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Strict Mittag-Leffler Conditions and Gorenstein Modules

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Abstract In this paper, firstly, we characterize some rings by strict Mittag-Leffler conditions. Then, we investigate when Gorenstein projective modules are Gorenstein flat by employing tilting modules and cotorsion pairs. Finally, we study the direct limits of Gorenstein projective modules.

Keywords Mittag-Leffler condition · Gorenstein module · Tilting module · Cotorsion pair · Direct limit

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1 Introduction

Mittag-Leffler conditions were first introduced by Grothendieck in [20], and deeply studied by some authors, for example, Raynaud and Gruson in [26] , Angeleri Hügel and Herbera in [1] and Herbera in [21]. It was shown that the conditions are closely related to many problems and theories, such as Baer splitting problem, telescope conjecture for module categories, tilting theories and homological algebra, etc. Among these research, we note that

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some rings can be characterized with these conditions. For example, Herbera and Trlifaj characterized noetherian rings in [22, Corollary 2.13], and Raynaud and Gruson also gave some characterizations of perfect rings with the conditions. Inspired by these, in Section 2 of this paper, we characterize some classical rings with strict Mittag-Leffler conditions, including IF rings and coherent rings (see Propositions 2.6 and 2.7 and Theorem 2.9). After that we concentrate on how to apply strict Mittag-Leffler conditions to the Gorenstein homological algebra.

As an extension of homological algebra, Gorenstein homological algebra was introduced by Auslander and Bridger [4], and Enochs and Jenda [16, 17]. It was developed rapidly in the past few years, see [9, 11, 15, 23, 24, 30, 31]. Especially, some of the authors paid their attention to the analogies between homological algebra and Gorenstein homological algebra. For example, it is well known that all projective modules are flat, a module is flat if and only if it is the direct limit of finitely generated projective modules, and the class of projective left R-modules is closed under direct limits if the ring R is left perfect. But the things seems to be more complicated in the case of Gorenstein modules. The question whether Gorenstein projective modules are Gorenstein flat in general case and whether the class of Gorenstein projective modules are closed under direct limits provided the ground ring is perfect still remain open. And there exist rings over which not all Gorenstein flat modules can be expressed as direct limits of finitely generated Gorenstein projective modules (Cf. [24]).

Recently, Emmanouil in [14, Theorem 2.2] found that one of above questions is closely related to strict Mittag-Leffler conditions. He proved that the assertion that all Gorenstein projective modules are Gorenstein flat is equivalent to the assertion that all Gorenstein projective module are strict Mittag-Leffler over R. And we generalized his result by strongly Gorenstein projective modules, see [32, Theorem 3.1].

Continuing this study, in Section 3, we generalize the above results with the notion of tilting modules, see Theorem 3.1. Then we give some conditions under which Gorenstein projective modules are Gorenstein flat by employing the theory of cotorsion pairs, see Proposition 3.8 and Theorem 3.9. At the end of this section, we show that some of well-known results concerning the question when Gorenstein projective modules are Gorenstein flat are obvious corollaries of our results.

In Section 4, we investigate the direct limits of Gorenstein projective modules. The main results of this section are the followings:

(1) If $\mathcal{S}$ is a class of finitely presented Gorenstein projective modules satisfying $\mathcal{S} \subseteq \mathcal{GF}(R)$, then every direct limit of a countable direct system of modules from $\mathcal{S}$ is Gorenstein flat, see Proposition 4.1.
(2) If R is a left perfect and right coherent ring, then the class of Gorenstein projective modules is closed under direct limits, see Corollary 4.8. As we stated above, the class of projective left R-modules is closed under direct limits if the ring R is left perfect, so Corollary 4.8 generalizes this result to the Gorenstein modules' case on some extent.

Throughout this paper, R is an associative ring with an identity. All modules are left R-modules unless stated otherwise. Denote by $\mathcal{I}(R)$, $\mathcal{P}(R)$ and $\mathcal{F}(R)$ the classes of injective, projective and flat left R-modules, respectively. For the definitions of Gorenstein projective, injective and flat modules, we refer the readers to [15]. We use $\mathcal{GP}(R)$ and $\mathcal{GF}(R)$ to denote the Gorenstein projective and Gorenstein flat left R-modules, respectively. Moreover, we denote by $\mathrm{Add}(\mathcal{M})$ the class of modules isomorphic to direct summands of direct sums of modules from $\mathcal{M}$. The category of all left (right) R-modules is denoted by R-Mod (Mod-R). We assume that all direct and inverse systems are indexed by directed sets.

2 Strict Mittag-Leffler Conditions

In this section, we recall the notions of strict Mittag-Leffler conditions and strict Mittag-Leffler modules, and characterize some rings with these notions. We begin with the following definition:

Definition 2.1 Let R be a ring and $\mathcal{A} = (A_\alpha, u_{\alpha\beta} : A_\beta \to A_\alpha)_{\beta\geq\alpha\in I}$ be an inverse system of modules with $A = \varprojlim A_\alpha$.

(1) $\mathcal{A}$ is said to satisfy the Mittag-Leffler condition provided that for each $\alpha \in I$ there exists an index $\gamma = \gamma(\alpha)$ with $\gamma \geq \alpha$ such that $u_{\alpha\beta}(A_\beta) = u_{\alpha\gamma}(A_\gamma)$ for any $\beta \geq \gamma$.

(2) Let u_α denote the canonical map $A \to A_\alpha$. The inverse system $\mathcal{A}$ is said to satisfy the strict Mittag-Leffler condition if for any index $\alpha \in I$ there exists an index $\gamma = \gamma(\alpha)$ with $\gamma \geq \alpha$, such that $u_{\alpha\beta}(A_\beta) = u_\alpha(A)$ for any $\beta \geq \gamma$.

Remark 2.2 If the index set I of the inverse system $\mathcal{A}$ is countable, then the Mittag-Leffler condition coincides with the strict Mittag-Leffler condition, see [1, Lemma 3.3].

In order to examine the exactness of the inverse limit functor, Grothendieck in [20] introduced the Mittag-Leffler condition for countable inverse systems. Raynaud and Gruson in [26] made an intensive study of the connection between the Mittag-Leffler condition and the Mittag-Leffler module. These notions were thorough and systematic studied in [1] and [21]. In the past few years, (strict) Mittag-Leffler conditions and modules were employed to solve many different problems in the homological algebra and the representation theory [2, 7, 8, 12–14, 22, 28]. In order to introduce the notion of strict Mittag-Leffler modules, we need the following theorem, whose proof can be found in works of Angeleri Hügel and Herbera [1, 8.11] and Emmanouil [13, Theorem 1.3].

Theorem 2.3 *Let M and N be R-modules. The following statements are equivalent:*

(1) *There is a direct system of finitely presented modules $(F_\alpha, u_{\beta\alpha})_{\alpha,\beta\in I}$ with $M = \varinjlim F_\alpha$, such that the inverse system*

$$(Hom_R(F_\alpha, N), Hom_R(u_{\beta\alpha}, N))_{\alpha,\beta\in I}$$

satisfies the strict Mittag-Leffler condition.

(2) *Every direct system of finitely presented modules $(F_\alpha, u_{\beta\alpha})_{\alpha,\beta\in I}$ with $M = \varinjlim F_\alpha$ has the property that the inverse system*

$$(Hom_R(F_\alpha, N), Hom_R(u_{\beta\alpha}, N))_{\alpha,\beta\in I}$$

satisfies the strict Mittag-Leffler condition.

(3) *For any divisible abelian group D, the natural transformation*

$$\Phi : Hom_Z(N, D) \otimes_R M \longrightarrow Hom_Z(Hom_R(M, N), D)$$

defined by $\Phi(f \otimes m) : g \mapsto f(g(m))$, $f \in Hom_Z(N, D), m \in M$ and $g \in Hom_R(M, N)$ is a monomorphism.

A left R-module M is said to be a strict Mittag-Leffler module over N provided that the equivalent conditions in the above theorem are satisfied. Following Emmanouil's notations

[13], we denote by SML(N) the class of strict Mittag-Leffler modules over N. Let $\mathcal{N}$ be a class of modules. We set SML($\mathcal{N}$) $= \{M | M \in$ SML(N) for any $N \in \mathcal{N}\}$, M is called a strict Mittag-Leffler module if $\mathcal{N}$=R-Mod

Remark 2.4 (1) SML(N) is precisely the class of strict N-stationary modules in [1], and the strict Mittag-Leffler module is called as locally pure projective modules in [5].

(2) Following from Theorem 2.3(3), it is clear that the class SML(N) is closed under direct sums and direct summands (cf. [1, Lemma 8.9]). Moreover, SML(N) is closed under pure submodules. And if $M \in$ SML(N), then $M \in$ SML(Add(N)) by [1, Proposition 8.4]. We also discuss some closure property of SML(R) in [32, Remark 3.2(2)]. Moreover, we will characterize IF-rings with some properties of SML(R).

(3) If M is finitely presented, then the natural transformation in Theorem 2.3(3) is an isomorphism. Thus all finitely presented modules are strict Mittag-Leffler over R-Mod. So Add(R) $\subseteq$ SML (R-Mod). This shows that $\mathcal{P}(R) \subseteq$ SML (R-Mod).

Recall that an R-module M is called FP-injective [27] if $\mathrm{Ext}^1_R(F, M) = 0$ for any finitely presented module F. The FP-injective dimension of M, denoted by FP-id(M), is defined to be the smallest nonnegative integer n such that $\mathrm{Ext}^{n+1}_R(F, M) = 0$ for every finitely presented R-module F. A ring R is said to be right(left) IF, if all injective right(left) R-modules are flat. In order to characterize IF rings with SML(R), we need a lemma.

By applying the homology to a projective resolution $\cdots \to P_1 \to P_0 \to M \to 0$ of an R-module M, the natural transformation Φ in Theorem 2.3(3) induces additive maps

$$\Phi^{(n)}_M : \mathrm{Tor}^R_n(\mathrm{Hom}_Z(N, D), M) \longrightarrow \mathrm{Hom}_Z(\mathrm{Ext}^n_R(M, N), D).$$

The following lemma is due to Emmanouil, and it is a key step in [14, Theorem 2.2] to examine the condition when all Gorenstein projective modules are Gorenstein flat. We denote by $\Omega^n(M) = \mathrm{Ker}(P_{n-1} \to P_{n-2})$, the n-th syzygy module of M.

Lemma 2.5 [13, Proposition1.5] *The following are equivalent for left R-modules M and N:*

(1) *The map*

$$\Phi^{(n)}_M : Tor^R_n(Hom_Z(N, D), M) \longrightarrow Hom_Z(Ext^n_R(M, N), D)$$

is monomorphic for any divisible abelian group D.

(2) $\Omega^n(M) \in$ SML(N)

Now we can give the following proposition:

Proposition 2.6 *Let R be a ring. If R has FP-injective dimension $\leq n$, then injective right R-modules have flat dimension $\leq n$ if and only if $\Omega^{n+1}(F) \in$ SML(R) for all (finitely presented) left R-modules F.*

Proof "⇒". Note that $\mathrm{Tor}^R_{n+1}(\mathrm{Hom}_Z(R, D), F) = 0$ for any left R-module F and any divisible abelian group D. Then Lemma 2.5 implies that $\Omega^{n+1}(F) \in$ SML(R).

"⇐". Assume that $\Omega^{n+1}(F) \in$ SML(R) for any finitely presented left R-module F. Following from Lemma 2.5, the map

$$\Phi^{(n+1)}_F : \mathrm{Tor}^R_{n+1}(\mathrm{Hom}_Z(R, D), F) \longrightarrow \mathrm{Hom}_Z(\mathrm{Ext}^{n+1}_R(F, R), D)$$

is monomorphic for any divisible abelian group D. Since FP-id$(R) \leq n$, we have that $\mathrm{Ext}_R^{n+1}(F, R) = 0$, and hence $\mathrm{Tor}_{n+1}^R(\mathrm{Hom}_Z(R, D), F) = 0$. Since every left R-module M can be expressed as the direct limit of a direct system $(F_i)_i$ of finitely presented left R-modules and

$$\mathrm{Tor}_{n+1}^R(\mathrm{Hom}_Z(R, D), \varinjlim F_i) \cong \varinjlim \mathrm{Tor}_{n+1}^R(\mathrm{Hom}_Z(R, D), F_i),$$

we obtain that $\mathrm{Tor}_{n+1}^R(\mathrm{Hom}_Z(R, D), M) = 0$ for any left R-module M. This shows that the injective right R-module $\mathrm{Hom}_Z(R, D)$ has flat dimension $\leq n$. Observe that every injective right R-module is a direct summand of a module of the form $\mathrm{Hom}_Z(R, D)$ for some suitable abelian group D, so every injective right R-module has flat dimension $\leq n$. □

Provided that R is FP-injective as a left R-module, Proposition 2.6 implies that R is a right IF ring if and only if $\Omega^1(F) \in \mathrm{SML}(R)$ for all (finitely presented) left R-modules F. Furthermore, if the ring R is self-injective, we can improve the above result. To this end, we need to recall that a class $\mathcal{C}$ of left R-modules is called projectively resolving if $\mathcal{P}(R) \subseteq \mathcal{C}$, and for every short exact sequence of left R-modules $0 \to A \to B \to C \to 0$ with $C \in \mathcal{C}$, the conditions $A \in \mathcal{C}$ and $B \in \mathcal{C}$ are equivalent.

Proposition 2.7 *Let R be a ring. If R is injective as a left R-module, then the following statements are equivalent.*

(1) *R is a right IF ring.*
(2) $\mathrm{SML}(R)$ *is projectively resolving.*

Proof (1) $\Rightarrow$ (2). Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of left R-modules. By the assumption on the ring R , the sequence $0 \to M' \to M \to M'' \to 0$ is left exact when applying the functors $\mathrm{Hom}_Z(R, D) \otimes_R -$ and $\mathrm{Hom}_R(-, R)$. Then we get the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_Z(R,D)\otimes_R M' \arrow[r] \arrow[d, "\Phi_{M'}"'] & \mathrm{Hom}_Z(R,D)\otimes_R M \arrow[r] \arrow[d, "\Phi_M"'] & \mathrm{Hom}_Z(R,D)\otimes_R M'' \arrow[r] \arrow[d, "\Phi_{M''}"'] & 0 \\
0 \arrow[r] & \mathrm{Hom}_Z(\mathrm{Hom}_R(M',R),D) \arrow[r] & \mathrm{Hom}_Z(\mathrm{Hom}_R(M,R),D) \arrow[r] & \mathrm{Hom}_Z(\mathrm{Hom}_R(M'',R),D) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. So $\Phi_{M'}$ is monomorphic if Φ_M is monomorphic, and Φ_M is monomorphic if $\Phi_{M'}$ and $\Phi_{M''}$ are both monomorphic. These show that $\mathrm{SML}(R)$ is closed under extensions and kernels of epimorphisms. Note that $\mathcal{P}(R) \subseteq \mathrm{SML}(R)$, thus $\mathrm{SML}(R)$ is projectively resolving.

(2) $\Rightarrow$ (1). Assume that the class $\mathrm{SML}(R)$ is projectively resolving. For any finitely presented left R-module F, there is an exact sequence of left R-modules $0 \to K \to P \to F \to 0$ with P finitely generated free and K finitely generated. Note that P and F are both contained in $\mathrm{SML}(R)$, thus $K = \Omega^1(F) \in \mathrm{SML}(R)$ by the assumption. Since injective modules are FP-injective, the proof follows from Proposition 2.6. □

Example 2.8 (1) A ring R is said to be an n-FC ring [10] if R is left and right coherent with FP-id$({}_RR) \leq n$ and FP-id$(R_R) \leq n$ for some integer $n \geq 0$. Let R be a 0-FC ring, [30, Lemma 3.1.4] shows that $\mathrm{Hom}_Z(R, Q/Z)$ is a flat left R-module and also a flat right R-module. Since every injective left R-module (right R-module, respectively) is a direct summand of a suitable direct product of copies of the injective left R-module (right R-module, respectively) $\mathrm{Hom}_Z(R, Q/Z)$. Thus every injective left R-module (right R-module, respectively) is flat. This implies that R is self-FP-injective and also a left and right IF-ring.

(2) Provided that R is 0-Gorenstein ring, that is left and right noehterian and self-injective as left and right R-module, then R is self-injective and example 2.8(1) shows that R is also a left and right IF ring.

At the end of this section, we will characterize coherent rings by some properties of strict Mittag-Leffler modules over some special classes of modules. Let $\mathcal{FI}(R)$ be the class of FP-injective left R-modules.

Theorem 2.9 *The following statements are equivalent for a ring R.*

(1) *R is left coherent.*
(2) $\mathrm{SML}(\mathcal{I}(R))$ *is projectively resolving.*
(3) $\mathrm{SML}(\mathcal{I}(R))$ *is closed under submodules.*
(4) *All submodules of projective modules are contained in* $\mathrm{SML}(\mathcal{FI}(R))$.

Proof (1) $\Rightarrow$ (2). For any injective left R-module I and any divisible abelian group D, we note that the functor $\mathrm{Hom}_R(-, I)$ is exact and $\mathrm{Hom}_Z(I, D)$ is a flat right R-module. The proof is similar to that of Proposition 2.7.

(2) $\Rightarrow$ (1). Let I be a finitely generated left ideal of R. We only need to prove that I is finitely presented as a left R-module. Consider the exact sequence $0 \to I \to R \to R/I \to 0$. By the assumption in (2), I is a strict Mittag-Leffler module over $R^+ = \mathrm{Hom}_Z(R, Q/Z)$, thus I is R-Mittag-Leffler by [1, Proposition 8.14], i.e., $\tau : (R^\Lambda) \otimes I \to I^\Lambda$ defined by $\tau((r_\lambda)_\lambda \otimes x) = (r_\lambda x)_\lambda$ is a monomorphism for any index set Λ. Note that I is finitely generated, and hence τ is epimorphic, thus τ is an isomorphism. The desired result follows from [15, Theorem 3.2.22].

(1) $\Rightarrow$ (3) Let M' be a submodule of M. For each $I \in \mathcal{I}(R)$, we consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_Z(I, D) \otimes_R M' \arrow[r] \arrow[d, "\Phi_{M'}"'] & \mathrm{Hom}_Z(I, D) \otimes_R M \arrow[d, "\Phi_M"'] \\
0 \arrow[r] & \mathrm{Hom}_Z(\mathrm{Hom}_R(M', I), D) \arrow[r] & \mathrm{Hom}_Z(\mathrm{Hom}_R(M, I), D)
\end{tikzcd}$$

with exact rows. It is easily seen that $\Phi_{M'}$ is monomorphic if Φ_M is monomorphic. Thus $\mathrm{SML}(\mathcal{I}(R))$ is closed under submodules.

(3) $\Rightarrow$ (1) is similar to that of (2) implying (1).

(1) $\Rightarrow$ (4). Provided that the ring R is left coherent. Given any divisible abelian group D, it is known that $\mathrm{Hom}_Z(F, D)$ is a flat right R-module for any FP-injective left R-module F. The proof is similar to that of (1) implying (3).

(4) $\Rightarrow$ (5) is trivial.

(5) $\Rightarrow$ (1). Note that $\mathrm{SML}(\mathcal{FI}(R)) \subseteq \mathrm{SML}(\mathcal{I}(R))$, then we can adopt the same method in the proof of (2) implying (1). $\square$

3 Gorenstein Projective and Gorenstein Flat Modules

In this section, we concentrate on the question when Gorenstein projective modules are Gorenstein flat. The main notions we employ here are strict Mittag-Leffler conditions, cotorsion pairs, tilting modules and strongly Gorenstein projective modules. The notions of strongly Gorenstein projective, injective and flat modules were introduced by Bennis and Mahdou in [9]. They proved that a module is Gorenstein projective, if and only if, it is a direct summand of a strongly Gorenstein projective module. This means that we can construct Gorenstein projective modules from strongly Gorenstein projective modules, which shows that strongly Gorenstein modules are more fundamental in the Gorenstein homological algebra. We denote the class of strongly Gorenstein projective modules by $\mathcal{SGP}(R)$.

Recall that a left R-module T is said to be n-tilting provided that
(T1) The projective dimension of T is at most n.
(T2) $\mathrm{Ext}^i_R(T, T^{(I)})$ for each $i \geq 1$ and any index set I.
(T3) There are $r \geq 0$ and a long exact sequence $0 \to R \to T_0 \to \cdots \to T_r \to 0$ such that $T_i \in \mathrm{Add}(T)$ for all $i \leq r$.

Emmanouil [14, Theorem 2.2] proved that $\mathcal{GP}(R) \subseteq \mathcal{GF}(R)$ if and only if $\mathcal{GP}(R) \subseteq \mathrm{SML}(R)$. Note that R can be viewed as 0-tilting module, so we generalize this result to the following:

Theorem 3.1 *The following conditions are equivalent for a ring R:*

(1) $\mathcal{GP}(R) \subseteq \mathcal{GF}(R)$.
(2) *There exists a tilting module T such that* $\mathcal{GP}(R) \subseteq \mathrm{SML}(T)$.
(3) *For any left R-module T of finite projective dimension we have* $\mathcal{SGP}(R) \subseteq \mathrm{SML}(T)$.

Proof (1) $\Rightarrow$ (3). Let $M \in \mathcal{SGP}(R)$. There is an exact sequence $0 \to M \to P \to M \to 0$ with P projective. Given a left R-module T with $\mathrm{pd}_R(T) \leq n$, It is easy to see that $\mathrm{id}_R(\mathrm{Hom}_Z(T, D)) \leq n$ for any divisible abelian group D. Note that M is also Gorenstein flat, and so $\mathrm{Tor}^R_1(\mathrm{Hom}_Z(T, D), M) \cong \mathrm{Tor}^R_{n+1}(\mathrm{Hom}_Z(T, D), M) = 0$. This shows that

$$\Phi^{(1)}_M : \mathrm{Tor}^R_1(\mathrm{Hom}_Z(T, D), M) \longrightarrow \mathrm{Hom}_Z(\mathrm{Ext}^1_R(M, T), D)$$

is monomorphic. Following from Lemma 2.5, we get that $M \in \mathrm{SML}(T)$.

(3) $\Rightarrow$ (2). Let T be a tilting module. Note that each Gorenstein projective module is a direct summand of a strongly Gorenstein projective module. It follows that $\mathcal{GP}(R) \subseteq \mathrm{SML}(T)$ by Remark 2.4(2).

(2) $\Rightarrow$ (1). Let $M \in \mathcal{GP}(R)$ and let T be an n-tilting module such that $\mathcal{GP}(R) \subseteq \mathrm{SML}(T)$. Then $M \in \mathrm{SML}(T)$, and hence $M \in \mathrm{SML}(\mathrm{Add}T)$ by Remark 2.4(2). On the other hand, there is a long exact sequence $0 \to R \to T_0 \to T_1 \to \cdots \to T_r \to 0$ where $T_i \in \mathrm{Add}(T)$ for all $i \leq r$. Since M is Gorenstein projective and T_i has finite projective dimension for all $i \leq r$, we have that $\mathrm{Ext}^k_R(M, T_i) = 0$ for all $i \leq r$ and all $k \geq 1$.

Next we will prove that $\mathcal{GP}(R) \subseteq \mathrm{SML}(R)$. For $M \in \mathcal{GP}(R)$, there is a long exact sequence $\cdots \to P_1 \to P_0 \to M \to 0$ such that $P_i \in \mathcal{P}(R)$ for all $i \geq 0$ and $\Omega^k(M) \in$

$\mathcal{GP}(R)$ for all $k \geq 1$. It follows that $\Omega^k(M) \in \mathrm{SML}(T) = \mathrm{SML}(Add(T))$ and hence the natural transformations

$$\mathrm{Tor}_k^R(\mathrm{Hom}_Z(T_i, D), M) \longrightarrow \mathrm{Hom}_Z(\mathrm{Ext}_R^k(M, T_i), D)$$

is monomorphic for all $0 \leq i \leq r$ and all $k \geq 1$. Thus $\mathrm{Tor}_k^R(\mathrm{Hom}_Z(T_i, D), M) = 0$ for all $0 \leq i \leq r$ and all $k \geq 1$. Applying $- \otimes_R M$ to the exact sequence

$$0 \to \mathrm{Hom}_Z(T_r, D) \to \cdots \to \mathrm{Hom}_Z(T_1, D) \to \mathrm{Hom}_Z(T_0, D) \to \mathrm{Hom}_Z(R, D) \to 0,$$

we obtain that $\mathrm{Tor}_{r+k}^R(\mathrm{Hom}_Z(R, D), M) \cong \mathrm{Tor}_k^R(\mathrm{Hom}_Z(T_r, D), M) = 0$ for all $k \geq 1$. By the definition of Gorenstein projective modules, M can be viewed as the $(r+1)$-th syzygy $\Omega^{r+1}(M')$ of a Gorenstein projective module M'. Thus the map

$$\mathrm{Tor}_{r+1}^R(\mathrm{Hom}_Z(R, D), M') \longrightarrow \mathrm{Hom}_Z(\mathrm{Ext}_{r+1}^k(M', R), D)$$

is monomorphic, so we conclude that $M \in \mathrm{SML}(R)$. The desired result follows from [14, Theorem 2.2]. □

Remark 3.2 (1) In [32, Theorem 3.1], we have proven that $\mathcal{GP}(R) \subseteq \mathcal{GF}(R)$ if and only if $\mathcal{SGP}(R) \subseteq \mathrm{SML}(R)$, so we also generalize this result by giving the above theorem. In fact, we obtain that $\mathcal{GP}(R) \subseteq \mathcal{GF}(R)$ if and only if $\mathcal{GP}(R) \subseteq \mathrm{SML}(R)$ if and only if $\mathcal{SGP}(R) \subseteq \mathrm{SML}(R)$ if and only if $\mathcal{GP}(R) \subseteq \mathrm{SML}(T)$ for some tilting module T if and only if $\mathcal{SGP}(R) \subseteq \mathrm{SML}(T)$ for any left R-module T of finite projective dimension.

(2) Following from the proof of Theorem 3.1, $\mathcal{GP}(R) \subseteq \mathrm{SML}(T)$ holds true for all tilting modules if $\mathcal{GP}(R) \subseteq \mathcal{GF}(R)$.

(3) An R-module is said to be countably presented if it is the cokernel of a homomorphism between two countably generated free modules. It is well-known that any countably presented module can be expressed as the direct limit of a countable direct system of finitely presented modules. Let M be a countably presented module and P an R-module. Then $\mathrm{Ext}_R^1(M, P^{(\mathbb{N})}) = 0$ implies that $M \in \mathrm{SML}(P)$ by [2, Example 2.4(4)] or [12, Proposition 2.3]. Therefore, for any module T of finite projective dimension, all countably presented Gorenstein projective R-module are strict Mittag-Leffler over T, and as an application of this fact, we have proven that over a left $\aleph_0$-Noetherian ring R, every countably generated Gorenstein projective left R-module is Gorenstein flat, see [32, Theorem 3.4].

In the remaining part of this section, we continue our study by employing the notion of cotorsion pairs. We note that the use of cotorsion pairs in the study of the relation between Gorenstein projective and Gorenstein flat modules is also used by Wang and Liang [29, Section 3]. Let's first recall some definitions. For a class $\mathcal{D}$ of left R-modules, we define its left and right orthogonal classes by ${}^{\perp}\mathcal{D} = \{M \in R\text{-Mod} \mid \mathrm{Ext}_R^1(M, D) = 0 \text{ for all } D \in \mathcal{D}\}$ and $\mathcal{D}^{\perp} = \{M \in R\text{-Mod} \mid \mathrm{Ext}_R^1(D, M) = 0 \text{ for all } D \in \mathcal{D}\}$ respectively. A pair of classes of modules $(\mathcal{A}, \mathcal{B})$ is said to be a cotorsion pair if $\mathcal{A} = {}^{\perp}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp}$. If moreover $\mathrm{Ext}_R^i(A, B) = 0$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$ and all $i \geq 1$, then $(\mathcal{A}, \mathcal{B})$ is called a hereditary cotorsion pair. The cotorsion pair $(\mathcal{A}, \mathcal{B})$ is said to be generated by a class of modules $\mathcal{C}$ provided that $\mathcal{B} = \mathcal{C}^{\perp}$. We say that $(\mathcal{A}, \mathcal{B})$ is of countable (finite) type if there exists a class $\mathcal{C}$ of countably (finitely) presented modules such that $\mathcal{B} = \mathcal{C}^{\perp}$. Note that the definition of countable (finite) type of cotorsion pairs here is different from [19, Definition 5.2.1].

Let $\mathcal{D}$ be a filter-closed class (see [3, Section 2] or [28, Section 2]). [3, Lemma 4.2] shows that ${}^{\perp}\mathcal{D} \subseteq \mathrm{SML}(\mathcal{D})$. Šaroch and Šťovíček in [28, Lemma 2.2] proved that a class $\mathcal{D}$ of modules is filter-closed if $\mathcal{D}$ is closed under direct products and unions of well-ordered chains. Observe that a class $\mathcal{D}$ of modules is closed under direct dlimits if it is closed under unions of well-ordered chains(Cf. [19, Lemma 1.2.10]). Thus we obtain that:

Lemma 3.3 *Let $\mathcal{D}$ be a class of R-modules such that $\mathcal{D}$ is closed under products and direct limits. Then ${}^{\perp}\mathcal{D} \subseteq \mathrm{SML}(\mathcal{D})$*

There are numerous classes of modules closed under direct products and direct limits.

Example 3.4 (1) Let R be a left noetherian ring and let $\mathcal{I}_n$ be the class of modules of injective dimension at most n. Then $\mathcal{I}_n$ is closed under direct products and direct limits.

(2) If R is right coherent, then the class $\mathcal{FI}_n$ of the modules of FP-injective dimension at most n and the class $\mathcal{F}(R)$ are both closed under direct products and direct limits.

Let $\mathcal{X}$ be a class of left R-modules and M a left R-module. M is said to have an $\mathcal{X}$-resolution if there is a long exact sequence $\cdots \to X_n \to \cdots X_1 \to X_0 \to M \to 0$ such that $X_n \in \mathcal{X}$ for all $n \geq 0$. We denote by $res.dim_{\mathcal{X}}(M)$ the $\mathcal{X}$-resolution dimension of M, which is defined as the least number n such that there exists an $\mathcal{X}$-resolution as above with $X_i = 0$ for all $i \geq n+1$. Otherwise we set $res.dim_{\mathcal{X}}(M) = \infty$. The $\mathcal{X}$-coresolution and the $\mathcal{X}$-coresolution dimension are defined dually. We denote $\mathcal{X}$-coresolution dimension of M by $cores.dim_{\mathcal{X}}(M)$.

Assuming that n is the minimal number such that $\mathrm{Tor}_i^R(M, X)$ =0 for any $X \in \mathcal{X}$ and all $i \geq n+1$, we say that M has flat dimension n with respect to $\mathcal{X}$ and denote by $fd_{\mathcal{X}}(M) = n$. Obviously, $fd_{\mathcal{X}}(M)$ coincides with flat dimension of M if $\mathcal{X} = R$-Mod.

Proposition 3.5 *Let $(\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair such that $\mathcal{B}$ is closed under direct limits. Then for any left R-module N, $cores.dim_{\mathcal{B}}(N) \leq n$ implies that $fd_{\mathcal{A}}(Hom_R(N, D)) \leq n$ for any divisible abelian group D.*

Proof Following from Lemma 3.3, $\mathcal{A} \subseteq \mathrm{SML}(\mathcal{B})$ since $\mathcal{B}$ is closed under direct products and direct limits. Note that $(\mathcal{A}, \mathcal{B})$ is hereditary, and so $\mathcal{A}$ is resolving. Thus $\Omega^i(M) \in \mathrm{SML}(\mathcal{B})$ for each $M \in \mathcal{A}$ and all $i \geq 1$. By Lemma 2.5, for any divisible abelian group D and any $B \in \mathcal{B}$, the maps

$$\Phi_M^{(i)} : \mathrm{Tor}_i^R(\mathrm{Hom}_Z(B, D), M) \longrightarrow \mathrm{Hom}_Z(\mathrm{Ext}_R^i(M, B), D)$$

are monomorphic for all $i \geq 1$. It follows that $\mathrm{Tor}_i^R(\mathrm{Hom}_Z(B, D), M) = 0$ for all $i \geq 1$.

Given a left R-module N satisfying $cores.dim_{\mathcal{B}}(N) \leq n$, let $\mathcal{E} : 0 \to N \to B_0 \to B_1 \to \cdots \to B_n \to 0$ be a $\mathcal{B}$-coresolution of N. We obtain an exact sequence $0 \to \mathrm{Hom}_Z(B_n, D) \to \cdots \to \mathrm{Hom}_Z(B_1, D) \to \mathrm{Hom}_Z(B_0, D) \to \mathrm{Hom}_Z(N, D) \to 0$ by applying the functor $\mathrm{Hom}_Z(-, D)$ to the exact sequence $\mathcal{E}$. So $\mathrm{Tor}_i^R(\mathrm{Hom}_Z(N, D), M) = 0$ for all $i \geq n+1$ and for all $M \in \mathcal{A}$. This shows that $fd_{\mathcal{A}}(\mathrm{Hom}_Z(N, D)) \leq n$. □

Using the method in the proof of proposition 3.5, we have the following result:

Proposition 3.6 *Let $\mathcal{B}$ be a class of modules closed under direct products and direct limits. If $\mathcal{SGP}(R) \subseteq {}^{\perp}\mathcal{B}$ and $cores.dim_{\mathcal{B}}(R) < \infty$, then $\mathcal{GP}(R) \subseteq \mathcal{GF}(R)$.*

Proof Suppose that $cores.dim_{\mathcal{B}}(R) \leq n$ for some integer n. Similar to the proof of the above proposition, we obtain that $fd_{\mathcal{SGP}(R)}(\mathrm{Hom}_R(R, D)) \leq n$. This shows that

$\mathrm{Tor}_i^R(\mathrm{Hom}_Z(R, D), M) = 0$ for each $M \in \mathcal{SGP}(R)$ and all $i \geq n+1$. By the definition of strongly Gorenstein projective modules, we have that $\mathrm{Tor}_i^R(\mathrm{Hom}_Z(R, D), M) = 0$ for all $i \geq 1$. Thus $M \in \mathrm{SML}(R)$. Therefore $\mathcal{GP}(R) \subseteq \mathrm{SML}(R)$ by Remark 3.2(1), and so $\mathcal{GP}(R) \subseteq \mathcal{GF}(R)$. $\square$

Let $\mathcal{S}$ be a class of left R-modules such that $\mathcal{S} \subseteq \mathrm{SML}(\mathcal{B})$ and $\mathcal{B} \subseteq \mathcal{S}^\perp$. Angeleri Hügel and Herbera in [1, Proposition 8.13] proved that any module isomorphic to a direct summand of an $\mathcal{S} \bigcup Add(R)$-filtered module is strict Mittag-Leffler over $\mathcal{B}$. Furthermore, we assume that $\mathcal{S}$ containing R, [19, Corollary 3.2.4] shows that the class ${}^\perp(\mathcal{S}^\perp)$ consists of all direct summands of $\mathcal{S}$-filtered modules. This gives the following result.

Lemma 3.7 *Let $\mathcal{S}$ be a class of left R-modules such that $R \in \mathcal{S}$, $\mathcal{S} \subseteq \mathrm{SML}(\mathcal{B})$ and $\mathcal{B} \subseteq \mathcal{S}^\perp$. Then ${}^\perp(\mathcal{S}^\perp) \subseteq \mathrm{SML}(\mathcal{B})$.*

Proposition 3.8 *Let $\mathcal{S}$ be the class of countably generated strongly Gorenstein projective modules and let $(\mathcal{A}, \mathcal{B})$ be the cotorsion pair generated by $\mathcal{S}$. If $\mathcal{SGP}(R) \subseteq \mathcal{A}$, then $\mathcal{GP}(R) \subseteq \mathcal{GF}(R)$.*

Proof It is obvious that $(\mathcal{A}, \mathcal{B}) = ({}^\perp(\mathcal{S}^\perp), \mathcal{S}^\perp)$ is a hereditary cotorsion pair. For any $N \in \mathcal{GP}(R)$, there exists $M \in \mathcal{SGP}(R)$ and an R-module N' such that $M \cong N \oplus N'$. Since $M \in \mathcal{A}$ and so $N \in \mathcal{A}$. Now we proves that $\mathcal{GP}(R) \subseteq \mathrm{SML}(R)$. Follows from Remark 3.2(3), all countably generated strongly Gorenstein projective modules are strict Mittag-Leffler over R, i.e., $\mathcal{S} \subseteq \mathrm{SML}(R)$. Note that $R \in \mathcal{S}^\perp$, then $\mathcal{A} = {}^\perp(\mathcal{S}^\perp) \subseteq \mathrm{SML}(R)$ by Lemma 3.7. It follows that $\mathcal{GP}(R) \subseteq \mathrm{SML}(R)$, as desired. $\square$

If the cotorsion pair is just generated by the class $\mathcal{SGP}(R)$, then we have:

Theorem 3.9 *Let R be right coherent and let $(\mathcal{A}, \mathcal{B})$ be the cotorsion pair generated by the class $\mathcal{SGP}(R)$. If the cotorsion pair $(\mathcal{A}, \mathcal{B})$ is of countable type, then $\mathcal{GP}(R) \subseteq \mathcal{GF}(R)$.*

Proof By the assumption, there is a class $\mathcal{S}$ of countably presented modules such that $\mathcal{B} = \mathcal{S}^\perp$. Then for any $M \in \mathcal{S}$, there is a countable direct system

$$F_1 \overset{f_1}{\to} F_2 \overset{f_2}{\to} F_3 \to \cdots \to F_n \overset{f_n}{\to} F_{n+1} \to \cdots$$

of finitely presented modules such that $M = \varinjlim F_n$. Note that $\mathcal{P}(R) \subseteq \mathcal{B}$, so $M \in \mathrm{SML}(\mathcal{P}(R))$ by Remark 3.2(3).

Next we will prove that $\mathcal{F}(R) \subseteq \mathcal{B}$. Let $F \in \mathcal{F}(R)$, we consider a pure exact sequence $0 \to K \to P \to F \to 0$ with P projective. Applying the functor $\mathrm{Hom}_R(F_n, -)$ to the pure exact sequence $0 \to K \to P \to F \to 0$, we obtain an inverse system of exact sequence of the form

$$0 \to \mathrm{Hom}_R(F_n, K) \to \mathrm{Hom}_R(F_n, P) \to \mathrm{Hom}_R(F_n, F) \to 0.$$

Since F_n is finitely presented and K is flat, we have that

$$\mathrm{Hom}_R(F_n, K) \cong \mathrm{Hom}_R(F_n, R) \otimes_R K.$$

Note that $M \in \mathrm{SML}(R)$ implies that the inverse system

$$(\mathrm{Hom}_R(F_n, R) \otimes_R K, \mathrm{Hom}_R(f_n, R) \otimes_R K)$$

satisfies the strict Mittag-Leffler condition. So, the inverse system of abelian groups $(\mathrm{Hom}_R(F_n, K), \mathrm{Hom}_R(f_n, K))$ satisfies the strict Mittag-Leffler condition. Then we have an exact sequence

$$0 \to \varprojlim \mathrm{Hom}_R(F_n, K) \to \varprojlim \mathrm{Hom}_R(F_n, P) \to \varprojlim \mathrm{Hom}_R(F_n, F) \to 0.$$

This shows that the sequence

$$0 \to \mathrm{Hom}_R(\varinjlim F_n, K) \to \mathrm{Hom}_R(\varinjlim F_n, P) \to \mathrm{Hom}_R(\varinjlim F_n, F) \to 0$$

is exact. Thus we have that the sequence

$$0 \to \mathrm{Hom}_R(M, K) \to \mathrm{Hom}_R(M, P) \to \mathrm{Hom}_R(M, F) \to 0$$

is exact. The exactness of $0 \to \mathrm{Ext}^1_R(M, K) \to \mathrm{Ext}^1_R(M, P) = 0$ implies that $\mathrm{Ext}^1_R(M, K) = 0$. Note that the cotorsion pair $(\mathcal{A}, \mathcal{B})$ is hereditary. It follows that $F \in \mathcal{B}$.

Now we have proven that $\mathcal{F}(R) \subseteq \mathcal{B}$, which shows that $\mathcal{SGP}(R) \subseteq {}^{\perp}\mathcal{F}(R)$. Since R is right coherent, thus $\mathcal{F}(R)$ is closed under direct products. Following from Lemma 3.3, $\mathcal{SGP}(R) \subseteq \mathrm{SML}(\mathcal{F}(R)) \subseteq \mathrm{SML}(R)$. Therefore $\mathcal{GP}(R) \subseteq \mathcal{GF}(R)$. □

Now we present some well-known results, which can be easily proven with (strict) Mittag-Leffler conditions.

Remark 3.10 (1). Recall that the left finitistic projective dimension of R is the supremum of the projective dimensions of those left R-modules possessing finite projective dimension. Jensen [25, proposition 6] proved that each flat left R-module has finite projective dimension if R has finite left finitistic projective dimension. In this case, it is easily seen that $\mathcal{GP}(R) \subseteq {}^{\perp}\mathcal{F}(R)$. Moreover, if the ring is right coherent, then $\mathcal{F}(R)$ is closed under direct products and direct limits. Therefore, if R is right coherent with finite left finitistic projective dimension, we have $\mathcal{GP}(R) \subseteq \mathrm{SML}(R)$ by Remark 3.2(1) and Lemma 3.3, and so $\mathcal{GP}(R) \subseteq \mathcal{GF}(R)$. This was also proven by Holm in [23, propostition 3.4]

(2). It is well known that a ring R is left perfect if and only if every flat left R-module is projective. Note that $\mathcal{P}(R)$ is closed under direct products and direct limits if R is right coherent and left perfect, and hence, in this case $\mathcal{GP}(R) \subseteq {}^{\perp}\mathcal{P}(R) \subseteq \mathrm{SML}(\mathcal{P}(R))$ by Lemma 3.3, therefore we have that $\mathcal{GP}(R) \subseteq \mathcal{GF}(R)$ by Theorem 3.1. This result was also given by Ding in [11, Corollary 2.12].

(3). A ring R is called an n-Gorenstein ring if R is both left and right noetherian with $\mathrm{id}({}_R R) \leq n$ and $\mathrm{id}(R_R) \leq n$ for some integer $n \geq 0$. It is known that $\mathcal{GP}(R) \subseteq \mathcal{GF}(R)$ if R is n-Gorenstein [15, Corollary 10.3.10]. However, this is immediate consequence of Example 3.4(1).

4 Direct Limits of Gorenstein Projective Modules

In [32], we discussed when the direct limit of a direct system of Gorenstein projective modules is Gorenstein flat. We proved that, over a $\aleph_0$-Noetherian ring, this holds true for a countable direct system of countably generated Gorenstein projective modules. In this section, we continue this study, and moreover, we investigate when the class of Gorenstein projective modules is closed under direct limits. We begin with the following proposition:

Proposition 4.1 *Let $\mathcal{S}$ be a class of finitely presented Gorenstein projective modules such that $\mathcal{S} \subseteq \mathcal{GF}(R)$. Then every direct limit of countable direct system of modules from $\mathcal{S}$ is Gorenstein flat.*

Proof Let $(F_i)_{i \in \mathbb{N}}$ be a direct system of modules from $\mathcal{S}$. Note that each F_i is a finitely presented Gorenstein projective left R-module. By [33, Proposition 1.4], for each $i \in I$, there exists a long exact sequence

$$0 \to F_i \to P_i^0 \to P_i^1 \to \cdots$$

with each P_i^k finitely generated projective for $k \geq 0$ such that $\mathrm{Hom}_R(-, P)$ leaves it exact whenever P is projective. We denote the kernel of $P_i^k \to P_i^{k+1}$ by K_i^k, it is easy to see that each K_i^k is finitely presented for all $k \geq 0$. Thus

$$\mathrm{Tor}_j^R(\mathrm{Hom}_Z(R, D), K_i^k) \cong \mathrm{Hom}_Z(\mathrm{Ext}_R^j(K_i^k, R), D) = 0$$

for $1 \leq j \leq k$ and any divisible abelian group D. Now we conclude that the exact sequence

$$0 \to F_i \to P_i^0 \to P_i^1 \to \cdots$$

is left exact by $I \otimes_R -$ whenever I is an injective right R-module since each injective right R-module is a direct summand of a module of the form $\mathrm{Hom}_Z(R, D)$ for some divisible abelian group D. Note that each $F_i \in \mathcal{GF}(R)$, in fact we prove that each K_i^k is Gorenstein flat for all $k \geq 0$.

Since each K_i^k is Gorenstein projective for all $k \geq 0$. we have the following commutative diagram

$$\begin{tikzcd}
G_0 =: \arrow[d, dashed] & 0 \arrow[r] & F_0 \arrow[r] \arrow[d] & P_0^0 \arrow[r] \arrow[d, dashed] & P_0^1 \arrow[r] \arrow[d, dashed] & P_0^2 \arrow[r] \arrow[d, dashed] & \cdots \\
G_1 =: \arrow[d, dashed] & 0 \arrow[r] & F_1 \arrow[r] \arrow[d] & P_1^0 \arrow[r] \arrow[d, dashed] & P_1^1 \arrow[r] \arrow[d, dashed] & P_1^2 \arrow[r] \arrow[d, dashed] & \cdots \\
G_2 =: \arrow[d, dashed] & 0 \arrow[r] & F_2 \arrow[r] \arrow[d] & P_2^0 \arrow[r] \arrow[d, dashed] & P_2^1 \arrow[r] \arrow[d, dashed] & P_2^2 \arrow[r] \arrow[d, dashed] & \cdots \\
\vdots & & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

with exact rows and each P_i^k finitely generated projective, and $K_i^k = \mathrm{Ker}(P_i^k \to P_i^{k+1})$ Gorenstein flat. Then each column in the above diagram is again a direct system. Now we obtain an exact sequence

$$\varinjlim G_i =: 0 \to \varinjlim F_i \to \varinjlim P_i^0 \to \varinjlim P_i^1 \to \cdots$$

where each $\varinjlim P_i^k$ is flat for all $k \geq 0$. Since the tensor product functor commutes with direct limits and $\varinjlim$ is an exact functor, we obtain that the exact sequence $\varinjlim G_n$ is $I \otimes_R -$ exact whenever I is an injective right R-module. Note that each F_i is Gorenstein flat, then $\mathrm{Tor}_i^R(I, \varinjlim F_i) \cong \varinjlim \mathrm{Tor}_i^R(I, F_i) = 0$ for all $i \geq 1$ and all injective right R-module I. This shows that $\varinjlim F_i \in \mathcal{GF}(R)$. □

It is known that, over coherent rings, all finitely presented Gorenstein projective module are Gorenstein flat (Cf. [15, Proposition 10.3.2]). Thus the following Corollary is an immediate consequence of Proposition 4.1.

Corollary 4.2 *Let R be a left coherent ring. Then any direct limit of countable direct system of finitely presented Gorenstein projective modules is Gorenstein flat.* □

For the class of strongly Gorenstein projective modules, we have the following result:

Proposition 4.3 *Let $\mathcal{S} = \mathcal{SGP}(R) \bigcap \mathrm{SML}(R)$. Then any direct limit of countable system of modules from $\mathcal{S}$ is Gorenstein flat.*

Proof By the definition of strongly Gorenstein projective modules and Remark 3.2(1), it is easy to see that $\mathcal{S} \subseteq \mathcal{GF}(R)$. The rest proof is similar to that of Proposition 4.1. □

Remark 4.4 (1). Note that each countably generated strongly Gorenstein projective module is strict Mittag-Leffler over R, and hence any direct limit of countable direct system of countably generated strongly Gorenstein projective modules is Gorenstein flat [32, Proposition 3.5].

(2). Provided that $\mathcal{GP}(R) \subseteq \mathcal{GF}(R)$, then any direct limit of countable direct system of Gorenstein projective modules is Gorenstein flat. The proof is similar to that of Proposition 4.1.

In the remaining part of this paper, we explore the conditions under which the class of Gorenstein projective modules is closed under direct limits. To this end, we need two lemmas.

Lemma 4.5 *Let R be a ring. Then R is left perfect if and only if $\mathcal{GF}(R) \subseteq \mathrm{SML}(R)$.*

Proof (⇒). Let R be a left perfect ring. Then for any $M \in \mathcal{GF}(R)$, there is a short exact sequence $0 \to M \to F \to M' \to 0$ such that $F \in \mathcal{P}(R)$ and $M' \in \mathcal{GF}(R)$. This implies that the map

$$\Phi^{(1)}_{M'} : \mathrm{Tor}^R_1(\mathrm{Hom}_Z(R, D), M') \longrightarrow \mathrm{Hom}_Z(\mathrm{Ext}^1_R(M', R), D)$$

is monomorphic for any divisible abelian group D. Follows from Lemma 2.5, $M \in \mathrm{SML}(R)$.

(⇐). Note that $\mathcal{F}(R) \subseteq \mathcal{GF}(R) \subseteq \mathrm{SML}(R)$. It is clear that R is left perfect by [1, Corollary 5.5] and [6, Corollary 2]. □

Lemma 4.6 *(*[18, Proposition 2.3]*) Let $\mathcal{S}$ be a class of R-modules. If $\mathcal{S}$ is closed under well-ordered direct limits, then it is closed under arbitrary direct limits.*

Theorem 4.7 *Let R be a ring such that all left R-modules are strict Mittag-Leffler over R. Then $\mathcal{GP}(R)$ is closed under direct limits.*

Proof By Lemma 4.6, it suffices to prove that $\mathcal{GP}(R)$ is closed under well ordered direct limits. Let $(M_\alpha)_{\alpha<\lambda}$ be a well ordered direct system of Gorenstein projective modules.

If $\lambda < \omega$, there is nothing to prove.

If $\lambda = \omega$, we consider the following diagram

$$\begin{tikzcd}
G_0 =: \arrow[d, dashed] & 0 \arrow[r] & M_0 \arrow[r] \arrow[d] & P_0^0 \arrow[r] \arrow[d, dashed] & P_0^1 \arrow[r] \arrow[d, dashed] & P_0^2 \arrow[r] \arrow[d, dashed] & \cdots \\
G_1 =: \arrow[d, dashed] & 0 \arrow[r] & M_1 \arrow[r] \arrow[d] & P_1^0 \arrow[r] \arrow[d, dashed] & P_1^1 \arrow[r] \arrow[d, dashed] & P_1^2 \arrow[r] \arrow[d, dashed] & \cdots \\
G_2 =: \arrow[d, dashed] & 0 \arrow[r] & M_2 \arrow[r] \arrow[d] & P_2^0 \arrow[r] \arrow[d, dashed] & P_2^1 \arrow[r] \arrow[d, dashed] & P_2^2 \arrow[r] \arrow[d, dashed] & \cdots \\
\vdots & & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

with exact rows and each P_i^j projective, and each $K_i^j = \mathrm{Ker}(P_i^j \to P_i^{j+1})$ Gorenstein projective for all $i \geq 0$ and $j \geq 0$. Since each M_n is Gorenstein projective, each map $M_i \to M_{i+1}$ gives rise to a chain map $G_i \to G_{i+1}$. Note that each column in the above diagram is again direct system. By the exactness of the functor $\varinjlim$, we obtain an exact sequence

$$\varinjlim G_i =: 0 \to \varinjlim M_i \to \varinjlim P_i^0 \to \varinjlim P_i^1 \to \varinjlim P_i^2 \to \cdots .$$

Since R is left perfect by Lemma 4.5, $\varinjlim P_i^j$ is projective for all $j \geq 0$. Now we need to prove that $\mathrm{Hom}_R(\varinjlim G_i, P)$ is exact whenever $P \in \mathcal{P}(R)$. Following from [34, Theorem 3.8], R-Mod $= \mathrm{SML}(R)$ if and only if R is $\sum$-pure injective as a left R-module, that is $R^{(I)}$ is pure injective for any set I. By [19, Lemma 3.3.4],

$$\mathrm{Ext}_R^1(\varinjlim K_i^j, R^{(I)}) \cong \varprojlim \mathrm{Ext}_R^1(K_i^j, R^{(I)}) = 0$$

for all $j \geq 0$. Then $\mathrm{Ext}_R^1(\varinjlim K_i^j, P) = 0$ for all $j \geq 0$. This shows that $\mathrm{Hom}_R(\varinjlim G_i, P)$ is exact. Since each $M_i, i = 0, 1, 2 \cdots$ is Gorenstein projective, it follows that

$$\mathrm{Ext}_R^n(\varinjlim M_i, R^{(I)}) \cong \varprojlim \mathrm{Ext}_R^n(M_i, R^{(I)}) = 0$$

for all $n \geq 1$. and so $\mathrm{Ext}_R^n(\varinjlim M_i, P) = 0$. Thus $\varinjlim M_i$ is Gorenstein projective.

Now we assume that $\lambda > \omega$. The rest of proof is similar to that of [31, Lemma 3.1], we reindex the modules $M_0, M_1, \cdots, M_\omega, M_{\omega+1}, \cdots$ such that $M_\omega = \varinjlim M_i$ and $M_{\omega+k+1}$ is the old of $M_{\omega+k}$ for each $k \geq 0$. Repeating this procedure for each limit ordinal, we have that $M_\beta = \varinjlim M_\alpha$ $(\alpha < \beta)$ whenever β is a limit ordinal with $\beta < \lambda$. Thus we may assume that the direct system $(M_\alpha)_{\alpha<\lambda}$ is continuous. Therefore the proof of the case $\lambda = \omega$ presented above can easily be extended to arbitrary limit ordinal, one can prove that M_β is Gorenstein projective for each limit ordinal β. By transfinite induction on λ, it is not difficult to check that $\varinjlim M_\alpha(\alpha < \lambda)$ is Gorenstein projective, and hence $\mathcal{GP}(R)$ is closed under direct limits. $\square$

As a corollary, we obtain that:

Corollary 4.8 *If R is a left perfect and right coherent ring, then $\mathcal{GP}(R)$ is closed under direct limits.*

Proof We note that $Add(R)$ is closed under products if R is left perfect and right coherent. So R is $\sum$-pure injective module as a left R-module by [19, Lemma 1.2.23]. Thus R-Mod $=$ SML(R), then Theorem 4.7 applies. □

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