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Cluster Structure on Generalized Weyl Algebras

Ibrahim Saleh[1]

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Abstract We introduce a class of non-commutative algebras that carry non-commutative cluster structure which are generated by identical copies of generalized Weyl algebras. Equivalent conditions for the finiteness of the set of the cluster variables of these cluster structures are provided. Mutations along with some combinatorial data, called *cluster strands,* arising from the cluster structure are used to construct representations of generalized Weyl algebras.

Keywords Non-commutative cluster algebras · Representations of generalized Weyl algebras

Mathematics Subject Classification (2010) Primary 13F60 · Secondary 17B10 · 17B37

1 Introduction

Cluster algebras were introduced by S. Fomin and A. Zelevinsky in [2, 8–10, 16]. A cluster algebra is a commutative algebra with a distinguished set of generators called cluster variables and particular type of relations called mutations. A quantum version was introduced in [3] and [5–7]. The original motivation was to create a combinatorial algebraic framework to study total positivity and dual canonical basis in coordinate rings of certain semisimple algebraic groups.

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✉ Ibrahim Saleh
ibrahim.saleh@uwc.edu

1 University of Wisconsin-Marathon, Wausau, WI, USA

Generalized Weyl algebras were first introduced by V. Bavula in [1] and separately as Hyperbolic algebras by A. Rosenberg in [13]. Their motivation was to find a ring theoretical frame work to study the representation theory of some important "small algebras" such as the first Heisenberg algebra, Weyl algebras, the universal enveloping algebra of the Lie algebra $sl(2)$. A complete list of "small algebras" can be found in [13]. Also, in [13] Rosenberg has obtained the representation theory of all "small algebras" using the Hyperbolic algebra as a frame work.

In this paper, we show that by relaxing the commutativity between cluster variables and some frozen variables (coefficients variables) we can extend the theory of cluster algebras to include some non-commutative algebras which are generated by isomorphic copies of generalized Weyl algebras. To achieve this goal, we introduced particular non-commutative seed-like combinatorial data called *presseds*, each preseed of rank n is defined by iteration from a rank one preseed. Every Fomin-Zelevinsky (coefficient free) rank one seed $(\{x\}, \cdot_x)$ gives rise to a preseed of rank one by attaching a valued star quiver with center at the vertex $\cdot_x$ and assigning a set of frozen variables, one frozen variable at each vertex of the star quiver. Here the frozen variables associated with the exchange variable x do not necessary commute with it. Every preseed of rank n is defined through an increasing set of $n-1$ (nested) preseeds of ranks $1, \ldots, n-1$ respectively, Definition 3.2. A valued star quiver is called *balanced* if the (componentwise) sum of the valuations of the arrows point in toward the center vertex equals the sum of the valuations of the arrows point out equals (a, a), for some non-negative integer a. A preseed is called balanced if each of its star quivers is balanced.

The set of all cluster variables produced from a preseed is not necessarily finite, even if the underlying quiver is of Dynkin type, Examples 3.9 and 3.10. In this paper we provide equivalent conditions on a preseed for its set of cluster variables to be finite, such as in Theorems 3.15 and Corollary 4.9 which are rephrased respectively as follows

Theorem 1.1 *Let p be a balanced preseed in the ambient division ring $\mathcal{D}$. If ϕ is a $\mathcal{D}$-automorphism that fixes the frozen variables such that for every frozen variable f associated to the initial cluster variable x we have*

$$fx = \phi(x)f.$$

Then the set of all cluster variables of p is finite if and only if ϕ is of finite order.

Corollary 1.2 *Let p be a balanced preseed with a non zero element q in the field K such that for every initial cluster variable x, we have*

$$fx = qxf, \textit{for each frozen variable } f \textit{ associated to } x.$$

Then the set of all cluster variables of p is finite if and only if q is an m^{th}-root of unity, for some natural number m.

Although Corollary 1.2 can be seen as a consequence of Theorem 1.1, in this paper we provide independent proofs for both of them.

Every generalized Weyl algebra of rank n gives rise to a preseed p of rank n endowed with an automorphism θ over the coefficients ring (the group ring of the group generated by the frozen variables), Example 4.5. In Theorem 4.12 we show that the associated cluster algebra $\mathcal{H}(p)$ satisfies the following

(1) The algebra $\mathcal{H}(p)$ is generated by (possibly) infinite isomorphic copies of the associated generalized Weyl algebra, each vertex in the exchange graph of p_n gives rise to two copies of them;
(2) There are n rank one preseeds $p_1(x_1), \ldots, p_1(x_n)$ such that

$$\mathcal{H}(p) = \mathcal{H}(p_1(x_1)) \otimes \cdots \otimes \mathcal{H}(p_1(x_n)). \tag{1.1}$$

Let V_n be the $K(f_1, \ldots, f_n)$-span of the cluster monomials of $\mathcal{H}(p)$, where $f_1, \ldots, f_n$ are the frozen variables of p. In Definition 5.5 we use right and left mutations, given in Definition 3.3, to introduce an action of generalized Weyl algebra in V_n. The combinatorial structure of the cluster monomials gives rise to combinatorial datum called *cluster strands*, which are particular elements of V_n, Definition 5.8. Some properties of the cluster strands are provided in Lemma 5.12. The submodules generated by cluster strands are called *strand submodules*. The properties of the strand submodules are studied in Proposition 5.15 and 5.16 and Corollary 5.17.

Conjecture 1.3 *Strand submodules are indecomposable.*

The paper is organized as follows. Section 2 is devoted to basic definitions of cluster algebras associated with valued quivers. In Section 3, we introduce the notion of preseeds and their mutations. Examples and properties of preseeds are also given. In the same section we provide equivalent conditions for a preseed to produce a finite set of cluster variables, Theorems 3.15. In Theorem 3.17, we introduce a class of $\mathcal{D}$-automorphisms that preserve the set of cluster variables. Weyl cluster algebras are defined in Section 4. The main results of Section 4 are Corollary 4.9 and Theorem 4.12 which give some basic properties of Weyl cluster algebras. Section 5 is where we introduce an action of generalized Weyl algebras on the space of cluster monomials. In the same section we introduce the cluster strands. Some of their basic properties are in Lemma 5.12. Some Properties of strand submodules are given in Proposition 5.15 and 5.16.

Through out the paper, K is a field of zero characteristic and the notation $[1, k]$ stands for the set $\{1, \ldots, k\}$.

2 Cluster Algebras Associated with Valued Quivers

For more details about the material of this section refer to [2, 8, 11, 15, 16].

2.1 Valued Quivers

- *A valued quiver* of rank n is a quadruple $Q = (Q_0, Q_1, V, d)$, where
  - Q_0 is a set of n vertices labeled by numbers from the set $[1, n]$;
  - Q_1 is called the *set of arrows* of Q and consists of ordered pairs of vertices, that is $Q_1 \subset Q_0 \times Q_0$;
  - V is a function $V : Q_1 \to \mathbb{N}\times\mathbb{N}$, $(i, j) \mapsto (v_{ij}, v_{ji})$, V is called the *valuation* of Q;
  - $d = (d_1, \cdots, d_n)$, where d_i is a positive integer for each i, such that $d_i v_{ij} = v_{ji} d_j$, for every $i, j \in Q_0$.

  In the case of $(i, j) \in Q_1$, then there is an arrow oriented from i to j and in notation we shall use the symbol $\cdot_i \xrightarrow{(v_{ij}, v_{ji})} \cdot_j$. If $v_{ij} = v_{ji} = 1$ we simply write $\cdot_i \longrightarrow \cdot_j$.

In this paper, we moreover assume that $(i,i) \notin Q_1$ for every $i \in Q_0$, and if $(i,j) \in Q_1$ then $(j,i) \notin Q_1$. The vector of positive integers $d = (d_1, \cdots, d_n)$ does not play any role in the context of this paper, so it will be ignored from now on.

- If $v_{ij} = v_{ji}$ for every $(v_{ij}, v_{ji}) \in V$ then Γ is called *equally valued quiver*.
- We say that the valued quiver $\Gamma = (Q_0, Q_1, V)$ is *connected*, if for every $v, v' \in Q_0$, there is a sequence of vertices $v = v_1, \cdots, v_l = v'$ such that for $t = 1, \cdots, l-1$, either (v_t, v_{t+1}) or (v_{t+1}, v_t) is in Q_1, in other words, any pair of subsequent vertices v_t and v_{t+1} are connected by an arrow.

Remarks 2.1 (1) Every (non valued) quiver Q without loops nor 2-cycles corresponds to an equally valued quiver which has an arrow (i,j) if there is at least one arrow directed from i to j in Q and with the valuation $(v_{ij}, v_{ji}) = (m,m)$, where m is the number of arrows from i to j.

(2) Every valued quiver of rank n corresponds to a skew symmetrizable integer matrix $B(Q) = (b_{ij})_{i,j\in[1,n]}$ given by

$$b_{ij} = \begin{cases} v_{ij}, & \text{if } (i,j) \in Q_1, \\ 0, & \text{if neither } (i,j) \text{ nor } (j,i) \text{ is in } Q_1, \\ -v_{ij}, & \text{if } (j,i) \in Q_1. \end{cases} \tag{2.1}$$

Conversely, given a skew symmetrizable $n \times n$ matrix B, a valued quiver Q_B can be easily defined such that $B(Q_B) = B$. This gives rise to a bijection between the skew-symmetrizabke $n \times n$ integral matrices B and the valued quivers with set of vertices $[1,n]$, up to isomorphism fixing the vertices.

Definition 2.2 (*Valued quivers mutations*) Let Q be a valued quiver. The mutation $\mu_k(Q)$ at a vertex k is defined through Fomin-Zelevinsky's mutation of the associated skew-symmetrizable matrix. The mutation of a skew symmetrizable matrix $B = (b_{ij})$ on the direction $k \in [1,n]$ is given by $\mu_k(B) = (b'_{ij})$, where

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } k \in \{i,j\}, \\ b_{ij} + \text{sign}(b_{ik}) \max(0, b_{ik}b_{kj}), & \text{otherwise.} \end{cases} \tag{2.2}$$

Remarks 2.3 (1) Let $Q = (Q_0, Q_1, V)$ be a valued quiver. The new valued quiver $\mu_k(Q) = (Q_0, Q'_1, V')$, obtained from Q by applying mutation at the vertex k, can be described using the mutation of $B(Q)$ as follows: We obtain Q'_1 and V' by altering Q_1 and V, based on the following rules

(a) replace the pairs (i,k) and (k,j) with (k,i) and (j,k) respectively and switch the components of the ordered pairs of their valuations;

(b) if $(i,k), (k,j) \in Q_1$, such that at least i or j is in Q_0 but $(j,i) \notin Q_1$ and $(i,j) \notin Q_1$ (respectively $(i,j) \in Q_1$) add the pair (i,j) to Q'_1, and give it the valuation $(v_{ik}v_{kj}, v_{ki}v_{jk})$ (respectively change its valuation to $(v_{ij} + v_{ik}v_{kj}, v_{ji} + v_{ki}v_{jk})$);

(c) if (i,k), (k,j) and (j,i) in Q_1, then we have three cases

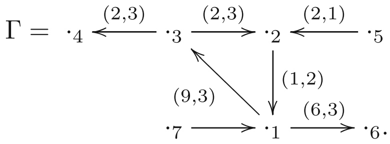
  (i) if $v_{ik}v_{kj} < v_{ij}$, then keep (j,i) and change its valuation to $(v_{ji} - v_{jk}v_{ki}, |-v_{ij} + v_{ik}v_{kj}|)$;

  (ii) if $v_{ik}v_{kj} > v_{ij}$, then replace (j,i) with (i,j) and change its valuation to $(-v_{ij} + v_{ik}v_{kj}, |v_{ji} - v_{jk}v_{ki}|)$;

  (iii) if $v_{ik}v_{kj} = v_{ij}$, then remove (j,i) and its valuation.

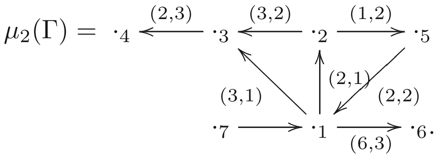
(2) One can see that; $\mu_k^2(Q) = Q$ and $\mu_k(B(Q)) = B(\mu_k(Q))$ at each vertex k.

Example 2.4 Let



(2.3)

One can see that Γ is a valued quiver with $d = (1, 2, 3, 2, 1, 2, 1)$. Applying mutation at the vertex 2, produces the following valued quiver



2.2 Cluster Algebras

Zelevinsky [16] Let $\mathcal{F}$ be an ambient field of rational functions in n independent variables over $\mathbb{Q}(t_1, \ldots, t_m)$. A *seed* in $\mathcal{F}$ is a pair (X, Q), where

- $X = \{x_1, \ldots, x_n\}$ forms a free generating set of $\mathcal{F}$, and
- $Q = (Q_0, Q_1, V)$ is a valued quiver with $Q_0 = \{1, \ldots, n, n+1, \ldots, n+m\}$, where vertices $1, \ldots, n$ are called *exchange vertices* and $n+1, \ldots, n+m$ are the called *frozen vertices*.

The variables $x_1, \ldots, x_n$ are associated with the exchange vertices and they are called *exchange cluster variables* and the variables $t_1, \ldots, t_m$ are associated with the frozen vertices and they are called *frozen variables*.

Definition 2.5 (Seed mutations) Let $p = (X, Q)$ be a seed in $\mathcal{F}$ and let $k \in [1, n]$. Applying the *seed mutation* μ_k on (X, Q) produces a new seed $\mu_k(X, Q) = (\mu_k(X), \mu_k(Q))$ by setting $\mu_k(X) = \{x_1, \ldots, x'_k, \ldots, x_n, t_{n+1}, \ldots, t_{n+m}\}$ where x'_k is defined by the so-called *exchange relations*:

$$x'_k x_k = \prod_{j,\ \cdot_{n+j} \longrightarrow \cdot_k} t_{n+j}^{v_{n+j,k}} \prod_{i,\ \cdot_i \longrightarrow \cdot_k} x_i^{v_{ik}} + \prod_{j,\ \cdot_k \longrightarrow \cdot_{n+j}} t_{n+j}^{v_{k,n+j}} \prod_{i,\ \cdot_k \longrightarrow \cdot_i} x_i^{v_{ki}}. \tag{2.4}$$

And $\mu_k(Q)$ is the mutation of Q at the vertex k, given in Definition 2.2 and Remarks 2.3. The elements of $\mathcal{F}$ obtained by applying iterated mutations on the elements $\{x_1, \ldots, x_n\}$ are called *cluster variables*.

Definitions 2.6 (Cluster algebra and exchange graph) (1) Let $\mathcal{X}$ be the set of all cluster variables of $\mathcal{F}$ produced from a seed (X, Q). The *cluster algebra* $\mathcal{A} = \mathcal{A}(X, Q)$ is the $\mathbb{Z}[\mathbb{P}]$-subalgebra of $\mathcal{F}$ generated by $\mathcal{X}$, where $\mathbb{P}$ is the (free) abelian group generated by the frozen variables written multiplicatively.

(2) The *exchange graph* of $\mathcal{A}(X, Q)$, denoted by $\mathbb{G}(X, Q)$, is the n-regular graph whose vertices are labeled by the seeds that can be obtained from (X, Q) by applying some sequence of mutations, and whose edges correspond to mutations. Two adjacent seeds in $\mathbb{G}$ can be obtained from each other by applying a mutation μ_k for some $k \in [1, n]$.

Theorem 2.7 *([8, Theorem 3.1, Laurent Phenomenon]) The cluster algebra $\mathcal{A}(X, Q)$ is contained in the integral ring of Laurent polynomials $\mathbb{Z}[\mathbb{P}][x_1^{\pm}, \ldots, x_n^{\pm}]$.*

3 Preseeds

Before introducing *preseeds*, we will introduce an increasing filtration of division rings of fractions by iteration and a particular type of quivers known as *star quivers*.

For each m in $[1, n]$, let $\mathbb{P}_m$ be a finitely generated free abelian group, written multiplicatively, with set of generators

$$F^m = \bigcup_{i=1}^{m} F_i \quad \text{where} \quad F_i = \{f_{i1}, \ldots, f_{im_i}\}. \tag{3.1}$$

Let $R_1 = K[\mathbb{P}_1]$ be the group ring of $\mathbb{P}_1$ over K. Let D_1 be an Ore domain containing R_1 such that there is $t_1 \in D_1$ so that $\{t_1^{\alpha_1}; \alpha_1 \in \mathbb{Z}\}$ form a basis for D_1 as a left R_1-module. Let $\mathcal{D}_1$ denote the set of right fractions ab^{-1} with $a, b \in D_1$, and $b \neq 0$; two such fractions ab^{-1} and cd^{-1} are identified if $af = cg$ and $bf = dg$ for some non-zero $f, g \in D_1$. The ring D_1 is embedded into $\mathcal{D}_1$ via $d \mapsto d \cdot 1^{-1}$. The addition and multiplication in D_1 extend to $\mathcal{D}_1$ so that $\mathcal{D}_1$ becomes a division ring. (Indeed, we can define $ab^{-1}+cd^{-1} = (ae+cf)g^{-1}$ where non-zero elements e, f , and g of D_1 are chosen so that $be = df = g$; similarly, $ab^{-1} \cdot cd^{-1} = ae \cdot (df)^{-1}$, where non-zero $e, f \in D_1$ are chosen so that $cf = be$). In such case we say $\mathcal{D}_1$ is the division ring of fractions in t_1 of D_1 over R_1. Now, for $i \in [2, n]$, let $R_i = K[\mathbb{P}_i]$ and D_i be an Ore domain containing (as sub rings) R_i and D_{i-1} such that there is $t_i \in D_i$ so that

$$t_i t_j = t_j t_i \ \text{ and } \ t_i f_{jr} = f_{jr} t_i, \ \text{ for every } i, j \in [1, n], \ j < i, \ \text{ for all } \ r \in [1, m_j]; \tag{3.2}$$

and $\{t_i^{\alpha_i}; \alpha_i \in \mathbb{Z}\}$ form a basis for D_i as a left R_i-module. Let $\mathcal{D}_i$ be the division ring of fractions in t_i of D_i over R_i. For each $i \in [1, n]$, the elements of the set F_i are called *frozen variables*. More details about Ore domains can be found in [12] and [2]. The following diagram is meant to help readers understand the relations between the rings R_i, D_i and $\mathcal{D}_i$, $i = 1, \cdots, n$.

$$\begin{array}{ccccccc} R_1 & \subset & R_2 & \subset & \cdots & \subset & R_n \\ \cap & & \cap & & \cdots & & \cap \\ D_1 & \subset & D_2 & \subset & \cdots & \subset & D_n \\ \cap & & \cap & & \cdots & & \cap \\ \mathcal{D}_1 & \subset & \mathcal{D}_2 & \subset & \cdots & \subset & \mathcal{D}_n. \end{array}$$

Definitions 3.1 (Valued star quivers) • A valued quiver $\Gamma = (Q_0, Q_1, V)$ is called a *valued star quiver* with center at $k \in Q_0$ if we have

$$Q_1 \subset (\{k\} \times Q_0) \cup (Q_0 \times \{k\}).$$

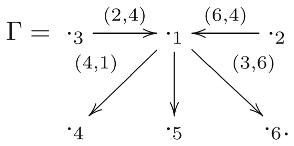
Furthermore, Γ is called a balanced star quiver if

$$\Big(\sum_{j,\ \cdot_k \longrightarrow \cdot_j} v_{kj}, \sum_{j,\ \cdot_k \longrightarrow \cdot_j} v_{jk}\Big) = \Big(\sum_{i,\ \cdot_i \longrightarrow \cdot_k} v_{ik}, \sum_{i,\ \cdot_i \longrightarrow \cdot_k} v_{ki}\Big) = (a_k, a_k), \tag{3.3}$$

and in this case the non-negative integer a_k is called the *frozen component* of Γ.

- A set of n star quivers $\Gamma = \{\Gamma_1, \dots, \Gamma_n\}$ is said be *balanced* with *frozen rank* $(a_1, \dots, a_n)$ if each Γ_k is balanced with frozen component a_k, $k = 1, \dots, n$.

The following valued quiver is an example of a balanced valued star quiver of frozen rank 8 and $d = (6, 4, 12, 24, 6, 3)$



Although, in general, presseds can be defined using any valued quiver, here they are defined using valued star quivers which serves best the purpose of the paper. From now on we will omit the word valued from the term valued star quiver.

Definition 3.2 (Preseeds) • A *preseed* p_1 of rank 1 in $\mathcal{D}_1$ is the triple $(\{F_1\}, \{x_1\}, \{\Gamma_1\})$, where

(1) F_1 is as described in Eq. 3.1;
(2) x_1 is an element of $\mathcal{D}_1$ such that there is an R_1- linear automorphism on $\mathcal{D}_1$ that fixes the frozen variables and sends t_1 to x_1. The element x_1 is called an *exchange cluster variable* and the set

$$\widetilde{X} := \{f_{11}, \dots, f_{1m_1}, x_1\}$$

is called the *extended cluster* of p_1;
(3) Γ_1 is a star quiver of rank $m_1 + 1$. The center vertex $\cdot_1$ of Γ_1 is called *exchange vertex* and all other vertices are called *frozen vertices*.

- A *preseed* p_n of rank n in $\mathcal{D}_n$ is the triple (F, X, Γ), where $F = \{F_1, \dots, F_n\}$ (as given in Eq. 3.1), $X = \{x_1, \dots, x_n\}$ and $\Gamma = \{\Gamma_1, \dots, \Gamma_n\}$ such that $p_k = (\{F_k\}, \{x_k\}, \{\Gamma_k\}\}$ is a preseed of rank 1 in $\mathcal{D}_k$, for every $k \in [1, n]$. The following set

$$\widetilde{X} = \{f_{11}, \dots, f_{1m_1}, \dots, f_{n1}, \dots, f_{nm_n}, x_1, \dots, x_n\}$$

is called the *extended cluster* of p_n. Furthermore, p_n is called *balanced preseed* if Γ is a balanced set of star quivers and the *frozen rank* of p_n is the same as the frozen rank of Γ.

Definition 3.3 (Preseeds mutations) Let $p_n = (F, X, \Gamma)$ be a preseed in $\mathcal{D}_n$. For each $k \in [1, n]$, two new triples $\mu_k^R(p_n) = (F, \mu_k^R(X), \mu_k(\Gamma))$ and $\mu_k^L(p_n) = (F, \mu_k^L(X), \mu_k\Gamma))$ can be obtained from p_n as follows

- (Right mutation)

$$\mu_k^R(x_i) = \begin{cases} (\prod_{j,i_j \to i} f_{ij}^{v_{i_j i}} \prod_{j \to i} x_j^{v_{ji}} + \prod_{j,i \to i_j} f_{ij}^{v_{i i_j}} \prod_{j,i \to j} x_j^{v_{ij}}) x_i^{-1}, & i = k; \\ x_i, & i \neq k. \end{cases} \tag{3.4}$$

- (Left mutation)

$$\mu_k^L(x_i) = \begin{cases} x_i^{-1}(\prod_{j,j \to i} x_j^{v_{ji}} \prod_{j,i_j \to i} f_{ij}^{v_{i_j i}} + \prod_{j,i \to j} x_j^{v_{ij}} \prod_{j,i \to i_j} f_{ij}^{v_{i i_j}}), & i = k; \\ x_i, & i \neq k. \end{cases} \tag{3.5}$$

- The mutation $\mu_k(\Gamma)$ is as defined in Definition 2.2 and Remarks 2.3.

Proposition 3.4 *Let $p_n = (F, X, \Gamma)$ be a preseed in $\mathcal{D}_n$. Then the following are true*

(1) *For any sequence of right mutations (respectively left) $\mu^R_{i_1}\mu^R_{i_2}\ldots\mu^R_{i_q}$, we have $\mu^R_{i_1}\mu^R_{i_2}\ldots\mu^R_{i_q}(p_n)$ (respectively $\mu^L_{i_1}\mu^L_{i_2}\ldots\mu^L_{i_q}(p_n)$) is again a preseed .*

(2) *For every $k \in [1, n]$,*

$$\mu^R_k\mu^L_k(p_n) = \mu^L_k\mu^R_k(p_n) = p_n. \tag{3.6}$$

Proof We prove part (1) for $\mu^R_k(p_n)$, (respectively $\mu^L_k(p_n)$) and the proof for an arbitrary sequence of right (respectively left) mutations is by induction on the length of the sequence. From Eq. 3.4 (respectively (3.5) one has $\mu^R_k(x_k)$ (respectively $\mu^L_k(x_k)$) is an expression in the elements of the set $\{x_k^{-1}\} \cup F_k$. Then Eq. 3.2 guarantees that $\{x_1, \ldots, x_{k-1}, \mu^R_k(x_k), x_{k+1}, \ldots, x_n\}$ (respectively $\{x_1, \ldots, x_{k-1}, \mu^L_k(x_k), x_{k+1}, \ldots, x_n\}$) is a commutative set. The commutativity of the elements of the set $\{\mu^R_k(x_k)\} \cup F_j$ (respectively the elements of the set $\mu^L_k(x_k) \cup F_j$) for $j \neq k$ is again due to that the expression of $\mu^R_k(x_k)$ (respectively $\mu^L_k(x_k)$) contains only elements of $\{x_k^{-1}\} \cup F_k$ which is by Eq. 3.2 commute with elements of F_j. Part (2) is immediate using the Eqs. (3.2), (3.4) and (3.5) and the fact that mutation is involutive on valued quivers. □

Definition 3.5 (Cluster sets) Let p_n be a preseed in $\mathcal{D}_n$. An element $y \in \mathcal{D}_n$ is said to be a *cluster variable* if y is a cluster variable in some seed q_n, where q_n is obtained from p_n by applying some sequence of (right or left) mutations. The set of all cluster variables of p_n is called *the cluster set* of p_n and is denoted by $\mathcal{X}(p_n)$.

Remark 3.6 (1) From the definition of preseeds, each exchange vertex is connected only to its associated frozen vertices. Then from the proof of Part 1 of Proposition 3.4, one can see that every cluster variable in $\mathcal{D}_n$, can be written as a Laurent expression in exactly one cluster variable and the frozen variables associated to it in some pressed. Which is a major difference between cluster variables produced from preseeds and cluster variables produced from other non-commutative seeds such as quantum seeds introduced in [3].

(2) Mutations of preseeds are not involutive but they are invertible, in the sense of Part 2 Proposition 3.4, however mutations of classical or quantum seeds are involutive.

Definition 3.7 A quadruple (F, X, Γ, φ) is said to be φ-commutative preseed in $\mathcal{D}_n$ if (F, X, Γ) is a preiseed and φ is an R_n-linear automorphism of $\mathcal{D}_n$, such that the following equations are satisfied

$$f x_i = \varphi(x_i) f, \quad \forall f \in F_i, \quad \forall i \in [1, n]. \tag{3.7}$$

One can see Eq. 3.7 induces the equations

$$f^a x_i = \varphi^a(x_i) f^a, \forall f \in F_i, \forall i \in [1, n], a \in \mathbb{Z}_{\geq 0}. \tag{3.8}$$

And

$$x_i f^a = f^a \varphi^{-a}(x_i), \forall f \in F_i, \forall i \in [1,n], a \in \mathbb{Z}_{\geq 0}. \tag{3.9}$$

An example of a φ-commutative preseed is given in next section, Example 4.5.

Proposition 3.8 *The properties of balanced and φ-commutative of preseeds are invariant under preseeds mutations.*

Proof One can see that the mutation of balanced preseed is again a balanced preseed with the same frozen rank.

Now we show that φ-commutativity of preseeds is invariant under right mutations and for left mutation is quite similar. For every $k \in [1,n]$, the right mutation μ_k^R of p_n gives rise to an R_n-automorphism $\psi : \mathcal{D}_n \to \mathcal{D}_n$ induced by

$$\psi(x_j) := \mu_k^R(x_j), \ \forall j \in [1,n]. \tag{3.10}$$

We will show that $\mu_k^R(p_n)$ is $\psi\varphi\psi^{-1}$-commutative. Let $i \in [1,n]$, $f \in F_i$. We have

$$\begin{aligned} f\mu_k(x_i) &= \psi(f x_i) \\ &= \psi(\varphi(x_i f)) \\ &= \psi\varphi\psi^{-1}(\mu_k(x_i))f. \end{aligned}$$

□

Let $p = (\{x_1\}, \cdot_{x_1})$ be a coefficient free seed of rank 1 in the field of fractions $K(t)$. In this case there is only one more seed $(\{\frac{2}{x_1}\}, \cdot_{\frac{2}{x_1}})$ which is mutation equivalent to p. The Fomin-Zelevinsky (commutative) cluster algebra of p is the algebra of polynomials with integral coefficients $\mathcal{A} = \mathbb{Z}[x_1, \frac{2}{x_1}]$. In the following we will see two examples of attaching star quivers at vertex $\cdot_{x_1}$ to produce preseeds.

Example 3.9 **The simplest non balanced preseed** . Let p_1 be the seed $(\{F_1\}, \{x_1\}, \{\Gamma_1\})$ where $F_1 = \{f_{11}\}$ and Γ_1 is the following star quiver

$$\cdot_{f_{11}} \longleftarrow \cdot_{x_1} \ .$$

Applying mutation at the vertex $\cdot_{x_1}$, we obtain the following cluster variables

$$\begin{aligned} x_1 &\overset{\mu_1^L}{\Rightarrow} x_1^{-1}(f_{11}+1) \\ &\overset{\mu_1^L}{\Rightarrow} (f_{11}+1)^{-1}x_1(f_{11}+1) \\ &\overset{\mu_1^L}{\Rightarrow} (f_{11}+1)^{-1}x_1^{-1}(f_{11}+1)^2 \\ &\overset{\mu_1^L}{\Rightarrow} (f_{11}+1)^{-2}x_1(f_{11}+1)^{+2} \\ &\dots \\ &\overset{\mu_1^L}{\Rightarrow} (f_{11}+1)^{-k}x_1^{-1}(f_{11}+1)^{k+1} \\ &\overset{\mu_1^L}{\Rightarrow} (f_{11}+1)^{-(k+1)}x_1(f_{11}+1)^{k+1} \\ &\dots, \end{aligned}$$

and

$$\begin{aligned} x_1 &\overset{\mu_1^R}{\Rightarrow} (f_{11}+1)x_1^{-1} \\ &\overset{\mu_1^R}{\Rightarrow} (f_{11}+1)x_1(f_{11}+1)^{-1} \\ &\overset{\mu_1^R}{\Rightarrow} (f_{11}+1)^2x_1^{-1}(f_{11}+1)^{-1} \\ &\overset{\mu_1^R}{\Rightarrow} (f_{11}+1)^2x_1(f_{11}+1)^{-2} \\ &\dots \\ &\overset{\mu_1^R}{\Rightarrow} (f_{11}+1)^{k+1}x_1^{-1}(f_{11}+1)^{-k} \\ &\overset{\mu_1^R}{\Rightarrow} (f_{11}+1)^{k+1}x_1(f_{11}+1)^{-(k+1)} \\ &\dots . \end{aligned}$$

Then, we have the infinite cluster set $\mathcal{X}(p_1) = \{x_1, (1+f_{11})^{k+1}x_1^{-1}(1+f_{11})^{-k}, (1+f_{11})^k x_1(1+f_{11})^{-k}, (1+f_{11})^{-k}x_1^{-1}(1+f_{11})^{k+1}, (1+f_{11})^{-k}x_1^{-1}(1+f_{11})^k, k \in \mathbb{Z}\}$. Later in this article, we will see that this seed is related to first Weyl algebra.

Example 3.10 **The cluster set of the simplest (nontrivial) balanced φ-commutative preseed**. Consider the seed $p_1 = (\{F_1\}, \{x_1\}, \{\Gamma_1\}, \varphi)$ where $F_1 = \{f_{11}, f_{12}\}$ and Γ_1 is the following star quiver with frozen rank is (1)

$$\cdot_{f_{11}} \longleftarrow \cdot_{x_1} \longleftarrow \cdot_{f_{12}} \ .$$

If φ be a R-linear automorphism of $\mathcal{D}_1$ satisfying the conditions (3.7). Then this seed produces the cluster set $\mathcal{X}(p_1)$ given by

$\{(f_{11}+f_{12})x_1^{-1}, x_1^{-1}(f_{11}+f_{12}), \varphi^k(x_1), (f_{11}+f_{12})\varphi^{-k}(x_1), \varphi^{-k}(x_1)(f_{11}+f_{12}); k \in \mathbb{Z}\}$.

One can see that $\mathcal{X}(p_1)$ is a finite set if and only if φ is of finite order.

Remark 3.11 Examples 3.9 and 3.10 show that the Fomin-Zelevinsky finite type classification [9] does not work in the preseed case in general.

Lemma 3.12 *Let $p_n = (F, X, \Gamma, \varphi)$ be a φ-commutative preseed. If Γ_k is a balanced star quiver, then we have*

$$(\mu_k^R)^2(x_k) = \varphi^{a_k}(x_k) \ \textit{for some nonnegative integer} \ a_k; \tag{3.11}$$

and

$$(\mu_k^L)^2(x_k) = \varphi^{-a_k}(x_k) \ \textit{for some nonnegative integer} \ a_k. \tag{3.12}$$

Proof Since $\mu_k^R(\Gamma_k) = -\Gamma_k$. Then, one has

$$\begin{aligned} (\mu_k^R)^2(x_k) &= \mu_k^R((\prod_{i,\cdot_i\to\cdot_k} f_{ki}^{v_{ik}} + \prod_{i,\cdot_k\to\cdot_i} f_{ki}^{v_{ki}}))x_k^{-1}) \\ &= (\prod_{i,\cdot_i\to\cdot_k} f_{ki}^{v_{ik}} + \prod_{i,\cdot_k\to\cdot_i} f_{ki}^{v_{ki}})x_k(\prod_{i,\cdot_i\to\cdot_k} f_{ki}^{v_{ik}} + \prod_{i,\cdot_k\to\cdot_i} f_{ki}^{v_{ki}})^{-1} \\ &= \varphi^{a_k}(x_k). \end{aligned}$$

The last equation is by the commutativity of $x_1, \ldots, x_k, \ldots x_n$ and applying (3.8), noticing that Γ_k is a balanced star quiver with frozen rank a_k. This finishes the proof of Eq. 3.11. The proof of Eq. 3.12 is quite similar except for using the commutation relations (3.9) instead of (3.8). □

Corollary 3.13 *Let $p_n = (F, X, \Gamma, \varphi)$ be a φ-commutative preseed with φ be a finite order ring homomorphism. If Γ_k is a balanced star quiver, then there is a non negative integer r such that*

$$(\mu_k^R)^{2r}(p_n) = (\mu_k^L)^{2r}(p_n) = p_n. \tag{3.13}$$

Proof Assume that $\varphi^r = id_{\mathcal{D}_n}$ for some non negative integer r. Then using (3.11) (r-times) we get

$$(\mu_k^R)^{2r}(x_k) = \varphi^{ra_k}(x_k) = x_k. \tag{3.14}$$

And, we already have $(\mu_k^R)^{2r}(x_j) = x_j$ for $j \neq k$, and $(\mu_k^R)^{2r}(\Gamma) = \Gamma$, which finishes the proof. □

Question 3.14 For which preseed $p_n = (F, X, \Gamma)$, the set of cluster variables $\mathcal{X}(p_n)$ is finite?

In the following we provide equivalent conditions on φ-commutative preseed to produce a finite type cluster algebra.

Theorem 3.15 *Let $p_n = (F, X, \Gamma, \varphi)$ be a balanced, φ-commutative preseed. Then, the set of all cluster variables $\mathcal{X}(p_n)$ is a finite set if and only if φ is of finite order.*

Proof Let $\mu_{i_i}, \ldots, \mu_{i_t}$ be a sequence of mutations containing j copies of μ_k. Then, by the definition of preseeds mutation, we have

$$\mu_{i_i} \cdots \mu_{i_t}(x_k) = \mu_k^j(x_k).$$

Hence, for any preseed (F, X, Γ), we have

$$\mathcal{X}(p_n) = \bigcup_{k=1}^{n} \mathcal{X}(p_1(x_k)), \text{ where } p_1(x_k) = (\{F_k\}, \{x_k\}, \{\Gamma_k\}). \tag{3.15}$$

So, from Eq. 3.11 one has, for every $k \in [1, n]$ the set of all cluster variables of $p_1(x_k)$ contains the set of the cluster variables of the form $\{\varphi^{la_k}(x_k);\ l \in \mathbb{N}\}$, where a_k is the frozen rank of $p_1(x_k)$. The set $\{\varphi^{la_k}(x_k);\ l \in \mathbb{N}\}$ is an infinite set if φ is not of finite order. Which implies that if the set of cluster variables of p_n is finite then φ must be of finite order. Now assume that, φ is of finite order. Then from Eq. 3.13, the preseed $p_1(x_k)$ will be reproduced after applying μ_k, $2r$-times which means that the set of cluster variables of $p_1(x_k)$ is finite for every $k \in [1, n]$ and then so is the set of cluster variables of p_n. □

Saleh [15] Let f be a R_n-linear automorphism over $\mathcal{D}_n$. Then f is said to be a *cluster variable preserver* of the preseed p_n if it keeps the cluster set $\mathcal{X}$ of p_n invariant.

Question 3.16 Given a preseed $p_n = (F, X, \Gamma)$ describe the set of all cluster preservers of p_n.

Theorem 3.17 *Let $p_n = (F, X, \Gamma, \varphi)$ be a balanced φ-commutative preseed with frozen rank $(a_1, \dots, a_n)$. Let ϕ_l be the R_n-linear automorphisms of $\mathcal{D}_n$ induced by*

$$\phi_l(t) = t, \quad \forall t \in R_n \ \text{ and } \ \phi_l(x_k) = \varphi^{la_k}(x_k), \quad \forall k \in [1, n]. \tag{3.16}$$

Then, for every $l \in \mathbb{Z}$, ϕ_l is a cluster variables preserver for p_n.

Proof Notice that, by definition of ϕ_l, it depends on the frozen rank of p_n, which is invariant under mutation, thanks to Proposition 3.8.

First, for nonnegative integers. Let $l = 1$. Equations (3.11) assure that, the action of the automorphism ϕ_1 on the cluster variables of p_n is identified with the action of the sequence of the mutations $\prod_{i=1}^{n}(\mu_i^R)^2$.

Let x be an element of $\mathcal{X}(p_n)$, without loss of generality, we assume that x is a cluster variable of some seed q_n, that can be obtain from p_n by applying some sequence of only right mutations say $\mu_{i_1}^R \dots \mu_{i_d}^R$. Then, $\phi_1(x)$ must be a cluster variable in the seed $\prod_{i=1}^{n}(\mu_i^R)^2(q_n) = \prod_{i=1}^{n}(\mu_i^R)^2 \mu_{i_1}^R \dots \mu_{i_d}^R(p_n)$.

For $l \geq 2$, again using (3.11), the action of ϕ_l is identified with the action of the sequence of mutations $(\prod_{i=1}^{n}(\mu_i^R)^2)^l(p_n)$. Proving that ϕ_l permutes the elements of $\mathcal{X}(p_n)$ is quite similar to the case of $l = 1$ with the obvious changes.

The case, when l is a negative integer, is similar, with using Eq. 3.12 instead of Eq. 3.11. □

4 Weyl Cluster Algebras

4.1 Definition of Generalized Weyl Algebras

Definition 4.1 (Generalized Weyl algebra (1, 13, 14)) Let $\{\xi_1, \dots, \xi_n\}$ be a fixed set of elements of a commutative ring R and $\theta = \{\theta_1, \dots, \theta_n\}$ be a set of ring automorphisms such that $\theta_i(\xi_j) = \xi_j$ for all $i \neq j$. *The generalized Weyl algebra* of degree n, denoted by $R(\theta, \xi, n)$, is defined to be the ring extension of R generated by the $2n$ indeterminates $x_1, \dots, x_n, y_1, \dots, y_n$ modulo the commutation relations:

$$x_i r = \theta_i(r) x_i \ \text{ and } \ y_i r = \theta_i^{-1}(r) y_i, \quad \text{for any } i \in [1, n] \text{ and for any } r \in R, \tag{4.1}$$

$$x_i y_i = \xi_i, \ y_i x_i = \theta^{-1}(\xi_i), \ x_i y_j = y_j x_i, \ x_i x_j = x_j x_i \text{ and } y_i y_j = y_j y_i \ \forall i \neq j \in [1, n]. \tag{4.2}$$

We warn the reader that $x_i y_i \neq y_i x_i$ in general. The variables $x_1, \dots, x_n, y_1, \dots, y_n$ are called *Weyl variables*.

Example 4.2 [4, 13, 14] Let A_n be the n^{th} Weyl algebra generated by the $2n$ variables $x_1, \dots, x_n, y_1 \dots, y_n$ over K with the relations

$$x_i y_i - y_i x_i = 1, \text{ and } x_i x_j = x_j x_i, \quad y_i y_j = y_j y_i \ \text{ for } i \neq j, \ \forall i, j \in [1, n]. \tag{4.3}$$

Let $\xi_i = y_i x_i + 1$, R be the ring of polynomials $K[\xi_1, \dots, \xi_n]$ and $\theta_i : R \to R$, induced by $\xi_i \mapsto \xi_i + 1, \xi_j \mapsto \xi_j, j \neq i$, for all $i, j \in [1, n]$. It is known that A_n is isomorphic to the generalized Weyl algebra $R(\theta, \xi, n)$.

Example 4.3 [13, 14] The coordinate algebra $A(SL_q(2,k))$ of algebraic quantum group $SL_q(2,k)$ is the K-algebra generated by x, y, u, and v subject to the following relations

$$qux = xu, \quad qvx = xv, \quad qyu = uy, \quad qyv = vy, \quad uv = vu, \quad q \in K^* \tag{4.4}$$

$$xy = quv + 1, \quad \text{and} \quad yx = q^{-1}uv + 1. \tag{4.5}$$

$A(SL_q(2,k))$ is isomorphic to the generalized Weyl algebra $R(\xi, \theta, 1)$, where R is the algebra of polynomials $K[u, v]$; $\xi = 1 + q^{-1}uv$ and θ is an automorphism of R, defined by $\theta(f(u, v)) = f(qu, qv)$ for any polynomial $f(u, v)$.

Definition 4.4 (Weyl preseeds and q-commutative preseeds) Let $p_n = (F, X, \Gamma)$ be a preseed of rank n in $\mathcal{D}_n$. A quadruple (F, X, Γ, θ) is said to be a *Weyl preeseed* if there is a set $\theta = \{\theta_1, \ldots, \theta_n\}$ of ring automorphisms of R_n, such that for every $i \in [1, n]$, θ_i fixes all the exchange cluster variables and satisfies

$$x_i^{\pm 1} f = \theta_i^{\pm 1}(f) x_i^{\pm 1}, \quad \forall f \in F_i, \forall i \in [1, n]. \tag{4.6}$$

If there is a fixed scalar $q \in K^*$ such that θ_i satisfies

$$\theta_i(f_i) = qf_i, \quad \text{for every} \quad i \in [1, n]. \tag{4.7}$$

In such special case, $p_n = (F, X, \Gamma, q)$ is called q-commutative preseed.

Let $p_n = (F, X, \Gamma)$ be a preseed and let

$$\xi_k = \prod_{\cdot i \to \cdot k} f_{ki}^{v_{ik}} + \prod_{\cdot k \to \cdot i} f_{ki}^{v_{ki}}, \quad k \in [1, n]. \tag{4.8}$$

Then Relations 4.6 can be extended to $\xi_k^{\pm 1}$ as follows

$$\theta_k^{\pm 1}(\xi_k^{\mp 1}) x_k^{\pm 1} = x_k^{\pm 1} \xi_k^{\mp 1}, \quad k \in [1, n]. \tag{4.9}$$

Example 4.5 Let $R(\theta, \xi, n)$ be a generalized Weyl algebra. Consider the quintuple $p_n = (F, Y, \Gamma, \varphi, \theta)$, where $F = \{F_i\}_{i=1}^n$, $F_i = \{f_i;\, f_i = y_i x_i\}$, $Y = \{y_1, \ldots, y_n\}$, $\Gamma = \{\Gamma_i\}_{i=1}^n$ such that for $i \in [1, n]$, Γ_i is the quiver

$$\cdot f_i \longleftarrow \cdot y_i \; ,$$

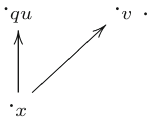
and φ is given by

$$\varphi(y_i) = \xi_i y_i \xi_i^{-1}, \text{ where } \xi_i = 1 + f_i, \; i \in [1, n]. \tag{4.10}$$

A short calculation shows that φ satisfies Eq. (3.7), hence $p_n = (F, X, \Gamma, \varphi, \theta)$ is a φ-commutative preseed. Also, from the properties of the R-automorphisms $\theta = (\theta_1, \ldots, \theta_n)$ given in Eqs. (4.1) and (4.2) one can see that θ_i satisfies Eq. (4.6) for each $i \in [1, n]$ which makes p_n a Weyl preseed, then p_n is φ-commutative Weyl preseed.

In this case the iterated division rings $\mathcal{D}_i$, $i = 1, \ldots, n$, attached with p_n, are subrings of the division ring of rational functions in $y_1, \ldots, y_n$ over the ring R. In particular, in the case of the n^{th} Weyl algebra A_n, the ring R is the ring of polynomials $K[\xi_1, \ldots, \xi_n]$. One can see that this ambient division ring of rational functions is an Ore domain. For information about Ore domains we refer to [3, 12].

Example 4.6 Recall the coordinate algebra $A(SL_q(2,k))$ of the algebraic quantum group $SL_q(2,k)$. Consider the preseed $p_1 = (\{F_1\}, \{x\}, \{\Gamma_1\}, \{\theta_1\})$, where $F_1 = \{qu, v\}$, $\theta_1 : R \to R$ given by $\theta_1(f(u,v)) = f(qu, qv)$ and Γ_1 is given by



(4.11)

One can see that p_1 is a q-commutative preseed. Let $\zeta = quv + 1$. The cluster set of p_1 is given by

$$\mathcal{X}(p_1) = \{x, \zeta^j x \zeta^{-j}, \zeta^{j+1} x^{-1} \zeta^{-j-1}, j \in \mathbb{N}\} \bigcup \{y, \zeta^j y \zeta^{-j}, \zeta^{j+1} y^{-1} \zeta^{-j-1}, j \in \mathbb{N}\}.$$

Remark 4.7 If $p_n = (F, X, \Gamma, \theta)$ is a Weyl preseed, then the two quadruples $(F, \mu_i^R(X), -\Gamma, \theta^{-1})$ and $(F, \mu_i^L(X), -\Gamma, \theta^{-1})$ where $\theta = \{\theta_1^{-1}, \ldots, \theta_n^{-1}\}$ are again Weyl preseeds, for every $i \in [1, n]$.

Lemma 4.8 *Let p_n be a q-commutative preseed with q being an m^{th} root of unity, for some natural number m and let Γ_k be balanced star quiver. Then, we have*

$$(\mu_k^R)^{2m}(p_n) = (\mu_k^L)^{2m}(p_n) = p_n. \tag{4.12}$$

Proof

$$\begin{aligned}
(\mu_k^R)^2(x_k) &= \mu_k^R\Big(\big(\prod_{i,\cdot_i \to \cdot_k} y_i^{v_{ik}} + \prod_{i,\cdot_k \to \cdot_i} y_i^{v_{ki}}\big)x_k^{-1}\Big)\\
&= \big(\prod_{i,\cdot_i \to \cdot_k} y_i^{v_{ik}} + \prod_{i,\cdot_k \to \cdot_i} y_i^{v_{ki}}\big)x_k\big(\prod_{i,\cdot_i \to \cdot_k} y_i^{v_{ik}} + \prod_{i,\cdot_k \to \cdot_i} y_i^{v_{ki}}\big)^{-1}\\
&= \big(\prod_{i,\cdot_i \to \cdot_k} y_i^{v_{ik}} + \prod_{i,\cdot_k \to \cdot_i} y_i^{v_{ki}}\big)x_k\big(\prod_{i,\cdot_i \to \cdot_k} y_i^{v_{ik}} + \prod_{i,\cdot_k \to \cdot_i} y_i^{v_{ki}}\big)^{-1}\\
&= \big(\prod_{i,\cdot_i \to \cdot_k} y_i^{v_{ik}} + \prod_{i,\cdot_k \to \cdot_i} y_i^{v_{ki}}\big)\theta_k\Big(\big(\prod_{i,\cdot_i \to \cdot_k} y_i^{v_{ik}} + \prod_{i,\cdot_k \to \cdot_i} y_i^{v_{ki}}\big)^{-1}\Big)x_k\\
&= \big(\prod_{i,\cdot_i \to \cdot_k} y_i^{v_{ik}} + \prod_{i,\cdot_k \to \cdot_i} y_i^{v_{ki}}\big)\Big(\theta_k\big(\prod_{i,\cdot_i \to \cdot_k} y_i^{v_{ik}} + \prod_{(i,\cdot_k \to \cdot_i} y_i^{v_{ki}}\big)\Big)^{-1}x_k\\
&= q^{-a_k}x_k, \text{ where } a_k \text{ is the frozen component of } \Gamma_k.
\end{aligned}$$

Then

$$(\mu_k^R)^{2m}(x_k) = q^{-ma_k}x_k = x_k.$$

And since we already have $(\mu_k^R)^{2m}(\Gamma) = \Gamma$, which completes the proof. □

Corollary 4.9 *Let $p_n = (F, X, \Gamma, q)$ be a q-commutative balanced preseed. Then the set of of all cluster variables $\mathcal{X}(p_n)$ is finite if and only if q is an m^{th}-root of unity, for some natural number m.*

Proof From Lemma 4.8 and Eq. 3.15 we have

$$\{q^{(-a_k)l}x_k;\ l \in \mathbb{N}\} \subset \mathcal{X}(p_1(x_k)) \subset \mathcal{X}(p_n), \forall k \in [1, n].$$

If for every natural number m, $q^m \neq 1$, then for each $k \in [1, n]$, the set $\{q^{(-a_k)l} x_k;\ l \in \mathbb{N}\}$ is an infinite set. So, if $\mathcal{X}(p_n)$ is a finite set then q must be an m^{th}-root of unity, for some natural number m. Now, assume that $q^m = 1$, for some natural number m, then again using Lemma 4.8, for each k in [1, n], the seed $p_1(x_k)$ will be reproduced after applying μ_k^R, $2m$-times. Then $\mathcal{X}(p_1(x_k))$ is a finite set for each k in $[1, n]$ and hence again from Eq. 3.15, the set $\mathcal{X}(p_n)$ must be a finite set. □

Definition 4.10 (Weyl cluster algebras) Let $p_n = (F, X, \Gamma, \theta)$ be a Weyl preseed. The Weyl cluster algebra $\mathcal{H}(p_n)$ is defined to be the R_n-subalgebra of $\mathcal{D}_n$ generated by the cluster set $\mathcal{X}(p_n)$.

The following remark and theorem shed some light on the structure of the Weyl cluster algebra $\mathcal{H}(p_n)$. Remark 4.11 and first part of the Theorem 4.12 can be phrased as, the Weyl cluster algebra $\mathcal{H}(p_n)$ is generated by R_n and many (could be infinite) isomorphic copies of generalized Weyl algebras, each vertex in the exchange graph of p_n gives rise to two copies of them. The second part of the theorem is the Laurent phenomenon, Theorem 2.8, in the Weyl preseeds case. The third part of the same theorem is simply saying that $\mathcal{H}(p_n)$ is isomorphic to the tensor product of the n Weyl cluster algebras of rank one associated to the n iterated rank one Weyl preseeds associated to p_n.

Remark and Definition 4.11 Let $p_n = (F, X, \Gamma, \theta)$ be a Weyl preseed and $R = K[\xi_1, \ldots, \xi_n]$ be the ring of polynomials in $\xi_1, \ldots, \xi_n$ where ξ_i, $i = 1, \ldots, n$ are as defined in Eq. 4.8. Then p_n gives rise to two copies of generalized Weyl algebras of rank n, as follows

(a) $H^R(p_n)$ is the ring extension of R generated by $\mu_1^R(x_1), \ldots, \mu_n^R(x_n), x_1, \ldots, x_n$.
(b) $H^L(p_n)$ is the ring extension of R generated by $x_1, \ldots, x_n, \mu_1^L(x_1), \ldots, \mu_n^L(x_n)$.
(b) In particular, if $p_n = (F, Y, \Gamma, \varphi, \theta)$ is the preseed given in Example 4.5, then each of $H^R(p_n)$ and $H^L(p_n)$ are isomorphic to $R(\theta, \xi, n)$ as generalized Weyl algebras. In the case of $H^R(p_n)$ (respectively $H^L(p_n)$) the isomorphism is defined by sending the cluster variable $\mu_i^R(x_i)$ to the Weyl variable x_i and the cluster variable x_i to the Weyl variable y_i of $R(\theta, \xi, n)$ (respectively by sending the cluster variable x_i to the Weyl variable x_i and $\mu_i^R(x_i)$ to the Weyl variable y_i) for $i = 1, \ldots, n$. Details for the case $n = 1$ are given in Example 4.14.

Theorem 4.12 *Let $p_n = (F, X, \Gamma, \theta)$ be a Weyl preseed in $\mathcal{D}_n$. Then the following are true*

(1) *Right and left mutations on p_n induce isomorphisms between the generalized Weyl algebras $H^R(p_n)$ and $H^R(\mu_k^R(p_n))$ (respectively $H^L(p_n)$ and $H^L(\mu_k^L(p_n))$).*
(2) *The Weyl cluster algebra $\mathcal{H}(p_n)$ is a subring of the (non-commutative) ring of Laurent polynomials in the initial exchange cluster variables with coefficients from ring of polynomials $R_n[\theta_1^{\pm 1}(\xi_1^{-1}), \ldots, \theta_n^{\pm 1}(\xi_n^{-1})]$.*
(3) *Let $p_1(x_k)$ be the rank one preseed $(F_k, \{x_k\}, \{\Gamma_k\}, \theta_k\})$. Then*

$$\mathcal{H}(p_n) \cong \mathcal{H}(p_1(x_1)) \otimes \cdots \otimes \mathcal{H}(p_1(x_n)). \tag{4.13}$$

Proof To prove part (1), consider the R_n-linear automorphism of $\mathcal{D}_n$, denoted by $T_{p_n,k}^R$: $\mathcal{D}_n \to \mathcal{D}_n$ induced by $x_k \mapsto \mu_k^R(x_k)$, $k \in [1, n]$. The restriction of this automorphism on $H^R(p_n)$ induces the algebras isomorphism $\widehat{T}_{p_n,k}^R : H^R(p_n) \to H^R(\mu_k^R(p_n))$ given by

$r \mapsto r, \forall r \in \mathcal{R}$, and $x_k \mapsto \mu_k^R(x_k) = \xi_k x_k^{-1}, \forall k \in [1,n]$. Which implies $\mu_k^R(x_k) \mapsto \xi_k x_k \xi_k^{-1} = \mu_k^R(\mu_k^R(x_k))$. Finally, it is easy to see that the generalized Weyl commutation relations (4.1) are invariant under $\widehat{T}^R_{p_n,k}$. (The argument for $H^L(p_n)$ is quite similar).

For part (2), let $y \in \mathcal{X}(p_n)$. Without loss of generality, using (3.15) we can assume that y is an element of $\mathcal{X}(p_1(x_k))$ for some $k \in [1,n]$. Hence, y can be obtained from x_k by applying some sequence of mutations on $p_1(x_k)$. Let l be the length of a shortest such sequence of mutations. By Eq. 3.6 we have that every non-trivial sequence of mutations can be reduced to either only right mutations or only left mutations. Then, by mathematical induction on l, one can show

$$y = \begin{cases} \xi_k^{\frac{l+1}{2}} x_k^{-1} \xi_k^{-(\frac{l+1}{2}-1)} \text{ or } \xi_k^{-(\frac{l+1}{2}-1)} x_k^{-1} \xi_k^{\frac{l+1}{2}}, \text{ if } l \text{ is an odd number;} \\ \xi_k^{\frac{l}{2}} x_k \xi_k^{-\frac{l}{2}} \text{ or } \xi_k^{-\frac{l}{2}} x_k \xi_k^{\frac{l}{2}}, \text{ if } l \text{ is an even number.} \end{cases} \tag{4.14}$$

Now, let m be a monomial in the elements of $\mathcal{X}(p_n)$. Then again using (3.15), and the identities (4.14), (4.9) and the commutations relations (4.2), one can write m as rm' where r is a monomial in the elements from the set $F^n \bigcup \{\theta_1^{\pm1}(\xi_1^{-1}), \ldots, \theta_n^{\pm1}(\xi_n^{-1})\}$ and m' is a monomial of elements from $\{x_k^{\pm1},\ k \in [1,n]\}$. Finally, the elements of $\mathcal{H}(p_n)$ are finite sum of finite product of monomials from the elements of $\mathcal{X}(p_n)$ which finishes the proof of Part (2).

For Part (3), by the definition of Weyl cluster algebra and the proof of Part (2) above one can see that the Weyl cluster algebra $\mathcal{H}(p_1(x_k))$ is generated as a K-vector space by the monomials

$$\mathfrak{m}_k = \{f_{k1}^{\alpha_{k1}} \cdots f_{km_k}^{\alpha_{km_k}} (\theta_k^{\pm1}(\xi_k^{-1}))^{\alpha'_k} x_k^{\beta}; \alpha_{kj}, \alpha'_k, \beta \in \mathbb{Z}, \forall j \in [1, m_k]\}. \tag{4.15}$$

Then, the Weyl cluster algebra $\mathcal{H}(p_n)$ is generated as a vector space by $\mathfrak{m}(p_n) = \{m_1 \cdots m_n;\ m_k \in \mathfrak{m}_k, k \in [1,n]\}$, where $m_i m_j = m_j m_i$ for $m_i \in \mathfrak{m}_i$ and $m_j \in \mathfrak{m}_j$ for every $i \neq j \in [1,n]$. Now we will show that $\mathfrak{m}(p_n)$ consists of linearly independent elements. Let $E = R_n[\theta_1^{\pm1}(\xi_1^{-1}), \ldots, \theta_n^{\pm1}(\xi_n^{-1})][t_1^{\pm1}, \cdots, t_n^{\pm1}]$, consider the linear endomorphisms of E given by $X_k^{\pm1}(f) = t^{\pm1} f, f \in E$. The map $\sigma : \mathcal{H}(p_n) \longrightarrow End(E)$ induced by $x_k^{\pm1} \longmapsto X_k^{\pm1}, k \in [1,n]$ defines an algebras homomorphism. One can see that the endomorphisms

$$f_{11}^{\alpha_{11}} \cdots f_{1m_1}^{\alpha_{1m_1}} \cdots f_{n1}^{\alpha_{n1}} \cdots f_{nm_j}^{\alpha_{nm_n}} (\theta_1^{\pm1}(\xi_1^{-1}))^{\alpha'_{k1}} \ldots (\theta_n^{\pm1}(\xi_n^{-1}))^{\alpha'_{km_k}} X_1^{\beta_1} \cdots X_n^{\beta_n};$$

$$\alpha_{ji}, \alpha'_{ji}, \beta_j \in \mathbb{Z}, i \in [1, m_j], j \in [1,n]$$

are linearly independent elements of $End(E)$ over K. Hence, $\mathfrak{m}(p_n)$ consists of linearly independent elements which makes it a basis for $\mathcal{H}(p_n)$ as a K-vector space and σ is an injective algebra homomorphism. Then the map that sends $m_1 \cdots m_n$ onto $m_1 \otimes \cdots \otimes m_n, m_k \in \mathfrak{m}_k, k \in [1,n]$ defines an isomorphism from $\mathcal{H}(p_n)$ to $\mathcal{H}(p_1(x_1)) \otimes \cdots \otimes \mathcal{H}(p_1(x_n))$. □

From the Proof of Part (2) of Theorem 4.12, we have the following remark.

Remark 4.13 The Weyl cluster algebra $\mathcal{H}(p_n)$ is finitely generated algebra.

Example 4.14 (Weyl cluster algebra associated to first Weyl algebra) Recall the n^{th} Weyl algebra given in Example 4.2 and the associated preseed given in Example 4.5. Let A_1 be the first Weyl algebra and consider the preseed $p_1 = (\{f\}, \{y\}, \{\cdot_y \longrightarrow \cdot_f\})$. Here $R_1 = K[\mathbb{P}]$, where $\mathbb{P}$ is the cyclic group generated by $f = yx$. Then We have the following exchange graph

- $\mathbb{G}(p_1)$

$$\cdots \overset{R}{\underset{L}{\longleftrightarrow}} \overset{y_{-3}}{\cdot} \overset{R}{\underset{L}{\longleftrightarrow}} \overset{y_{-2}}{\cdot} \overset{R}{\underset{L}{\longleftrightarrow}} \overset{y_{-1}}{\cdot} \overset{R}{\underset{L}{\longleftrightarrow}} \overset{y_0=y}{\cdot} \overset{R}{\underset{L}{\longleftrightarrow}} \overset{y_1}{\cdot} \overset{R}{\underset{L}{\longleftrightarrow}} \overset{y_2}{\cdot} \overset{R}{\underset{L}{\longleftrightarrow}} \overset{y_3}{\cdot} \overset{R}{\underset{L}{\longleftrightarrow}} \cdots, \tag{4.16}$$

(here $\cdot \xrightarrow{R}$is left mutation and $\xleftarrow[L]{} \cdot$is right mutation). Which can be encoded by the following equations

$$y_{2k\pm1}y_{2k} = y_{2k}y_{2k\pm1} + 1, \quad \text{for } k \in \mathbb{Z}. \tag{4.17}$$

The Weyl cluster algebra $\mathcal{H}(p_1(y))$ is the R_1-subalgebra of $\mathcal{D}_1$ generated by the set of cluster variables $\{y_k, k \in \mathbb{Z}\}$. Relations (4.17) can be interpreted as follows, each arrow in $\mathbb{G}(p_1)$ corresponds to a copy of first Weyl algebra, denoted by $A_1^k = K\langle y_k, y_{k+1}\rangle, k \in \mathbb{Z}$ and right (respectively left) mutations define isomorphisms between the adjacent copies, given by $T_k : A_1^k \to A_1^{k+1}, \ \ y_k \mapsto y_{k+1}$ for $k \in \mathbb{Z}$ (respectively to the inverses of $T_k, k \in \mathbb{Z}$).

The *adjunction isomorphism* $\Theta : R(\theta^{-1}, \theta^{-2}(\xi), 1) \to R(\theta, \xi, 1)$ given by $r \mapsto \theta^{-1}(r), x \mapsto y$ and $y \mapsto x$. In [13], the adjunction isomorphism played an important role in describing the representations theory of generalized Weyl algebra $R(\theta, \xi)$.

Remark 4.15 Consider the preseed $p_1 = (F, Y, \Gamma)$ associated to the generalized Weyl algebra $R(\xi, \theta, 1)$, given in Example 4.5. The action of the adjunction isomorphism Θ on the exchange cluster variables of any two adjacent seeds on the exchange graph of p_1 coincides with the action of the right and left mutations.

5 Representations Arising from Weyl Cluster Structure

5.1 Space of Representations V_n

In the following, let $p_n = (F, Y, \Gamma, \theta)$ be the generalized Weyl preseed associated to the generalized Weyl algbera $R(\theta, \xi, n)$, as given in Example 4.5. A *cluster monomial* in $\mathcal{H}(p_n)$ is a product of non negative powers of exchange cluster variables belonging to the same cluster. To visualize that, the monomial $m = z_1^{\beta_1} \cdots z_n^{\beta_n}, \beta_i \in \mathbb{Z}_{\geq 0}, i \in [1, n]$ is a cluster monomial if $\{z_1, \ldots, z_n\}$ is the set of the exchange cluster variables of some seed in the exchange graph of p_n.

Definition 5.1 The *space of representations* V_n is defined to be the $K(f_1, \ldots, f_n)$-left span by the set of all cluster monomials.

Lemma 5.2 *The space of representations V_n is independent of p_n and depends only on the exchange graph $\mathbb{G}(p_n)$.*

Proof The statement of the lemma is equivalent to the fact that "the set of all cluster monomials of every seed in $\mathbb{G}(p_n)$ is the same" which is equivalent to "any two seeds in $\mathbb{G}(p_n)$ have the same exchange graph" which is an immediate result of the fact that the set of all seeds in $\mathbb{G}(p_n)$ form an equivalent class under (left and right) mutations as equivalent relation which is due to Part (2) of Proposition 3.4. □

Proposition 5.3 *If p_n is a preseed, then the following are true*

(1) *For any set of n (or less) different cluster variables, not including two variables produced from the same initial cluster variable, there is at least one preseeed in $\mathbb{G}(p_n)$ which contains all of them;*
(2) *For any two cluster variables z_1 and z_2, produced from the same initial cluster variable, there are two cases for their product*

- *if z_2 can be obtained from z_1 by applying some sequence of mutations of an odd length, then $z_1 z_2 \in K(f_1, \ldots, f_n)$;*
- *if z_2 can be obtained from z_1 by applying some sequence of mutations of an even length, then $z_1 z_2$ can be written as $g z_1^2$, for some $g \in K(f_1, \ldots, f_n)$.*

Proof Every cluster variable can be traced back to one of the initial cluster variables. More precisely, for any $y \in \mathcal{X}(p_n)$ there is $k \in [1, n]$ such that $y \in \mathcal{X}(p_1(x_k))$, thanks to Eq. 3.15. Hence, there is a sequence of mutations μ^y such that $y = \mu^y(x_k)$. Now, let $\{y_1, \ldots, y_t\}$ be a subset of $\mathcal{X}(p_n)$ such that $t \in [1, n]$. Then, one can see that the cluster of the seed $\mu^{y_1} \cdots \mu^{y_t}(p_n)$ contains the set $\{y_1, \ldots, y_t\}$. Part (2) is immediate from Eq. 4.14. □

Let $Y = \{y_1, \ldots, y_n\}$ be the cluster of the preseed p_n. For $t \in \mathbb{Z}$, $y_{i,t}$ denotes the cluster variable obtained from the initial cluster variable y_i by applying one of the following sequence of mutations $(\mu_i^R)^t$ if $t \geq 0$ or $(\mu_i^L)^{-t}$ if $t < 0$.

Using Proposition 5.3 and the above notation, a typical element of V_n can be written as a sum of elements of the form

$$v = r(f_1, \ldots, f_n) y_{1,m_1}^{\beta_1} \cdots y_{n,m_n}^{\beta_n}, \tag{5.1}$$

where $r(f_1, \ldots, f_n) \in K(f_1, \ldots, f_n)$, $(\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n_{\geq 0}$ and $(m_1, \ldots, m_n) \in \mathbb{Z}^n$.

Example 5.4 Consider the Weyl preseed $p_n = (F, Y, \Gamma, \theta)$, as given in Example 4.5. The i^{th} branch of the exchange graph $\mathbb{G}(p_n)$ is as follows

$$\cdots \overset{(y_{1,m_1},\ldots,y_{i,m_i-1},\ldots,y_{n,m_n})}{\cdot} \underset{L}{\overset{R}{\longleftrightarrow}} \overset{(y_{1,m_1},\ldots,y_{i,m_i},\ldots,y_{n,m_n})}{\cdot} \underset{L}{\overset{R}{\longleftrightarrow}} \overset{(y_{1,m_1},\ldots,y_{i,m_i+1},\ldots,y_{n,m_n})}{\cdot} \cdots$$

For the sake of simplicity, we labeled each vertex by the clusters only. The space of representations V_n is the left $K(\xi_1, \ldots, \xi_n)$-linear span by the set

$$\{y_{1,m_1}^{\beta_1} \cdots y_{n,m_n}^{\beta_n} | \text{ for } m = (m_1, \ldots, m_n) \in \mathbb{Z}^n, \text{and } \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n_{\geq 0}\}. \tag{5.2}$$

Definition 5.5 (Representations of $R(\theta, \xi, n)$ on V_n) An action of the generators $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ on the (a generic) element v (given in (5.1)), is given by

$$y_i(v) := r(f_1, \ldots, f_{i-1}, \theta_i^{-1}(f_i), \ldots, f_n) y_{1,m_1}^{\beta_1} \cdots y_{i-1,m_{i-1}}^{\beta_{i-1}} y_{i,m_i-1}^{\beta_i} y_{i+1,m_{i+1}}^{\beta_{i+1}} \cdots y_{n,m_n}^{\beta_n}; \tag{5.3}$$

and

$$x_i(v) := \theta_i(f_i) r(f_1, \ldots, f_{i-1}, \theta_i(f_i), \ldots, f_n) y_{1,m_1}^{\beta_1} \cdots y_{i-1,m_{i-1}}^{\beta_{i-1}} y_{i,m_i+1}^{\beta_i} y_{i+1,m_{i+1}}^{\beta_{i+1}} \cdots y_{n,m_n}^{\beta_n}. \tag{5.4}$$

Lemma 5.6 *The actions given in definition 5.5 define a fully faithful left module structure of $R(\theta, \xi, n)$ on V_n.*

Proof The module structure of $R(\theta, \xi, n)$ on V_n is defined by extending Eqs. (5.3) and (5.4) to random elements of $R(\theta, \xi, n)$. It is obvious to see that the actions given in Eqs. 5.3 and 5.4 are compatible with Relations (4.1). In the following we show that Relations (4.2) are satisfied on the generic element v, given in (5.1). We have

$$\begin{aligned} x_i y_i(v) &= x_i(r(f_1, \ldots, \theta_i^{-1}(f_i), \ldots, f_n) y_{1,m_1}^{\beta_1} \cdots y_{i-1,m_{i-1}}^{\beta_{i-1}} y_{i,m_i-1}^{\beta_i} y_{i+1,m_i+1}^{\beta_{i+1}} \cdots y_{n,m_n}^{\beta_n})) \\ &= \theta_i(f) r(f_1, \ldots, \theta_i^{-1}(\theta(f_i)), \ldots, f_n) y_{1,m_1}^{\beta_1} \cdots y_{i-1,m_{i-1}}^{\beta_{i-1}} y_{i,m_i}^{\beta_i} y_{i+1,m_i+1}^{\beta_{i+1}} \cdots y_{n,m_n}^{\beta_n}) \\ &= \theta_i(f_i)(v) \\ &= \xi_i v. \end{aligned}$$

And

$$\begin{aligned} y_i x_i(v) &= y_i(\theta_i(f_i) r(f_1, \ldots, \theta_i(f_i), \ldots, f_n) y_{1,m_1}^{\beta_1} \cdots y_{i-1,m_{i-1}}^{\beta_{i-1}} y_{i,m_i+1}^{\beta_i} y_{i+1,m_i+1}^{\beta_{i+1}} \cdots y_{n,m_n}^{\beta_n})) \\ &= \theta_i(\theta_i^{-1}(f_i)) r(f_1, \ldots, \theta_i(\theta^{-1}(f_i)), \ldots, f_n) y_{1,m_1}^{\beta_1} \cdots y_{i-1,m_{i-1}}^{\beta_{i-1}} y_{i,m_i}^{\beta_i} y_{i+1,m_i+1}^{\beta_{i+1}} \cdots y_{n,m_n}^{\beta_n}) \\ &= f_i v. \end{aligned}$$

In a similar way, one can get the rest of the Relations (4.2). The property of fully faithful module is a straightforward from the definitions of the actions given in Eqs. 5.3 and 5.4. □

Proposition 5.7 *The module structure given in Definition 5.5 can be extended to the Weyl cluster algebra associated to* p_n.

Proof To upgrade the representations of $R(\theta, \xi, n)$ on V_n to the Weyl cluster algebra associated to p_n, we introduce the action of y_i^{-1} on the element v, by $y_i^{-1}(v) = \theta^{-1}(\xi_i) x_i(v)$. The action of a random element of the Weyl cluster algebra $\mathcal{H}(p_n)$ will be induced from the action of both of y_i and y_i^{-1} for $i = 1, \ldots, n$, thanks to Part (2) of Theorem 4.12. □

5.2 Cluster Strands and the Strand Submodules of V_n

Before introducing the cluster strands we need to introduce the following notations. For $t \in \mathbb{Z}$, let

$$\theta^t(-) = \begin{cases} \overbrace{\theta(\theta(\ldots\theta(-)))}^{t-times}, & \text{if } t > 0, \\ id_R, & \text{if } t = 0, \\ \overbrace{\theta^{-1}(\theta^{-1}(\ldots\theta^{-1}(-)))}^{|t|-times}, & \text{if } t < 0. \end{cases}$$

Consider the following three sets of monomials in the elements $\{\theta^t(z); t \in \mathbb{Z}\}$

(1)

$$M^+(z) := \{1, \theta^t(z^{\pm 1})\theta^{t+1}(z^{\pm 1}) \cdots \theta^{t+q}(z^{\pm 1}) | q, t \in \mathbb{Z}_{\geq 0}\};$$

(2)

$$M^-(z) := \{1, \theta^t(z^{\pm 1})\theta^{t-1}(z^{\pm 1}) \cdots \theta^{t-q}(z^{\pm 1}) | q \in \mathbb{Z}_{\geq 0}, t \in \mathbb{Z}_{<0}\};$$

(3)

$$M(z) := \{m_1 m_2 | m_1 \in M^+(z) \text{ and } m_2 \in M^-(z)\}. \tag{5.5}$$

For every $h \in K(f_1, \ldots, f_n)$ and $t = (t_1, \ldots, t_n) \in \mathbb{Z}^n$ we associate a subset of $K(f_1, \ldots, f_n)$ as follows

$$c(h, t) := \{\alpha_1 \cdots \alpha_n h(\theta_1^{t_1}(f_1), \ldots, \theta_n^{t_n}(f_n))|\, \alpha_i \in M(f_i), \forall i \in [1, n]\}. \tag{5.6}$$

Definition 5.8 (*Cluster strands*) Fix a natural number l and a one to one map $\sigma : [1, l] \to \mathbb{Z}^n_{\geq 0} \times \mathbb{Z}^n$. Let $\beta = (\beta_1, \ldots, \beta_l) \in \overbrace{\mathbb{Z}^n_{\geq 0} \times \cdots \times \mathbb{Z}^n_{\geq 0}}^{l-\text{times}}$ and $m = (m_1, \ldots, m_l) \in \overbrace{\mathbb{Z}^n \times \cdots \times \mathbb{Z}^n}^{l-\text{times}}$ such that $\sigma(j) = (\sigma_1(j), \sigma_2(j)) = (\beta_j, m_j)$ where $\beta_j = (\beta_{j1}, \ldots, \beta_{jn})$ and $m_j = (m_{j1}, \ldots, m_{jn})$, $j \in [1, l]$. Let $r = (r_1, \ldots, r_l)$ such that $r_j \in K(f_1, \ldots, f_n)$ for $j \in [1, l]$. Consider the following subset of V_n

$$S_l(\sigma, r) := \left\{ \sum_{j=1}^{l} g_j y^{\beta_{j1}}_{1,m_{j1}+t_{j1}} \cdots y^{\beta_{jn}}_{n,m_{jn}+t_{jn}} | g_j \in c(r_j, t_j), t_j = (t_{j1}, \ldots, t_{jn}) \in \mathbb{Z}^n, j \in [1, l] \right\}. \tag{5.7}$$

With the above data, $S_l(\sigma, r)$ is called a *cluster strand of length l, with respect to r and* σ. Furthermore, $S_l(\sigma, r)$ is called a *full cluster strand* if $\sigma_1(j) \in \mathbb{Z}^n_{>0}$ for every $j \in [1, l]$.

Example 5.9 (A cluster strands of length 2 in V_3) Let $l = 2$, $\sigma_1(1) = (0, 3, 0)$, $\sigma_1(2) = (1, 0, 2)$, $\sigma_2(1) = (1, 1, 0)$, $\sigma_2(2) = (0, 1, 1)$, and $r = (f_1^2 + f_2, f_1 f_3)$. For $t_j = (t_{j1}, t_{j2}, t_{j3}) \in \mathbb{Z}^3$, $j \in [1, 2]$, we have

$$c(f_1^2 + f_2, t_1) = \{\alpha_1\alpha_2\alpha_3((\theta_1^{t_{11}}(f_1))^2 + \theta_2^{t_{12}}(f_2))|\, \alpha_i \in M(f_i), i \in [1, 3]\},$$

and

$$c(f_1 f_3, t_2) = \{\alpha_1\alpha_2\alpha_3\theta_1^{t_{21}}(f_1)\theta_3^{t_{23}}(f_3)|\, \alpha_i \in M(f_i), i \in [1, 3]\}.$$

With the above data we have

$$S_3(\sigma, r) = \left\{g_1 y^3_{2,1+t_{12}} + g_2 y_{1,0+t_{21}} y^2_{3,1+t_{23}} | g_1 \in c(f_1^2 + f_2, t_1), g_2 \in c(f_1 f_2, t_2), t_1, t_2 \in \mathbb{Z}^3\right\}.$$

Proposition 5.10 *Each element of V_n gives rise to a cluster strand.*

Proof For every element v of V_n, one can find $r_1, \ldots, r_l$ elements of $K(f_1, \ldots, f_n)$ such that v can be written uniquely as follows

$$v = r_1(f_1, \ldots, f_n) y^{\beta_{11}}_{1,m_{11}} \cdots y^{\beta_{1n}}_{n,m_{1n}} + \ldots + r_l(f_1, \ldots, f_n) y^{\beta_{l1}}_{1,m_{l1}} \cdots y^{\beta_{ln}}_{n,m_{ln}}.$$

Such that a $1 - 1$ map $\sigma : [1, l] \to \mathbb{Z}^n_{\geq 0} \times \mathbb{Z}^n$ can be defined with $\sigma(j) = (\sigma_1(j), \sigma_2(j))$, where $\sigma_1(j) = (\beta_{j1}, \ldots, \beta_{jn})$ and $\sigma_2(j) = (m_{j1}, \ldots, m_{jn})$, $j \in [1, l]$. Using Definition 5.8, one can introduce a cluster strand $S_l(\sigma, r)$ with $r = (r_1, \ldots, r_l)$ and σ as defined above. □

We denote the cluster strand associated to v by $S_l(\sigma, r)(v)$.

Question 5.11 Does the cluster strand $S_l(\sigma, r)(v)$ depend on the choices of r or σ?

The following lemma and Remarks 5.14 provide some basic properties of the cluster strands.

Lemma 5.12 (1) *Let v be an element of $S_l(r,\sigma)$. Then $S_l(r,\sigma)(v) = S_l(r,\sigma)$;*
(2) *We have $\mathrm{S}_l(\sigma, g) = \mathrm{S}_l(\sigma, f)$ if and only if for every $i \in [1,n]$, either $g_i \in c(f_i, t_i)$ for some $t_i \in \mathbb{Z}^n$ or $f_i \in c(g_i, t_i')$ for some $t_i' \in \mathbb{Z}^n$;*
(3) *Let $g = (g_1, \ldots, g_n)$ such that $g_i \in c(f_i, t_i)$ for some $t_i \in \mathbb{Z}^n$, $i \in [1,n]$. Then $\mathrm{S}_l(\sigma', g) = \mathrm{S}_l(\sigma, f)$ if and only if $\forall j \in [1,l]$, $\sigma_1'(j) = \sigma_1(j)$ and $\sigma_2'(j) = \sigma_2(j) + q_j$ for some $q_j \in \mathbb{Z}^n$.*

Proof For Part (1). Fix $v = \sum_{j=1}^{l} g_j y_{1,m'_{j1}}^{\beta_{j1}} \cdots y_{n,m'_{jn}}^{\beta_{jn}} \in S_l(\sigma, r)$. Then we must have, for every $j \in [1,l]$, $g_j = \alpha'_{j1} \cdots \alpha'_{jn} r_j(\theta_1^{t'_{j1}}(f_1), \ldots, \theta_n^{t'_{jn}}(f_n)) \in c(r_j, t'_j)$, for some $t'_j = (t'_{j1}, \ldots, t'_{jn}) \in \mathbb{Z}^n$. A typical element of $c(r_j, t_j)$ would be of the form $\alpha_{j1} \cdots \alpha_{jn} r_j(\theta_1^{t_{j1}}(f_1), \ldots, \theta_n^{t_{jn}}(f_n))$ with $\alpha_{ji} \in M(f_i)$, $\forall i \in [1,n]$, which can be written as

$$\alpha_{j1} \cdots \alpha_{jn} \alpha'_{jn}(\theta_1^{t_{j1}-t'_{j1}}(f_1^{-1})) \cdots \alpha'_{j1}(\theta_n^{t_{jn}-t'_{jn}}(f_n^{-1})) g_j(\theta_1^{t_{j1}-t'_{j1}}(f_1), \ldots, \theta_n^{t_{jn}-t'_{jn}}(f_n))$$

which is an element of $c(g_j, t_j - t'_j)$. Thus, any element of the following form $\sum_{j=1}^{l} r_j y_{1,m_{j1}+t_{j1}}^{\beta_{j1}} \cdots y_{n,m_{jn}+t_{jn}}^{\beta_{jn}}$ is in fact an element of $S_l(g, \sigma')$, where $\sigma_1'(j) = \sigma_1(j)$ and $\sigma_2'(j) = \sigma_2(j) + t_j$, $j \in [1,l]$. Then $S_l(\sigma, r) \subseteq S_l(\sigma', g)$. But from the Proof of Proposition 5.10, one can see that $S_l(\sigma', g) = S_l(\sigma', g)(v)$. Again from the proof of Proposition 5.10, one can see that $S_l(\sigma, g)(v) \subseteq S_l(\sigma, r)$. Therefore, $S_l(\sigma, r) = S_l(\sigma, g)(v)$.

For Part (2). ($\Rightarrow$) is Obvious. For the other direction ($\Leftarrow$). Without loss of generality, let $g_j \in c(f_j, t_j)$ for some $t_j = (t_{j1}, \ldots, t_{jn}) \in \mathbb{Z}^n$. Then for every $j \in [1,l]$, there are $\alpha_{ji} \in M(f_i)$, $i \in [1,n]$ such that $g_j = \alpha_{j1} \cdots \alpha_{jn} f_j(\theta_1^{t_{j1}}(f_1), \ldots, \theta_n^{t_{jn}}(f_n))$. Now, let $v \in S_l(g, \sigma)$. Hence, we have

$$\begin{aligned} v &= \sum_{j=1}^{l} g_j y_{1,m'_{j1}}^{\beta_{j1}} \cdots y_{n,m'_{jn}}^{\beta_{jn}} \\ &= \sum_{j=1}^{l} \alpha_{j1} \cdots \alpha_{jn} f_j(\theta_1^{t_{j1}}(f_1), \ldots, \theta_n^{t_{jn}}(f_n)) y_{1,m'_{j1}}^{\beta_{j1}} \cdots y_{n,m'_{jn}}^{\beta_{jn}} \in S_l(f, \sigma). \end{aligned}$$

Therefore $S_l(\sigma, f) = S_l(\sigma, g)(v) = S_l(\sigma, g)$ thanks to Part (1) of this lemma.

For Part (3). First for ($\Rightarrow$). One can see that, if $\sigma_1'(j) = \sigma_1(j)$, $\forall j \in [1,l]$, then $\sigma' = \sigma + (0, q)$, $q \in \mathbb{Z}^n$. Now, assume that $\sigma'(j_0) \neq \sigma(j_0) + (0, q_j)$ for some $j_0 \in [1,l]$ and for every $q \in \mathbb{Z}^n$. Hence $\sigma_1'(j_0) \neq \sigma_1(j_0)$. Then the element

$$v_0 = g_{j_0} y_{1,m_{j_01}+t_{j_01}}^{\beta_{j_01}} \cdots y_{n,m_{j_0n}+t_{j_0n}}^{\beta_{j_0n}} + \sum_{j \in [1,l] \setminus \{j_0\}} g_j y_{1,m_{j1}+t_{j1}}^{\beta_{j1}} \cdots y_{n,m_{jn}+t_{jn}}^{\beta_{jn}}$$

is an element of $\mathrm{S}_l(\sigma', g)$ with $\sigma_1'(j) = (\beta_{j1}, \ldots, \beta_{jn})$. However, v_0 is not an element of $\mathrm{S}_l(\sigma, f)$. ($\Leftarrow$) is immediate. □

Definition 5.13 Any submodule of V_n generated by a cluster strand $\mathrm{S}_l(\sigma, r)$ is called a strand submodule and is denoted by $W_l(\sigma, r)$.

In the occasions, when we want to emphasis on a certain element v of V_n, we will denote the strand submodule associated to the cluster strand $S_l(\sigma, r)(v)$ by $W_l(\sigma, r)(v)$ or, for the sake of simplicity, by $W_l(v)$.

Let $\mathcal{M}(E)$ be the set of all monomials formed from elements of the set $E = \{x_1, \dots, x_n, y_1, \dots, y_n\}$. *A special cluster strand* is defined to be a subset of a full cluster strand $S_l(\sigma, r)$ of the form

$$\widehat{S_l}(\sigma, r) := \Big\{ \sum_{j=1}^{l} g_j y_{1,m_{j1}+t_1}^{\beta_{j1}} \cdots y_{n,m_{jn}+t_n}^{\beta_{jn}} | t = (t_1, \dots, t_n) \in \mathbb{Z}^n, g_j \in c(r_j, t), j \in [1, l] \Big\}. \tag{5.8}$$

The submodule of $W_l(\sigma, r)$ generated by the special cluster strand $\widehat{S_l}(\sigma, r)$ is called *special stand module* and is denoted by $\widehat{W_l}(\sigma, r)$.

Remarks 5.14 (1) Let $\widehat{S_l}(\sigma, h)$ be a special cluster strand. Then

(a)
$$\widehat{S_l}(\sigma, h) \text{ is a proper subset of } S_l(\sigma, h);$$

(b)
$$\widehat{S_l}(\sigma, h) = \mathcal{M}(E)w, \text{ for every } w \in \widehat{S_l}(\sigma, h).$$

(2) There is a bijection between the set of all cyclic submodules of V_n and the set of all special strand submodules.

Proof Part (1) is straight forward. For Part (2), let W be a cyclic module generated by w with associated cluster strand $S_l(\sigma, r)(w)$. Then by the definition of special cluster strands, we have $\widehat{W_l}(\sigma, r)$ is a submodule of W. One can realize that W is a submodule of $\widehat{W_l}(\sigma, r)$ too, if we recall that W is cyclic module generated by w which is an element of $\widehat{S_l}(\sigma, r)$. The bijection is defined to send W to $\widehat{S_l}(\sigma, r)$. □

Proposition 5.15 (1) *Every strand submodule $W_l(f, \sigma)$ can be identified with a sum of (identical) copies of the cluster strand $S_l(h, \sigma)$.*

(2) *Every submodule W of V_n is a sum of some strand submodules. In particular, W is generated by a set of cluster strands.*

Proof (1) First we show that the extensions of the action of the elements of $\mathcal{M}(E)$, induced by Eqs. 5.3 and 5.4, keeps the cluster strands invariant. One can see that for every $g \in K(f_1, \dots, f_n)$ and $t \in \mathbb{Z}^n$ the coefficients set $c(g, t)$ is invariant under the actions of the elements of E and then under elements of $\mathcal{M}(E)$. Now, let $v = \sum_{j=1}^{l} g_j y_{1,m_{j1}}^{\beta_{j1}} \cdots y_{n,m_{jn}}^{\beta_{jn}}$ be an element of the cluster strand $S_l(h, \sigma)$. Recalling that the actions given in Eqs. 5.3 and 5.4 define a fully faithful representation, one can see that under the actions of elements of E the length l stays unchanged with respect to h and σ, which will stay unchanged too. Hence for any monomial $m \in \mathcal{M}(E)$, we have $m(v) \in S_l(h, \sigma)$, more precisely

$$m(v) = \sum_{j=1}^{l} \alpha_{j1} \cdots \alpha_{jn} g_j(\theta_1^{t_{j1}}(f_1), \dots, \theta_1^{t_{jn}}(f_n)) y_{1,m_{j1}+t_{j1}}^{\beta_{j1}} \cdots y_{n,m_{jn}+t_{jn}}^{\beta_{jn}}. \tag{5.9}$$

Where $\alpha_{ji} \in M(f_i)$ and $t_{ji} \in \mathbb{Z}$, $\forall i \in [1,n], j \in [1,l]$. Recall that, elements of $W_l(h,\sigma)$ are finite sums of finite products of elements of $R(\theta,\xi,n)$ acting on an element of $S_l(f,\sigma)$. But every element of $R(\theta,\xi,n)$ can be written as a $K(f_1,\ldots,f_n)$-linear combination of elements of $\mathcal{M}(E)$. Then from Eq. (5.9) elements of $W_l(h,\sigma)$ are finite sum of elements of $S_l(h,\sigma)$. In the same time one can obviously see that every sum of elements of $S_l(h,\sigma)$ must be an element of $W_l(h,\sigma)$.

(2) We first notice that, from Part (1) of Lemma 5.12, we conclude that every two cluster strands are either identical or have zero intersection. So we can introduce the following equivalence relation on V_n

$$\forall s, s' \in V_n,\ s \sim s' \text{ if and only if } s \text{ and } s' \text{ belong to the same cluster strand.} \tag{5.10}$$

Let W be a submodule of V_n and $W^* = W/\sim$. Here, every $w^* \in W^*$, is the intersection of W with the cluster strand $S_l(\sigma, f)(w)$. If $W_l^*(w)$ denote the submodule of W generated by w^*. Then we have the following identity

$$W = \sum_{w^* \in W^*} W_l^*(w). \tag{5.11}$$

□

Proposition 5.16 *Let $S_l(\sigma,h)$ be a full cluster strand. Then any two strand submodules of $W_l(\sigma,h)$ have a non-zero intersection.*

Proof Let $W_1 = W_{l_1}(\sigma^1,h^1)(w_1)$ and $W_2 = W_{l_2}(\sigma^2,h^2)(w_2)$ be any two proper strand submodules of $W_l(\sigma,h)$. From Proposition 5.15 and Proposition 5.10, one can see that the cluster stands $S_{l_1}(\sigma^1,h^1)(w_1)$ and $S_{l_2}(\sigma^2,h^2)(w_2)$ satisfy the following

- There are two natural numbers d_1 and d_2 such that $l_i = d_i l, i = 1,2$;
- $h^1 = (h_{11},\ldots,h_{1l},\ldots,h_{d_1 1},\ldots,h_{d_1 l})$ and $h^2 = (h'_{11},\ldots,h'_{1l},\ldots,h'_{d_2 1},\ldots,h'_{d_2 l})$ where there are $t_{j_2} \in \mathbb{Z}^n$ such that $h_{j_1 j_2}, h'_{j_1 j_2} \in c(h_{j_2}, t_{j_2})$ for every $j_1 \in [1,d_i]$, $j_2 \in [1,l]$;
- For $i = 1,2$ we have $\sigma^i : \{11,12,\ldots,1l,\ldots,d_i 1,\ldots,d_i l\} \to \mathbb{Z}^n_{>0} \times \mathbb{Z}^n$, such that $\sigma^i = (\sigma_1^i,\sigma_2^i)$ where $\sigma_1^i(j_1 j_2) = \sigma_1(j_2)$ and $\sigma_2^i(j_1 j_2) = \sigma_2(j_2) + t_{j_1 j_2}$ for some $t_{j_1 j_2} \in \mathbb{Z}^n$, for all $j_2 \in [1,l]$.

Now we will show that the sum of any d_i-elements of $S_l(\sigma,h)$ is an element of $S_{l_i}(\sigma^i,h^i)(w_i)$, for $i = 1,2$. Consider the following two elements

$$w_{j_1} = \sum_{j_2=1}^{l} h_{j_1 j_2} y_{1,m_{j_2 1}+t^1_{j_1 j_2}}^{\beta_{j_2 1}} \cdots y_{n,m_{j_2 n}+t^n_{j_1 j_2}}^{\beta_{j_2 n}}$$

and

$$w'_{j_1} = \sum_{j_2=1}^{l} h'_{j_1 j_2} y_{1,m_{j_2 1}+t^1_{j_1 j_2}}^{\beta_{j_2 1}} \cdots y_{n,m_{j_2 n}+t^n_{j_1 j_2}}^{\beta_{j_2 n}}.$$

One can see that the elements w_{j_1} and w'_{j_1} are elements of $S_l(\sigma,h)$ for every $j_1 \in [1,d_i]$ for $i = 1,2$. Then the cluster strands associated to w_{j_1} and w'_{j_1} coincide with $S_l(\sigma,h)$ for every $j_1 \in [1,d_i]$, $i = 1,2$, thanks to Part (1) of Lemma 5.12.

For every $j_1 \in [1, d_i]$, we have $s \in S(w_{j_1}) = S(w'_{j_1})$ for every $s \in S(w)$. Let l' be the least common multiple of l_1 and l_2. So, $l' = n_i l_i$, for some $n_i \in \mathbb{N}, i = 1, 2$. Consider the element

$$w' = \sum_{i=1}^{l'} s_i, \text{ where } s_i \in S(w) \setminus \{s_1, \ldots, s_{i-1}\}, \forall i \in [1, l'].$$

One can see that, w' is in deed a sum of $n_i d_i$-elements of the cluster strand of w_{j_i} and elements of $S_{l_i}(\sigma^i, h^i)(w_i)$ are sums of d_i-elements of the cluster strand w_{j_i}, $i = 1, 2$. Then w' is a sum of d_i-elements of the cluster stared $S_{l_i}(\sigma^i, h^i)(w_i)$, $i = 1, 2$. Therefore from Part (1) of Proposition 5.15, we have

$$w' \in W(w_1) \cap W(w_2).$$

□

The following corollary is a consequence of the proof of Proposition 5.16. Let $S_l(\sigma, h)$ be a cluster strand with a strand module $W_l(\sigma, h)$. For every natural number j we introduce a subset of $S_l(\sigma, h)$ give by

$$S_l^j(\sigma, h) = \{s_1 + \cdots + s_j; \quad s_i \in S_l(\sigma, h) \setminus \{s_1, \ldots, s_{i-1}\}, \forall i \in [1, j]\}.$$

Corollary 5.17 (1) *For every $w^j, s^j \in S_l^j(\sigma, h)$, the cluster strands $S_{jl}(\sigma^1, h^2)(s^j)$ and $S_{jl}(\sigma^2, h^2)(w^j)$, defined in the Proof of Proposition 5.16 are coincide and with length of jl.*

(2) *Let $W(j)$ denote the strand module of $S_{jl}(\sigma^1, h^2)(s^j)$. Then we have the following descending chain of strand modules*

$$W_l(\sigma, h) \supseteq W(j) \supset W(2j) \supset \ldots \supset W(nj) \supset \ldots, \forall j \in \mathbb{N}.$$

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