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Graded Limits of Minimal Affinizations over the Quantum Loop Algebra of Type G_2

Jian-Rong Li[1,2] · Katsuyuki Naoi[3]

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Abstract The aim of this paper is to study the graded limits of minimal affinizations over the quantum loop algebra of type G_2. We show that the graded limits are isomorphic to multiple generalizations of Demazure modules, and obtain defining relations of them. As an application, we obtain a polyhedral multiplicity formula for the decomposition of minimal affinizations of type G_2 as a $U_q(\mathfrak{g})$-module, by showing the corresponding formula for the graded limits. As another application, we prove a character formula of the least affinizations of generic parabolic Verma modules of type G_2 conjectured by Mukhin and Young.

Keywords Minimal affinizations · Quantum loop algebras · Current algebras

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✉ Jian-Rong Li
lijr07@gmail.com; lijr@lzu.edu.cn

Katsuyuki Naoi
naoik@cc.tuat.ac.jp

[1] School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, People's Republic of China

[2] Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem 9190401, Israel

[3] Institute of Engineering, Tokyo University of Agriculture and Technology, 3-8-1 Harumi-cho, Fuchu-shi, Tokyo, Japan

1 Introduction

Let $\mathfrak{g}$ be a simple Lie algebra, $\mathbf{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ the corresponding loop algebra, and $U_q(\mathbf{L}\mathfrak{g})$ the corresponding quantum loop algebra. Minimal affinizations of representations of quantum groups are an important family of simple $U_q(\mathbf{L}\mathfrak{g})$-modules introduced in [1]. Minimal affinizations are natural generalizations of the celebrated Kirillov-Reshetikhin modules, which have several applications and are studied intensively during the past few decades. Minimal affinizations are important from the physical point of view, see for example, [1, 11, 13].

Graded limits of minimal affinizations, which are graded analogs of the classical limits defined over the current algebra $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$, were studied in [2, 4, 18, 19, 23, 24].

Minimal affinizations over the quantum loop algebra of type G_2 were studied in [1, 5, 16, 20, 25]. The aim of this paper is to study the graded limits of minimal affinizations over the quantum loop algebra of type G_2.

Assume that $\mathfrak{g}$ is of type G_2. Let $L(m)$ be the graded limit of a minimal affinization with highest weight λ, and let $M(\lambda)$ be the $\mathfrak{g}[t]$-module generated by a nonzero vector v_λ with certain relations. Our first main result (Theorem 3.2) is that $M(\lambda) \cong L(m) \cong T(\lambda)$, where $T(\lambda)$ is some generalized Demazure module. These isomorphisms were previously conjectured by Moura in [18].

Let ω_1 (resp. ω_2) be the fundamental weight with respect to the long (resp. short) simple root, and assume that $\lambda = k\omega_1 + l\omega_2$. Using the above isomorphisms, we obtain the following polyhedral multiplicity formula as a $\mathfrak{g}$-module (Theorem 3.3)

$$L(m) \cong \bigoplus_{(a_1,\ldots,a_5)\in S_\lambda} V\big((k - a_1 + a_3 + a_4 - a_5)\omega_1 + (l - a_2 - 3a_3 - 3a_4)\omega_2\big),$$

where

$$S_\lambda = \big\{(a_1,\ldots,a_5) \in \mathbb{Z}_+^5 \,\big|\, a_1 \le k,\ a_1 - a_3 + a_5 \le k,\ 2a_2 + 3a_3 + 3a_4 \le l,\ 2a_2 + 3a_4 + 3a_5 \le l\big\}.$$

Here $V(\mu)$ denotes the simple $\mathfrak{g}$-module with highest weight μ. As an immediate corollary, we obtain a similar formula for the multiplicity of minimal affinizations as a $U_q(\mathfrak{g})$-module (Corollary 3.4). This formula is a generalization of the one given in [5], in which the formula for Kirillov-Reshetikhin modules (i.e. the case $k = 0$ or $l = 0$) is given.

We also give a formula for the limit of normalized characters (Corollary 3.5), which yields the character formula of least affinizations of generic parabolic Verma modules of type G_2 conjectured by Mukhin and Young [21, Conjecture 6.3].

The paper is organized as follows. In Section 2, we give some background information about the quantum loop algebra of type G_2. In Section 3, we describe our main results in this paper. In Section 4, we prove Theorem 3.2. In Section 5, we prove Theorem 3.3.

2 Background

Let $\mathbb{Z}$ be the set of integers, and $\mathbb{Z}_+$ the set of nonnegative integers. In this paper, we take $\mathfrak{g}$ to be the complex simple Lie algebra of type G_2. Let $\mathfrak{h}$ be a Cartan subalgebra and $\mathfrak{b}$ a Borel subalgebra containing $\mathfrak{h}$. Let $I = \{1, 2\}$. We choose simple roots α_1, α_2 and scalar product $(\cdot,\cdot)$ such that

$$(\alpha_1, \alpha_1) = 6,\ (\alpha_1, \alpha_2) = -3,\ (\alpha_2, \alpha_2) = 2.$$

Therefore α_1 is the long simple root and α_2 is the short simple root. The set of long positive roots is

$$\{\alpha_1,\ \alpha_1+3\alpha_2,\ 2\alpha_1+3\alpha_2\}.$$

The set of short positive roots is

$$\{\alpha_2,\ \alpha_1+\alpha_2,\ \alpha_1+2\alpha_2\}.$$

Denote by Δ the root system of $\mathfrak{g}$, and by Δ_+ the set of positive roots. Let W denote the Weyl group with simple reflections s_i $(i \in I)$. Denote by $\mathfrak{g}_\alpha$ $(\alpha \in \Delta)$ the corresponding root space, and for each $\alpha \in \Delta_+$ fix nonzero elements $e_\alpha \in \mathfrak{g}_\alpha$, $f_\alpha \in \mathfrak{g}_{-\alpha}$ and $\alpha^\vee \in \mathfrak{h}$ such that

$$[e_\alpha, f_\alpha] = \alpha^\vee, \quad [\alpha^\vee, e_\alpha] = 2e_\alpha, \quad [\alpha^\vee, f_\alpha] = -2f_\alpha.$$

We also use the notation $e_i = e_{\alpha_i}$, $f_i = f_{\alpha_i}$ for $i \in I$, and $e_{-\alpha} = f_\alpha$ for $\alpha \in \Delta_+$. Set $\mathfrak{n}_\pm = \bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_{\pm\alpha}$.

Let ω_i $(i \in I)$ be the fundamental weight. We have $\omega_1 = 2\alpha_1 + 3\alpha_2$, $\omega_2 = \alpha_1 + 2\alpha_2$. Let P be the weight lattice, and

$$P_+ = \sum_{i\in I} \mathbb{Z}_+\omega_i \subseteq P, \quad Q_+ = \sum_{i\in I} \mathbb{Z}_+\alpha_i \subseteq P.$$

Note that P coincides with the root lattice $\sum_{i\in I}\mathbb{Z}\alpha_i$, but $P_+ \neq Q_+$. We write $\lambda \leq \mu$ for $\lambda, \mu \in P$ if $\mu - \lambda \in Q_+$. For $\lambda \in P_+$, denote by $V(\lambda)$ the simple $\mathfrak{g}$-module with highest weight λ.

Let $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$ be the affine Kac-Moody Lie algebra associated with $\mathfrak{g}$, where K is the canonical central element and d is the degree operator. Let $\widehat{I} = \{0, 1, 2\}$, and

$$e_0 = f_{2\alpha_1+3\alpha_2} \otimes t, \quad f_0 = e_{2\alpha_1+3\alpha_2} \otimes t^{-1}.$$

In this paper, we put $\widehat{\ }$ to denote the objects associated with $\widehat{\mathfrak{g}}$. For example, $\widehat{P}$ and $\widehat{Q}$ denote the weight and root lattices of $\widehat{\mathfrak{g}}$ respectively, and so on. Let $\delta \in \widehat{P}$ be the null root, and denote by $\Lambda_0 \in \widehat{P}_+$ the unique dominant integral weight of $\widehat{\mathfrak{g}}$ satisfying

$$\langle \alpha_i^\vee, \Lambda_0\rangle = 0 \text{ for } i \in I, \quad \langle K, \Lambda_0\rangle = 1, \quad \langle d, \Lambda_0\rangle = 0.$$

Let $\mathbf{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ and $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$ be the loop algebra and the current algebra associated with $\mathfrak{g}$ respectively, whose Lie algebra structures are given by

$$[x \otimes f(t), y \otimes g(t)] = [x, y] \otimes f(t)g(t).$$

Note that $\mathfrak{g}[t]$ is naturally considered as a Lie subalgebra of $\widehat{\mathfrak{g}}$.

The quantum loop algebra $U_q(\mathbf{L}\mathfrak{g})$ in Drinfeld's new realization is a $\mathbb{C}(q)$-algebra generated by $x_{i,n}^\pm$ $(i \in I, n \in \mathbb{Z})$, $k_i^{\pm 1}$ $(i \in I)$, $h_{i,n}$ $(i \in I, n \in \mathbb{Z}\setminus\{0\})$, subject to certain relations, see [9]. Denote by $U_q(\mathfrak{g})$ the subalgebra of $U_q(\mathbf{L}\mathfrak{g})$ generated by $x_{i,0}^\pm$ $(i \in I)$, $k_i^{\pm1}$ $(i \in I)$, which is isomorphic to the quantized enveloping algebra associated with $\mathfrak{g}$. For $\lambda \in P_+$, let $V_q(\lambda)$ denote the finite-dimensional simple $U_q(\mathfrak{g})$-module of type 1 with highest weight λ.

Simple $U_q(\mathbf{L}\mathfrak{g})$-modules are parametrized by dominant monomials in $\mathbb{Z}[Y_{i,a}^{\pm1}]_{i\in I, a\in\mathbb{C}(q)^\times}$, where $Y_{i,a}^{\pm1}$'s are formal variables, and a monomial $m = \prod_{i\in I, a\in\mathbb{C}(q)^\times} Y_{i,a}^{u_{i,a}}$ is dominant if $u_{i,a} \geq 0$ for all i and a (see [8], or [11] for the present formulation). For a dominant monomial m, denote by $L_q(m)$ the corresponding simple $U_q(\mathbf{L}\mathfrak{g})$-module. Let $\mathcal{P}_+$ be the monoid generated by $\{Y_{i,a} | i \in I, a \in \mathbb{C}^\times q^{\mathbb{Z}}\}$.

Let $\lambda = k\omega_1 + l\omega_2$, $k, l \in \mathbb{Z}_+$. A simple $U_q(\mathbf{L}\mathfrak{g})$-module $L_q(m)$ is a *minimal affinization* of $V_q(\lambda)$ if and only if m is one of the following monomials

$$\left(\prod_{i=0}^{k-1} Y_{1,aq^{6i}}\right)\left(\prod_{i=0}^{l-1} Y_{2,aq^{6k+2i+1}}\right), \qquad \left(\prod_{i=0}^{l-1} Y_{2,aq^{2i}}\right)\left(\prod_{i=0}^{k-1} Y_{1,aq^{2l+6i+5}}\right), \tag{2.1}$$

for some $a \in \mathbb{C}(q)^\times$, see [7].

3 Main Results

The aim of this paper is to study the graded limits of minimal affinizations in type G_2. So let us recall briefly the definition of the graded limits (see [23] for example, for more details).

Let $\lambda = k\omega_1 + l\omega_2$, and m be one of the monomials in Eq. 2.1. Without loss of generality, we may assume that $a \in \mathbb{C}^\times$. Let $\mathbf{A} = \mathbb{C}[q, q^{-1}]$, $U_\mathbf{A}(\mathbf{L}\mathfrak{g})$ be the $\mathbf{A}$-lattice of $U_q(\mathbf{L}\mathfrak{g})$ (see [6]), and $L_\mathbf{A}(m) = U_\mathbf{A}(\mathbf{L}\mathfrak{g})v_m$ where v_m is a highest ℓ-weight vector of $L_q(m)$. Then

$$\overline{L_q(m)} = L_\mathbf{A}(m) \otimes_\mathbf{A} \mathbb{C}$$

becomes a finite-dimensional $\mathbf{L}\mathfrak{g}$-module called the *classical limit* of $L_q(m)$, where we identify $\mathbb{C}$ with $\mathbf{A}/\langle q-1\rangle$. Define a Lie algebra automorphism $\varphi_a \colon \mathfrak{g}[t] \to \mathfrak{g}[t]$ by

$$\varphi_a\big(x \otimes f(t)\big) = x \otimes f(t-a) \quad \text{for } x \in \mathfrak{g},\ f \in \mathbb{C}[t].$$

Now we consider $\overline{L_q(m)}$ as a $\mathfrak{g}[t]$-module by restriction, and define a $\mathfrak{g}[t]$-module $L(m)$ by the pull-back $\varphi_a^*\big(\overline{L_q(m)}\big)$. We call $L(m)$ the *graded limit* of $L_q(m)$. By the construction we have for every $\mu \in P_+$ that

$$\Big[L_q(m) : V_q(\mu)\Big] = \Big[L(m) : V(\mu)\Big], \tag{3.1}$$

where the left- and right-hand sides are the multiplicities as a $U_q(\mathfrak{g})$-module and $\mathfrak{g}$-module, respectively.

Now we shall state our first main theorem, which gives isomorphisms between $L(m)$ and other two $\mathfrak{g}[t]$-modules. Let $M(\lambda)$ be the $\mathfrak{g}[t]$-module generated by a nonzero vector v_M with relations

$$\begin{aligned} &\mathfrak{n}_+[t]v_M = 0, \quad (h \otimes t^k)v_M = \delta_{k0}\langle h, \lambda\rangle v_M \text{ for } h \in \mathfrak{h}, \quad f_i^{\langle \alpha_i^\vee, \lambda\rangle + 1} v_M = 0 \text{ for } i \in I, \\ &(f_{\alpha_1} \otimes t)v_M = 0, \quad (f_{\alpha_2} \otimes t)v_M = 0, \quad (f_{\alpha_1+\alpha_2} \otimes t)v_M = 0. \end{aligned} \tag{3.2}$$

The other $\mathfrak{g}[t]$-module is a multiple generalization of a Demazure module defined as follows. Let $\xi_1, \dots, \xi_p$ be a sequence of elements of $\widehat{P}$, and assume for each $1 \le i \le p$ that there exists $\Lambda^i \in \widehat{P}_+$ such that ξ_i belongs to the affine Weyl group orbit $\widehat{W}\Lambda^i$ of Λ^i. Let $\widehat{V}(\Lambda^i)$ denote the simple highest weight $\widehat{\mathfrak{g}}$-module with highest weight Λ^i, and $v_{\xi_i} \in \widehat{V}(\Lambda^i)_{\xi_i}$ be an extremal weight vector with weight ξ_i. We define a $\widehat{\mathfrak{b}}$-module $D(\xi_1, \dots, \xi_p)$ by

$$D(\xi_1, \dots, \xi_p) = U(\widehat{\mathfrak{b}})(v_{\xi_1} \otimes \cdots \otimes v_{\xi_p}) \subseteq \widehat{V}(\Lambda^1) \otimes \cdots \otimes \widehat{V}(\Lambda^p). \tag{3.3}$$

Here $\widehat{\mathfrak{b}} = \mathfrak{b} \oplus \mathbb{C}K \oplus \mathbb{C}d \oplus t\mathfrak{g}[t]$ is the standard Borel subalgebra of $\widehat{\mathfrak{g}}$.

Remark 3.1 For any $c_1, \dots, c_p \in \mathbb{Z}$, it obviously holds that

$$D(\xi_1 + c_1\delta, \dots, \xi_p + c_p\delta) \cong D(\xi_1, \dots, \xi_p)$$

as $\big(\mathfrak{b} \oplus t\mathfrak{g}[t]\big)$-modules.

Now write $l = 3r + s$ with $r \in \mathbb{Z}_+$, $s \in \{0, 1, 2\}$, and set

$$T(\lambda) = \begin{cases} D\big(k(-\omega_1 + \Lambda_0), r(-3\omega_2 + \Lambda_0)\big) & \text{if } s = 0, \\ D\big(k(-\omega_1 + \Lambda_0), r(-3\omega_2 + \Lambda_0), -s\omega_2 + \Lambda_0\big) & \text{otherwise.} \end{cases}$$

Note that $T(\lambda)$ is extended to a module over $\mathfrak{g}[t] \oplus \mathfrak{K} \oplus \mathfrak{d}$, and as a $\mathfrak{g}[t]$-module $T(\lambda)$ is generated by the one-dimensional weight space $T(\lambda)_\lambda$.

Our first main theorem is the following.

Theorem 3.2 *As a* $\mathfrak{g}[t]$*-module, we have*

$$M(\lambda) \cong L(m) \cong T(\lambda).$$

The second main theorem gives a multiplicity formula for $L(m)$ as a $\mathfrak{g}$-module. For $\lambda = k\omega_1 + l\omega_2$, define a subset $S_\lambda \subseteq \mathbb{Z}_+^5$ by

$$S_\lambda = \Big\{(a_1, \dots, a_5) \;\Big|\; a_1 \le k,\ a_1 - a_3 + a_5 \le k,\ 2a_2 + 3a_3 + 3a_4 \le l,\ 2a_2 + 3a_4 + 3a_5 \le l\Big\}.$$

Theorem 3.3 *As a* $\mathfrak{g}$*-module,*

$$L(m) \cong \bigoplus_{(a_1,\dots,a_5)\in S_\lambda} V\big((k - a_1 + a_3 + a_4 - a_5)\omega_1 + (l - a_2 - 3a_3 - 3a_4)\omega_2\big).$$

By Eq. 3.1, we immediately obtain the following corollary.

Corollary 3.4 *As a* $U_q(\mathfrak{g})$*-module,*

$$L_q(m) \cong \bigoplus_{(a_1,\dots,a_5)\in S_\lambda} V_q\big((k - a_1 + a_3 + a_4 - a_5)\omega_1 + (l - a_2 - 3a_3 - 3a_4)\omega_2\big).$$

From Theorem 3.2, we also obtain the following formula for the limit of the (normalized) characters of minimal affinizations.

Corollary 3.5 *Let J be a subset of I, and suppose that $\lambda_1, \lambda_2, \dots$ is an infinite sequence of elements of P_+ such that*

$$\lim_{n\to\infty} \langle \alpha_i^\vee, \lambda_n\rangle = \infty \textit{ for all } i \in J \textit{ and } \langle \alpha_i^\vee, \lambda_n\rangle = 0 \textit{ for all } i \notin J,\ n \in \mathbb{Z}_{>0}.$$

Let $m_1, m_2, \dots$ be an infinite sequence of elements of $\mathcal{P}_+$ such that $L_q(m_n)$ is a minimal affinization of $V_q(\lambda_n)$. Then $\lim_{n\to\infty} e^{-\lambda_n} \mathrm{ch}\, L_q(m_n)$ *exists, and*

$$\lim_{n\to\infty} e^{-\lambda_n} \mathrm{ch}\, L_q(m_n) = \prod_{\alpha\in\Delta_+} \left(\frac{1}{1 - e^{-\alpha}}\right)^{\max_{j\in J}\langle\omega_j^\vee, \alpha\rangle}.$$

Proof This result follows from Theorem 3.2, and the proof is the same as one given in [23, Corollary 4.13]. □

This corollary, together with [21, Corollary 5.6], yields the character formula of the least affinizations of generic parabolic Verma modules of type G_2 conjectured by Mukhin and Young [21, Conjecture 6.3].

4 Proof of Theorem 3.2

Throughout the rest of this paper, we fix $\lambda = k\omega_1 + l\omega_2 \in P_+$ and set $r \in \mathbb{Z}_+$ and $s \in \{0, 1, 2\}$ to be such that $l = 3r + s$. Let m be one of the monomials in Eq. 2.1 with $a \in \mathbb{C}^\times$. In this section, we shall prove one by one the existence of three surjective homomorphisms

$$M(\lambda) \twoheadrightarrow L(m), \quad L(m) \twoheadrightarrow T(\lambda), \quad T(\lambda) \twoheadrightarrow M(\lambda),$$

which completes the proof of Theorem 3.2.

4.1 Proof of $M(\lambda) \twoheadrightarrow L(m)$

Let v_m be a highest ℓ-weight vector of $L_q(m)$, and $W = U_q(\mathfrak{g})v_m \subseteq L_q(m)$ the simple $U_q(\mathfrak{g})$-submodule generated by v_m. It follows from [1, Proposition 5.5] that $\bigoplus_{\mu \geq \lambda-\alpha_1-\alpha_2} L_q(m)_\mu \subseteq W$, where $L_q(m)_\mu$ denotes the weight space with weight μ. Hence we have

$$x^-_{\alpha_1,1}v_m \in W, \quad x^-_{\alpha_2,1}v_m \in W, \quad [x^-_{\alpha_1,1}, x^-_{\alpha_2,0}]v_m \in W.$$

Then it is proved from the definition of the graded limit that the vector $\bar{v}_m = v_m \otimes_{\mathbf{A}} 1 \in L(m)$ satisfies

$$(f_{\alpha_1} \otimes t)\bar{v}_m = (f_{\alpha_2} \otimes t)\bar{v}_m = (f_{\alpha_1+\alpha_2} \otimes t)\bar{v}_m = 0$$

(see [23, Subsection 4.1]). The other relations in Eq. 3.2 are easily checked from the construction. Hence $M(\lambda) \twoheadrightarrow L(m)$ follows.

4.2 Proof of $L(m) \twoheadrightarrow T(\lambda)$

Here we only consider the case where the monomial m is of the form $\prod_{i=0}^{k-1} Y_{1,aq^{6i}} \cdot \prod_{i=0}^{l-1} Y_{2,aq^{6k+2i+1}}$. The proof of the other case is similar.

Set

$$m_1 = \prod_{i=0}^{k-1} Y_{1,aq^{6i}}, \quad m_2 = \prod_{i=0}^{l-1} Y_{2,aq^{6k+2i+1}}.$$

By [3, Theorem 5.1] (or more precisely, the dualized statement of it), there exists an injective homomorphism

$$L_q(m) \hookrightarrow L_q(m_1) \otimes L_q(m_2)$$

mapping a highest ℓ-weight vector to the tensor product of highest ℓ-weight vectors. Then by the definition of graded limits, we obtain a $\mathfrak{g}[t]$-module homomorphism

$$L(m) \to L(m_1) \otimes L(m_2)$$

mapping a highest weight vector to the tensor product of highest weight vectors. Now the existence of a surjection $L(m) \twoheadrightarrow T(\lambda)$ is proved from the following lemma.

Lemma 4.1 *(i) $L(m_1)$ is isomorphic to $D\big(k(-\omega_1 + \Lambda_0)\big)$ as a $\mathfrak{g}[t]$-module.*
(ii) $L(m_2)$ is isomorphic to $D\big(r(-3\omega_2 + \Lambda_0)\big)$ (resp. $D\big(r(-3\omega_2 + \Lambda_0), -s\omega_2 + \Lambda_0\big)$) if $s = 0$ (resp. $s = 1, 2$) as a $\mathfrak{g}[t]$-module.

Proof The graded limit $L(m_1)$ is isomorphic to the Kirillov-Reshetikhin module $KR(k\omega_1)$ for $\mathfrak{g}[t]$ defined in [4, 5], which is proved from the facts that there exists a surjection $KR(k\omega_1) \twoheadrightarrow L(m_1)$ (see Section 4.1) and the characters of two modules are the same (see [5, 12, 14]). Hence the assertion (i) follows from [10, Theorem 4]. Similarly $L(m_2)$

is isomorphic to $KR(l\omega_2)$, and hence by [5, Corollary 2.3] it is isomorphic to the $\mathfrak{g}[t]$-submodule of $KR(3r\omega_2) \otimes KR(s\omega_2)$ generated by the tensor product of highest weight vectors. Now $KR(3r\omega_2) \cong D\big(r(-3\omega_2 + \Lambda_0)\big)$ follows from [10, Theorem 4], and $KR(s\omega_2) \cong D(-s\omega_2 + \Lambda_0)$ is verified by the Demazure character formula (see [10]). Hence the assertion (ii) is proved. □

4.3 Proof of $T(\lambda) \twoheadrightarrow M(\lambda)$

First we introduce the following notation, as in [23, 24]. Assume that V is a $\widehat{\mathfrak{g}}$-module and D is a $\widehat{\mathfrak{b}}$-submodule of V. For $i \in \widehat{I}$ let $\widehat{\mathfrak{p}}_i$ denote the parabolic subalgebra $\widehat{\mathfrak{b}} \oplus \mathfrak{f}_i \subseteq \widehat{\mathfrak{g}}$, and set $F_i D = U(\widehat{\mathfrak{p}}_i)D \subseteq V$ to be the $\widehat{\mathfrak{p}}_i$-submodule generated by D. It is easily seen that, if $\xi_1, \dots, \xi_p \in \widehat{W}(\widehat{P}_+)$ satisfy $\langle \alpha_i^\vee, \xi_j \rangle \geq 0$ for all $1 \leq j \leq p$, then

$$F_i D(\xi_1, \dots, \xi_p) = D(s_i\xi_1, \dots, s_i\xi_p) \tag{4.1}$$

(see [23, Lemma 2.4]).

Let $\widehat{\Delta}^{\mathrm{re}} = \Delta + \mathbb{Z}\delta$ be the set of real roots of $\widehat{\mathfrak{g}}$, and $\widehat{\Delta}_+^{\mathrm{re}} = \Delta_+ \sqcup (\Delta + \mathbb{Z}_{>0}\delta)$ the set of positive real roots. For $\gamma = \alpha + p\delta \in \widehat{\Delta}^{\mathrm{re}}$, set

$$\gamma^\vee = \alpha^\vee + \frac{6p}{(\alpha, \alpha)} K,$$

and define a number $\rho(\gamma)$ by

$$\rho(\gamma) = \max\big\{0, -\langle \gamma^\vee, k(\omega_1 + \Lambda_0)\rangle\big\} + \max\big\{0, -\langle \gamma^\vee, r(3\omega_2 + \Lambda_0)\rangle\big\} + \max\big\{0, -\langle \gamma^\vee, s\omega_2 + \Lambda_0\rangle\big\}.$$

The explicit values of $\rho(\gamma)$ for $\gamma \in \widehat{\Delta}_+^{\mathrm{re}}$ are given as follows:

$$\begin{aligned}
\rho\big(-(\alpha_1 + 2\alpha_2) + \delta\big) &= 3r + \delta_{s2}, \\
\rho\big(-(\alpha_1 + 3\alpha_2) + \delta\big) &= 2r + \delta_{s2}, \\
\rho\big(-(2\alpha_1 + 3\alpha_2) + \delta\big) &= k + 2r + \delta_{s2}, \\
\rho\big(-(\alpha_1 + 3\alpha_2) + 2\delta\big) &= \rho\big(-(2\alpha_1 + 3\alpha_2) + 2\delta\big) = r,
\end{aligned}$$

and $\rho(\gamma) = 0$ for all the other $\gamma \in \widehat{\Delta}_+^{\mathrm{re}}$. Here δ_{s2} denotes the Kronecker's delta. For $\alpha + p\delta \in \widehat{\Delta}^{\mathrm{re}}$ set $x_{\alpha+p\delta} = e_\alpha \otimes t^p$.

Recall that v_ξ denotes an extremal weight vector in $\widehat{V}(\Lambda)$ with weight ξ, where $\Lambda \in \widehat{P}_+$ is the element satisfying $\xi \in \widehat{W}\Lambda$. Let $v_T \in T(\lambda)$ be the tensor product of the extremal weight vectors:

$$v_T = \begin{cases} v_{k(\omega_1+\Lambda_0)} \otimes v_{r(3\omega_2+\Lambda_0)} & s = 0, \\ v_{k(\omega_1+\Lambda_0)} \otimes v_{r(3\omega_2+\Lambda_0)} \otimes v_{s\omega_2+\Lambda_0} & s = 1, 2. \end{cases}$$

Note that $T(\lambda)$ is generated by v_T as a $\mathfrak{g}[t]$-module. Throughout the rest of this paper, we will abbreviate $X \otimes t^p$ as Xt^p to shorten the notation.

Lemma 4.2 *We have*

$$\mathrm{Ann}_{U(\widehat{\mathfrak{n}}_+)}(v_T) = U(\widehat{\mathfrak{n}}_+)\Big(\bigoplus_{\gamma \in \widehat{\Delta}_+^{\mathrm{re}}} x_\gamma^{\rho(\gamma)+1} + f_{\alpha_1+3\alpha_2}t^2(f_{\alpha_1+2\alpha_2}t)^{3r-2} + t\mathfrak{h}[t]\Big),$$

where $f_{\alpha_1+3\alpha_2}t^2(f_{\alpha_1+2\alpha_2}t)^{3r-2}$ *is omitted if* $r = 0$.

Proof First assume that $s = 0$, and set $\Lambda = r(-2\omega_1 + 3\omega_2 + \Lambda_0)$. Note that

$$F_0 D(k\Lambda_0, \Lambda) \cong D\big(k(\omega_1 + \Lambda_0), r(3\omega_2 + \Lambda_0)\big)\big(= U(\widehat{\mathfrak{b}})v_T\big)$$

holds by Eq. 4.1, and we have

$$\mathrm{Ann}_{U(\widehat{\mathfrak{n}}_+)}(v_{k\Lambda_0}\otimes v_\Lambda)=\mathrm{Ann}_{U(\widehat{\mathfrak{n}}_+)}(v_\Lambda)$$

since $\widehat{\mathfrak{n}}_+$ acts trivially on $v_{k\Lambda_0}$. We shall check that $D(k\Lambda_0,\Lambda)$ satisfies the conditions (i) – (iii) (for T) in [23, Lemma 5.3]. Note that the condition (iii) holds by [15, Theorem 5]. By [17, Lemma 26], we have

$$\begin{aligned}\mathrm{Ann}_{U(\widehat{\mathfrak{n}}_+)}(v_\Lambda) &= U(\widehat{\mathfrak{n}}_+)\Big(\bigoplus_{\gamma\in\widehat{\Delta}_+^{\mathrm{re}}}\mathfrak{x}_\gamma^{\max\{0,-\Lambda(\gamma^\vee)\}+1}+t\mathfrak{h}[t]\Big)\\ &= U(\widehat{\mathfrak{n}}_+)e_0+U(\widehat{\mathfrak{n}}_+)\Big(\bigoplus_{\gamma\in\widehat{\Delta}_+^{\mathrm{re}}\setminus\{\alpha_0\}}\mathfrak{x}_\gamma^{\max\{0,-\Lambda(\gamma^\vee)\}+1}+t\mathfrak{h}[t]\Big).\end{aligned}$$

It follows that

$$\max\{0,-\Lambda(\gamma^\vee)\}=\begin{cases}3r & \gamma=\alpha_1+\alpha_2,\\ 2r & \gamma=\alpha_1,\\ r & \gamma=\alpha_1+\delta \text{ or } 2\alpha_1+3\alpha_2,\\ 0 & \text{otherwise.}\end{cases}$$

Let $\widehat{\mathfrak{n}}_0$ be the Lie subalgebra $\bigoplus_{\gamma\in\widehat{\Delta}_+^{\mathrm{re}}\setminus\{\alpha_0\}}\mathfrak{x}_\gamma\oplus t\mathfrak{h}[t]$ of $\widehat{\mathfrak{n}}_+$, and define a left $U(\widehat{\mathfrak{n}}_0)$-ideal $\mathcal{I}$ by

$$\mathcal{I}=U(\widehat{\mathfrak{n}}_0)\Big(\bigoplus_{\gamma\in\widehat{\Delta}_+^{\mathrm{re}}\setminus\{\alpha_0\}}\mathfrak{x}_\gamma^{\max\{0,-\Lambda(\gamma^\vee)\}+1}+\mathbb{C}e_{\alpha_1}te_{\alpha_1+\alpha_2}^{3r-2}+t\mathfrak{h}[t]\Big).$$

It is directly checked for every $p\in\mathbb{Z}_+$ that

$$\mathrm{ad}(e_0)(e_{\alpha_1+\alpha_2}^p)\in\mathbb{C}^\times e_{\alpha_1+\alpha_2}^{p-1}f_{\alpha_1+2\alpha_2}t+\mathbb{C}^\times e_{\alpha_1+\alpha_2}^{p-2}f_{\alpha_2}t+\mathbb{C}^\times e_{\alpha_1+\alpha_2}^{p-3}e_{\alpha_1}t,$$

where we set $e_{\alpha_1+\alpha_2}^q=0$ if $q<0$. Using this we see that $\mathcal{I}$ is $\mathrm{ad}(e_0)$-invariant, and

$$\mathrm{Ann}_{U(\widehat{\mathfrak{n}}_+)}(v_\Lambda)=U(\widehat{\mathfrak{n}}_+)e_0+U(\widehat{\mathfrak{n}}_+)\mathcal{I}.$$

Now the assertion (for $s=0$) follows by [23, Lemma 5.3].

The case $s=1$ is easily proved from the case $s=0$ since $\widehat{\mathfrak{n}}_+$ acts trivially on $v_{\omega_2+\Lambda_0}$ and hence

$$\mathrm{Ann}_{U(\widehat{\mathfrak{n}}_+)}\big(v_{k(\omega_1+\Lambda_0)}\otimes v_{r(3\omega_2+\Lambda_0)}\otimes v_{\omega_2+\Lambda_0}\big)=\mathrm{Ann}_{U(\widehat{\mathfrak{n}}_+)}\big(v_{k(\omega_1+\Lambda_0)}\otimes v_{r(3\omega_2+\Lambda_0)}\big).$$

For the case $s=2$, notice by Eq. 4.1 that

$$D\big(r(3\omega_2+\Lambda_0),2\omega_2+\Lambda_0\big)\cong F_0F_1F_2F_1F_0D(r\Lambda_0,\omega_2+\Lambda_0).$$

Then this is isomorphic to

$$F_0F_1F_2F_1F_0D\big(\omega_2+(r+1)\Lambda_0\big)\cong D\big((3r+2)\omega_2+(r+1)\Lambda_0\big)$$

since the $\widehat{\mathfrak{g}}$-submodule of $\widehat{V}(r\Lambda_0)\otimes\widehat{V}(\omega_2+\Lambda_0)$ generated by the tensor product of highest weight vectors is isomorphic to $\widehat{V}\big(\omega_2+(r+1)\Lambda_0\big)$. Hence we have

$$D\big(k(\omega_1+\Lambda_0),r(3\omega_2+\Lambda_0),2\omega_2+\Lambda_0\big)\cong D\big(k(\omega_1+\Lambda_0),(3r+2)\omega_2+(r+1)\Lambda_0\big).$$

Using this isomorphism, the assertion for $s=2$ is proved in almost the same way with the case $s=0$. □

Now Lemma 4.2 and the following proposition yield a $(\mathfrak{h}\oplus\widehat{\mathfrak{n}}_+)$-module homomorphism from $U(\mathfrak{h}\oplus\widehat{\mathfrak{n}}_+)v_T$ to $M(\lambda)$ sending v_T to v_M since their weights are both λ, and then the existence of a surjection $T(\lambda)\twoheadrightarrow M(\lambda)$ is proved by the same argument with [23, two paragraphs below Lemma 5.2].

Proposition 4.3 *The vector $v_M \in M(\lambda)$ satisfies the relations*

$$x_\gamma^{\rho(\gamma)+1} v_M = 0 \text{ for } \gamma \in \widehat{\Delta}_+^{\mathrm{re}}, \quad t\mathfrak{h}[t]v_M = 0, \quad f_{\alpha_1+3\alpha_2} t^2 (f_{\alpha_1+2\alpha_2} t)^{3r-2} v_M = 0,$$

where the last one is omitted when $r = 0$.

The rest of this subsection is devoted to prove Proposition 4.3. For simplicity *we assume that $s = 0$ in the rest of this subsection*, and prove the proposition only in this case. The proof of the other cases are almost the same. Note that the relations $x_\gamma v_M = 0$ for

$$\gamma \notin \{-(\alpha_1+2\alpha_2)+\delta, -(\alpha_1+3\alpha_2)+\delta, -(2\alpha_1+3\alpha_2)+\delta, -(\alpha_1+3\alpha_2)+2\delta, -(2\alpha_1+3\alpha_2)+2\delta\}$$

and $t\mathfrak{h}[t]v_M = 0$ are easily proved from the definition. For example when $\gamma = -(\alpha_1 + 2\alpha_2) + 2\delta$, $x_\gamma v_M = 0$ follows since $[x_{-(\alpha_1+\alpha_2)+\delta}, x_{-\alpha_2+\delta}]v_M = 0$.

For computational convenience, we assume from now on that the root vectors are normalized so that

$$\left[e_{\alpha_2}, f_{\alpha_1+3\alpha_2}\right] = f_{\alpha_1+2\alpha_2}, \ \left[e_{\alpha_2}, f_{\alpha_1+2\alpha_2}\right] = f_{\alpha_1+\alpha_2}, \ \left[e_{\alpha_2}, f_{\alpha_1+\alpha_2}\right] = f_{\alpha_1},$$
$$\left[f_{\alpha_1+\alpha_2}, f_{\alpha_1+2\alpha_2}\right] = 6 f_{2\alpha_1+3\alpha_2}.$$

For an element X in an algebra and $p \in \mathbb{Z}_+$ denote by $X^{(p)}$ the divided power $X^p/p!$, and set $X^{(p)} = 0$ if $p < 0$.

Lemma 4.4 *(i) For $q \in \mathbb{Z}_+$, we have*

$$e_{\alpha_2} f_{\alpha_1+2\alpha_2}^{(q)} \equiv 3 f_{2\alpha_1+3\alpha_2} f_{\alpha_1+2\alpha_2}^{(q-2)} \mod U(\mathfrak{g})\big(e_{\alpha_2} \oplus f_{\alpha_1} \oplus f_{\alpha_1+\alpha_2}\big).$$

(ii) For $p, q \in \mathbb{Z}_+$, we have

$$e_{\alpha_2}^{(p)} f_{\alpha_1+3\alpha_2}^{(q)} \equiv \sum_i f_{2\alpha_1+3\alpha_2}^{(i)} f_{\alpha_1+3\alpha_2}^{(q-p+i)} f_{\alpha_1+2\alpha_2}^{(p-3i)} \mod U(\mathfrak{g})\big(e_{\alpha_2} \oplus f_{\alpha_1} \oplus f_{\alpha_1+\alpha_2}\big),$$

where i runs over the set of integers such that $\max\{0, p-q\} \le i \le p/3$.

Proof We have

$$e_{\alpha_2} f_{\alpha_1+2\alpha_2}^{(q)} \equiv \frac{1}{q!} \sum_{i=1}^{q} f_{\alpha_1+2\alpha_2}^{i-1} f_{\alpha_1+\alpha_2} f_{\alpha_1+2\alpha_2}^{q-i} \equiv \frac{1}{q!} \sum_{i=1}^{q} 6(q-i) f_{2\alpha_1+3\alpha_2} f_{\alpha_1+2\alpha_2}^{q-2}$$
$$= \frac{1}{q!} \cdot 3q(q-1) f_{2\alpha_1+3\alpha_2} f_{\alpha_1+2\alpha_2}^{q-2} = 3 f_{2\alpha_1+3\alpha_2} f_{\alpha_1+2\alpha_2}^{(q-2)},$$

and the assertion (i) holds. The assertion (ii) with $p = 1$ is immediate. Then we have by induction and (i) that

$$(p+1) e_{\alpha_2}^{(p+1)} f_{\alpha_1+3\alpha_2}^{(q)} \equiv e_{\alpha_2} \sum_i f_{2\alpha_1+3\alpha_2}^{(i)} f_{\alpha_1+3\alpha_2}^{(q-p+i)} f_{\alpha_1+2\alpha_2}^{(p-3i)}$$
$$\equiv \sum_i f_{2\alpha_1+3\alpha_2}^{(i)} \Big(f_{\alpha_1+3\alpha_2}^{(q-p+i-1)} f_{\alpha_1+2\alpha_2} f_{\alpha_1+2\alpha_2}^{(p-3i)} + 3 f_{2\alpha_1+3\alpha_2} f_{\alpha_1+3\alpha_2}^{(q-p+i)} f_{\alpha_1+2\alpha_2}^{(p-3i-2)} \Big)$$
$$= \sum_i (p-3i+1) f_{2\alpha_1+3\alpha_2}^{(i)} f_{\alpha_1+3\alpha_2}^{(q-p+i-1)} f_{\alpha_1+2\alpha_2}^{(p-3i+1)} + \sum_i 3(i+1) f_{2\alpha_1+3\alpha_2}^{(i+1)} f_{\alpha_1+3\alpha_2}^{(q-p+i)} f_{\alpha_1+2\alpha_2}^{(p-3i-2)}$$
$$= (p+1) \sum_i f_{2\alpha_1+3\alpha_2}^{(i)} f_{\alpha_1+3\alpha_2}^{(q-p+i-1)} f_{\alpha_1+2\alpha_2}^{(p-3i+1)}.$$

Hence the assertion (ii) holds. □

By Lemma 4.4 (ii), we also see that

$$e_{\alpha_2}^{(p)}(f_{\alpha_1+3\alpha_2}t)^{(q)} \equiv \sum_{i=\max\{0,p-q\}}^{\lfloor p/3\rfloor} (f_{2\alpha_1+3\alpha_2}t^2)^{(i)}(f_{\alpha_1+3\alpha_2}\quad t)^{(q-p+i)}(f_{\alpha_1+2\alpha_2}t)^{(p-3i)} \tag{4.2}$$

$$\mod \quad U(\mathfrak{g})\big(\mathfrak{e}_{\alpha_2} \oplus \mathfrak{f}_{\alpha_1}t \oplus \mathfrak{f}_{\alpha_1+\alpha_2}t\big).$$

Lemma 4.5 *The relations* $(f_{\alpha_1+3\alpha_2}t)^{2r+1}v_M = 0$ *and* $(f_{2\alpha_1+3\alpha_2}t)^{k+2r+1}v_M = 0$ *hold.*

Proof We have

$$\big\langle \alpha_2^\vee, \mathrm{wt}\big((f_{\alpha_1+3\alpha_2}t)^{2r+1}v_M\big)\big\rangle = \langle \alpha_2^\vee, \lambda - (2r+1)(\alpha_1+3\alpha_2)\rangle = -(3r+3).$$

On the other hand, it follows from Eq. 4.2 that

$$e_{\alpha_2}^{3r+3}(f_{\alpha_1+3\alpha_2}t)^{2r+1}v_M = 0,$$

and hence we have $(f_{\alpha_1+3\alpha_2}t)^{2r+1}v_M = 0$ since $M(\lambda)$ is an integrable $\mathfrak{g}$-module. Now it is an elementary fact that this relation and $f_{\alpha_1}^{k+1}v_M = 0$ imply $(f_{2\alpha_1+3\alpha_2}t)^{k+2r+1}v_M = 0$ (for example, see [22, Lemma 4.5]). □

Lemma 4.6 *The relations* $(f_{2\alpha_1+3\alpha_2}t^2)^{r+1}v_M = 0$ *and* $(f_{\alpha_1+3\alpha_2}t^2)^{r+1}v_M = 0$ *hold.*

Proof By Lemma 4.5 and Eq. 4.2, we have

$$0 = e_{\alpha_2}^{(3r+3)}(f_{\alpha_1+3\alpha_2}t)^{(2r+2)}v_M = (f_{2\alpha_1+3\alpha_2}t^2)^{(r+1)}v_M,$$

and hence $(f_{2\alpha_1+3\alpha_2}t^2)^{r+1}v_M = 0$ follows. From this we see that

$$0 = e_{\alpha_1}^{r+1}(f_{2\alpha_1+3\alpha_2}t^2)^{r+1}v_M = c(f_{\alpha_1+3\alpha_2}t^2)^{r+1}v_M$$

with some nonzero c. Hence $(f_{\alpha_1+3\alpha_2}t^2)^{r+1}v_M = 0$ also holds. □

Lemma 4.7 *The relation* $(f_{\alpha_1+2\alpha_2}t)^{3r+1}v_M = 0$ *holds.*

Proof By Lemma 4.5 and Eq. 4.2, we have for $p \geq 2r+1$ that

$$0 = e_{\alpha_2}^{(p)}(f_{\alpha_1+3\alpha_2}t)^{(p)}v_M = \sum_{i=0}^{\lfloor p/3\rfloor} \frac{1}{(p-3i)!}(f_{2\alpha_1+3\alpha_2}t^2)^{(i)}(f_{\alpha_1+3\alpha_2}t)^{(i)}(f_{\alpha_1+2\alpha_2}t)^{p-3i}v_M.$$

When $2r+1 \leq p \leq 3r+1$, by multiplying $(f_{\alpha_1+2\alpha_2}t)^{3r+1-p}$ to this we obtain r linear relations

$$\sum_{i=0}^{\lfloor p/3\rfloor} \frac{1}{(p-3i)!}(f_{2\alpha_1+3\alpha_2}t^2)^{(i)}(f_{\alpha_1+3\alpha_2}t)^{(i)}(f_{\alpha_1+2\alpha_2}t)^{3r+1-3i}v_M = 0.$$

Hence in order to prove $(f_{\alpha_1+2\alpha_2}t)^{3r+1}v_M = 0$, it is enough to show that the matrix $A = (a_{ij})_{0\leq i,j\leq r}$ with

$$a_{ij} = \begin{cases} \frac{1}{(3r+1-3i-j)!} & \text{if } 3r+1-3i-j \geq 0, \\ 0 & \text{otherwise} \end{cases}$$

is invertible. Assume that $v_0, v_1, \dots, v_r$ satisfy $\sum_i a_{ij} v_i = 0$ for all j, and consider the polynomial

$$f(x) = \frac{v_0}{(3r+1)!}x^{3r+1} + \frac{v_1}{(3r-2)!}x^{3r-2} + \cdots + \frac{v_i}{(3r+1-3i)!}x^{3r+1-3i} + \cdots + \frac{v_r}{1!}x.$$

Then $\frac{d^j f}{dx^j}(1) = 0$ holds for all $0 \le j \le r$, which implies that $f(x)$ is divisible by $(x-1)^{r+1}$. Since $f(\zeta x) = \zeta f(x)$ holds where ζ is a third primitive root of unity, we see that $f(x)$ is divisible by $(x^3-1)^{r+1}$. By the degree consideration we have $f(x) = 0$, and the proof is complete. □

Now the following lemma completes the proof of Proposition 4.3.

Lemma 4.8 *The relation* $f_{\alpha_1+3\alpha_2}t^2(f_{\alpha_1+2\alpha_2}t)^{3r-2}v_M = 0$ *holds when* $r \neq 0$.

Proof Let $p \geq 2r-1$. By Lemma 4.5, we have

$$\begin{aligned} 0 &= e_{\alpha_1+3\alpha_2}(f_{\alpha_1+3\alpha_2}t)^{(p+2)}v_M = \frac{1}{(p+2)!}\sum_{i=0}^{p+1}(f_{\alpha_1+3\alpha_2}t)^{p-i+1}(\alpha_1+3\alpha_2)^{\vee}t(f_{\alpha_1+3\alpha_2}t)^i v_M \\ &= \frac{1}{(p+2)!}\sum_{i=0}^{p+1} -2i(f_{\alpha_1+3\alpha_2}t)^p f_{\alpha_1+3\alpha_2}t^2 v_M = -(f_{\alpha_1+3\alpha_2}t)^{(p)} f_{\alpha_1+3\alpha_2}t^2 v_M. \end{aligned} \tag{4.3}$$

We easily see that all the elements e_{α_2}, $f_{\alpha_1}t$, $f_{\alpha_1+\alpha_2}t$ annihilate the vector $f_{\alpha_1+3\alpha_2}t^2 v_M$, and hence we have from Eqs. 4.2 and 4.3 that

$$\begin{aligned} 0 &= e_{\alpha_2}^{(p)}(f_{\alpha_1+3\alpha_2}t)^{(p)} f_{\alpha_1+3\alpha_2}t^2 v_M \\ &= \sum_{i=0}^{\lfloor p/3 \rfloor} \frac{1}{(p-3i)!} f_{\alpha_1+3\alpha_2}t^2(f_{2\alpha_1+3\alpha_2}t^2)^{(i)}(f_{\alpha_1+3\alpha_2}t)^{(i)}(f_{\alpha_1+2\alpha_2}t)^{p-3i}v_M. \end{aligned}$$

Now the lemma is proved by a similar argument as in the proof of Lemma 4.7. □

5 Proof of Theorem 3.3

5.1 A Basis of the Space of Highest Weight Vectors

For $a = (a_1, a_2, a_3, a_4, a_5) \in \mathbb{Z}_+^5$, set

$$\boldsymbol{f_a} = (f_{2\alpha_1+3\alpha_2}t^2)^{(a_5)}(f_{\alpha_1+3\alpha_2}t^2)^{(a_4)}(f_{\alpha_1+3\alpha_2}t)^{(a_3)}(f_{\alpha_1+2\alpha_2}t)^{(a_2)}(f_{2\alpha_1+3\alpha_2}t)^{(a_1)},$$

and

$$\begin{aligned} \mathrm{wt}(a) &= (2a_1+a_2+a_3+a_4+2a_5)\alpha_1 + (3a_1+2a_2+3a_3+3a_4+3a_5)\alpha_2 \\ &= (a_1-a_3-a_4+a_5)\omega_1 + (a_2+3a_3+3a_4)\omega_2 \in Q_+. \end{aligned}$$

Note that $\mathrm{wt}(\boldsymbol{f_a}) = -\mathrm{wt}(a)$. In this section, we denote by v a highest weight vector of $L(m)$. Since $L(m) \cong M(\lambda)$, we easily see from Proposition 4.3 and the PBW theorem that

$$L(m) = \sum_{a \in \mathbb{Z}_+^5} U(\mathfrak{g})\boldsymbol{f_a}v.$$

Let $\alpha \in Q_+$, and set $L(m)_{>\lambda-\alpha} = \bigoplus_{\mu>\lambda-\alpha} L(m)_\mu$. The $\mathfrak{g}$-submodule $U(\mathfrak{g})L(m)_{>\lambda-\alpha}$ of $L(m)$ coincides with the sum of simple $\mathfrak{g}$-submodules whose highest weights are larger than $\lambda-\alpha$. Hence we see that the multiplicity of $V(\lambda-\alpha)$ in $L(m)$ is equal to the dimension of the weight space of the quotient $\mathfrak{g}$-module $L(m)\Big/U(\mathfrak{g})L(m)_{>\lambda-\alpha}$ with weight $\lambda-\alpha$, that is

$$\Big[L(m) : V(\lambda-\alpha)\Big] = \dim\Big(L(m)\Big/U(\mathfrak{g})L(m)_{>\lambda-\alpha}\Big)_{\lambda-\alpha}.$$

Therefore, in order to prove Theorem 3.3 it suffices to show the following proposition, which is proved in the next subsections.

Proposition 5.1 *For every $\alpha \in Q_+$, the projection images of $\{\boldsymbol{f_a}v \mid a \in S_\lambda,\ \mathrm{wt}(a) = \alpha\}$ form a basis of $\Big(L(m)\Big/U(\mathfrak{g})L(m)_{>\lambda-\alpha}\Big)_{\lambda-\alpha}$.*

5.2 The Space is Spanned by the Vectors

For $\alpha \in Q_+$, set

$$\mathbb{Z}_+^5[\alpha] = \{a \in \mathbb{Z}_+^5 \mid \mathrm{wt}(a) = \alpha\}, \quad S_\lambda[\alpha] = S_\lambda \cap \mathbb{Z}_+^5[\alpha].$$

In this subsection, we shall show the following.

Lemma 5.2 *For every $\alpha \in Q_+$, the projection images of $\{\boldsymbol{f_a}v \mid a \in S_\lambda[\alpha]\}$ span the space $\Big(L(m)\Big/U(\mathfrak{g})L(m)_{>\lambda-\alpha}\Big)_{\lambda-\alpha}$.*

We denote by $\leq$ the lexicographic order on $\mathbb{Z}_+^5$, that is, $(a_1,\dots,a_5) < (b_1,\dots,b_5)$ if and only if there exists i such that $a_j = b_j$ for $j < i$ and $a_i < b_i$. Fix $\alpha \in Q_+$. Following [5, Subsection 3.5], we define a finite sequence $r_1,\dots,r_t$ of elements of $\mathbb{Z}_+^5[\alpha]$ inductively as follows. Set r_1 to be the least element (with respect to the lexicographic order) of $\mathbb{Z}_+^5[\alpha]$ such that $\boldsymbol{f_{r_1}}v \notin U(\mathfrak{g})L(m)_{>\lambda-\alpha}$. Assume that $r_1,\dots,r_p$ are defined. We set r_{p+1} to be the least element of $\mathbb{Z}_+^5[\alpha]$ such that

$$\boldsymbol{f_{r_{p+1}}}v \notin \sum_{i=1}^{p} \mathbb{C}\boldsymbol{f_{r_i}}v + U(\mathfrak{g})L(m)_{>\lambda-\alpha}$$

if such an element exists, and otherwise we set $t = p$.

Set $K[\alpha] = \{r_1,\dots,r_t\}$. By the definition the projection images of $\{\boldsymbol{f_a}v \mid a \in K[\alpha]\}$ span $\Big(L(m)\Big/U(\mathfrak{g})L(m)_{>\lambda-\alpha}\Big)_{\lambda-\alpha}$, and every $r \in K[\alpha]$ satisfies that

$$\boldsymbol{f_r}v \notin \sum_{\substack{a\in\mathbb{Z}_+^5[\alpha],\\ a<r}} \mathbb{C}\boldsymbol{f_a}v + U(\mathfrak{g})L(m)_{>\lambda-\alpha}. \tag{5.1}$$

It is enough to show that every $r = (r_1,\dots,r_5) \in K[\alpha]$ satisfies

$$r_1 \leq k, \quad r_1 - r_3 + r_5 \leq k, \quad 2r_2 + 3r_3 + 3r_4 \leq l, \quad 2r_2 + 3r_4 + 3r_5 \leq l,$$

since this implies $K[\alpha] \subseteq S_\lambda[\alpha]$.

Fix $r = (r_1,\dots,r_5) \in K[\alpha]$, and first assume that $r_1 > k$. The Lie subalgebra of $\mathfrak{g}[t]$

spanned by f_{α_1}, $f_{\alpha_1+3\alpha_2}t$, and $f_{2\alpha_1+3\alpha_2}t$ is isomorphic to the 3-dimensional Heisenberg algebra. Then [5, Lemma 1.5] and $f_{\alpha_1}^{k+1}v = 0$ imply that

$$(f_{\alpha_1+3\alpha_2}t)^{r_3}(f_{2\alpha_1+3\alpha_2}t)^{r_1}v \in \sum_{0<p,0\le q,0\le s\le k} \mathfrak{f}_{\alpha_1}^{p}(f_{\alpha_1+3\alpha_2}t)^{q}(f_{2\alpha_1+3\alpha_2}t)^{s}v.$$

From this we easily see that

$$\boldsymbol{f_r}v \in \sum_{\substack{a\in\mathbb{Z}_+^5[\alpha],\\ a<r}} \mathbb{C}\boldsymbol{f_a}v + U(\mathfrak{g})L(m)_{>\lambda-\alpha},$$

which contradicts (5.1).

Next assume that $r_1 - r_3 + r_5 > k$. Let e_i $(1 \le i \le 5)$ denote the standard basis of $\mathbb{Z}^5$, and set $s = r - r_4e_4 + r_4e_5$. We easily see that

$$e_{\alpha_1}^{r_4}\boldsymbol{f_s}v \in \mathbb{C}^\times \boldsymbol{f_r}v + \sum_{\substack{a\in\mathbb{Z}_+^5[\alpha],\\ a<r}} \mathbb{C}\boldsymbol{f_a}v. \tag{5.2}$$

Note that

$$\mathrm{wt}(\boldsymbol{f_s}v) = \lambda - \alpha - r_4\alpha_1 = (k - r_1 + r_3 - r_4 - r_5)\omega_1 + (l - r_2 - 3r_3)\omega_2,$$

and hence we have

$$s_1\mathrm{wt}(\boldsymbol{f_s}v) = \lambda - \alpha + (r_1 - r_3 + r_5 - k)\alpha_1 > \lambda - \alpha,$$

which implies $\boldsymbol{f_s}v \in U(\mathfrak{g})L(m)_{>\lambda-\alpha}$. Then this and (5.2) contradict (5.1).

The inequality $2r_2 + 3r_3 + 3r_4 \le l$ is proved in the same way as in [5, Subsection 3.5].

Finally assume that $2r_2 + 3r_4 + 3r_5 > l$. Then $r_5 > r_3$ follows, since otherwise we have $2r_2 + 3r_4 + 3r_5 \le 2r_2 + 3r_3 + 3r_4 \le l$. Set

$$s_j = (r_1, 0, r_2 + r_3 + 2r_5 - 2j, r_4, j) \quad \text{for } 0 \le j \le r_3.$$

We have

$$\mathrm{wt}(\boldsymbol{f_{s_j}}v) = \lambda - \alpha - (r_2 + 3r_5 - 3j)\alpha_2, \quad \langle \mathrm{wt}(\boldsymbol{f_{s_j}}v), \alpha_2^\vee\rangle = l - 3r_2 - 3r_3 - 3r_4 - 6r_5 + 6j.$$

Then by a similar argument as in the proof of $r_1 - r_3 + r_5 \le k$, we can show that

$$\boldsymbol{f_{s_j}}v \in U(\mathfrak{g})L(m)_{>\lambda-\alpha} \quad \text{for all } 0 \le j \le r_3. \tag{5.3}$$

It follows from Eq. 4.2 that

$$\begin{aligned}
e_{\alpha_2}^{(r_2+3r_5-3j)}\boldsymbol{f_{s_j}}v &= \sum_{i=\max\{0,r_5-r_3-j\}}^{r_5-j+\lfloor r_2/3\rfloor} \binom{i+j}{j} \boldsymbol{f}(r_1, r_2 + 3r_5 - 3i - 3j, r_3 - r_5 + i + j, r_4, i + j)v \\
&= \sum_{i=-\lfloor r_2/3\rfloor}^{\min\{r_5-j,r_3\}} \binom{r_5-i}{j} \boldsymbol{f}(r_1, r_2 + 3i, r_3 - i, r_4, r_5 - i)v \\
&\in \sum_{i=0}^{\min\{r_5-j,r_3\}} \binom{r_5-i}{j} \boldsymbol{f}r + i(3e_2 - e_3 - e_5)v + \sum_{\substack{a\in\mathbb{Z}_+^5[\alpha],\\ a<r}} \mathbb{C}\boldsymbol{f_a}v,
\end{aligned}$$

and then by Eq. 5.3 we have for every $0 \leq j \leq r_3$ that

$$\sum_{i=0}^{\min\{r_5-j,r_3\}} \binom{r_5-i}{j} f r + i(3e_2 - e_3 - e_5)v \in \sum_{\substack{a\in\mathbb{Z}_+^5[\alpha],\\ a<r}} \mathbb{C} \boldsymbol{f_a} v + U(\mathfrak{g})L(m)_{>\lambda-\alpha}.$$

From this we can show that

$$\boldsymbol{f_r} v \in \sum_{\substack{a\in\mathbb{Z}_+^5[\alpha],\\ a<r}} \mathbb{C} \boldsymbol{f_a} v + U(\mathfrak{g})L(m)_{>\lambda-\alpha}$$

by a similar argument as in Lemma 4.7, in which we use a polynomial

$$f(x) = v_0 x^{r_5} + v_1 x^{r_5-1} + \cdots + v_{r_3} x^{r_5-r_3}$$

instead. Now this contradicts (5.1).

5.3 Linearly Independence

Proposition 5.1 is proved from the following lemma, together with Lemma 5.2.

Lemma 5.3 *For every $\alpha \in Q_+$, the images of $\{\boldsymbol{f_a} v \mid a \in S_\lambda[\alpha]\}$ under the canonical projection $L(m) \twoheadrightarrow L(m)/U(\mathfrak{g})L(m)_{>\lambda-\alpha}$ are linearly independent.*

Fix $\alpha \in Q_+$. Let $\overline{L(m)} = L(m)/U(\mathfrak{g})L(m)_{>\lambda-\alpha}$, and pr denote the canonical projection $L(m) \twoheadrightarrow \overline{L(m)}$. We shall show the lemma by the induction on k. The case $k = 0$ is proved in [5].

Assume that $k > 0$, and a sequence $\{c_a\}_{a\in S_\lambda[\alpha]}$ of complex numbers satisfies

$$\sum_{a\in S_\lambda[\alpha]} c_a \mathrm{pr}(\boldsymbol{f_a} v) = 0. \tag{5.4}$$

First we shall show that

$$c_a = 0 \text{ for all } a \in S_\lambda[\alpha] \text{ such that } a_1 > 0. \tag{5.5}$$

Let L_1 and L_2 be the graded limits of minimal affinizations of $V_q(\omega_1)$ and $V_q(\lambda - \omega_1)$ respectively, and v_1, v_2 be respective highest weight vectors. Set $\lambda_2 = \lambda - \omega_1$. It follows that

$$L(m) \cong T(\lambda) \hookrightarrow T(k\omega_1) \otimes T(l\omega_2) \hookrightarrow T(\omega_1) \otimes T\big((k-1)\omega_1\big) \otimes T(l\omega_2),$$

and from this we see that $L(m) \cong U(\mathfrak{g}[t])(v_1 \otimes v_2) \subseteq L_1 \otimes L_2$. It is known that

$$L_1 = U(\mathfrak{g})v_1 \oplus U(\mathfrak{g})\boldsymbol{f e_1} v_1 \cong V(\omega_1) \oplus V(0)$$

as a $\mathfrak{g}$-module, and $\boldsymbol{f_a} v_1 = 0$ if $a \notin \{0, e_1\}$.

Let $\mathrm{pr}^1 \colon L_1 \twoheadrightarrow V(0)$ be the projection with respect to the $\mathfrak{g}$-module decomposition, and $\mathrm{pr}^2_{\lambda-\alpha} \colon L_2 \twoheadrightarrow L_2/U(\mathfrak{g})(L_2)_{>\lambda-\alpha}$ the canonical projection. Since

$$(L_1 \otimes L_2)_{>\lambda-\alpha} = \bigoplus_{\mu\in P} (L_1)_\mu \otimes (L_2)_{>\lambda-\alpha-\mu} \subseteq V(0) \otimes (L_2)_{>\lambda-\alpha} \oplus V(\omega_1) \otimes L_2,$$

we have

$$U(\mathfrak{g})(L_1 \otimes L_2)_{>\lambda-\alpha} \subseteq V(0) \otimes U(\mathfrak{g})(L_2)_{>\lambda-\alpha} \oplus V(\omega_1) \otimes L_2.$$

Hence the composition

$$\kappa \colon L(m) \hookrightarrow L_1 \otimes L_2 \overset{\mathrm{pr}^1\otimes \mathrm{pr}^2_{\lambda-\alpha}}{\twoheadrightarrow} V(0) \otimes \Big(L_2/U(\mathfrak{g})(L_2)_{>\lambda-\alpha}\Big) \cong L_2/U(\mathfrak{g})(L_2)_{>\lambda-\alpha}$$

induces a $\mathfrak{g}$-module homomorphism $\overline{\kappa}\colon \overline{L(m)} \to L_2/U(\mathfrak{g})(L_2)_{>\lambda-\alpha}$. It is easily seen for $a = (a_1, \ldots, a_5)$ that

$$\boldsymbol{f_a}(v_1 \otimes v_2) = \begin{cases} v_1 \otimes \boldsymbol{f_a} v_2 + \boldsymbol{f}\boldsymbol{e_1} v_1 \otimes \boldsymbol{f_{a-e_1}} v_2 & \text{if } a_1 > 0, \\ v_1 \otimes \boldsymbol{f_a} v_2 & \text{otherwise.} \end{cases} \tag{5.6}$$

Hence we see from the definition of κ that (5.4) yields

$$0 = \overline{\kappa}\Big(\sum_{a \in S_\lambda[\alpha]} c_a \mathrm{pr}(\boldsymbol{f_a} v)\Big) = \sum_{a \in S_\lambda[\alpha]} c_a \kappa(\boldsymbol{f_a} v) = \sum_{a \in S_\lambda[\alpha]:\, a_1 > 0} c_a \mathrm{pr}^2_{\lambda-\alpha}(\boldsymbol{f_{a-e_1}} v_2).$$

Since $\lambda - \alpha = \lambda_2 - (\alpha - \omega_1)$ and $\{a - e_1 \mid a \in S_\lambda[\alpha], a_1 > 0\} \subseteq S_{\lambda_2}[\alpha - \omega_1]$, Eq. (5.5) follows from the induction hypothesis, as required.

Set

$S^0_\lambda[\alpha] = \{a \in S_\lambda[\alpha] \mid a_1 = 0\}$ and $S^{0,k}_\lambda[\alpha] = \{a \in S_\lambda[\alpha] \mid a_1 = 0, -a_3 + a_5 = k\} \subseteq S^0_\lambda[\alpha]$.

It is easily checked that

$$S^0_\lambda[\alpha] = S^0_{\lambda_2}[\alpha] \sqcup S^{0,k}_\lambda[\alpha]. \tag{5.7}$$

Next we would like to prove that

$$c_a = 0 \ \text{ for all } a \in S^0_{\lambda_2}[\alpha], \tag{5.8}$$

and in order to do that we will first prove that

$$\boldsymbol{f_a} v_2 \in \mathbb{C}^\times f_{\alpha_1} \boldsymbol{f_{a+(e_4-e_5)}} v_2 + U(\mathfrak{g})(L_2)_{>\lambda_2-(\alpha-\alpha_1)} \ \text{ if } a \in S^{0,k}_\lambda[\alpha]. \tag{5.9}$$

Assume that $r = (0, r_2, r_3, r_4, r_3 + k) \in S^{0,k}_\lambda[\alpha]$. We see by a direct calculation that

$$e^{r_4}_{\alpha_1} \boldsymbol{f_{r+r_4(e_5-e_4)}} v_2 \in \mathbb{C}^\times \boldsymbol{f_r} v_2 \ \text{ and } \ e^{r_4+1}_{\alpha_1} \boldsymbol{f_{r+r_4(e_5-e_4)}} v_2 \in \mathbb{C}^\times \boldsymbol{f_{r+(e_4-e_5)}} v_2. \tag{5.10}$$

Since

$$\mathrm{wt}(f_{\alpha_1} \boldsymbol{f_{r+r_4(e_5-e_4)}} v_2) = -(r_4 + 3)\omega_1 + (l - r_2 - 3r_3 + 3)\omega_2,$$

it follows that

$$s_1 \mathrm{wt}(f_{\alpha_1} \boldsymbol{f_{r+r_4(e_5-e_4)}} v_2) = \mathrm{wt}(\boldsymbol{f_r} v_2) + 2\alpha_1 > \lambda_2 - (\alpha - \alpha_1),$$

which implies $f_{\alpha_1} \boldsymbol{f_{r+r_4(e_5-e_4)}} v_2 \in U(\mathfrak{g})(L_2)_{>\lambda_2-(\alpha-\alpha_1)}$. Hence it follows that

$$\begin{aligned} f_{\alpha_1} e^{r_4+1}_{\alpha_1} \boldsymbol{f_{r+r_4(e_5-e_4)}} v_2 &= (e^{r_4+1}_{\alpha_1} f_{\alpha_1} + [f_{\alpha_1}, e^{r_4+1}_{\alpha_1}]) \boldsymbol{f_{r+r_4(e_5-e_4)}} v_2 \\ &\in \mathbb{C}^\times e^{r_4}_{\alpha_1} \boldsymbol{f_{r+r_4(e_5-e_4)}} v_2 + U(\mathfrak{g})(L_2)_{>\lambda_2-(\alpha-\alpha_1)}, \end{aligned}$$

which together with (5.10) imply (5.9). Let $\mathrm{pr}^2_{\lambda_2-\alpha}\colon L_2 \twoheadrightarrow L_2/U(\mathfrak{g})(L_2)_{>\lambda_2-\alpha}$ be the canonical projection. Since $U(\mathfrak{g})(L_1 \otimes L_2)_{>\lambda-\alpha} \subseteq L_1 \otimes U(\mathfrak{g})(L_2)_{>\lambda_2-\alpha}$, the composition

$$L(m) \hookrightarrow L_1 \otimes L_2 \twoheadrightarrow L_1 \otimes \big(L_2/U(\mathfrak{g})(L_2)_{>\lambda_2-\alpha}\big)$$

induces a $\mathfrak{g}$-module homomorphism $\overline{L(m)} \to L_1 \otimes \big(L_2/U(\mathfrak{g})(L_2)_{>\lambda_2-\alpha}\big)$. We see from Eq. 5.9 that $\mathrm{pr}^2_{\lambda_2-\alpha}(\boldsymbol{f_a} v_2) = 0$ if $a \in S^{0,k}_\lambda[\alpha]$, and then Eqs. 5.5, 5.6, 5.7 and the induced homomorphism yield

$$v_1 \otimes \Big(\sum_{a \in S^0_{\lambda_2}[\alpha]} c_a \mathrm{pr}^2_{\lambda_2-\alpha}(\boldsymbol{f_a} v_2)\Big) = 0.$$

By the induction hypothesis this implies (5.8), as required.

We have

$$\sum_{a \in S^{0,k}_\lambda[\alpha]} c_a \mathrm{pr}(\boldsymbol{f_a} v) = 0 \tag{5.11}$$

by Eqs. 5.4, 5.5 and 5.8. It remains to show that $c_a = 0$ for $a \in S_\lambda^{0,k}[\alpha]$. Fix $r = (r_1, \dots, r_5) \in S_\lambda^{0,k}[\alpha]$, and set $s = r + e_4 - e_5$. We define a $\mathfrak{g}$-submodule L_2' of L_2 by

$$L_2' = \sum_{\substack{a \in S_{\lambda_2} \\ \mathrm{wt}(a) < \alpha,\, a \neq s}} U(\mathfrak{g}) \boldsymbol{f_a} v_2.$$

We have $(L_2)_{>\lambda_2-\alpha} \subseteq \mathbb{C}\boldsymbol{f_s} v_2 + L_2'$ by Lemma 5.2, and from this we see that

$$\begin{aligned}(L_1 \otimes L_2)_{>\lambda-\alpha} &= \mathrm{y}_1 \otimes (L_2)_{>\lambda_2-\alpha} \oplus \bigoplus_{\beta>0} (L_1)_{\omega_1-\beta} \otimes (L_2)_{>\lambda_2-\alpha+\beta} \\ &\subseteq \mathrm{y}_1 \otimes \boldsymbol{f_s} v_2 + L_1 \otimes L_2',\end{aligned}$$

which implies $U(\mathfrak{g})(L_1 \otimes L_2)_{>\lambda-\alpha} \subseteq U(\mathfrak{g})(v_1 \otimes \boldsymbol{f_s} v_2) + L_1 \otimes L_2'$. Hence the composition

$$\rho\colon L(m) \hookrightarrow L_1 \otimes L_2 \twoheadrightarrow (L_1 \otimes L_2)\Big/\big(U(\mathfrak{g})(v_1 \otimes \boldsymbol{f_s} v_2) + L_1 \otimes L_2'\big)$$

induces a $\mathfrak{g}$-module homomorphism

$$\overline{\rho}\colon \overline{L(m)} \to (L_1 \otimes L_2)\Big/\big(U(\mathfrak{g})(v_1 \otimes \boldsymbol{f_s} v_2) + L_1 \otimes L_2'\big).$$

If $a \in S_\lambda^{0,k}[\alpha] \setminus \{r\}$, then we have $a + e_4 - e_5 \in S_{\lambda_2}[\alpha - \alpha_1] \setminus \{s\}$ and hence it follows by Eq. 5.9 that

$$\boldsymbol{f_a}(v_1 \otimes v_2) = v_1 \otimes \boldsymbol{f_a} v_2 \in L_1 \otimes L_2'.$$

Hence we have from Eq. 5.11 that

$$0 = \overline{\rho}\Big(\sum_{a \in S_\lambda^{0,k}[\alpha]} c_a \mathrm{pr}(\boldsymbol{f_a} v)\Big) = \sum_{a \in S_\lambda^{0,k}[\alpha]} c_a \rho(\boldsymbol{f_a} v) = c_r \rho(\boldsymbol{f_r} v).$$

Assume that $c_r \neq 0$, which implies $\rho(\boldsymbol{f_r} v) = 0$. Let pr_2' denote the canonical projection $L_2 \to L_2/L_2'$. We easily see that $\rho(\boldsymbol{f_r} v) = 0$ is equivalent to

$$v_1 \otimes \mathrm{pr}_2'(\boldsymbol{f_r} v_2) \in U(\mathfrak{g})\big(v_1 \otimes \mathrm{pr}_2'(\boldsymbol{f_s} v_2)\big). \tag{5.12}$$

Note that $\mathrm{pr}_2'(\boldsymbol{f_s} v_2) \neq 0$ by the induction hypothesis, and this also implies $\mathrm{pr}_2'(\boldsymbol{f_r} v_2) \neq 0$ since $e_{\alpha_1} \mathrm{pr}_2'(\boldsymbol{f_r} v_2) \in \mathbb{C}^\times \mathrm{pr}_2'(\boldsymbol{f_s} v_2)$ by Eq. 5.10. Since

$$\mathfrak{n}_+\big(v_1 \otimes \mathrm{pr}_2'(\boldsymbol{f_s} v_2)\big) = 0 \text{ and } \mathrm{wt}\big(v_1 \otimes \mathrm{pr}_2'(\boldsymbol{f_r} v_2)\big) = \mathrm{wt}\big(v_1 \otimes \mathrm{pr}_2'(\boldsymbol{f_s} v_2)\big) - \alpha_1,$$

Equation 5.12 implies

$$v_1 \otimes \mathrm{pr}_2'(\boldsymbol{f_r} v_2) \in \mathfrak{f}_{\alpha_1}\big(v_1 \otimes \mathrm{pr}_2'(\boldsymbol{f_s} v_2)\big).$$

However this contradicts $f_{\alpha_1} v_1 \neq 0$. Hence $c_r = 0$ holds, as required.

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