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A New Approach to Representations of 3-Lie Algebras and Abelian Extensions

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Abstract In this paper, we introduce the notion of generalized representation of a 3-Lie algebra, by which we obtain a generalized semidirect product 3-Lie algebra. Moreover, we develop the corresponding cohomology theory. Various examples of generalized representations of 3-Lie algebras and computation of 2-cocycles of the new cohomology are provided. Also, we show that a split abelian extension of a 3-Lie algebra is isomorphic to a generalized semidirect product 3-Lie algebra. Furthermore, we describe general abelian extensions of 3-Lie algebras using Maurer-Cartan elements.

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1 Introduction

The notion of a Filippov algebra, or a n-Lie algebra was introduced in [11]. Ternary Lie algebras are related to Nambu mechanics [18], generalizing Hamiltonian mechanics by using more than one hamiltonian. The algebraic formulation of this theory is due to Takhtajan [22], see also [13]. Moreover, 3-Lie algebras appeared in String Theory. Fuzzy sphere (noncommutative space) arises naturally in the description of D1-branes ending on D3-branes in Type IIB superstring theory and the effective dynamics of this system is described by the Nahm equations. In order to find an appropriate description of the lift of this configuration to M-theory, one can study supergravity solutions describing M2-branes ending on M5-branes. In [6], Basu and Harvey suggested to replace the Lie algebra appearing in the Nahm equation by a 3-Lie algebra for the lifted Nahm equations. Furthermore, in the context of Bagger-Lambert-Gustavsson model of multiple M2-branes, Bagger-Lambert managed to construct, using a ternary bracket, an $N = 2$ supersymmetric version of the worldvolume theory of the M-theory membrane, see [2]. An extensive literatures are related to this pioneering work, see [3, 14, 15, 20].

In mathematics, the first appearance of ternary operation went back to 19th century, where Cayley considered cubic matrices. In these last decades, several algebraic aspects of 3-Lie algebras, or more generally, of n-Lie algebras were studied. See [1, 4, 5] for the construction, realization and classifications of 3-Lie algebras and n-Lie algebras. In particular, representation theory of n-Lie algebras was first introduced by Kasymov in [16]. The adjoint representation is defined by the ternary bracket in which two elements are fixed. Through fundamental objects one may also represent a 3-Lie algebra and more generally an n-Lie algebra by a Leibniz algebra [7]. Following this approach, deformations of 3-Lie algebras and n-Lie algebras are studied in [10, 23], see [17] for a review. In [21], the author defined a graded Lie algebra structure on the cochain complex of a n-Leibniz algebra and describe a n-Leibniz structure as a canonical structure. See the review article [8] for more details.

In this paper, we provide a new approach to representation theory of 3-Lie algebras. We define generalized representations of a 3-Lie algebra $\mathfrak{g}$ on a vector space V via canonical structures in the differential graded Lie algebra associated to $\mathfrak{g} \oplus V$ given in [7]. A generalized representation leads also to a new 3-Lie algebra, which we call a generalized semidirect product. We also develop the corresponding cohomology theory. As applications, we study abelian extensions of 3-Lie algebras and provide several examples. We show that a split abelian extension is isomorphic to a generalized semidirect product. Note that even split abelian extensions cannot be studied via the usual representations (in the sense of Kasymov [16]). This justifies the usage of our approach. Furthermore, we use Maurer-Cartan elements to describe non-split abelian extensions.

The paper is organized as follows. In Section 2, we give a review of representations of 3-Lie algebras, the differential graded Lie algebras associated to 3-Lie algebras and canonical structures. In Section 3, we first explain the usual representations of a 3-Lie algebra using canonical structures, then we introduce the notion of generalized representation of a 3-Lie algebra. We show that a generalized representation of a 3-Lie algebra gives rise to a generalized semidirect product 3-Lie algebra. Various examples of generalized representations are given. In Section 4, we provide the corresponding cohomology theory associated to generalized representations. Moreover, we study in detail 1-cocycles, 2-cocycles and provide examples. Section 5 deals with abelian extensions of 3-Lie algebras. First, we show that a split abelian extension is isomorphic to a generalized semidirect product. Unlike the case

of Lie algebras, one cannot use cocycles to describe general abelian extensions of 3-Lie algebras. Alternatively, we can use Maurer-Cartan elements to describe them.

2 3-Lie Algebras and Their Representations

In this paper, we work over an algebraically closed field $\mathbb{K}$ of characteristic 0 and all the vector spaces are over $\mathbb{K}$.

Definition 2.1 [11] A **3-Lie algebra** is a vector space $\mathfrak{g}$ together with a skew-symmetric linear map $[\cdot, \cdot, \cdot] : \otimes^3 \mathfrak{g} \to \mathfrak{g}$ such that the following **Fundamental Identity (FI)** holds:

$$
\begin{aligned}
&F_{x_1, x_2, x_3, x_4, x_5} \\
&\triangleq [x_1, x_2, [x_3, x_4, x_5]] - [[x_1, x_2, x_3], x_4, x_5] - [x_3, [x_1, x_2, x_4], x_5] - [x_3, x_4, [x_1, x_2, x_5]] \\
&= 0.
\end{aligned}
\tag{1}
$$

Elements in $\wedge^2 \mathfrak{g}$ are called **fundamental objects** of the 3-Lie algebra $(\mathfrak{g}, [\cdot, \cdot, \cdot])$. There is a bilinear operation $[\cdot, \cdot]_F$ on $\wedge^2 \mathfrak{g}$, which is given by

$$
[\mathfrak{X}, \mathfrak{Y}]_F = [x_1, x_2, y_1] \wedge y_2 + y_1 \wedge [x_1, x_2, y_2], \quad \forall \mathfrak{X} = x_1 \wedge x_2, \mathfrak{Y} = y_1 \wedge y_2.
\tag{2}
$$

It is well-known that $(\wedge^2 \mathfrak{g}, [\cdot, \cdot]_F)$ is a Leibniz algebra [7], which plays an important role in the theory of 3-Lie algebras.

Definition 2.2 [16] A **representation** ρ of a 3-Lie algebra $\mathfrak{g}$ on a vector space V is a linear map $\rho : \wedge^2 \mathfrak{g} \longrightarrow \text{End}(V)$, such that

$$
\begin{aligned}
\rho(x_1, x_2)\rho(x_3, x_4) &= \rho([x_1, x_2, x_3], x_4) + \rho(x_3, [x_1, x_2, x_4]) + \rho(x_3, x_4)\rho(x_1, x_2), \\
\rho(x_1, [x_2, x_3, x_4]) &= \rho(x_3, x_4)\rho(x_1, x_2) - \rho(x_2, x_4)\rho(x_1, x_3) + \rho(x_2, x_3)\rho(x_1, x_4).
\end{aligned}
$$

It is straightforward to obtain

Lemma 2.3 *Let $\mathfrak{g}$ be a 3-Lie algebra, V a vector space and $\rho : \wedge^2 \mathfrak{g} \to \mathfrak{gl}(V)$ a skew-symmetric linear map. Then $(V; \rho)$ is a representation of $\mathfrak{g}$ if and only if there is a 3-Lie algebra structure (called the semidirect product) on the direct sum of vector spaces $\mathfrak{g} \oplus V$, defined by*

$$
[x_1 + v_1, x_2 + v_2, x_3 + v_3]_\rho = [x_1, x_2, x_3] + \rho(x_1, x_2)v_3 + \rho(x_3, x_1)v_2 + \rho(x_2, x_3)v_1, \tag{3}
$$

for $x_i \in \mathfrak{g}, v_i \in V, 1 \leq i \leq 3$. We denote this semidirect product 3-Lie algebra by $\mathfrak{g} \ltimes_\rho V$.

A p-cochain on $\mathfrak{g}$ with coefficients in a representation $(V; \rho)$ is a linear map

$$
\alpha : \wedge^2 \mathfrak{g} \otimes \overset{(p-1)}{\cdots} \otimes \wedge^2 \mathfrak{g} \wedge \mathfrak{g} \longrightarrow V.
$$

Denote the space of p-cochains by $C^{p-1}(\mathfrak{g}, V)$. The coboundary operator $\delta_\rho : C^{p-1}(\mathfrak{g}, V)$ $\longrightarrow C^p(\mathfrak{g}, V)$ is given by

$$
\begin{aligned}
&(\delta_\rho \alpha)(\mathfrak{X}_1, \cdots, \mathfrak{X}_p, z) \\
&= \sum_{1 \le j < k} (-1)^j \alpha(\mathfrak{X}_1, \cdots, \hat{\mathfrak{x}}_j, \cdots, \mathfrak{X}_{k-1}, [\mathfrak{X}_j, \mathfrak{X}_k]_F, \mathfrak{X}_{k+1}, \cdots, \mathfrak{X}_p, z) \\
&+ \sum_{j=1}^p (-1)^j \alpha(\mathfrak{X}_1, \cdots, \hat{\mathfrak{x}}_j, \cdots, \mathfrak{X}_p, [\mathfrak{X}_j, z]) \\
&+ \sum_{j=1}^p (-1)^{j+1} \rho(\mathfrak{X}_j) \alpha(\mathfrak{X}_1, \cdots, \hat{\mathfrak{x}}_j, \cdots, \mathfrak{X}_p, z) \\
&+ (-1)^{p+1} \Big(\rho(y_p, z) \alpha(\mathfrak{X}_1, \cdots, \mathfrak{X}_{p-1}, x_p) + \rho(z, x_p) \alpha(\mathfrak{X}_1, \cdots, \mathfrak{X}_{p-1}, y_p) \Big), \quad (4)
\end{aligned}
$$

for all $\mathfrak{X}_i = (x_i, y_i) \in \wedge^2 \mathfrak{g}$ and $z \in \mathfrak{g}$. An element $\alpha \in C^{p-1}(\mathfrak{g}, V)$ is called a p-cocycle if $\delta_\rho \alpha = 0$; It is called a p-coboundary if there exists some $\beta \in C^{p-2}(\mathfrak{g}, V)$ such that $\alpha = \delta_\rho \beta$. Denote by $Z^p(\mathfrak{g}; V)$ and $B^p(\mathfrak{g}; V)$ the set of p-cocycles and the set of p-coboundaries respectively. Then the p-th cohomology group is

$$H^p(\mathfrak{g}; V) = Z^p(\mathfrak{g}; V)/B^p(\mathfrak{g}; V). \quad (5)$$

See [9] for more details about cohomology of n-Lie algebras.

In [21], the author constructed a graded Lie algebra structure by which one can describe a n-Leibniz algebra structure as a canonical structure. Here, we give the precise formulas for the 3-Lie algebra case.

Set $L_p = C^p(\mathfrak{g}, \mathfrak{g}) = \mathrm{Hom}(\wedge^2 \mathfrak{g} \otimes \overset{(p)}{\cdots} \otimes \wedge^2 \mathfrak{g} \wedge \mathfrak{g}, \mathfrak{g})$ and $L = \oplus_{p \ge 0} L_p$. Let $\alpha \in C^p(\mathfrak{g}, \mathfrak{g}), \beta \in C^q(\mathfrak{g}, \mathfrak{g}), \quad p, q \ge 0$. Let $\mathfrak{X}_i = x_i \wedge y_i \in \wedge^2 \mathfrak{g}$ for $i = 1, 2, \cdots, p + q$ and $x \in \mathfrak{g}$. For each subset $J = \{j_1, \cdots, j_{q+1}\}_{j_1 < \cdots < j_{q+1}} \subset N \triangleq \{1, 2, \cdots, p + q + 1\}$, let $I = \{i_1, \cdots, i_p\}_{i_1 < \cdots < i_p} = N/J$. Then we have

Theorem 2.4 [21] *The graded vector space L equipped with the graded commutator bracket*

$$[\alpha, \beta]^{3\mathrm{Lie}} = (-1)^{pq} \alpha \circ \beta - \beta \circ \alpha, \quad (6)$$

is a graded Lie algebra and $\alpha \circ \beta \in L^{p+q}$ is defined by

$$
\begin{aligned}
\alpha \circ \beta(\mathfrak{X}_1, \cdots, \mathfrak{X}_{p+q}, x) &= \sum_{J, j_{q+1} < p+q+1} (-1)^{(J,I)} (-1)^k \\
&\Big(\alpha(\mathfrak{X}_{i_1}, \cdots, \mathfrak{X}_{i_k}, \beta(\mathfrak{X}_{j_1}, \cdots, \mathfrak{X}_{j_q}, x_{j_{q+1}}) \wedge y_{j_{q+1}}, \mathfrak{X}_{i_{k+1}}, \cdots, X_{i_{p-1}}, x) \\
&+ \alpha(\mathfrak{X}_{i_1}, \cdots, \mathfrak{X}_{i_k}, x_{j_{q+1}} \wedge \beta(\mathfrak{X}_{j_1}, \cdots, \mathfrak{X}_{j_q}, y_{j_{q+1}}), \mathfrak{X}_{i_{k+1}}, \cdots, X_{i_{p-1}}, x) \Big) \\
&+ \sum_{J, j_{q+1}=p+q+1} (-1)^{(J,I)} (-1)^p \alpha(\mathfrak{X}_{i_1}, \cdots, \mathfrak{X}_{i_p}, \beta(\mathfrak{X}_{j_1}, \cdots, \mathfrak{X}_{j_q}, x)),
\end{aligned}
$$

where k is uniquely determined by the condition $i_k < j_{q+1} < i_{k+1}$ and if $j_{q+1} < i_1$, i.e. $j_{q+1} = q + 1, i_1 = q + 2$ then $k = 0$; if $j_{q+1} > i_{p-1}$, i.e. $j_{q+1} = p + q$ then $k = p - 1$.

We can use the graded Lie algebra structure $(L, [\cdot, \cdot]^{3\mathrm{Lie}})$ to describe 3-Lie algebra structures as well as coboundary operators.

Lemma 2.5 *The map* $\pi : \wedge^3\mathfrak{g} \longrightarrow \mathfrak{g}$ *defines a 3-Lie bracket if and only if* $[\pi, \pi]^{3\text{Lie}} = 0$, *i.e.* π *is a canonical structure.*

Let $\mathfrak{g}$ be a 3-Lie algebra. Given $x_1, x_2 \in \mathfrak{g}$, define ad $: \wedge^2\mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ by

$$\text{ad}_{x_1,x_2} y = [x_1, x_2, y].$$

Then, the map ad defines a representation of the 3-Lie algebra $\mathfrak{g}$ on itself, which we call **adjoint representation** of $\mathfrak{g}$. The coboundary operator associated to this representation is denoted by $\delta_\mathfrak{g}$.

Lemma 2.6 *If* $\pi : \wedge^3\mathfrak{g} \longrightarrow \mathfrak{g}$ *is a 3-Lie bracket, then we have*

$$[\pi, \alpha]^{3\text{Lie}} = \delta_\mathfrak{g}(\alpha), \quad \forall \alpha \in C^p(\mathfrak{g}, \mathfrak{g}), p \geq 0. \tag{7}$$

3 Generalized Representations of 3-Lie Algebras

In this section, we introduce a concept of generalized representation of a 3-Lie algebra using canonical structures. First, we show that a representation of a 3-Lie algebra will give rise to a canonical structure.

Let $\rho : \wedge^2\mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be a linear map. Then, it induces a linear map $\bar\rho : \wedge^3(\mathfrak{g} \oplus V) \longrightarrow \mathfrak{g} \oplus V$ defined by

$$\bar\rho(x+u, y+v, z+w) = \rho(x, y)(w) + \rho(y, z)(u) + \rho(z, x)(v), \quad \forall x, y, z \in \mathfrak{g}, u, v, w \in V. \tag{8}$$

Consider the graded Lie algebra given in Theorem 2.4 associated to the vector space $\mathfrak{g} \oplus V$.

Proposition 3.1 *A linear map* $\rho : \wedge^2\mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ *is a representation of the 3-Lie algebra* $\mathfrak{g}$ *on* V *if and only if* $\pi + \bar\rho$ *is a canonical structure in the graded Lie algebra associated to* $\mathfrak{g} \oplus V$, *i.e.*

$$[\pi + \bar\rho, \pi + \bar\rho]^{3\text{Lie}} = 0.$$

Proof By Lemma 2.3, $\rho : \wedge^2\mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is a representation of $\mathfrak{g}$ if and only if $\mathfrak{g} \oplus V$ is a 3-Lie algebra, where the 3-Lie bracket is exactly given by

$$\begin{aligned}
[x + u, y + v, z + w]_\rho &= [x, y, z] + \rho(x, y)(w) + \rho(y, z)(u) + \rho(z, x)(v) \\
&= (\pi + \bar\rho)(x + u, y + v, z + w).
\end{aligned}$$

Thus, by Lemma 2.5, $\rho : \wedge^2\mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is a representation of $\mathfrak{g}$ if and only if $\pi + \bar\rho$ is a canonical structure. $\qquad\square$

Now, we define a new concept of representation of a 3-Lie algebra, generalizing the usual one. A so called generalized representation of a 3-Lie algebra involves two linear maps.

Definition 3.2 A **generalized representation** of a 3-Lie algebra $\mathfrak{g}$ on a vector space V consists of linear maps $\rho : \wedge^2\mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ and $\nu : \mathfrak{g} \longrightarrow \text{Hom}(\wedge^2 V, V)$ such that

$$[\pi + \bar\rho + \bar\nu, \pi + \bar\rho + \bar\nu]^{3\text{Lie}} = 0, \tag{9}$$

where $\bar\nu : \wedge^3(\mathfrak{g} \oplus V) \longrightarrow \mathfrak{g} \oplus V$ is induced by ν via

$$\bar\nu(x+u, y+v, z+w) = \nu(x)(v \wedge w) + \nu(y)(w \wedge u) + \nu(z)(u \wedge v), \quad \forall x, y, z \in \mathfrak{g}, u, v, w \in V.$$

We will refer to a generalized representation by a triple $(V; \rho, \nu)$.

Remark 3.3 If $\nu = 0$, then we recover the usual definition of a representation of a 3-Lie algebra on a vector space (in the sense of Kasymov [16]). If the dimension of the vector space V is 1, then ν must be zero. In this case, we only have the usual representation.

Given linear maps $\rho : \wedge^2 \mathfrak{g} \longrightarrow \text{End}(V)$ and $\nu : \mathfrak{g} \longrightarrow \text{Hom}(\wedge^2 V, V)$, define a trilinear bracket operation on $\mathfrak{g} \oplus V$ by

$$
\begin{aligned}
[x + u, y + v, z + w]_{(\rho,\nu)} = {} & [x, y, z] + \rho(x, y)(w) + \rho(y, z)(u) + \rho(z, x)(v) \\
& + \nu(x)(v \wedge w) + \nu(y)(w \wedge u) + \nu(z)(u \wedge v).
\end{aligned}
\tag{10}
$$

Theorem 3.4 *Let $(\mathfrak{g}, [\cdot, \cdot, \cdot])$ be a 3-Lie algebra and $(V; \rho, \nu)$ a generalized representation of $\mathfrak{g}$. Then $(\mathfrak{g} \oplus V, [\cdot, \cdot, \cdot]_{(\rho,\nu)})$ is a 3-Lie algebra, where $[\cdot, \cdot, \cdot]_{(\rho,\nu)}$ is given by* Eq. 10.

We call the 3-Lie algebra $(\mathfrak{g} \oplus V, [\cdot, \cdot, \cdot]_{(\rho,\nu)})$ the **generalized semidirect product** of $\mathfrak{g}$ and V.

Proof It follows from

$$
[x + u, y + v, z + w]_{(\rho,\nu)} = (\pi + \bar{\rho} + \bar{\nu})(x + u, y + v, z + w)
$$

and Lemma 2.5. $\qquad\square$

In the following, we give a characterization of a generalized representation of a 3-Lie algebra.

Proposition 3.5 *Linear maps $\rho : \wedge^2 \mathfrak{g} \longrightarrow \text{End}(V)$ and $\nu : \mathfrak{g} \longrightarrow \text{Hom}(\wedge^2 V, V)$ give rise to a generalized representation of a 3-Lie algebra $\mathfrak{g}$ on a vector space V if and only if for all $x_i \in \mathfrak{g}$, $v_j \in V$, the following equalities hold:*

$$
\rho(x_1, x_2)\rho(x_3, x_4) = \rho([x_1, x_2, x_3], x_4) + \rho(x_3, [x_1, x_2, x_4]) + \rho(x_3, x_4)\rho(x_1, x_2),
\tag{11}
$$

$$
\rho(x_1, [x_2, x_3, x_4]) = \rho(x_3, x_4)\rho(x_1, x_2) - \rho(x_2, x_4)\rho(x_1, x_3) + \rho(x_2, x_3)\rho(x_1, x_4),
\tag{12}
$$

$$
\begin{aligned}
\rho(x_1, x_2)(\nu(x_3)(v_1, v_2)) = {} & \nu([x_1, x_2, x_3])(v_1, v_2) + \nu(x_3)(\rho(x_1, x_2)(v_1), v_2) \\
& + \nu(x_3)(v_1, \rho(x_1, x_2)(v_2)),
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
\nu(x_1)(v_1, \rho(x_2, x_3)(v_2)) = {} & \nu(x_3)(v_2, \rho(x_2, x_1)(v_1)) + \nu(x_2)(\rho(x_3, x_1)(v_1), v_2) \\
& + \rho(x_2, x_3)(\nu(x_1)(v_1, v_2)),
\end{aligned}
\tag{14}
$$

$$
\nu(x_1)(v_1, \nu(x_2)(v_2, v_3)) = \nu(x_2)(\nu(x_1)(v_1, v_2), v_3) + \nu(x_2)(v_2, \nu(x_1)(v_1, v_3)),
\tag{15}
$$

$$
\nu(x_1)(\nu(x_2)(v_1, v_2), v_3) = \nu(x_2)(\nu(x_1)(v_1, v_2), v_3).
\tag{16}
$$

Proof The triple $(V; \rho, \nu)$ is a generalized representation if and only if $[\pi + \bar{\rho} + \bar{\nu}, \pi + \bar{\rho} + \bar{\nu}]^{3\text{Lie}} = 0$. By straightforward computations,

$$
[\pi + \bar{\rho} + \bar{\nu}, \pi + \bar{\rho} + \bar{\nu}]^{3\text{Lie}}(x_1, x_2, x_3, x_4, v) = 0
$$

is equivalent to Eq. 11; And

$$
[\pi + \bar{\rho} + \bar{\nu}, \pi + \bar{\rho} + \bar{\nu}]^{3\text{Lie}}(x_1, v, x_2, x_3, x_4) = 0
$$

is equivalent to Eq. 12. Other identities can be proved similarly. The details are omited. $\quad\square$

Remark 3.6 By Eqs. 11 and 12, the map ρ in a generalized representation $(V; \rho, \nu)$ gives rise to a usual representation in the sense of Definition 2.2. Conversely, for any representation ρ, $(V; \rho, \nu = 0)$ is a generalized representation.

Remark 3.7 Equation 13 may be written as $\nu([x_1, x_2, x_3]) = [\rho(x_1, x_2), \nu(x_3)]_{NR}$, where $[\cdot, \cdot]_{NR}$ is the Nijenhuis-Richardson bracket on the graded vector space $\oplus_k \text{Hom}(\wedge^k V, V)$; see [19] for more details about Nijenhuis-Richardson bracket.

Remark 3.8 One may obtain from previous conditions the following identity

$$\nu([x_1, x_2, x_3])(v_1, v_2) = \rho(x_2, x_3)(\nu(x_1)(v_1, v_2)) + \rho(x_3, x_1)(\nu(x_2)(v_1, v_2))$$
$$+ \rho(x_1, x_2)(\nu(x_3)(v_1, v_2)). \tag{17}$$

In the following, we provide a series of examples to illustrate the new concept of generalized representation.

Example 3.9 Let $\mathfrak{g}$ be an abelian 3-Lie algebra. Define $\rho = 0$, and $\nu = \xi \otimes \pi$, where $\xi \in \mathfrak{g}^*$ and $\pi \in \text{Hom}(\wedge^2 V \otimes V)$ is a Lie algebra structure on V. Then $(V; \rho, \nu)$ is a generalized representation. In fact, since $\mathfrak{g}$ is abelian and $\rho = 0$, Eqs. 11–14 hold naturally. Since π satisfies the Jacobi identity, Eqs. 15 and 16 also hold.

Example 3.10 Let $\mathfrak{g}$ be the 3-dimensional 3-Lie algebra defined with respect to a basis $\{x_1, x_2, x_3\}$ by the skew-symmetric bracket $[x_1, x_2, x_3] = x_1$.

Let V be a 2-dimensional vector space and $\{v_1, v_2\}$ its basis. The following map ρ, given with respect to bases by

$$\rho(x_1, x_2)(v_1) = 0, \ \rho(x_1, x_2)(v_2) = r_1 v_1, \ \rho(x_1, x_3)(v_1) = 0,$$
$$\rho(x_1, x_3)(v_2) = r_2 v_1, \ \rho(x_2, x_3)(v_1) = r_3 v_1, \ \rho(x_2, x_3)(v_2) = r_4 v_1 + (r_3 - 1) v_2,$$

where $r_1, \cdots, r_4$ are parameters in $\mathbb{K}$, defines a representation of $\mathfrak{g}$ on V (in the usual sense, Eqs. 11 and 12 are satisfied).

It defines together with a map ν a generalized representation of $\mathfrak{g}$ on V in the following cases.

1. $\rho(x_1, x_2)(v_1) = 0, \ \rho(x_1, x_2)(v_2) = r_1 v_1, \ \rho(x_1, x_3)(v_1) = 0,$
   $\rho(x_1, x_3)(v_2) = r_2 v_1, \ \rho(x_2, x_3)(v_1) = v_1, \ \rho(x_2, x_3)(v_2) = r_4 v_1,$
   $\nu(x_1)(v_1, v_2) = 0, \ \nu(x_2)(v_1, v_2) = s_1 v_1, \ \nu(x_3)(v_1, v_2) = \frac{s_1 r_2}{r_1} v_1.$
2. $\rho(x_1, x_2)(v_1) = 0, \ \rho(x_1, x_2)(v_2) = 0, \ \rho(x_1, x_3)(v_1) = 0,$
   $\rho(x_1, x_3)(v_2) = r_2 v_1, \ \rho(x_2, x_3)(v_1) = v_1, \ \rho(x_2, x_3)(v_2) = r_4 v_2,$
   $\nu(x_1)(v_1, v_2) = 0, \ \nu(x_2)(v_1, v_2) = 0, \ \nu(x_3)(v_1, v_2) = s_1 v_1.$
3. $\rho(x_1, x_2)(v_1) = 0, \ \rho(x_1, x_2)(v_2) = 0, \ \rho(x_1, x_3)(v_1) = 0,$
   $\rho(x_1, x_3)(v_2) = 0, \ \rho(x_2, x_3)(v_1) = 0, \ \rho(x_2, x_3)(v_2) = r_4 v_1 - v_2,$
   $\nu(x_1)(v_1, v_2) = 0, \ \nu(x_2)(v_1, v_2) = -s_1 r_4 v_1 + s_1 v_2, \ \nu(x_3)(v_1, v_2) = -s_2 r_4 v_1 + s_2 v_2.$
4. $\rho(x_1, x_2)(v_1) = 0, \ \rho(x_1, x_2)(v_2) = 0, \ \rho(x_1, x_3)(v_1) = 0,$
   $\rho(x_1, x_3)(v_2) = 0, \ \rho(x_2, x_3)(v_1) = v_1, \ \rho(x_2, x_3)(v_2) = r_4 v_1,$
   $\nu(x_1)(v_1, v_2) = 0, \ \nu(x_2)(v_1, v_2) = s_1 v_1, \ \nu(x_3)(v_1, v_2) = s_2 v_1.$

where r_i and s_j are parameters in $\mathbb{K}$.

Example 3.11 We may also consider, for the previous 3-dimensional 3-Lie algebra, the following map ρ which defines a representation of $\mathfrak{g}$ on V (in the usual sense).

$$\rho(x_1, x_2)(v_1) = r_1 r_4 v_1 + r_1 v_2, \quad \rho(x_1, x_2)(v_2) = -r_1 r_4^2 v_1 - r_1 r_4 v_2,$$

$$\rho(x_1, x_3)(v_1) = r_2 r_4 v_1 + r_2 v_2,$$

$$\rho(x_1, x_3)(v_2) = -r_2 r_4^2 v_1 - r_2 r_4 v_2, \quad \rho(x_2, x_3)(v_1) = r_3 v_1,$$

$$\rho(x_2, x_3)(v_2) = r_4 v_1 + (1 + r_3) v_2.$$

It defines together with a map ν a generalized representation of $\mathfrak{g}$ on V in the following cases.

1. $\rho(x_1, x_2)(v_1) = r_1 r_4 v_1 + r_1 v_2, \quad \rho(x_1, x_2)(v_2) = -r_1 r_4^2 v_1 - r_1 r_4 v_2,$
   $\rho(x_1, x_3)(v_1) = r_2 r_4 v_1 + r_2 v_2,$
   $\rho(x_1, x_3)(v_2) = -r_2 r_4^2 v_1 - r_2 r_4 v_2, \quad \rho(x_2, x_3)(v_1) = 0, \quad \rho(x_2, x_3)(v_2) = r_4 v_1 + v_2,$
   $\nu(x_1)(v_1, v_2) = 0, \quad \nu(x_2)(v_1, v_2) = s_1 r_4 v_1 + s_1 v_2, \quad \nu(x_3)(v_1, v_2) = \frac{s_1 r_2 r_4}{r_1} v_1 + \frac{s_1 r_2}{r_1} v_2,$
2. $\rho(x_1, x_2)(v_1) = 0, \quad \rho(x_1, x_2)(v_2) = 0, \quad \rho(x_1, x_3)(v_1) = r_2 r_4 v_1 + r_2 v_2,$
   $\rho(x_1, x_3)(v_2) = -r_2 r_4^2 v_1 - r_2 r_4 v_2, \quad \rho(x_2, x_3)(v_1) = 0, \quad \rho(x_2, x_3)(v_2) = r_4 v_1 + v_2,$
   $\nu(x_1)(v_1, v_2) = 0, \quad \nu(x_2)(v_1, v_2) = 0, \quad \nu(x_3)(v_1, v_2) = r_4 s_1 v_1 + s_1 v_2,$
3. $\rho(x_1, x_2)(v_1) = 0 \ \rho(x_1, x_2)(v_2) = 0, \quad \rho(x_1, x_3)(v_1) = 0,$
   $\rho(x_1, x_3)(v_2) = 0, \quad \rho(x_2, x_3)(v_1) = 0, \quad \rho(x_2, x_3)(v_2) = r_4 v_1 + v_2,$
   $\nu(x_1)(v_1, v_2) = 0, \quad \nu(x_2)(v_1, v_2) = s_1 r_4 v_1 + s_1 v_2, \quad \nu(x_3)(v_1, v_2) = s_2 r_4 v_1 + s_2 v_2,$
4. $\rho(x_1, x_2)(v_1) = 0, \quad \rho(x_1, x_2)(v_2) = 0, \quad \rho(x_1, x_3)(v_1) = 0,$
   $\rho(x_1, x_3)(v_2) = 0, \quad \rho(x_2, x_3)(v_1) = -v_1, \quad \rho(x_2, x_3)(v_2) = r_4 v_1,$
   $\nu(x_1)(v_1, v_2) = 0, \quad \nu(x_2)(v_1, v_2) = s_1 v_1, \quad \nu(x_3)(v_1, v_2) = s_2 v_1,$
5. $\rho(x_1, x_2)(v_1) = 0, \quad \rho(x_1, x_2)(v_2) = 0, \quad \rho(x_1, x_3)(v_1) = r_2 v_2,$
   $\rho(x_1, x_3)(v_2) = 0, \quad \rho(x_2, x_3)(v_1) = 0, \quad \rho(x_2, x_3)(v_2) = v_2,$
   $\nu(x_1)(v_1, v_2) = 0, \quad \nu(x_2)(v_1, v_2) = 0, \quad \nu(x_3)(v_1, v_2) = s_1 v_2,$
6. $\rho(x_1, x_2)(v_1) = r_1 v_2, \quad \rho(x_1, x_2)(v_2) = 0, \quad \rho(x_1, x_3)(v_1) = 0,$
   $\rho(x_1, x_3)(v_2) = 0, \quad \rho(x_2, x_3)(v_1) = 0, \quad \rho(x_2, x_3)(v_2) = v_2,$
   $\nu(x_1)(v_1, v_2) = 0, \quad \nu(x_2)(v_1, v_2) = s_1 v_2, \quad \nu(x_3)(v_1, v_2) = 0,$

where r_i and s_j are parameters in $\mathbb{K}$.

In the following, we consider generalized representations where the usual representation vanishes.

Proposition 3.12 *Let $\mathfrak{g}$ be an n-dimensional 3-Lie algebra such that $[\mathfrak{g}, \mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ ($\mathfrak{g}$ is said to be perfect). If the generalized representation $(V; \rho, \nu)$ is such that $\rho = 0$ then $\nu = 0$.*

Proof It follows from condition (13). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 3.13 *Let $\mathfrak{g}$ be the 4-dimensional simple 3-Lie algebra defined with respect to a basis $\{x_1, x_2, x_3, x_4\}$ by the skew-symmetric brackets*

$$[x_1, x_2, x_3] = x_4, \ [x_1, x_2, x_4] = x_3, \ [x_1, x_3, x_4] = x_2, \ [x_2, x_3, x_4] = x_1.$$

If a n-dimensional generalized representation $(V; \rho, \nu)$ of $\mathfrak{g}$ satisfies $\rho = 0$, then $\nu = 0$.

Now, we show some examples where $[\mathfrak{g}, \mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$.

Example 3.14 We provide for the 3-dimensional 3-Lie algebra, defined with respect to a basis $\{x_1, x_2, x_3\}$ by the skew-symmetric bracket $[x_1, x_2, x_3] = x_1$, all the 2-dimensional

generalized representations $(V; \rho, \nu)$ with $\rho = 0$. Every generalized representation on a 2-dimensional vector space V with a trivial ρ is given by one of the following maps ν defined, with respect to a basis $\{v_1, v_2\}$ of V, by

1. $\nu(x_1)(v_1, v_2) = 0, \ \nu(x_2)(v_1, v_2) = s_1 v_1 + s_2 v_2, \ \nu(x_3)(v_1, v_2) = s_3 v_1 + \frac{s_2 s_3}{s_1} v_2,$
2. $\nu(x_1)(v_1, v_2) = 0, \ \nu(x_2)(v_1, v_2) = s_1 v_2, \ \nu(x_3)(v_1, v_2) = s_2 v_2,$
3. $\nu(x_1)(v_1, v_2) = 0, \ \nu(x_2)(v_1, v_2) = 0, \ \nu(x_3)(v_1, v_2) = s_1 v_1 + s_2 v_2,$

where s_1, s_2 are parameters in $\mathbb{K}$.

Example 3.15 Let $\mathfrak{g}$ be the 4-dimensional 3-Lie algebra defined, with respect to a basis $\{x_1, x_2, x_3, x_4\}$, by the skew-symmetric brackets

$$[x_1, x_2, x_4] = x_3, \ [x_1, x_3, x_4] = x_2, \ [x_2, x_3, x_4] = x_1.$$

Every generalized representation $(V; \rho, \nu)$, on a 2-dimensional vector space V with a trivial ρ, of $\mathfrak{g}$ is given by one of the following maps ν, defined with respect to a basis $\{v_1, v_2\}$ of V, by

$$\nu(x_1)(v_1, v_2) = 0, \ \nu(x_2)(v_1, v_2) = 0, \ \nu(x_3)(v_1, v_2) = 0, \ \nu(x_4)(v_1, v_2) = s_1 v_1 + s_2 v_2,$$

where s_1, s_2 are parameters in $\mathbb{K}$.

Example 3.16 Let $\mathfrak{g}$ be the 4-dimensional 3-Lie algebra defined, with respect to a basis $\{x_1, x_2, x_3, x_4\}$, by $[x_2, x_3, x_4] = x_1$. Every generalized representation $(V; \rho, \nu)$, on a 2-dimensional vector space V with trivial ρ, of $\mathfrak{g}$ is given by one of the following maps ν defined, with respect to a basis $\{v_1, v_2\}$ of V, by

1. $\nu(x_1)(v_1, v_2) = 0, \ \nu(x_2)(v_1, v_2) = s_1 v_1 + s_2 v_2,$
   $\nu(x_3)(v_1, v_2) = s_3 v_1 + \frac{s_2 s_3}{s_1} v_2, \ \nu(x_4)(v_1, v_2) = s_4 v_1 + \frac{s_2 s_4}{s_1} v_2,$
2. $\nu(x_1)(v_1, v_2) = 0, \ \nu(x_2)(v_1, v_2) = 0,$
   $\nu(x_3)(v_1, v_2) = s_1 v_1 + s_2 v_2, \ \nu(x_4)(v_1, v_2) = s_3 v_1 + \frac{s_2 s_4}{s_1} v_2,$

where s_1, s_2, s_3, s_4 are parameters in $\mathbb{K}$.

Now, we characterize the notion of equivalent generalized representations of 3-Lie algebras.

Definition 3.17 Let $(V_1; \rho_1, \nu_1)$ and $(V_2; \rho_2, \nu_2)$ be two generalized representations of a 3-Lie algebra $(\mathfrak{g}, [\cdot, \cdot, \cdot])$. They are said to be **equivalent** if there exists an isomorphism of vector spaces $T : V_1 \longrightarrow V_2$ such that

$$T\rho_1(x, y)(u) = \rho_2(x, y)(Tu), \quad T\nu_1(x)(u, v) = \nu_2(x)(Tu, Tv), \quad \forall x, y \in \mathfrak{g}, u, v \in V_1.$$

In terms of diagrams, we have

4 New Cohomology Complex of 3-Lie Algebras

Based on the generalized representations defined in the previous section, we introduce a new type of cohomology for 3-Lie algebras.

Let $(\mathfrak{g}, [\cdot, \cdot, \cdot])$ be a 3-Lie algebra and $(V; \rho, \nu)$ a generalized representation of $\mathfrak{g}$. We set $C^p_>(\mathfrak{g} \oplus V, V)$ to be the set of $(p+1)$-cochains, which is defined as a subset of $C^p(\mathfrak{g} \oplus V, V)$ such that

$$C^p(\mathfrak{g} \oplus V, V) = C^p_>(\mathfrak{g} \oplus V, V) \oplus C^p(V, V). \tag{18}$$

Recall that $C^p(\mathfrak{g} \oplus V, V) = \{\alpha : \wedge^2(\mathfrak{g} \oplus V) \otimes \overset{(p \text{ times})}{\cdots} \otimes \wedge^2(\mathfrak{g} \oplus V) \wedge (\mathfrak{g} \oplus V) \longrightarrow V\}$.

By direct calculation, we have

$$[\pi + \bar{\rho} + \bar{\nu}, C^\bullet_>(\mathfrak{g} \oplus V, V)]^{3\text{Lie}} \subseteq C^{\bullet+1}_>(\mathfrak{g} \oplus V, V).$$

Define $\mathrm{d} : C^p_>(\mathfrak{g} \oplus V, V) \longrightarrow C^{p+1}_>(\mathfrak{g} \oplus V, V)$ by

$$\mathrm{d}(\alpha) := [\pi + \bar{\rho} + \bar{\nu}, \alpha]^{3\text{Lie}}, \quad \alpha \in C^p_>(\mathfrak{g} \oplus V, V). \tag{19}$$

Theorem 4.1 *Let $(V; \rho, \nu)$ be a generalized representation of a 3-Lie algebra $\mathfrak{g}$. Then $\mathrm{d} \circ \mathrm{d} = 0$. Thus, we obtain a new cohomology complex, where the space of p-cochains is given by $C^{p-1}_>(\mathfrak{g} \oplus V, V)$.*

Proof By the graded Jacobi identity, for any $\alpha \in C^{p-1}_>(\mathfrak{g} \oplus V, V)$, we have

$$\mathrm{d} \circ \mathrm{d}(\alpha) := [\pi + \bar{\rho} + \bar{\nu}, [\pi + \bar{\rho} + \bar{\nu}, \alpha]^{3\text{Lie}}]^{3\text{Lie}} = \frac{1}{2}[[\pi + \bar{\rho} + \bar{\nu}, \pi + \bar{\rho} + \bar{\nu}]^{3\text{Lie}}, \alpha]^{3\text{Lie}} = 0,$$

which finishes the proof. $\square$

An element $\alpha \in C^{p-1}_>(\mathfrak{g} \oplus V, V)$ is called a *p-cocycle* if $\mathrm{d}(\alpha) = 0$; It is called a *p-coboundary* if there exists $\beta \in C^{p-2}_>(\mathfrak{g} \oplus V, V)$ such that $\alpha = \mathrm{d}(\beta)$. Denote by $\mathcal{Z}^p(\mathfrak{g}; V)$ and $\mathcal{B}^p(\mathfrak{g}; V)$ the sets of *p*-cocycles and *p*-coboundaries respectively. By Theorem 4.1, we have $\mathcal{B}^p(\mathfrak{g}; V) \subset \mathcal{Z}^p(\mathfrak{g}; V)$. We define the *p*-th cohomolgy group $\mathcal{H}^p(\mathfrak{g}; V)$ to be $\mathcal{Z}^p(\mathfrak{g}; V)/\mathcal{B}^p(\mathfrak{g}; V)$.

A relationship between this new cohomology and the one given by Eq. 5, is stated in the following result.

Proposition 4.2 *There is a forgetful map from $\mathcal{H}^p(\mathfrak{g}; V)$ to $H^p(\mathfrak{g}; V)$.*

Proof It is obvious that $C^p(\mathfrak{g}, V) \subseteq C^p_>(\mathfrak{g} \oplus V, V)$. By direct calculation, for $\mathfrak{X}_i \in \wedge^2 \mathfrak{g}$, $z \in \mathfrak{g}$, we have

$$\mathrm{d}(\alpha)(\mathfrak{X}_1, \cdots, \mathfrak{X}_{p+1}, z) = \delta_\rho(\alpha)(\mathfrak{X}_1, \cdots, \mathfrak{X}_{p+1}, z), \quad \alpha \in C^p(\mathfrak{g}, V),$$

where δ_ρ is the coboundary operator given by Eq. 4. Thus, the natural projection from $C^p_>(\mathfrak{g} \oplus V, V)$ to $C^p(\mathfrak{g}, V)$ induces a forgetful map from $\mathcal{H}^p(\mathfrak{g}; V)$ to $H^p(\mathfrak{g}; V)$. $\square$

In the sequel, we give some characterization of low dimensional cocycles.

Proposition 4.3 *A linear map* $\alpha \in \mathrm{Hom}(\mathfrak{g}, V)$ *is a 1-cocycle if only if for all* $x_1, x_2, x_3 \in \mathfrak{g}, v \in V$, *the following identities hold:*

$$\nu(x_1)(\alpha(x_2), v) - \nu(x_2)(\alpha(x_1), v) = 0,$$

$$\alpha([x_1, x_2, x_3]) - \rho(x_1, x_2)(\alpha(x_3)) - \rho(x_2, x_3)(\alpha(x_1)) - \rho(x_3, x_1)(\alpha(x_2)) = 0.$$

Proof For $\alpha \in \mathrm{Hom}(\mathfrak{g}, V)$, we have

$$\mathrm{d}(\alpha)(x_1, x_2, v) = \nu(x_1)(\alpha(x_2), v) - \nu(x_2)(\alpha(x_1), v),$$

and

$$\mathrm{d}(\alpha)(x_1, x_2, x_3) = \delta_\rho(\alpha)(x_1, x_2, x_3)$$
$$= \rho(x_1, x_2)(\alpha(x_3)) + \rho(x_2, x_3)(\alpha(x_1)) + \rho(x_3, x_1)(\alpha(x_2)) - \alpha([x_1, x_2, x_3]),$$

which finishes the proof. $\qquad\square$

Proposition 4.4 *A 2-cochain* $\alpha_1 + \alpha_2 + \alpha_3 \in C^1_>(\mathfrak{g} \oplus V, V)$, *where* $\alpha_1 \in \mathrm{Hom}(\wedge^2 V \wedge \mathfrak{g}, V), \alpha_2 \in \mathrm{Hom}(\wedge^2 \mathfrak{g} \wedge V, V), \alpha_3 \in \mathrm{Hom}(\wedge^3 \mathfrak{g}, V)$, *is a 2-cocycle if and only if for all* $x_i \in \mathfrak{g}, v_j \in V$ *and* $v \in V$, *the following identities hold:*

$$
\begin{aligned}
0 = {} & -\rho(x_1, x_2)(\alpha_3(x_3, x_4, x_5)) - \alpha_3(x_1, x_2, [x_3, x_4, x_5]) + \rho(x_4, x_5)(\alpha_3(x_1, x_2, x_3)) \\
& + \alpha_3([x_1, x_2, x_3], x_4, x_5) + \rho(x_5, x_3)(\alpha_3(x_1, x_2, x_4)) + \alpha_3(x_3, [x_1, x_2, x_4], x_5) \\
& + \rho(x_3, x_4)(\alpha_3(x_1, x_2, x_5)) + \alpha_3(x_3, x_4, [x_1, x_2, x_5]),
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
0 = {} & \nu(x_4)(\alpha_3(x_1, x_2, x_3), v) + \nu(x_3)(v, \alpha_3(x_1, x_2, x_4)) + \rho(x_1, x_2)(\alpha_2(x_3, x_4, v)) \\
& - \rho(x_3, x_4)(\alpha_2(x_1, x_2, v)) - \alpha_2([x_1, x_2, x_3], x_4, v) - \alpha_2(x_3, [x_1, x_2, x_4], v),
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
0 = {} & \nu(x_1)(v, \alpha_3(x_2, x_3, x_4)) + \rho(x_3, x_4)(\alpha_2(x_1, x_2, v)) - \rho(x_2, x_4)(\alpha_2(x_1, x_3, v)) \\
& + \rho(x_2, x_3)(\alpha_2(x_1, x_4, v)) + \alpha_2(x_3, x_4, \rho(x_1, x_2)(v)) - \alpha_2(x_2, x_4, \rho(x_1, x_3)(v)) \\
& + \alpha_2(x_2, x_3, \rho(x_1, x_4)(v)) - \alpha_2(x_1, [x_2, x_3, x_4], v),
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
0 = {} & \nu(x_3)(v_2, \alpha_2(x_1, x_2, v_1)) + \nu(x_3)(\alpha_2(x_1, x_2, v_2), v_1) + \alpha_2(x_1, x_2, \nu(x_3)(v_1, v_2)) \\
& + \rho(x_1, x_2)(\alpha_1(v_1, v_2, x_3)) - \alpha_1(\rho(x_1, x_2)(v_1), v_2, x_3) - \alpha_1(v_1, \rho(x_1, x_2)(v_2), x_3) \\
& - \alpha_1(v_1, v_2, [x_1, x_2, x_3]),
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
0 = {} & \nu(x_3)(v_2, \alpha_2(x_2, x_1, v_1)) + \nu(x_2)(\alpha_2(x_3, x_1, v_1), v_2) - \nu(x_1)(v_1, \alpha_2(x_2, x_3, v_2)) \\
& + \alpha_2(x_2, x_3, \nu(x_1)(v_1, v_2)) + \rho(x_2, x_3)(\alpha_1(v_1, v_2, x_1)) + \alpha_1(\rho(x_1, x_2)(v_1), v_2, x_3) \\
& - \alpha_1(v_1, \rho(x_2, x_3)(v_2), x_1) + \alpha_1(v_2, \rho(x_1, x_3)(v_1), x_2),
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
0 = {} & \alpha_2(x_1, x_3, \nu(x_2)(v_1, v_2)) - \alpha_2(x_2, x_3, \nu(x_1)(v_1, v_2)) - \alpha_2(x_1, x_2, \nu(x_3)(v_1, v_2)) \\
& + \rho(x_1, x_3)(\alpha_1(v_1, v_2, x_2)) - \rho(x_1, x_2)(\alpha_1(v_1, v_2, x_3)) - \rho(x_2, x_3)(\alpha_1(v_1, v_2, x_1)) \\
& + \alpha_1(v_1, v_2, [x_1, x_2, x_3]),
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
0 = {} & -\nu(x_2)(\alpha_1(v_1, v_2, x_1), v_3) - \nu(x_2)(v_2, \alpha_1(v_1, v_3, x_1)) + \nu(x_1)(v_1, \alpha_1(v_2, v_3, x_2)) \\
& - \alpha_1(\nu(x_1)(v_1, v_2), v_3, x_2) - \alpha_1(v_2, \nu(x_1)(v_1, v_3), x_2) + \alpha_1(v_1, \nu(x_2)(v_2, v_3), x_1),
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
0 = {} & \nu(x_2)(\alpha_1(v_1, v_2, x_1), v_3) - \nu(x_1)(v_3, \alpha_1(v_1, v_2, x_2)) + \alpha_1(\nu(x_1)(v_1, v_2), v_3, x_2) \\
& - \alpha_1(\nu(x_2)(v_1, v_2), v_3, x_1).
\end{aligned} \tag{27}
$$

Proof For $\alpha_3 \in \mathrm{Hom}(\wedge^3 \mathfrak{g}, V)$, we have

$$
\begin{aligned}
\mathrm{d}(\alpha_3)(x_1, x_2, x_3, x_4, x_5) &= \rho(x_1, x_2)(\omega(x_3, x_4, x_5)) + \omega(x_1, x_2, [x_3, x_4, x_5]) \\
&\quad -\rho(x_4, x_5)(\omega(x_1, x_2, x_3)) - \omega([x_1, x_2, x_3], x_4, x_5) \\
&\quad -\rho(x_5, x_3)(\omega(x_1, x_2, x_4)) - \omega(x_3, [x_1, x_2, x_4], x_5) \\
&\quad -\rho(x_3, x_4)(\omega(x_1, x_2, x_5)) - \omega(x_3, x_4, [x_1, x_2, x_5]), \\
\mathrm{d}(\alpha_3)(x_1, x_2, x_3, x_4, v) &= \nu(x_4)(\alpha_3(x_1, x_2, x_3), v) + \nu(x_3)(v, \alpha_3(x_1, x_2, x_4)), \\
\mathrm{d}(\alpha_3)(x_1, v, x_2, x_3, x_4) &= \nu(x_1)(v, \alpha_3(x_2, x_3, x_4)).
\end{aligned}
$$

For $\alpha_2 \in \mathrm{Hom}(\wedge^2 \mathfrak{g} \wedge V, V)$, we have

$$
\begin{aligned}
\mathrm{d}(\alpha_2)(x_1, x_2, x_3, x_4, v) &= \rho(x_1, x_2)(\alpha_2(x_3, x_4, v)) - \rho(x_3, x_4)(\alpha_2(x_1, x_2, v)) \\
&\quad -\alpha_2([x_1, x_2, x_3], x_4, v) - \alpha_2(x_3, [x_1, x_2, x_4], v), \\
\mathrm{d}(\alpha_2)(x_1, v, x_2, x_3, x_4) &= \rho(x_3, x_4)(\alpha_2(x_1, x_2, v)) - \rho(x_2, x_4)(\alpha_2(x_1, x_3, v)) \\
&\quad +\rho(x_2, x_3)(\alpha_2(x_1, x_4, v)) + \alpha_2(x_3, x_4, \rho(x_1, x_2)(v)) \\
&\quad -\alpha_2(x_2, x_4, \rho(x_1, x_3)(v)) + \alpha_2(x_2, x_3, \rho(x_1, x_4)(v)) \\
&\quad -\alpha_2(x_1, [x_2, x_3, x_4], v),
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{d}(\alpha_2)(x_1, x_2, v_1, v_2, x_3) &= \nu(x_3)(v_2, \alpha_2(x_1, x_2, v_1)) + \nu(x_3)(\alpha_2(x_1, x_2, v_2), v_1) \\
&\quad +\alpha_2(x_1, x_2, \nu(x_3)(v_1, v_2)), \\
\mathrm{d}(\alpha_2)(x_1, v_1, x_2, v_2, x_3) &= \nu(x_3)(v_2, \alpha_2(x_2, x_1, v_1)) + \nu(x_2)(\alpha_2(x_3, x_1, v_1), v_2) \\
&\quad -\nu(x_1)(v_1, \alpha_2(x_2, x_3, v_2)) + \alpha_2(x_2, x_3, \nu(x_1)(v_1, v_2)), \\
\mathrm{d}(\alpha_2)(v_1, v_2, x_1, x_2, x_3) &= \alpha_2(x_1, x_3, \nu(x_2)(v_1, v_2)) - \alpha_2(x_2, x_3, \nu(x_1)(v_1, v_2)) \\
&\quad -\alpha_2(x_1, x_2, \nu(x_3)(v_1, v_2)).
\end{aligned}
$$

For $\alpha_1 \in \mathrm{Hom}(\wedge^2 V \wedge \mathfrak{g}, V)$, we have

$$
\begin{aligned}
\mathrm{d}(\alpha_1)(x_1, x_2, v_1, v_2, x_3) &= \rho(x_1, x_2)(\alpha_1(v_1, v_2, x_3)) - \alpha_1(\rho(x_1, x_2)(v_1), v_2, x_3) \\
&\quad -\alpha_1(v_1, \rho(x_1, x_2)(v_2), x_3) - \alpha_1(v_1, v_2, [x_1, x_2, x_3]), \\
\mathrm{d}(\alpha_1)(x_1, v_1, x_2, v_2, x_3) &= \rho(x_2, x_3)(\alpha_1(v_1, v_2, x_1)) + \alpha_1(\rho(x_1, x_2)(v_1), v_2, x_3) \\
&\quad -\alpha_1(v_1, \rho(x_2, x_3)(v_2), x_1) + \alpha_1(v_2, \rho(x_1, x_3)(v_1), x_2), \\
\mathrm{d}(\alpha_1)(v_1, v_2, x_1, x_2, x_3) &= \rho(x_1, x_3)(\alpha_1(v_1, v_2, x_2)) - \rho(x_1, x_2)(\alpha_1(v_1, v_2, x_3)) \\
&\quad -\rho(x_2, x_3)(\alpha_1(v_1, v_2, x_1)) + \alpha_1(v_1, v_2, [x_1, x_2, x_3]), \\
\mathrm{d}(\alpha_1)(x_1, v_1, v_2, v_3, x_2) &= -\nu(x_2)(\alpha_1(v_1, v_2, x_1), v_3) - \nu(x_2)(v_2, \alpha_1(v_1, v_3, x_1)) \\
&\quad +\nu(x_1)(v_1, \alpha_1(v_2, v_3, x_2)) - \alpha_1(\nu(x_1)(v_1, v_2), v_3, x_2) \\
&\quad -\alpha_1(v_2, \nu(x_1)(v_1, v_3), x_2) + \alpha_1(v_1, \nu(x_2)(v_2, v_3), x_1), \\
\mathrm{d}(\alpha_1)(v_1, v_2, x_1, x_2, v_3) &= \nu(x_2)(\alpha_1(v_1, v_2, x_1), v_3) - \nu(x_1)(v_3, \alpha_1(v_1, v_2, x_2)) \\
&\quad +\alpha_1(\nu(x_1)(v_1, v_2), v_3, x_2) - \alpha_1(\nu(x_2)(v_1, v_2), v_3, x_1).
\end{aligned}
$$

Thus, $\mathrm{d}(\alpha_1 + \alpha_2 + \alpha_3) = 0$ if and only if Eqs. 20–27 hold.   □

In the following we provide two examples of computation of 2-cocycles of the 3-dimensional ternary algebra defined with respect to a basis $\{x_1, x_2, x_3\}$ by the bracket $[x_1, x_2, x_3] = x_1$. We consider two different generalized representations on a 2-dimensional vector space V, with basis $\{v_1, v_2\}$.

Example 4.5 (Example 1) We consider the generalized representation (V, ρ, ν), where ρ and ν are defined with respect to the basis by

$$\rho(x_1, x_2)(v_1) = 0, \ \rho(x_1, x_2)(v_2) = 0, \ \rho(x_1, x_3)(v_1) = s_1 s_2 v_1 + s_1 v_2,$$
$$\rho(x_1, x_3)(v_2) = -s_1 s_2^2 v_1 - s_1 s_2 v_2, \ \rho(x_2, x_3)(v_1) = 0, \ \rho(x_2, x_3)(v_2) = s_2 v_1 + v_2,$$

$$\nu(x_1)(v_1, v_2) = 0, \ \nu(x_2)(v_1, v_2) = 0, \ \nu(x_3)(v_1, v_2) = s_3 s_2 v_1 + s_3 v_2,$$

with s_1, s_2, s_3 parameters in $\mathbb{K}$.

The 2-cocycles are given by $\alpha_1 = 0$, $\alpha_3 = 0$ and α_2 defined as

$$\alpha_2(x_1, x_2, v_1) = 0, \qquad \alpha_2(x_1, x_2, v_2) = 0,$$
$$\alpha_2(x_1, x_3, v_1) = s_2 p_1 v_1 + p_1 v_2, \ \alpha_2(x_1, x_3, v_2) = -s_2^2 p_1 v_1 - s_2 p_1 v_2,$$
$$\alpha_2(x_2, x_3, v_1) = s_2 p_2 v_1 + p_2 v_2, \ \alpha_2(x_2, x_3, v_2) = -s_2^2 p_2 v_1 - s_2 p_2 v_2,$$

where p_1, p_2 are parameters in $\mathbb{K}$.

Example 4.6 (Example 2) Now, we consider the generalized representation (V, ρ, ν), where ρ is trivial and ν is given with respect to the basis by

$$\nu(x_1)(v_1, v_2) = 0, \ \nu(x_2)(v_1, v_2) = s_1 v_1 + s_2 v_2, \ \nu(x_3)(v_1, v_2) = s_3 v_1 + \frac{s_2 s_3}{s_1} v_2,$$

with s_1, s_2, s_3 parameters in $\mathbb{K}$.

The 2-cocycles are given by $\alpha_1 = 0$ and α_2, α_3 defined as

$$\alpha_2(x_1, x_2, v_1) = -s_1 p_2 v_1 - s_2 p_2 v_2, \ \alpha_2(x_1, x_2, v_2) = s_1 p_1 v_1 + s_2 p_1 v_2,$$
$$\alpha_2(x_1, x_3, v_1) = -s_3 p_2 v_1 - \frac{s_2 s_3 p_2}{s_1} v_2, \ \alpha_2(x_1, x_3, v_2) = s_3 p_1 v_1 + \frac{s_2 s_3 p_1}{s_1} v_2,$$
$$\alpha_2(x_2, x_3, v_1) = p_3 v_1 + \frac{s_2 p_3}{s_1} v_2, \qquad \alpha_2(x_2, x_3, v_2) = p_4 v_1 + \frac{s_2 p_4}{s_1} v_2,$$

$$\alpha_3(x_1, x_2, x_3) = p_1 v_1 + p_2 v_2,$$

where p_1, p_2, p_3, p_4 are parameters in $\mathbb{K}$.

5 Abelian Extensions of 3-Lie Algebras

In this section, first we study the split abelian extension of 3-Lie algebras, which is isomorphic to a generalized semidirect product 3-Lie algebra. This provides a good motivation for the introduction of the generalized representation. Then, we study non-split abelian extensions of 3-Lie algebras. Unlike the case of Lie algebras, they cannot be described by 2-cocycles. Finally, we describe non-split abelian extensions via Maurer-Cartan elements.

Consider an exact sequence of 3-Lie algebras:

$$0 \longrightarrow V \xrightarrow{\ i\ } \hat{\mathfrak{g}} \xrightarrow{\ p\ } \mathfrak{g} \longrightarrow 0 , \tag{28}$$

then $\hat{\mathfrak{g}}$ is said to be an abelian extension of the 3-Lie algebra $\mathfrak{g}$ by V if V is abelian. A linear map $\sigma : \mathfrak{g} \longrightarrow \hat{\mathfrak{g}}$ is called a splitting of $\hat{\mathfrak{g}}$ if it satisfies $p \circ \sigma = id_{\mathfrak{g}}$. If there exists a splitting which is also a homomorphism between 3-Lie algebras, we say that the abelian extension is **split**.

Let $\hat{\mathfrak{g}}$ be a split abelian extension and $\sigma : \mathfrak{g} \longrightarrow \hat{\mathfrak{g}}$ the corresponding splitting. Define $\rho : \wedge^2 \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ and $\nu : \mathfrak{g} \longrightarrow \mathrm{Hom}(\wedge^2 V, V)$ by

$$\rho(x, y)(u) = [\sigma(x), \sigma(y), u]_{\hat{\mathfrak{g}}},$$
$$\nu(x)(u, v) = [\sigma(x), u, v]_{\hat{\mathfrak{g}}}.$$

Then, we can transfer the 3-Lie algebra structure on $\hat{\mathfrak{g}}$ to that on $\mathfrak{g} \oplus V$ in terms of ρ and ν:

$$[x + u, y + v, z + w]_{(\rho, \nu)} = [x, y, z] + \rho(x, y)(w) + \rho(y, z)(u) + \rho(z, x)(v)$$
$$+ \nu(x)(v \wedge w) + \nu(y)(w \wedge u) + \nu(z)(u \wedge v).$$

Note that the Fundamental Identity gives the character of ρ and ν. However, by Theorem 3.4, it is straightforward to obtain the following proposition.

Proposition 5.1 *Any split abelian extension of 3-Lie algebras is isomorphic to a generalized semidirect product 3-Lie algebra.*

Now, for non-split abelian extensions, we can further define $\omega : \wedge^3 \mathfrak{g} \longrightarrow V$ by

$$\omega(x, y, z) = [\sigma(x), \sigma(y), \sigma(z)]_{\hat{\mathfrak{g}}} - \sigma[x, y, z]_{\mathfrak{g}}.$$

Then, we also transfer the 3-Lie algebra structure on $\hat{\mathfrak{g}}$ to that on $\mathfrak{g} \oplus V$ in terms of ρ, ν and ω :

$$[x_1 + v_1, x_2 + v_2, x_3 + v_3]_{(\rho, \nu, \omega)} = [x_1, x_2, x_3]_{\mathfrak{g}} + \rho(x_1, x_2)(v_3) + \rho(x_3, x_1)(v_2) + \rho(x_2, x_3)(v_1)$$
$$+ \nu(x_1)(v_2, v_3) + \nu(x_2)(v_3, v_1) + \nu(x_3)(v_1, v_2) + \omega(x_1, x_2, x_3).$$

The Fundamental Identity gives the character of ρ, ν and ω.

Theorem 5.2 *With above notations, $(\mathfrak{g} \oplus V, [\cdot, \cdot, \cdot]_{(\rho, \nu, \omega)})$ is a 3-Lie algebra if and only if for all $x_1, x_2, x_3, x_4, x_5 \in \mathfrak{g}$ and $v, v_1, v_2, v_3 \in V$, Eqs. 13–16 and the following identities hold:*

$$\begin{aligned} 0 = &-\rho(x_1, x_2)(\omega(x_3, x_4, x_5)) - \omega(x_1, x_2, [x_3, x_4, x_5]) + \rho(x_4, x_5)(\omega(x_1, x_2, x_3)) \\ &+ \omega([x_1, x_2, x_3], x_4, x_5) + \rho(x_5, x_3)(\omega(x_1, x_2, x_4)) + \omega(x_3, [x_1, x_2, x_4], x_5) \\ &+ \rho(x_3, x_4)(\omega(x_1, x_2, x_5)) + \omega(x_3, x_4, [x_1, x_2, x_5]), \end{aligned} \tag{29}$$

$$\begin{aligned} 0 = &\nu(x_1)(v, \omega(x_2, x_3, x_4)) + \rho([x_2, x_3, x_4], x_1)(v) + \rho(x_3, x_4)\rho(x_1, x_2)(v) \\ &- \rho(x_2, x_4)\rho(x_1, x_3)(v) + \rho(x_2, x_3)\rho(x_1, x_4)(v), \end{aligned} \tag{30}$$

$$\begin{aligned} 0 = &\rho(x_1, x_2)\rho(x_3, x_4)(v) - \rho(x_3, x_4)\rho(x_1, x_2)(v) - \rho([x_1, x_2, x_3], x_4)(v) \\ &- \nu(x_4)(v, \omega(x_1, x_2, x_3)) - \rho(x_3, [x_1, x_2, x_4])(v) + \nu(x_3)(v, \omega(x_1, x_2, x_4)). \end{aligned} \tag{31}$$

Proof The pair $(\mathfrak{g} \oplus V, [\cdot, \cdot, \cdot]_{(\rho, \nu, \omega)})$ defines a 3-Lie algebra if and only if for all $e_1, \cdots, e_5 \in \mathfrak{g} \oplus V$,

$$F_{e_1, e_2, e_3, e_4, e_5} = 0.$$

By $F_{x_1, x_2, x_3, x_4, x_5} = 0$, we have

$$\begin{aligned} [x_1, x_2, [x_3, x_4, x_5]_{(\rho, \nu, \omega)}]_{(\rho, \nu, \omega)} - [[x_1, x_2, x_3]_{(\rho, \nu, \omega)}, x_4, x_5]_{(\rho, \nu, \omega)} - [x_3, [x_1, x_2, x_4]_{(\rho, \nu, \omega)}, x_5]_{(\rho, \nu, \omega)} \\ - [x_3, x_4, [x_1, x_2, x_5]_{(\rho, \nu, \omega)}]_{(\rho, \nu, \omega)} = 0, \end{aligned}$$

which gives Eq. 29.

Similarly, $F_{x_1,v,x_2,x_3,x_4} = 0$ gives Eq. 30. $F_{x_1,x_2,v,x_3,x_4} = 0$ gives Eq. 31. $F_{x_1,x_2,v_1,v_2,x_3} = 0$ gives Eq. 13. $F_{v_1,x_1,v_2,x_2,x_3} = 0$ gives Eq. 14. $F_{v_1,x_1,v_2,v_3,x_2} = 0$ gives Eq. 15. $F_{v_1,v_2,v_3,x_1,x_2} = 0$ gives Eq. 16.

Conversely, if Eqs. 13–16 and Eqs. 29–31 hold, it is straightforward to see that for all $e_1, \cdots, e_5 \in \mathfrak{g} \oplus V$, $F_{e_1,e_2,e_3,e_4,e_5} = 0$. Thus, $(\mathfrak{g} \oplus V, [\cdot, \cdot, \cdot]_{(\rho, \nu, \omega)})$ is a 3-Lie algebra. □

Remark 5.3 For an abelian extension of 3-Lie algebras, we have seen that $(V; \rho, \nu)$ is not a generalized representation, but the failure is controlled by ω. This is totally different from the case of Lie algebras. For Lie algebras, abelian extension will give us a representation and a 2-cocycle. Only when we consider nonabelian extensions of Lie algebras [12], the phenomenon that ρ is not a representation will occur. But now, even for abelian extension of 3-Lie algebras, this phenomenon has already occurred.

Example 5.4 Let $\mathfrak{g}$ be the 3-dimensional 3-Lie algebra defined, with respect to a basis $\{x_1, x_2, x_3\}$, by the skew-symmetric bracket $[x_1, x_2, x_3] = x_1$.

Let V be a 2-dimensional vector space and $\{v_1, v_2\}$ its basis. The following maps ρ, ν and ω define an abelian extension of the 3-Lie algebra $\mathfrak{g}$ on V (according to Theorem 5.2).

$$\rho(x_1, x_2)(v_1) = r_1 v_1 + \frac{s_2 r_1}{s_1} v_2, \quad \rho(x_1, x_2)(v_2) = \frac{s_1 r_2}{s_2} v_1 + r_2 v_2,$$

$$\rho(x_1, x_3)(v_1) = \frac{s_3 r_1}{s_1} v_1 + \frac{s_2 s_3 r_1}{s_1^2} v_2, \quad \rho(x_1, x_3)(v_2) = \frac{s_3 r_2}{s_2} v_1 + \frac{s_3 r_2}{s_1} v_2,$$

$$\rho(x_2, x_3)(v_1) = \frac{s_2 r_1 r_3}{s_1 r_2} v_1 + \frac{s_2^2 r_1 r_3}{s_1^2 r_2} v_2, \quad \rho(x_2, x_3)(v_2) = r_3 v_1 + \frac{s_2 r_3}{s_1} v_2,$$

$$\nu(x_1)(v_1, v_2) = 0, \quad \nu(x_2)(v_1, v_2) = s_1 v_1 + s_2 v_2, \quad \nu(x_3)(v_1, v_2) = s_3 v_1 + \frac{s_2 s_3}{s_1} v_2,$$

$$\omega(x_1, x_2, x_3) = \frac{r_2}{s_2} v_1 - \frac{r_1}{s_1} v_2,$$

where r_i and s_j are parameters in $\mathbb{K}$.

Notice that the map ρ is not a representation in the usual sense (identities Eqs. 11 and 12 are not satisfied). It becomes a generalized representation if and only if $r_1 = r_2 = 0$.

Now, we describe non-split abelian extensions using Maurer-Cartan elements. The set $MC(L)$ of **Maurer-Cartan elements** of a DGLA $(L, [\cdot, \cdot], d)$ is defined by

$$MC(L) \triangleq \left\{ P \in L_1 \mid dP + \frac{1}{2}[P, P] = 0 \right\}.$$

Let $(\mathfrak{g}, [\cdot, \cdot, \cdot]_{\mathfrak{g}})$ be a 3-Lie algebra and V a vector space. Let $\mathfrak{g} \oplus V$ be the 3-Lie algebra direct sum of $\mathfrak{g}$ and V, where the bracket is defined by $[x+u, y+v, z+w] = [x, y, z]_{\mathfrak{g}}$. Then there is a DGLA $(C(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V), [\cdot, \cdot, \cdot]^{3\text{Lie}}, \delta)$, where $C(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V) = \oplus_{p \geq 0} C^p(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V)$ and δ is the coboundary operator for the 3-Lie algebra $\mathfrak{g} \oplus V$ with the coefficients in the adjoint representation. It is not difficult to see that $(C_>(\mathfrak{g} \oplus V, V), [\cdot, \cdot, \cdot]^{3\text{Lie}}, \delta)$ is a sub-DGLA of $(C(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V), [\cdot, \cdot, \cdot]^{3\text{Lie}}, \delta)$, where $C_>(\mathfrak{g} \oplus V, V) = \oplus_{p \geq 0} C_>^p(\mathfrak{g} \oplus V, V)$ is defined by Eq. 18.

Proposition 5.5 *The following two statements are equivalent:*

(a) $(\mathfrak{g} \oplus V, [\cdot, \cdot, \cdot]_{(\rho, \nu, \omega)})$ *is a 3-Lie algebra, which is a non-split abelian extension of* $\mathfrak{g}$ *by* V;

(b) $\bar{\rho} + \bar{\nu} + \omega$ is a Maurer-Cartan element of the DGLA $(C_>(\mathfrak{g} \oplus V, V), [\cdot, \cdot, \cdot]^{3\text{Lie}}, \delta)$.

Proof By Lemma 2.5, $(\mathfrak{g} \oplus V, [\cdot, \cdot, \cdot]_{(\rho, \nu, \omega)})$ is a 3-Lie algebra if and only if

$$[\pi + \bar{\rho} + \bar{\nu} + \omega, \pi + \bar{\rho} + \bar{\nu} + \omega]^{3\text{Lie}} = 0,$$

which can be rewritten as

$$[\pi, \bar{\rho} + \bar{\nu} + \omega]^{3\text{Lie}} + \frac{1}{2}[\bar{\rho} + \bar{\nu} + \omega, \bar{\rho} + \bar{\nu} + \omega]^{3\text{Lie}} = 0.$$

Since $[\pi, \bar{\rho} + \bar{\nu} + \omega]^{3\text{Lie}} = \delta(\bar{\rho} + \bar{\nu} + \omega)$, we have

$$\delta(\bar{\rho} + \bar{\nu} + \omega) + \frac{1}{2}[\bar{\rho} + \bar{\nu} + \omega, \bar{\rho} + \bar{\nu} + \omega]^{3\text{Lie}} = 0,$$

which means that $\bar{\rho} + \bar{\nu} + \omega$ is a Maurer-Cartan element. □

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