mathbb{Z}\times\mathbb{Z}. \quad (26)$$

Definition 15 The grading of $\mathrm{Diff}_{m,r}(\mathcal{F})$ defined by Eq. 26 we will call the operator bigrading of $\mathrm{Diff}(\mathcal{F})$.

Example 13

$$X_1 = X(e^{pu}) = e^{pu}\sum_{n=1}^{+\infty} B_{n-1}(u_1,\dots,u_{n-1})\frac{\partial}{\partial u_n} \in \mathrm{Diff}_{p,1}(\mathcal{F}).$$

Remark 6 Although $X_0 = \frac{\partial}{\partial u} \notin \mathrm{Diff}\mathcal{F}$ its adjoint $\mathrm{ad}X_0$ defines a derivation of $\mathrm{Diff}\mathcal{F}$

$$\mathrm{ad}X_0(X) = [X_0, X] = \sum_{j=1}^{+\infty}\frac{\partial Q_j}{\partial u}\frac{\partial}{\partial u_j},\ \ X = \sum_{j=1}^{+\infty} Q_j\frac{\partial}{\partial u_j}. \quad (27)$$

One can see that a subspace $V_p = \oplus_{n\in\mathbb{Z}}\mathrm{Diff}_{p,n}(\mathcal{F})$ is an eigensubspace of $\mathrm{ad}X_0$ which corresponds to the eigenvalue $\lambda = p$. We have the decomposition of the Lie algebra $\mathrm{Diff}\mathcal{F}$ into a direct sum of eigensubspaces of the operator $\mathrm{ad}X_0$.

$$\mathrm{Diff}\mathcal{F} = \bigoplus_{p\in\mathbb{Z}} V_p = \bigoplus_{p\in\mathbb{Z}} \left(\oplus_{n\in\mathbb{Z}}\mathrm{Diff}_{p,n}(\mathcal{F})\right).$$

Definition 16 Define a Lie algebra

$$\hat{\mathrm{Diff}}\mathcal{F} = C^{\omega}(\Omega)X_0 \oplus_{\ltimes} \mathrm{Diff}\mathcal{F}$$

as a semidirect sum of the Lie algebra $C^{\omega}(\Omega)X_0 = \{g(u)X_0, g(u) \in C^{\omega}(\Omega)\}$ acting on $\mathrm{Diff}\mathcal{F}$ by the formula (27). It is possible to extend the operator bigrading to the whole algebra $\hat{\mathrm{Diff}}\mathcal{F}$ by setting its value on the element X_0 equal to $(0, 0)$.

9 The Sinh-Gordon Equation

Theorem 2 *The characteristic Lie algebra $\chi(\sinh u)$ of the sinh-Gordon equation*

$$u_{xy} = \sinh u$$

is isomorphic to the non-negative part

$$\mathcal{L}(\mathfrak{sl}(2,\mathbb{K}))^{\geq 0} = \mathfrak{sl}(2,\mathbb{K})\otimes\mathbb{K}[t],$$

of the loop algebra $\mathcal{L}(\mathfrak{sl}(2,\mathbb{K})) = \mathfrak{sl}(2,\mathbb{K})\otimes\mathbb{K}[t,t^{-1}]$.

It is generated by three elements X'_0, X'_1, X'_2 that satisfy the following relations

$$[X'_0, X'_1] = X'_1,\ [X_0, X'_2] = -X'_2, \tag{28}$$

$$\left[X'_1, [X'_1, [X'_1, X'_2]]\right] = 0,\ \left[X'_2, [X'_2, [X'_2, X'_1]]\right] = 0. \tag{29}$$

It particular it means that the subalgebra $\chi(\sinh u)^{+}$ generated by X'_1 and X'_2 is a codimension one ideal in $\chi(\sinh u)$ and it is isomorphic to the (nilpotent) positive part $N(A_1^{(1)})$ of the Kac-Moody algebra $A_1^{(1)} = \hat{\mathcal{L}}(\mathfrak{sl}(2,\mathbb{K})) = \mathcal{L}(\mathfrak{sl}(2,\mathbb{K})) \oplus \mathbb{K}c \oplus \mathbb{K}d$.

Proof We denote

$$X_0 = \frac{\partial}{\partial u},\ X_1 = \sum_{n=1}^{+\infty} D^{n-1}(\sinh u)\frac{\partial}{\partial u_n}.$$

The construction of the characteristic Lie algebra has an inductive nature. We start with the first order differential operators X_0, X_1 and then consider the commutators of higher orders with the participation of generators X_0, X_1.

Consider a linear span $\langle X_0, X_1, Y_1\rangle$, where $Y_1 = [X_0, X_1]$. Choose a new basis in $\langle X_0, X_1, Y_1\rangle$

$$X'_0 = X_0, X'_1 = X_1 + Y_1, X'_2 = X_1 - Y_1.$$

It means that

$$X'_1 = \sum_{n=1}^{+\infty} D^{n-1}(e^u)\frac{\partial}{\partial u_n}, \quad X'_2 = -\sum_{n=1}^{+\infty} D^{n-1}(e^{-u})\frac{\partial}{\partial u_n}.$$

We have

$$X'_1 = e^u \textstyle\sum_{n=1}^{+\infty} B_{n-1}(u_1,\dots,u_{n-1})\frac{\partial}{\partial u_n},$$
$$X'_2 = -e^{-u} \textstyle\sum_{n=1}^{+\infty} B_{n-1}(-u_1,\dots,-u_{n-1})\frac{\partial}{\partial u_n}.$$

The elements X_1', X_2' are of operator bidegrees $(1,1), (1,-1)$ respectively. Obviously

$$[X_0', X_1'] = X_1', \; [X_0', X_2'] = -X_2'.$$

It's easy to calculate the first terms of the commutator $[X_1', X_2']$

$$X_3' = [X_1', X_2'] = 2\left(\frac{\partial}{\partial u_2} + u_1^2\frac{\partial}{\partial u_4} + 5u_1u_2\frac{\partial}{\partial u_5} + \dots\right)$$

The operator X_3' has operator bidegree $(2,0)$ (it means in particular that all its coefficients do not depend on variable u) and hence

$$[X_0', X_3'] = \left[\frac{\partial}{\partial u}, X_3'\right] = 0.$$

Now we consider $X_4' = -[X_1', X_3']$ of operator bidegree $(3,1)$ and we also can write out some of its first terms

$$X_4' = -[X_1', X_3'] = 2e^u\left(\frac{\partial}{\partial u_3} + u_1\frac{\partial}{\partial u_4} + (2u_1^2 + u_2)\frac{\partial}{\partial u_5} + \dots\right)$$

Evidently $[X_0', X_4'] = X_4'$.

We define an operator X_5' of operator bidegree $(3,-1)$ as

$$X_5' = [X_2', X_3'] = -2e^{-u}\left(\frac{\partial}{\partial u_3} - u_1\frac{\partial}{\partial u_4} + (2u_1^2 - u_2)\frac{\partial}{\partial u_5} + \dots\right).$$

Obviously

$$[X_0', X_5'] = -X_5'.$$

Now we need to involve the operator D in our play. It has operator bidegree $(-1,0)$. We start with an obvious remark that $[D, X_0'] = 0$. It follows from Eq. 10 that

$$[D, X_1'] = -e^u X_0', \; [D, X_2'] = e^{-u} X_0'. \tag{30}$$

Hence we have

$$\begin{aligned} &[D, X_3'] = \left[D, [X_1', X_2']\right] = \left[[D, X_1'], X_2'\right] + \left[X_1', [D, X_2']\right] = \\ &= -\left[e^u X_0', X_2'\right] + \left[X_1', e^{-u} X_0'\right] = e^u X_2' - e^{-u} X_1'; \\ &[D, X_4'] = -\left[D, [X_1', X_3']\right] = -\left[[D, X_1'], X_3'\right] - \left[X_1', [D, X_3']\right] = \\ &= \left[e^u X_0', X_3'\right] - \left[X_1', e^u X_2' + e^{-u} X_1'\right] = -e^u X_3'. \end{aligned} \tag{31}$$

□

Proposition 4 $[X_1', X_4'] = [X_2', X_5'] = 0.$

Proof

$$\begin{aligned} &\left[D, [X_1', X_4']\right] = \left[[D, X_1'], X_4'\right] + \left[X_1', [D, X_4']\right] = \\ &= -\left[e^u X_0', X_4'\right] - \left[X_1', e^u X_3'\right] = -e^u X_4' + e^u X_4' = 0. \end{aligned}$$

Also we have

$$\begin{aligned} &[D, X_5'] = \left[D, [X_2', X_3']\right] = \left[[D, X_2'], X_3'\right] + \left[X_2', [D, X_3']\right] = \\ &= -e^{-u}\left[X_0', X_3'\right] + \left[X_2', e^u X_2' + e^{-u} X_1'\right] = -e^{-u} X_3'. \end{aligned}$$

This implies

$$\begin{aligned} &[D, [X_2', X_5']] = \left[[D, X_2'], X_5'\right] + \left[X_2', [D, X_5']\right] = \\ &= -e^{-u}\left[X_0', X_5'\right] - e^{-u}\left[X_2', X_3'\right] = 0. \end{aligned}$$

It follows from Lemma 1 that both brackets $[X_1', X_4']$ and $[X_2', X_5']$ vanish. □

Now we define recursively

$$X'_{3k+1} = -[X'_1, X'_{3k}],\, X'_{3k+2} = [X'_2, X'_{3k}],\, X'_{3k+3} = [X'_1, X'_{3k+2}],\, k \geq 1.$$

$X'_{3k+1}, X'_{3k+2}, X'_{3k+3}$ have bidegrees $(2k+1, 1)$, $(2k+1, -1)$, $(2k+2, 0)$ respectively.

$$[X'_0, X'_{3k+1}] = X'_{3k+1},\, [X'_0, X'_{3k+2}] = -X'_{3k+2},\, [X'_0, X'_{3k}] = 0. \tag{32}$$

Lemma 3 *First order differential operators* $X'_{3k+1}, X'_{3k+2}, X'_{3k+3}, k \geq 0$, *are all non-trivial and satisfy the following relations*

$$\begin{aligned}&[D, X'_{3k+1}] = -e^u X'_{3k},\, [D, X'_{3k+2}] = e^{-u} X'_{3k},\\ &[D, X'_{3k+3}] = -e^{-u} X'_{3k+1} + e^u X'_{3k+2};\end{aligned} \tag{33}$$

Proof

$$\begin{aligned}&[D, X'_{3k+1}] = -\left[D, [X'_1, X'_{3k}]\right] = -\left[[D, X'_1], X'_{3k}\right] - \left[X'_1, [D, X'_{3k}]\right] =\\ &= \left[e^u X'_0, X'_{3k}\right] - \left[X'_1, -e^{-u} X'_{3(k-1)+1} + e^u X'_{3(k-1)+2}\right] = -e^u X'_{3k},\end{aligned}$$

Second relation from Eq. 33 can be proved completely analogously. The third assertion is verified below

$$\begin{aligned}&[D, X'_{3k+3}] = \left[D, [X'_1, X'_{3k+2}]\right] = \left[[D, X'_1], X'_{3k+2}\right] + \left[X'_1, [D, X'_{3k+2}]\right] =\\ &= -\left[e^u X'_0, X'_{3k+2}\right] + \left[X'_1, e^{-u} X'_{3k}\right] = e^u X'_{3k+2} - e^{-u} X'_{3k+1},\end{aligned}$$

Non-triviality of $X'_{3k+1}, X'_{3k+2}, X'_{3k+3}, k \geq 0$, follows from Lemma 1 and Eq. 33. □

Lemma 4 *Differential operators* $X'_0, X'_1, X'_2, \ldots$ *satisfy the following commutation relations*

$$\begin{aligned}&[X_{3l+1}, X_{3k+1}] = 0,\;\; [X_{3l+2}, X_{3k+2}] = 0,\;\; [X_{3l}, X_{3k}] = 0,\\ &[X_{3l+1}, X_{3k+2}] = X_{3(k+l)+3},\;\; [X_{3l}, X_{3k+1}] = X_{3(k+l)+1},\\ &[X_{3l}, X_{3k+2}] = -X_{3(k+l)+2},\;\; k, l \geq 0;\end{aligned} \tag{34}$$

Proof We prove (34) by recursion on $N = k + l$. The basis of recursion is $k + l = 1$.

$$\begin{aligned}&[X'_1, X'_4] = 0, [X'_2, X'_5] = 0, [X'_0, X'_3] = 0,\\ &[X'_1, X'_2] = X'_3, [X'_0, X'_1] = X'_1, [X'_0, X'_2] = -X'_2,\\ &[X'_1, X'_5] = X'_6, [X'_0, X'_4] = X'_4, [X'_0, X'_5] = -X'_5,\\ &[X'_4, X'_2] = X'_6, [X'_3, X'_1] = X'_4, [X'_3, X'_2] = -X'_5,\end{aligned}$$

We have already checked out almost all of these formulas. It only remains to verify the equality $[X'_4, X'_2] = X'_6$. Indeed

$$\begin{aligned}&\left[D, [X'_4, X'_2] - X'_6\right] = \left[[D, X'_4], X'_2\right] + \left[X'_4, [D, X'_2]\right] - \left[D, X'_6\right] =\\ &= \left[-e^u X'_3, X'_2\right] + \left[X'_4, e^{-u} X'_0\right] + e^{-u} X'_4 - e^u X'_5 = 0.\end{aligned}$$

Suppose that relations (34) have already been established for $k + l = N$, we now prove them for $k + l = N + 1$.

$$\begin{aligned}&\left[D, [X'_{3l+1}, X'_{3k+1}]\right] = \left[[D, X'_{3l+1}], X'_{3k+1}\right] + \left[X'_{3l+1}, [D, X'_{3k+1}]\right] =\\ &= -e^u \left[X'_{3l}, X'_{3k+1}\right] - e^u \left[X'_{3l+1}, X'_{3k}\right] = 0.\end{aligned}$$

Thereby it follows from Lemma 1 that $[X'_{3l+1}, X'_{3k+1}]$=0. The relations $[X_{3l+2}, X_{3k+2}]$=0 and $[X_{3l}, X_{3k}]$=0 can be verified absolutely analogously to the previous case.

Now we turn to the second group of relations (34)

$$\Big[D, [X'_{3l+1}, X'_{3k+2}] - X'_{3(k+l)+3}\Big] =$$
$$= \big[[D, X'_{3l+1}], X'_{3k+2}\big] + \big[X'_{3l+1}, [D, X'_{3k+2}]\big] - \Big[D, X'_{3(k+l)+3}\Big] =$$
$$= \big[-e^u X'_{3l}, X'_{3k+2}\big] + \big[X'_{3l+1}, e^{-u} X'_{3k}\big] + e^{-u} X'_{3(k+l)+1} - e^u X'_{3(k+l)+2} = 0.$$

We leave to the reader in the form of an exercise the proof of the relation $[X_{3l}, X_{3k+1}] = X_{3(k+l)+1}$. We finish the proof of our Lemma by verifying the last equality in Eq. 34.

$$\Big[D, [X'_{3l}, X'_{3k+2}] + X'_{3(k+l)+2}\Big] =$$
$$= \big[[D, X'_{3l}], X'_{3k+2}\big] + \big[X'_{3l}, [D, X'_{3k+2}]\big] + \Big[D, X'_{3(k+l)+2}\Big] =$$
$$= \Big[e^u X'_{3(l-1)+2} - e^{-u} X'_{3l-2}, X'_{3k+2}\Big] + e^{-u}\big[X'_{3l}, X'_{3k}\big] + e^{-u} X'_{3(k+l)} = 0.$$

□

Corollary 2 *1) The characteristic Lie algebra $\chi(\sin u)$ of the sin-Gordon equation $u_{xy} = \sin u$ is isomorphic to the non-negative part*

$$\mathcal{L}(\mathfrak{so}(2,1), \mathbb{K})_{\geq 0} = \mathfrak{so}(2,1) \otimes \mathbb{K}[t]$$

of the loop algebra $\mathcal{L}(\mathfrak{so}(2,1), \mathbb{K}) = \mathfrak{so}(2,1) \otimes \mathbb{K}[t, t^{-1}]$.

2) the loop algebras $\mathcal{L}(\mathfrak{so}(2,1), \mathbb{K})$ and $\mathcal{L}(\mathfrak{sl}(2,\mathbb{K}))$ are non-isomorphic over $\mathbb{K}=\mathbb{R}$ and are isomorphic over $\mathbb{K}=\mathbb{C}$.

Corollary 3 *It follows now from Theorem 2 and from Lemma 2 that the characteristic Lie algebra $\chi(\sin u)$ of the sin-Gordon equation has the same growth as the Lie algebra $\mathfrak{n}_1$.*

10 The Tzitzeica Equation

Theorem 3 *The characteristic Lie algebra $\chi(e^u + e^{-2u})$ of the Tzitzeica equation*

$$u_{xy} = e^u + e^{-2u}$$

is isomorphic to the non-negative part

$$\mathcal{L}(\mathfrak{sl}(3,\mathbb{K}), \mu)^{\geq 0} = \bigoplus_{j=0}^{+\infty} \mathfrak{g}_{j\,(mod\,2)} \otimes t^j,\ \mathfrak{sl}(3,\mathbb{K}) = \mathfrak{g}_0 \oplus \mathfrak{g}_1,\ [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta\,(mod\,2)},$$

of the twisted loop algebra $\mathcal{L}(\mathfrak{sl}(3,\mathbb{K}), \mu) = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_{j\,(mod\,2)} \otimes t^j$, where μ is a diagram automorphism of $\mathfrak{sl}(3,\mathbb{K})$, $\mu^2 = Id$, and $\mathfrak{g}_0$, $\mathfrak{g}_1$ are eigen-spaces of μ corresponding to eigen-values $1, -1$ respectively. In particular $\mathfrak{g}_0$ is a subalgebra in $\mathfrak{sl}(3,\mathbb{K})$ isomorphic to $\mathfrak{so}(3,\mathbb{K})$ [9].

The Lie algebra $\chi(e^u + e^{-2u})$ is generated by three elements Y'_0, Y'_1, Y'_2 that satisfy the following relations

$$[Y'_0, Y'_1] = Y'_1,\ [Y_0, Y'_2] = -2Y'_2, \quad (35)$$
$$\big[Y'_1, [Y'_1, [Y'_1, [Y'_1, [Y'_1, Y'_2]]\ldots\big] = 0,\ \big[Y'_2, [Y'_2, Y'_1]\big] = 0. \quad (36)$$

It is a pro-solvable infinite-dimensional Lie algebra. Its subalgebra $\chi(e^u + e^{-2u})^+$ generated by two elements Y'_1, Y'_2 is isomorphic to the (nilpotent) positive part $N(A_2^{(2)})$ of the Kac-Moody algebra $A_2^{(2)} = \hat{\mathcal{L}}(\mathfrak{sl}(3,\mathbb{K}), \mu) = \mathcal{L}(\mathfrak{sl}(3,\mathbb{K}), \mu) \oplus \mathbb{K}c \oplus \mathbb{K}d$.

Proof Our proof will consist in constructing an infinite basis $Y'_1, Y'_2, Y'_3, Y'_4, \dots$ of $\chi(e^u+e^{-2u})$ and verifying that basic fields $Y_n, n \geq 1$, satisfy the commutation relations (15) of $\mathcal{L}(\mathfrak{sl}(3,\mathbb{K}),\mu)^{\geq 0}$.

We recall that by definition the characteristic Lie algebra $\chi(e^u+e^{-2u})$ is the Lie algebra generated by two operators

$$X_0 = \frac{\partial}{\partial u},\ \ X_1 = \sum_{n=1}^{+\infty} D^{n-1}(e^u+e^{-2u})\frac{\partial}{\partial u_n}.$$

Consider a linear span $\langle X_0, X_1, Y_1\rangle$, where $Y_1 = [X_0, X_1]$. Let introduce a new basis Y'_0, Y'_1, Y'_2 in $\langle X_0, X_1, Y_1\rangle$, where

$$Y'_0 = X_0,\ Y'_1 = \frac{2}{3}X_1 + \frac{1}{3}Y_1,\ Y'_2 = \frac{1}{3}X_1 - \frac{1}{3}Y_1.$$

We recall explicit expressions for Y'_1 and Y'_2 in terms of Bell polynomials

$$\begin{aligned} Y'_1 &= \textstyle\sum_{n=1}^{+\infty} D^{n-1}(e^u)\frac{\partial}{\partial u_n} = e^u \sum_{n=1}^{+\infty} B_{n-1}(u_1,\dots,u_{n-1})\frac{\partial}{\partial u_n}, \\ Y''_2 &= \textstyle\sum_{n=1}^{+\infty} D^{n-1}(e^{-2u})\frac{\partial}{\partial u_n} = e^{-2u} \sum_{n=1}^{+\infty} B_{n-1}(-2u_1,\dots,-2u_{n-1})\frac{\partial}{\partial u_n}. \end{aligned} \tag{37}$$

Obviously we have

$$[Y'_0, Y'_1] = Y'_1,\ \ [Y'_0, Y'_2] = -2Y'_2.$$

It's easy to calculate the first terms of the expansion for $[Y'_1, Y'_2]$

$$Y'_3 = [Y'_1, Y'_2] = -3e^{-u}\left(\frac{\partial}{\partial u_2} - 2u_1\frac{\partial}{\partial u_3} + (5u_1^2 - 3u_2)\frac{\partial}{\partial u_4} + \dots\right)$$

The operator Y'_3 has operator bidegree $(2, -1)$ and

$$[Y'_0, Y'_3] = \left[\frac{\partial}{\partial u}, Y'_3\right] = -Y'_3.$$

Now we consider $Y'_4 = [Y'_1, Y'_3]$ (it has operator bidegree $(3, 0)$) and we can write down the first terms of the expansion of Y'_4

$$Y'_4 = [Y'_1, Y'_3] = 9\left(\frac{\partial}{\partial u_3} - 2u_1\frac{\partial}{\partial u_4} + (5u_1^2 - 5u_2)\frac{\partial}{\partial u_5} + \dots\right)$$

All the coefficients of the differential operator Y'_4 do not depend on the variable u and hence $[Y'_0, Y'_4] = 0$. We define an operator Y'_5 of bidegree $(4, 1)$ by

$$Y'_5 = -\frac{1}{3}[Y'_1, Y'_4] = 9e^u\left(\frac{\partial}{\partial u_4} - u_1\frac{\partial}{\partial u_5} + (4u_1^2 - 6u_2)\frac{\partial}{\partial u_6} + \dots\right).$$

Obviously $[Y'_0, Y'_5] = Y'_5$. We recall that $[D, Y'_0] = 0$. Then we deduce that

$$[D, Y'_1] = \sum_{i=1}^{+\infty} D(D^{i-1}(e^u))\frac{\partial}{\partial u_i} - e^u\frac{\partial}{\partial u} - \sum_{i=1}^{+\infty} D^i(e^u)\frac{\partial}{\partial u_i} = -e^u\frac{\partial}{\partial u} = -e^u Y'_0.$$

Similarly, we conclude that $[D, Y'_2] = -e^{-2u}Y'_0$. It holds that

$$\begin{aligned} [D, Y'_3] &= \left[D, [Y'_1, Y'_2]\right] = \left[[D, Y'_1], Y'_2\right] + \left[Y'_1, [D, Y'_2]\right] = \\ &= -\left[e^u Y'_0, Y'_2\right] - \left[Y'_1, e^{-2u}Y'_0\right] = 2e^u Y'_2 + e^{-2u}Y'_1. \end{aligned}$$

□

Proof □

Proposition 5 $[Y_2', Y_3'] = [Y_2', Y_4'] = 0.$

$$\left[D, [Y_2', Y_3']\right] = \left[[D, Y_2'], Y_3'\right] + \left[Y_2', [D, Y_3']\right] = \\ = -\left[e^{-2u}Y_0', X_3'\right] + \left[Y_2', 2e^u Y_2' + e^{-2u}Y_1'\right] = e^{-2u}Y_3' + e^{-2u}\left[Y_2', Y_1'\right] = 0.$$

Consider the second commutator $[Y_2', Y_4']$

$$[D, Y_4'] = \left[D, [Y_1', Y_3']\right] = \left[[D, Y_1'], Y_3'\right] + \left[Y_1', [D, Y_3']\right] = \\ = -e^u\left[Y_0', Y_3'\right] + \left[Y_1', 2e^u Y_2' + e^{-2u}Y_1'\right] = e^u Y_3' + 2e^{2u}Y_3' = 3e^u Y_3'.$$

Hence it implies that

$$[D, [Y_2', Y_4']] = \left[[D, Y_2'], Y_4'\right] + \left[Y_2', [D, Y_4']\right] = -e^{-2u}\left[Y_0', Y_4'\right] + 3e^{2u}\left[Y_2', Y_3'\right] = 0.$$

It follows from Lemma 1 that both brackets $[Y_2', Y_4']$ and $[Y_2', Y_3']$ vanish.

Now it's the turn of $[D, Y_5']$.

$$-3[D, Y_5'] = \left[[D, Y_1'], Y_4'\right] + \left[Y_1', [D, Y_4']\right] = -e^u\left[Y_0', Y_4'\right] + \left[Y_1', 3e^u Y_3'\right] = 3e^u Y_4'.$$

We define the sixth element Y_6' of our basis with operator bidegree (5, 2)

$$Y_6' = -\frac{1}{2}[Y_1', Y_5'] = -9e^{2u}\left(\frac{\partial}{\partial u_5} + u_1\frac{\partial}{\partial u_6} + \dots\right).$$

Obviously $[Y_0', Y_6'] = 2Y_6'$.

Proposition 6 $[Y_1', Y_6'] = 0.$

Proof

$$-2[D, Y_6'] = \left[D, [Y_1', Y_5']\right] = \left[[D, Y_1'], Y_5'\right] + \left[Y_1', [D, Y_5']\right] = \\ = -e^u\left[Y_0', Y_5'\right] - e^u\left[Y_1', Y_4'\right] = 2e^u Y_5'.$$

After that we can calculate the commutator $[D, [Y_1', Y_6']]$

$$[D, [Y_1', Y_6']] = \left[[D, Y_1'], Y_6'\right] + \left[Y_1', [D, Y_6']\right] = \\ = -e^u\left[Y_0', Y_6'\right] - \left[Y_1', e^u Y_5'\right] = -2e^u Y_6' - e^u\left[Y_1', Y_5'\right] = 0.$$

Hence $[Y_1', Y_6']$ vanishes. □

We define $Y_7' = [Y_2', Y_5']$. The operator Y_7' has operator bidegree $(5, -1)$. One can verify that

$$[Y_0', Y_7'] = -Y_7', \ [D, Y_7'] = -e^{-2u}Y_5'.$$

Indeed

$$[D, Y_7'] = \left[D, [Y_2', Y_5']\right] = \left[[D, Y_2'], Y_5'\right] + \left[Y_2', [D, Y_5']\right] = \\ = -e^{-2u}\left[Y_0', Y_5'\right] - \left[Y_2', e^u Y_4'\right] = -e^{-2u}Y_5'.$$

Remark that

$$\left[D, [Y_3', Y_4']\right] = \left[[D, Y_3'], Y_4'\right] + \left[Y_3', [D, Y_4']\right] = \\ = \left[e^{-2u}Y_1' + 2e^u Y_2', Y_4'\right] - \left[Y_2', 3e^u Y_3'\right] = e^{-2u}\left[Y_1', Y_4'\right] = -3e^{-2u}Y_5'.$$

It follows from Lemma 1 that $[Y_3', Y_4'] = 3Y_7'$. We set

$$Y_8' = [Y_1', Y_7'].$$

The operator Y_8' has bidegree (6, 0). We need also the following two relations

$$[Y_0', Y_8'] = 0, \ [D, Y_8'] = 2e^{-2u} Y_6' + e^u Y_7'.$$

Let prove them

$$\begin{gathered}[D, Y_8'] = \left[D, [Y_1', Y_7']\right] = \left[[D, Y_1'], Y_7'\right] + \left[Y_1', [D, Y_7']\right] = \\ = -e^u \left[Y_0', Y_7'\right] - \left[Y_1', e^{-2u} Y_5'\right] = 2e^{-2u} Y_6' + e^u Y_7'.\end{gathered}$$

Besides this

$$\left[D, [Y_2', Y_7']\right] = \left[[D, Y_2'], Y_7'\right] + \left[Y_2', [D, Y_7']\right] = -e^{-2u} \left[Y_0', Y_7'\right] + \left[Y_2', -e^{-2u} Y_5'\right] = 0.$$

We sum up the first results of our calculations and collect the obtained relations

$$\begin{array}{l}[D, Y_0'] = 0, [D, Y_1'] = -e^u Y_0', [D, Y_2'] = -e^{-2u} Y_0', \\ [D, Y_3'] = 2e^u Y_2' + e^{-2u} Y_1', [D, Y_4'] = 3e^u Y_3', [D, Y_5'] = -e^u Y_4', \\ [D, Y_6'] = -e^u Y_5', [D, Y_7'] = -e^{-2u} Y_5', [D, Y_8'] = 2e^{-2u} Y_6' + e^u Y_7'; \\ [Y_2', Y_3'] = [Y_2', Y_4'] = [Y_2', Y_7'] = 0.\end{array} \tag{38}$$

It is time to define all the vectors of our infinite basis. We do this with the help of recursive formulas (we recall that Y_1', Y_2' are defined by Eq. 37)

$$\begin{array}{l}Y_{8k+3}' = [Y_1', Y_{8k+2}'], Y_{8k+4}' = [Y_1', Y_{8k+3}'], Y_{8k+5}' = -\frac{1}{3}[Y_1', Y_{8k+4}'], \\ Y_{8k+6}' = -\frac{1}{2}[Y_1', Y_{8k+5}'], Y_{8k+7}' = [Y_2', Y_{8k+5}'], Y_{8k+8}' = [Y_1', Y_{8k+7}'], \\ Y_{8k+9}' = -[Y_1', Y_{8k+8}'], Y_{8k+10}' = \frac{1}{2}[Y_2', Y_{8k+8}'], \quad k \geq 0.\end{array} \tag{39}$$

By induction, it is easy to establish that they are eigenvectors of the operator $\mathrm{ad} Y_0'$

$$\begin{array}{l}[Y_0', Y_{8k+1}'] = Y_{8k+1}', [Y_0', Y_{8k+2}'] = -2Y_{8k+2}', [Y_0', Y_{8k+3}'] = -Y_{8k+3}', \\ [Y_0', Y_{8k+4}'] = 0, [Y_0', Y_{8k+5}'] = Y_{8k+5}', [Y_0', Y_{8k+6}'] = 2Y_{8k+6}', \\ [Y_0', Y_{8k+7}'] = -Y_{8k+7}', [Y_0', Y_{8k+8}'] = 0.\end{array} \tag{40}$$

Lemma 5 *Operators Y_n', $n \geq 1$, defined by Eq. 39 are all non-trivial. More precisely they satisfy the following relations*

$$\begin{array}{l}[D, Y_{8k+1}'] = -e^u Y_{8k}', [D, Y_{8k+2}'] = -e^{-2u} Y_{8k}', \\ [D, Y_{8k+3}'] = e^{-2u} Y_{8k+1}' + 2e^u Y_{8k+2}', [D, Y_{8k+4}'] = 3e^u Y_{8k+3}', \\ [D, Y_{8k+5}'] = -e^u Y_{8k+4}', [D, Y_{8k+6}'] = -e^u Y_{8k+5}', \\ [D, Y_{8k+7}'] = -e^{-2u} Y_{8k+5}', [D, Y_{8k+8}'] = e^u Y_{8k+7}' + 2e^{-2u} Y_{8k+6}'; \\ [Y_2', Y_{8k+2}'] = [Y_2', Y_{8k+3}'] = [Y_2', Y_{8k+4}'] = [Y_2', Y_{8k+7}'] = 0, \ k \geq 0.\end{array} \tag{41}$$

Proof We prove the lemma and Eq. 41 by induction on k. We have allready verified the case $k = 0$ (see Eq. 38). Suppose that the formulas (41) are true for all $l \leq k - 1$, we prove them for k.

$$\begin{gathered}[D, Y_{8k+1}'] = -\left[D, [Y_1', Y_{8k}']\right] = -\left[[D, Y_1'], Y_{8k}'\right] - \left[Y_1', [D, Y_{8k}']\right] = \\ = [e^u Y_0', Y_{8k}'] - [Y_1', e^u Y_{8k-1}' + 2e^{-2u} Y_{8k-2}'] = -e^u Y_{8k}'; \\ [D, Y_{8k+2}'] = \left[D, [Y_2', Y_{8k}']\right] = \left[[D, Y_2'], Y_{8k}'\right] + \left[Y_2', [D, Y_{8k}']\right] = \\ = [-e^u Y_0', Y_{8k}'] + [Y_2', e^u Y_{8k-1}' + 2e^{-2u} Y_{8k-2}'] = [Y_2', 2e^{-2u} Y_{8k-2}'] = -e^{-2u} Y_{8k}'.\end{gathered}$$

We skip some evident steps in our calculations and continue

$$\begin{gathered}[D, Y_{8k+3}'] = [-e^u X_0', Y_{8k+2}'] + [Y_1', -e^{-2u} Y_{8k}'] = 2e^u Y_{8k+2}' + e^{-2u} Y_{8k+1}'; \\ \left[D, [Y_2', Y_{8k+2}']\right] = [-e^{-2u} Y_0', Y_{8k+2}'] + [Y_2', -e^{-2u} Y_{8k}'] = 0; \\ [D, Y_{8k+4}'] = [-e^u Y_0', Y_{8k+3}'] + [Y_1', 2e^u Y_{8k+2}' + e^{-2u} Y_{8k+1}'] = 3e^u Y_{8k+3}'.\end{gathered}$$

The relations $\left[D, [Y_2', Y_{8k+3}']\right] = \left[D, [Y_2', Y_{8k+4}']\right] = 0$. are verified completely analogously. Next two steps are

$$-3[D, Y_{8k+5}'] = \left[D, [Y_1', X_{8k+4}']\right] = \left[[D, Y_1'], Y_{8k+4}'\right] + \left[X_1', [D, Y_{8k+4}']\right] =$$
$$= [-e^u Y_0', Y_{8k+4}'] + [X_1', 3e^u Y_{8k+3}'] = 3e^u Y_{8k+4}',$$
$$-2[D, Y_{8k+6}'] = \left[D, [Y_1', Y_{8k+5}']\right] = \left[[D, Y_1'], X_{8k+5}'\right] + \left[X_1', [D, Y_{8k+5}']\right] =$$
$$= [-e^u Y_0', Y_{8k+5}'] + [Y_1', -e^u Y_{8k+4}'] = -e^u Y_{8k+5}' + 3e^u Y_{8k+5}' = 2e^u Y_{8k+5}'.$$

We leave the verifying of the following two relations as an exercise to a reader.

$$[D, Y_{8k+7}'] = -e^{-2u} Y_{8k+5}', \ [D, Y_{8k+8}'] = e^u Y_{8k+7}' + 2e^{-2u} Y_{8k+6}'.$$

We finish the proof of the lemma by

$$\left[D, [Y_2', Y_{8k+7}']\right] = \left[[D, Y_2'], X_{8k+7}'\right] + \left[Y_2', [D, X_{8k+7}']\right] =$$
$$= [-e^{-2u} Y_0', Y_{8k+7}'] + [Y_2', -e^{-2u} Y_{8k+5}'] = e^{-2u} Y_{8k+7}' - e^{-2u} Y_{8k+7}' = 0.$$

□

Lemma 6 *The operators Y_n', $n \geq 1$, satisfy the relations (15)*

$$[Y_q', Y_l'] = d_{ql} Y_{q+l}', \ q, l \in \mathbb{N},$$

where structure constants $d_{ql} = -d_{lq}$ are taken from the Table 1.

Proof We are going to apply the formulas (41) obtained in the previous Lemma.

$$\left[D, [Y_{8q}', Y_{8l}']\right] = \left[[D, Y_{8q}'], Y_{8l}'\right] + \left[Y_{8q}', [D, Y_{8l}']\right] =$$
$$= [e^u Y_{8q-1}' + 2e^{-2u} Y_{8q-2}, Y_{8l}'] + [Y_{8q}', e^u Y_{8l-1}' + 2e^{-2u} Y_{8l-2}] = 0.$$

Hence $[Y_{8q}', Y_{8l}'] = 0$. Next relation is

$$\left[D, [Y_{8q}', Y_{8l+1}']\right] = \left[[D, Y_{8q}'], Y_{8l+1}'\right] + \left[Y_{8q}', [D, Y_{8l+1}']\right] =$$
$$= [e^u Y_{8q-1}' + 2e^{-2u} Y_{8q-2}, Y_{8l+1}'] + [Y_{8q}', -e^u Y_{8l}'] = -e^u Y_{8(q+l)}'.$$

It follows that $[Y_{8q}', Y_{8l+1}'] = Y_{8(q+l)+1}$ because $[D, Y_{8(q+l)+1}] = -e^u Y_{8(q+l)}'$. Then

$$\left[D, [Y_{8q}', Y_{8l+2}']\right] = \left[[D, Y_{8q}'], Y_{8l+2}'\right] + \left[Y_{8q}', [D, Y_{8l+2}']\right] =$$
$$= [e^u Y_{8q-1}' + 2e^{-2u} Y_{8q-2}, Y_{8l+2}'] + [Y_{8q}', -e^{-2u} Y_{8l}'] = 2e^{-2u} Y_{8(q+l)}'.$$

Recall that $[D, Y_{8(q+l)+1}'] = -e^{-2u} Y_{8(q+l)}'$. Hence $[Y_{8q}', Y_{8l+2}'] = -2Y_{8(q+l)+2}'$. We leave the reader, as an exercise, to prove the relations from the first row of Table 1.

$$[Y_{8q}', Y_{8l+3}'] = -Y_{8(q+l)+3}, [Y_{8q}', Y_{8l+4}'] = 0, [Y_{8q}', Y_{8l+5}'] = Y_{8(q+l)+5},$$
$$[Y_{8q}', Y_{8l+6}'] = 2Y_{8(q+l)+6}, [Y_{8q}', Y_{8l+7}'] = -Y_{8(q+l)+7}.$$

Now we switch to the second row of the Table 1. We start with

$$\left[D, [Y_{8q+1}', X_{8l+1}']\right] = \left[[D, Y_{8q+1}'], Y_{8l+1}'\right] + \left[Y_{8q+1}', [D, Y_{8l+1}']\right] =$$
$$= [-e^u Y_{8q}', Y_{8l+1}'] + [Y_{8q+1}', -e^u Y_{8l}'] = 0,$$
$$\left[D, [Y_{8q+1}', Y_{8l+2}']\right] = \left[[D, Y_{8q+1}'], Y_{8l+2}'\right] + \left[X_{8q+1}', [D, Y_{8l+2}']\right] =$$
$$= [-e^u Y_{8q}', Y_{8l+2}'] + [Y_{8q+1}', -e^{-2u} Y_{8l}'] = 2e^u Y_{8(q+l)+2}' + e^{-2u} Y_{8(q+l)+1}'.$$

Hence $[Y'_{8q+1}, Y'_{8l+2}] = Y'_{8(q+l)+3}$ as $[D, Y'_{8(q+l)+3}]{=}2e^u Y'_{8(q+l)+2}{+}e^{-2u} Y'_{8(q+l)+1}$.

$$\Big[D, [Y'_{8q+1}, Y'_{8l+3}]\Big] = \Big[[D, Y'_{8q+1}], Y'_{8l+3}\Big] + \Big[Y'_{8q+1}, [D, Y'_{8l+3}]\Big] =$$
$$=[-e^u Y'_{8q}, Y'_{8l+3}]{+}[Y'_{8q+1}, e^{-2u} Y'_{8q+1}{+}2e^u Y'_{8l+2}]{=}3e^u Y'_{8(q+l)+3}{=}[D, Y_{8(q+l)+4}].$$

We conclude that $[Y'_{8q+1}, Y'_{8l+3}] = Y'_{8(q+l)+4}$. Continuing in the same way and calculating step by step commutators $[Y'_{8q+r}, Y'_{8l+s}]$ with $1 \leq r \leq s \leq 7$ we obtain all structure relations (15). □

We define a Lie algebra isomorphism $\varphi : \chi(e^u + e^{-2u}) \to \tilde{\mathfrak{n}}_2$ by setting

$$\varphi(Y'_n) = f_n, n \geq 0.$$

Corollary 4 *It follows now from Theorem 3 and from Lemma 2 that* $\chi(e^u{+}e^{-2u})$ *has the same growth as the Lie algebra* $\mathfrak{n}_2$.

11 Final Remarks

The characteristic Lie algebra $\chi(\sinh u)$ of sinh-Gordon equation $u_{xy} = \sinh u$ was studied by Murtazina and Zhiber in [26]. An infinite basis of $\chi(\sinh u)$ was constructed there and commutation relations were found. But the very important Lie algebras isomorphism

$$\chi(\sinh u) \cong \mathcal{L}(\mathfrak{sl}(2, \mathbb{K}))^{\geq 0}, \mathbb{K} = \mathbb{R}, \mathbb{C},$$

was missed there as well as different gradings of $\chi(\sinh u)$.

Sakieva examined the characteristic Lie algebra $\chi(e^u{+}e^{-2u})$ of Tzitzeica equation in [22]. An infinite basis and commutation relations were found it this case also. But again the very important Lie algebras isomorphism

$$\chi(e^u{+}e^{-2u}) \cong \mathcal{L}(\mathfrak{sl}(3, \mathbb{K}), \mu)^{\geq 0}, \mathbb{K} = \mathbb{R}, \mathbb{C},$$

was missed.

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Appendix: The correspondence tables of different gradings for $\mathfrak{n}_1$ and $\mathfrak{n}_2$

It follows from the proof of Theorem 2 that the weighted bigrading of Diff$\mathcal{F}$ induces the $\mathbb{Z}_{\geq 0}{\times}\mathbb{Z}_3$-grading on the Lie algebra $\tilde{\mathfrak{n}}_1$. The corresponding bidegrees of its basic elements X'_n are presented in the Table 2.

Table 2 The correspondence table of different gradings of $\mathfrak{n}_1$

width one	X'_0	X'_1	X'_2	X'_{3k}	X'_{3k+1}	X'_{3k+2}
natural		1	1	$2k$	$2k{+}1$	$2k{+}1$
canonical	$(0, 0)$	$(1, 0)$	$(0, 1)$	(k, k)	$(k{+}1, k)$	$(k, k{+}1)$
$\mathbb{Z}_{\geq 0}{\times}\mathbb{Z}_3$	$(0, 0)$	$(1, 1)$	$(1, -1)$	$(k, 0)$	$(k, 1)$	$(k, -1)$

Table 3 The correspondence table of different gradings of $\mathfrak{n}_2$

basis	Y'_{8k}	Y'_{8k+1}	Y'_{8k+2}	Y'_{8k+3}	Y'_{8k+4}	Y'_{8k+5}	Y'_{8k+6}	Y'_{8k+7}
natural	$6k$	$6k+1$	$6k+1$	$6k+2$	$6k+3$	$6k+4$	$6k+5$	$6k+5$
canon.	$\binom{4k}{2k}$	$\binom{4k+1}{2k}$	$\binom{4k}{2k+1}$	$\binom{4k+1}{2k+1}$	$\binom{4k+2}{2k+1}$	$\binom{4k+3}{2k+1}$	$\binom{4k+4}{2k+1}$	$\binom{4k+3}{2k+2}$
$\mathbb{Z}_{\geq 0}\times\mathbb{Z}_5$	$\binom{6k}{0}$	$\binom{6k+1}{1}$	$\binom{6k+1}{-2}$	$\binom{6k+2}{-1}$	$\binom{6k+3}{0}$	$\binom{6k+4}{1}$	$\binom{6k+5}{2}$	$\binom{6k+5}{-1}$

The Lie algebra $\tilde{\mathfrak{n}}_2$ is $\mathbb{Z}_{\geq 0}\times\mathbb{Z}_5$-graded (see the proof of Theorem 3). We list the corresponding bidegrees of its basic elements Y'_n in the Table 3.

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