

Non-Archimedean Duality: Algebras, Groups, and Multipliers

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Abstract We develop a duality theory for multiplier Banach-Hopf algebras over a non-Archimedean field \mathbb{K} . As examples, we consider algebras corresponding to discrete groups and zero-dimensional locally compact groups with \mathbb{K} -valued Haar measure, as well as algebras of operators generated by regular representations of discrete groups.

Keywords Non-Archimedean field · Locally compact group · Group algebra · Dual object · Multiplier Banach-Hopf algebra

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1 Introduction

Non-Archimedean harmonic analysis, the Fourier analysis of functions $f : G \rightarrow \mathbb{K}$ where G is a group, \mathbb{K} is a non-Archimedean valued field, was initiated in the thesis [19] by W. H. Schikhof; this subject should not be confused with the study of complex-valued functions on non-Archimedean structures started in another great thesis, by J. Tate. For Abelian groups admitting a \mathbb{K} -valued Haar measure (this class was described by Monna and Springer [15]), Schikhof proved an analog of Pontryagin's duality theorem; see also [17, 18]. Duality theorems for compact groups and related group algebras in the non-Archimedean setting were proved later by Schikhof [20] and Diarra [3]; for related subjects see [5].

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In this paper we develop a duality theory for general multiplier Banach-Hopf algebras over \mathbb{K} , including a description of a dual object and a biduality theorem. This covers three main examples: 1) the algebra $A_1 = c_0(G_1)$ of \mathbb{K} -valued functions on a discrete group G_1 tending to zero by the filter of complements of finite sets, with pointwise operations; 2) the algebra $A_2 = C_0(G_2)$ of \mathbb{K} -valued continuous functions on a zero-dimensional locally compact group G_2 admitting a \mathbb{K} -valued Haar measure, and 3) the algebra $A_3 = A(G_3)$ generated by the right regular representation of a discrete group G_3 . The first two examples describe, as much as possible, the group duality (groups without a \mathbb{K} -valued Haar measure possess quite pathological properties [4] and are not accessible so far).

The third algebra $A(G_3)$ appeared [12] as a group algebra belonging, for some classes of groups, to the class of operator algebras with Baer reductions, a class seen as a non-Archimedean counterpart of von Neumann algebras [11]. By its properties [12], this algebra can be seen simultaneously as an analog of the reduced group C^* -algebra. In this paper we see its third face – as a Banach-Hopf algebra, it is isomorphic to the object dual to A_1 and can be considered as a non-Archimedean compact quantum group. For another approach to non-Archimedean quantum groups see [22].

Our methods and framework are very close to the purely algebraic setting developed by Van Daele [26, 27]; in the group situation this theory deals with complex-valued functions on a group different from zero at finitely many points. It is interesting that the transition from the algebraic duality theory to the non-Archimedean topological theory is easier than the similar, in principle, transitions to the cases of C^* -algebras and von Neumann algebras over \mathbb{C} [13, 14, 25, 28]. The main reason is the fact that, for example, $c_0(G_1)$ is simultaneously the basic Banach space; no Hilbert space is needed, and a reasonable analog of the latter does not exist in the non-Archimedean case where there are no natural involutions. In contrast to the classical situations, in all the above examples (and in our general results) the counit and antipode are everywhere defined bounded mappings.

The contents of the paper are as follows. In Section 2, we present some preliminaries from non-Archimedean analysis; we also discuss the notion of a multiplier and properties of multiplier algebras. In Section 3, we define and study non-Archimedean multiplier Banach-Hopf algebras. Section 4 is devoted to the dual object and the biduality theorem. In Section 5, we describe in detail the above three examples.

The proofs of many results about non-Archimedean multiplier Banach-Hopf algebras are very similar to their classical counterparts; in such cases we will just refer refer to papers by Van Daele [26, 27] and their expositions in [23, 24].

2 Preliminaries

2.1 Non-Archimedean Banach Spaces [1, 16, 17]

A non-Archimedean valued field is a field \mathbb{K} with a nonnegative real absolute value (or valuation) $|\lambda|$, $\lambda \in \mathbb{K}$, such that $|\lambda| = 0$ if and only if $\lambda = 0$, $|\lambda\mu| = |\lambda| \cdot |\mu|$, $|\lambda + \mu| \leq \max(|\lambda|, |\mu|)$, $\lambda, \mu \in \mathbb{K}$. Below we assume that the valuation is nontrivial, that is we exclude the case where $|\lambda| = 1$ for every $\lambda \neq 0$. The field with a nontrivial valuation is a nondiscrete totally disconnected topological field with respect to the topology induced by the ultrametric $(\lambda, \mu) \mapsto |\lambda - \mu|$. Typical examples are the field \mathbb{Q}_p of p -adic numbers (p is a prime number) and the field \mathbb{C}_p , the completion of an algebraic closure of \mathbb{Q}_p with respect to a valuation obtained by extension of the one from \mathbb{Q}_p . On \mathbb{Q}_p , the valuation is discrete – the

absolute value takes values from the set $\{p^n, n \in \mathbb{Z}\}$, while the valuation on \mathbb{C}_p is dense, its possible values are $p^v, v \in \mathbb{Q}$. Many properties of these fields are quite different.

Let E be a \mathbb{K} -vector space. A *norm* on E is a map $\|\cdot\| : E \rightarrow [0, \infty)$, such that $\|x\| = 0$ if and only if $x = 0$; $\|\lambda x\| = |\lambda| \cdot \|x\|$; $\|x + y\| \leq \max(\|x\|, \|y\|)$ for all $x, y \in E, \lambda \in \mathbb{K}$. E is called a (non-Archimedean) Banach space, if E is complete with respect to the ultrametric $(x, y) \mapsto \|x - y\|$. Under this ultrametric, all the sets $\{x \in E : \|x - x_0\| \leq c\}, \{x \in E : \|x - x_0\| < c\}, \{x \in E : \|x - x_0\| = c\}$ ($x_0 \in e, c > 0$) are both open and closed. A normed space E is a Banach space, if and only if every sequence converging to zero is summable.

The above phenomena are of purely non-Archimedean nature. Meanwhile most of the notions and results of classical functional analysis have their non-Archimedean counterparts. Here we will touch the notions of non-Archimedean orthogonality, separation property and tensor product.

We say that vectors x and y of a normed space E are *orthogonal*, if, for all $\lambda, \mu \in \mathbb{K}$,

$$\|\lambda x + \mu y\| = \max(\|\lambda x\|, \|\mu y\|). \tag{2.1}$$

If $\|x\| = \|y\| = 1$, the equality (2.1) turns into the orthonormality relation

$$\|\lambda x + \mu y\| = \max(|\lambda|, |\mu|).$$

In an obvious way, this property is extended to any finite number of elements.

There is also a weaker property of t -orthogonality of a system $\{e_1, \dots, e_n\}, 0 < t \leq 1$, defined by the property that

$$\|\lambda_1 e_1 + \dots + \lambda_n e_n\| \geq t \max_{1 \leq i \leq n} \|\lambda_i e_i\|$$

for any $\lambda_1, \dots, \lambda_n \in \mathbb{K}$. The 1-orthogonality coincides with the above orthogonality.

A sequence $e_1, e_2, \dots \in E$ is called a t -orthogonal basis, if any its finite subset is t -orthogonal and every $x \in E$ has a convergent expansion $x = \sum_{n=1}^{\infty} \lambda_n e_n, \lambda_n \in \mathbb{K}$. The basis is called *orthogonal*, if $t = 1$, and *orthonormal* if, in addition, $\|e_n\| = 1$ for all $n \in \mathbb{N}$. See [1, 16, 17] regarding classes of Banach spaces possessing orthonormal bases.

It was a nontrivial problem whether an analog of the Hahn-Banach theorem holds in the non-Archimedean case. We will not need its solution in this paper (see [16, 17]); note only that it holds for spaces over \mathbb{Q}_p and typically does not hold for spaces over \mathbb{C}_p . However we will need the following weaker property resembling some classical applications of the Hahn-Banach theorem.

A Banach space E over \mathbb{K} is called *polar*, if it has the following property. Let B be the unit ball in E , that is $B = \{x \in E : \|x\| \leq 1\}$, and let $y \notin B$. Then there exists such a linear continuous functional $f : E \rightarrow \mathbb{K}$ that $|f(x)| \leq 1$ for $x \in B$ and $|f(y)| > 1$; see Lemma 4.4.4 in [16] regarding the equivalence of this definition to some others.

If a Banach space E is polar, then linear continuous functionals separate points of E . All the spaces encountered in this paper are polar (see Section 4.4 in [16]).

Let E and F be Banach spaces over \mathbb{K} . On the algebraic tensor product $E \otimes F$, we define a norm setting

$$\|g\| = \inf \left(\max_{1 \leq i \leq r} \|x_i\| \cdot \|y_i\| \right) \tag{2.2}$$

where the infimum runs over all possible representations

$$g = \sum_{i=1}^r x_i \otimes y_i, \quad x_i \in E, y_i \in F.$$

The proof that (2.2) is a norm indeed, employs essentially the non-Archimedean properties of E, F ; see [16], Corollary 10.2.10. Note also that $\|x \otimes y\| = \|x\| \cdot \|y\|$ (see [17], Theorem 4.27).

The completion $E \widehat{\otimes} F$ of $E \otimes F$ with respect to this norm is called *the completed tensor product* of E and F .

Lemma 2.1 *Let $w \in E \widehat{\otimes} F$, $0 < t < 1$. Then there exists a t -orthogonal sequence $\{a_i\}_{i=1}^\infty \subset E$, and a sequence $\{b_i\}_{i=1}^\infty \subset F$, such that $\lim_{i \rightarrow \infty} \|a_i\| \cdot \|b_i\| = 0$,*

$$w = \sum_{i=1}^\infty a_i \otimes b_i \tag{2.3}$$

and $\|w\| = \inf \left\{ \max_i \|a_i\| \cdot \|b_i\| \right\}$ where the infimum runs over all possible representations (2.3). Once the t -orthogonal sequence $\{a_i\}$ is chosen, the sequence $\{b_i\}$ is unique and $t \sup_{i \geq 1} \|a_i\| \cdot \|b_i\| \leq \|w\| \leq \sup_{i \geq 1} \|a_i\| \cdot \|b_i\|$.

For the proof see [17], Lemma 4.30 and Corollary 4.31, and [16], Corollary 10.2.10.

As in the classical case, given linear bounded mappings of Banach spaces, it is possible to define their tensor product acting on the completed tensor product of the spaces; see [16], Theorem 10.3.7.

2.2 Banach Algebras

A \mathbb{K} -algebra A is called a *Banach algebra*, if A is simultaneously a Banach space with the norm $\|\cdot\|$, and $\|xy\| \leq \|x\| \cdot \|y\|$, $x, y \in A$. If A is an algebra with unit e , then we assume that $\|e\| = 1$. We will often denote a unit by 1, if that does not lead to a confusion. We call a Banach algebra polar, if such is the underlying Banach space.

If A and B are Banach algebras, then $A \widehat{\otimes} B$ is also a Banach algebra with the multiplication extending the relation $(a' \otimes b')(a'' \otimes b'') = (a'a'') \otimes (b'b'')$; see [1], Section 3.1.1. If A and B are polar, then $A \widehat{\otimes} B$ is polar too ([16], Theorem 10.2.7).

We say that the product in an algebra A is *nondegenerate* if the equality $ab = 0$ for all b implies $a = 0$, and the equality $ab = 0$ for all a implies $b = 0$.

Lemma 2.2 *If A and B are polar Banach algebras with nondegenerate products, then the product in $A \widehat{\otimes} B$ is nondegenerate.*

Proof Let $w \in A \widehat{\otimes} B$, $(c \otimes d)w = 0$ for all $c \in A, d \in B$. Writing w in the form of Eq. 2.3, where the sequences $\{a_i\}, \{b_i\}$ are such as stated in Lemma 2.1, we find that

$$\sum_{i=1}^\infty (ca_i) \otimes (db_i) = 0. \tag{2.4}$$

Let us apply to both sides of Eq. 2.4 the mapping $\text{id} \otimes \chi$ where χ is a linear continuous functional on B . We get

$$c \sum_{i=1}^{\infty} a_i \chi(db_i) = 0.$$

Since c is arbitrary and the product in A is nondegenerate, we have $\sum_{i=1}^{\infty} a_i \chi(db_i) = 0$, and the t -orthogonality of $\{a_i\}$ implies the equalities $\chi(db_i) = 0$ for all i . Since χ is arbitrary and B is polar, we get $db_i = 0$, thus $b_i = 0$ and $w = 0$. Similarly, if $w(c \otimes d) = 0$ for any c, d , then $w = 0$. □

We will often use the notion of a bounded approximate identity in a Banach algebra A without a unit. By definition, a bounded approximate identity is a net $\{e_\lambda\}_{\lambda \in \Lambda} \subset A$, contained within some ball, such that, for any $x \in A$, $e_\lambda x \rightarrow x$ and $x e_\lambda \rightarrow x$ in the topology of A .

2.3 Multipliers

Let A be a polar Banach algebra over \mathbb{K} with nondegenerate product.

A *left multiplier* on A is a linear continuous mapping $\rho_l : A \rightarrow A$, such that $\rho_l(ab) = \rho_l(a)b$ for all $a, b \in A$. Similarly, a *right multiplier* is defined by the property $\rho_r(ab) = a\rho_r(b)$, $a, b \in A$. A *multiplier* on A is a couple (ρ_l, ρ_r) of a left multiplier ρ_l and a right multiplier ρ_r , such that

$$\rho_r(a)b = a\rho_l(b) \quad \text{for all } a, b \in A. \tag{2.5}$$

If ρ_l and ρ_r are linear continuous mappings satisfying (2.5), then (ρ_l, ρ_r) is a multiplier. Indeed, it follows from Eq. 2.5 that $\rho_r(ac)b = ac\rho_l(b)$ for any $a, b, c \in A$, and also that $\rho_r(c)b = c\rho_l(b)$, so that $a\rho_r(c)b = ac\rho_l(b)$. Subtracting we have

$$[\rho_r(ac) - a\rho_r(c)]b = 0 \quad \text{for all } b.$$

It follows from the nondegeneracy that ρ_r is a right multiplier. Similarly we show that ρ_l is a left multiplier.

Denote by $L(A)$, $R(A)$, and $M(A)$ the sets of left, right, and multipliers on A respectively. The sets $L(A)$ and $R(A)$ are unital Banach algebras with respect to the usual norms of operators. $M(A)$ is a unital Banach algebra with the multiplication

$$(\rho'_l, \rho'_r)(\rho''_l, \rho''_r) = (\rho'_l \rho''_l, \rho''_r \rho'_r)$$

and the norm $\|(\rho_l, \rho_r)\| = \max(\|\rho_l\|, \|\rho_r\|)$. Examples of multiplier algebras will be given in Section 5 below.

For brevity, we will often write $ca = \rho_l(a)$ and $ac = \rho_r(a)$ for a multiplier $c = (\rho_l, \rho_r)$ and $a \in A$. In other words, in order to define a multiplier c , it suffices to define continuous multiplications ac and ca and to prove that $(ac)b = a(cb)$. This notation agrees with the obvious imbeddings $A \subset L(A)$, $A \subset R(A)$, and $A \subset M(A)$. When A is unital, all the three multiplier algebras are isomorphic to A . Therefore below we consider non-unital algebras.

Let A, B be Banach algebras over \mathbb{K} with nondegenerate product. A continuous homomorphism $\Phi : A \rightarrow M(B)$ is called *nondegenerate* if the linear span of each of the sets $\{\Phi(a)b : a \in A, b \in B\}$ and $\{b\Phi(a) : a \in A, b \in B\}$ is dense in B .

Theorem 2.3 *If A and B have nondegenerate products, A possesses a bounded approximate identity $\{e_\lambda\}_{\lambda \in \Lambda}$, and $\Phi : A \rightarrow M(B)$ is a nondegenerate continuous homomorphism, then Φ has a unique extension to a continuous homomorphism $\Phi_1 : M(A) \rightarrow M(B)$.*

Proof Let $(L, R) \in M(A)$. Let us first define the operators $\Phi_1(L)$ and $\Phi_1(R)$ on linear sets $B_l = \text{span}\{\Phi(a)b : a \in A, b \in B\}$ and $B_r = \text{span}\{b\Phi(a) : a \in A, b \in B\}$ respectively.

If $\Phi_1(L)$ is a required continuation onto B_l , then $\Phi(La)b = \Phi_1(L)\Phi(a)b, a \in A, b \in B$. This implies the uniqueness of the continuation. Similarly $b\Phi(aR) = b\Phi(a)\Phi_1(R)$, so that we get the uniqueness of the continuation $\Phi_1(R)$.

If there is a finite sum $\sum \Phi(a_i)b_i = 0$, then for any $d \in A, e \in B$,

$$e\Phi(d) \sum \Phi(La_i)b_i = e \sum \Phi(dLa_i)b_i = e\Phi(dR) \sum \Phi(a_i)b_i = 0,$$

so that $\sum \Phi(La_i)b_i = 0$. Therefore it is legitimate to define $\Phi_1(L)$ setting

$$\Phi_1(L)[\Phi(a)b] = \Phi(La)b.$$

Similarly, $\Phi_1(R)$ is defined as follows:

$$b\Phi(a)\Phi_1(R) = b\Phi(aR).$$

Then we set $\Phi_1(L, R) = (\Phi_1(L), \Phi_1(R))$. Obviously, Φ_1 is an extension of Φ . Also,

$$\Phi_1(L_1L_2)\Phi(a)b = \Phi(L_1L_2a)b = \Phi_1(L_1)\Phi(L_2a)b = \Phi_1(L_1)\Phi_1(L_2)\Phi(a)b.$$

This relation, together with a similar one for right multipliers, shows that Φ_1 is a homomorphism.

We have not yet checked that Φ_1 sends multipliers to multipliers. Let

$$z_1 = \sum_i \Phi(a_i)b_i, \quad z_2 = \sum_i \Phi(c_i)d_i.$$

Then

$$\begin{aligned} \Phi_1(L)(z_1z_2) &= \Phi_1(L) \left(\sum_{i,j} [\Phi(a_i)b_i] \cdot [\Phi(c_j)d_j] \right) = \sum_{i,j} \Phi_1(L)(\Phi(a_i)(b_i(\Phi(c_j)d_j))) \\ &= \sum_{i,j} \Phi(La_i)(b_i \cdot (\Phi(c_j)d_j)) = \sum_{i,j} [\Phi(La_i)b_i] \cdot \Phi(c_j)d_j = \Phi_1(L)(z_1)z_2. \end{aligned}$$

Similarly we can get the required property for right multipliers.

Since $\Phi(Le_\lambda)\Phi(x) = \Phi(Le_\lambda x) \rightarrow \Phi(Lx)$ in $M(B)$ for every $x \in A$, we find that

$$\Phi(Le_\lambda)z \rightarrow \Phi_1(L)z \quad \text{for all } z \in B_l.$$

The net $\{e_\lambda\}$ is bounded, so that $\|\Phi(Le_\lambda)z\|_B \leq C\|z\|_B$ and $\|\Phi_1(L)z\|_B \leq C\|z\|_B$. Thus, we have proved that $\Phi_1(L)$ is a bounded operator on B_l . Therefore it admits a unique extension to a bounded operator on B . Its multiplier property follows via the extension by continuity.

If $L_m \rightarrow 0$ in the uniform operator topology, then $L_me_\lambda \rightarrow 0$ uniformly in $\lambda \in \Lambda$. Therefore $\Phi(L_me_\lambda)\xi \rightarrow 0$ in B , uniformly with respect to $\lambda \in \Lambda$ and $\xi \in B, \|\xi\| \leq 1$. We may take a limit in λ and obtain that $\Phi_1(L_m)\xi \rightarrow 0$ uniformly in $\xi \in B, \|\xi\| \leq 1$, so that $\|\Phi_1(L_m)\| \rightarrow 0$. This means the continuity of Φ_1 . The proof for right multipliers is similar. □

Similarly one can obtain the extension property for anti-homomorphisms, that is for mappings Φ satisfying the identity $\Phi(ab) = \Phi(b)\Phi(a)$, $a, b \in A$. The only difference is that an anti-homomorphism transforms a left multiplier into a right one, and vice versa:

$$b\Phi(La) = b\Phi(a)\Phi_1(L)$$

etc.

Remark The above proof follows [26] and [8]. A proof of the extension property given within a general theory of multipliers of complex Banach algebras [2] is based on the Cohen-Hewitt factorization theorem. It is not known whether the latter is valid in the non-Archimedean situation, but all its available proofs (see [10]) fail in this case.

2.4 Strict Topology

Below we will use systematically the multiplier algebra $M(A\widehat{\otimes}A)$. The imbedding $A \subset M(A)$ implies the imbeddings $A\widehat{\otimes}A \subset M(A)\widehat{\otimes}M(A) \subset M(A\widehat{\otimes}A)$.

Let us consider the subset $M_0(A\widehat{\otimes}A) \subset M(A\widehat{\otimes}A)$ consisting of such multipliers $x \in M(A\widehat{\otimes}A)$ that $x(1 \otimes a)$, $x(a \otimes 1)$, $(1 \otimes a)x$, and $(a \otimes 1)x$ belong to $A\widehat{\otimes}A$ for all $a \in A$. $M_0(A\widehat{\otimes}A)$ is a \mathbb{K} -vector subspace of $M(A\widehat{\otimes}A)$. The set of all seminorms $x \mapsto \|x(1 \otimes a)\|_{A\widehat{\otimes}A}$, $x \mapsto \|x(a \otimes 1)\|_{A\widehat{\otimes}A}$, $x \mapsto \|(1 \otimes a)x\|_{A\widehat{\otimes}A}$, $x \mapsto \|(a \otimes 1)x\|_{A\widehat{\otimes}A}$, $a \in A$ endow $M_0(A\widehat{\otimes}A)$ with a locally convex topology, which will be called the *strict topology*.

Proposition 2.4 *If A possesses a bounded approximate identity $\{e_\lambda\}_{\lambda \in \Lambda}$, then $A \otimes A$ is dense in $M_0(A\widehat{\otimes}A)$ in the strict topology. In particular, if $x \in M_0(A\widehat{\otimes}A)$, then $x(1 \otimes e_\lambda) \rightarrow x$, $x(e_\lambda \otimes 1) \rightarrow x$, $(e_\lambda \otimes 1)x \rightarrow x$, $(1 \otimes e_\lambda)x \rightarrow x$ in the strict topology.*

Proof The first assertion follows from the second one, since, for example, $x(1 \otimes e_\lambda)$ belongs to $A\widehat{\otimes}A$ and can be approximated by elements from $A \otimes A$ in the topology of $A\widehat{\otimes}A$, thus also in the strict topology.

If $y \in A \otimes A$, $y = \sum_{i=1}^n p_i \otimes q_i$, $p_i, q_i \in A$, then $y - y(1 \otimes e_\lambda) = \sum_{i=1}^n p_i \otimes (q_i - q_i e_\lambda)$, so that

$$\|y - y(1 \otimes e_\lambda)\|_{A\widehat{\otimes}A} \leq \max_i [\|p_i\| \cdot \|q_i - q_i e_\lambda\|] \rightarrow 0$$

in $\lambda \in \Lambda$.

Let $\|e_\lambda\| \leq C$. Then $\|(1 \otimes e_\lambda)\|_{M(A\widehat{\otimes}A)} \leq C$. If $X \in A\widehat{\otimes}A$, then for any $\epsilon > 0$, there exists $y \in A \otimes A$, such that $\|X - y\|_{A\widehat{\otimes}A} < \min(\epsilon, \frac{\epsilon}{C})$. We get

$$\|X - X(1 \otimes e_\lambda)\|_{A\widehat{\otimes}A} \leq \max\{\epsilon, \|y - y(1 \otimes e_\lambda)\|_{A\widehat{\otimes}A}, \|(X - y)(1 \otimes e_\lambda)\|_{A\widehat{\otimes}A}\}.$$

Choosing $\lambda_0 \in \Lambda$, such that $\|y - y(1 \otimes e_\lambda)\|_{A\widehat{\otimes}A} < \epsilon$ for $\lambda \geq \lambda_0$, we find that for these λ ,

$$\|X - X(1 \otimes e_\lambda)\|_{A\widehat{\otimes}A} < \epsilon. \tag{2.6}$$

In particular, this is true for $X = (a \otimes 1)x$, $a \in A$, so that

$$(a \otimes 1)[x(1 \otimes e_\lambda) - x] \longrightarrow 0.$$

Next,

$$\begin{aligned} x(1 \otimes e_\lambda)(1 \otimes a) &= x(1 \otimes (e_\lambda a)) = x(1 \otimes (ae_\lambda)) + x(1 \otimes (e_\lambda a - ae_\lambda)) \\ &= x(1 \otimes a)(1 \otimes e_\lambda) + x(1 \otimes (e_\lambda a - ae_\lambda)). \end{aligned}$$

The first summand tends to $x(1 \otimes a)$ by Eq. 2.6, while the second summand tends to 0. The proofs of the remaining limit relations are similar. \square

3 Multiplier Banach-Hopf Algebras

3.1 Main Notions

Let A be a polar Banach algebra over \mathbb{K} with a nondegenerate product possessing a bounded approximate identity.

A nondegenerate continuous homomorphism $\Delta : A \rightarrow M(A\widehat{\otimes}A)$ is called a *comultiplication* on A , if:

- (i) For any $a, b \in A$, the elements from $M(A\widehat{\otimes}A)$ in the right-hand sides of the formulas

$$T_1(a \otimes b) = \Delta(a)(1 \otimes b); \quad T_2(a \otimes b) = (a \otimes 1)\Delta(b);$$

$$T_3(a \otimes b) = (1 \otimes b)\Delta(a); \quad T_4(a \otimes b) = \Delta(b)(a \otimes 1)$$

actually belong to $A\widehat{\otimes}A$, so that $\Delta(a) \in M_0(A\widehat{\otimes}A)$ for any $a \in A$.

- (ii) The homomorphism Δ satisfies the following coassociativity condition – the diagram

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & M(A\widehat{\otimes}A) \\ \downarrow \Delta & & \downarrow \text{id} \otimes \Delta \\ M(A\widehat{\otimes}A) & \xrightarrow{\Delta \otimes \text{id}} & M(A\widehat{\otimes}A\widehat{\otimes}A) \end{array}$$

in which the nondegenerate homomorphisms $\Delta, \Delta \otimes \text{id}$, and $\text{id} \otimes \Delta$, are extended to the appropriate multiplier algebras, – is commutative.

A pair (A, Δ) is called a *regular multiplier Banach-Hopf algebra*, if the expressions T_1, T_2, T_3 , and T_4 defined in (i) are extended to bijective linear isometric mappings $T_1, T_2, T_3, T_4 : A\widehat{\otimes}A \rightarrow A\widehat{\otimes}A$. In this paper we will not consider non-regular algebras, thus the word “regular” will be dropped.

Let Δ' be the opposite comultiplication obtained from Δ by composing it with the flip σ on $A\widehat{\otimes}A, \sigma(a \otimes b) = b \otimes a$. Then (A, Δ') is also a multiplier Banach-Hopf algebra. Note that σ is an isometry ([1], Section 2.1, Proposition 6(ii)) and can be extended to multipliers. For example, if $l \in L(A\widehat{\otimes}A)$, then $l_\sigma = \sigma(l)$ acts as follows:

$$l_\sigma(a \otimes b) = \sigma(l(\sigma(a \otimes b))) = \sigma(l(b \otimes a)),$$

and the mapping $l \mapsto l_\sigma$ extends the flip operation on $A \otimes A$.

Let $m : A\widehat{\otimes}A \rightarrow A$ be an extension by linearity and continuity of the multiplication operator, $m(a \otimes b) = ab$; see Proposition 2.1.7.1 in [1]. Our construction of the counit follows [26]. We define a continuous mapping $E : A \rightarrow L(A)$ setting

$$E(a)b = mT_1^{-1}(a \otimes b).$$

It is checked directly that $E(a)$ is indeed a left multiplier, and the identity

$$(\text{id} \otimes E)((a \otimes 1)\Delta(b)) = (ab) \otimes 1 \tag{3.1}$$

is valid for any $a, b \in A$.

The proof of Eq. 3.1 is identical with that of Lemma 3.2 in [26], with a single difference – instead of a finite sum in the representation of $a \otimes b$ as an element from the range of T_1

we have to write a limit of an infinite sequence:

$$a \otimes b = \lim_{j \rightarrow \infty} \sum_{i=1}^{m_j} \Delta(a_i^{(j)}) (1 \otimes b_i^{(j)}) \tag{3.2}$$

using the continuity of Δ and the polarity of A . In order to simplify notation, instead of the expressions like (3.2), we will always write

$$a \otimes b = \lim \sum \Delta(a_i)(1 \otimes b_i), \tag{3.3}$$

always remembering that (3.3) is just a shorthand for (3.2).

It follows from the surjectivity of T_2 that, for arbitrary elements $c, d \in A$,

$$c \otimes d = T_2 \left(\lim \sum a_j \otimes b_j \right) = \lim \sum (a_j \otimes 1) \Delta(b_j),$$

and by the identity (3.1),

$$(\text{id} \otimes E)(c \otimes d) = \lim \sum (a_i b_i) \otimes 1$$

where the existence of a limit $x = \lim \sum a_i b_i \in A \widehat{\otimes} A$ follows from the continuity of m .

Thus $c \otimes E(d) = (\text{id} \otimes E)(c \otimes d) = x \otimes 1$ for any $c, d \in A$, so that $E(A) \subset \mathbb{K} \cdot 1$. Now we can define the counit $\varepsilon : A \rightarrow \mathbb{K}$ setting $\varepsilon(a)1 = E(a)$.

Theorem 3.1 *The counit ε is a continuous homomorphism $A \rightarrow \mathbb{K}$, such that*

$$\begin{aligned} (\text{id} \otimes \varepsilon)((a \otimes 1) \Delta(b)) &= ab; & (\varepsilon \otimes \text{id})(\Delta(a)(1 \otimes b)) &= ab; \\ (\text{id} \otimes \varepsilon)(\Delta(a)(b \otimes 1)) &= ab; & (\varepsilon \otimes \text{id})((1 \otimes a) \Delta(b)) &= ab. \end{aligned} \tag{3.4}$$

For the proof see [26].

By the above construction, $\|\varepsilon\| \leq 1$.

The antipode $S : A \rightarrow M(A)$ is defined by the formula

$$S(a)b = (\varepsilon \otimes \text{id})T_1^{-1}(a \otimes b), \quad a, b \in A. \tag{3.5}$$

Theorem 3.2 *The antipode is a continuous anti-homomorphism $A \rightarrow M(A)$ satisfying the identities*

$$m(S \otimes \text{id})(\Delta(a)(1 \otimes b)) = \varepsilon(a)b; \tag{3.6}$$

$$m(\text{id} \otimes S)((a \otimes 1) \Delta(b)) = \varepsilon(b)a. \tag{3.7}$$

The identities (3.6) and (3.7) define the antipode in a unique way. In addition, elements $S(a), a \in A$, belong actually to A .

Proof Since T_1 is an isometry and $\|a \otimes b\| = \|a\| \cdot \|b\|$, we have

$$\|S(a)b\| \leq \|\varepsilon \otimes \text{id}\|_{A \widehat{\otimes} A \rightarrow A} \|a\| \cdot \|b\|,$$

so that $\|S(a)\| \leq C\|a\|$, which proves the continuity of S . The algebraic properties are proved just as in [26] and [23]. \square

Let ε' and S' be the counit and antipode corresponding to the opposite comultiplication Δ' . Again, just as in the algebraic theory [26], we prove that $\varepsilon' = \varepsilon, S' : A \rightarrow A$,

$$SS' = S'S = \text{id}, \tag{3.8}$$

and

$$(1 \otimes Sb) \Delta(Sa) = (S \otimes S)(\Delta'(a)(1 \otimes b)). \tag{3.9}$$

The relation (3.8) also means that S has a bounded inverse.

Proposition 3.3 *The antipode S can be extended to a continuous anti-homomorphism $M(A) \rightarrow M(A)$.*

Proof We have to check that S is nondegenerate, that is the set

$$\text{span}\{S(a)b : a, b \in A\}$$

is dense in A . Since T_1 is bijective, the set $\text{span}\{T_1^{-1}(a \otimes b) : a, b \in A\}$ is dense in $A \widehat{\otimes} A$. Let us choose $y \in A$ in such a way that $\varepsilon(y) = 1$. Then $(\varepsilon \otimes \text{id})(y \otimes x) = x$ for every $x \in A$. This means that every element $x \in A$ can be approximated by linear combinations of elements $(\varepsilon \otimes \text{id})T_1^{-1}(a \otimes b)$ where the corresponding linear combinations of elements $a \otimes b$ are chosen to approximate $T_1(y \otimes x)$. \square

3.2 Invariant Functionals

Let ω be a linear continuous mapping $A \rightarrow \mathbb{K}$, a be an element from A . Define a left multiplier ρ_l setting

$$\rho_l(b) = (\omega \otimes \text{id})(\Delta(a)(1 \otimes b))$$

and a right multiplier

$$\rho_r(b) = (\omega \otimes \text{id})((1 \otimes b)\Delta(a)).$$

The isometricity of T_1 and T_3 implies the continuity of ρ_l and ρ_r , and it is checked easily that $(\rho_l, \rho_r) \in M(A)$. As in [27], we denote

$$(\rho_l, \rho_r) = (\omega \otimes \text{id})\Delta(a).$$

Similarly, using T_2 and T_4 we define $(\text{id} \otimes \omega)\Delta(a) \in M(A)$.

The nonzero functional ω is called *left-invariant*, if

$$(\text{id} \otimes \omega)\Delta(a) = \omega(a)1_{M(A)}, \tag{3.10}$$

and *right-invariant*, if

$$(\omega \otimes \text{id})\Delta(a) = \omega(a)1_{M(A)}. \tag{3.11}$$

In this section and in Section 4, we assume the existence of left-invariant and right-invariant functionals. In fact, it suffices to know the existence of one of them – if φ is a left-invariant functional, and $\psi = \varphi \circ S$, then ψ is right-invariant (see [27]). In the examples of Section 5, the invariant functionals will be constructed explicitly.

Below we will prove the uniqueness of invariant functionals (up to a scalar factor). The above notation, φ for a left-invariant functional, and ψ for a right-invariant one, will be retained throughout the paper.

Let us study some properties of invariant functionals. First of all, they are faithful, that is the equality $\varphi(ba) = 0$ (or the equality $\varphi(ab) = 0$) for all $b \in A$ implies the equality $a = 0$. The functional ψ possesses a similar property. For the proof see [27], Proposition 3.4.

In the following lemma, often used in the sequel, the infinite sums (3.3) are understood in the sense of (3.2).

Lemma 3.4 *Let $p_i, q_i \in A$. Suppose that there exists the limit $\lim \sum \Delta(p_j)(q_j \otimes 1)$ in $A \widehat{\otimes} A$. Then, for any linear continuous functional $\chi : A \rightarrow \mathbb{K}$, there exists the limit $\lim \sum p_j \chi(q_j)$ in A .*

Proof We have $T_4(q_i \otimes p_i) = \Delta(p_i)(q_i \otimes 1)$,

$$\sum p_i \otimes q_i = \sigma \circ T_4^{-1} \left(\sum \Delta(p_i)(q_i \otimes 1) \right).$$

Therefore there exists $\lim \sum p_i \otimes q_i$, thus also $\lim \sum p_j \chi(q_j)$. □

Similar reasoning works for sums related to T_1, T_2, T_3 .

Returning to properties of invariant functionals, we begin with the following result similar to Lemma 3.6 in [27]. We consider only nonzero invariant functionals.

Lemma 3.5 *The following sets of functionals on A coincide:*

$$\{\psi(\cdot a) : a \in A\} = \{\varphi(\cdot a) : a \in A\}; \tag{3.12}$$

$$\{\psi(a \cdot) : a \in A\} = \{\varphi(a \cdot) : a \in A\}. \tag{3.13}$$

In particular, these sets do not depend on the choice of φ, ψ .

Proof Let us prove the inclusion $\{\psi(\cdot a) : a \in A\} \subseteq \{\varphi(\cdot a) : a \in A\}$. Then the remaining three inclusions can be proved in a similar way.

In fact, we have to prove that for a given left-invariant functional φ , a given a , and for any right-invariant functional ψ , there exists such $c \in A$ that $\varphi(\cdot a) = \psi(\cdot c)$.

By definition, we have $(\text{id} \otimes \varphi)\Delta(a) = \varphi(a)1_{M(A)}$. Choose $b \in A$ so that $\psi(b) = 1$. Then, for any $x \in A$,

$$\varphi(xa) = \psi(b)\varphi(xa) = (\psi \otimes \varphi)(\Delta(x)\Delta(a)(b \otimes 1)).$$

Since T_1 is surjective, we may write

$$\Delta(a)(b \otimes 1) = \lim \sum \Delta(c_i)(1 \otimes d_i), \quad c_i, d_i \in A.$$

Now

$$\varphi(xa) = \lim \sum (\psi \otimes \varphi)(\Delta(x)\Delta(c_i)(1 \otimes d_i)) = \lim \sum \psi(xc_i)\varphi(d_i) = \psi(xc)$$

where $c = \lim \sum c_i\varphi(d_i)$. The existence of this limit follows from Lemma 3.4. □

Theorem 3.6 *The left-invariant and right-invariant functionals are unique, up to multiplication by scalars.*

The *proof*, based on Lemma 3.4 and Lemma 3.5, repeats the reasoning from [27].

In fact, just as in [27], all the four sets of functionals listed in (3.12), (3.13) coincide (below this set of functionals will have the meaning of the dual object \hat{A}). The *proof*, identical with that of Proposition 3.11 in [27], uses also the following lemma needed later.

Lemma 3.7 *Let $a, b, a_i, b_i \in A$. The following properties are equivalent:*

- (i) $\Delta(a)(1 \otimes b) = \lim \sum \Delta(a_i)(b_i \otimes 1)$;
- (ii) $a \otimes S^{-1}b = \lim \sum (a_i \otimes 1)\Delta(b_i)$;
- (ii) $(1 \otimes a)\Delta(b) = \lim \sum (Sb_i) \otimes a_i$.

For the *proof* see [26] (Lemma 5.5).

3.3 Modular Element. Modular Automorphism

A modular element δ , a general version of the classical modular function defined on locally compact groups, is described in the next proposition.

Proposition 3.8 *There exists such a multiplier $\delta \in M(A)$ that $(\varphi \otimes \text{id})\Delta(a) = \varphi(a)\delta$, that is*

$$(\varphi \otimes \text{id})((1 \otimes b)\Delta(a)) = \varphi(a)b\delta; \tag{3.14}$$

$$(\varphi \otimes \text{id})(\Delta(a)(1 \otimes b)) = \varphi(a)\delta b. \tag{3.15}$$

Proof For each $a \in A$, define a multiplier $\delta_a \in M(A)$ setting $\delta_a = (\varphi \otimes \text{id})\Delta(a)$, that is

$$\delta_a b = (\varphi \otimes \text{id})(\Delta(a)(1 \otimes b)),$$

$$b\delta_a = (\varphi \otimes \text{id})((1 \otimes b)\Delta(a)).$$

For a continuous linear functional ω on A , set

$$\varphi_1(a) = (\varphi \otimes \omega)((1 \otimes b)\Delta(a)) = \omega(b\delta_a).$$

Let us check that the functional φ_1 is left invariant. Since A is polar, it is sufficient to prove that for any linear continuous functional f on A and any $c, d \in A$,

$$(f \otimes \varphi_1)(\Delta(c)(d \otimes 1)) = \varphi_1(c)f(d); \tag{3.16}$$

$$(f \otimes \varphi_1)((d \otimes 1)\Delta(c)) = \varphi_1(c)f(d); \tag{3.17}$$

see [23], Remark 6.1.3.

By Proposition 2.4, we can write $\Delta(c)$ as a strict limit

$$\Delta(c) = \lim \sum c'_i \otimes c''_i, \quad c'_i, c''_i \in A. \tag{3.18}$$

Writing the coassociativity relation for the opposite comultiplication Δ' :

$$(a \otimes 1 \otimes 1)(\Delta' \otimes \text{id})(\Delta'(b)(1 \otimes c)) = (\text{id} \otimes \Delta')((a \otimes 1)\Delta'(b))(1 \otimes 1 \otimes c)$$

and applying the flip to both sides, we get the identity

$$(\Delta \otimes \text{id})((1 \otimes a)\Delta(b))(c \otimes 1 \otimes 1) = (1 \otimes 1 \otimes a)(\text{id} \otimes \Delta)(\Delta(b)(c \otimes 1)), \quad a, b, c \in A. \tag{3.19}$$

Using Eqs. 3.18 and 3.19 we find that

$$\begin{aligned} (f \otimes \varphi_1)(\Delta(c)(d \otimes 1)) &= \lim \sum f(c'_i d)\varphi_1(c''_i) = \lim \sum f(c'_i d)\omega(b\delta_{c''_i}) \\ &= \omega(\lim \sum (f \otimes \varphi \otimes 1)((c'_i d) \otimes ((1 \otimes b)\Delta(c''_i)))) = \omega((f \otimes \varphi \otimes 1)((1 \otimes 1 \otimes b)(\text{id} \otimes \Delta)(\Delta(c)(d \otimes 1)))) \\ &= \omega((f \otimes \varphi \otimes 1)((\Delta \otimes \text{id})((1 \otimes b)\Delta(c)))(d \otimes 1 \otimes 1)). \end{aligned}$$

Now we use Eq. 3.18 again, and then the left invariance of φ . Thus,

$$\begin{aligned} (f \otimes \varphi_1)(\Delta(c)(d \otimes 1)) &= \lim \sum (f \otimes \varphi)(\Delta(c'_i)(d \otimes 1))\omega(bc''_i) = \lim \sum \varphi(c'_i)f(d)\omega(bc''_i) \\ &= f(d)(\varphi \otimes \omega)((1 \otimes b)\Delta(c)) = f(d)\varphi_1(c), \end{aligned}$$

so that we have proved (3.16).

In order to prove (3.17), we use the following coassociativity-like identity:

$$(1 \otimes 1 \otimes b)(\text{id} \otimes \Delta)((c \otimes 1)\Delta(a)) = (c \otimes 1 \otimes 1)(\Delta \otimes \text{id})((1 \otimes b)\Delta(a)) \tag{3.20}$$

obtained from the main identity $(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta$, where $\Delta : M(A) \rightarrow M(\widehat{A \otimes A})$ while $\Delta \otimes \text{id}$ and $\text{id} \otimes \Delta$ are extended to homomorphisms $M(\widehat{A \otimes A}) \rightarrow M(\widehat{A \otimes A \otimes A})$. Note that $(\Delta \otimes \text{id})(1 \otimes b) = 1 \otimes 1 \otimes b$, $(\text{id} \otimes \Delta)(c \otimes 1) = c \otimes 1 \otimes 1$.

Now, using Eqs. 3.18 and 3.20 we get

$$\begin{aligned}
 (f \otimes \varphi_1)((d \otimes 1)\Delta(c)) &= \lim \sum f(dc'_i)\varphi_1(c''_i) = \lim \sum f(dc'_i)\omega(b\delta_{c''_i}) \\
 &= \omega\left(\lim \sum (f \otimes \varphi \otimes \text{id})((dc'_i) \otimes (1 \otimes b)\Delta(c''_i))\right) = \omega((f \otimes \varphi \otimes \text{id})(1 \otimes 1 \otimes b)(\text{id} \otimes \Delta)((d \otimes 1)\Delta(c))) \\
 &= \omega((f \otimes \varphi \otimes \text{id})((d \otimes 1 \otimes 1)(\Delta \otimes \text{id})((1 \otimes b)\Delta(c)))) = \lim \sum \omega((f \otimes \varphi \otimes \text{id})((d \otimes 1)\Delta(c'_i)) \otimes (bc''_i)) \\
 &= \lim \sum (f \otimes \varphi)((d \otimes 1)\Delta(c'_i))\omega(bc''_i) = \lim \sum \varphi(c'_i)f(d)\omega(bc''_i) = f(d)(\varphi \otimes \omega)((1 \otimes b)\Delta(c)) \\
 &= f(d)\varphi_1(c),
 \end{aligned}$$

and we have proved (3.17).

By Theorem 3.6, $\varphi_1(a) = \lambda\varphi(a)$, where $\lambda \in \mathbb{K}$ does not depend on a . This means that

$$\omega(b\delta_a)\varphi(c) = \omega(b\delta_c)\varphi(a)$$

for any $a, b, c \in A$ and any functional ω . Since A is polar, this implies the identity $\varphi(c)\delta_a = \varphi(a)\delta_c$. Choose such an element c that $\varphi(c) = 1$ and set $\delta = \delta_c$. Then $\delta_a = \varphi(a)\delta$, so that δ is the required multiplier. □

The next proposition contains some identities for the modular element δ .

Proposition 3.9 *The modular element is invertible and satisfies the following equalities:*

$$\Delta(\delta) = \delta \otimes \delta; \quad \varepsilon(\delta) = 1; \quad S(\delta) = \delta^{-1}; \tag{3.21}$$

$$\varphi(S(a)) = \varphi(a\delta) \quad \text{for any } a \in A. \tag{3.22}$$

In these formulas, Δ, ε , and S are extended onto multipliers.

Proof Applying Δ to both sides of the equality $\varphi(a)\delta = (\varphi \otimes \text{id})\Delta(a)$ we come to the identity

$$\varphi(a)\Delta(\delta) = (\varphi \otimes \text{id} \otimes \text{id})((\text{id} \otimes \Delta)\Delta(a)). \tag{3.23}$$

Indeed, $(\varphi \otimes \text{id})\Delta(a)$ is the left multiplier

$$Lb = (\varphi \otimes \text{id})(\Delta(a)(1 \otimes b)), \quad b \in A. \tag{3.24}$$

Let us calculate the value of $\Delta(L)$ on the set of elements $\Delta(b)(v \otimes w)$, $v, w \in A$, whose span is dense in $A \widehat{\otimes} A$. We have

$$\Delta(L)(\Delta(b)(v \otimes w)) = \Delta(Lb)(v \otimes w).$$

Writing $\Delta(a)(1 \otimes b) = \lim \sum a'_i \otimes a''_i$ we get

$$Lb = \lim \sum \varphi(a'_i)a''_i, \quad \Delta(Lb) = \lim \sum \varphi(a'_i)\Delta(a''_i),$$

so that

$$\begin{aligned}
 \Delta(L)(\Delta(b)(v \otimes w)) &= \lim \sum \varphi(a'_i)\Delta(a''_i)(v \otimes w) \\
 &= \lim \sum (\varphi \otimes \text{id} \otimes \text{id})(a'_i \otimes (\Delta(a''_i)(v \otimes w))) = \lim \sum (\varphi \otimes \text{id} \otimes \text{id})(((\text{id} \otimes \Delta)(a'_i \otimes a''_i))(v \otimes w)) \\
 &= (\varphi \otimes \text{id} \otimes \text{id})((\text{id} \otimes \Delta)(\Delta(a)(1 \otimes b)))(v \otimes w) = (\varphi \otimes \text{id} \otimes \text{id})((\text{id} \otimes \Delta)(\Delta(a))(\Delta(b)(v \otimes w))).
 \end{aligned}$$

A similar equality holds for the appropriate right multipliers. Thus, Eq. 3.23 has been proved.

Next, in Eq. 3.23 we can use the coassociativity and obtain the equality

$$\varphi(a)\Delta(\delta) = (\varphi \otimes \text{id} \otimes \text{id})((\Delta \otimes \text{id})\Delta(a)). \tag{3.25}$$

Multiplying both sides from the right by the multiplier $1 \otimes b$ and using again the definition of δ we find that

$$\begin{aligned} & (\varphi \otimes \text{id} \otimes \text{id})((\Delta \otimes \text{id})\Delta(a))(1 \otimes b) = \lim \sum (\varphi \otimes \text{id} \otimes \text{id}) (\Delta(a'_i) \otimes a''_i) \\ & = \lim \sum \varphi(a'_i)\delta \otimes a''_i = \delta \otimes [(\varphi \otimes \text{id})(\Delta(a)(1 \otimes b))] = \delta \otimes [\varphi(a)\delta b] = \varphi(a)(\delta \otimes \delta)(1 \otimes b), \end{aligned}$$

and by Eq. 3.25,

$$(\Delta(\delta) - \delta \otimes \delta)(1 \otimes b) = 0$$

for any $b \in A$. Multiplying from the right by $c \otimes 1$, $c \in A$, we prove that $\Delta(\delta) - \delta \otimes \delta$ vanishes on a dense subset of $A \widehat{\otimes} A$. This proves the first equality from Eq. 3.21.

Similarly, we calculate $\varepsilon(L)$ (for L given in Eq. 3.24) using (3.4):

$$\begin{aligned} \varepsilon(L)\varepsilon(b) &= \varepsilon(Lb) = \lim \sum \varphi(a'_i)\varepsilon(a''_i) = (\varphi \otimes \varepsilon)(\Delta(a)(1 \otimes b)) = \varphi((\text{id} \otimes \varepsilon)(\Delta(a)(1 \otimes b))) = \varphi(ab), \end{aligned}$$

so that

$$\varepsilon(L)\varepsilon(b) = \varphi(ab). \tag{3.26}$$

Let us substitute for b the approximate identity e_λ . We have $\varphi(ae_\lambda) \rightarrow \varphi(a)$, $\varepsilon(ze_\lambda) = \varepsilon(z)\varepsilon(e_\lambda)$ and $\varepsilon(ze_\lambda) \rightarrow \varepsilon(z)$ for each $z \in A$, so that $\varepsilon(e_\lambda) \rightarrow 1$. Passing to the limit in Eq. 3.26 we find that $\varepsilon(L) = \varphi(a)$.

On the other hand, it follows from the definition of δ that $\varepsilon(L) = \varphi(a)\varepsilon(\delta)$. Since a is arbitrary, we get $\varepsilon(\delta) = 1$.

To prove the identity for $S(\delta)$, we take $a, b \in A$ and write the identity

$$\Delta(a\delta)(1 \otimes b) = \Delta(a)(\delta \otimes (\delta b)).$$

Let us apply $S \otimes \text{id}$ to both sides and use the anti-homomorphism property of S . We obtain that

$$(S \otimes \text{id})\Delta(a)(\delta \otimes (\delta b)) = (S(\delta) \otimes 1)(S \otimes \text{id})(\Delta(a)(1 \otimes \delta b)).$$

Taking an arbitrary $c \in A$ and using (3.7) we get the identity

$$c\varepsilon(a\delta)b = S(\delta)c\varepsilon(a)\delta b. \tag{3.27}$$

In Eq. 3.27, we specify $c = e_\lambda$ and pass to the limit using the fact that $\varepsilon(\delta) = 1$. This results in the equality $b = S(\delta)\delta b$, so that $S(\delta)\delta = 1$.

Similarly we can write

$$(c \otimes 1)\Delta(\delta a) = (1 \otimes \delta)(c\delta \otimes 1)\Delta(a)$$

whence

$$(\text{id} \otimes S)((c \otimes 1)\Delta(\delta a)(1 \otimes b)) = (\text{id} \otimes S)((c\delta \otimes 1)\Delta(a)(1 \otimes b))(1 \otimes S(\delta)).$$

Applying the mapping m and using the identity (3.6) we find that $c\varepsilon(\delta a)b = c\delta\varepsilon(a)bS(\delta)$, so that $cb = c\delta bS(\delta)$. Here we set $b = e_\lambda$, pass to the limit and use the arbitrariness of c . As a result, $\delta S(\delta) = 1$. This means that δ is invertible and $S(\delta) = \delta^{-1}$.

The proof of the identity (3.22) is similar to that of Proposition 3.10 in [27]. □

Let \mathfrak{S} be a topology on A generated by the set of seminorms $x \mapsto |\Phi(x)|$ where Φ is a functional from the family (3.12)–(3.13).

Proposition 3.10 *There exists an \mathfrak{S} -continuous automorphism β of the algebra A , such that*

$$\varphi(ab) = \varphi(b\beta(a)) \quad \text{for all } a, b \in A. \tag{3.28}$$

The functional φ is β -invariant.

Proof Note first of all that $\text{span}\{ab : a, b \in A\}$ is dense in A . Indeed, we assume that the mapping $T_1(a \otimes b) = \Delta(a)(1 \otimes b)$ extends to a bijection of $A \widehat{\otimes} A$ onto itself. On the other hand,

$$(\varepsilon \otimes \text{id})(\Delta(a)(1 \otimes b)) = ab.$$

Choose $c \in A$ in such a way that $\varepsilon(c) = 1$. For any $d \in A$, $(\varepsilon \otimes \text{id})(c \otimes d) = d$. Since T_1 is a bijection, we can write $c \otimes d = \lim \sum \Delta(a_i)(1 \otimes b_i)$, $d = \lim \sum a_i b_i$.

Define β by the relation (3.28); that is possible since φ is faithful. For any $a, b, c \in A$,

$$\varphi(abc) = \varphi(c\beta(ab)) = \varphi(bc\beta(a)) = \varphi(c\beta(a)\beta(b)),$$

and the faithfulness of φ implies the equality $\beta(ab) = \beta(a)\beta(b)$. Similarly, the linearity of φ implies the linearity of β .

If $\beta(a_1) = \beta(a_2)$, $a_1, a_2 \in A$, then

$$\varphi((a_1 - a_2)b) = \varphi(b(\beta(a_1) - \beta(a_2))) = 0$$

for any $b \in A$, thus $a_1 = a_2$. The surjectivity of β follows from the above coincidence of the families of functionals (3.12) and (3.13).

Let us prove the \mathfrak{S} -continuity of β . Let $a_\lambda \rightarrow a$ be a \mathfrak{S} -convergent net in A . Then $\varphi(a_\lambda b) \rightarrow \varphi(ab)$ for any $b \in A$, so that

$$\varphi(b[\beta(a_\lambda) - \beta(a)]) = \varphi(a_\lambda b) - \varphi(ab) \rightarrow 0,$$

which means the \mathfrak{S} -continuity of β .

To prove that φ is β -invariant, we write (3.28) with $b = e_\lambda$ and pass to the limit. □

As in [27] (page 340), we get also the relation

$$\varphi(\delta^{-1}a\delta) = \tau\varphi(a) \quad a \in A, \tag{3.29}$$

with some constant $\tau \in \mathbb{K}$.

Let β' be a similar automorphism associated with the right-invariant functional ψ . Then $S\beta' = \beta^{-1}S$. For the proof see [27], Proposition 3.13.

Following [27] (Propositions 3.14 and 3.15) and [24], Section 2.2.4, we prove, for any $a \in A$, the relations

$$\Delta(\beta(a)) = (S^2 \otimes \beta)\Delta(a); \quad \Delta(\beta'(a)) = (\beta' \otimes S^{-2})\Delta(a);$$

$$\beta(\delta) = \beta'(\delta) = \frac{1}{\tau}\delta; \quad \delta\beta(a) = \beta'(a)\delta$$

where $\tau \in \mathbb{K}$ is the constant appearing in Eq. 3.29.

4 The Dual Object

4.1 The Dual Algebra

Denote by \hat{A} the set of linear continuous functionals on A of the form $\varphi(\cdot a)$, $a \in A$. Below we assume that the left-invariant functional φ is such that $\|\varphi\| \leq 1$ and possesses the *norm reproducing property*

$$\|a\| = \sup_{x \neq 0} \frac{|\varphi(xa)|}{\|x\|}, \quad a \in A. \tag{4.1}$$

Equivalently, functionals from \hat{A} can be represented as $\varphi(b\cdot)$, $\psi(\cdot c)$, $\psi(\cdot d)$ where $b, c, d \in A$, ψ is a right-invariant functional.

\hat{A} is obviously a Banach space over \mathbb{K} with respect to the norm $\|\varphi(\cdot a)\| = \|a\|$, well-defined since φ is faithful. By Eq. 4.1, this norm coincides with the standard norm of the functional $\varphi(\cdot a)$ on A . The functional $\varphi(\cdot a)$ can be seen as the Fourier transform of an element a . Then the identity (4.1) can be interpreted as a kind of the Plancherel formula.

The product in \hat{A} is defined as follows. If $\omega_1 = \varphi(\cdot a_1)$, $\omega_2 = \varphi(\cdot a_2)$, $a_1, a_2 \in A$, then, by definition,

$$(\omega_1 \omega_2)(x) = (\omega_1 \otimes \omega_2)\Delta(x) \tag{4.2}$$

where

$$(\omega_1 \otimes \omega_2)\Delta(x) = (\varphi \otimes \varphi)(\Delta(x)(a_1 \otimes a_2)). \tag{4.3}$$

The right-hand side of Eq. 4.3 is well-defined for all $x \in A$; it defines a linear continuous functional.

More specifically, writing $a_1 \otimes a_2 = \lim \sum \Delta(p_i)(q_i \otimes 1)$, $p_i, q_i \in A$, we have

$$(\omega_1 \omega_2)(x) = \lim \sum \varphi(x p_i)\varphi(q_i) = \varphi(xb) \tag{4.4}$$

where $b = \lim \sum p_i \varphi(q_i) \in A$ exists by virtue of Lemma 3.4. Therefore $\omega_1 \omega_2 \in \hat{A}$.

This product is nondegenerate (see [27], page 346) and associative (see the proof in [26] for more general functionals); this associativity is based on the coassociativity of Δ .

Looking at the element b in Eq. 4.4 we find, as in the proof of Lemma 3.4, that

$$\begin{aligned} b &= (\text{id} \otimes \varphi)(\lim \sum p_i \otimes q_i) = (\text{id} \otimes \varphi)(\sigma \circ T_4^{-1}(\lim \sum \Delta(p_i)(q_i \otimes 1))) \\ &= (\text{id} \otimes \varphi)(\sigma \circ T_4^{-1}(a_1 \otimes a_2)). \end{aligned}$$

Since $\|\varphi\| \leq 1$ and T_4 is an isometry, we find that $\|b\| \leq \|a_1 \otimes a_2\| = \|a_1\| \cdot \|a_2\|$, so that $\|\omega_1 \omega_2\| \leq \|\omega_1\| \cdot \|\omega_2\|$, and \hat{A} is indeed a Banach algebra.

4.2 Comultiplication on \hat{A}

In order to define $\hat{\Delta} : \hat{A} \rightarrow M(\hat{A} \hat{\otimes} \hat{A})$, we first define, for any $\omega_1, \omega_2 \in \hat{A}$, the linear continuous functionals $\hat{\Delta}(\omega_1)(1 \otimes \omega_2)$ and $(\omega_1 \otimes 1)\hat{\Delta}(\omega_2)$ on $A \hat{\otimes} A$ as follows:

$$(\hat{\Delta}(\omega_1)(1 \otimes \omega_2))(x \otimes y) = (\omega_1 \otimes \omega_2)((x \otimes 1)\Delta(y)), \tag{4.5}$$

$$((\omega_1 \otimes 1)\hat{\Delta}(\omega_2))(x \otimes y) = (\omega_1 \otimes \omega_2)(\Delta(x)(1 \otimes y)), \tag{4.6}$$

$x, y \in A$.

Lemma 4.1 *If $\omega_1, \omega_2, \omega_3 \in \hat{A}$, then the functionals $\hat{\Delta}(\omega_2)(1 \otimes \omega_3)$ and $(\omega_1 \otimes 1)\hat{\Delta}(\omega_2)$ defined in Eqs. 4.5, 4.6 belong to $\hat{A} \hat{\otimes} \hat{A}$, and*

$$((\omega_1 \otimes 1)\hat{\Delta}(\omega_2))(1 \otimes \omega_3) = (\omega_1 \otimes 1)(\hat{\Delta}(\omega_2)(1 \otimes \omega_3)). \tag{4.7}$$

Proof Let $\omega_j = \varphi(\cdot a_j)$, $j = 2, 3$. Writing $a_2 \otimes a_3 = \lim \sum \Delta(b_i)(c_i \otimes 1)$, $b_i, c_i \in A$, we find that

$$(\hat{\Delta}(\omega_2)(1 \otimes \omega_3))(x \otimes y) = (\varphi \otimes \varphi)((x \otimes 1)\Delta(y)(a_2 \otimes a_3)) = \lim \sum \varphi(x c_i)\varphi(y b_i).$$

Thus,

$$\hat{\Delta}(\omega_2)(1 \otimes \omega_3) = \lim \sum \varphi(\cdot c_i)\varphi(\cdot b_i) \tag{4.8}$$

(the convergence of functionals in the sense of $\hat{A} \hat{\otimes} \hat{A}$ is proved as in Lemma 3.4).

Similarly, using the representation $\omega_j = \psi(a'_j \cdot)$, $j = 1, 2$, we get

$$(\omega_1 \otimes 1)\hat{\Delta}(\omega_2) = \lim \sum \psi(c'_j \cdot)\psi(b'_j \cdot). \tag{4.9}$$

Let us write (4.8) as a convergent limit relation for functionals:

$$\hat{\Delta}(\omega_2)(1 \otimes \omega_3) = \lim \sum \xi'_i \otimes \xi''_i, \quad \xi'_i, \xi''_i \in \hat{A}.$$

We have

$$\begin{aligned} ((\omega_1 \otimes 1)(\hat{\Delta}(\omega_2)(1 \otimes \omega_3)))(x \otimes y) &= \lim \sum ((\omega_1 \xi'_i) \otimes \xi''_i)(x \otimes y) \\ &= \lim \sum ((\omega_1 \otimes \xi'_i)\Delta(x))\xi''_i(y) = \lim \sum ((\omega_1 \otimes \xi'_i \otimes \xi''_i)(\Delta(x) \otimes y)). \end{aligned}$$

Let us approximate $\Delta(x) = \lim \sum \mu'_j \otimes \mu''_j$ in the topology of $M_0(A \hat{\otimes} A)$. The convergence in i is uniform with respect to j because the convergence in $M_0(A \hat{\otimes} A)$ means the convergence of multipliers in strong operator topology, which implies their uniform boundedness ([16], Theorem 2.1.20). Therefore we may change the order of convergence:

$$\begin{aligned} ((\omega_1 \otimes 1)(\hat{\Delta}(\omega_2)(1 \otimes \omega_3)))(x \otimes y) &= \lim \sum \lim_i \sum \omega_1(\mu'_j)(\xi'_i \otimes \xi''_i)(\mu''_j \otimes y) \\ &= \lim \sum \omega_1(\mu'_j)(\hat{\Delta}(\omega_2)(1 \otimes \omega_3))(\mu''_j \otimes y) = \lim \sum \omega_1(\mu'_j)(\omega_2 \otimes \omega_3)((\mu''_j \otimes 1)\Delta(y)). \end{aligned}$$

As above, we approximate $\Delta(y) = \lim \sum v'_k \otimes v''_k$ in $M_0(A \hat{\otimes} A)$ and get

$$((\omega_1 \otimes 1)(\hat{\Delta}(\omega_2)(1 \otimes \omega_3)))(x \otimes y) = \lim \sum \lim \sum \omega_1(\mu'_j)\omega_2(\mu''_j v'_k)\omega_3(v''_k). \tag{4.10}$$

In a similar way, we transform the left-hand side of Eq. 4.7 and obtain for it the expression identical with the one in the right-hand side of Eq. 4.10. This proves the identity (4.7). \square

The identity (4.7) means that the mappings $\omega_1 \otimes 1 \mapsto (\omega_1 \otimes 1)\hat{\Delta}(\omega_2) = \hat{T}_2(\omega_1 \otimes \omega_2)$ and $1 \otimes \omega_3 \mapsto \hat{\Delta}(\omega_2)(1 \otimes \omega_3) = \hat{T}_1(\omega_2 \otimes \omega_3)$ extend to the right multiplier $\hat{\Delta}(\omega_2)_r$ and the left multiplier $\hat{\Delta}(\omega_2)_l$ respectively, and this pair defines a multiplier $\hat{\Delta}(\omega_2) \in M(\hat{A} \hat{\otimes} \hat{A})$.

By Eq. 4.4, $\hat{T}_1(\omega_1 \otimes \omega_2)(x \otimes y) = (\omega_1 \otimes \omega_2)(T_2(x \otimes y))$, for any $x, y \in A$, $\omega_1, \omega_2 \in \hat{A}$. As before, we write $\omega_1 = \varphi(\cdot a_1)$, $\omega_2 = \varphi(\cdot a_2)$, but here it will be convenient to represent

$$\Delta(a_2)(a_1 \otimes 1) = \lim \sum p_i \otimes q_i, \quad p_i, q_i \in A.$$

The limit is in the topology of $A \hat{\otimes} A$. Then, for any $x, y \in A$,

$$\begin{aligned} ((\omega_1 \otimes \omega_2)(x \otimes y) &= \varphi(x a_1)\varphi(y a_2) = (\varphi \otimes \varphi)((x \otimes 1)\Delta(y a_2)(a_1 \otimes 1)) \\ &= \lim \sum (\varphi \otimes \varphi)((x \otimes 1)\Delta(y)(p_i \otimes q_i)). \end{aligned}$$

Writing $\omega'_i = \varphi(\cdot p_i)$, $\omega''_i = \varphi(\cdot q_i)$, we find that

$$(\omega_1 \otimes \omega_2)(x \otimes y) = \lim \sum (\omega'_i \otimes \omega''_i)((x \otimes 1)\Delta(y)) = \lim \sum (\hat{\Delta}(\omega'_i)(1 \otimes \omega''_i))(x \otimes y),$$

where the limit is in the sense of $\hat{A} \hat{\otimes} \hat{A}$. This proves the surjectivity of \hat{T}_1 .

Extending by linearity and continuity we find that

$$\hat{T}_1(\omega_1 \otimes \omega_2)(p) = (\omega_1 \otimes \omega_2)(T_2 p)$$

for all $p \in \hat{A} \hat{\otimes} \hat{A}$. Since T_2 is an isometry, we have

$$\left| \hat{T}_1(\omega_1 \otimes \omega_2)(p) \right| \leq \|\omega_1 \otimes \omega_2\|_{\hat{A} \hat{\otimes} \hat{A}} \cdot \|p\|,$$

so that $\|\hat{T}_1(\omega_1 \otimes \omega_2)\|_{\hat{A} \hat{\otimes} \hat{A}} \leq \|\omega_1 \otimes \omega_2\|_{\hat{A} \hat{\otimes} \hat{A}}$.

Similarly we write $(\omega_1 \otimes \omega_2)(p) = \hat{T}_1(\omega_1 \otimes \omega_2)(T_2^{-1} p)$ and obtain the inverse inequality. Therefore \hat{T}_1 is an isometry. Note the importance of the norm reproducing property (4.1). Here we interpreted the norm on \hat{A} as the standard norm of functionals, while the proof that \hat{A} is a Banach algebra was based on the fact that the norm of the functional $\varphi(\cdot a)$ equals $\|a\|$.

In a similar way we define the mappings \hat{T}_2, \hat{T}_3 , and \hat{T}_4 , and prove their isometry and surjectivity properties. In fact, we followed [23] (pages 90, 91). The difference from the purely algebraic case is the need to check the possibility to change the order of limits, and that is done as in the proof of Lemma 4.1.

Proposition 4.2 *The mapping $\hat{\Delta}$ is a continuous homomorphism $\hat{A} \rightarrow M(\hat{A} \hat{\otimes} \hat{A})$ satisfying the coassociativity condition. The counit $\hat{\varepsilon} : \hat{A} \rightarrow \mathbb{K}$ and the antipode $\hat{S} : \hat{A} \rightarrow \hat{A}$ given by the formulas*

$$\hat{\varepsilon}(\varphi(\cdot a)) = \hat{\varepsilon}(\varphi(a \cdot)) = \varphi(a), \tag{4.11}$$

$$\hat{\varepsilon}(\psi(\cdot a)) = \hat{\varepsilon}(\psi(a \cdot)) = \psi(a), \tag{4.12}$$

$$\hat{S}(\omega)(a) = \omega(S(a)), \quad a \in A, \omega \in \hat{A}, \tag{4.13}$$

are continuous homomorphisms satisfying, for any $\omega_1, \omega_2 \in \hat{A}$, the identities

$$(\hat{\varepsilon} \otimes \text{id})(\hat{\Delta}(\omega_1)(1 \otimes \omega_2)) = \omega_1 \omega_2 \tag{4.14}$$

and three other identities similar to Eq. 3.4,

$$m(\hat{S} \otimes \text{id})(\hat{\Delta}(\omega_1)(1 \otimes \omega_2)) = \hat{\varepsilon}(\omega_1) \omega_2, \tag{4.15}$$

$$m(\text{id} \otimes \hat{S})(\omega_1 \otimes 1) \hat{\Delta}(\omega_2) = \hat{\varepsilon}(\omega_2) \omega_1. \tag{4.16}$$

Proof The proof of the statement about $\hat{\Delta}$ is similar to that of Proposition 4.6 from [27] and Proposition 7.1.7 from [23].

Let $\omega \in \hat{A}$. Then we can write

$$\omega = \varphi(a \cdot) = \varphi(\cdot a') = \psi(b \cdot) = \psi(\cdot b')$$

for some $a, a', b, b' \in A$. In order to justify the definitions (4.11)–(4.12), we have to check that $\varphi(a) = \varphi(a') = \psi(b) = \psi(b')$. Since $\varphi(ax) = \varphi(x\beta(a))$ (see Eq. 3.28) and $\varphi = \varphi \circ \beta$, we find that $a' = \beta(a)$ and $\varphi(a') = \varphi(a)$. In order to compare $\varphi(a)$ and $\psi(b)$, we write $\varphi(ax) = \psi(bx)$, set $x = e_\lambda$, and pass to the limit.

Let us prove (4.14). Writing $\omega_1 = \varphi(\cdot a_1)$, $\omega_2 = \varphi(\cdot a_2)$, we get

$$(\hat{\Delta}(\omega_1)(1 \otimes \omega_2))(x \otimes y) = (\omega_1 \otimes \omega_2)((x \otimes 1)\Delta(y)) = (\varphi \otimes \varphi)((x \otimes 1)\Delta(y)(a_1 \otimes a_2)).$$

Let $a_1 \otimes a_2 = \lim \sum \Delta(p_i)(q_i \otimes 1)$. Then

$$(\hat{\Delta}(\omega_1)(1 \otimes \omega_2))(x \otimes y) = \lim \sum \varphi(xq_i)\varphi(yp_i) \tag{4.17}$$

due to the left invariance of φ .

On the other hand,

$$(\omega_1 \omega_2)(y) = (\varphi \otimes \varphi)(\Delta(y)(a_1 \otimes a_2)) = \lim \sum \varphi(q_i)\varphi(yp_i). \tag{4.18}$$

Applying $\hat{\varepsilon} \otimes \text{id}$ to both sides of Eq. 4.17 and comparing with (4.18), we prove (4.14). The proofs of other identities for $\hat{\varepsilon}$ are similar.

Note that (4.14) implies the homomorphism property of $\hat{\varepsilon}$. Indeed, let us take $\omega_3 \in \hat{A}$ and write (4.14) as

$$(\hat{\varepsilon} \otimes \text{id})(\hat{\Delta}(\omega_1\omega_3)(1 \otimes \omega_2)) = \omega_1\omega_3\omega_2. \tag{4.19}$$

The left-hand side of Eq. 4.19 equals

$$(\hat{\varepsilon} \otimes \text{id})(\hat{\Delta}(\omega_1)(\hat{\Delta}(\omega_3)(1 \otimes \omega_2))),$$

while the right-hand side is equal to

$$\omega_1(\omega_3\omega_2) = \omega_1(\hat{\varepsilon} \otimes \text{id})(\hat{\Delta}(\omega_3)(1 \otimes \omega_2)).$$

Due to the surjectivity of \hat{T}_1 , we may substitute in both sides $\omega_3 \otimes \omega_2$ for $\hat{\Delta}(\omega_3)(1 \otimes \omega_2)$. We obtain that

$$(\hat{\varepsilon} \otimes \text{id})(\hat{\Delta}(\omega_1)(\omega_3 \otimes \omega_2)) = \omega_1(\hat{\varepsilon} \otimes \text{id})(\omega_3 \otimes \omega_2) = \hat{\varepsilon}(\omega_3)\omega_1\omega_2,$$

so that

$$(\hat{\varepsilon} \otimes \text{id})(\hat{\Delta}(\omega_1)(1 \otimes \omega_2)(\omega_3 \otimes 1)) = \hat{\varepsilon}(\omega_3)(\hat{\varepsilon} \otimes \text{id})(\hat{\Delta}(\omega_1)(1 \otimes \omega_2)).$$

Here we substitute $\omega_1 \otimes \omega_2$ in both sides for $\hat{\Delta}(\omega_1)(1 \otimes \omega_2)$. This results in the equality

$$(\hat{\varepsilon} \otimes \text{id})((\omega_1\omega_3) \otimes \omega_2) = \hat{\varepsilon}(\omega_3)(\hat{\varepsilon} \otimes \text{id})(\omega_1 \otimes \omega_2),$$

thus $\hat{\varepsilon}(\omega_1\omega_3)\omega_2 = \hat{\varepsilon}(\omega_1)\hat{\varepsilon}(\omega_3)\omega_2$, and since the algebra \hat{A} is nondegenerate, $\hat{\varepsilon}(\omega_1\omega_3) = \hat{\varepsilon}(\omega_1)\hat{\varepsilon}(\omega_3)$.

Turning to the antipode \hat{S} on \hat{A} defined by Eq. 4.13, we use the standard identities for the antipode S on A extended onto $M(A)$, see [29], Section 2:

$$T_1^{-1}(a \otimes b) = ((\text{id} \otimes S)\Delta(a))(1 \otimes b); \tag{4.20}$$

$$T_2^{-1}(a \otimes b) = (a \otimes 1)((S \otimes \text{id})\Delta(b)), \tag{4.21}$$

$a, b \in A$.

Let $\omega_1, \omega_2 \in \hat{A}$, $\omega_1 = \varphi(a_1 \cdot)$, $a_1 \in A$. Then by Eqs. 4.5 and (4.21), for any $x \in A$,

$$\begin{aligned} (m(\hat{S} \otimes \text{id})(\hat{\Delta}(\omega_1)(1 \otimes \omega_2)))(x) &= (\hat{\Delta}(\omega_1)(1 \otimes \omega_2))((S \otimes \text{id})\Delta(x)) \\ &= (\omega_1 \otimes \omega_2)(T_2((S \otimes \text{id})\Delta(x))) = (\varphi \otimes \omega_2)((a_1 \otimes 1)T_2((S \otimes \text{id})\Delta(x))) \\ &= (\varphi \otimes \omega_2)(T_2((a_1 \otimes 1)(S \otimes \text{id})\Delta(x))) = (\varphi \otimes \omega_2)(a_1 \otimes x) = \varphi(a_1)\omega_2(x) \\ &= \hat{\varepsilon}(\omega_1)\omega_2(x), \end{aligned}$$

so that we have obtained the first defining identity (4.15) for the antipode.

Similarly, using Eqs. 4.6 and 4.20, we prove the second defining identity (4.16). As we mentioned in Theorem 3.2, the identities (4.15) and (4.16) are sufficient to define the antipode.

The continuity properties of the mappings treated in this proposition are obvious. □

4.3 Invariant Functionals

The right-invariant continuous linear functional $\hat{\psi}$ and the left-invariant continuous linear functional $\hat{\varphi}$ on \hat{A} are given by the formulas

$$\hat{\psi}(\omega) = \varepsilon(a) \quad \text{for } \omega = \varphi(\cdot a);$$

$$\hat{\varphi}(\omega) = \varepsilon(a) \quad \text{for } \omega = \psi(a \cdot);$$

The proof is similar to that in [27] (Proposition 4.8).

The following important lemma is proved just like its algebraic counterpart ([27], Lemma 4.11).

Lemma 4.3 *Let $\omega, \omega_1 \in \hat{A}$, $\omega = \varphi(\cdot a)$. Then $\hat{\psi}(\omega_1\omega) = \omega_1(S^{-1}(a))$.*

4.4 Biduality Theorem

Details of the proof of the following biduality property, fundamental for the duality theory, were kindly provided to the author by L. Vainerman.

For any $a \in A$, define a functional $\Gamma(a) : \hat{A} \rightarrow \mathbb{K}$ setting

$$\Gamma(a)(\omega) = \omega(a), \quad \omega \in \hat{A}.$$

By Lemma 4.3, if we write $\omega = \varphi(\cdot S(a))$, then $\Gamma(a) = \hat{\psi}(\cdot\omega)$, so that $\Gamma(a) \in \hat{\hat{A}}$ where $\hat{\hat{A}}$ denotes the dual object to \hat{A} . Obviously, the mapping $a \mapsto \Gamma(a)$ is a continuous isomorphism of Banach spaces.

Below we assume that the Banach algebra \hat{A} possesses a bounded approximate identity, so that the results on extension of homomorphism onto multipliers are applicable. We also assume the norm reproducing property for \hat{A} . It would be interesting to find some sufficient conditions for these properties formulated in terms of A .

Theorem 4.4 *Γ is an isomorphism of multiplier Banach-Hopf algebras, that is*

$$\Gamma(a_1 \cdot a_2) = \Gamma(a_1) \cdot \Gamma(a_2), \tag{4.22}$$

for any $a_1, a_2 \in A$, and

$$\hat{\Delta}(\Gamma(a)) = (\Gamma \otimes \Gamma)(\Delta(a)), \tag{4.23}$$

for any $a \in A$. Here $\hat{\Delta}$ is the coproduct map in $\hat{\hat{A}}$.

Proof First, one can equivalently define Γ as follows:

$$\Gamma(a)(\omega) = \hat{\psi}(\omega \cdot \widehat{S(a)}), \tag{4.24}$$

for all $a \in A, \omega \in \hat{A}$; here $\widehat{S(a)} = \varphi(\cdot S(a))$ is the Fourier transform of $S(a)$. Indeed, the right-hand side of Eq. 4.24 is equal to $\omega(S^{-1}(S(a))) = \omega(a)$ due to Lemma 4.3.

Then we find, using Lemma 4.3 again, that

$$\begin{aligned} (\Gamma(a_1) \cdot \Gamma(a_2))(\omega) &= (\Gamma(a_1) \otimes \Gamma(a_2))(\hat{\Delta}(\omega)) = (\hat{\psi} \otimes \hat{\psi})(\hat{\Delta}(\omega)(\widehat{S(a_1)} \otimes \widehat{S(a_2)})) \\ &= \hat{\psi} \left(\left[(\text{id} \otimes \hat{\psi})(\hat{\Delta}(\omega)(1 \otimes \widehat{S(a_2)})) \right] \cdot \widehat{S(a_1)} \right) = \left[(\text{id} \otimes \hat{\psi})(\hat{\Delta}(\omega)(1 \otimes \widehat{S(a_2)})) \right] (a_1). \end{aligned}$$

Thus, in order to prove the needed equality for Γ , it suffices to show that the functional in the square brackets is equal to $\omega(\cdot a_2)$. Now, we can choose ω in the form of $\hat{b} = \varphi(\cdot b)$. Then everything we need can be formulated as the following equality:

$$(\text{id} \otimes \hat{\psi})(\hat{\Delta}(\hat{b})(1 \otimes \widehat{S(a)})) = \hat{a}b, \tag{4.25}$$

for any $a, b \in A$.

To prove (4.25), we use the reasoning from the proof of Proposition 4.8 in [27] – write $b \otimes S(a) = \lim \sum \Delta(p_i)(q_i \otimes 1)$, then the calculation in the beginning of that proof means that

$$\hat{\Delta}(\hat{b})(1 \otimes \widehat{S(a)}) = \lim \sum (\hat{q}_i \otimes \hat{p}_i),$$

so that

$$(\text{id} \otimes \hat{\psi})(\hat{\Delta}(\hat{b})(1 \otimes \widehat{S(a)})) = \lim \sum \hat{\psi}(\hat{p}_i)\hat{q}_i = \lim \sum \varepsilon(p_i)\hat{q}_i.$$

It remains to show that the last expression equals ab . Using properties of the antipode we can write the above equality for $b \otimes S(a)$ as

$$a \otimes b = \lim \sum (S^{-1} \otimes \text{id})(\sigma \Delta(p_i))(1 \otimes q_i).$$

Looking at this equality as an equality in the opposite multiplier Banach-Hopf algebra whose coproduct is $\sigma \Delta$, the antipode is S^{-1} , the multiplication and the counit being the same, applying to both its sides the multiplication map m and using the identity (3.6), we finish the proof of Eq. 4.22.

The equality (4.23) is an equality in $M(\hat{\hat{A}} \hat{\otimes} \hat{\hat{A}})$, and the map in the right-hand side is understood as follows. We have seen that the map $\Gamma : A \rightarrow \hat{A}$ is multiplicative, so that the map $\Gamma \otimes \Gamma : A \hat{\otimes} A \rightarrow \hat{\hat{A}} \hat{\otimes} \hat{\hat{A}}$ is a well-defined multiplicative map. It can be seen as a multiplicative map from $A \hat{\otimes} A$ to $M(\hat{\hat{A}} \hat{\otimes} \hat{\hat{A}})$, which has a canonical extension to a multiplicative map from $M(A \hat{\otimes} A)$ to $M(\hat{\hat{A}} \hat{\otimes} \hat{\hat{A}})$. This extension is exactly the map in the right-hand side of Eq. 4.23.

As we know, the multiplier $\hat{\Delta}(\Gamma(a))$ is defined by the elements $\hat{\Delta}(\Gamma(a))(1 \otimes \Gamma(b))$ and $(\Gamma(b) \otimes 1)\hat{\Delta}(\Gamma(a))$ of $\hat{\hat{A}} \hat{\otimes} \hat{\hat{A}}$. The equality of multipliers in Eq. 4.23 is equivalent to the system of the following equalities on $\hat{\hat{A}} \hat{\otimes} \hat{\hat{A}}$:

$$\hat{\Delta}(\Gamma(a))(1 \otimes \Gamma(b)) = (\Gamma \otimes \Gamma)(\Delta(a))(1 \otimes \Gamma(b)); \tag{4.26}$$

$$(\Gamma(b) \otimes 1)\hat{\Delta}(\Gamma(a)) = (\Gamma(b) \otimes 1)(\Gamma \otimes \Gamma)(\Delta(a)), \tag{4.27}$$

for any $a, b \in A$. Let us prove (4.26); the proof of Eq. 4.27 is similar.

The right-hand side of Eq. 4.26 evaluated on $\omega_1 \otimes \omega_2 \in \hat{A} \otimes \hat{A}$ gives $(\omega_1 \otimes \omega_2)(\Delta(a)(1 \otimes b))$ by the definition of Γ and by its multiplicativity. The left-hand side of Eq. 4.26 evaluated on $\omega_1 \otimes \omega_2$ can be rewritten, using Eq. 4.5, as $(\Gamma(a) \otimes \Gamma(b))(\omega_1 \otimes 1)\hat{\Delta}(\omega_2)$, which is equal, by the definition of Γ , to $((\omega_1 \otimes 1)\hat{\Delta}(\omega_2))(a \otimes b)$. Finally, the identity (4.6) shows that this expression is equal to $(\omega_1 \otimes \omega_2)(\Delta(a)(1 \otimes b))$. This proves (4.26). \square

4.5 Multiplier Banach-Hopf Algebras of Compact and Discrete Types

A multiplier Banach-Hopf algebra (A, Δ) with invariant functionals is said to be of compact type, if A is a unital algebra, and to be of discrete type, if there is such a nonzero element $h \in A$ that $ah = \varepsilon(a)h$ for all $a \in A$.

Theorem 4.5 *If (A, Δ) is of discrete type, then the dual object $(\hat{A}, \hat{\Delta})$ is of compact type. If (A, Δ) is of compact type, then $(\hat{A}, \hat{\Delta})$ is of discrete type.*

The proof is identical to the one given in [27] (Proposition 5.3) for the purely algebraic case.

5 Examples

In this section, we describe the three examples listed in Introduction. For each case, we give explicit descriptions of the multiplication, comultiplication, counit and antipode, invariant functionals, identify the bounded approximate identities and prove the norm reproducing property for the initial and dual algebras.

5.1 Discrete Groups

Let G_1 be a discrete group. Let $A_1 = c_0(G_1)$ be the commutative Banach algebra of \mathbb{K} -valued functions on G_1 tending to zero by the filter of complements to finite sets, with the pointwise operations and sup-norm.

Lemma 5.1 *The multiplier algebra $M(A_1)$ is isomorphic to the algebra $l^\infty(G_1)$ of all bounded functions on G_1 , with the pointwise operations and sup-norm.*

Proof It is obvious that $l^\infty(G_1) \subseteq M(A_1)$. Conversely, let $\rho_l \in L(A_1)$. Fix $s \in G_1$. There exists such a function $a \in A_1$ that $a(s) = 1$. Denote $f(s) = \rho_l(a)(s)$. This element does not depend on the choice of a – if also $a'(s) = 1$, then

$$f(s) = f(s)a'(s) = \rho_l(a)(s)a'(s) = \rho_l(aa')(s) = \rho_l(a')(s)a(s) = \rho_l(a')(s).$$

Therefore we may consider s as a variable obtaining the function $f_l : G_1 \rightarrow \mathbb{K}$.

For $b \in A_1$,

$$\rho_l(b)(s) = \rho_l(b)(s)a(s) = \rho_l(a)(s)b(s) = f_l(s)b(s),$$

so that ρ_l is the operator of multiplication by f_l .

Note that $f_l \in l^\infty(G_1)$. Otherwise there would exist such a sequence $\{g_n\} \subset G_1$ that $0 \neq |f_l(g_n)| \rightarrow \infty$. Choose a function $b \in A_1$, equal to 0 everywhere outside this sequence and equal to $\frac{1}{f_l(g_n)}$ on it. Then the function $f_l b$ does not belong to $c_0(G_1)$, and we have come to a contradiction.

Similarly, every right multiplier is an operator of multiplication by a function $f_r \in l^\infty(G_1)$. The consistency condition (2.5) means that $f_r a b = f_l a b$ for any $a, b \in A_1$, so that $f_r = f_l$, thus $M(A_1) \subseteq l^\infty(G_1)$. □

In order to define a comultiplication, we need a description of $M(A_1 \widehat{\otimes} A_1)$.

Lemma 5.2 *There are the isomorphisms $A_1 \widehat{\otimes} A_1 = c_0(G_1 \times G_1)$, $M(A_1 \widehat{\otimes} A_1) = l^\infty(G_1 \times G_1)$.*

Proof For any \mathbb{K} -Banach space X , the space $A_1 \widehat{\otimes} X$ is isomorphic to the space of vector-valued sequences $c_0(G_1, X)$ (see [21]). For $X = A_1$, this space consists of functions of two variables $f(s, t)$, $s, t \in G_1$, such that: 1) for any $\epsilon > 0$, there exists a finite set $H_\epsilon \subset G_1$, for which $|f(s, t)| < \epsilon$ for all $s \in G_1, t \in G_1 \setminus H_\epsilon$; 2) for every $t \in G_1$, there exists such a finite set $S_{t,\epsilon} \subset G_1$ that $|f(s, t)| < \epsilon$ for $s \in G_1 \setminus S_{t,\epsilon}, t \in G_1$.

In particular, for $t \in H_\epsilon$, since the set H_ϵ is finite, one can choose $S_{t,\epsilon} = S_\epsilon$ independent of t .

Therefore, for $t \notin H_\epsilon$ we have $|f(s, t)| < \epsilon$ for all $s \in G_1$, while for $t \in H_\epsilon$ we have $|f(s, t)| < \epsilon$ for all $s \notin S_\epsilon$. In other words, $|f(s, t)| < \epsilon$ for $(s, t) \notin S_\epsilon \times H_\epsilon$. This means that $A_1 \widehat{\otimes} A_1 = c_0(G_1 \times G_1)$. By Lemma 5.1, $M(A_1 \widehat{\otimes} A_1) = l^\infty(G_1 \times G_1)$. □

Define a comultiplication $\Delta : c_0(G_1) \rightarrow l^\infty(G_1 \times G_1)$ setting

$$(\Delta f)(s, t) = f(st), \quad f \in c_0(G_1), s, t \in G_1. \tag{5.1}$$

Then, for any $a, b \in A_1, s, t \in G_1$,

$$T_1(a \otimes b)(s, t) = T_3(a \otimes b)(s, t) = a(st)b(t); \tag{5.2}$$

$$T_2(a \otimes b)(s, t) = T_4(a \otimes b)(s, t) = a(s)b(st). \tag{5.3}$$

It is obvious that the right-hand sides in Eqs. 5.2 and 5.3 belong to $c_0(G_1 \times G_1)$.

Moreover, the above mappings are isometric. To prove the isometry property of an operator, it is sufficient to check that the operator transforms an orthonormal basis into an orthonormal basis. Such a basis of $A_1 = c_0(G_1)$ is formed by functions $\delta_\xi(t) = \begin{cases} 1, & \text{if } t = \xi, \\ 0, & \text{if } t \neq \xi. \end{cases}$. An orthonormal basis in $A_1 \widehat{\otimes} A_1 = c_0(G_1 \times G_1)$ is given by the system of functions $(s, t) \mapsto \delta_\xi(s)\delta_\eta(t)$, $(\xi, \eta) \in G_1 \times G_1$, that is the system $\{\delta_\xi \otimes \delta_\eta\}$.

We have

$$T_1(\delta_\xi \otimes \delta_\eta)(s, t) = \delta_\xi(st)\delta_\eta(t) = \delta_{\xi\eta^{-1}}(s)\delta_\eta(t),$$

so that the image under T_1 of the above orthonormal basis is a rearrangement of the latter. Similarly we check the isometry property of T_2 . This reasoning proves also the surjectivity of these mappings.

As in [27], it is easy to calculate that

$$\varepsilon(f) = f(e), \quad (S(f))(t) = f(t^{-1}), \quad f \in A_1, t \in G_1, \tag{5.4}$$

where e is the identity element in G_1 . Note that $a\delta_e = a(e)\delta_e = \varepsilon(a)\delta_e$ for all $a \in A_1$, that is A_1 is a multiplier Banach-Hopf algebra of discrete type, and its dual algebra is unital.

The extension of the above homomorphisms to multipliers is given explicitly by the same formulas written for $f \in l^\infty(G_1)$. However the general results are applicable too. We have only to construct a bounded approximate identity $\{e_\lambda\}_{\lambda \in \Lambda}$.

Let Λ be the set of all finite subsets of G_1 ordered by inclusion. For any $\lambda \in \Lambda$, define a function $e_\lambda \in A_1$ setting

$$e_\lambda(s) = \begin{cases} 1, & \text{if } s \in \lambda, \\ 0, & \text{if } s \notin \lambda. \end{cases}$$

For any $f \in A_1$,

$$(fe_\lambda)(s) = \begin{cases} f(s), & \text{if } s \in \lambda, \\ 0, & \text{if } s \notin \lambda, \end{cases}$$

so that

$$(fe_\lambda)(s) - f(s) = \begin{cases} 0, & \text{if } s \in \lambda, \\ -f(s), & \text{if } s \notin \lambda, \end{cases} \tag{5.5}$$

For every $\epsilon > 0$, there exists such $\lambda_0 \in \Lambda$ that $|f(s)| < \epsilon$ for $s \notin \lambda$, if $\lambda \supseteq \lambda_0$. By Eq. 5.5, $|(fe_\lambda)(s) - f(s)| < \epsilon$ for all $s \in G_1$, if $\lambda \supseteq \lambda_0$, so that $fe_\lambda \rightarrow f$.

Following [27] once more, we find that the left- and simultaneously right-invariant functional on A_1 is given by the equality

$$\varphi(f) = \psi(f) = \sum_{s \in G_1} f(s), \quad f \in A_1. \tag{5.6}$$

Note however that the convergence in Eq. 5.6 is a purely non-Archimedean phenomenon; see Theorem 2.5.1 in [16] regarding the summation of possibly uncountable sequences.

In order to check for this case the equality (4.1), note first that

$$\sup_{x \neq 0} \frac{|\varphi(xa)|}{\|x\|} \leq \|a\|$$

by the ultrametric inequality. On the other hand, $\|a\| = |a(s_0)|$ for some $s_0 \in G_1$. If $x = \delta_{s_0}$, then $\varphi(xa) = a(s_0)$ and

$$\sup_{x \neq 0} \frac{|\varphi(xa)|}{\|x\|} \geq \frac{|\varphi(\delta_{s_0}a)|}{\|\delta_{s_0}\|} = |a(s_0)| = \|a\|,$$

as desired.

Let us consider the dual object \hat{A}_1 . Let $\omega_1 = \varphi(\cdot a_1)$, $\omega_2 = \varphi(\cdot a_2) \in \hat{A}_1$. For any $b \in A_1$,

$$(\omega_1 \omega_2)(b) = (\omega_1 \otimes \omega_2)\Delta(b) = \sum_{s \in G_1} a_1(s) \sum_{t \in G_1} a_2(t)b(st),$$

so that

$$(\omega_1 \omega_2)(b) = \sum_{\tau \in G_1} \left[\sum_{s \in G_1} a_1(s)a_2(s^{-1}\tau) \right] b(\tau) \tag{5.7}$$

and

$$\hat{\varphi}(\omega_1 \omega_2) = \sum_{s \in G_1} a_1(s)a_2(s^{-1}). \tag{5.8}$$

The convolution structure in Eq. 5.7 shows that $\varphi(\cdot \delta_e)$ is the unit in \hat{A}_1 . The equality (5.8) implies the norm reproducing property for \hat{A}_1 . The proof is similar to the above proof for the case of A_1 .

By the definition of comultiplication in the dual object, we find that for every $x, y \in A_1$,

$$(\hat{\Delta}(\omega_1)(1 \otimes \omega_2))(x \otimes y) = \sum_{s,t \in G_1} a_1(s)a_2(s^{-1}t)x(s)y(t), \tag{5.9}$$

so that in the correspondence of $\omega_j \in \hat{A}_1$ with $a_j \in A_1$, $\hat{\Delta}(\omega_1)(1 \otimes \omega_2)$ corresponds to $a_1(s)a_2(s^{-1}t)$. A similar formula can be written for right multipliers. We will return to these formulas in Section 5.3 below.

5.2 Zero-Dimensional Groups

Let G_2 be a zero-dimensional Hausdorff locally compact topological group. We assume that G_2 carries a \mathbb{K} -valued left-invariant measure μ_l . By [15, 17], this happens if either the residue field of \mathbb{K} has characteristic zero, or this characteristic equals $p \neq 0$ and G_2 has a p -free compact open subgroup O , that is no open subgroup in O has an index divisible by p . The well-known example: $\mathbb{K} = \mathbb{Q}_p$, $G_2 = \mathbb{Q}_l$ with $l \neq p$. Under the same conditions, a \mathbb{K} -valued right-invariant measure μ_r exists too.

The commutative Banach algebra $A_2 = C_0(G_2)$, with the sup-norm and pointwise operations, consists of continuous functions $f : G_2 \rightarrow \mathbb{K}$, such that for any $\epsilon > 0$, the set $\{s \in G_2 : |f(s)| \geq \epsilon\}$ is compact.

Lemma 5.3 *The multiplier algebra $M(A_2)$ is isomorphic to the algebra $C_b(G_2)$ of all bounded continuous functions from G_2 to \mathbb{K} with the sup-norm and pointwise operations.*

Proof As in the proof of Lemma 5.1, for each $\rho_l \in L(A_2)$ we find a function $f_l : G_2 \rightarrow \mathbb{K}$, such that $(\rho_l(b))(s) = f_l(s)b(s)$, for all $b \in A_2, s \in G_2$. For a fixed s , $f_l(s) = \rho_l(a)(s)$ where $a \in A_2$ is chosen in such a way that $a(s) = 1$. The existence of such a function $a \in A_2$ is proved in [16], Theorem 2.5.32.

The function f_l is continuous as a ratio of two continuous functions. Let us prove its boundedness (we follow the method from [30]). For any point $s \in G_2$, we set

$$C_s = \sup_{0 \neq \gamma \in A_2} \frac{|\gamma(s)|}{\|\gamma\|} = \sup_{\gamma \in A_2, \|\gamma\|=1} |\gamma(s)|$$

(the above expressions are equal, since in this case the sup-norm takes the same values as the absolute value $|\cdot|$). Then $0 \leq C_s \leq 1$ and $|\gamma(s)| \leq C_s \|\gamma\|$. In particular,

$$|f_l(s)b(s)| \leq C_s \|f_l b\| = C_s \|\rho_l(b)\| \leq C_s \|\rho_l\| \cdot \|b\|$$

for any $b \in A_2$, in particular, for any b with $\|b\| = 1$. Therefore

$$|f_l(s)| \leq \inf_{\|b\|=1} \frac{C_s \|\rho_l\|}{|b(s)|} = \frac{C_s \|\rho_l\|}{\sup_{\|b\|=1} |b(s)|} = \|\rho_l\|,$$

so that f_l is bounded, $\|f_l\| \leq \|\rho_l\|$. □

It is known [9] that $A_2 \widehat{\otimes} A_2 = C_0(G_2 \times G_2)$. Therefore $M(A_2 \widehat{\otimes} A_2) = C_b(G_2 \times G_2)$.

The comultiplication, counit and antipode are defined by the same formulas (5.1) and (5.4) as in the case of discrete groups. The isometry of the mappings (5.2) and (5.3) are proved in this case as follows.

It is known ([16], Theorem 2.5.34) that the Banach space $C_0(G_2)$ has an orthonormal basis $\{\chi_v\}$ consisting of characteristic functions of open compact sets. Then $\{\chi_v(s)\chi_w(t)\}$ is an orthogonal basis in $C_0(G_2 \times G_2)$; see [16], Corollary 10.2.10 and Theorem 10.3.16. It is obvious that the characteristic functions of two sets are orthogonal if and only if neither of the sets contains the other. It is straightforward to check that the mappings $T_1 = T_3$ and $T_2 = T_4$ given by Eqs. 5.2 and 5.3 maintain this property. Thus, they transform the above orthonormal basis into an orthonormal system of functions, which implies the isometry property.

To prove the surjectivity of T_1 (T_2 is considered in a similar way), note that any function of x and y from $C_0(G_2 \times G_2)$ can be approximated uniformly as $\lim \sum a_i(x)b_i(y)$. In particular, we may write

$$u(xy^{-1}, y) = \lim \sum a_i(x)b_i(y)$$

(note that the mapping $(s, t) \mapsto (st, t)$ is continuous, hence this function belongs to $C_0(G_2 \times G_2)$), and then set $x = st, y = t, t \in G_2$, which results in the uniform limit

$$u(s, t) = \lim \sum T_1(a_i \otimes b_i)(s, t).$$

The invariant functionals on A_2 are defined as follows:

$$\varphi(f) = \int_{G_2} f(s)\mu_l(ds); \quad \psi(f) = \int_{G_2} f(s)\mu_r(ds).$$

It is known [15] that the \mathbb{K} -valued Haar measure can be normalized in such a way that $\|\varphi\| \leq 1$ and the identity (4.1) is valid. The proofs of some properties of non-Archimedean integrals in [15] are given for integrands with compact supports, but are easily extended using standard approximation arguments.

In order to use the above general results, we need the existence of bounded approximate identities.

Lemma 5.4 *The Banach algebras A_2 and \hat{A}_2 possess bounded approximate identities.*

Proof Note first of all that G_2 is paracompact (see [7], Theorem 8.13), so that we can use Lemma 28.10 from [6] stating that there exists a family $H_\lambda, \lambda \in \Lambda$ (Λ is a directed set), of

compact open subsets of G_2 , such that $H_{\lambda'} \subset H_{\lambda''}$ for $\lambda' < \lambda''$ and $G_2 = \bigcup_{\lambda \in \Lambda} H_\lambda$. Let e_λ be the \mathbb{K} -valued characteristic function of H_λ . This function belongs to A_2 , and for any $u \in A_2$,

$$(ue_\lambda - u)(x) = \begin{cases} 0, & \text{for } x \in H_\lambda, \\ -u(x), & \text{elsewhere.} \end{cases}$$

For any $\epsilon > 0$, there exists such a compact set F_ϵ that $|u(x)| < \epsilon$, if $x \notin F_\epsilon$. The family H_λ forms an open covering of F_ϵ ; there exists its finite subcovering, and since the family H_λ is increasing, we find such $\lambda_\epsilon \in \Lambda$ that $F_\epsilon \subset H_{\lambda_\epsilon}$ for all $\lambda \geq \lambda_\epsilon$. Therefore $|u(x)| < \epsilon$, if $x \notin H_{\lambda_\epsilon}$, so that $\|ue_{\lambda_\epsilon} - u\| < \epsilon$, if $\lambda \geq \lambda_\epsilon$. Thus, $\{e_\lambda\}$ is a bounded (by one) approximate identity in A_2 .

The dual object \hat{A}_2 consists of functionals

$$\omega_a(x) = \int_{G_2} a(s)x(s)\mu_1(ds), \quad a \in A_2,$$

with the multiplication

$$\begin{aligned} (\omega_{a'}\omega_{a''})(x) &= (\omega_{a'} \otimes \omega_{a''}) \Delta(x) = \int_{G_2} a'(s)\mu_1(ds) \int_{G_2} a''(t)x(st)\mu_1(dt) \\ &= \int_{G_2} x(\theta)(a' \star a'')(\theta)\mu_1(d\theta) \end{aligned}$$

where the convolution $a' \star a''$ has the form

$$(a' \star a'')(\theta) = \int_{G_2} a'(s)a''(s^{-1}\theta)\mu_1(ds)$$

(for an analog of Fubini’s theorem for \mathbb{K} -valued measures see [15]).

With this multiplication, \hat{A}_2 is isomorphic to the group Banach algebra $L(G_2)$ studied in [18] where, in particular, a bounded approximate identity for this algebra is constructed. Note that the right multiplier algebra of $L(G_2)$ is isomorphic to the algebra of improper measures on G_2 ; see Exercise 8.B.v in [17].

As we know, $\hat{\varphi}(\omega_a) = \varepsilon(a) = a(e)$, so that

$$\hat{\varphi}(\omega_{a'}\omega_{a''}) = \int_{G_2} a'(s)a''(s^{-1})\mu_1(ds).$$

The norm reproducing property for \hat{A}_2 follows from the result from [15] already used above. □

5.3 Algebras Generated by Regular Representations

Let G_3 be a discrete group. On the \mathbb{K} -Banach space $c_0(G_3)$, we consider the right regular representation of G_3 : $(R_a f)(s) = f(sa)$, $s, a \in G_3$. Denote by \mathbf{R} the set of all the operators R_a , $a \in G_3$. The closure of the linear span of \mathbf{R} in the strong operator topology coincides with its closure in the uniform operator topology and is isomorphic to the Banach algebra $A_3 = A(G_3)$ equal, as a Banach space, to $c_0(G_3)$ but, in contrast to A_1 , endowed with the product

$$(u * v)(d) = \sum_{l \in G_3} u(l)v(l^{-1}d), \quad d \in G_3; \tag{5.10}$$

for the details see [12].

As we know (Section 5.1), $\{\delta_s\}_{s \in G_3}$ is an orthonormal basis in A_3 . The element δ_e , where e is the unit in G_3 , is the unit in A_3 , so that A_3 is equal to its multiplier algebra.

Define a comultiplication in A_3 setting $\Delta(\delta_s) = \delta_s \otimes \delta_s$ and extending this by linearity and continuity. In other words, writing an arbitrary $u \in A_3$ as

$$\Delta(u) = \sum_{s \in G_3} u(s)\delta_s$$

we have

$$u = \sum_{s \in G_3} u(s)\delta_s \otimes \delta_s.$$

By Lemma 5.2, $A_3 \widehat{\otimes} A_3$ can be identified with $c_0(G_3 \times G_3)$ as a Banach space; this identification extends to the identity of Banach algebras, if we endow $c_0(G_3 \times G_3)$ with the coordinate-wise convolution product. In this setting, for any $u \in A_3$,

$$(\Delta(u))(s, t) = \begin{cases} u(s), & \text{if } s = t; \\ 0, & \text{if } s \neq t, \end{cases} \quad s, t \in G_3,$$

so that

$$T_1(u \otimes v)(s, t) = (\Delta(u)(1 \otimes v))(s, t) = \sum_{l \in G_3} (\Delta(u))(s, l)v(l^{-1}t) = u(s)v(s^{-1}t). \quad (5.11)$$

Obviously, $T_1(u \otimes v)$ belongs to $A_3 \widehat{\otimes} A_3$. Since any function $F(s, st)$, $F \in A_3 \widehat{\otimes} A_3$, can be approximated uniformly by the expressions $\sum c_j u_j(s)v_j(t)$, $c_j \in \mathbb{K}$, $u_j, v_j \in A_3$, for $t = s^{-1}\tau$ this gives the uniform approximation of any function $F(s, \tau)$, thus the surjectivity of T_1 .

More generally, we can write

$$T_1(F)(s, t) = F(s, s^{-1}t)$$

for any $F \in A_3 \widehat{\otimes} A_3$. This implies, in particular, that T_1 is an isometry. Similarly, T_2, T_3 , and T_4 can be considered.

It is easy to find the counit and antipode:

$$\varepsilon(u) = \sum_{s \in G_3} u(s), \quad S(u)(t) = u(t^{-1}),$$

and the invariant functionals:

$$\varphi(u) = \psi(u) = u(e).$$

The equality (4.1) is checked just as its counterpart for the algebra A_1 .

Comparing (5.10) with (5.7), and (5.11) with (5.9), we see that A_3 is isomorphic to \hat{A}_1 .

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