Algebr Represent Theor (2017) 20:487–529
DOI 10.1007/s10468-016-9652-1

Gorenstein Homological Aspects of Monomorphism Categories via Morita Rings

Nan Gao[1] · **Chrysostomos Psaroudakis**[2]

Received: 3 July 2016 / Accepted: 4 October 2016 / Published online: 3 November 2016

Abstract In this paper we construct Gorenstein-projective modules over Morita rings with zero bimodule homomorphisms and we provide sufficient conditions for such rings to be Gorenstein Artin algebras. This is the first part of our work which is strongly connected with monomorphism categories. In the second part, we investigate monomorphisms where the domain has finite projective dimension. In particular, we show that the latter category is a Gorenstein subcategory of the monomorphism category over a Gorenstein algebra. Finally, we consider the category of coherent functors over the stable category of this Gorenstein subcategory and show that it carries a structure of a Gorenstein abelian category.

Keywords Monomorphism categories · Morita rings · Homological embeddings · Gorenstein artin algebras · Gorenstein-projective modules · Gorenstein (sub)categories · Coherent functors

Mathematics Subject Classification (2010) 16E10 · 16E65 · 16G · 16G50 · 16S50

Presented by Jon F. Carlson.

✉ Chrysostomos Psaroudakis
chpsaroud@gmail.com

Nan Gao
nangao@shu.edu.cn

[1] Department of Mathematics, Shanghai University, Shanghai, 200444, People's Republic of China

[2] Department of Mathematical Sciences, Norwegian University of Science and Technology, Trondheim, 7491, Norway

1 Introduction and Main Results

This article deals with Gorenstein homological aspects of Morita rings with zero bimodule homomorphisms and monomorphism categories. The class of Morita rings is a natural generalization on the one hand of triangular matrix rings and on the other hand it covers many trivial extension rings [22]. In what follows, we first give some motivation and present the main result in the paper.

For an abelian category $\mathscr{A}$ we denote by $\mathsf{Mor}\,\mathscr{A}$ the category of morphisms over $\mathscr{A}$. The monomorhism category $\mathsf{Mon}\,\mathscr{A}$ is by definition the full subcategory of $\mathsf{Mor}\,\mathscr{A}$ consisting of all monomorphisms in $\mathscr{A}$. If R is a ring and $\mathscr{A}$ is the category $\mathsf{Mod}\text{-}R$ of left R-modules, then the category of monomorphisms $\mathsf{Mon}\,(\mathsf{Mod}\text{-}R)$ can be considered as a full subcategory of the module category $\mathsf{Mod}\text{-}\mathsf{T}_2(R)$, where $\mathsf{T}_2(R) = \left(\begin{smallmatrix} R & R \\ 0 & R \end{smallmatrix}\right)$, since it is known that the morphism category $\mathsf{Mor}\,(\mathsf{Mod}\text{-}R)$ is equivalent to $\mathsf{Mod}\text{-}\mathsf{T}_2(R)$. Note that $\mathsf{Mon}\,\mathscr{A}$ is an extension closed subcategory of the abelian category $\mathsf{Mor}\,\mathscr{A}$ and therefore is an exact category in the sense of Quillen. Monomorphism categories appear quite naturally in various settings and are omnipresent in representation theory. In fact, there are connections with classification problems (Ringel, Schmidmeier [40–42], Xiong, Zhang, Zhang [45]), with weighted projective lines (Kussin, Lenzing, Meltzer [30]), and with aspects of Gorenstein homological algebra (Beligiannis [10, 11], Zhang [48], Chen [17]). In a series of papers [31, 46, 47], the Gorenstein-projective modules over the triangular matrix algebra $\left(\begin{smallmatrix} A & N \\ 0 & B \end{smallmatrix}\right)$ were determined under some conditions on the bimodule ${}_AN_B$. In the special case where $A = N = B$ and A is a Gorenstein Artin algebra, i.e. $\mathsf{T}_2(A) = \left(\begin{smallmatrix} A & A \\ 0 & A \end{smallmatrix}\right)$, the authors in [31] showed that a module over $\mathsf{T}_2(A)$, i.e. a triple (X, Y, f), is Gorenstein-projective if and only if (X, Y, f) belongs to the monomorphism category $\mathsf{Mon}(\mathsf{mod}\text{-}A)$ and the modules X, Y and $\mathsf{Coker}\, f$ are Gorenstein-projective.

A natural extension of triangular matrix rings is the class of Morita rings. Recall that Morita rings are 2×2 matrix rings associated to Morita contexts ([6, 21]). We refer to [33, 43] for the terminology of Morita rings, and to [25] for a thorough discussion of Morita rings as well as examples and situations where Morita rings appear. A particular case of interest is the Morita ring $\Delta_{(0,0)}$ with entries the same associative unital ring R and bimodule homomorphisms zero. The reason is that there is a full embedding $\mathsf{Mod}\text{-}\mathsf{T}_2(R) \to \mathsf{Mod}\text{-}\Delta_{(0,0)}$. The main problem considered in this paper is:

Problem Construct Gorenstein-projective modules over Morita rings with zero bimodule homomorphisms.

The solution of this problem provides a link between monomorphism categories and Morita rings (Section 2.3). This problem and the important role that monomorphism categories as well as Morita rings play in different contexts, provide a strong motivation for studying these using homological and representation-theoretic tools. Our aim in this paper is two-fold and can be summarized as follows:

(i) Solve the above problem and provide sufficient conditions for such rings to be Gorenstein algebras.
(ii) Construct Gorenstein abelian categories from exact subcategories of the monomorphism category.

The organization and the main results of the paper are as follows. In Section 2 we collect preliminary notions and results on Morita rings and monomorphism categories that will be useful throughout the paper and we fix notation. Moreover we introduce the double morphism category of an abelian category and we define the monomorphism category in this general setting. The rest of the paper is divided into two parts.

The first part consists of Sections 3 and 4. For an Artin algebra Λ, we denote by $\mathsf{Gproj}\,\Lambda$ the category of all finitely generated Gorenstein-projective Λ-modules. Let $\Lambda_{(0,0)} = \left(\begin{smallmatrix} A & {}_AN_B \\ {}_BM_A & B \end{smallmatrix}\right)$ be a Morita ring which is an Artin algebra and has zero bimodule homomorphisms. In order to construct Gorenstein-projective modules over $\Lambda_{(0,0)}$, we need to assume some natural conditions on the bimodules ${}_AN_B$ and ${}_BM_A$, similar to the conditions considered by Zhang [47] in the triangular matrix case. We refer to these conditions as the *compatibility conditions* on ${}_BM_A$ and ${}_AN_B$ (Section 3.1). These conditions have a nice interpretation via finiteness of the projective dimension of the bimodules N and M (Corollary 3.11). Our first main result is Theorem A (i), which provides a method to construct Gorenstein-projective modules over Morita rings with zero bimodule homomorphisms. We refer to Theorem 3.10 for the proof as well as its dual version. Moreover, we give sufficient conditions for a Morita ring $\Lambda_{(0,0)}$ with zero bimodule homomorphisms to be a Gorenstein Artin algebra. This constitutes our second main result and is Theorem A (ii), see Theorem 4.13. Recall that Gorensteinness of an algebra Λ is determined by the finiteness of the dimensions $\mathsf{spli}\,\Lambda$ and $\mathsf{silp}\,\Lambda$ (Section 4.2). Our second main result is closely related to the property of the natural embeddings $\mathsf{Mod}\text{-}B \longrightarrow \mathsf{Mod}\text{-}\Lambda_{(0,0)}$ and $\mathsf{Mod}\text{-}A \longrightarrow \mathsf{Mod}\text{-}\Lambda_{(0,0)}$ being homological. In this connection, we characterize the Morita rings such that the above functors are homological embeddings (Proposition 4.1). Our main results in this part are summarized in the following theorem.

Theorem A Let $\Lambda_{(0,0)}$ be a Morita ring which is an Artin algebra and has zero bimodule homomorphisms.

(i) (Theorem 3.10: **Gorenstein-projectives**) Assume that the bimodules ${}_BM_A$ and ${}_AN_B$ satisfy the compatibility conditions (see 3.1) Let Z be a Gorenstein-projective B-module with a monomorphism $s\colon N \otimes_B Z \longrightarrow X$, for some A-module X, such that $\mathsf{Coker}\,s$ lies in $\mathsf{Gproj}\,A$ and there is a monomorphism $t\colon M \otimes_A \mathsf{Coker}\,s \longrightarrow Y$ with $\mathsf{Coker}\,t = Z$ and Y an B-module. We set π_X, resp. π_Y, for the map $M \otimes_A X \longrightarrow \mathsf{Coker}\,s$, resp. $N \otimes_B Y \longrightarrow \mathsf{Coker}\,t$. Then the tuple:

$$\big(X, Y, (\mathrm{Id}_M \otimes \pi_X) \circ t, (\mathrm{Id}_N \otimes \pi_Y) \circ s\big) \in \mathsf{Gproj}\,\Lambda_{(0,0)}$$

(ii) (Theorem 4.13: **Gorenstein algebras**) Assume that the following conditions hold:

(a) M_A is projective and $\mathsf{pd}\,{}_BM < \infty$.
(b) N_B is projective and $\mathsf{pd}\,{}_AN < \infty$.
(c) The functors Z_A and Z_B are homological embeddings.

If $\mathsf{silp}\,A < \infty$ and $\mathsf{silp}\,B < \infty$, then $\mathsf{silp}\,\Lambda_{(0,0)} < \infty$.

As an application of Theorem A (i), we construct Gorenstein-projective modules over the Morita ring $\Delta_{(0,0)}$ that lie in the monomorphism category $\mathsf{mono}(\Lambda)$ (Corollary 3.6). Also, from Theorem A (ii) we get examples of Morita rings which are Gorenstein algebras (Corollary 4.15).

In the second part, which is Section 5, we study the subcategory $\mathscr{C}$ of $\mathsf{mono}(\Lambda)$, where Λ is an Artin algebra, consisting of all monomorphisms $f\colon X \longrightarrow Y$ such that the projective dimension of X is finite. Our third main result is Theorem B (i) where assuming that Λ is Gorenstein we show that $\mathscr{C}$ is a Gorenstein subcategory of $\mathsf{mono}(\Lambda)$, see Definition 5.2 and

Theorem 5.3. Moreover, inspired by recent work of Matsui and Takahashi [32] we consider the category of coherent functors $\mathsf{mod}\text{-}\underline{\mathscr{C}}$ over the stable category $\underline{\mathscr{C}}$ of $\mathscr{C}$. Also, we define the subcategory $\Omega^n(\mathscr{C})$ of $\mathscr{C}$ consisting of all nth syzygies of objects in $\mathscr{C}$ (Section 5.2). In this context, the fourth main result of this paper is Theorem B (ii), see Corollary 5.8, which shows that the category of coherent functors over $\underline{\mathscr{C}}$ is a Gorenstein abelian category in the sense of [12]. Finally, using a result of Beligiannis [8] we realize the singularity category [32] of $\mathsf{mod}\text{-}\underline{\mathscr{C}}$ as the stable category of Cohen-Macaulay objects over $\mathsf{mod}\text{-}\underline{\mathscr{C}}$.

Theorem B (**Gorenstein categories**) Let Λ be an n-Gorenstein Artin algebra.

(i) (Theorem 5.3) The category $\mathscr{C} = \{(X, Y, f, 0) \in \mathsf{mono}(\Lambda) \mid \mathsf{pd}_\Lambda X < \infty\}$ is an n-Gorenstein subcategory of $\mathsf{mono}(\Lambda)$.

(ii) (Corollary 5.8) For the category of coherent functors over $\underline{\mathscr{C}}$ and $\underline{\Omega^n(\mathscr{C})}$ the following statements hold:

(a) $\mathsf{mod}\text{-}\underline{\mathscr{C}}$ is a $3n$-Gorenstein abelian category.

(b) $\mathsf{mod}\text{-}\underline{\Omega^n(\mathscr{C})}$ is a Frobenius abelian category.

Moreover, there are the following triangle equivalences:

$$\mathsf{D}_{\mathsf{sg}}(\mathsf{mod}\text{-}\underline{\mathscr{C}}) \xrightarrow{\simeq} \underline{\mathsf{Gproj}}(\mathsf{mod}\text{-}\underline{\mathscr{C}}) \quad \text{and} \quad \mathsf{D}_{\mathsf{sg}}(\mathsf{mod}\text{-}\underline{\Omega^n(\mathscr{C})}) \xrightarrow{\simeq} \underline{\mathsf{mod}\text{-}\underline{\Omega^n(\mathscr{C})}}$$

Statement (ii) above is a consequence of Theorem 5.6 which provides sufficient conditions on a subcategory $\mathscr{B}$ of an exact category $\mathscr{A}$ with enough projectives such that $\mathsf{mod}\text{-}\underline{\mathscr{B}}$ is a Gorenstein abelian category. It should be noted that this result generalizes, and is inspired by, a result of Matsui and Takahashi [32].

Conventions and Notation We compose morphisms in a given category in a diagrammatic order. Our subcategories are assumed to be closed under isomorphisms and direct summands. For a ring R we usually work with left R-modules and the corresponding category is denoted by $\mathsf{Mod}\text{-}R$. By a module over an Artin algebra Λ, we mean a finitely generated left Λ-module and we denote by $\mathsf{mod}\text{-}\Lambda$ the category of finitely generated left Λ-modules. In this paper, for simplicity we work over Artin algebras. All Morita rings in Sections 3, 4 and 5 are Artin algebras. For all unexplained notions and results concerning the representation theory of Artin algebras we refer to [5].

2 Morita Rings and Monomorphism Categories

In this section we fix notation and we collect several preliminary results on Morita rings and monomorphism categories that will be used throughout the paper.

2.1 Morita Rings

Let A and B be two rings, ${}_AN_B$ an A-B-bimodule, ${}_BM_A$ a B-A-bimodule, and $\phi\colon M \otimes_A N \longrightarrow B$ a B-B-bimodule homomorphism, and $\psi\colon N \otimes_B M \longrightarrow A$ an A-A-bimodule homomorphism. Then from the Morita context $\mathcal{M} = (A, N, M, B, \phi, \psi)$ we define the Morita ring:

$$\Lambda_{(\phi,\psi)} = \begin{pmatrix} A & {}_AN_B \\ {}_BM_A & B \end{pmatrix}$$

where the addition of elements of $\Lambda_{(\phi,\psi)}$ is componentwise and multiplication is given by

$$\begin{pmatrix} a & n \\ m & b \end{pmatrix} \cdot \begin{pmatrix} a' & n' \\ m' & b' \end{pmatrix} = \begin{pmatrix} aa' + \psi(n \otimes m') & an' + nb' \\ ma' + bm' & bb' + \phi(m \otimes n') \end{pmatrix}$$

We assume that $\phi(m \otimes n)m' = m\psi(n \otimes m')$ and $n\phi(m \otimes n') = \psi(n \otimes m)n'$ for all $m, m' \in M$ and $n, n' \in N$. This condition ensures that $\Lambda_{(\phi,\psi)}$ is an associative ring.

The description of the modules over a Morita ring $\Lambda_{(\phi,\psi)}$ is well known, see for instance [24], but for completeness and due to our needs we also include it here. We introduce the following category.

Let $\mathcal{M}(\Lambda)$ be the category whose objects are tuples (X, Y, f, g) where $X \in \mathsf{Mod}\text{-}A$, $Y \in \mathsf{Mod}\text{-}B$, $f \in \mathsf{Hom}_B(M \otimes_A X, Y)$ and $g \in \mathsf{Hom}_A(N \otimes_B Y, X)$ such that the following diagrams are commutative:

$$\begin{array}{ccc} N \otimes_B M \otimes_A X & \xrightarrow{\mathrm{Id}_N \otimes f} & N \otimes_B Y \\ \big\downarrow{\scriptstyle \psi \otimes \mathrm{Id}_X} & & \big\downarrow{\scriptstyle g} \\ A \otimes_A X & \xrightarrow{\cong} & X \end{array} \qquad \begin{array}{ccc} M \otimes_A N \otimes_B Y & \xrightarrow{\mathrm{Id}_M \otimes g} & M \otimes_A X \\ \big\downarrow{\scriptstyle \phi \otimes \mathrm{Id}_Y} & & \big\downarrow{\scriptstyle f} \\ B \otimes_B Y & \xrightarrow{\cong} & Y \end{array} \tag{2.1}$$

We denote by Ψ_X and Φ_Y the following compositions:

$$\Psi_X : N \otimes_B M \otimes_A X \xrightarrow{\psi \otimes \mathrm{Id}_X} A \otimes_A X \xrightarrow{\cong} X \qquad \Phi_Y : M \otimes_A N \otimes_B Y \xrightarrow{\phi \otimes \mathrm{Id}_Y} B \otimes_B Y \xrightarrow{\cong} Y$$

Let (X, Y, f, g) and (X', Y', f', g') be objects of $\mathcal{M}(\Lambda)$. Then a morphism $(X, Y, f, g) \longrightarrow (X', Y', f', g')$ in $\mathcal{M}(\Lambda)$ is a pair of homomorphisms (a, b), where $a\colon X \longrightarrow X'$ is an A-morphism and $b\colon Y \longrightarrow Y'$ is a B-morphism, such that the following diagrams are commutative:

$$\begin{array}{ccc} M \otimes_A X & \xrightarrow{f} & Y \\ \big\downarrow{\scriptstyle \mathrm{Id}_M \otimes a} & & \big\downarrow{\scriptstyle b} \\ M \otimes_A X' & \xrightarrow{f'} & Y' \end{array} \qquad \begin{array}{ccc} N \otimes_B Y & \xrightarrow{g} & X \\ \big\downarrow{\scriptstyle \mathrm{Id}_N \otimes b} & & \big\downarrow{\scriptstyle a} \\ N \otimes_B Y' & \xrightarrow{g'} & X' \end{array}$$

The relationship between $\mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$ and $\mathcal{M}(\Lambda)$ is given via the functor $\mathsf{F}\colon \mathcal{M}(\Lambda) \longrightarrow \mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$ which is defined on objects (X, Y, f, g) of $\mathcal{M}(\Lambda)$ as follows: $\mathsf{F}(X, Y, f, g) = X \oplus Y$ as abelian groups, with a $\Lambda_{(\phi,\psi)}$-module structure given by $\left(\begin{smallmatrix} a & n \\ m & b \end{smallmatrix}\right)(x, y) = (ax + g(n \otimes y), by + f(m \otimes x))$, for all $a \in A, b \in B, n \in N, m \in M, x \in X$ and $y \in Y$. If $(a, b)\colon (X, Y, f, g) \longrightarrow (X', Y', f', g')$ is a morphism in $\mathcal{M}(\Lambda)$ then $\mathsf{F}(a, b) = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)\colon X \oplus Y \longrightarrow X' \oplus Y'$. Then the functor F turns out to be an equivalence of categories, see [24, Theorem 1.5]. From now on we identify the modules over $\Lambda_{(\phi,\psi)}$ with the objects of $\mathcal{M}(\Lambda)$.

Throughout the paper we deal mainly with Morita rings which are Artin algebras. Then it is easy to observe that, see [25, Proposition 2.2], a Morita ring $\Lambda_{(\phi,\psi)}$ is an Artin algebra if and only if there is a commutative artin ring R such that A and B are Artin R-algebras and M and N are finitely generated over R which acts centrally both on M and N. We summarize in the next remark some properties of $\mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$ that we need in the sequel. We refer to [35, Chapter 3] for a thorough discussion on the abelian structure of Morita rings in a more general setting.

Remark 2.1 Let $\Lambda_{(\phi,\psi)} = \left(\begin{smallmatrix} A & {}_AN_B \\ {}_BM_A & B \end{smallmatrix}\right)$ be a Morita ring.

(i) A sequence of tuples $0 \longrightarrow (X_1, Y_1, f_1, g_1) \longrightarrow (X_2, Y_2, f_2, g_2) \longrightarrow (X_3, Y_3, f_3, g_3) \longrightarrow 0$ is exact in $\mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$ if and only if the sequences $0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow 0$ and $0 \longrightarrow Y_1 \longrightarrow Y_2 \longrightarrow Y_3 \longrightarrow 0$ are exact in $\mathsf{Mod}\text{-}A$ and $\mathsf{Mod}\text{-}B$ respectively.

(ii) Let $(a, b)\colon (X, Y, f, g) \longrightarrow (X', Y', f', g')$ be a morphism in $\mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$ and consider the maps $c\colon \mathsf{Ker}\, a \longrightarrow X$ and $d\colon \mathsf{Ker}\, b \longrightarrow Y$. Then the kernel of (a, b) is the object $(\mathsf{Ker}\, a, \mathsf{Ker}\, b, h, j)$ where the maps h and j are induced from the following commutative diagrams:

$$\begin{array}{ccccc} M\otimes_A \mathsf{Ker}\, a & \xrightarrow{\mathrm{Id}_M\otimes c} & M\otimes_A X & \xrightarrow{\mathrm{Id}_M\otimes a} & M\otimes_A X' \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ \mathsf{Ker}\, b & \xrightarrow{d} & Y & \xrightarrow{b} & Y' \end{array} \qquad \begin{array}{ccccc} N\otimes_B \mathsf{Ker}\, b & \xrightarrow{\mathrm{Id}_N\otimes d} & N\otimes_B Y & \xrightarrow{\mathrm{Id}_N\otimes b} & N\otimes_B Y' \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ \mathsf{Ker}\, a & \xrightarrow{c} & X & \xrightarrow{a} & X' \end{array} \tag{2.2}$$

Similarly, we derive a description for the cokernel of the morphism (a, b).

As in [25] we define the following functors:

(i) The functor $\mathsf{T}_A\colon \mathsf{Mod}\text{-}A \longrightarrow \mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$ is defined by $\mathsf{T}_A(X) = (X, M\otimes_A X, \mathrm{Id}_{M\otimes X}, \Psi_X)$ on the objects $X \in \mathsf{Mod}\text{-}A$ and given an A-morphism $a\colon X \longrightarrow X'$ then $\mathsf{T}_A(a) = (a, \mathrm{Id}_M \otimes a)$.

(ii) The functor $\mathsf{U}_A\colon \mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)} \longrightarrow \mathsf{Mod}\text{-}A$ is defined by $\mathsf{U}_A(X, Y, f, g) = X$ on the objects $(X, Y, f, g) \in \mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$ and given a morphism $(a, b)\colon (X, Y, f, g) \longrightarrow (X', Y', f', g')$ in $\mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$ then $\mathsf{U}_A(a, b) = a$.

(iii) The functor $\mathsf{T}_B\colon \mathsf{Mod}\text{-}B \longrightarrow \mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$ is defined by $\mathsf{T}_B(Y) = (N\otimes_B Y, Y, \Phi_Y, \mathrm{Id}_{N\otimes Y})$ on the objects $Y \in \mathsf{Mod}\text{-}B$ and given a B-morphism $b\colon Y \longrightarrow Y'$ then $\mathsf{T}_B(b) = (\mathrm{Id}_N \otimes b, b)$.

(iv) The functor $\mathsf{U}_B\colon \mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)} \longrightarrow \mathsf{Mod}\text{-}B$ is defined by $\mathsf{U}_B(X, Y, f, g) = Y$ on the $\Lambda_{(\phi,\psi)}$-modules (X, Y, f, g) and given a $\Lambda_{(\phi,\psi)}$-morphism $(a, b)\colon (X, Y, f, g) \longrightarrow (X', Y', f', g')$ then $\mathsf{U}_B(a, b) = b$.

(v) The functor $\mathsf{H}_A\colon \mathsf{Mod}\text{-}A \longrightarrow \mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$ is defined by $\mathsf{H}_A(X) = (X, \mathsf{Hom}_A(N, X), \delta'_{M\otimes X} \circ \mathsf{Hom}_A(N, \Psi_X), \epsilon'_X)$ on the objects $X \in \mathsf{Mod}\text{-}A$ and given an A-morphism $a\colon X \longrightarrow X'$ then $\mathsf{H}_A(a) = (a, \mathsf{Hom}_A(N, a))$.

(vi) The functor $\mathsf{H}_B\colon \mathsf{Mod}\text{-}B \longrightarrow \mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$ is defined by $\mathsf{H}_B(Y) = (\mathsf{Hom}_B(M, Y), Y, \epsilon_Y, \delta_{N\otimes Y} \circ \mathsf{Hom}_B(M, \Phi_Y))$ on the objects $Y \in \mathsf{Mod}\text{-}B$ and given a B-morphism $b\colon Y \longrightarrow Y'$ then $\mathsf{H}_B(b) = (\mathsf{Hom}_B(M, b), b)$.

(vii) Suppose that $\phi = 0 = \psi$. Then we define the functor $\mathsf{Z}_A\colon \mathsf{Mod}\text{-}A \longrightarrow \mathsf{Mod}\text{-}\Lambda_{(0,0)}$ by $\mathsf{Z}_A(X) = (X, 0, 0, 0)$ on the objects $X \in \mathsf{Mod}\text{-}A$ and if $a\colon X \longrightarrow X'$ is an A-morphism then $\mathsf{Z}_A(a) = (a, 0)$. Dually we define the functor $\mathsf{Z}_B\colon \mathsf{Mod}\text{-}B \longrightarrow \mathsf{Mod}\text{-}\Lambda_{(0,0)}$.

When a Morita ring is an Artin algebra we have the following description of the indecomposable projective and injective modules.

Proposition 2.2 [25, Propositions 3.1 and 3.2] *Let $\Lambda_{(\phi,\psi)}$ be a Morita ring. Then the indecomposable projective $\Lambda_{(\phi,\psi)}$-modules are objects of the form*:

$$\begin{cases} \mathsf{T}_A(P) = (P, M \otimes_A P, \mathrm{Id}_{M\otimes_A P}, \Psi_P) \\ \mathsf{T}_B(Q) = (N \otimes_B Q, Q, \Phi_Q, \mathrm{Id}_{N\otimes_B Q}) \end{cases}$$

where P is an indecomposable projective A-module and Q is an indecomposable projective B-module. Also, the indecomposable injective $\Lambda_{(\phi,\psi)}$-modules are objects of the form:

$$\begin{cases} \mathsf{H}_A(I) = (I, \mathsf{Hom}_A(N, I), \delta'_{M\otimes I} \circ \mathsf{Hom}_A(N, \Psi_I), \epsilon'_I) \\ \mathsf{H}_B(J) = (\mathsf{Hom}_B(M, J), J, \epsilon_J, \delta_{N\otimes J} \circ \mathsf{Hom}_B(M, \Phi_J)) \end{cases}$$

where I is an indecomposable injective A-module and J is an indecomposable injective B-module.

We continue now with examples of Morita rings which will be used in the sequel.

Example 2.3 (i) Let R be a ring with an idempotent element e. Then, from the Pierce decomposition of R with respect to the idempotents e and $f = 1_R - e$, it follows that R is the Morita ring with $A = eRe$, $B = fRf$, $N = eRf$, $M = fRe$ and the bimodule homomorphisms ϕ, ψ are induced by the multiplication in R.

(ii) Any pair (A, P_A), where A is a ring and P_A is a right A-module, induces a Morita ring as follows:

$$\Lambda_{(\phi,\psi)} = \begin{pmatrix} A & \mathsf{Hom}_A(P, A) \\ P & \mathsf{End}_A(P) \end{pmatrix}$$

with bimodule homomorphisms $\phi\colon P \otimes_A \mathsf{Hom}_A(P, A) \longrightarrow \mathsf{End}_A(P)$, $p \otimes f \mapsto \phi(p \otimes f)(p') = pf(p')$ and $\psi\colon \mathsf{Hom}_A(P, A) \otimes_{\mathsf{End}_A(P)} P \longrightarrow A$, $f \otimes p \mapsto \psi(f \otimes p) = f(p)$. It is well known that if the A-module P is a progenerator, then the rings A and $\mathsf{End}_A(P)$ are Morita equivalent.

(iii) Let $\Lambda_{(0,0)} = \left(\begin{smallmatrix} A & {}_AN_B \\ {}_BM_A & B \end{smallmatrix}\right)$ be a Morita ring with ϕ and ψ zero. Then we have an isomorphism of rings between $\Lambda_{(0,0)}$ and $(A \times B) \ltimes M \oplus N$, where $(A \times B) \ltimes M \oplus N$ is the trivial extension ring of $A \times B$ by the $(A \times B)$-$(A \times B)$-bimodule $M \oplus N$. For the notion of trivial extension of rings and for the above isomorphism we refer to [22], see also [25, Proposition 2.5].

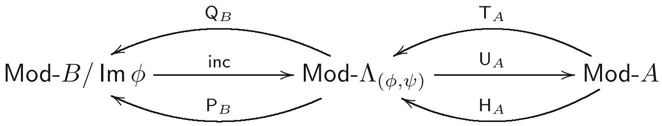
(iv) Suppose that we have the following Morita ring:

$$\Lambda_{(\phi,\psi)} = \begin{pmatrix} A & A \\ A & A \end{pmatrix}$$

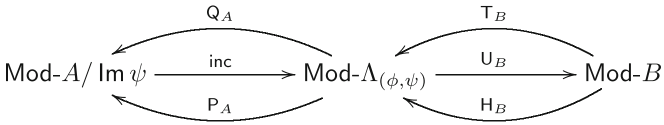
where every entry is a ring A. Then, it follows from the associativity of the multiplication that $\phi = \psi$, see [25, Corollary 2.13] for more details. A special case is when $\phi = 0$, that is $\Delta_{(0,0)} := \left(\begin{smallmatrix} \Lambda & \Lambda \\ \Lambda & \Lambda \end{smallmatrix}\right) \cong (\Lambda \times \Lambda) \ltimes \Lambda \oplus \Lambda$. In the next subsection we analyze the module category of $\Delta_{(0,0)}$ via recollements of abelian categories.

We close this subsection with the next result, which shows that always a Morita ring gives rise to a recollement situation. This provides a way to relate the module category of a Morita ring with the module categories of its underlying rings. For the proof see [25, Proposition 2.4] and for more details on recollements of abelian categories we refer to [23, 36].

Proposition 2.4 *Let $\Lambda_{(\phi,\psi)}$ be a Morita ring. Then the following diagrams:*



and

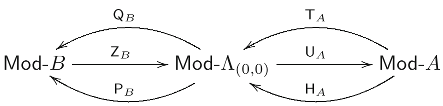


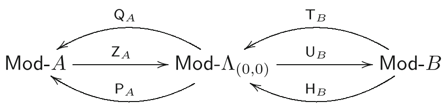
are recollements of abelian categories, that is:

(*i*) $(\mathsf{T}_A, \mathsf{U}_A, \mathsf{H}_A)$ *is an adjoint triple.*
(*ii*) *The functors* T_A *and* H_A *are fully faithful.*
(*iii*) $\operatorname{Ker}\mathsf{U}_A = \text{Mod-}B/\operatorname{Im}\phi$.

(*i*) $(\mathsf{T}_B, \mathsf{U}_B, \mathsf{H}_B)$ *is an adjoint triple.*
(*ii*) *The functors* T_B *and* H_B *are fully faithful.*
(*iii*) $\operatorname{Ker}\mathsf{U}_B = \text{Mod-}A/\operatorname{Im}\psi$.

In particular, if $\phi = 0 = \psi$ then we have the following recollements of module categories:





Our aim next is to analyze the recollement of the module category of the Morita ring $\Delta_{(0,0)} = \left(\begin{smallmatrix} R & R \\ R & R \end{smallmatrix}\right)$, where R is a unital associative ring. For this reason, we introduce in the next subsection the double morphism category of an abelian category. This construction can be considered as an abstract model for the category of modules over $\Delta_{(0,0)}$.

2.2 The Double Morphism Category

Let $\mathscr{A}$ be an abelian category. The **double morphism category** of $\mathscr{A}$, denoted by $\mathsf{DMor}(\mathscr{A})$, has as objects diagrams of the form:

$$X \underset{f}{\overset{g}{\leftrightarrows}} Y$$

where f and g are morphisms in $\mathscr{A}$ such that $f \circ g = 0$ and $g \circ f = 0$. We simply denote the objects as tuples (X, Y, f, g). A morphism $(X, Y, f, g) \longrightarrow (X', Y', f', g')$ in $\mathsf{DMor}(\mathscr{A})$

is a pair (a, b) of morphisms in $\mathscr{A}$, where $a\colon X \longrightarrow X'$ and $b\colon Y \longrightarrow Y'$, such that the following diagram commutes:

$$\begin{array}{ccc} X & \overset{g}{\underset{f}{\rightleftarrows}} & Y \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\ X' & \overset{g'}{\underset{f'}{\rightleftarrows}} & Y' \end{array}$$

that is, $g \circ a = b \circ g'$ and $f \circ b = a \circ f'$. We show that the double morphism category $\mathsf{DMor}(\mathscr{A})$ is an abelian category and that there is a recollement which relates $\mathsf{DMor}(\mathscr{A})$ and $\mathscr{A}$. In order to give an abelian structure on $\mathsf{DMor}(\mathscr{A})$, we provide another description of $\mathsf{DMor}(\mathscr{A})$. In particular, we show that there is an equivalence of categories between $\mathsf{DMor}(\mathscr{A})$ and $(\mathscr{A} \times \mathscr{A}) \ltimes H$, where $(\mathscr{A} \times \mathscr{A}) \ltimes H$ is the trivial extension of $\mathscr{A} \times \mathscr{A}$ by an endofunctor H, see Fossum-Griffith-Reiten [22].

We define the functor $H\colon \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{A} \times \mathscr{A}$, $H(X, Y) = (Y, X)$, and given a morphism $(a, b)\colon (X, Y) \longrightarrow (X', Y')$ then $H(a, b) = (b, a)$. Then we can define the trivial extension $(\mathscr{A} \times \mathscr{A}) \ltimes H$, where the objects are morphisms $\alpha\colon H(X, Y) \longrightarrow (X, Y)$ such that $H(\alpha) \circ \alpha = 0$, and if $\alpha\colon H(X, Y) \longrightarrow (X, Y)$ and $\beta\colon H(X', Y') \longrightarrow (X', Y')$ are two objects in $(\mathscr{A} \times \mathscr{A}) \ltimes H$, then a morphism between the objects α and β is a morphism $\gamma\colon (X, Y) \longrightarrow (X', Y')$ such that the diagram

$$\begin{array}{ccc} H(X,Y) & \xrightarrow{H(\gamma)} & H(X',Y') \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ (X,Y) & \xrightarrow{\gamma} & (X',Y') \end{array}$$

is commutative, where $\alpha = (a_1, a_2)$, $\beta = (b_1, b_2)$ and $\gamma = (c_1, c_2)$. Since the endofunctor H is (right) exact, it follows from [22] that the trivial extension $(\mathscr{A} \times \mathscr{B}) \ltimes H$ is an abelian category.

Proposition 2.5 *Let $\mathscr{A}$ be an abelian category.*

(i) *There is an equivalence of categories between $\mathsf{DMor}(\mathscr{A})$ and $(\mathscr{A} \times \mathscr{A}) \ltimes H$. In particular, the double morphism category $\mathsf{DMor}(\mathscr{A})$ is abelian.*
(ii) *There is a recollement of abelian categories:*

$$\begin{array}{ccccc} & \xleftarrow{\mathsf{Q}_{\mathscr{A}}} & & \xleftarrow{\mathsf{T}_{\mathscr{A}}} & \\ \mathscr{A} & \xrightarrow{\mathsf{inc}} & \mathsf{DMor}(\mathscr{A}) & \xrightarrow{\mathsf{U}_{\mathscr{A}}} & \mathscr{A} \\ & \xleftarrow[\mathsf{P}_{\mathscr{A}}]{} & & \xleftarrow[\mathsf{H}_{\mathscr{A}}]{} & \end{array} \tag{2.3}$$

Proof (i) Let (X, Y, f, g) be an object of $\mathsf{DMor}(\mathscr{A})$. We define the functor

$$\mathfrak{F}\colon \mathsf{DMor}(\mathscr{A}) \longrightarrow (\mathscr{A} \times \mathscr{A}) \ltimes H\ ,\ \mathfrak{F}(X,Y,f,g) = H(X,Y) \xrightarrow{(g,f)} (X,Y)$$

and given a morphism $(a, b)\colon (X, Y, f, g) \longrightarrow (X', Y', f', g')$ in $\mathsf{DMor}(\mathscr{A})$ then $\mathcal{F}(a, b) = H(a, b)$. The functor $\mathcal{F}$ is well defined since the following composition

$$H^2(X,Y) \xrightarrow{(f,g)} H(X,Y) \xrightarrow{(g,f)} (X,Y)$$

is zero, i.e. the object $\mathcal{F}(X, Y, f, g)$ lies in $(\mathscr{A} \times \mathscr{A}) \ltimes H$. It is clear that the functor $\mathcal{F}$ is faithful. Let

$$\big[(Y,X) \xrightarrow{(g,f)} (X,Y)\big] \xrightarrow{(a,b)} \big[(Y',X') \xrightarrow{(g',f')} (X',Y')\big]$$

be a morphism in $(\mathscr{A} \times \mathscr{A}) \ltimes H$. Then the following commutative diagram

$$\begin{array}{ccc} (Y,X) & \xrightarrow{(b,a)} & (Y',X') \\ {\scriptstyle (g,f)}\downarrow & & \downarrow{\scriptstyle (g',f')} \\ (X,Y) & \xrightarrow{(a,b)} & (X',Y') \end{array}$$

implies that $(a, b)\colon (X, Y, f, g) \longrightarrow (X', Y', f', g')$ is a morphism in $\mathsf{DMor}(\mathscr{A})$ and $\mathcal{F}(a, b) = H(a, b)$. Thus the functor $\mathcal{F}$ is full. Finally, if $(a_1, a_2)\colon H(X, Y) \longrightarrow (X, Y)$ is an object of $(\mathscr{A} \times \mathscr{A}) \ltimes H$, then since $H(a_1, a_2) \circ (a_1, a_2) = 0$ we infer that $(X, Y, a_2, a_1) \in \mathsf{DMor}(\mathscr{A})$ such that $\mathcal{F}(X, Y, a_2, a_1) = (a_1, a_2)$. This shows that the functor $\mathcal{F}$ is essentially surjective. Hence, the categories $\mathsf{DMor}(\mathscr{A})$ and $(\mathscr{A} \times \mathscr{A}) \ltimes H$ are equivalent and therefore the double morphism category $\mathsf{DMor}(\mathscr{A})$ is abelian.

(ii) The functors appearing in diagram (2.3) were defined in Section 2.1 for the module category of a Morita ring. In this case, if A is an object in $\mathscr{A}$ then $\mathsf{T}_{\mathscr{A}}(A) = (A, A, \mathrm{Id}_A, 0)$, $\mathsf{H}_{\mathscr{A}}(A) = (A, A, 0, \mathrm{Id}_A)$ and for a tuple (X, Y, f, g) in $\mathsf{DMor}(\mathscr{A})$ we have $\mathsf{U}_{\mathscr{A}}(X, Y, f, g) = X$. Similarly with Section 2.1, we get a description of these functors on morphisms. Then, it is easy to check that $(\mathsf{T}_{\mathscr{A}}, \mathsf{U}_{\mathscr{A}}, \mathsf{H}_{\mathscr{A}})$ is an adjoint triple with $\mathsf{T}_{\mathscr{A}}$ (equivalently, $\mathsf{H}_{\mathscr{A}}$) fully faithful and the kernel of $\mathsf{U}_{\mathscr{A}}$ is equivalent with $\mathscr{A}$, see also Proposition 2.4. We infer that $(\mathscr{A}, \mathsf{DMor}(\mathscr{A}), \mathscr{A})$ is a recollement of abelian categories. □

In the following remark we collect some properties of the recollement diagram (2.3).

Remark 2.6 Let $\mathsf{DMor}(\mathscr{A})$ be the double morphism category of an abelian category $\mathscr{A}$.

(i) The functors $\mathsf{T}_{\mathscr{A}}\colon \mathscr{A} \longrightarrow \mathsf{DMor}(\mathscr{A})$ and $\mathsf{H}_{\mathscr{A}}\colon \mathscr{A} \longrightarrow \mathsf{DMor}(\mathscr{A})$ are exact. Thus, the recollement (2.3) of $\mathsf{DMor}(\mathscr{A})$ has the property that the left and right adjoint of the quotient functor $\mathsf{U}_{\mathscr{A}}\colon \mathsf{DMor}(\mathscr{A}) \longrightarrow \mathscr{A}$ are exact. In general, this property doesn't hold in a recollement situation.

(ii) Let (X, Y, f, g) be an object in $\mathsf{DMor}(\mathscr{A})$. Then the tuple (Y, X, g, f) is also an object in $\mathsf{DMor}(\mathscr{A})$ since the composition of morphisms is still zero. This gives a functor $\mathcal{F}\colon \mathsf{DMor}(\mathscr{A}) \longrightarrow \mathsf{DMor}(\mathscr{A})$, $\mathcal{F}(X, Y, f, g) = (Y, X, g, f)$, which turns out to be an auto-equivalence.

(iii) For an object (X, Y, f, g) in $\mathsf{DMor}(\mathscr{A})$ we define the exact functor $\mathsf{U}'_{\mathscr{A}}\colon \mathsf{DMor}(\mathscr{A}) \longrightarrow \mathscr{A}$ given by $\mathsf{U}'_{\mathscr{A}}(X, Y, f, g) = Y$, and $\mathsf{U}'_{\mathscr{A}}(a, b) = b$ for a morphism (a,b) in $\mathsf{DMor}(\mathscr{A})$. It is easy to check that $\mathsf{U}'_{\mathscr{A}}$ is the middle functor of the adjoint triple $(\mathsf{H}_{\mathscr{A}}, \mathsf{U}'_{\mathscr{A}}, \mathsf{T}_{\mathscr{A}})$ and therefore we obtain a recollement of abelian categories $(\mathscr{A}, \mathsf{DMor}(\mathscr{A}), \mathscr{A})$. Then the following commutative diagram:

$$\begin{array}{ccc} \mathsf{DMor}(\mathscr{A}) & \xrightarrow{\mathsf{U}_{\mathscr{A}}} & \mathscr{A} \\ {\scriptstyle \mathcal{F}}\downarrow \simeq & & \downarrow {\scriptstyle \mathrm{Id}_{\mathscr{A}}} \\ \mathsf{DMor}(\mathscr{A}) & \xrightarrow{\mathsf{U}'_{\mathscr{A}}} & \mathscr{A} \end{array}$$

shows that there is a natural equivalence of functors between $\mathsf{U}'_{\mathscr{A}}\mathcal{F}$ and $\mathsf{U}_{\mathscr{A}}$. Thus, from [37, Definition 4.1, Lemma 4.2] we infer that the two recollements of $\mathsf{DMor}(\mathscr{A})$, i.e. the recollement (2.3) and the recollement $(\mathscr{A}, \mathsf{DMor}(\mathscr{A}), \mathscr{A})$ given by $\mathsf{U}'_{\mathscr{A}}$, are equivalent. From now on, we fix the recollement diagram (2.3) for the double morphism category $\mathsf{DMor}(\mathscr{A})$.

Example 2.7 Let R be a ring and consider the Morita ring $\Delta_{(0,0)} = \left(\begin{smallmatrix} R & R \\ R & R \end{smallmatrix}\right)$, see Example 2.3 (iv). From Section 2.1, the category $\mathsf{Mod}\text{-}\Delta_{(0,0)}$ is equivalent to the double morphism category $\mathsf{DMor}\,(\mathsf{Mod}\text{-}R)$ of $\mathsf{Mod}\text{-}R$. Then, from Proposition 2.5 (ii) we have the following recollement:

$$\mathsf{Mod}\text{-}R \;\underset{\overset{\curvearrowleft}{\mathsf{Ker}}}{\overset{\overset{\mathsf{Cok}}{\curvearrowleft}}{\xrightarrow{\ \mathsf{Z}_2\ }}}\; \mathsf{Mod}\text{-}\Delta_{(0,0)} \;\underset{\overset{\curvearrowleft}{\mathsf{H}_1}}{\overset{\overset{\mathsf{T}_1}{\curvearrowleft}}{\xrightarrow{\ \mathsf{U}_1\ }}}\; \mathsf{Mod}\text{-}R \tag{2.4}$$

For later use, we fix the above notation for the functors of the recollement of $\mathsf{Mod}\text{-}\Delta_{(0,0)}$. In particular, and relative to Section 2.1 and Proposition 2.5, we have:

(i) The functor $\mathsf{T}_1\colon \mathsf{Mod}\text{-}R \longrightarrow \mathsf{Mod}\text{-}\Delta_{(0,0)}$ is given by $\mathsf{T}_1(X) = (X, X, \mathrm{Id}_X, 0)$ on the objects $X \in \mathsf{Mod}\text{-}R$ and for an R-morphism $a\colon X \longrightarrow X'$ then $\mathsf{T}_1(a) = (a,a)$. Moreover, the functor T_1 is exact. Similarly, the functor $\mathsf{T}_2\colon \mathsf{Mod}\text{-}R \longrightarrow \mathsf{Mod}\text{-}\Delta_{(0,0)}$ is given by $\mathsf{T}_2(X) = (X, X, 0, \mathrm{Id}_X)$ on the objects $X \in \mathsf{Mod}\text{-}R$ and for an R-morphism $a\colon X \longrightarrow X'$ then $\mathsf{T}_2(a) = (a,a)$. Note that in this case T_2 is precisely the functor H_1 appearing in the recollement (2.4).

(ii) The functor $\mathsf{U}_{\mathscr{A}}$ of Proposition 2.5 is now denoted by $\mathsf{U}_1\colon \mathsf{Mod}\text{-}\Delta_{(0,0)} \longrightarrow \mathsf{Mod}\text{-}R$.

(iii) The functor $\mathsf{Z}_2\colon \mathsf{Mod}\text{-}R \longrightarrow \mathsf{Mod}\text{-}\Delta_{(0,0)}$, given by $\mathsf{Z}_2(X) = (0, X, 0, 0)$ for $X \in \mathsf{Mod}\text{-}R$, is the functor Z_B defined in Section 2.1.

(vi) The cokernel functor $\mathsf{Cok}\colon \mathsf{Mod}\text{-}\Delta_{(0,0)} \longrightarrow \mathsf{Mod}\text{-}R$ is given by $\mathsf{Cok}(X,Y,f,g) = \mathsf{Coker}\,f$ on the objects $(X,Y,f,g) \in \mathsf{Mod}\text{-}\Delta_{(0,0)}$ and for a $\Delta_{(0,0)}$-morphism $(a,b)\colon (X,Y,f,g) \longrightarrow (X',Y',f',g')$ we have $\mathsf{Cok}(a,b) = c$, where $c\colon \mathsf{Coker}\,f \longrightarrow \mathsf{Coker}\,f'$ is the induced morphism such that $b \circ \pi' = \pi \circ c$, where $\pi : Y \longrightarrow \mathsf{Coker}\,f$ and $\pi'\colon Y' \longrightarrow \mathsf{Coker}\,f'$. This is the functor $\mathsf{Q}_{\mathscr{A}}$ in Proposition 2.5.

(v) The kernel functor $\mathsf{Ker}\colon \mathsf{Mod}\text{-}\Delta_{(0,0)} \longrightarrow \mathsf{Mod}\text{-}R$ is given by $\mathsf{Ker}(X,Y,f,g) = \mathsf{Ker}\,g$ on the objects (X,Y,f,g) of $\mathsf{Mod}\text{-}\Delta_{(0,0)}$ and for a $\Delta_{(0,0)}$-morphism $(a,b)\colon (X,Y,f,g) \longrightarrow (X',Y',f',g')$ we have $\mathsf{Ker}(a,b) = c$, where c is the restriction map of b to $\mathsf{Ker}\,g$. This is the functor $\mathsf{P}_{\mathscr{A}}$ in Proposition 2.5.

2.3 Monomorphism Categories

Let $\mathscr{A}$ be an abelian category. We denote by $\mathsf{Mor}\mathscr{A}$ the category of morphisms of $\mathscr{A}$. Recall that the objects of $\mathsf{Mor}\mathscr{A}$ are triples (X, Y, f), where $f : X \longrightarrow Y$ is a morphism in $\mathscr{A}$, and given two objects (X, Y, f) and (X', Y', f') then a morphism is a pair (a, b) of maps in $\mathscr{A}$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X' & \xrightarrow{\ f'\ } & Y' \end{array}$$

Since the morphism category $\mathsf{Mor}\mathscr{A}$ is a special case of a trivial extension of abelian categories [22], it follows that $\mathsf{Mor}\mathscr{A}$ is an abelian category. The monomorphism category $\mathsf{Mon}\mathscr{A}$ of $\mathscr{A}$, which is the full subcategory of $\mathsf{Mor}\mathscr{A}$ consisting of monomorphisms in $\mathscr{A}$, is an extension closed additive subcategory of $\mathsf{Mor}\mathscr{A}$. This implies that $\mathsf{Mon}\mathscr{A}$ is an exact category in the sense of Quillen [39]. Given an additive category $\mathscr{A}$, recall that a pair of composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ called **exact**, if f is the kernel of g and g is the cokernel of f. Let $\mathscr{E}$ be a class of exact pairs which is closed under isomorphisms. A pair (f, g) in $\mathscr{E}$ is called a **conflation**, while the map f is called an **inflation** and the map g is called a **deflation**. Then the class $\mathscr{E}$ is an **exact structure** of $\mathscr{A}$ and $(\mathscr{A}, \mathscr{E})$ is called an **exact category**, if a series of axioms are satisfied. We refer to [28, Appendix A], see also [16], for the precise definition and for all the notions/results on exact categories needed in this paper (Section 5).

We now return to the double morphism category. For an abelian category $\mathscr{A}$, define the monomorphism categories of $\mathsf{DMor}(\mathscr{A})$ as follows: $\mathsf{Mono}_1(\mathscr{A}) = \{(X, Y, f, 0) \mid f : X \longrightarrow Y \text{ monomorphism in } \mathscr{A}\}$ and $\mathsf{Mono}_2(\Lambda) = \{(X, Y, 0, g) \mid g : Y \longrightarrow X$ monomorphism in $\mathscr{A}\}$. It is straightforward to show that the monomorphism categories $\mathsf{Mon}(\mathscr{A})$, $\mathsf{Mono}_1(\mathscr{A})$ and $\mathsf{Mono}_2(\mathscr{A})$ are equivalent as exact categories.

From now on the **monomorphism category** of an abelian category $\mathscr{A}$, denoted by $\mathsf{Mono}(\mathscr{A})$, is the category $\mathsf{Mono}_1(\mathscr{A})$. Note that when $\mathscr{A}$ is exact, the monomorphism category of $\mathscr{A}$ is the *inflation category* of $\mathscr{A}$, that is, the morphisms of $\mathscr{A}$ which are inflations. We continue to call this category the monomorphism category of $\mathscr{A}$ and we denote it by $\mathsf{Mono}(\mathscr{A})$ as well. The next result provides a description of the projective and injective objects in $\mathsf{Mono}(\mathscr{A})$. The proof follows similarly to [17, Lemma 2.1], so it is left to the reader.

Lemma 2.8 *Let $\mathscr{A}$ be an exact (abelian) category with enough projective and injective objects. Then the monomorphism category $\mathsf{Mono}(\mathscr{A})$ has enough projective and injective objects, in particular:*

(i) $\mathsf{Proj}(\mathsf{Mono}(\mathscr{A})) = \mathsf{add}\{\mathsf{T}_1(P) \oplus \mathsf{Z}_2(Q) \mid P, Q \in \mathsf{Proj}\,\mathscr{A}\}$, *and*
(ii) $\mathsf{Inj}(\mathsf{Mono}(\mathscr{A})) = \mathsf{add}\{\mathsf{T}_1(I) \oplus \mathsf{Z}_2(J) \mid I, J \in \mathsf{Inj}\,\mathscr{A}\}$.

Let Λ be an Artin algebra and consider the Morita ring $\Delta_{(0,0)}$ as an Artin algebra. In this case, the monomorphism category of Λ is the following full subcategory of $\mathsf{mod}\text{-}\Delta_{(0,0)}$, i.e. of $\mathsf{DMor}(\mathsf{mod}\text{-}\Lambda)$:

$$\mathsf{mono}(\Lambda) = \{(X, Y, f, 0) \mid f : X \longrightarrow Y \text{ is a monomorphism}\} \tag{2.5}$$

In the next result we collect some useful properties of $\mathsf{mono}(\Lambda)$ that we need in the sequel.

Lemma 2.9 *Let Λ be an Artin algebra. Then the following statements hold.*

(i) *The monomorphism category* $\mathsf{mono}(\Lambda)$ *is an exact category which is closed under kernels.*
(ii) *We have the adjoint triples* $(\mathsf{Cok}, \mathsf{Z}_2, \mathsf{U}_2)$ *and* $(\mathsf{U}_2, \mathsf{T}_1, \mathsf{U}_1)$:

$$\mathsf{mod}\text{-}\Lambda \underset{\mathsf{U}_2}{\overset{\mathsf{Cok}}{\leftleftarrows}} \xrightarrow{\ \mathsf{Z}_2\ } \mathsf{mono}(\Lambda) \xleftarrow{\ \mathsf{T}_1\ } \underset{\mathsf{U}_1}{\overset{\mathsf{U}_2}{\rightrightarrows}} \mathsf{mod}\text{-}\Lambda$$

The above functors are exact and preserve projective objects, and Z_2 *and* T_1 *are fully faithful.*

Proof (i) The monomorphism category $\mathsf{mono}(\Lambda)$ is exact, since it is an extension closed subcategory of $\mathsf{mod}\text{-}\Delta_{(0,0)}$. Let $(a, b)\colon (X, Y, f, 0) \longrightarrow (X', Y', f', 0)$ be a morphism in $\mathsf{mod}\text{-}\Delta_{(0,0)}$ with $(X, Y, f, 0)$ and $(X', Y', f', 0)$ in $\mathsf{mono}(\Lambda)$. Consider the following exact commutative diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathsf{Ker}\, a & \xrightarrow{\ i\ } & X & \xrightarrow{\ a\ } & X' \\
& \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\
0 \longrightarrow & \mathsf{Ker}\, b & \xrightarrow{\ j\ } & Y & \xrightarrow{\ b\ } & Y'
\end{array}$$

Since the composition $f \circ i$ is a monomorphism, it follows that the map h is a monomorphism. Then $\mathsf{Ker}(a, b) = (\mathsf{Ker}\, a, \mathsf{Ker}\, b, h, 0)$ lies in $\mathsf{mono}(\Lambda)$. We infer that $\mathsf{mono}(\Lambda)$ is closed under kernels.

(ii) It is easy to check that the above functors form adjoint pairs, see Proposition 2.4 and Example 2.7. Since Cok, Z_2, U_2 and T_1 are left adjoint functors of exact functors it follows that they preserve projective objects. The functor U_1 preserves projectives by the description of $\mathsf{proj}(\mathsf{mono}(\Lambda))$ given in Lemma 2.8, see also Example 2.7. Moreover, it follows easily from the definition that the functors Z_2, U_2, T_1 and U_1 are exact, and moreover that Z_2 and T_1 are fully faithful, see again Example 2.7. It remains to show that the cokernel functor $\mathsf{Cok}\colon \mathsf{mono}(\Lambda) \longrightarrow \mathsf{mod}\text{-}\Lambda$ is exact. Let $(X_1, Y_1, f_1, 0) \longrightarrow (X_2, Y_2, f_2, 0) \longrightarrow (X_3, Y_3, f_3, 0)$ be a conflation in $\mathsf{mono}(\Lambda)$. Then we have the following exact commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\
0 & \longrightarrow & Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_3 & \longrightarrow & 0
\end{array}$$

where the maps f_1, f_2 and f_3 are monomorphisms. From the Snake Lemma in the above diagram, it follows that the functor $\mathsf{Cok}_1\colon \mathsf{mono}(\Lambda) \longrightarrow \mathsf{mod}\text{-}\Lambda$ is exact. □

3 Gorenstein-Projective Modules over Morita Rings

In this section we provide a method for constructing Gorenstein-projective modules over Morita rings, which are Artin algebras and satisfy certain conditions, from Gorenstein-projective modules of the underlying algebras. This section is divided into two subsections and the main result is stated in the second subsection. We start by recalling the notion of Gorenstein-projective modules and we also fix notation.

Let Λ be an Artin algebra. An acyclic complex of projective Λ-modules $\mathsf{P}^\bullet\colon \cdots \longrightarrow P^{i-1} \longrightarrow P^i \longrightarrow P^{i+1} \longrightarrow \cdots$ is called **totally acyclic**, if the complex $\mathsf{Hom}_\Lambda(\mathsf{P}^\bullet, \Lambda)$ is acyclic. Then, a Λ-module X is called **Gorenstein-projective**, if it is of the form $X = \mathsf{Coker}\,(P^{-1} \longrightarrow P^0)$ for some totally acyclic complex $\mathsf{P}^\bullet$ of projective Λ-modules. We denote by $\mathsf{Gproj}\,\Lambda$ the full subcategory of $\mathsf{mod}\text{-}\Lambda$ consisting of the finitely generated Gorenstein-projective Λ-modules. Moreover, we denote $\{X \in \mathsf{mod}\text{-}\Lambda \mid \mathsf{Ext}^1_\Lambda(\mathsf{Gproj}\,\Lambda, X) = 0\}$ by $(\mathsf{Gproj}\,\Lambda)^\perp$. Recall also from [10, 11], that an Artin algebra Λ is said to be of **finite Cohen-Macaulay type**, if the category $\mathsf{Gproj}\,\Lambda$ is of finite representation type, i.e. the set of isomorphism classes of indecomposable finitely generated Gorenstein-projective modules is finite. Finally, for a Λ-module X we denote by $\mathsf{add}\,X$ the full subcategory of $\mathsf{mod}\text{-}\Lambda$ consisting of all direct summands of finite direct sums of X.

3.1 Lifting Gorenstein-Projective Modules

From Proposition 2.4 it follows that the functors $\mathsf{T}_A\colon \mathsf{mod}\text{-}A \longrightarrow \mathsf{mod}\text{-}\Lambda_{(\phi,\psi)}$ and $\mathsf{T}_B\colon \mathsf{mod}\text{-}B \longrightarrow \mathsf{mod}\text{-}\Lambda_{(\phi,\psi)}$ preserve projective modules. In this subsection we investigate when the functors T_A and T_B preserve Gorenstein-projective modules. The first step towards this problem, is to examine when the above functors preserve totally acyclic complexes. Under some conditions, this is achieved in the next result.

Proposition 3.1 *Let $\Lambda_{(\phi,\psi)} = \begin{pmatrix} A & {}_AN_B \\ {}_BM_A & B \end{pmatrix}$ be a Morita ring.*

(i) *Assume that the functor $M \otimes_A -\colon \mathsf{mod}\text{-}A \longrightarrow \mathsf{mod}\text{-}B$ sends acyclic complexes of projective A-modules to acyclic complexes of B-modules and $\mathsf{add}\,{}_AN \subseteq (\mathsf{Gproj}\,A)^\perp$. Then a complex $\mathsf{P}^\bullet$ in $\mathsf{mod}\text{-}A$ is totally acyclic if and only if the complex $\mathsf{T}_A(\mathsf{P}^\bullet)$ is totally acyclic in $\mathsf{mod}\text{-}\Lambda_{(\phi,\psi)}$.*

(ii) *Assume that the functor $N \otimes_B -\colon \mathsf{mod}\text{-}B \longrightarrow \mathsf{mod}\text{-}A$ sends acyclic complexes of projective B-modules to acyclic complexes of A-modules and $\mathsf{add}\,{}_BM \subseteq (\mathsf{Gproj}\,B)^\perp$. Then a complex $\mathsf{P}^\bullet$ in $\mathsf{mod}\text{-}B$ is totally acyclic if and only if the complex $\mathsf{T}_B(\mathsf{P}^\bullet)$ is totally acyclic in $\mathsf{mod}\text{-}\Lambda_{(\phi,\psi)}$.*

Proof We prove only (i), the statement (ii) is dual. Assume that

$$\mathsf{P}^\bullet\colon \quad \cdots \longrightarrow P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} P^1 \longrightarrow \cdots$$

is a totally acyclic complex of projectives in $\mathsf{mod}\text{-}A$. Then, by the assumption on the functor $M \otimes_A -$ and Remark 2.1 (ii), we obtain that the following complex:

$$\mathsf{T}_A(\mathsf{P}^\bullet)\colon \quad \cdots \longrightarrow \mathsf{T}_A(P^{-1}) \xrightarrow{\mathsf{T}_A(d^{-1})} \mathsf{T}_A(P^0) \xrightarrow{\mathsf{T}_A(d^0)} \mathsf{T}_A(P^1) \longrightarrow \cdots$$

is exact, where each $\mathsf{T}_A(P^i)$ lies in $\mathsf{proj}\,\Lambda_{(\phi,\psi)}$ by Proposition 2.4. We show now that the complex $\mathsf{Hom}_{\Lambda_{(\phi,\psi)}}(\mathsf{T}_A(\mathsf{P}^\bullet), (X, Y, f, g))$ is acyclic for all (X, Y, f, g) in

$\mathsf{proj}\,\Lambda_{(\phi,\psi)}$. In fact, from Proposition 2.2 (i) it is enough to consider only the complexes $\mathsf{Hom}_{\Lambda_{(\phi,\psi)}}(\mathsf{T}_A(\mathsf{P}^\bullet), \mathsf{T}_A(P))$ and $\mathsf{Hom}_{\Lambda_{(\phi,\psi)}}(\mathsf{T}_A(\mathsf{P}^\bullet), \mathsf{T}_B(Q))$, where P lies in $\mathsf{proj}\,A$ and Q lies in $\mathsf{proj}\,B$. In the first case, the complex $\mathsf{Hom}_{\Lambda_{(\phi,\psi)}}(\mathsf{T}_A(\mathsf{P}^\bullet), \mathsf{T}_A(P))$ is acyclic since the complex $\mathsf{Hom}_A(\mathsf{P}^\bullet, P)$ is acyclic and from Proposition 2.4 the functor T_A is fully faithful. Let Q be a projective B-module. Then, by using the adjoint pair $(\mathsf{T}_A, \mathsf{U}_A)$, we have the following commutative diagram:

$$\begin{array}{ccccccc}
\cdots \to \mathsf{Hom}_{\Lambda_{(\phi,\psi)}}(\mathsf{T}_A(P^1), \mathsf{T}_B(Q)) & \to & \mathsf{Hom}_{\Lambda_{(\phi,\psi)}}(\mathsf{T}_A(P^0), \mathsf{T}_B(Q)) & \to & \mathsf{Hom}_{\Lambda_{(\phi,\psi)}}(\mathsf{T}_A(P^{-1}), \mathsf{T}_B(Q)) \to \cdots \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
\cdots \longrightarrow \mathsf{Hom}_A(P^1, N\otimes_B Q) & \longrightarrow & \mathsf{Hom}_A(P^0, N\otimes_B Q) & \longrightarrow & \mathsf{Hom}_A(P^{-1}, N\otimes_B Q) \longrightarrow \cdots
\end{array}$$

Since $N \otimes_B Q$ is a direct sum of summands of N and $\mathsf{add}\,_AN \subseteq (\mathsf{Gproj}\,A)^\perp$, it follows that the complex $\mathsf{Hom}_A(\mathsf{P}^\bullet, N \otimes_B Q)$ is acyclic and therefore the complex $\mathsf{Hom}_{\Lambda_{(\phi,\psi)}}(\mathsf{T}_A(\mathsf{P}^\bullet), \mathsf{T}_B(Q))$ is also acyclic. We infer that the complex $\mathsf{T}_A(\mathsf{P}^\bullet)$ is totally acyclic.

Conversely, assume that $\mathsf{P}^\bullet$ is a complex of A-modules such that $\mathsf{T}_A(\mathsf{P}^\bullet)$ is totally acyclic. If we apply the functor U_A to the complex $\mathsf{T}_A(\mathsf{P}^\bullet)$, we get that the complex $\mathsf{P}^\bullet\colon \cdots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$ is acyclic. Note that since the functor T_A is right exact and fully faithful it follows that each P^i lies in $\mathsf{proj}\,A$, see Proposition 2.4. Then, for every projective A-module P, we derive as above that the complex $\mathsf{Hom}_A(\mathsf{P}^\bullet, P)$ is acyclic. We remark that in this direction we did not make use of our assumptions. □

We refer to the above conditions as the **compatibility conditions** on the bimodules $_AN_B$ and $_BM_A$.

Example 3.2 Let $\Lambda_{(\phi,\psi)} = \left(\begin{smallmatrix} A & {}_AN_B \\ {}_BM_A & B \end{smallmatrix}\right)$ be a Morita ring.

(i) Assume that M_A is projective as a right A-module and $_AN$ is projective as a left A-module. Then the functor $M \otimes_A -\colon \mathsf{mod}\text{-}A \longrightarrow \mathsf{mod}\text{-}B$ is exact and the subcategory $\mathsf{add}\,_AN$ lies in $(\mathsf{Gproj}\,A)^\perp$. Hence, from Proposition 3.1 (i) it follows that a complex $\mathsf{P}^\bullet$ is totally acyclic in $\mathsf{mod}\text{-}A$ if and only if the complex $\mathsf{T}_A(\mathsf{P}^\bullet)$ is totally acyclic in $\mathsf{mod}\text{-}\Lambda_{(\phi,\psi)}$. Similarly, if N_B is projective as a right B-module and $_BM$ is projective as a left B-module, then the statement of Proposition 3.1 (ii) holds. In particular, consider the case of the Morita ring $\Delta_{(\phi,\phi)} = \left(\begin{smallmatrix} \Lambda & \Lambda \\ \Lambda & \Lambda \end{smallmatrix}\right)$, see Example 2.7. Then it follows that a complex $\mathsf{P}^\bullet$ in $\mathsf{mod}\text{-}\Lambda$ is totally acyclic if and only if the complex $\mathsf{T}_1(\mathsf{P}^\bullet)$ is totally acyclic in $\mathsf{mod}\text{-}\Delta_{(\phi,\phi)}$ if and only if the complex $\mathsf{T}_2(\mathsf{P}^\bullet)$ is totally acyclic in $\mathsf{mod}\text{-}\Delta_{(\phi,\phi)}$.

(ii) Assume that $\mathsf{pd}\,M_A < \infty$ and $\mathsf{pd}\,_AN < \infty$ (or $\mathsf{id}\,_AN < \infty$). Then from [47, Proposition 1.3], it follows that the functor $M \otimes_A -\colon \mathsf{mod}\text{-}A \longrightarrow \mathsf{mod}\text{-}B$ sends acyclic complexes of projective A-modules to acyclic complexes of B-modules and $\mathsf{add}\,_AN \subseteq (\mathsf{Gproj}\,A)^\perp$. Hence, if $\Lambda_{(\phi,\psi)}$ is a Morita ring which is an Artin algebra such that $\mathsf{pd}\,M_A < \infty$ and $\mathsf{pd}\,_AN < \infty$ (or $\mathsf{id}\,_AN < \infty$), then from Proposition 3.1 (i) we get that the functor T_A preserves totally acyclic complexes. Dually, if we assume that $\mathsf{pd}\,N_B < \infty$ and $\mathsf{pd}\,_BM < \infty$ (or $\mathsf{id}\,_BM < \infty$), then the conditions of Proposition 3.1 (ii) are satisfied and therefore the functor T_B preserves totally acyclic complexes. Note that this example generalizes the situation mentioned in (i).

As a consequence of Proposition 3.1 we have the following result, which provides sufficient conditions such that the functors T_A and T_B lift Gorenstein-projective modules. In particular, we derive that Cohen-Macaulay finiteness of the Morita ring is inherited to the underlying algebras as well.

Corollary 3.3 *Let $\Lambda_{(\phi,\psi)}$ be a Morita ring.*

(i) *Assume that the functor $M \otimes_A -\colon \mathsf{mod}\text{-}A \longrightarrow \mathsf{mod}\text{-}B$ sends acyclic complexes of projective A-modules to acyclic complexes of B-modules and $\mathsf{add}\,_A N \subseteq (\mathsf{Gproj}\, A)^{\perp}$.*

- (a) *If $X \in \mathsf{Gproj}\, A$ then $\mathsf{T}_A(X) \in \mathsf{Gproj}\, \Lambda_{(\phi,\psi)}$.*
- (b) *If $\Lambda_{(\phi,\psi)}$ is of finite Cohen-Macaulay type, then A is also of finite Cohen-Macaulay type.*

(ii) *Assume that the functor $N \otimes_B -\colon \mathsf{mod}\text{-}B \longrightarrow \mathsf{mod}\text{-}A$ sends acyclic complexes of projective B-modules to acyclic complexes of A-modules and $\mathsf{add}\,_B M \subseteq (\mathsf{Gproj}\, B)^{\perp}$.*

- (a) *If $Y \in \mathsf{Gproj}\, B$ then $\mathsf{T}_B(Y) \in \mathsf{Gproj}\, \Lambda_{(\phi,\psi)}$.*
- (b) *If $\Lambda_{(\phi,\psi)}$ is of finite Cohen-Macaulay type, then B is also of finite Cohen-Macaulay type.*

Now we turn our attention to the algebra $\Delta_{(\phi,\phi)} = \left(\begin{smallmatrix} \Lambda & \Lambda \\ \Lambda & \Lambda \end{smallmatrix}\right)$. We recall the following.

Proposition 3.4 *Let Λ be an Artin algebra and let $n \geq 0$ be a natural number.*

(i) [25, Corollary 6.4] *Λ is n-Gorenstein if and only if the Morita ring $\Delta_{(\phi,\phi)}$ is n-Gorenstein algebra.*
(ii) [25, Corollary 6.6] *Assume that Λ is Gorenstein. Then a $\Delta_{(\phi,\phi)}$-module (X, Y, f, g) is Gorenstein-projective if and only if X and Y are Gorenstein-projective Λ-modules.*

In the next result we show the one direction of Proposition 3.4 (ii) without assuming Λ to be Gorenstein.

Lemma 3.5 *Let Λ be an Artin algebra and let $\Delta_{(\phi,\phi)} = \left(\begin{smallmatrix} \Lambda & \Lambda \\ \Lambda & \Lambda \end{smallmatrix}\right)$. If (X, Y, f, g) is an object in $\mathsf{Gproj}\, \Delta_{(\phi,\phi)}$ then the Λ-modules X and Y lie in $\mathsf{Gproj}\, \Lambda$.*

Proof Let (X, Y, f, g) be a Gorenstein-projective $\Delta_{(\phi,\phi)}$-module. Then from Proposition 2.2, there exists a totally acyclic complex of the following form:

$$\mathsf{T}^{\bullet}\colon \quad \cdots \longrightarrow \mathsf{T}_1(P^{-1}) \oplus \mathsf{T}_2(Q^{-1}) \dashrightarrow \mathsf{T}_1(P^{0}) \oplus \mathsf{T}_2(Q^{0}) \longrightarrow \cdots$$
$$\mathsf{T}_1(P^{-1}) \oplus \mathsf{T}_2(Q^{-1}) \twoheadrightarrow (X, Y, f, g) \rightarrowtail \mathsf{T}_1(P^{0}) \oplus \mathsf{T}_2(Q^{0})$$

where P^i and Q^i are projective Λ-modules. Then, if we apply the exact functor $\mathsf{U}_1\colon \mathsf{mod}\text{-}\Delta_{(\phi,\phi)} \longrightarrow \mathsf{mod}\text{-}\Lambda$, we get the exact sequence of projective Λ-modules:

$$\mathsf{P}^{\bullet}\colon \quad \cdots \longrightarrow P^{-1} \oplus (\Lambda \otimes_{\Lambda} Q^{-1}) \dashrightarrow P^{0} \oplus (\Lambda \otimes_{\Lambda} Q^{0}) \longrightarrow \cdots$$
$$P^{-1} \oplus (\Lambda \otimes_{\Lambda} Q^{-1}) \twoheadrightarrow X \rightarrowtail P^{0} \oplus (\Lambda \otimes_{\Lambda} Q^{0})$$

We claim that the above complex is totally acyclic. Let P be a projective Λ-module. Then from Example 2.7, we have the following isomorphisms:

$$\begin{aligned}\mathsf{Hom}_\Lambda(P^i \oplus (\Lambda \otimes_\Lambda Q^i), P) &\cong \mathsf{Hom}_{\Delta_{(\phi,\phi)}}\big(\mathsf{T}_1(P^i) \oplus \mathsf{T}_2(Q^i), \mathsf{H}_1(P)\big)\\ &\cong \mathsf{Hom}_{\Delta_{(\phi,\phi)}}\big(\mathsf{T}_1(P^i) \oplus \mathsf{T}_2(Q^i), \mathsf{T}_2(P)\big)\end{aligned}$$

Since the complex $\mathsf{Hom}_{\Delta_{(\phi,\phi)}}(\mathsf{T}^\bullet, \mathsf{T}_2(P))$ is acyclic, it follows from the above isomorphisms that the complex $\mathsf{Hom}_\Lambda(\mathsf{P}^\bullet, P)$ is also acyclic. We infer that the complex $\mathsf{P}^\bullet$ is totally acyclic and therefore the Λ-module X is Gorenstein-projective. Similarly we show that Y is a Gorenstein-projective Λ-module. □

As a consequence of Corollary 3.3 and Lemma 3.5 we obtain the following. Note that if Λ is Gorenstein, Proposition 3.4 (ii) gives a direct proof of the next result.

Corollary 3.6 *Let Λ be an Artin algebra and let $\Delta_{(\phi,\phi)} = \left(\begin{smallmatrix} \Lambda & \Lambda \\ \Lambda & \Lambda \end{smallmatrix}\right)$. Then for a Λ-module X the following statements are equivalent*:

(i) $X \in \mathsf{Gproj}\, A$.
(i) $\mathsf{T}_1(X) \in \mathsf{Gproj}\, \Delta_{(\phi,\phi)}$.
(iii) $\mathsf{T}_2(X) \in \mathsf{Gproj}\, \Delta_{(\phi,\phi)}$.

In the special case where $\phi = 0$, the Gorenstein-projective modules $\mathsf{T}_1(X)$ lie in the monomorphism category $\mathsf{mono}(\Lambda)$ as defined in Section 2.3. We close this subsection with the next example.

Example 3.7 Let $\mathbb{K}$ be a field and $R = \mathbb{K}[[X_1, X_2]]/(X_1X_2)$. Consider the Morita ring $\Delta_{(0,0)} = \left(\begin{smallmatrix} R & R \\ R & R \end{smallmatrix}\right)$. By [19, Example 4.1.5] the R-modules $\overline{X}_1$ and $\overline{X}_2$ are Gorenstein-projective, where $\overline{X}_i$ is the residue class in R of X_i for $i = 1, 2$. Thus, from Corollary 3.6 and for $i = 1, 2$ it follows that the objects $\mathsf{T}_1(\overline{X}_i) = (\overline{X}_i, \overline{X}_i, \mathrm{Id}_{\overline{X}_i}, 0)$ and $\mathsf{T}_2(\overline{X}_i) = (\overline{X}_i, \overline{X}_i, 0, \mathrm{Id}_{\overline{X}_i})$ are Gorenstein-projective $\Delta_{(0,0)}$-modules.

3.2 Constructing Gorenstein-Projective Modules

Before we proceed to the main result of this subsection (Theorem 3.10), we need some preparations.

Lemma 3.8 *Let $\Lambda_{(0,0)} = \left(\begin{smallmatrix} A & {}_AN_B \\ {}_BM_A & B \end{smallmatrix}\right)$ be a Morita ring. Then for every A-module X and B-module Y we have the following exact sequences in $\mathsf{Mod}\text{-}\Lambda_{(0,0)}$*:

$$\begin{cases} 0 \longrightarrow \mathsf{Z}_B(M \otimes_A X) \longrightarrow \mathsf{T}_A(X) \longrightarrow \mathsf{Z}_A(X) \longrightarrow 0 \\ 0 \longrightarrow \mathsf{Z}_A(N \otimes_B Y) \longrightarrow \mathsf{T}_B(Y) \longrightarrow \mathsf{Z}_B(Y) \longrightarrow 0 \end{cases}$$

$$\begin{cases} 0 \longrightarrow \mathsf{Z}_A(X) \longrightarrow \mathsf{H}_A(X) \longrightarrow \mathsf{Z}_B(\mathsf{Hom}_A(N, X)) \longrightarrow 0 \\ 0 \longrightarrow \mathsf{Z}_B(Y) \longrightarrow \mathsf{H}_B(Y) \longrightarrow \mathsf{Z}_A(\mathsf{Hom}_B(M, Y)) \longrightarrow 0 \end{cases}$$

Proof Let X be an A-module. Then the map $(\mathrm{Id}_X, 0)\colon \mathsf{T}_A(X) \longrightarrow \mathsf{Z}_A(X)$ is an epimorphism in the category $\mathsf{Mod}\text{-}\Lambda_{(0,0)}$, where $\mathsf{T}_A(X) = (X, M \otimes_A X, \mathrm{Id}_{M\otimes_A X}, 0)$, $\mathsf{Z}_A(X) = (X, 0, 0, 0)$, and the kernel of the morphism $(\mathrm{Id}_X, 0)$ is the object $\mathsf{Z}_B(M \otimes_A X) = (0, M \otimes_A X, 0, 0)$. We infer that the sequence $0 \longrightarrow \mathsf{Z}_B(M \otimes_A X) \longrightarrow \mathsf{T}_A(X) \longrightarrow \mathsf{Z}_A(X) \longrightarrow 0$ is exact. In the same way we derive that the rest sequences are exact, the details are left to the reader. □

Lemma 3.9 *Let* $\Lambda_{(0,0)}$ *be a Morita ring. Then for every* $X, X' \in \mathsf{Mod}\text{-}A$ *and* $Y, Y' \in \mathsf{Mod}\text{-}B$*, we have the following isomorphisms*:

$$\mathsf{Hom}_{\Lambda_{(0,0)}}\big(\mathsf{T}_A(X) \oplus \mathsf{T}_B(Y), \mathsf{Z}_A(X')\big) \cong \mathsf{Hom}_A(X, X')$$

and

$$\mathsf{Hom}_{\Lambda_{(0,0)}}\big(\mathsf{T}_A(X) \oplus \mathsf{T}_B(Y), \mathsf{Z}_B(Y')\big) \cong \mathsf{Hom}_B(Y, Y')$$

Proof We show the first isomorphism. From Proposition 2.4, we have the adjoint pair $(\mathsf{Q}_A, \mathsf{Z}_A)$ and from the recollement $(\mathsf{Mod}\text{-}A, \mathsf{Mod}\text{-}\Lambda_{(0,0)}, \mathsf{Mod}\text{-}B)$ it follows that $\mathsf{Q}_A\mathsf{T}_B = 0$. Then, we have the isomorphism

$$\mathsf{Hom}_{\Lambda_{(0,0)}}\big(\mathsf{T}_A(X) \oplus \mathsf{T}_B(Y), \mathsf{Z}_A(X')\big) \cong \mathsf{Hom}_A\big(\mathsf{Q}_A\mathsf{T}_A(X), X'\big)$$

and it remains to compute the object $\mathsf{Q}_A\mathsf{T}_A(X)$. From the counit of the adjoint pair $(\mathsf{T}_B, \mathsf{U}_B)$ we have the following exact sequence in $\mathsf{Mod}\text{-}\Lambda_{(0,0)}$:

$$\mathsf{T}_B\mathsf{U}_B\big(\mathsf{T}_A(X)\big) \xrightarrow{(0,\mathrm{Id}_{M\otimes X})} \mathsf{T}_A(X) \longrightarrow \mathsf{Z}_A\mathsf{Q}_A(\mathsf{T}_A(X)) \longrightarrow 0$$

see Proposition 2.4, where $\mathsf{T}_B\mathsf{U}_B(\mathsf{T}_A(X)) = (N \otimes_B M \otimes_A X, M \otimes_A X, 0, \mathrm{Id}_{N\otimes M\otimes X})$ and $\mathsf{Z}_A\mathsf{Q}_A(\mathsf{T}_A(X)) \cong \mathsf{Coker}\,(0, \mathrm{Id}_{M\otimes X}) \cong \mathsf{Z}_A(X)$. This implies that $\mathsf{Q}_A\mathsf{T}_A(X) \cong X$ and therefore we have the isomorphism $\mathsf{Hom}_{\Lambda_{(0,0)}}(\mathsf{T}_A(X) \oplus \mathsf{T}_B(Y), \mathsf{Z}_A(X')) \cong \mathsf{Hom}_A(X, X')$. The second isomorphism follows similarly by using the adjoint pair $(\mathsf{Q}_B, \mathsf{Z}_B)$. □

We are ready to prove the main result of this section which constructs Gorenstein-projective modules over Morita rings $\Lambda_{(0,0)}$. This result constitutes the first part of Theorem A presented in the Introduction.

Theorem 3.10 *Let* $\Lambda_{(0,0)}$ *be a Morita ring such that the bimodules* ${}_AN_B$ *and* ${}_BM_A$ *satisfy the compatibility conditions, that is, the following conditions hold*:

(i) *The functor* $M \otimes_A -\colon \mathsf{mod}\text{-}A \longrightarrow \mathsf{mod}\text{-}B$ *sends acyclic complexes of projective* A*-modules to acyclic complexes of* B*-modules.*
(ii) $\mathsf{add}\,{}_AN \subseteq (\mathsf{Gproj}\,A)^{\perp}$.
(iii) *The functor* $N \otimes_B -\colon \mathsf{mod}\text{-}B \longrightarrow \mathsf{mod}\text{-}A$ *sends acyclic complexes of projective* B*-modules to acyclic complexes of* A*-modules.*
(iv) $\mathsf{add}\,{}_BM \subseteq (\mathsf{Gproj}\,B)^{\perp}$.

(α) *Assume that there exists a Gorenstein-projective* B*-module* Z *with a monomorphism* $s\colon N \otimes_B Z \longrightarrow X$*, for some* A*-module* X*, such that* $\mathsf{Coker}\,s$ *lies in* $\mathsf{Gproj}\,A$ *and there is a* *monomorphism* $t\colon M \otimes_A \mathsf{Coker}\, s \longrightarrow Y$ *with* $\mathsf{Coker}\, t = Z$ *and for some* B*-module* Y*. Then the tuple*

$$\big(X, Y, (\mathrm{Id}_M \otimes \pi_X) \circ t, (\mathrm{Id}_N \otimes \pi_Y) \circ s\big) \tag{3.1}$$

is a Gorenstein-projective $\Lambda_{(0,0)}$*-module, where* $\pi_X\colon X \longrightarrow \mathsf{Coker}\, s$ *and* $\pi_Y\colon Y \longrightarrow \mathsf{Coker}\, t$.

(β) *Assume that there exists a Gorenstein-projective* A*-module* Z *with a monomorphism* $t\colon M \otimes_A Z \longrightarrow Y$*, for some* B*-module* Y*, such that* $\mathsf{Coker}\, t$ *lies in* $\mathsf{Gproj}\, B$ *and there is a monomorphism* $s\colon N \otimes_B \mathsf{Coker}\, t \longrightarrow X$ *with* $\mathsf{Coker}\, s = Z$ *and for some* A*-module* X*. Then the tuple*

$$\big(X, Y, (\mathrm{Id}_M \otimes \pi_X) \circ t, (\mathrm{Id}_N \otimes \pi_Y) \circ s\big) \tag{3.2}$$

is a Gorenstein-projective $\Lambda_{(0,0)}$*-module, where* $\pi_X\colon X \longrightarrow \mathsf{Coker}\, s$ *and* $\pi_Y\colon Y \longrightarrow \mathsf{Coker}\, t$.

Proof We prove (α), the statement (β) is dual. The proof for (α) is divided into four steps. In the first two steps we construct (co)resolutions of X and Y by objects coming from the totally acyclic complexes of $\mathsf{Coker}\, s$ and Z. Then in the third step we lift this (co)resolutions to $\mathsf{mod}\text{-}\Lambda_{(0,0)}$ and in the final step we show that this construction is indeed a totally acyclic complex of the object $(X, Y, (\mathrm{Id}_M \otimes \pi_X) \circ t, (\mathrm{Id}_N \otimes \pi_Y) \circ s)$.

Step 1: Since the A-module $\mathsf{Coker}\, s$ is Gorenstein-projective, there exists a totally acyclic complex of projective A-modules:

$$\mathsf{P}^\bullet\colon \quad \cdots \longrightarrow P^{-1} \xrightarrow{d_P^{-1}} P^0 \xrightarrow{d_P^0} P^1 \longrightarrow \cdots$$

such that $\mathsf{Ker}\, d_P^0 = \mathsf{Coker}\, s$ and let $d_P^{-1} = \lambda_P^{-1} \circ \kappa_P^{-1}$ be the canonical factorization through $\mathsf{Coker}\, s$. Also, since the B-module Z is Gorenstein-projective there exists a totally acyclic complex of projective B-modules:

$$\mathsf{Q}^\bullet\colon \quad \cdots \longrightarrow Q^{-1} \xrightarrow{d_Q^{-1}} Q^0 \xrightarrow{d_Q^0} Q^1 \longrightarrow \cdots$$

such that $\mathsf{Ker}\, d_Q^0 = Z$ and let $d_Q^{-1} = \lambda_Q^{-1} \circ \kappa_Q^{-1}$ be the canonical factorization through Z. Then from the assumption (iii), it follows that the complex of A-modules $N \otimes_B \mathsf{Q}^\bullet$ is acyclic. Applying to the exact sequence:

$$0 \longrightarrow N \otimes_B Z \xrightarrow{s} X \xrightarrow{\pi_X} \mathsf{Coker}\, s \longrightarrow 0$$

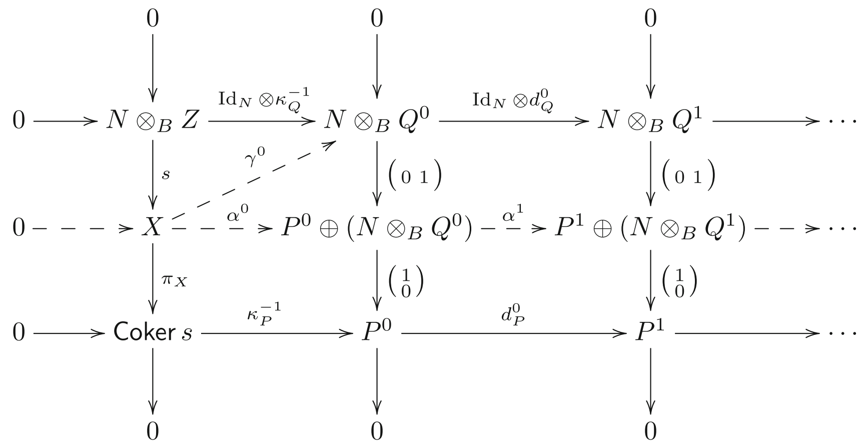
the functor $\mathsf{Hom}_A(-, N \otimes_B Q^0)$, we get the exact sequence:

$$0 \longrightarrow \mathsf{Hom}_A(\mathsf{Coker}\, s, N \otimes_B Q^0) \longrightarrow \mathsf{Hom}_A(X, N \otimes_B Q^0) \longrightarrow \mathsf{Hom}_A(N \otimes_B Z, N \otimes_B Q^0) \longrightarrow 0$$

since $\mathsf{Coker}\, s \in \mathsf{Gproj}\, A$ and $N \otimes_B Q^0 \in (\mathsf{Gproj}\, A)^\perp$ from the assumption (ii). This implies that there is a map $\gamma^0\colon X \longrightarrow N \otimes_B Q^0$ such that $s \circ \gamma^0 = \mathrm{Id}_N \otimes \kappa_Q^{-1}$ and therefore we obtain the map

$$\alpha^0 := \big(\pi_X \circ \kappa_P^{-1} \;\; \gamma^0\big)\colon X \longrightarrow P^0 \oplus (N \otimes_B Q^0)$$

Then from the Horseshoe Lemma, see also [47, Lemma 1.6], we obtain the exact commutative diagram:



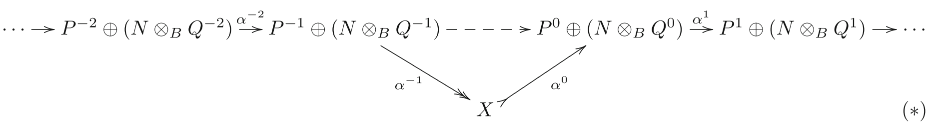
where for all $i \geq 1$ we have $\alpha^i = \begin{pmatrix} d_P^{i-1} & 0 \\ \gamma^i & \mathrm{Id}_N \otimes d_Q^{i-1} \end{pmatrix} \colon P^{i-1} \oplus (N \otimes_B Q^{i-1}) \longrightarrow P^i \oplus (N \otimes_B Q^i)$ and $\gamma^i \colon P^{i-1} \longrightarrow N \otimes_B Q^i$. Note that the existence of the maps γ^i follow by using the assumption (ii). In the same way, we can construct a resolution of X by objects of the form $P^i \oplus (N \otimes_B Q^i)$ but now we use that the modules P^{-i}, $i \geq 1$, are projective. In particular, we get the map

$$\alpha^{-1} = \begin{pmatrix} \gamma^{-1} \\ (\mathrm{Id}_N \otimes \lambda_Q^{-1}) \circ s \end{pmatrix} \colon P^{-1} \oplus (N \otimes_B Q^{-1}) \longrightarrow X$$

where $\gamma^{-1} \colon P^{-1} \longrightarrow X$ such that $\gamma^{-1} \circ \pi_X = \lambda_P^{-1}$, and for every $i \geq 2$ we have the maps:

$$\alpha^{-i} = \begin{pmatrix} d_P^{-i} & 0 \\ \gamma^{-i} & \mathrm{Id}_N \otimes d_Q^{-i} \end{pmatrix} \colon P^{-i} \oplus (N \otimes_B Q^{-i}) \longrightarrow P^{-i+1} \oplus (N \otimes_B Q^{-i+1})$$

similarly as described above. Thus, summarizing the construction so far, we have constructed the following exact sequence:



$(*)$

Step 2: We construct an exact sequence similar to $(*)$ for the B-module Y. Since $\mathsf{add}_B M \subseteq (\mathsf{Gproj}\, B)^{\perp}$ (assumption (iv)) we have as in Step 1 the following exact commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M\otimes_A \mathsf{Coker}\, s & \xrightarrow{\mathrm{Id}_M\otimes\kappa_P^{-1}} & M\otimes_A P^0 & \xrightarrow{\mathrm{Id}_M\otimes d_P^0} & M\otimes_A P^1 & \longrightarrow & \cdots \\
 & & \downarrow t & \nearrow^{\delta^0} & \downarrow \begin{pmatrix}1 & 0\end{pmatrix} & & \downarrow \begin{pmatrix}1 & 0\end{pmatrix} & & \\
0 & \dashrightarrow & Y & \overset{\beta^0}{\dashrightarrow} & (M\otimes_A P^0)\oplus Q^0 & \overset{\beta^1}{\dashrightarrow} & (M\otimes_A P^1)\oplus Q^1 & \dashrightarrow & \cdots \\
 & & \downarrow \pi_Y & & \downarrow \begin{pmatrix}0 \\ 1\end{pmatrix} & & \downarrow \begin{pmatrix}0 \\ 1\end{pmatrix} & & \\
0 & \longrightarrow & Z & \xrightarrow{\kappa_Q^{-1}} & Q^0 & \xrightarrow{d_Q^0} & Q^1 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \qquad (**)$$

where $\beta^0 := \begin{pmatrix}\delta^0 & \pi_Y\circ\kappa_Q^{-1}\end{pmatrix} \colon Y \longrightarrow (M\otimes_A P^0)\oplus Q^0$ and for all $i \geq 1$ we have:

$$\beta^i = \begin{pmatrix}\mathrm{Id}_M\otimes d_P^{i-1} & \delta^i \\ 0 & d_Q^{i-1}\end{pmatrix} \colon (M\otimes_A P^{i-1})\oplus Q^{i-1} \longrightarrow (M\otimes_A P^i)\oplus Q^i$$

for some $\delta^i \colon Q^{i-1} \longrightarrow M\otimes_A P^i$. Then, as in Step 1 we construct a resolution of Y by objects of the form $(M\otimes_A P^i)\oplus Q^i$ and putting together these, we obtain the following exact sequence:

$$\begin{array}{ccccccc}
\cdots \to (M\otimes_A P^{-2})\oplus Q^{-2} & \xrightarrow{\beta^{-2}} & (M\otimes_A P^{-1})\oplus Q^{-1} & \dashrightarrow & (M\otimes_A P^0)\oplus Q^0 & \xrightarrow{\beta^1} & (M\otimes_A P^1)\oplus Q^1 \longrightarrow \cdots \\
 & & {\scriptstyle\beta^{-1}}\searrow & & \nearrow{\scriptstyle\beta^0} & & \\
 & & & Y & & &
\end{array} \qquad (**)$$

Step 3: We glue together the exact sequences $(*)$ and $(**)$ and we derive the following sequence:

$$\mathsf{T}^{\bullet}: \quad \begin{array}{ccccc}
\cdots \xrightarrow{(\alpha^{-2},\beta^{-2})} & \mathsf{T}_A(P^{-1})\oplus\mathsf{T}_B(Q^{-1}) & \dashrightarrow & \mathsf{T}_A(P^0)\oplus\mathsf{T}_B(Q^0) & \xrightarrow{(\alpha^1,\beta^1)} \cdots \\
 & {\scriptstyle(\alpha^{-1},\beta^{-1})}\searrow & & \nearrow{\scriptstyle(\alpha^0,\beta^0)} & \\
 & & (X,Y,f,g) & &
\end{array}$$

We claim that the sequence $\mathsf{T}^\bullet$ is exact in $\mathsf{mod}\text{-}\Lambda_{(0,0)}$. First, since the following diagrams are commutative

$$\begin{array}{ccc}
(M \otimes_A P^i) \oplus (M \otimes_A N \otimes_B Q^i) & \xrightarrow{\left(\begin{smallmatrix} \mathrm{Id}_{M\otimes P^i} & 0 \\ 0 & 0 \end{smallmatrix}\right)} & (M \otimes_A P^i) \oplus Q^i \\
{\scriptstyle \left(\begin{smallmatrix} \mathrm{Id}_M \otimes d_P^i & 0 \\ \mathrm{Id}_M \otimes \gamma^{i+1} & \mathrm{Id}_{M\otimes N} \otimes d_Q^i \end{smallmatrix}\right)} \downarrow & & \downarrow {\scriptstyle \left(\begin{smallmatrix} \mathrm{Id}_M \otimes d_P^i & \delta^{i+1} \\ 0 & d_Q^i \end{smallmatrix}\right)} \\
(M \otimes_A P^{i+1}) \oplus (M \otimes_A N \otimes_B Q^{i+1}) & \xrightarrow{\left(\begin{smallmatrix} \mathrm{Id}_{M\otimes P^{i+1}} & 0 \\ 0 & 0 \end{smallmatrix}\right)} & (M \otimes_A P^{i+1}) \oplus Q^{i+1}
\end{array}$$

and

$$\begin{array}{ccc}
(N \otimes_B M \otimes_A P^i) \oplus (N \otimes_B Q^i) & \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 0 & \mathrm{Id}_{N\otimes Q^i} \end{smallmatrix}\right)} & P^i \oplus (N \otimes_B Q^i) \\
{\scriptstyle \left(\begin{smallmatrix} \mathrm{Id}_{N\otimes M} \otimes d_P^i & \mathrm{Id}_N \otimes \delta^{i+1} \\ 0 & \mathrm{Id}_N \otimes d_Q^i \end{smallmatrix}\right)} \downarrow & & \downarrow {\scriptstyle \left(\begin{smallmatrix} d_P^i & 0 \\ \gamma^{i+1} & \mathrm{Id}_N \otimes d_Q^i \end{smallmatrix}\right)} \\
(N \otimes_B M \otimes_A P^{i+1}) \oplus (N \otimes_B Q^{i+1}) & \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 0 & \mathrm{Id}_{N\otimes Q^{i+1}} \end{smallmatrix}\right)} & P^{i+1} \oplus (N \otimes_B Q^{i+1})
\end{array}$$

it follows that the maps

$$(\alpha^i, \beta^i) \colon \mathsf{T}_A(P^i) \oplus \mathsf{T}_B(Q^i) \longrightarrow \mathsf{T}_A(P^{i+1}) \oplus \mathsf{T}_B(Q^{i+1})$$

are morphisms in $\mathsf{mod}\text{-}\Lambda_{(0,0)}$. Since the complexes $(*)$ and $(**)$ are acyclic, it follows from Remark 2.1 (i) that $\mathsf{T}^\bullet$ is an exact sequence in $\mathsf{mod}\text{-}\Lambda_{(0,0)}$. Moreover, the object (X, Y, f, g) arises as the kernel of the morphism (α^1, β^1) and by Remark 2.1 (ii) we observe that $f = (\mathrm{Id}_M \otimes \pi_X) \circ t$ and $g = (\mathrm{Id}_N \otimes \pi_Y) \circ s$.

$\mathsf{Step\ 4}$: The final step of the proof is devoted to show that the acyclic complex $\mathsf{T}^\bullet$ is totally acyclic. From Proposition 2.2, it is enough to show that the complexes $\mathsf{Hom}_{\Lambda_{(0,0)}}(\mathsf{T}^\bullet, \mathsf{T}_A(P))$ and $\mathsf{Hom}_{\Lambda_{(0,0)}}(\mathsf{T}^\bullet, \mathsf{T}_B(Q))$ are acyclic, where P is a projective A-module and Q is a projective B-module. From Lemma 3.8 we have the exact sequence $0 \longrightarrow \mathsf{Z}_B(M \otimes_A P) \longrightarrow \mathsf{T}_A(P) \longrightarrow \mathsf{Z}_A(P) \longrightarrow 0$ and since each term of the complex $\mathsf{T}^\bullet$ is a projective $\Lambda_{(0,0)}$-module, it follows that the following sequence:

$$0 \longrightarrow \mathsf{Hom}_{\Lambda_{(0,0)}}(\mathsf{T}^\bullet, \mathsf{Z}_B(M \otimes_A P)) \longrightarrow \mathsf{Hom}_{\Lambda_{(0,0)}}(\mathsf{T}^\bullet, \mathsf{T}_A(P)) \longrightarrow \mathsf{Hom}_{\Lambda_{(0,0)}}(\mathsf{T}^\bullet, \mathsf{Z}_A(P)) \longrightarrow 0 \tag{3.3}$$

is an exact sequence of complexes. Then, from Lemma 3.9 we have the isomorphism $\mathsf{Hom}_{\Lambda_{(0,0)}}(\mathsf{T}^\bullet, \mathsf{Z}_A(P)) \cong \mathsf{Hom}_A(\mathsf{P}^\bullet, P)$ and since $\mathsf{P}^\bullet$ is totally acyclic we infer that $\mathsf{Hom}_{\Lambda_{(0,0)}}(\mathsf{T}^\bullet, \mathsf{Z}_A(P))$ is acyclic. Also, from Lemma 3.9 we have $\mathsf{Hom}_{\Lambda_{(0,0)}}(\mathsf{T}^\bullet, \mathsf{Z}_B(M \otimes_A P)) \cong \mathsf{Hom}_B(\mathsf{Q}^\bullet, M \otimes_A P)$ and since $M \otimes_A P$ lies in $(\mathsf{Gproj}\, B)^\perp$ by assumption (iv), it follows that the complex $\mathsf{Hom}_B(\mathsf{Q}^\bullet, M \otimes_A P)$ is acyclic. Then, the complex $\mathsf{Hom}_{\Lambda_{(0,0)}}(\mathsf{T}^\bullet, \mathsf{Z}_B(M \otimes_A P))$ is acyclic and therefore from the exact sequence (3.3), we infer that the complex $\mathsf{Hom}_{\Lambda_{(0,0)}}(\mathsf{T}^\bullet, \mathsf{T}_A(P))$ is acyclic. Similarly, using the exact sequence $0 \longrightarrow \mathsf{Z}_A(N \otimes_B Q) \longrightarrow \mathsf{T}_B(Q) \longrightarrow \mathsf{Z}_B(Q) \longrightarrow 0$ we derive that the complex $\mathsf{Hom}_{\Lambda_{(0,0)}}(\mathsf{T}^\bullet, \mathsf{T}_B(Q))$ is acyclic.

In conclusion, the $\Lambda_{(0,0)}$-module $(X, Y, (\mathrm{Id}_M \otimes \pi_X) \circ t, (\mathrm{Id}_N \otimes \pi_Y) \circ s)$ is Gorenstein-projective. □

Corollary 3.11 *Let $\Lambda_{(0,0)}$ be a Morita ring such that the conditions* (1) *or* (3)*, and* (2) *or* (4) *hold*:

(1) $\mathsf{pd}\, M_A < \infty$ *and* $\mathsf{pd}\, {}_A N < \infty$. (3) $\mathsf{pd}\, M_A < \infty$ *and* $\mathsf{id}\, {}_A N < \infty$.

(2) $\mathsf{pd}\, N_B < \infty$ *and* $\mathsf{pd}\, {}_B M < \infty$. (4) $\mathsf{pd}\, N_B < \infty$ *and* $\mathsf{id}\, {}_B M < \infty$.

(α) *Assume that for an A-module X there exists an exact sequence*

$$0 \longrightarrow N \otimes_B Z \xrightarrow{s} X \xrightarrow{\pi_X} \mathsf{Coker}\, s \longrightarrow 0$$

with $Z \in \mathsf{Gproj}\, B$ *and* $\mathsf{Coker}\, s \in \mathsf{Gproj}\, A$*, such that there is an exact sequence*

$$0 \longrightarrow M \otimes_A \mathsf{Coker}\, s \xrightarrow{t} Y \xrightarrow{\pi_Y} Z \longrightarrow 0$$

for some B*-module* Y*. Then the objects*: $(X, Y, (\mathrm{Id}_M \otimes \pi_X) \circ t, (\mathrm{Id}_N \otimes \pi_Y) \circ s)$, $\mathsf{T}_A(\mathsf{Coker}\, s)$, $\mathsf{T}_B(Z)$ *are Gorenstein-projective* $\Lambda_{(0,0)}$*-modules.*

(β) *Assume that for a* B*-module* Y *there exists an exact sequence*

$$0 \longrightarrow M \otimes_A Z \xrightarrow{t} Y \xrightarrow{\pi_Y} \mathsf{Coker}\, t \longrightarrow 0$$

with $Z \in \mathsf{Gproj}\, A$ *and* $\mathsf{Coker}\, t \in \mathsf{Gproj}\, B$*, such that there is an exact sequence*

$$0 \longrightarrow N \otimes_B \mathsf{Coker}\, t \xrightarrow{s} X \xrightarrow{\pi_X} Z \longrightarrow 0$$

for some A*-module* X*. Then the objects*: $(X, Y, (\mathrm{Id}_M \otimes \pi_X) \circ t, (\mathrm{Id}_N \otimes \pi_Y) \circ s)$, $\mathsf{T}_A(Z)$, $\mathsf{T}_B(\mathsf{Coker}\, t)$ *are Gorenstein-projective* $\Lambda_{(0,0)}$*-modules.*

Proof From Example 3.2 the conditions (i) – (iv) of Theorem 3.10 are satisfied. Then the result follows from Corollary 3.3 and Theorem 3.10. □

The next result is a consequence of Theorem 3.10 for the Morita ring $\Delta_{(0,0)}$. Recall that mod-$\Delta_{(0,0)}$ is the double morphism category $\mathsf{DMor}(\text{mod-}\Lambda)$ that we studied in Section 2.2.

Corollary 3.12 *Let* Λ *be an Artin algebra and consider the algebra* $\Delta_{(0,0)} = \left(\begin{smallmatrix} \Lambda & \Lambda \\ \Lambda & \Lambda \end{smallmatrix}\right)$*. Let* (X, Y, f, g) *be a* $\Delta_{(0,0)}$*-module such that there exist exact sequences*

$$0 \longrightarrow Z \xrightarrow{s} X \xrightarrow{\pi_X} W \longrightarrow 0$$

$$0 \longrightarrow W \xrightarrow{t} Y \xrightarrow{\pi_Y} Z \longrightarrow 0$$

with $Z, W \in \mathsf{Gproj}\, \Lambda$ *and set* $f := \pi_X \circ t$*,* $g := \pi_Y \circ s$*. Then the objects* (X, Y, f, g) *and* (Y, X, g, f) *are Gorenstein-projective* $\Delta_{(0,0)}$*-modules.*

Remark 3.13 Let (X, Y, f, g) be a $\Lambda_{(0,0)}$-module. Assume that for X and Y the conditions of Theorem 3.10 (α) are satisfied. Then, we cannot infer in general from Theorem 3.10 that (X, Y, f, g) lies in $\mathsf{Gproj}\, \Delta_{(0,0)}$. In other words, Theorem 3.10 does not provide us with sufficient conditions for a tuple (X, Y, f, g) to be Gorenstein-projective. We explain now where is the problem. Following the construction of Theorem 3.10, we conclude that the object $(X, Y, (\mathrm{Id}_M \otimes \pi_X) \circ t, (\mathrm{Id}_N \otimes \pi_Y) \circ s)$ is Gorenstein-projective. From Remark 2.1 (ii) we know that the maps $(\mathrm{Id}_M \otimes \pi_X) \circ t$ and $(\mathrm{Id}_N \otimes \pi_Y) \circ s$ are uniquely determined and satisfy the corresponding commutative diagrams (2.2). But since f and g are arbitrary maps, we don't know in general if they satisfy the diagrams (2.2). If f and g satisfy these diagrams, then from uniqueness it follows that $f = (\mathrm{Id}_M \otimes \pi_X) \circ t$, $g = (\mathrm{Id}_N \otimes \pi_Y) \circ s$ and therefore (X, Y, f, g) is Gorenstein-projective. Hence, we cannot conclude from Theorem 3.10 that (X, Y, f, g) is Gorenstein-projective.

The next example shows how we can apply Corollary 3.12 to construct Gorenstein-projective modules over $\Delta_{(0,0)}$ from Gorenstein-projective modules over the underlying triangular matrix algebras.

Example 3.14 Let Λ be an Artin algebra and consider the Morita ring $\Delta_{(0,0)} = \left(\begin{smallmatrix} \Lambda & \Lambda \\ \Lambda & \Lambda \end{smallmatrix}\right)$.

(i) Consider the lower triangular matrix algebra $\mathsf{T}_2(\Lambda) = \left(\begin{smallmatrix} \Lambda & 0 \\ \Lambda & \Lambda \end{smallmatrix}\right)$. From [47, Theorem 1.4], a triple (X, Y, f) is a Gorenstein-projective Γ-module if and only if there is an exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{\pi} \mathsf{Coker}\, f \longrightarrow 0 \tag{3.4}$$

such that the Λ-modules X and $\mathsf{Coker}\, f$ are Gorenstein-projective. Let (X, Y, f) be a Gorenstein-projective Γ-module. Thus, we have the sequence (3.4) and we also form the split exact sequence:

$$0 \longrightarrow \mathsf{Coker}\, f \xrightarrow{\left(1\ 0\right)} \mathsf{Coker}\, f \oplus X \xrightarrow{\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)} X \longrightarrow 0$$

Then, Corollary 3.12 yields that the objects

$$\left(Y, \mathsf{Coker}\, f \oplus X, \pi \circ \left(1\ 0\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \circ f\right) \quad \text{and} \quad \left(\mathsf{Coker}\, f \oplus X, Y, \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \circ f, \pi \circ \left(1\ 0\right)\right)$$

are Gorenstein-projective $\Delta_{(0,0)}$-modules. Consider now the upper triangular matrix algebra $\Sigma = \left(\begin{smallmatrix} \Lambda & \Lambda \\ 0 & \Lambda \end{smallmatrix}\right)$ and let $(Z, W, g) \in \mathsf{Gproj}\, \Sigma$. Then, from [47, Theorem 1.4] there is an exact sequence:

$$0 \longrightarrow W \xrightarrow{g} Z \xrightarrow{\rho} \mathsf{Coker}\, g \longrightarrow 0$$

such that the Λ-modules W and $\mathsf{Coker}\, g$ lie in $\mathsf{Gproj}\, \Lambda$, and we also have the split exact sequence:

$$0 \longrightarrow \mathsf{Coker}\, g \xrightarrow{\left(1\ 0\right)} \mathsf{Coker}\, g \oplus W \xrightarrow{\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)} W \longrightarrow 0$$

Hence, by Corollary 3.12 it follows that the following objects:

$$\left(Z, \mathsf{Coker}\, g \oplus W, \rho \circ \left(1\ 0\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \circ g\right) \quad \text{and} \quad \left(\mathsf{Coker}\, g \oplus W, Z, \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \circ g, \rho \circ \left(1\ 0\right)\right)$$

are Gorenstein-projective $\Delta_{(0,0)}$-modules.

(ii) Let X be a Gorenstein-projective Λ-module. From (i) the objects $(X, X, 0, \mathrm{Id}_X)$ and $(X, X, \mathrm{Id}_X, 0)$ are Gorenstein-projective $\Delta_{(0,0)}$-modules. Note that this was also observed in Corollary 3.6.

The above example shows that using Theorem 3.10, we obtain non-trivial examples of Gorenstein-projective modules over the Morita ring $\Delta_{(0,0)}$ from Gorenstein-projective modules of the triangular matrix algebras Γ and Σ. It should be noted that we don't know if all Gorenstein-projective modules over $\Delta_{(0,0)}$ arises in this way, as well as how many objects from $\mathsf{Gproj}\, \Delta_{(0,0)}$ we finally obtain.

We close this subsection with the following consequence of Corollary 3.12 and an example. We mention that Example 3.16 provides an interesting connection between our main result (Theorem 3.10) and the class of strongly Gorenstein-projective modules.

Corollary 3.15 *Let Λ be an Artin algebra and consider the algebra $\Delta_{(0,0)} = \left(\begin{smallmatrix} \Lambda & \Lambda \\ \Lambda & \Lambda \end{smallmatrix}\right)$. Let (X, Y, f, g) be a $\Delta_{(0,0)}$-module such that* $\mathsf{Im}\, f = \mathsf{Ker}\, g$, $\mathsf{Im}\, g = \mathsf{Ker}\, f$ *and assume that* $\mathsf{Im}\, f$

lies in $\mathsf{Gproj}\,\Lambda$. *Then* $(X, Y, f, g) \in \mathsf{Gproj}\,\Delta_{(0,0)}$ *if and only if* $X, Y \in \mathsf{Gproj}\,\Lambda$ *if and only if* $(Y, X, g, f) \in \mathsf{Gproj}\,\Delta_{(0,0)}$.

Proof Suppose first that X and Y are Gorenstein-projective Λ-modules. Then, from our assumptions the following complex:

$$\cdots \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} X \xrightarrow{f} Y \longrightarrow \cdots$$

is acyclic. Thus, we have the short exact sequences $0 \longrightarrow \mathsf{Im}\,g \longrightarrow X \longrightarrow \mathsf{Im}\,f \longrightarrow 0$ and $0 \longrightarrow \mathsf{Im}\,f \longrightarrow Y \longrightarrow \mathsf{Im}\,g \longrightarrow 0$. Since $\mathsf{Gproj}\,\Lambda$ is closed under kernels of epimorphisms, it follows that $\mathsf{Im}\,f \in \mathsf{Gproj}\,\Lambda$ if and only if $\mathsf{Im}\,g \in \mathsf{Gproj}\,\Lambda$. Then, for $Z = \mathsf{Im}\,g$ in Corollary 3.12, we get that the module (X, Y, f, g) is Gorenstein-projective. Note that, in this case, the maps of the tuple that we obtain from Corollary 3.12 are precisely f and g. Similarly, if $Z = \mathsf{Im}\,f$ then the tuple (Y, X, g, f) is Gorenstein-projective. The converse directions follow from Lemma 3.5. □

Example 3.16 Let Λ be an Artin algebra and consider the matrix algebra $\Delta_{(0,0)} = \begin{pmatrix} \Lambda & \Lambda \\ \Lambda & \Lambda \end{pmatrix}$. Let

$$\cdots \longrightarrow P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \longrightarrow \cdots$$

be a totally acyclic complex of projective Λ-modules. Then, from Corollary 3.15 it follows that (P, P, f, f) is a Gorenstein-projective $\Delta_{(0,0)}$-module. In this case, the Λ-module $\mathsf{Im}\,f$ is called strongly Gorenstein-projective. We refer to [13] for more details on this class of modules.

As a particular example, let $\mathbb{K}$ be a field, $\Lambda = \mathbb{K}[X]/(X^2)$ and consider the matrix algebra $\Delta_{(0,0)}$. Denote by $\overline{X}$ the residue class of X in Λ. Then by [13, Example 2.5] the following sequence

$$\cdots \longrightarrow \Lambda \xrightarrow{x} \Lambda \xrightarrow{x} \Lambda \xrightarrow{x} \Lambda \longrightarrow \cdots$$

is a totally acyclic complex of projective Λ-modules and $\overline{X} = \mathsf{Im}\,x = \mathsf{Ker}\,x$ is a strongly Gorenstein-projective Λ-module. We infer that (Λ, Λ, x, x) is a Gorenstein-projective $\Delta_{(0,0)}$-module.

Remark 3.17 By Corollary 3.15 we can instantly derive Example 3.14 (i). Indeed, let $f : X \longrightarrow Y$ be a monomorphism with $\mathsf{Coker}\,f$ in $\mathsf{Gproj}\,\Lambda$. Consider the maps $\begin{pmatrix} 1 & 0 \end{pmatrix} : \mathsf{Coker}\,f \longrightarrow \mathsf{Coker}\,f \oplus X$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix} : \mathsf{Coker}\,f \oplus X \longrightarrow X$. Then by Corollary 3.15 we get that $(Y, \mathsf{Coker}\,f \oplus X, \pi \circ \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \circ f)$ is a Gorenstein projective $\Delta_{(0,0)}$-module if and only if Y and $\mathsf{Coker}\,f \oplus X$ are Gorenstein-projectives if and only if Y and X are Gorenstein-projectives.

4 Homological Embeddings and Gorenstein Artin Algebras

Our purpose in this section is to provide a method for constructing Morita rings $\Lambda_{(0,0)} = \begin{pmatrix} A & {}_AN_B \\ {}_BM_A & B \end{pmatrix}$ which are Gorenstein Artin algebras. It turns out that our construction is strongly connected with the property of the functors $\mathsf{Z}_B : \mathsf{Mod}\text{-}B \longrightarrow \mathsf{Mod}\text{-}\Lambda_{(0,0)}$ and $\mathsf{Z}_A : \mathsf{Mod}\text{-}A \longrightarrow \mathsf{Mod}\text{-}\Lambda_{(0,0)}$ being homological embeddings. This section is divided into two subsections and the main result is stated in the second one.

4.1 Homological Embeddings

Let $\Lambda_{(\phi,\psi)} = \begin{pmatrix} A & {}_AN_B \\ {}_BM_A & B \end{pmatrix}$ be a Morita ring. Associated with the Morita ring $\Lambda_{(\phi,\psi)}$ are the following recollements of abelian categories (see Proposition 2.4):

$$\mathsf{Mod}\text{-}B/\operatorname{\mathsf{Im}}\phi \xrightarrow{\mathsf{I}_B} \mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)} \xrightarrow{\mathsf{U}_A} \mathsf{Mod}\text{-}A \qquad (\mathsf{T}_A,\ \mathsf{H}_A) \qquad\qquad \mathsf{Mod}\text{-}A/\operatorname{\mathsf{Im}}\psi \xrightarrow{\mathsf{I}_A} \mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)} \xrightarrow{\mathsf{U}_B} \mathsf{Mod}\text{-}B \qquad (\mathsf{T}_B,\ \mathsf{H}_B)$$

Using the idempotent elements $e = \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix}$ and $f = \begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix}$ of $\Lambda_{(\phi,\psi)}$, we obtain easily that $\mathsf{Mod}\text{-}\Lambda/\Lambda e\Lambda \simeq \mathsf{Mod}\text{-}B/\operatorname{\mathsf{Im}}\phi$, $\mathsf{Mod}\text{-}e\Lambda e \simeq \mathsf{Mod}\text{-}A$, $\mathsf{Mod}\text{-}\Lambda/\Lambda f\Lambda \simeq \mathsf{Mod}\text{-}A/\operatorname{\mathsf{Im}}\psi$ and $\mathsf{Mod}\text{-}f\Lambda f \simeq \mathsf{Mod}\text{-}B$. Note that for simplicity we denote the Morita ring $\Lambda_{(\phi,\psi)}$ by Λ.

In this subsection we investigate when the ideals $\langle e\rangle = \Lambda e\Lambda$ and $\langle f\rangle = \Lambda f\Lambda$ are stratifying. We recall first the notion of stratifying ideals due to Cline-Parshall-Scott [20].

Let R be a ring and e an idempotent element of R. Then we have the exact sequence

$$0 \longrightarrow \operatorname{\mathsf{Ker}}\mu_R \longrightarrow Re \otimes_{eRe} eR \xrightarrow{\mu_R} R \longrightarrow R/ReR \longrightarrow 0$$

where $\operatorname{\mathsf{Im}}\mu_R = ReR$ and $\operatorname{\mathsf{Ker}}\mu_R$ lies in $\mathsf{Mod}\text{-}R/ReR$. The ideal $\langle e\rangle = ReR$ is called **stratifying**, if the following two conditions hold:

(i) The multiplication map $Re \otimes_{eRe} eR \longrightarrow ReR$ is an isomorphism.
(ii) $\mathsf{Tor}^i_{eRe}(Re, eR) = 0$, for all $i > 0$.

The surjective ring homomorphism $R \longrightarrow R/ReR$ induces a fully faithful functor $\mathsf{I}_R\colon \mathsf{Mod}\text{-}R/ReR \longrightarrow \mathsf{Mod}\text{-}R$. Then it is known from [20] that the ideal $\langle e\rangle$ is stratifying if and only if the functor I_R is a **homological embedding** [36], i.e. the exact functor I_R induces an isomorphism $\mathsf{Ext}^n_{R/ReR}(X, Y) \cong \mathsf{Ext}^n_R(X, Y)$ for all X, Y in $\mathsf{Mod}\text{-}R/ReR$ and $n \geq 0$. For more on homological embeddings between abelian categories we refer to [36].

We now characterize when the ideals $\langle e\rangle$ and $\langle f\rangle$ are stratifying, or equivalently when the functors $\mathsf{I}_B\colon \mathsf{Mod}\text{-}B/\operatorname{\mathsf{Im}}\phi \longrightarrow \mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$ and $\mathsf{I}_A\colon \mathsf{Mod}\text{-}A/\operatorname{\mathsf{Im}}\psi \longrightarrow \mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}$ are homological embeddings.

Proposition 4.1 *Let $\Lambda_{(\phi,\psi)}$ be a Morita ring.*

(i) *The ideal $\langle e\rangle$ is stratifying if and only if the map $\phi\colon M \otimes_A N \longrightarrow B$ is a monomorphism and $\mathsf{Tor}^A_i(M, N) = 0$ for all $i > 0$.*
(ii) *The ideal $\langle f\rangle$ is stratifying if and only if the map $\psi\colon N \otimes_B M \longrightarrow A$ is a monomorphism and $\mathsf{Tor}^B_i(N, M) = 0$ for all $i > 0$.*

Proof We only prove (i) since part (ii) is dual. For simplicity we write Λ for the Morita ring $\Lambda_{(\phi,\psi)}$. An easy computation shows that $f\Lambda e = M$ and $e\Lambda f = N$. Then, since $\Lambda e = e\Lambda e \oplus f\Lambda e$ and $e\Lambda = e\Lambda e \oplus e\Lambda f$, we have the following isomorphisms:

$$\mathsf{Tor}^i_{e\Lambda e}(\Lambda e, e\Lambda) \cong \mathsf{Tor}^i_{e\Lambda e}(e\Lambda e \oplus f\Lambda e, e\Lambda e \oplus e\Lambda f) \cong \mathsf{Tor}^i_{e\Lambda e}(f\Lambda e, e\Lambda f) \cong \mathsf{Tor}^i_A(M, N)$$

Also, the canonical map $\mu_\Lambda\colon \Lambda e \otimes_{e\Lambda e} e\Lambda \longrightarrow \Lambda$ is a monomorphism if and only if the map $f\Lambda e \otimes_{e\Lambda e} e\Lambda \longrightarrow f\Lambda$ is a monomorphism if and only if the map $f\Lambda e \otimes_{e\Lambda e} e\Lambda f \longrightarrow f\Lambda f$ is a monomorphism, i.e. the map $\phi\colon M \otimes_A N \longrightarrow B$ is a monomorphism. Hence, we infer that the ideal $\langle e \rangle$ is stratifying if and only if $\mathsf{Tor}_i^A(M, N) = 0$ for all $i > 0$ and the map $\phi\colon M \otimes_A N \longrightarrow B$ is a monomorphism. $\square$

We provide examples of Morita rings where the conditions of Proposition 4.1 are satisfied.

Example 4.2 Let $\Lambda_{(\phi,\psi)}$ be a Morita ring. If $M = 0$ we have the upper triangular matrix ring $\Lambda = \left(\begin{smallmatrix} A & {}_AN_B \\ 0 & B \end{smallmatrix}\right)$ and the recollements $(\mathsf{Mod}\text{-}B, \mathsf{Mod}\text{-}\Lambda, \mathsf{Mod}\text{-}A)$ and $(\mathsf{Mod}\text{-}A, \mathsf{Mod}\text{-}\Lambda, \mathsf{Mod}\text{-}B)$, see [36, Example 2.12]. Then we obtain immediately from Proposition 4.1, that the functors $\mathsf{Z}_B\colon \mathsf{Mod}\text{-}B \longrightarrow \mathsf{Mod}\text{-}\Lambda$ and $\mathsf{Z}_A\colon \mathsf{Mod}\text{-}A \longrightarrow \mathsf{Mod}\text{-}\Lambda$ are homological embeddings. The same considerations hold when $N = 0$.

Example 4.3 Let $\Lambda_{(0,0)}$ be a Morita ring such that ${}_AN_B$ has an A-tight projective $\Lambda_{(0,0)}$-resolution and ${}_BM_A$ has a B-tight projective $\Lambda_{(0,0)}$-resolution, in the sense of [25]. This means that we have projective resolutions $\cdots \longrightarrow {}_AP_1 \longrightarrow {}_AP_0 \longrightarrow {}_AN \longrightarrow 0$ and $\cdots \longrightarrow {}_BQ_1 \longrightarrow {}_BQ_0 \longrightarrow {}_BM \longrightarrow 0$, such that $M \otimes_A P_i = 0$ and $N \otimes_B Q_i = 0$. Then, if we apply the functor $M \otimes_A -$ to the projective resolution of N we obtain that $M \otimes_A N = 0$ and $\mathsf{Tor}_i^A(M, N) = 0$ for all $i > 0$. Similarly, by applying the functor $N \otimes_B -$ to the projective resolution of M, it follows that $N \otimes_B M = 0$ and $\mathsf{Tor}_i^B(N, M) = 0$ for all $i > 0$. Hence, from Propositions 4.1 we infer that the functors $\mathsf{Z}_A\colon \mathsf{Mod}\text{-}A \longrightarrow \mathsf{Mod}\text{-}\Lambda_{(0,0)}$ and $\mathsf{Z}_B\colon \mathsf{Mod}\text{-}B \longrightarrow \mathsf{Mod}\text{-}\Lambda_{(0,0)}$ are homological embeddings. We refer to [25] for examples of Morita rings with tight resolutions.

Example 4.4 Let Λ be an Artin algebra with primitive idempotents $\{e_1, \dots, e_n\}$. Let $\{S_1, \dots, S_n\}$ be the corresponding simple Λ-modules. Assume that $S := S_1$ is localizable, i.e. $\mathsf{pd}_\Lambda S \leq 1$ and $\mathsf{Ext}^1_\Lambda(S, S) = 0$. If we consider the idempotent element $\alpha = e_2 + \cdots + e_n$, then it is easy to see that $\alpha(S_1) = 0$. This shows that $\mathsf{add}\, S$ is the kernel of the exact functor $\alpha\Lambda \otimes_\Lambda -\colon \mathsf{mod}\text{-}\Lambda \longrightarrow \mathsf{mod}\text{-}\alpha\Lambda\alpha$, in particular the category $\mathsf{mod}\text{-}\Lambda/\Lambda\alpha\Lambda$ is precisely the additive closure $\mathsf{add}\, S$ of S. From the short exact sequence $0 \longrightarrow \Lambda\alpha\Lambda \longrightarrow \Lambda \longrightarrow \Lambda/\Lambda\alpha\Lambda \longrightarrow 0$ and since $\mathsf{pd}_\Lambda \Lambda/\Lambda\alpha\Lambda \leq 1$, it follows that $\Lambda\alpha\Lambda$ is a projective Λ-module. Then by [29, Remark 3.2] we get that $\alpha\Lambda$ is a projective left $\alpha\Lambda\alpha$-module and the map $\Lambda\alpha \otimes_{\alpha\Lambda\alpha} \alpha\Lambda \longrightarrow \Lambda\alpha\Lambda$ is an isomorphism. This implies that the map $e_1\Lambda\alpha \otimes_{\alpha\Lambda\alpha} \alpha\Lambda e_1 \longrightarrow e_1\Lambda e_1$ is a monomorphism and $\mathsf{Tor}^i_{\alpha\Lambda\alpha}(e_1\Lambda\alpha, \alpha\Lambda e_1) = 0$ for all $i > 0$. Note that we view Λ as the Morita ring with $A = \alpha\Lambda\alpha$, $B = e_1\Lambda e_1$, $N = \alpha\Lambda e_1$ and $M = e_1\Lambda\alpha$, see Example 2.3 (i). Hence, from Proposition 4.1 we infer that the ideal $\Lambda\alpha\Lambda$ is stratifying. The above claim, that $\Lambda\alpha\Lambda$ being projective implies that $\Lambda\alpha\Lambda$ is a stratifying ideal, can be proved in a different way. We refer to [36, Example 3.14] for more details.

We restrict now to the case where the bimodule homomorphisms ϕ and ψ are zero, that is $\Lambda_{(0,0)}$ is the trivial extension $(A \times B) \ltimes M \oplus N$ and we have the recollements $(\mathsf{Mod}\text{-}A, \mathsf{Mod}\text{-}\Lambda_{(0,0)}, \mathsf{Mod}\text{-}B)$ and $(\mathsf{Mod}\text{-}B, \mathsf{Mod}\text{-}\Lambda_{(0,0)}, \mathsf{Mod}\text{-}A)$, see Proposition 2.4 and Example 2.3 (iii). The following result, which is due to Beligiannis [9, Corollary 4.4], shows that under some conditions we can compute the extension groups induced by the functors $\mathsf{Z}_A\colon \mathsf{Mod}\text{-}A \longrightarrow \mathsf{Mod}\text{-}\Lambda_{(0,0)}$ and $\mathsf{Z}_B\colon \mathsf{Mod}\text{-}B \longrightarrow \mathsf{Mod}\text{-}\Lambda_{(0,0)}$.

Lemma 4.5 *Let $\Lambda_{(0,0)}$ be a Morita ring. Assume that the right modules M_A and N_B are projective.*

(i) *For every A-modules X, X' and $n \geq 0$ there are the following isomorphisms:*

(a) *For* $n = 0, 1$: $\mathsf{Ext}^n_{\Lambda_{(0,0)}}(\mathsf{Z}_A(X), \mathsf{Z}_A(X')) \cong \mathsf{Ext}^n_A(X, X')$.

(b) *For* $n = 2k$: $\mathsf{Ext}^n_{\Lambda_{(0,0)}}(\mathsf{Z}_A(X), \mathsf{Z}_A(X')) \cong \mathsf{Ext}^{2k}_A(X, X') \oplus \mathsf{Ext}^{2(k-1)}_A(N \otimes_B M \otimes_A X, X') \oplus \mathsf{Ext}^{2(k-2)}_A((N \otimes_B M)^{\otimes^2} \otimes_A X, X') \oplus \cdots \oplus \mathsf{Hom}_A((N \otimes_B M)^{\otimes^k} \otimes_A X, X')$.

(c) *For* $n = 2k+1$: $\mathsf{Ext}^n_{\Lambda_{(0,0)}}(\mathsf{Z}_A(X), \mathsf{Z}_A(X')) \cong \mathsf{Ext}^{2k+1}_A(X, X') \oplus \mathsf{Ext}^{2k-1}_A(N \otimes_B M \otimes_A X, X') \oplus \mathsf{Ext}^{2k-3}_A((N \otimes_B M)^{\otimes^2} \otimes_A X, X') \oplus \cdots \oplus \mathsf{Ext}^1_A((N \otimes_B M)^{\otimes^k} \otimes_A X, X')$.

(ii) *For every B-modules Y, Y' and $n \geq 0$ there are the following isomorphisms*:

(a) *For* $n = 0, 1$: $\mathsf{Ext}^n_{\Lambda_{(0,0)}}(\mathsf{Z}_B(Y), \mathsf{Z}_B(Y')) \cong \mathsf{Ext}^n_B(Y, Y')$.

(b) *For* $n = 2k$: $\mathsf{Ext}^n_{\Lambda_{(0,0)}}(\mathsf{Z}_B(Y), \mathsf{Z}_B(Y')) \cong \mathsf{Ext}^{2k}_B(Y, Y') \oplus \mathsf{Ext}^{2(k-1)}_B(M \otimes_A N \otimes_B Y, Y') \oplus \mathsf{Ext}^{2(k-2)}_B((M \otimes_A N)^{\otimes^2} \otimes_B Y, Y') \oplus \cdots \oplus \mathsf{Hom}_B((M \otimes_A N)^{\otimes^k} \otimes_B Y, Y')$.

(c) *For* $n = 2k+1$: $\mathsf{Ext}^n_{\Lambda_{(0,0)}}(\mathsf{Z}_B(Y), \mathsf{Z}_B(Y')) \cong \mathsf{Ext}^{2k+1}_B(Y, Y') \oplus \mathsf{Ext}^{2k-1}_B(M \otimes_A N \otimes_B Y, Y') \oplus \mathsf{Ext}^{2k-3}_B((M \otimes_A N)^{\otimes^2} \otimes_B Y, Y') \oplus \cdots \oplus \mathsf{Ext}^1_B((M \otimes_A N)^{\otimes^k} \otimes_B Y, Y')$.

Proof We only sketch the proof of (ii), statement (i) follows similarly. From Proposition 2.4 the functor $\mathsf{Z}_B \colon \mathsf{Mod}\text{-}B \longrightarrow \mathsf{Mod}\text{-}\Lambda_{(0,0)}$ is fully faithful and from [36, Remark 3.7] we always have the isomorphism $\mathsf{Ext}^1_B(Y, Y') \cong \mathsf{Ext}^1_{\Lambda_{(0,0)}}(\mathsf{Z}_B(Y), \mathsf{Z}_B(Y'))$ for all B-modules Y and Y'. We explain now how we obtain the rest isomorphisms from [9, Corollary 4.4]. First, from Example 2.3 (iii) or Proposition 2.5 (i), the module category $\mathsf{Mod}\text{-}\Lambda_{(0,0)}$ is equivalent to the trivial extension $(\mathsf{Mod}\text{-}A \times \mathsf{Mod}\text{-}B) \ltimes H$, where H is the endofunctor $H \colon \mathsf{Mod}\text{-}A \times \mathsf{Mod}\text{-}B \longrightarrow \mathsf{Mod}\text{-}A \times \mathsf{Mod}\text{-}B$, $H(X, Y) = (N \otimes_B Y, M \otimes_A X)$. We refer to [22] for more details on trivial extensions of abelian categories. We compute only $\mathsf{Ext}^2_{\Lambda_{(0,0)}}(\mathsf{Z}_B(Y), \mathsf{Z}_B(Y'))$. Using the description of $\mathsf{Mod}\text{-}\Lambda_{(0,0)}$ as a trivial extension and [9, Corollary 4.4], it follows that $\mathsf{Ext}^2_{\Lambda_{(0,0)}}(\mathsf{Z}_B(Y), \mathsf{Z}_B(Y'))$ is isomorphic with the direct sum $\oplus_{i=0}^2 \mathsf{Ext}^i_{A \times B}(H^{2-i}(0, Y), (0, Y'))$. The latter extension group is isomorphic with $\mathsf{Ext}^2_B(Y, Y') \oplus \mathsf{Hom}_B(M \otimes_A N \otimes_B Y, Y')$, since $\mathsf{Ext}^1_{A \times B}(H(0, Y), (0, Y')) = \mathsf{Ext}^1_{A \times B}((N \otimes_B Y, 0), (0, Y')) = 0$. Hence, $\mathsf{Ext}^2_{\Lambda_{(0,0)}}(\mathsf{Z}_B(Y), \mathsf{Z}_B(Y')) \cong \mathsf{Ext}^2_B(Y, Y') \oplus \mathsf{Hom}_B(M \otimes_A N \otimes_B Y, Y')$. The rest isomorphisms follow in the same way, the details are left to the reader. □

As a consequence of Lemma 4.5 we have the next result. Note that it also follows from Proposition 4.1.

Corollary 4.6 *Let $\Lambda_{(0,0)}$ be a Morita ring such that the modules M_A and N_B are projective modules.*

(i) *The following are equivalent*:

(a) *The functor* $\mathsf{Z}_A \colon \mathsf{Mod}\text{-}A \longrightarrow \mathsf{Mod}\text{-}\Lambda_{(0,0)}$ *is a homological embedding.*

(b) $N \otimes_B M = 0$.

(ii) *The following are equivalent*:

(a) *The functor* $\mathsf{Z}_B \colon \mathsf{Mod}\text{-}B \longrightarrow \mathsf{Mod}\text{-}\Lambda_{(0,0)}$ *is a homological embedding.*

(b) $M \otimes_A N = 0$.

Proof (i) (a) $\Longrightarrow$ (b) If the functor Z_A is a homological embedding, then from Lemma 4.5 (i) we get that $\mathsf{Hom}_A(N \otimes_B M \otimes_A X, X') = 0$ for every A-module X and X'. We infer that $N \otimes_B M = 0$.

(b) $\Longrightarrow$ (a) If $N \otimes_B M = 0$, then from Lemma 4.5 (i) it follows that $\mathsf{Ext}^n_A(X, X') \cong \mathsf{Ext}^n_{\Lambda_{(0,0)}}(\mathsf{Z}_A(X), \mathsf{Z}_A(X'))$ for every A-module X, X' and $n \geq 0$.

(ii) This follows as in (i) using Lemma 4.5 (ii). □

The next result provides another reason for investigating stratifying ideals. It is a consequence of Proposition 4.1 and the well known result of Cline-Parshal-Scott [20] which relates stratifying ideals and recollements of derived module categories. For the notion of recollement of triangulated categories see [7], and for more details on deriving recollements of abelian categories we refer to [38].

Corollary 4.7 *Let $\Lambda_{(\phi,\psi)}$ be a Morita ring.*

(i) *If the map $\phi\colon M \otimes_A N \longrightarrow B$ is a monomorphism and $\mathsf{Tor}^A_i(M, N) = 0$ for all $i > 0$, then we have the following recollement of derived categories*:

$$\mathsf{D}(\mathsf{Mod}\text{-}B/\operatorname{Im}\phi) \xrightarrow{\mathsf{D}(\mathsf{I}_A)} \mathsf{D}(\mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}) \xrightarrow{\mathsf{D}(\mathsf{U}_A)} \mathsf{D}(\mathsf{Mod}\text{-}A)$$

(ii) *If the map $\psi\colon N \otimes_B M \longrightarrow A$ is a monomorphism and $\mathsf{Tor}^B_i(N, M) = 0$ for all $i > 0$, then we have the following recollement of derived categories*:

$$\mathsf{D}(\mathsf{Mod}\text{-}A/\operatorname{Im}\psi) \xrightarrow{\mathsf{D}(\mathsf{I}_B)} \mathsf{D}(\mathsf{Mod}\text{-}\Lambda_{(\phi,\psi)}) \xrightarrow{\mathsf{D}(\mathsf{U}_B)} \mathsf{D}(\mathsf{Mod}\text{-}B)$$

4.2 Gorenstein Algebras

Recall from [4, 27] that an Artin algebra Λ is called **Gorenstein** if $\mathsf{id}\,_\Lambda\Lambda < \infty$ and $\mathsf{id}\,\Lambda_\Lambda < \infty$. Equivalently, Λ is Gorenstein if and only if $\mathsf{spli}\,\Lambda = \sup\{\mathsf{pd}\,_\Lambda I \mid I \in \mathsf{inj}\,\Lambda\} < \infty$ and $\mathsf{silp}\,\Lambda = \sup\{\mathsf{id}\,_\Lambda P \mid P \in \mathsf{proj}\,\Lambda\} < \infty$, i.e. $\mathsf{mod}\text{-}\Lambda$ is a Gorenstein abelian category in the sense of [12].

We start with the next result which, under some conditions, gives isomorphisms between the extension groups induced from the adjoint pairs $(\mathsf{T}_A, \mathsf{U}_A)$ and $(\mathsf{T}_B, \mathsf{U}_B)$. It follows from [36, Theorem 3.10], but for completeness we give a direct proof.

Lemma 4.8 *Let $\Lambda_{(\phi,\psi)}$ be a Morita ring. Let X be an A-module and let Y be a B-module.*

(i) *Assume that the module M_A is projective. Then for every $\Lambda_{(\phi,\psi)}$-module (X', Y', f', g') and $n \geq 0$ we have an isomorphism*:

$$\mathsf{Ext}^n_{\Lambda_{(\phi,\psi)}}(\mathsf{T}_A(X), (X', Y', f', g')) \xrightarrow{\cong} \mathsf{Ext}^n_A(X, X')$$

(ii) *Assume that the module N_B is projective. Then for every $\Lambda_{(\phi,\psi)}$-module (X', Y', f', g') and $n \geq 0$ we have an isomorphism*:

$$\mathsf{Ext}^n_{\Lambda_{(\phi,\psi)}}(\mathsf{T}_B(Y), (X', Y', f', g')) \xrightarrow{\cong} \mathsf{Ext}^n_B(Y, Y')$$

Proof (i) Let X be an A-module and let $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$ be a projective resolution of X. Since the functor $M \otimes_A -$ is exact, it follows from Proposition 2.2 and Remark 2.1 that the sequence $\cdots \longrightarrow \mathsf{T}_A(P_1) \longrightarrow \mathsf{T}_A(P_0) \longrightarrow \mathsf{T}_A(X) \longrightarrow 0$ is a projective resolution of $\mathsf{T}_A(X)$. Let (X', Y', f', g') be a $\Lambda_{(\phi,\psi)}$-module. Then, using the adjoint pair $(\mathsf{T}_A, \mathsf{U}_A)$ we have the following commutative diagram:

$$\begin{array}{ccccccc}
(\mathsf{T}_A(X), (X', Y', f', g')) & \rightarrowtail & (\mathsf{T}_A(P_0), (X', Y', f', g')) & \longrightarrow & (\mathsf{T}_A(P_1), (X', Y', f', g')) & \longrightarrow & \cdots \\
\downarrow{\cong} & & \downarrow{\cong} & & \downarrow{\cong} & & \\
\mathsf{Hom}_A(X, X') & \rightarrowtail & \mathsf{Hom}_A(P_0, X') & \longrightarrow & \mathsf{Hom}_A(P_1, X') & \longrightarrow & \cdots
\end{array}$$

This implies that $\mathsf{Ext}^n_{\Lambda_{(\phi,\psi)}}(\mathsf{T}_A(X), (X', Y', f', g')) \cong \mathsf{Ext}^n_A(X, X')$ for every $n \geq 0$.

(ii) This follows similarly as in (i). □

Lemma 4.9 *Let $\Lambda_{(\phi,\psi)}$ be a Morita ring.*

(i) *Assume that M_A and N_B are projective modules. If* $\mathsf{id}\,_{\Lambda_{(\phi,\psi)}}\Lambda_{(\phi,\psi)} < \infty$, *then*:

$$\begin{cases} \mathsf{id}\,_A A < \infty, \quad \mathsf{id}\,_B B < \infty. \\ \mathsf{id}\,_A N < \infty, \quad \mathsf{id}\,_B M < \infty. \end{cases}$$

(ii) *Assume that $_B M$ and $_A N$ are projective modules. If* $\mathsf{id}\,\Lambda_{(\phi,\psi)}{}_{\Lambda_{(\phi,\psi)}} < \infty$, *then*:

$$\begin{cases} \mathsf{id}\,A_A < \infty, \quad \mathsf{id}\,B_B < \infty. \\ \mathsf{id}\,N_B < \infty, \quad \mathsf{id}\,M_A < \infty. \end{cases}$$

Proof (i) From Proposition 2.2 we have $\mathsf{id}\,_{\Lambda_{(\phi,\psi)}}\mathsf{T}_A(A) < \infty$ and $\mathsf{id}\,_{\Lambda_{(\phi,\psi)}}\mathsf{T}_B(B) < \infty$. Then, from Lemma 4.8 (i) we have the following isomorphisms for every A-module X and $n \geq 0$:

$$\mathsf{Ext}^n_{\Lambda_{(\phi,\psi)}}(\mathsf{T}_A(X), \mathsf{T}_A(A)) \cong \mathsf{Ext}^n_A(X, A) \quad \text{and} \quad \mathsf{Ext}^n_{\Lambda_{(\phi,\psi)}}(\mathsf{T}_A(X), \mathsf{T}_B(B)) \cong \mathsf{Ext}^n_A(X, N)$$

These isomorphisms imply that $\mathsf{id}\,_A A \leq \mathsf{id}\,_{\Lambda_{(\phi,\psi)}}\mathsf{T}_A(A) < \infty$ and $\mathsf{id}\,_A N \leq \mathsf{id}\,_{\Lambda_{(\phi,\psi)}}\mathsf{T}_B(B) < \infty$. Similarly, for every B-module Y and $n \geq 0$ we have from Lemma 4.8 (ii) the following isomorphisms:

$$\mathsf{Ext}^n_{\Lambda_{(\phi,\psi)}}(\mathsf{T}_B(Y), \mathsf{T}_A(A)) \cong \mathsf{Ext}^n_B(Y, M) \quad \text{and} \quad \mathsf{Ext}^n_{\Lambda_{(\phi,\psi)}}(\mathsf{T}_B(Y), \mathsf{T}_B(B)) \cong \mathsf{Ext}^n_B(Y, B)$$

Hence, $\mathsf{id}\,_B M < \infty$ and $\mathsf{id}\,_B B < \infty$.

(ii) In this part we use right modules. If X is a right A-module, then $\mathsf{T}_A(X) = (X, X \otimes_A N, \mathrm{Id}_{X \otimes N}, \Psi_X)$ and since $_A N$ is projective it follows that the functor $- \otimes_A N\colon \mathsf{mod}\text{-}A \longrightarrow \mathsf{mod}\text{-}B$ is exact and therefore T_A is exact. Similarly, since $_B M$ is projective we obtain that the functor T_B is exact. Then using the isomorphisms of Lemma 4.8, but for right modules now, the result follows as in case (i). □

It should be clear from the proof of the above result, that the assumption of M_A, resp. N_B, being projective, implies that $\mathsf{id}_A A < \infty$ and $\mathsf{id}_A N < \infty$, resp. $\mathsf{id}_B B < \infty$ and $\mathsf{id}_B M < \infty$. The same separation property also holds for part (ii). We continue with the next consequence of Lemma 4.9.

Corollary 4.10 *Let $\Lambda_{(\phi,\psi)}$ be a Morita ring which is a Gorenstein Artin algebra.*

(i) *If M_A is a projective right A-module and ${}_AN$ is a projective left A-module, then the algebra A is Gorenstein.*
(ii) *If N_B is a projective right B-module and ${}_BM$ is a projective left B-module, then the algebra B is Gorenstein.*

Let Λ be an Artin algebra and consider the Morita ring $\Delta_{(\phi,\phi)} = \left(\begin{smallmatrix} \Lambda & \Lambda \\ \Lambda & \Lambda \end{smallmatrix}\right)$. If $\Delta_{(\phi,\phi)}$ is Gorenstein, then by Corollary 4.10 the algebra Λ is also Gorenstein. We mention that this was observed in Proposition 3.4 (i), where the the converse also holds in this case. Hence, Corollary 4.10 generalizes the one direction of Proposition 3.4 (i). We give an example to show that the conditions in Corollary 4.10 are only sufficient.

Example 4.11 Let Λ be a bimodule d-Calabi-Yau noetherian algebra over a field k, where $d \geq 2$ is an integer. Let e be a non-trivial idempotent element of Λ such that $\Lambda/\Lambda e\Lambda$ is a finite dimensional k-algebra. By [1, Theorem 2.2] and it's proof, the algebra $e\Lambda e$ is Gorenstein and the $e\Lambda e$-module $e\Lambda$ is a non-projective Gorenstein-projective. Note that from [1, Proposition 2.4] the algebra Λ has finite global dimension and therefore Λ is Gorenstein.

In the rest of the subsection our aim is to consider the converse of Corollary 4.10, that is how the Gorensteinness of A and B should be inherited to the whole Morita ring. We first need the following preliminary result. As usual we denote by $\mathsf{D}\colon \mathsf{mod}\text{-}\Lambda \longrightarrow \mathsf{mod}\text{-}\Lambda^{\mathsf{op}}$ the duality for Artin algebras, see [5].

Lemma 4.12 *Let $\Lambda_{(0,0)} = \left(\begin{smallmatrix} A & {}_AN_B \\ {}_BM_A & B \end{smallmatrix}\right)$ be a Morita ring.*

(i) *Assume that $\mathsf{pd}_A N < \infty$. If $\mathsf{id}_{\Lambda_{(0,0)}} \mathsf{Z}_A(A) < \infty$ then $\mathsf{id}_{\Lambda_{(0,0)}} \mathsf{Z}_A(N) < \infty$.*
(ii) *Assume that $\mathsf{pd}_B M < \infty$. If $\mathsf{id}_{\Lambda_{(0,0)}} \mathsf{Z}_B(B) < \infty$ then $\mathsf{id}_{\Lambda_{(0,0)}} \mathsf{Z}_B(M) < \infty$.*
(iii) *Assume that $\mathsf{pd}\, M_A < \infty$. If $\mathsf{pd}_{\Lambda_{(0,0)}} \mathsf{Z}_A(\mathsf{D}(A)) < \infty$ then $\mathsf{pd}_{\Lambda_{(0,0)}} \mathsf{Z}_A\big(\mathsf{Hom}_B(M, \mathsf{D}(B))\big) < \infty$.*
(iv) *Assume that $\mathsf{pd}\, N_B < \infty$. If $\mathsf{pd}_{\Lambda_{(0,0)}} \mathsf{Z}_B(\mathsf{D}(B)) < \infty$ then $\mathsf{pd}_{\Lambda_{(0,0)}} \mathsf{Z}_B\big(\mathsf{Hom}_A(N, \mathsf{D}(A))\big) < \infty$.*

Proof (i) Let $0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow {}_AN \longrightarrow 0$ be a finite projective resolution of N. Then, since the functor $\mathsf{Z}_A\colon \mathsf{mod}\text{-}A \longrightarrow \mathsf{mod}\text{-}\Lambda_{(0,0)}$ is exact (see Proposition 2.4) and $\mathsf{id}_{\Lambda_{(0,0)}} \mathsf{Z}_A(A) < \infty$, it follows that $\mathsf{id}_{\Lambda_{(0,0)}} \mathsf{Z}_A(N) < \infty$. The proof of part (ii) is dual.

(iii) Since the projective dimension of M_A is finite if and only if the injective dimension of ${}_A\,\mathsf{D}(M)$ is finite and we have an isomorphism ${}_A\mathsf{Hom}_B({}_BM_A, {}_B\,\mathsf{D}(B)) \cong {}_A\mathsf{D}(M)$, then the result follows by applying the exact functor Z_A to a finite injective coresolution of ${}_A\,\mathsf{D}(M)$. The proof of part (iv) is dual. □

The following is the main result of this section which provides sufficient conditions for Morita rings $\Lambda_{(0,0)}$ with zero bimodule homomorphisms such that $\mathsf{silp}\, \Lambda_{(0,0)} < \infty$ and

$\mathsf{spli}\,\Lambda_{(0,0)} < \infty$. This result constitutes the second part of Theorem A presented in the Introduction.

Theorem 4.13 *Let* $\Lambda_{(0,0)} = \begin{pmatrix} A & {}_AN_B \\ {}_BM_A & B \end{pmatrix}$ *be a Morita ring.*

(i) *Assume the following conditions*:

(a) M_A *is projective and* $\mathsf{pd}\,{}_BM < \infty$.
(b) N_B *is projective and* $\mathsf{pd}\,{}_AN < \infty$.
(c) *The functors* Z_A, Z_B *are homological embeddings.*

If $\mathsf{silp}\,A < \infty$ *and* $\mathsf{silp}\,B < \infty$, *then* $\mathsf{silp}\,\Lambda_{(0,0)} < \infty$.

(ii) *Assume the following conditions*:

(a) ${}_BM$ *is projective and* $\mathsf{pd}\,M_A < \infty$.
(b) ${}_AN$ *is projective and* $\mathsf{pd}\,N_B < \infty$.
(c) *The functors* Z_A, Z_B *are homological embeddings.*

If $\mathsf{spli}\,A < \infty$ *and* $\mathsf{spli}\,B < \infty$, *then* $\mathsf{spli}\,\Lambda_{(0,0)} < \infty$.

Proof (i) From Proposition 2.2, it is enough to consider the projective $\Lambda_{(0,0)}$-modules $\mathsf{T}_A(A)$ and $\mathsf{T}_B(B)$. Assume that $\mathsf{pd}\,{}_BM = \kappa < \infty$ and $\mathsf{pd}\,{}_AN = \lambda < \infty$. From Lemma 3.8, we have the following exact sequences in $\mathsf{mod}\text{-}\Lambda_{(0,0)}$:

$$0 \longrightarrow \mathsf{Z}_B(M) \longrightarrow \mathsf{T}_A(A) \longrightarrow \mathsf{Z}_A(A) \longrightarrow 0 \tag{4.1}$$

and

$$0 \longrightarrow \mathsf{Z}_A(N) \longrightarrow \mathsf{T}_B(B) \longrightarrow \mathsf{Z}_B(B) \longrightarrow 0 \tag{4.2}$$

Thus, from Lemma 4.12 we have to show that $\mathsf{id}\,_{\Lambda_{(0,0)}}\mathsf{Z}_A(A) < \infty$ and $\mathsf{id}\,_{\Lambda_{(0,0)}}\mathsf{Z}_B(B) < \infty$. We first prove that $\mathsf{id}\,_{\Lambda_{(0,0)}}\mathsf{Z}_B(B) < \infty$. Let (X, Y, f, g) be a $\Lambda_{(0,0)}$-module. Then, from the morphism $(\mathrm{Id}_X, f)\colon \mathsf{T}_A(X) \longrightarrow (X, Y, f, g)$, we derive the following exact sequences in $\mathsf{mod}\text{-}\Lambda_{(0,0)}$:

$$0 \longrightarrow \mathsf{Z}_B(\mathsf{Ker}\,f) \longrightarrow \mathsf{T}_A(X) \longrightarrow (X, \mathsf{Im}\,f, k', 0) \longrightarrow 0 \tag{4.3}$$

and

$$0 \longrightarrow (X, \mathsf{Im}\,f, k', 0) \longrightarrow (X, Y, f, g) \longrightarrow \mathsf{Z}_B(\mathsf{Coker}\,f) \longrightarrow 0 \tag{4.4}$$

Applying the functor $\mathsf{Hom}_{\Lambda_{(0,0)}}(-, \mathsf{Z}_B(B))$ to the exact sequence (4.3), we obtain the following long exact Ext-sequence:

$$\cdots \to \mathsf{Ext}^n_{\Lambda_{(0,0)}}\big((X, \mathsf{Im}\,f, k', 0), \mathsf{Z}_B(B)\big) \to \mathsf{Ext}^n_{\Lambda_{(0,0)}}\big(\mathsf{T}_A(X), \mathsf{Z}_B(B)\big) \to \mathsf{Ext}^n_{\Lambda_{(0,0)}}\big(\mathsf{Z}_B(\mathsf{Ker}\,f), \mathsf{Z}_B(B)\big) \to \cdots$$

From Lemma 4.8 (i) it follows that $\mathsf{Ext}^n_{\Lambda_{(0,0)}}(\mathsf{T}_A(X), \mathsf{Z}_B(B)) = 0$ for every $n \geq 0$. Let $\mathsf{silp}\,B = \mu < \infty$. Since the functor $\mathsf{Z}_B\colon \mathsf{mod}\text{-}B \longrightarrow \mathsf{mod}\text{-}\Lambda_{(0,0)}$ is a homological embedding, we have $\mathsf{Ext}^n_{\Lambda_{(0,0)}}(\mathsf{Z}_B(\mathsf{Ker}\,f), \mathsf{Z}_B(B)) = 0$ for every $n \geq \mu + 1$. We infer that $\mathsf{Ext}^n_{\Lambda_{(0,0)}}((X, \mathsf{Im}\,f, k', 0), \mathsf{Z}_B(B)) = 0$ for every $n \geq \mu + 2$. Then from the following long exact sequence:

$$\cdots \to \mathsf{Ext}^n_{\Lambda_{(0,0)}}\big(\mathsf{Z}_B(\mathsf{Coker}\,f), \mathsf{Z}_B(B)\big) \to \mathsf{Ext}^n_{\Lambda_{(0,0)}}\big((X, Y, f, g), \mathsf{Z}_B(B)\big) \to \mathsf{Ext}^n_{\Lambda_{(0,0)}}\big((X, \mathsf{Im}\,f, k', 0), \mathsf{Z}_B(B)\big) \to \cdots$$

obtained from Eq. 4.4, it follows that $\mathsf{Ext}^n_{\Lambda_{(0,0)}}((X, Y, f, g), \mathsf{Z}_B(B)) = 0$ for every $n \geq \mu + 2$. Hence we have $\mathsf{id}_{\Lambda_{(0,0)}} \mathsf{Z}_B(B) \leq \mu + 1$ and therefore from Lemma 4.12 we infer that $\mathsf{id}_{\Lambda_{(0,0)}} \mathsf{Z}_B(M) \leq \kappa + \mu + 1$. Next, for the injective dimension of $\mathsf{Z}_A(A)$, we consider the following exact sequence:

$$0 \longrightarrow \mathsf{Z}_A(\mathsf{Ker}\, g) \longrightarrow \mathsf{T}_B(Y) \xrightarrow{(g, \mathrm{Id}_Y)} (X, Y, f, g) \longrightarrow \mathsf{Z}_A(\mathsf{Coker}\, g) \longrightarrow 0 \quad (4.5)$$

where $\mathsf{Im}\,(g, \mathrm{Id}_Y) = (\mathsf{Im}\, g, Y, 0, l')$. Let $\mathsf{silp}\, A = \nu < \infty$. Then, applying the functor $\mathsf{Hom}_{\Lambda_{(0,0)}}(-, \mathsf{Z}_A(A))$ to the two short exact sequences obtained from Eq. 4.5, we derive as above that $\mathsf{id}_{\Lambda_{(0,0)}} \mathsf{Z}_A(A) \leq \nu + 1$. Note that now we use Lemma 4.8 (ii) and that the functor $\mathsf{Z}_A \colon \mathsf{mod}\text{-}A \longrightarrow \mathsf{mod}\text{-}\Lambda_{(0,0)}$ is a homological embedding. Since $\mathsf{pd}_A N = \lambda < \infty$, it follows from Lemma 4.12 that $\mathsf{id}_{\Lambda_{(0,0)}} \mathsf{Z}_A(N) \leq \lambda + \nu + 1$. Hence, from the exact sequences (4.1) and (4.2) we have $\mathsf{id}_{\Lambda_{(0,0)}} \mathsf{T}_A(A) \leq \max\{\kappa + \mu, \nu\} + 1$ and $\mathsf{id}_{\Lambda_{(0,0)}} \mathsf{T}_B(B) \leq \max\{\lambda + \nu, \mu\} + 1$. We infer that $\mathsf{silp}\, \Lambda_{(0,0)} < \infty$.

(ii) This part follows by dual arguments but for completeness we sketch the proof. First, from Proposition 2.2 it is enough to consider the injective $\Lambda_{(0,0)}$-modules $\mathsf{H}_A(\mathsf{D}(A))$ and $\mathsf{H}_B(\mathsf{D}(B))$. Then, from Lemma 3.8 we have the exact sequences in $\mathsf{mod}\text{-}\Lambda_{(0,0)}$: $0 \longrightarrow \mathsf{Z}_A(\mathsf{D}(A)) \longrightarrow \mathsf{H}_A(\mathsf{D}(A)) \longrightarrow \mathsf{Z}_B(\mathsf{Hom}_A(N, \mathsf{D}(A))) \longrightarrow 0$ and $0 \longrightarrow \mathsf{Z}_B(\mathsf{D}(B)) \longrightarrow \mathsf{H}_B(\mathsf{D}(B)) \longrightarrow \mathsf{Z}_A(\mathsf{Hom}_B(M, \mathsf{D}(B))) \longrightarrow 0$. Thus, from Lemma 4.12 we have to show that $\mathsf{pd}_{\Lambda_{(0,0)}} \mathsf{Z}_A(\mathsf{D}(A)) < \infty$ and $\mathsf{pd}_{\Lambda_{(0,0)}} \mathsf{Z}_B(\mathsf{D}(B)) < \infty$. Also, for any $\Lambda_{(0,0)}$-module (X, Y, f, g) we obtain, from the units of the adjoint pairs $(\mathsf{U}_A, \mathsf{H}_A)$ and $(\mathsf{U}_B, \mathsf{H}_B)$, the exact sequences: $0 \longrightarrow \mathsf{Z}_A(\mathsf{Ker}\, \pi(f)) \longrightarrow (X, Y, f, g) \longrightarrow \mathsf{H}_B(Y) \longrightarrow \mathsf{Z}_A(\mathsf{Coker}\, \pi(f)) \longrightarrow 0$ and $0 \longrightarrow \mathsf{Z}_B(\mathsf{Ker}\, \rho(g)) \longrightarrow (X, Y, f, g) \longrightarrow \mathsf{H}_A(X) \longrightarrow \mathsf{Z}_B(\mathsf{Coker}\, \rho(g)) \longrightarrow 0$. Then, similarly with part (i) we show that $\mathsf{spli}\, \Lambda_{(0,0)} < \infty$. The details are left to the reader. □

As a consequence we have the next result on the finiteness of the global dimension of $\Lambda_{(0,0)}$.

Corollary 4.14 *Let* $\Lambda_{(0,0)} = \begin{pmatrix} A & {}_AN_B \\ {}_BM_A & B \end{pmatrix}$ *be a Morita ring such that the modules* M_A, N_B *are projective and the functors* Z_A, Z_B *are homological embeddings. If* $\mathsf{gl.dim}\, A < \infty$ *and* $\mathsf{gl.dim}\, B < \infty$, *then* $\mathsf{gl.dim}\, \Lambda_{(0,0)} < \infty$.

Proof By the proof of Theorem 4.13 we have that $\mathsf{id}_{\Lambda_{(0,0)}} \mathsf{Z}_A(N) < \infty$ and $\mathsf{id}_{\Lambda_{(0,0)}} \mathsf{Z}_B(M) < \infty$. Since $\mathsf{spli}\, \Lambda_{(0,0)} < \infty$ it follows that $\mathsf{pd}_{\Lambda_{(0,0)}} \mathsf{Z}_A(N) < \infty$ and $\mathsf{pd}_{\Lambda_{(0,0)}} \mathsf{Z}_B(M) < \infty$. Using that $\Lambda_{(0,0)}$ is the trivial extension ring $(A \times B) \ltimes M \oplus N$ (Example 2.3 (iii)) and [25, Proposition 5.19], we infer that the global dimension of $\Lambda_{(0,0)}$ is finite. □

We continue with the following result which gives us a class of Morita rings, in particular a class of trivial extension rings (Example 2.3 (iii)), where Theorem 4.13 can be applied.

Corollary 4.15 *Let* $\Lambda_{(0,0)} = \begin{pmatrix} A & {}_AN_A \\ {}_AN_A & A \end{pmatrix}$ *be a Morita ring. Assume the following conditions:*

(a) N_A *and* ${}_AN$ *are projective.*
(b) $N \otimes_A N = 0$.

If A *is Gorenstein, then the ring* $\Lambda_{(0,0)}$ *is Gorenstein.*

Proof The condition $N \otimes_A N = 0$ implies that the functor $\mathsf{Z}_A \colon \mathsf{mod}\text{-}A \longrightarrow \mathsf{mod}\text{-}\Lambda_{(0,0)}$ is a homological embedding, see Corollary 4.16. The second functor we have to check that is a homological embedding is $\mathsf{Z}'_A \colon \mathsf{mod}\text{-}A \longrightarrow \mathsf{mod}\text{-}\Lambda_{(0,0)}$ given by $\mathsf{Z}'_A(X) = (0, X, 0, 0)$. Note that this is the functor Z_2 in the notation of Section 2.2. Exactly in the same way with Remark 2.6 (iii), we show that the two recollements of $\mathsf{mod}\text{-}\Lambda_{(0,0)}$ are equivalent. Both of them are of the form $(\mathsf{mod}\text{-}A, \mathsf{mod}\text{-}\Lambda_{(0,0)}, \mathsf{mod}\text{-}A)$, see Proposition 2.4. Using this equivalence it follows that Z_A is a homological embedding if and only if Z'_A is a homological embedding. Then the result follows from Theorem 4.13. □

The above method for constructing Gorenstein algebras is illustrated in the next example.

Example 4.16 Let A be a finite dimensional Gorenstein k-algebra, where k is a field, and let e and f be two idempotents elements of A such that $fAe = 0$. Consider the A-A-bimodule $N := Ae \otimes_k fA$. Then it follows easily that $N \otimes_A N = 0$ and therefore from Corollary 4.15 we get the Gorenstein algebra:

$$\Lambda_{(0,0)} = \begin{pmatrix} A & {}_AN_A \\ {}_AN_A & A \end{pmatrix}$$

Note that $\Lambda_{(0,0)}$ is the trivial extension algebra $(A \times A) \ltimes N \oplus N$, see Example 2.3 (iii).

We close this section with an example of a Morita ring which is a Gorenstein algebra and the conditions of Theorem 4.13 are not satisfied.

Example 4.17 Let A be a ring and M be a right A-module. Then from Example 2.3 (ii) we have the Morita ring

$$\Lambda_{(\phi,\psi)} = \begin{pmatrix} B & {}_BM_A \\ M^* & A \end{pmatrix}$$

where $B = \mathsf{End}_A(M)$ and $M^* = {}_A\mathsf{Hom}_A(M, A)_B$. Note that this Morita ring is the Auslander context, in the sense of Buchweitz [15], defined by the pair (A, M). If M_A is a finitely generated projective right A-module, then from [15, Proposition 2.6, Corollary 1.10] it follows that the rings A and $\Lambda_{(\phi,\psi)}$ are Morita equivalent, and therefore A is Gorenstein if and only if $\Lambda_{(\phi,\psi)}$ is Gorenstein. Hence, if A is a Gorenstein algebra and M_A is a finitely generated projective module, then the Morita ring $\Lambda_{(\phi,\psi)}$ is Gorenstein. By Example 2.3 (ii), the bimodule homomorphisms of this Morita ring are not zero, and also the rest assumptions of Theorem 4.13 are not satisfied in general.

5 Gorenstein Subcategories and Coherent Functors

In this section we study the monomorphism category $\mathsf{mono}(\Lambda)$, see Eq. 2.5 in Section 2.3. In particular, we investigate the full subcategory $\mathscr{C}$ of $\mathsf{mono}(\Lambda)$ consisting of all monomorphisms $f \colon X \longrightarrow Y$ such that the projective dimension of X is finite. In the first subsection, we show that it is a Gorenstein subcategory of $\mathsf{mono}(\Lambda)$ when Λ is a Gorenstein Artin algebra. In the second subsection, we prove that the category of coherent functors over the stable category of $\mathscr{C}$ is a Gorenstein abelian category.

5.1 The Gorenstein Subcategory of $\mathsf{mono}(\Lambda)$

Let $\mathscr{A}$ be an abelian category with enough projective and injective objects and let n be a non-negative integer. Recall from [12, Theorem 2.2, Chapter VII] that $\mathscr{A}$ is n-Gorenstein if and only if every object has Gorenstein-projective dimension at most n. For our purpose, we need the notion of a Gorenstein subcategory but now in the context of exact categories (see Section 2.3). Before that, we define Gorenstein-projective objects for exact categories.

Definition 5.1 Let $\mathscr{A} = (\mathscr{A}, \mathscr{E})$ be an exact category with enough projective objects. An object X in $\mathscr{A}$ is called **Gorenstein-projective** if there is an $\mathscr{E}$-acyclic complex of projective objects in $\mathscr{A}$:

$$\mathsf{P}^\bullet: \quad \cdots \longrightarrow P^{-1} \longrightarrow P^0 \xrightarrow{d^0} P^1 \longrightarrow P^2 \longrightarrow \cdots, \qquad \kappa: P^0 \to X, \quad \lambda: X \to P^1$$

such that $\mathsf{Hom}_{\mathscr{A}}(\mathsf{P}^\bullet, P)$ is acyclic for every object P in $\mathsf{Proj}\,\mathscr{A}$ and $d^0 = \lambda \circ \kappa$, where $\kappa\colon P^0 \longrightarrow X$ is a deflation and $\lambda\colon X \longrightarrow P^1$ is an inflation. We denote by $\mathsf{GProj}\,\mathscr{A}$ the full subcategory of Gorenstein-projective objects of $\mathscr{A}$.

For a complex being acyclic in an exact category we refer to [16, Definition 10.1]. From now on, when we write $\mathscr{A}$ for an exact category we fix a class $\mathscr{E}$ of exact pairs.

Definition 5.2 Let $\mathscr{A}$ be an exact category with enough projective objects. Then $\mathscr{A}$ is ***n*-Gorenstein** for some non-negative integer n if every object has Gorenstein-projective dimension at most n. Let $\mathscr{B}$ be an exact subcategory of $\mathscr{A}$. We call $\mathscr{B}$ an ***n*-Gorenstein subcategory** of $\mathscr{A}$, if for all X in $\mathscr{B}$ there exists an exact sequence $0 \longrightarrow G_n \longrightarrow \cdots \longrightarrow G_0 \longrightarrow X \longrightarrow 0$ in $\mathscr{B}$ such that $G_j \in \mathsf{GProj}\,\mathscr{A}$ for all $0 \leq j \leq n$.

Consider now the following subcategory of $\mathsf{mono}(\Lambda)$:

$$\mathscr{C} := \big\{(X, Y, f, 0) \in \mathsf{mono}(\Lambda) \mid \mathsf{pd}_{\Lambda} X < \infty\big\}. \tag{5.1}$$

We denote by $\mathscr{P}^{<\infty}(\Lambda)$ the full subcategory of $\mathsf{mod}\text{-}\Lambda$ consisting of all Λ-modules of finite projective dimension. Then $\mathscr{P}^{<\infty}(\Lambda)$ is an exact subcategory of $\mathsf{mod}\text{-}\Lambda$, since it is extension closed, and this implies that $\mathscr{C}$ is also an exact subcategory of $\mathsf{mono}(\Lambda)$. The first main result on the structure of $\mathscr{C}$ is as follows. This result constitutes the first part of Theorem B presented in the Introduction.

Theorem 5.3 *Let Λ be an n-Gorenstein algebra for some non-negative integer n. Then $\mathscr{C}$ is an n-Gorenstein subcategory of* $\mathsf{mono}(\Lambda)$.

Proof Let $(X, Y, f, 0)$ be an object in $\mathscr{C}$ and consider the following exact sequence in $\mathsf{mono}(\Lambda)$:

$$0 \longrightarrow \mathsf{T}_1(X) \xrightarrow{(\mathrm{Id}_X, f)} (X, Y, f, 0) \xrightarrow{(0,p)} \mathsf{Z}_2(\mathsf{Coker}\, f) \longrightarrow 0 \tag{5.2}$$

Since $(\mathsf{U}_2, \mathsf{T}_1)$ is an adjoint pair of exact functors and both functors preserve projective objects (Lemma 2.9), we have the isomorphism $\mathsf{Ext}^i_{\mathsf{mono}(\Lambda)}((G_1, G_2, f, 0), \mathsf{T}_1(X)) \cong \mathsf{Ext}^i_{\Lambda}(\mathsf{U}_2(G_1, G_2, f, 0), X) = \mathsf{Ext}^i_{\Lambda}(G_2, X)$ for all $i \geq 1$ and $(G_1, G_2, f, 0)$ in $\mathsf{Gproj}(\mathsf{mono}(\Lambda))$. Since $\mathsf{mono}(\Lambda)$ has the same projectives as $\mathsf{mod}\text{-}\mathsf{T}_2(\Lambda)$, it follows that it has the same Gorenstein-projective objects as $\mathsf{mod}\text{-}\mathsf{T}_2(\Lambda)$. Thus, [31, Theorem 1.1] yields that the Λ-module G_2 is Gorenstein-projective. Since $\mathsf{pd}_{\Lambda} X < \infty$, it follows that

$\mathsf{Ext}^i_\Lambda(G_2, X) = 0$ for all $i \geq 1$ (recall that $(\mathsf{Gproj}\,\Lambda, \mathscr{P}^{<\infty}(\Lambda))$ is a cotorsion pair in mod-Λ, see [12]). Hence, we have $\mathsf{Ext}^i_{\mathsf{mono}(\Lambda)}((G_1, G_2, f, 0), \mathsf{T}_1(X)) = 0$ for all $i \geq 1$ and $(G_1, G_2, f, 0)$ in $\mathsf{Gproj}(\mathsf{mono}(\Lambda))$. This implies that Eq. 5.2 remains exact after applying $\mathsf{Hom}_{\mathsf{mono}(\Lambda)}((G_1, G_2, f, 0), -)$, for every $(G_1, G_2, f, 0)$ in $\mathsf{Gproj}(\mathsf{mono}(\Lambda))$. Since the algebra Λ is n-Gorenstein, there exist the following two exact sequences of left Λ-modules:

$$0 \longrightarrow P_n \xrightarrow{a_n} \cdots \longrightarrow P_1 \xrightarrow{a_1} P_0 \xrightarrow{a_0} X \longrightarrow 0$$

and

$$0 \longrightarrow G_n \xrightarrow{b_n} \cdots \longrightarrow G_1 \xrightarrow{b_1} G_0 \xrightarrow{b_0} \mathsf{Coker}\, f \longrightarrow 0$$

where P_j and G_j are Gorenstein-projective Λ-modules for all $0 \leq j \leq n$. Applying the exact functors T_1 and Z_2, respectively, we get the exact sequences in $\mathscr{C}$:

$$0 \longrightarrow \mathsf{T}_1(P_n) \xrightarrow{\mathsf{T}_1(a_n)} \cdots \longrightarrow \mathsf{T}_1(P_1) \xrightarrow{\mathsf{T}_1(a_1)} \mathsf{T}_1(P_0) \xrightarrow{\mathsf{T}_1(a_0)} \mathsf{T}_1(X) \longrightarrow 0$$

and

$$0 \longrightarrow \mathsf{Z}_2(G_n) \xrightarrow{\mathsf{Z}_2(b_n)} \cdots \longrightarrow \mathsf{Z}_2(G_1) \xrightarrow{\mathsf{Z}_2(b_1)} \mathsf{Z}_2(G_0) \xrightarrow{\mathsf{Z}_2(b_0)} \mathsf{Z}_2(\mathsf{Coker}\, f) \longrightarrow 0$$

where $\mathsf{T}_1(P_j)$ and $\mathsf{Z}_2(G_j)$ belong to $\mathsf{Gproj}(\mathsf{mono}(\Lambda))$, for all $0 \leq j \leq n$, by [31, Theorem 1.1] again. Since the map $\mathsf{Hom}_{(\mathsf{mono}(\Lambda))}(\mathsf{Z}_2(G_0), (0, p))$ is surjective, we obtain from the Horseshoe Lemma the following exact commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathsf{T}_1(P_0) & \xrightarrow{(1\ 0)} & \mathsf{T}_1(P_0) \oplus \mathsf{Z}_2(G_0) & \xrightarrow{{}^t(0\ 1)} & \mathsf{Z}_2(G_0) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathsf{T}_1(a_0)} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \mathsf{Z}_2(b_0)} & & \\
0 & \longrightarrow & \mathsf{T}_1(X) & \xrightarrow{(\mathrm{Id}_X, f)} & (X, Y, f, 0) & \xrightarrow{(0,p)} & \mathsf{Z}_2(\mathsf{Coker}\, f) & \longrightarrow & 0
\end{array}$$

Now taking the exact sequence of the kernels and applying the functor $\mathsf{Hom}_{(\mathsf{mono}(\Lambda))}(\mathsf{Z}_2(G_1), -)$, we obtain that the map $\mathsf{Hom}_{(\mathsf{mono}(\Lambda))}(\mathsf{Z}_2(G_1), \mathsf{Ker}\, a_0) \longrightarrow \mathsf{Hom}_{\mathsf{mono}(\Lambda)}(\mathsf{Z}_2(G_1), \mathsf{Z}_2(\mathsf{Ker}\, b_0))$ is surjective. This follows since $\mathsf{Ext}^1_{\mathsf{mono}(\Lambda)}(\mathsf{Z}_2(G_1), \mathsf{T}_1(\mathsf{Ker}\, a_0)) \cong \mathsf{Ext}^1_\Lambda(G_1, \mathsf{Ker}\, a_0) = 0$. Then continuing in the same way we construct an exact sequence of $(X, Y, f, 0)$ by objects in $\mathsf{Gproj}(\mathsf{mono}(\Lambda))$ of length at most n. We infer that $\mathscr{C}$ is an n-Gorenstein subcategory of $\mathsf{mono}(\Lambda)$. □

5.2 Categories of Coherent Functors and Gorensteinness

It is known by [14, 34] that the singularity category $\mathsf{D}_{\mathsf{sg}}(\Lambda)$ of an algebra Λ is defined as the Verdier quotient $\mathsf{D}^{\mathsf{b}}(\text{mod-}\Lambda)/\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)$. When we deal with an additive category $\mathscr{A}$, the notion of singularity category can be extended using the category of coherent functors over $\mathscr{A}$. This approach was recently investigated by Matsui and Takahashi [32]. We now recall this. Let $\mathscr{A}$ be an additive category with weak kernels, that is, for each morphism $f: X \longrightarrow Y$ in $\mathscr{A}$ there exists a morphism $g: Z \longrightarrow X$ in $\mathscr{A}$ such that the sequence $\mathsf{Hom}_{\mathscr{A}}(-, Z) \longrightarrow \mathsf{Hom}_{\mathscr{A}}(-, X) \longrightarrow \mathsf{Hom}_{\mathscr{A}}(-, Y)$ is exact. We denote by mod-$\mathscr{A}$ the category of coherent functors over $\mathscr{A}$, i.e. functors $F: \mathscr{A}^{\mathsf{op}} \longrightarrow \mathscr{A}b$ such that there is an exact sequence $\mathsf{Hom}_{\mathscr{A}}(-, X) \longrightarrow \mathsf{Hom}_{\mathscr{A}}(-, Y) \longrightarrow F \longrightarrow 0$ with X and Y in $\mathscr{A}$. It is known that mod-$\mathscr{A}$ is an abelian category with enough projective objects. We refer to

[2, 3] for more details on coherent functors. Then, Matsui and Takahashi [32] considered the Verdier quotient

$$\mathsf{D}_{\mathsf{sg}}(\mathsf{mod}\text{-}\mathscr{A}) := \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\text{-}\mathscr{A})/\mathsf{K}^{\mathsf{b}}(\mathsf{proj}(\mathsf{mod}\text{-}\mathscr{A}))$$

and call it the **singularity category** of $\mathsf{mod}\text{-}\mathscr{A}$. We remark that this triangulated category is included in the general framework of the stabilization of an abelian or exact category studied by Beligiannis [8].

In what follows, we show that the singularity category of $\mathsf{mono}(\Lambda)$ is trivial. We write $\mathsf{mod}\text{-}\mathsf{mono}(\Lambda)$ for the category of coherent functors over the monomorphism category $\mathsf{mono}(\Lambda)$.

Proposition 5.4 *Let Λ be an Artin algebra. Then the following hold*

(i) *The category* $\mathsf{mod}\text{-}\mathsf{mono}(\Lambda)$ *is abelian.*
(ii) *We have*: $\mathsf{gl.dim}(\mathsf{mod}\text{-}(\mathsf{mod}\text{-}\Lambda)) \leq \mathsf{gl.dim}(\mathsf{mod}\text{-}\mathsf{mono}(\Lambda)) \leq 2$.
(iii) *The singularity category* $\mathsf{D}_{\mathsf{sg}}(\mathsf{mod}\text{-}\mathsf{mono}(\Lambda))$ *is trivial.*

Proof (i) Since $\mathsf{mono}(\Lambda)$ is closed under kernels by Lemma 2.9, it follows that $\mathsf{mono}(\Lambda)$ has weak kernels. Hence, the category of coherent functors $\mathsf{mod}\text{-}(\mathsf{mono}(\Lambda))$ is abelian.

(ii) Let F be a functor in $\mathsf{mod}\text{-}\mathsf{mono}(\Lambda)$, that is, there is an exact sequence:

$$\mathsf{Hom}_{\mathsf{mono}(\Lambda)}(-,(X_1,Y_1,f_1,0)) \xrightarrow{(-,(a,b))} \mathsf{Hom}_{\mathsf{mono}(\Lambda)}(-,(X_0,Y_0,f_0,0)) \longrightarrow F \longrightarrow 0$$

where $(X_1, Y_1, f_1, 0)$ and $(X_0, Y_0, f_0, 0)$ are objects in $\mathsf{mono}(\Lambda)$. Since we have the exact sequence

$$0 \longrightarrow (\mathsf{Ker}\, a, \mathsf{Ker}\, b, k, 0) \longrightarrow (X_1, Y_1, f_1, 0) \xrightarrow{(a,b)} (X_0, Y_0, f_0, 0),$$

and $\mathsf{Ker}\,(a,b) = (\mathsf{Ker}\, a, \mathsf{Ker}\, b, k, 0)$ lies in $\mathsf{mono}(\Lambda)$, we obtain the following exact sequence:

$$0 \longrightarrow (-, \mathsf{Ker}\,(a,b)) \longrightarrow (-,(X_1,Y_1,f_1,0)) \longrightarrow (-,(X_0,Y_0,f_0,0)) \longrightarrow F \longrightarrow 0$$

This implies that $\mathsf{gl.dim}(\mathsf{mod}\text{-}\mathsf{mono}(\Lambda)) \leq 2$. From Lemma 2.9 we know that $(\mathsf{T}_1, \mathsf{U}_1)$ is an adjoint pair between $\mathsf{mod}\text{-}\Lambda$ and $\mathsf{mono}(\Lambda)$ and the functor T_1 is fully faithful. Then [44, Theorem 3.1] yields that $\mathsf{gl.dim}(\mathsf{mod}\text{-}(\mathsf{mod}\text{-}\Lambda)) \leq \mathsf{gl.dim}(\mathsf{mod}\text{-}\mathsf{mono}(\Lambda))$. This completes the proof of (ii).

(iii) This statement follows immediately from (ii). □

Although that the singularity category $\mathsf{D}_{\mathsf{sg}}(\mathsf{mod}\text{-}\mathsf{mono}(\Lambda))$ is trivial, we show in Corollary 5.8 that if we restrict to the subcategory $\mathscr{C}$ of $\mathsf{mono}(\Lambda)$, then this singularity category is not at all trivial.

Before we get there we need some more definitions.

Definition 5.5 An additive subcategory $\mathscr{B}$ of $\mathscr{A}$ is called **quasi-resolving** if it contains $\mathsf{Proj}\,\mathscr{A}$ and given a conflation $X \longrightarrow Y \longrightarrow Z$ with Y and Z in $\mathscr{B}$ then the object X lies in $\mathscr{B}$. A quasi-resolving subcategory $\mathscr{B}$ is called **resolving** if it is closed under direct summands and extensions, i.e. given a conflation $X \longrightarrow Y \longrightarrow Z$ with X and Z in $\mathscr{B}$ then the object Y lies in $\mathscr{B}$.

Note that a resolving subcategory $\mathscr{B}$ of $\mathscr{A}$ is an exact subcategory of $\mathscr{A}$ since it is closed under extensions. Let X be an object in $\mathscr{A}$. Since $\mathscr{A}$ has enough projective objects there exists a deflation $g\colon P \longrightarrow X$ with $P \in \mathsf{Proj}\,\mathscr{A}$ (or a right $\mathsf{Proj}\,\mathscr{A}$-approximation). This means that there is an exact pair $K \longrightarrow P \longrightarrow X$, where the map $f\colon K \longrightarrow P$ is an inflation. The object K is called the first syzygy of X and is denoted by $\Omega(X)$. The nth syzygy $\Omega^n(X)$ of X is defined inductively as $\Omega(\Omega^{n-1}(X))$. We denote by $\Omega^n(\mathscr{A})$ the subcategory of $\mathscr{A}$ consisting of all nth syzygies of objects in $\mathscr{A}$. Assume that there is a left $\mathsf{Proj}\,\mathscr{A}$-approximation $f\colon X \longrightarrow P$, i.e. f is an inflation with $P \in \mathsf{Proj}\,\mathscr{A}$ such that the map $\mathsf{Hom}_{\mathscr{A}}(f, P')\colon \mathsf{Hom}_{\mathscr{A}}(P, P') \longrightarrow \mathsf{Hom}_{\mathscr{A}}(X, P')$ is surjective for all $P' \in \mathsf{Proj}\,\mathscr{A}$. Then we have the exact pair $X \longrightarrow P \longrightarrow L$, where the map $g\colon P \longrightarrow L$ is a deflation. The object L is called the first cosyzygy of X and is denoted by $\Omega^{-1}(X)$. The nth cosyzygy $\Omega^{-n}(X)$ of X is defined inductively as $\Omega^{-1}(\Omega^{-(n-1)}(X))$. We denote by $\Omega^{-n}(\mathscr{A})$ the subcategory of $\mathscr{A}$ consisting of all nth cosyzygies of objects in $\mathscr{A}$.

We are now ready to prove the second main result of this section which generalizes [32, Theorem 3.11] to the setting of exact categories.

Theorem 5.6 *Let $\mathscr{A}$ be an exact category with enough projective objects. Let $\mathscr{B}$ be a quasi-resolving subcategory of $\mathscr{A}$ such that $\Omega^n(\mathscr{B}) \subseteq \mathsf{GProj}\,\mathscr{A}$ for some non-negative integer n and is closed under Ω^{-1}. Then the following statements hold.*

(i) *$\mathsf{mod}\text{-}\underline{\Omega^n(\mathscr{B})}$ is a Frobenius abelian category.*
(ii) *$\mathsf{mod}\text{-}\underline{\mathscr{B}}$ is a $3n$-Gorenstein abelian category.*

Moreover, there are the following triangle equivalences:

$$\mathsf{D}_{\mathsf{sg}}(\mathsf{mod}\text{-}\underline{\mathscr{B}}) \xrightarrow{\simeq} \underline{\mathsf{Gproj}}(\mathsf{mod}\text{-}\underline{\mathscr{B}}) \quad \textit{and} \quad \mathsf{D}_{\mathsf{sg}}(\mathsf{mod}\text{-}\underline{\Omega^n(\mathscr{B})}) \xrightarrow{\simeq} \underline{\mathsf{mod}\text{-}}\underline{\Omega^n(\mathscr{B})}$$

Proof We divide the proof into four steps.

Step 1: We show that $\mathsf{mod}\text{-}\underline{\mathscr{B}}$ is an abelian category with enough projective objects. It suffices to show that $\underline{\mathscr{B}}$ has weak kernels. Let $m\colon M \longrightarrow N$ be a morphism in $\mathscr{B}$. Since $\mathscr{A}$ has enough projective objects and $\mathsf{Proj}\,\mathscr{A} \subseteq \mathscr{B}$, there is a deflation $p\colon P \longrightarrow N$ with $P \in \mathsf{Proj}\,\mathscr{A}$. Then, we have the pullback diagram

$$\begin{array}{ccc} L & \xrightarrow{p'} & M \\ \downarrow{\scriptstyle m'} & & \downarrow{\scriptstyle m} \\ P & \xrightarrow{p} & N \end{array}$$

such that the map p' is also a deflation. From [28, Proposition A.1] and since $\mathscr{B}$ is a quasi-resolving subcategory of $\mathscr{A}$ we obtain the following conflation in $\mathscr{B}$:

$$L \xrightarrow{f} M \oplus P \xrightarrow{g} N$$

where $f = \begin{pmatrix} p' \\ -u' \end{pmatrix}$ and $g = \begin{pmatrix} u & p \end{pmatrix}$. Thus, for an object X in $\mathscr{B}$ we have the exact sequence:

$$\mathsf{Hom}_{\mathscr{A}}(X, L) \longrightarrow \mathsf{Hom}_{\mathscr{A}}(X, M \oplus P) \longrightarrow \mathsf{Hom}_{\mathscr{A}}(X, N)$$

Let $u\colon X \longrightarrow M \oplus P$ be a morphism in $\mathscr{A}$ such that $\underline{u}$ is in the kernel of $\underline{\mathsf{Hom}}_{\mathscr{A}}(X, g)\colon \underline{\mathsf{Hom}}_{\mathscr{A}}(X, M \oplus P) \longrightarrow \underline{\mathsf{Hom}}_{\mathscr{A}}(X, N)$. Then $u \circ g$ is the composition of some morphisms $a\colon X \longrightarrow Q$ and $b\colon Q \longrightarrow N$ in $\mathscr{A}$, where $Q \in \mathsf{Proj}\,\mathscr{A}$. There is a morphism $c\colon Q \longrightarrow M \oplus P$ with $c \circ g = b$. So $(a \circ c - u) \circ g = 0$. This implies that there is a morphism $d\colon X \longrightarrow L$ such that $a \circ c - u = d \circ f$. We have $\underline{u} = \underline{d \circ f}$, which is in the image of $\underline{\mathsf{Hom}}_{\mathscr{A}}(X, f)\colon \underline{\mathsf{Hom}}_{\mathscr{A}}(X, L) \longrightarrow \underline{\mathsf{Hom}}_{\mathscr{A}}(X, M)$. Thus the next sequence is exact in $\mathsf{mod}\text{-}\underline{\mathscr{B}}$:

$$\underline{\mathsf{Hom}}_{\mathscr{A}}(-, L)|_{\underline{\mathscr{B}}} \longrightarrow \underline{\mathsf{Hom}}_{\mathscr{A}}(-, M)|_{\underline{\mathscr{B}}} \longrightarrow \underline{\mathsf{Hom}}_{\mathscr{A}}(-, N)|_{\underline{\mathscr{B}}}$$

Step 2: We show that for any object F in $\mathsf{mod}\text{-}\underline{\mathscr{B}}$ there is a conflation $A \longrightarrow B \longrightarrow C$ in $\mathscr{B}$ which induces a projective resolution as follows:

$$\begin{array}{l}
\cdots\cdots \longrightarrow \underline{\mathsf{Hom}}_{\mathscr{A}}(-, \Omega^2(A))|_{\underline{\mathscr{B}}} \longrightarrow \underline{\mathsf{Hom}}_{\mathscr{A}}(-, \Omega^2(B))|_{\underline{\mathscr{B}}} \longrightarrow \underline{\mathsf{Hom}}_{\mathscr{A}}(-, \Omega^2(C))|_{\underline{\mathscr{B}}} \\
\longrightarrow \underline{\mathsf{Hom}}_{\mathscr{A}}(-, \Omega(A))|_{\underline{\mathscr{B}}} \longrightarrow \underline{\mathsf{Hom}}_{\mathscr{A}}(-, \Omega(B))|_{\underline{\mathscr{B}}} \longrightarrow \underline{\mathsf{Hom}}_{\mathscr{A}}(-, \Omega(C))|_{\underline{\mathscr{B}}} \longrightarrow \underline{\mathsf{Hom}}_{\mathscr{A}}(-, A)|_{\underline{\mathscr{B}}} \\
\longrightarrow \underline{\mathsf{Hom}}_{\mathscr{A}}(-, B)|_{\underline{\mathscr{B}}} \longrightarrow \underline{\mathsf{Hom}}_{\mathscr{A}}(-, C)|_{\underline{\mathscr{B}}} \longrightarrow F \longrightarrow 0
\end{array} \tag{5.3}$$

Let F be a functor in $\mathsf{mod}\text{-}\underline{\mathscr{B}}$. Then there is an exact sequence $\underline{\mathsf{Hom}}_{\mathscr{A}}(-, B)|_{\underline{\mathscr{B}}} \overset{\phi}{\longrightarrow} \underline{\mathsf{Hom}}_{\mathscr{A}}(-, C)|_{\underline{\mathscr{B}}} \longrightarrow F \longrightarrow 0$ with $B, C \in \mathscr{B}$ and by Yoneda's Lemma the map ϕ is of the form $\underline{\mathsf{Hom}}_{\mathscr{A}}(-, u)|_{\underline{\mathscr{B}}}$ for some morphism $u\colon B \longrightarrow C$. As in Step 1, we obtain a conflation $A \longrightarrow B \oplus Q \longrightarrow C$ in $\mathscr{B}$ with Q in $\mathsf{Proj}\,\mathscr{A}$. Since $\mathscr{A}$ has enough projective objects, there is a deflation $P \longrightarrow C$ with P projective in $\mathscr{A}$. Note that any deflation ending at the object P splits. Then we can form the following pullback diagram

$$\begin{array}{ccccc}
 & & \Omega(C) & = & \Omega(C) \\
 & & \downarrow & & \downarrow \\
A & \longrightarrow & A \oplus P & \longrightarrow & P \\
\| & & \downarrow & & \downarrow \\
A & \longrightarrow & B \oplus Q & \longrightarrow & C
\end{array}$$

where every row or column is a conflation. In particular, we get the conflation $\Omega(C) \longrightarrow A \oplus P \longrightarrow B \oplus Q$. Consider now a deflation $Q' \longrightarrow B$ with Q' in $\mathsf{Proj}\,\mathscr{A}$. From the following commutative diagram

$$\begin{array}{ccccc}
 & & \Omega(B) \oplus Q & = & \Omega(B) \oplus Q \\
 & & \downarrow & & \downarrow \\
\Omega(C) & \longrightarrow & \Omega(C) \oplus Q' \oplus Q & \longrightarrow & Q' \oplus Q \\
\| & & \downarrow & & \downarrow \\
\Omega(C) & \longrightarrow & A \oplus P & \longrightarrow & B \oplus Q
\end{array}$$

we obtain the conflation $\Omega(B) \oplus Q \longrightarrow \Omega(C) \oplus P' \longrightarrow A \oplus P$ where $P' = Q' \oplus Q$. Iterating this procedure yields the conflations: $\Omega(B) \oplus Q \longrightarrow \Omega(C) \oplus P' \longrightarrow A \oplus P$, $\Omega(A) \oplus P \longrightarrow \Omega(B) \oplus P'' \longrightarrow \Omega(C) \oplus P'$, $\Omega^2(C) \oplus P' \longrightarrow \Omega(A) \oplus P''' \longrightarrow \Omega(B) \oplus P''$

and so on, where P', P'', P''' belong to $\mathsf{Proj}\,\mathscr{A}$. Putting these conflations together and using Step 1, the desired projective resolution of F follows immediately.

Step 3: We show that for any object F in $\mathsf{mod}\text{-}\underline{\mathscr{B}}$ we have $\mathsf{Ext}^i_{\mathsf{mod}\text{-}\underline{\mathscr{B}}}(F, \mathsf{Proj}(\mathsf{mod}\text{-}\underline{\mathscr{B}})) = 0$ for $i > 3n$. Firstly, given a conflation $A \longrightarrow B \longrightarrow C$ in $\mathscr{B}$ such that $\mathsf{Ext}^1_{\mathscr{A}}(C, \mathsf{Proj}\,\mathscr{A}) = 0$, it follows as in [32, Lemma 2.2 (2)] that the sequence $\underline{\mathsf{Hom}}_{\mathscr{A}}(C, X) \longrightarrow \underline{\mathsf{Hom}}_{\mathscr{A}}(B, X) \longrightarrow \underline{\mathsf{Hom}}_{\mathscr{A}}(A, X)$ is exact for every $X \in \mathscr{A}$. Since $\Omega^j(C)$ lies in $\mathsf{Gproj}\,\mathscr{A}$ for all $j \geq n$, we know that $\mathsf{Ext}^1_{\mathscr{A}}(\Omega^j(C), \mathsf{Proj}\,\mathscr{A}) = 0$ for all $j \geq n$. Thus by the above fact and Step 2 we obtain that $\underline{\mathsf{Hom}}_{\mathscr{A}}(\Omega^j C, Y) \longrightarrow \underline{\mathsf{Hom}}_{\mathscr{A}}(\Omega^j B, Y) \longrightarrow \underline{\mathsf{Hom}}_{\mathscr{A}}(\Omega^j A, Y)$ is exact for any $Y \in \mathscr{B}$ and all $j \geq n$. Then, applying the functor $(-, \underline{\mathsf{Hom}}_{\mathscr{A}}(-, Y)|_{\underline{\mathscr{B}}})$ to Eq. 5.3 and using Yoneda's Lemma, we infer that $\mathsf{Ext}^i_{\mathsf{mod}\text{-}\underline{\mathscr{B}}}(F, \mathsf{Proj}(\mathsf{mod}\text{-}\underline{\mathscr{B}})) = 0$ for $i > 3n$.

Step 4: Let F be an object in $\mathsf{mod}\text{-}\underline{\mathscr{B}}$. By Step 2 there is a conflation $A \longrightarrow B \longrightarrow C$ in $\mathscr{B}$ which induces a projective resolution of F as indicated in diagram (5.3). Set $G := \mathsf{Coker}(\underline{\mathsf{Hom}}_{\mathscr{A}}(-, \Omega^n(B))|_{\underline{\mathscr{B}}} \longrightarrow \underline{\mathsf{Hom}}_{\mathscr{A}}(-, \Omega^n(C))|_{\underline{\mathscr{B}}})$. We show that G is a Gorenstein-projective object in $\mathsf{mod}\text{-}\underline{\mathscr{B}}$. For simplicity, we write $L := \Omega^n(A)$, $M := \Omega^n(B)$ and $N := \Omega^n(C)$. Then from Step 2 we get a conflation $L \longrightarrow M \longrightarrow N$. Since $\Omega^n(\mathscr{B})$ is closed under Ω^{-1}, there is a left $\mathsf{Proj}\,\mathscr{A}$-approximation $L \longrightarrow Q$ with $Q \in \mathsf{Proj}\,\mathscr{A}$. We make the following pushout diagram:

$$\begin{array}{ccccc}
L & \longrightarrow & Q & \longrightarrow & \Omega^{-1}(L) \\
\downarrow & & \downarrow & & \| \\
M & \longrightarrow & N \oplus Q & \longrightarrow & \Omega^{-1}(L) \\
\downarrow & & \downarrow & & \\
N & = & N & &
\end{array}$$

Note that the middle vertical conflation splits, i.e. $\mathsf{Ext}^1_{\mathscr{A}}(N, Q) = 0$ since $N \in \mathsf{GProj}\,\mathscr{A}$ and $Q \in \mathsf{Proj}\,\mathscr{A}$. Thus we obtain the conflation $M \longrightarrow N \oplus Q \longrightarrow \Omega^{-1}(L)$. Iterating this procedure gives rise to the conflations: $N \oplus Q \longrightarrow \Omega^{-1}(L) \oplus Q' \longrightarrow \Omega^{-1}M$, $\Omega^{-1}(L) \oplus Q' \longrightarrow \Omega^{-1}(M) \oplus Q'' \longrightarrow \Omega^{-1}(N) \oplus Q$, $\Omega^{-1}(M) \oplus Q'' \longrightarrow \Omega^{-1}(N) \oplus Q''' \longrightarrow \Omega^{-2}(L) \oplus Q'$ and so on, where Q', Q'', Q''' are in $\mathsf{Proj}\,\mathscr{A}$. Thus by Step 1 we obtain an exact sequence: $\underline{\mathsf{Hom}}_{\mathscr{A}}(-, L)|_{\underline{\mathscr{B}}} \longrightarrow \underline{\mathsf{Hom}}_{\mathscr{A}}(-, M)|_{\underline{\mathscr{B}}} \longrightarrow \underline{\mathsf{Hom}}_{\mathscr{A}}(-, N)|_{\underline{\mathscr{B}}} \longrightarrow \underline{\mathsf{Hom}}_{\mathscr{A}}(-, \Omega^{-1}(L))|_{\underline{\mathscr{B}}} \longrightarrow \underline{\mathsf{Hom}}_{\mathscr{A})}(-, \Omega^{-1}(M))|_{\underline{\mathscr{B}}} \longrightarrow \underline{\mathsf{Hom}}_{\mathscr{A}}(-, \Omega^{-1}(N))|_{\underline{\mathscr{B}}} \longrightarrow \cdots$. Combining this with Eq. 5.3 we obtain an exact sequence of projective objects in $\mathsf{mod}\text{-}\underline{\mathscr{B}}$ as follows:

$$\cdots\cdots \longrightarrow \underline{\mathsf{Hom}}_{\mathscr{A}}(-, \Omega(N))|_{\underline{\mathscr{B}}} \longrightarrow \underline{\mathsf{Hom}}_{\mathscr{A}}(-, L)|_{\underline{\mathscr{B}}} \longrightarrow \underline{\mathsf{Hom}}_{\mathscr{A}}(-, M)|_{\underline{\mathscr{B}}} \longrightarrow \underline{\mathsf{Hom}}_{\mathscr{A}}(-, N)|_{\underline{\mathscr{B}}} \xrightarrow{\alpha} \underline{\mathsf{Hom}}_{\mathscr{A}}(-, \Omega^{-1}(L))|_{\underline{\mathscr{B}}} \longrightarrow \underline{\mathsf{Hom}}_{\mathscr{A}}(-, \Omega^{-1}(M))|_{\underline{\mathscr{B}}} \longrightarrow \cdots \tag{5.4}$$

where $\mathsf{Im}\,\alpha = G$. By the construction of the above pushout diagrams using left $\mathsf{Proj}\,\mathscr{A}$-approximations and since $\Omega^n(\mathscr{B}) \subseteq \mathsf{GProj}\,\mathscr{A}$, we get that $\mathsf{Ext}^1_{\mathscr{A}}(\Omega^i(\Omega^n(\mathscr{B})), \mathsf{Proj}\,\mathscr{A})) = 0$ for any $i \in \mathbb{Z}$. Let Y be an object in $\mathscr{B}$. Then applying the functor $(-, \underline{\mathsf{Hom}}_{\mathscr{A}}(-, Y)|_{\underline{\mathscr{B}}})$ to Eq. 5.4 and using [32, Lemma 2.2 (2)] as explained in Step 3, we obtain an acyclic complex: $\cdots \longrightarrow \underline{\mathsf{Hom}}_{\mathscr{A}}(\Omega^{-1}(L), Y) \longrightarrow \underline{\mathsf{Hom}}_{\mathscr{A}}(N, Y) \longrightarrow \underline{\mathsf{Hom}}_{\mathscr{A}}(M, Y) \longrightarrow \underline{\mathsf{Hom}}_{\mathscr{A}}(L, Y) \longrightarrow \underline{\mathsf{Hom}}_{\mathscr{A}}(\Omega(N), Y) \longrightarrow \cdots$. This implies that Eq. 5.4 is a totally acyclic complex, equivalently, G is a Gorenstein-projective object in mod-$\underline{\mathscr{B}}$. Hence, we have shown that for every object F in mod-$\underline{\mathscr{B}}$ there is a projective resolution as in Eq. 5.3 such that the nth syzygy G is Gorenstein-projective. We infer that mod-$\underline{\mathscr{B}}$ is a $3n$-Gorenstein abelian category. Moreover, from [8, Corollary 4.13] we obtain the desired triangle equivalence between $\mathsf{D}_{\mathsf{sg}}(\text{mod-}\underline{\mathscr{B}})$ and $\underline{\mathsf{GProj}}(\text{mod-}\underline{\mathscr{B}})$.

It remains to show that mod-$\underline{\Omega^n(\mathscr{B})}$ is a Frobenius abelian category, i.e mod-$\underline{\Omega^n(\mathscr{B})}$ is of Gorenstein dimension at most zero. From [32, Proposition 3.6] it suffices to show that the stable category $\underline{\Omega^n(\mathscr{B})}$ is triangulated. Recall that $\mathsf{GProj}\,\mathscr{A}$ is an exact Frobenius category and as in the abelian case it follows easily that $\mathsf{GProj}\,\mathscr{A}$ is extension closed. We claim that $\Omega^n(\mathscr{B})$ is an admissible subcategory of $\mathsf{GProj}\,\mathscr{A}$ (see [18]), that is, $\Omega^n(\mathscr{B})$ is an extension closed subcategory of $\mathsf{GProj}\,\mathscr{A}$ such that for each object B in $\Omega^n(\mathscr{B})$ there are conflations $B \longrightarrow P \longrightarrow \Omega^{-1}(B)$ and $\Omega(B) \longrightarrow Q \longrightarrow B$ with P, Q in $\mathsf{Proj}\,\mathscr{A}$. Note that $\Omega^n(\mathscr{B})$ being admissible implies that it is an exact Frobenius category and therefore from [26] it follows that $\underline{\Omega^n(\mathscr{B})}$ is triangulated. Since we have $\mathsf{Proj}\,\mathscr{A} \subseteq \Omega^n(\mathscr{B}) \subseteq \mathsf{GProj}\,\mathscr{A}$ we only have to show that $\Omega^n(\mathscr{B})$ is extension closed. For this, we first show that $\Omega^n(\mathscr{B}) = \mathscr{B} \cap \mathsf{GProj}\,\mathscr{A}$. Since $\mathscr{B}$ is quasi-resolving we have $\Omega^n(\mathscr{B}) \subseteq \mathscr{B}$. This implies that $\Omega^n(\mathscr{B}) \subseteq \mathscr{B} \cap \mathsf{GProj}\,\mathscr{A}$. Let X be an object in $\mathscr{B} \cap \mathsf{GProj}\,\mathscr{A}$. Then X is Gorenstein-projective, so $X \cong \Omega^{-n}(\Omega^n(X))$. Since $X \in \mathscr{B}$ and $\Omega^n(\mathscr{B})$ is closed under Ω^{-1}, we have that $\Omega^n(X) \in \Omega^n(\mathscr{B})$ and $\Omega^{-n}(\Omega^n(X)) \in \Omega^n(\mathscr{B})$. This shows that $X \in \Omega^n(\mathscr{B})$, i.e. $\mathscr{B} \cap \mathsf{GProj}\,\mathscr{A} \subseteq \Omega^n(\mathscr{B})$. We now show that $\Omega^n(\mathscr{B}) = \mathscr{B} \cap \mathsf{GProj}\,\mathscr{A}$ is extension closed. Consider a conflation $X \longrightarrow Y \longrightarrow Z$ with X and Z in $\mathscr{B} \cap \mathsf{GProj}\,\mathscr{A}$. Then there is a left-$\mathsf{Proj}\,\mathscr{A}$ approximation $X \longrightarrow P \longrightarrow \Omega^{-1}(X)$. Taking the pushout diagram of these two conflations and since $\mathsf{Ext}^1(Z, P) = 0$, we obtain the conflation $Y \longrightarrow P \oplus Z \longrightarrow \Omega^{-1}(X)$. The object $P \oplus Z$ lies in $\mathscr{B}$ and the object $\Omega^{-1}(X)$ lies also in $\mathscr{B}$ since we assume that $\Omega^n(B) = \mathscr{B} \cap \mathsf{GProj}\,\mathscr{A}$ is closed under Ω^{-1}. Since $\mathscr{B}$ is quasi-resolving, it follows that the object Y lies in $\mathscr{B}$. Since $\mathsf{GProj}\,\mathscr{A}$ is closed under extensions, we conclude that the object Y lies in $\mathscr{B} \cap \mathsf{GProj}(\mathscr{A})$. This completes the proof that mod-$\underline{\Omega^n(\mathscr{B})}$ is a Frobenius abelian category. Finally, from [8, Corollary 4.13] we get that a triangle equivalence between $\mathsf{D}_{\mathsf{sg}}(\text{mod-}\underline{\Omega^n(\mathscr{B})})$ and $\underline{\text{mod-}\underline{\Omega^n(\mathscr{B})}}$. □

Remark 5.7 The first part of the proof of Theorem 5.6 is devoted to show that the category of coherent functors mod-$\underline{\mathscr{B}}$ is abelian. This is similar to [32, Proposition 2.11(i)]. However, in the setting of exact categories we need to show how we obtain from the axioms the conflation which gives us the correct Hom-exact sequence in order to conclude that $\underline{\mathscr{B}}$ has weak kernels. Part two of our proof is proved in the same way as [32, Proposition 2.11(ii)], but again we need to make clear that this construction works in our setting. Similar comments hold for the rest of the proof. Moreover, as in [32, Theorem 5.4], we can deduce a triangle equivalence between $\mathsf{D}_{\mathsf{sg}}(\text{mod-}\underline{\mathscr{B}})$ and $\mathsf{D}_{\mathsf{sg}}(\text{mod-}\underline{\Omega^n(\mathscr{B})})$.

We return to the subcategory $\mathscr{C}$ of $\mathsf{mono}(\Lambda)$, see Eq. 5.1. Assuming that Λ is Gorenstein, it can be shown that: (i) $\mathscr{C}$ is a resolving subcategory of $\mathsf{mono}(\Lambda)$ (but not of mod-$\Delta_{(0,0)}$), (ii) the category $\Omega^n(\mathscr{C})$ is a Frobenius subcategory of $\mathsf{Gproj}(\mathsf{mono}(\Lambda))$ and $\underline{\Omega^n(\mathscr{C})}$ is a triangulated subcategory of $\underline{\mathsf{Gproj}}(\mathsf{mono}(\Lambda))$, and (iii) $\Omega^n(\mathscr{C})$ is closed under Ω^{-1}. We close this section with the following consequence of Theorem 5.6, which is the second part of Theorem B presented in the Introduction.

Corollary 5.8 *Let Λ be an n-Gorenstein Artin algebra for some integer $n \geq 0$. Then for the category of coherent functors over $\underline{\mathscr{C}}$ and $\underline{\Omega^n(\mathscr{C})}$, respectively, the following hold:*

(i) mod-$\underline{\mathscr{C}}$ *is a $3n$-Gorenstein abelian category.*

(ii) $\mathsf{mod}\text{-}\underline{\Omega^n(\mathscr{C})}$ *is a Frobenius abelian category.*

Moreover, there are the following triangle equivalences:

$$\mathsf{D}_{\mathsf{sg}}(\mathsf{mod}\text{-}\underline{\mathscr{C}}) \xrightarrow{\simeq} \underline{\mathsf{Gproj}}(\mathsf{mod}\text{-}\underline{\mathscr{C}}) \quad and \quad \mathsf{D}_{\mathsf{sg}}(\mathsf{mod}\text{-}\underline{\Omega^n(\mathscr{C})}) \xrightarrow{\simeq} \underline{\mathsf{mod}}\text{-}\underline{\Omega^n(\mathscr{C})}$$

Acknowledgments This work was developed during a stay of the authors to the University of Stuttgart in 2014. Both authors would like to express their gratitude to Steffen Koenig for the warm hospitality, his support, as well as his comments and suggestions on this project. The first author is supported by the National Natural Science Foundation of China (Grant No. 11101259) and the second author is supported by the Norwegian Research Council (NFR 221893) under the project *Triangulated categories in Algebra*. The authors are grateful to the referee for simplifying the proof of Theorem 5.3, and for many useful remarks and suggestions that improved significantly the exposition of the paper.

References

1. Amiot, C., Iyama, O., Reiten, I.: Stable categories of Cohen-Macaulay modules and cluster categories. Am. J. Math. **137**(3), 813–857 (2015)
2. Auslander, M.: Coherent functors. In: Proceeding Conference Categorical Algebra (La Jolla, Calif., 1965), pp. 189–231. Springer, New York (1966)
3. Auslander, M. Representation dimension of artin algebras, Queen Mary College Notes (1971)
4. Auslander, M., Reiten, I.: Cohen-Macaulay and Gorenstein Algebras. Progress in Math **95**, 221–245 (1991)
5. Auslander, M., Reiten, I., Smalø, S.: Representation Theory of Artin Algebras. Cambridge University Press (1995)
6. Bass, H. The Morita theorems, mimeographed notes, University of Oregon (1962)
7. Beilinson, A., Bernstein, J., Deligne, P.: Faisceaux Pervers, (French) [Perverse sheaves], Analysis and topology on singular spaces, I (Luminy, 1981), 5–171, Asterisque 100 Soc. Math. France, Paris (1982)
8. Beligiannis, A.: The Homological Theory of Contravariantly Finite Subcategories: Gorenstein Categories, Auslander-Buchweitz Contexts and (Co-)Stabilization. Comm. Algebra **28**, 4547–4596 (2000)
9. Beligiannis, A.: On the Relative Homology of Cleft Extensions of Rings and Abelian Categories. J. Pure Appl. Algebra **150**, 237–299 (2000)
10. Beligiannis, A.: Cohen-Macaulay modules, (co)torsion pairs and virtually Gorenstein algebras. J. Algebra **288**(1), 137–211 (2005)
11. Beligiannis, A.: On algebras of finite Cohen-Macaulay type. Adv. Math. **226**(2), 1973–2019 (2011)
12. Beligiannis, A., Reiten, I.: Homological and homotopical aspects of torsion theories. Mem. Amer. Math. Soc. **188**(883), viii+207 (2007)
13. Bennis, D., Mahdou, N.: Strongly Gorenstein projective, injective, and flat modules. J. Pure Appl. Algebra **210**, 437–445 (2007)
14. Buchweitz, R.-O.: Maximal Cohen-Macaulay modules and Tate-Cohomology over Gorenstein rings, unpublished manuscript, p. 155 (1987)
15. Buchweitz, R.-O.: Morita contexts, idempotents, and Hochschild cohomology - with applications to invariant rings, Contemp. Math. Amer. Math, Soc., Providence, RI **331**, 25–53 (2003)
16. Bühler, T.: Exact categories. Expo. Math. **28**(1), 1–69 (2010)
17. Chen, X.-W.: The stable monomorphism category of a Frobenius category. Math. Res. Lett. **18**(1), 125–137 (2011)
18. Chen, X.-W.: Three results on Frobenius categories. Math. Z. **270**(1-2), 43–58 (2012)
19. Christensen, L.W.: Gorenstein dimensions. In: Lecture Notes in math, vol. 1747. Springer, Berlin (2000)
20. Cline, E., Parshall, B., Scott, L.: Stratifying endomorphisms algebras. Mem. Amer. Math. Soc. **124**(591), viii+119 (1996)
21. Cohn, P.M. Morita equivalence and duality, Qeen Mary College Math. Notes (1966)
22. Fossum, R., Griffith, P., Reiten, I.: Trivial Extensions of Abelian Categories with Applications to Ring Theory, vol. 456. Springer L.N.M. (1975)
23. Franjou, V., Pirashvili, T.: Comparison of abelian categories recollements. Documenta Math. **9**, 41–56 (2004)

24. Green, E.L.: On the representation theory of rings in matrix form. Pacific. J. Math. **100**(1), 138–152 (1982)
25. Green, E.L., Psaroudakis, C.: On Artin algebras arising from Morita contexts. Algebr. Represent. Theory **17**(5), 1485–1525 (2014)
26. Happel, D.: Triangulated categories in the representation theory of finite dimensional algebras, London Math. Soc. Lecture Notes Ser., vol. 119. Cambridge University Press, Cambridge (1988)
27. Happel, D.: On Gorenstein algebras. In: Representation theory of finite groups and finite-dimensional algebras, Prog. Math., vol. 95, pp. 389–404 (1991)
28. Keller, B.: Chain complexes and stable categories. Manuscripta Math. **67**, 379–417 (1990)
29. Koenig, S., Nagase, H.: Hochschild cohomology and stratifying ideals. J. Pure Appl. Algebra **213**(5), 886–891 (2009)
30. Kussin, D., Lenzing, H., Meltzer, H.: Nilpotent operators and weighted projective lines. J. Reine Angew. Math. **685**, 33–71 (2013)
31. Li, Z.-W., Zhang, P.: A construction of Gorenstein-projective modules. J. Algebra **323**(6), 1802–1812 (2010)
32. Matsui, H., Takahashi, R. Singularity categories and singular equivalences for resolving subcategories, arXiv:1412.8061, Math. Z. (to appear)
33. McConnell, J.C., Robson, J.C.: Noncommutative Noetherian rings, Pure and Applied Mathematics (New York), A Wiley-Interscience Publication. Wiley, Chichester (1987)
34. Orlov, D.: Triangulated categories of singularities and D-branes in Landau-Ginzburg models. Tr. Mat. Inst. Steklova **246**, 240–262 (2004). English transl.: Proc. Steklov. Inst. Math. 246 (2204), no. 3, 227–248
35. Psaroudakis, C.: Representation Dimension, Cohen-Macaulay Modules and Triangulated Categories, Ph.D. thesis, p. 201. University of Ioannina, Greece (2013)
36. Psaroudakis, C.: Homological Theory of Recollements of Abelian Categories. J. Algebra **398**, 63–110 (2014)
37. Psaroudakis, C., Vitória, J.: Recollements of Module Categories. Appl. Categ. Structures **22**(4), 579–593 (2014)
38. Psaroudakis, C., Vitória, J. Realisation functors in tilting theory, arXiv:1511.02677
39. Quillen, D.: Higher algebraic K-theory. I, in Algebraic K-theory, I: higher K-theories, Seattle, WA, 1972. Lecture Notes in Mathematics, vol. 341, pp. 85–147. Springer, Berlin (1973)
40. Ringel, C.M., Schmidmeier, M.: Submodule categories of wild representation type. J. Pure Appl. Algebra **205**(2), 412–422 (2006)
41. Ringel, C.M., Schmidmeier, M.: The Auslander-Reiten translation in submodule categories. Trans. Amer. Math. Soc. **360**(2), 691–716 (2008)
42. Ringel, C.M., Schmidmeier, M.: Invariant subspaces of nilpotent linear operators. I, J. Reine Angew. Math. **614**, 1–52 (2008)
43. Rowen, L.H.: Ring Theory. Student edition. Academic Press, Inc., Doston (1991)
44. Xi, C.C.: Adjoint functors and representation dimensions. Acta. Math. Sin. **22**(2), 625–640 (2006)
45. Xiong, B.L., Zhang, P., Zhang, Y.H.: Auslander-Reiten translations in monomorphism categories. Forum Math. **26**, 863–912 (2014)
46. Xiong, B.L., Zhang, P.: Gorenstein-projective modules over triangular matrix Artin algebras. J. Algebra Appl. **11**(4), 14 (2012)
47. Zhang, P.: Gorenstein-projective modules and symmetric recollements. J. Algebra **388**, 65–80 (2013)
48. Zhang, P.: Monomorphism categories, cotilting theory, and Gorenstein-projective modules. J. Algebra **339**, 181–202 (2011)