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Extension Closedness of Syzygies and Local Gorensteinness of Commutative Rings

Shiro Goto[1] **· Ryo Takahashi**[2]

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Abstract We refine a well-known theorem of Auslander and Reiten about the extension closedness of nth syzygies over noether algebras. Applying it, we obtain the converse of a celebrated theorem of Evans and Griffith on Serre's condition (S_n) and the local Gorensteiness of a commutative ring in height less than n. This especially extends a recent result of Araya and Iima concerning a Cohen–Macaulay local ring with canonical module to an arbitrary local ring.

Keywords Syzygy · Extension closed subcategory · Gorenstein ring · Serre's condition

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1 Introduction

In this paper we are interested in the following theorem.

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✉ Ryo Takahashi
takahashi@math.nagoya-u.ac.jp

Shiro Goto
goto@math.meiji.ac.jp
http://www.math.nagoya-u.ac.jp/~takahashi/

1 Department of Mathematics, School of Science and Technology, Meiji University, 1-1-1 Higashi-mita, Tama-ku, Kawasaki, 214-8571, Japan

2 Graduate School of Mathematics, Nagoya University, Furocho, Chikusaku, Nagoya, 464-8602, Japan

Theorem 1.1 (Evans–Griffith, Araya–Iima) *Let $n \geq 0$ be an integer. Let R be a commutative noetherian ring satisfying Serre's condition* (S_n). *Consider the following conditions.*

(1) *One has* $\Omega^n(\mathsf{mod}\, R) = \mathsf{S}_n(R)$.
(2) *The local ring* $R_\mathfrak{p}$ *is Gorenstein for all* $\mathfrak{p} \in \mathsf{Spec}\, R$ *with* $\mathsf{ht}\, \mathfrak{p} < n$.

Then the implication (2) $\Rightarrow$ (1) *holds. The opposite implication* (1) $\Rightarrow$ (2) *holds if R is a Cohen–Macaulay local ring with canonical module.*

Let us explain some notation. For a right noetherian ring Λ, we denote by $\mathsf{mod}\, \Lambda$ the category of finitely generated right Λ-modules, and by $\Omega^n(\mathsf{mod}\, \Lambda)$ the full subcategory of nth syzygies. For a commutative noetherian ring R, we denote by $\mathsf{S}_n(R)$ the full subcategory of $\mathsf{mod}\, R$ consisting of modules satisfying Serre's condition (S_n). The first assertion of the theorem is a celebrated result of Evans and Griffith [7, Theorem 3.8], while the second assertion has recently been shown by Araya and Iima [1, Theorem 2.2].

It is natural to ask whether the implication (1) $\Rightarrow$ (2) in Theorem 1.1 holds for arbitrary commutative noetherian rings, and the main purpose of this paper is to give an answer to this question. Since $\mathsf{S}_n(R)$ is an extension closed subcategory of $\mathsf{mod}\, R$, the equality $\Omega^n(\mathsf{mod}\, R) = \mathsf{S}_n(R)$ implies the extension closedness of $\Omega^n(\mathsf{mod}\, R)$. We thus study when the subcategory $\Omega^n(\mathsf{mod}\, R)$ is extension closed, rather than when one has $\Omega^n(\mathsf{mod}\, R) = \mathsf{S}_n(R)$.

A *noether algebra* is by definition a module-finite algebra of a commutative noetherian ring. (Thus a noether algebra is a two-sided noetherian ring.) The extension closedness of syzygies has been investigated over a noether algebra by Auslander and Reiten [5, Theorem 0.1].

Theorem 1.2 (Auslander–Reiten) *Let Λ be a noether algebra. Then the following are equivalent for each nonnegative integer n.*

(1) $\Omega^i(\mathsf{mod}\, \Lambda)$ *is extension closed for all* $1 \leq i \leq n$.
(2) $\mathsf{grade}_\Lambda \mathsf{Ext}^i_{\Lambda^{\mathsf{op}}}(M, \Lambda) \geq i$ *for all* $1 \leq i \leq n$ *and* $M \in \mathsf{mod}\, \Lambda^{\mathsf{op}}$.

The first condition of this theorem is too strong for our purpose in that it requires the extension closedness of ith syzygies for *all* integers i with $1 \leq i \leq n$, while it is the extension closedness of nth syzygies that we want to deal with.

Our first main result is the following theorem on not-necessarily-commutative rings. This in fact provides a refinement of the implication (1) $\Rightarrow$ (2) in Theorem 1.2.

Theorem A *Let Λ be a noether algebra such that* $\Omega^n(\mathsf{mod}\, \Lambda)$ *is extension closed. Let M be a finitely generated Λ^{op}-module with* $\mathsf{grade}_\Lambda \mathsf{Ext}^i_{\Lambda^{\mathsf{op}}}(M, \Lambda) \geq i - 1$ *for all* $1 \leq i \leq n$. *Then* $\mathsf{grade}_\Lambda \mathsf{Ext}^n_{\Lambda^{\mathsf{op}}}(M, \Lambda) \geq n$.

This theorem enables us to achieve our main purpose stated above; applying it to commutative rings, we can prove the following theorem, which is the second main result of this paper. This extends Theorem 1.1 to arbitrary commutative noetherian local rings.

Theorem B *Let R be a commutative noetherian local ring satisfying* (S_n). *The following are equivalent.*

(1) $\Omega^n(\mathsf{mod}\, R)$ *is extension closed.*
(2) $\Omega^n(\mathsf{mod}\, R) = \mathsf{S}_n(R)$.
(3) $R_\mathfrak{p}$ *is Gorenstein for all* $\mathfrak{p} \in \mathsf{Spec}\, R$ *with* $\mathsf{ht}\, \mathfrak{p} < n$.

The organization of this paper is as follows. The next Section 2 is devoted to introducing several notions and basic properties that are necessary in the later sections. In Section 3, we consider extension closedness of syzygies. We prove Theorem A, and apply it to commutative rings to get a sufficient condition for local Gorensteinness in height $n-1$. In the final Section 4, we study local Gorensteinness of commutative rings. For each integer $t \geq 0$ we show the equivalence of local Gorensteinness in height equal to t and local Gorensteinness in height at most t. Combining this with a result obtained in Section 3, we finally give a proof of Theorem B.

2 Preliminaries

We start by stating our conventions.

Convention 2.1 Throughout the rest of this paper, let Λ be a two-sided noetherian ring, and let R be a commutative noetherian ring. We assume that all modules are finitely generated right ones, and that all subcategories are full ones.

Denote by $\mathsf{mod}\,\Lambda$ the category of (finitely generated right) Λ-modules, and by $\mathsf{proj}\,\Lambda$ the subcategory of projective modules. Define the functor $(-)^*$: $\mathsf{mod}\,\Lambda \to \mathsf{mod}\,\Lambda^{\mathsf{op}}$ by $(-)^* = \mathsf{Hom}_\Lambda(-, \Lambda)$. A subcategory $\mathcal{X}$ of $\mathsf{mod}\,\Lambda$ is said to be *extension closed* provided that for each exact sequence $0 \to L \to M \to N \to 0$ in $\mathsf{mod}\,\Lambda$, if L and N are in $\mathcal{X}$, then so is M.

We recall the definitions of minimal morphisms and approximations.

Definition 2.2 Let $\mathcal{C}$ be a category, and let $\mathcal{X}$ be a subcategory of $\mathcal{C}$. Let $\phi : C \to X$ be a morphism in $\mathcal{C}$ with $X \in \mathcal{X}$. We say that:

(1) ϕ is *left minimal* if all endomorphisms $f : X \to X$ with $\phi = f\phi$ are automorphisms.

(2) ϕ is a *left $\mathcal{X}$-approximation* if all morphisms from C to objects in $\mathcal{X}$ factor through ϕ.

A *right minimal* morphism and a *right $\mathcal{X}$-approximation* are defined dually.

A left (respectively, right) $\mathcal{X}$-approximation is sometimes called an *$\mathcal{X}$-preenvelope* (respectively, *$\mathcal{X}$-precover*), and a left (respectively, right) minimal one an *$\mathcal{X}$-envelope* (respectively, *$\mathcal{X}$-cover*). A homomorphism $\phi : M \to P$ of Λ-modules with P projective is a left $\mathsf{proj}\,\Lambda$-approximation if and only if the Λ-dual map $\phi^* : P^* \to M^*$ is surjective. For the details of minimal morphisms and approximations, see [3, Section 1] for instance.

Next we recall the definitions of an adjoint pair, a unit and a counit.

Definition 2.3 Let $S : \mathcal{C} \to \mathcal{D}$ and $T : \mathcal{D} \to \mathcal{C}$ be functors between categories $\mathcal{C}$ and $\mathcal{D}$. Suppose that for all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ there is a functorial isomorphism

$$\Phi_{XY} : \mathsf{Hom}_{\mathcal{D}}(SX, Y) \xrightarrow{\cong} \mathsf{Hom}_{\mathcal{C}}(X, TY).$$

Then we say that $(S, T) : \mathcal{C} \to \mathcal{D}$ (or more precisely, $(S, T, \Phi) : \mathcal{C} \to \mathcal{D}$) is an *adjoint pair*. Taking $\Phi_{X,SX}(1_{SX})$ and $\Phi^{-1}_{TY,Y}(1_{TY})$, one obtains natural transformations

$$u : \mathbf{1}_{\mathcal{C}} \to TS, \qquad c : ST \to \mathbf{1}_{\mathcal{D}},$$

which are called the *unit* and *counit* of the adjunction, respectively.

For each $X \in \mathcal{C}$, every morphism $X \to TY$ with $Y \in \mathcal{D}$ uniquely factors through $uX : X \to TSX$. Dually, for each $Y \in \mathcal{D}$, every morphism $SX \to Y$ with $X \in \mathcal{C}$ uniquely factors through $cY : STY \to Y$. In paricular, uX and cY are a left $\mathsf{Im}(T)$-approximation and a right $\mathsf{Im}(S)$-approximation, respectively. (Here, for a functor $F : \mathcal{X} \to \mathcal{Y}$ we denote by $\mathsf{Im}(F)$ the *essential image* of F, namely, the subcategory of $\mathcal{Y}$ consisting of objects N such that $N \cong FM$ for some $M \in \mathcal{X}$.) Also, the equalities $\Phi_{XY}(f) = Tf \cdot uX$ and $\Phi_{XY}^{-1}(g) = cY \cdot Sg$ hold for all morphisms $f : SX \to Y$ and $g : X \to TY$. Furthermore, the compositions of natural transformations $S \xrightarrow{Su} STS \xrightarrow{cS} S$ and $T \xrightarrow{uT} TST \xrightarrow{Tc} T$ are both identities. The details can be found in [11, Theorem IV.1.1].

Let us recall the definitions of syzygies, transposes and stable categories.

Definition 2.4 (1) Let M be a Λ-module. Let $\cdots \to P_n \xrightarrow{\partial_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{\partial_1} P_0 \to M \to 0$ be a projective resolution of M.

(a) The *nth syzygy* $\Omega^n M$ of M is defined as the image of the map $\partial_n : P_n \to P_{n-1}$.
(b) The *transpose* $\mathsf{Tr}M$ of M is defined to be the cokernel of the map $\partial_1^* : P_0^* \to P_1^*$.

(2) For Λ-modules M and N, let $\underline{\mathsf{Hom}}_\Lambda(M, N)$ be the quotient of $\mathsf{Hom}_\Lambda(M, N)$ by the Λ-homomorphisms $M \to N$ factoring through some projective Λ-modules. The residue class in $\underline{\mathsf{Hom}}_\Lambda(M, N)$ of an element $f \in \mathsf{Hom}_\Lambda(M, N)$ is denoted by $\underline{f}$.

(3) We denote by $\underline{\mathsf{mod}}\,\Lambda$ the *stable category* of $\mathsf{mod}\,\Lambda$. The objects of $\underline{\mathsf{mod}}\,\Lambda$ are precisely the (finitely generated right) Λ-modules, and the hom-set from a Λ-module M to a Λ-module N is given by $\underline{\mathsf{Hom}}_\Lambda(M, N)$.

The modules ΩM and $\mathsf{Tr}M$ are uniquely determined by M up to projective summands. The assignments $M \mapsto \Omega M$ and $M \mapsto \mathsf{Tr}M$ give rise to additive functors

$$\Omega : \underline{\mathsf{mod}}\,\Lambda \to \underline{\mathsf{mod}}\,\Lambda, \qquad \mathsf{Tr} : \underline{\mathsf{mod}}\,\Lambda \to \underline{\mathsf{mod}}\,\Lambda^{\mathsf{op}}.$$

Moreover, Tr is a duality, i.e., one has $\mathsf{Tr}\mathsf{Tr}M \cong M$ in $\underline{\mathsf{mod}}\,\Lambda$ for each $M \in \underline{\mathsf{mod}}\,\Lambda$. We define the functor $\mathsf{D}_n : \underline{\mathsf{mod}}\,\Lambda \to \underline{\mathsf{mod}}\,\Lambda^{\mathsf{op}}$ by $\mathsf{D}_n = \Omega^n \mathsf{Tr}$; one has $\mathsf{D}_2 M \cong M^*$ in $\underline{\mathsf{mod}}\,\Lambda^{\mathsf{op}}$ for each $M \in \underline{\mathsf{mod}}\,\Lambda$. We denote by $\Omega^n(\mathsf{mod}\,\Lambda)$ the subcategory of $\mathsf{mod}\,\Lambda$ consisting of nth syzygies X, that is, modules X admitting an exact sequence $0 \to X \to P_{n-1} \to \cdots \to P_1 \to P_0$ of Λ-modules with each P_i projective. Also, $\Omega^n(\underline{\mathsf{mod}}\,\Lambda)$ denotes the essential image of the functor $\Omega^n : \underline{\mathsf{mod}}\,\Lambda \to \underline{\mathsf{mod}}\,\Lambda$, which coincides with the essential image of $\Omega^n(\mathsf{mod}\,\Lambda)$ by the canonical functor $\mathsf{mod}\,\Lambda \to \underline{\mathsf{mod}}\,\Lambda$. We refer the reader to [2, Section 2] for the details of syzygies, transposes and stable categories.

Finally, we recall the definitions of grade, depth and Serre's condition.

Definition 2.5 (1) The *grade* of a Λ-module M, denoted by $\mathsf{grade}_\Lambda M$, is defined to be the infimum of nonnegative integers i such that $\mathsf{Ext}^i_\Lambda(M, \Lambda) \neq 0$.

(2) The *grade* of an ideal I of R is defined by $\mathsf{grade}\, I = \mathsf{grade}_R(R/I)$.

(3) When R is a local ring with maximal ideal $\mathfrak{m}$, the *depth* of an R-module M, denoted by $\mathsf{depth}_R M$, is defined as the infimum of nonnegative integers i with $\mathsf{Ext}^i_R(R/\mathfrak{m}, M) \neq 0$.

(4) Let n be a nonnegative integer. An R-module M is said to satisfy *Serre's condition* (S_n) if the inequality $\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \geq \inf\{n, \operatorname{ht}\mathfrak{p}\}$ holds for all prime ideals $\mathfrak{p}$ of R.

The maximal regular sequences on R in I (respectively, on M in $\mathfrak{m}$) have the same length, and this common length is equal to $\operatorname{grade} I$ (respectively, $\operatorname{depth}_R M$). If R satisfies (S_n), then $\operatorname{ht}\mathfrak{p} = \operatorname{grade}\mathfrak{p}$ for all prime ideals $\mathfrak{p}$ with $\operatorname{ht}\mathfrak{p} \leq n$. We denote by $\mathsf{S}_n(R)$ the subcategory of $\operatorname{mod} R$ consisting of modules satisfying (S_n). This is an extension closed subcategory. See [6, Sections 1 and 2] for the details of grade and depth, and [7] for Serre's condition (S_n).

3 Extension Closedness of Syzygies

Let X, Y be Λ-modules. Let $\cdots \to P_1 \to P_0 \to \operatorname{Tr}X \to 0$ and $\cdots \to Q_1 \to Q_0 \to Y \to 0$ be projective resolutions. Let $f : \operatorname{Tr}\Omega^n\operatorname{Tr}X \to Y$ be a homomorphism of Λ-modules. We extend f to a chain map of complexes as in the left below, and make a commutative diagram with exact rows as in the right below.

$$\begin{array}{ccccccccccc} P_0^* & \to & P_1^* & \to \cdots \to & P_{n+1}^* & \to & \operatorname{Tr}\Omega^n\operatorname{Tr}X & \to & 0 \\ \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_0 & & \downarrow f & & \\ Q_{n+1} & \to & Q_n & \to \cdots \to & Q_0 & \to & Y & \to & 0 \end{array} \qquad \begin{array}{ccccccc} P_0^* & \to & P_1^* & \to & X & \to & 0 \\ \downarrow f_{n+1} & & \downarrow f_n & & \downarrow \phi(f) & & \\ Q_{n+1} & \to & Q_n & \to & \Omega^n Y & \to & 0 \end{array}$$

Thus we get a homomorphism $\phi(f) : X \to \Omega^n Y$ of Λ-modules. Conversely, let $g : X \to \Omega^n Y$ be a homomorphism of Λ-modules. First, extend g to a commutative diagram with exact rows as in the left below, whose rows are finite projective presentations. Second, extend $h_0 := g_1^*$ and $h_1 := g_0^*$ to a chain map as in the middle below. Third, make a commutative diagram with exact rows as in the right below.

$$\begin{array}{ccccccc} P_0^* & \to & P_1^* & \to & X & \to & 0 \\ \downarrow g_1 & & \downarrow g_0 & & \downarrow g & & \\ Q_{n+1} & \to & Q_n & \to & \Omega^n Y & \to & 0 \end{array} \quad \begin{array}{ccccccc} Q_0^* & \to & Q_1^* & \to \cdots \to & Q_n^* & \to & Q_{n+1}^* \\ \downarrow h_{n+1} & & \downarrow h_n & & \downarrow h_1 & & \downarrow h_0 \\ P_{n+1} & \to & P_n & \to \cdots \to & P_1 & \to & P_0 \end{array} \quad \begin{array}{ccccccc} P_n^* & \to & P_{n+1}^* & \to & \operatorname{Tr}\Omega^n\operatorname{Tr}X & \to & 0 \\ \downarrow h_n^* & & \downarrow h_{n+1}^* & & \downarrow \psi(g) & & \\ Q_1 & \to & Q_0 & \to & Y & \to & 0 \end{array}$$

Thus we get a homomorphism $\psi(g) : \operatorname{Tr}\Omega^n\operatorname{Tr}X \to Y$ of Λ-modules. The following lemma holds; see [5, Corollary 3.3].

Lemma 3.1 *With the notation above, the assignments $\Phi : \underline{f} \mapsto \underline{\phi(f)}$ and $\Psi : \underline{g} \mapsto \underline{\psi(g)}$ make functorial isomorphisms*

$$\Phi : \underline{\operatorname{Hom}}_\Lambda(\operatorname{Tr}\Omega^n\operatorname{Tr}X, Y) \rightleftarrows \underline{\operatorname{Hom}}_\Lambda(X, \Omega^n Y) : \Psi$$

which are mutually inverse. In particular, one has an adjoint pair $(\operatorname{Tr}\Omega^n\operatorname{Tr}, \Omega^n) : \underline{\operatorname{mod}}\,\Lambda \to \underline{\operatorname{mod}}\,\Lambda$. *(The corresponding statement for Λ^{op} also holds.)*

Remark 3.2 Let $\mathcal{X}$ be a subcategory of $\operatorname{mod}\Lambda$. Let $f : M \to X$ be a homomorphism of Λ-modules with $X \in \mathcal{X}$. Let N be the image of f. We have an inclusion map $g : N \to X$ and a natural surjection $h : M \to N$, and f is the composition of these two maps. It is straightforward that the following statements hold.

(1) If f is a left $\mathcal{X}$-approximation in $\operatorname{mod}\Lambda$, then so is g.

(2) If $\underline{f}$ is left minimal in $\underline{\operatorname{mod}}\,\Lambda$, then so is $\underline{g}$.

Now we prove the following theorem, which is the main result of this section.

Theorem 3.3 (=Theorem A) *Let Λ be a noether algebra, M a module over Λ^{op}, and n a nonnegative integer. Suppose that $\Omega^n(\mathsf{mod}\,\Lambda)$ is an extension closed subcategory of $\mathsf{mod}\,\Lambda$. If $\mathsf{grade}_\Lambda\,\mathsf{Ext}^i_{\Lambda^{\mathsf{op}}}(M,\Lambda) \geq i-1$ for all $1 \leq i \leq n$, then $\mathsf{grade}_\Lambda\,\mathsf{Ext}^n_{\Lambda^{\mathsf{op}}}(M,\Lambda) \geq n$.*

Proof The assertion is trivial for $n=0$, and follows from Theorem 1.2 for $n=1$. So assume $n \geq 2$. We use the notation of the part preceding Lemma 3.1. Set $X := \mathsf{Tr}M$, $Y := \mathsf{Tr}\Omega^n\mathsf{Tr}X$, $Q_0 := P^*_{n+1}$, $Q_1 := P^*_n$ and let f be the identity map of Y. We get a chain map as in the left below and a commutative diagram as in the right below.

$$\begin{array}{ccccccccccccccc}
P_0^* & \xrightarrow{\alpha} & P_1^* & \xrightarrow{\beta} & \cdots & \to & P_{n-1}^* & \to & P_n^* & \to & P_{n+1}^* & \to & Y & \to & 0\\
\downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & & & \downarrow{\scriptstyle f_2} & & \|{\scriptstyle f_1} & & \|{\scriptstyle f_0} & & \|{\scriptstyle f} & & \\
Q_{n+1} & \to & Q_n & \to & \cdots & \to & Q_2 & \to & Q_1 & \to & Q_0 & \to & Y & \to & 0
\end{array}
\qquad
\begin{array}{ccccccc}
P_0^* & \xrightarrow{\alpha} & P_1^* & \xrightarrow{\pi} & X & \to & 0\\
\downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle \phi(f)} & & \\
Q_{n+1} & \to & Q_n & \to & \Omega^n Y & \to & 0
\end{array}$$

These diagrams induce a chain map

$$\begin{array}{lcccccccccccccccc}
A: & 0 & \to & X & \xrightarrow{\eta} & P_2^* & \to & P_3^* & \to \cdots \to & P_{n-2}^* & \to & P_{n-1}^* & \to & W & \to & 0\\
\downarrow{\scriptstyle \xi} & & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle f_{n-2}} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_2} & & \| & & \\
B: & \underbrace{0}_{0} & \to & \underbrace{V}_{1} & \to & \underbrace{Q_{n-1}}_{2} & \to & \underbrace{Q_{n-2}}_{3} & \to \cdots \to & \underbrace{Q_3}_{n-2} & \to & \underbrace{Q_2}_{n-1} & \to & \underbrace{W}_{n} & \to & \underbrace{0}_{n+1}
\end{array}$$

where we put $\rho := \phi(f)$, $V := \Omega^n Y = \Omega^n\mathsf{Tr}\Omega^n\mathsf{Tr}X$ and $W := \Omega^2\mathsf{Tr}\Omega^n\mathsf{Tr}X$. Note that for each $1 \leq i \leq n$ the ith homology $\mathsf{H}^i(A)$ of the cochain complex A is isomorphic to $\mathsf{Ext}^i_{\Lambda^{\mathsf{op}}}(\mathsf{Tr}X,\Lambda)$, while B is an exact complex. Take the mapping cone of the chain map ξ. It is easy to see that it is quasi-isomorphic to a complex

$$C: \quad \underbrace{0}_{-1} \to \underbrace{X}_{0} \xrightarrow{\binom{\rho}{\eta}} \underbrace{V \oplus P_2^*}_{1} \to \underbrace{Q_{n-1} \oplus P_3^*}_{2} \to \cdots \to \underbrace{Q_3 \oplus P_{n-1}^*}_{n-2} \to \underbrace{Q_2}_{n-1} \to \underbrace{0}_{n}$$

with $\mathsf{H}^i(C) \cong \mathsf{Ext}^{i+1}_{\Lambda^{\mathsf{op}}}(\mathsf{Tr}X,\Lambda) \cong \mathsf{Ext}^{i+1}_{\Lambda^{\mathsf{op}}}(M,\Lambda)$ for $0 \leq i \leq n-1$.

By Lemma 3.1 we have an adjoint pair $(S,T) : \underline{\mathsf{mod}}\,\Lambda \to \underline{\mathsf{mod}}\,\Lambda$ with $S = \mathsf{Tr}\Omega^n\mathsf{Tr}$ and $T = \Omega^n$. Let $u : \mathbf{1} \to TS$ be the unit of the adjunction. Then the morphism $\underline{\rho}$ is nothing but uX. We establish several claims.

Claim 1 The morphism $\binom{\rho}{\eta} : X \to V \oplus P_2^*$ in $\mathsf{mod}\,\Lambda$ is a left $\Omega^n(\mathsf{mod}\,\Lambda)$-approximation.

Proof of Claim As $\underline{\rho} = uX$, the morphism $\underline{\rho}$ in $\underline{\mathsf{mod}}\,\Lambda$ is a left $\Omega^n(\underline{\mathsf{mod}}\,\Lambda)$-approximation. We have $\beta = \eta\pi$, and the complex $P_2^{**} \xrightarrow{\beta^*} P_1^{**} \xrightarrow{\alpha^*} P_0^{**}$ is exact, since it is isomorphic to the complex $P_2 \to P_1 \to P_0$. It is observed from this that the map $\eta^* : P_2^{**} \to X^*$ is surjective, which implies that the morphism η in $\mathsf{mod}\,\Lambda$ is a left $\mathsf{proj}\,\Lambda$-approximation. The claim can now easily be shown. □

Claim 2 If $\mathsf{Ext}^i_{\Lambda^{\mathsf{op}}}(\mathsf{Tr}X,\Lambda) = 0$ for all $1 \leq i \leq n$ (i.e., X is *n-torsionfree*), then uX is an isomorphism in $\underline{\mathsf{mod}}\,\Lambda$. (The corresponding statement for Λ^{op} is also true.)

Proof of Claim The assumption implies that the complex $P_0^* \to P_1^* \to \cdots \to P_{n+1}^* \to \mathsf{Tr}\Omega^n\mathsf{Tr}X \to 0$ is exact. Hence we can take $Q_i := P_{n+1-i}^*$ and $f_i := 1_{P_{n+1-i}^*}$ for every $0 \le i \le n+1$. It follows that uX is an isomorphism. □

Claim 3 The morphism $u\mathsf{D}_nX : \mathsf{D}_nX \to \mathsf{D}_n^3X$ in $\underline{\mathsf{mod}}\,\Lambda$ is an isomorphism.

Proof of Claim Combining the assumption of the theorem with [2, Proposition (2.26)] yields that $\mathsf{D}_nX = \Omega^nM$ is n-torsionfree. The assertion follows from Claim 2. □

Claim 4 The equality $\underline{1_{\mathsf{D}_nZ}} = \mathsf{D}_nuZ \cdot u\mathsf{D}_nZ$ holds for all $Z \in \underline{\mathsf{mod}}\,\Lambda$. (The corresponding statement for Λ^{op} is also true.)

Proof of Claim Let $c : ST \to \mathbf{1}$ stand for the counit of the adjunction. The composition $Tc \cdot uT$ of natural transformations is an identity. It is straightforward to verify that c coincides with the composition $\mathsf{Tr}u\mathsf{Tr}$. The assertion is now easily deduced. □

Claim 5 The morphism $\underline{\rho} : X \to V$ in $\underline{\mathsf{mod}}\,\Lambda$ is left minimal.

Proof of Claim We have $\underline{\rho} = uX$ and $V = \mathsf{D}_n^2X$. Let us prove that $uX : X \to \mathsf{D}_n^2X$ is left minimal. Let $h : \mathsf{D}_n^2X \to \mathsf{D}_n^2X$ be an endomorphism in $\underline{\mathsf{mod}}\,\Lambda$ such that $uX = h \cdot uX$. Then $\mathsf{D}_nuX = \mathsf{D}_nuX \cdot \mathsf{D}_nh$. By Claims 3 and 4 the morphism D_nuX is an isomorphism. Hence D_nh is an automorphism, and so is D_n^2h. Using Claim 4 again, we have $\underline{1_{\mathsf{D}_n^2X}} = \mathsf{D}_nu\mathsf{D}_nX \cdot u\mathsf{D}_n^2X$. Applying Claim 3 again implies that $\mathsf{D}_nu\mathsf{D}_nX$ is an isomorphism. Hence $u\mathsf{D}_n^2X$ is also an isomorphism. It follows from the equality $\mathsf{D}_n^2h \cdot u\mathsf{D}_n^2X = u\mathsf{D}_n^2X \cdot h$ that h is an automorphism. Therefore, the morphism uX is left minimal. □

Let E be the cokernel of the map $\binom{\rho}{\eta}$. By virtue of [4, Corollary 1.8], the subcategory $\Omega^n(\mathsf{mod}\,\Lambda)$ of $\mathsf{mod}\,\Lambda$ is covariantly finite. Recall our assumption that Λ is a noether algebra and $\Omega^n(\mathsf{mod}\,\Lambda)$ is extension closed. In view of Claim 1, Claim 5 and Remark 3.2, all the assumptions of the result [5, Lemma 4.5], which is an analogue of Wakamatsu's lemma, are satisfied, and therefore we obtain $\mathsf{Ext}^1_\Lambda(E, \Omega^n(\mathsf{mod}\,\Lambda)) = 0$. In particular, $\mathsf{Ext}^1_\Lambda(E, \Lambda) = 0$.

Decomposing the complex C into short exact sequences

$$0 \to B^i \xrightarrow{p^i} Z^i \to H^i \to 0, \quad 0 \to Z^i \xrightarrow{q^i} C^i \to B^{i+1} \to 0 \quad (0 \le i \le n-1) \quad (3.1)$$

with $H^i = \mathsf{H}^i(C) \cong \mathsf{Ext}^{i+1}_{\Lambda^{\mathsf{op}}}(M, \Lambda)$. The assumption of the theorem implies $\mathsf{grade}_\Lambda H^i \ge i$ for all $0 \le i \le n-1$.

What we want to prove is that $\mathsf{grade}_\Lambda H^{n-1} \ge n$. Making the pushout diagram of the maps p^1 and q^1, we get an exact sequence $0 \to H^1 \to E \to B^2 \to 0$. When $n = 2$, we have $B^2 = 0$ and $H^1 = E$. Hence $\mathsf{Ext}^1_\Lambda(H^1, \Lambda) = 0$, which implies $\mathsf{grade}_\Lambda H^1 \ge 2$ and we are done. Let $n \ge 3$. The functor $(-)^*$ gives an exact sequence $0 = (H^1)^* \to \mathsf{Ext}^1_\Lambda(B^2, \Lambda) \to \mathsf{Ext}^1_\Lambda(E, \Lambda) = 0$, which shows $\mathsf{Ext}^1_\Lambda(B^2, \Lambda) = 0$. Let $2 \le i \le n-2$ be an integer. From Eq. 3.1 we get an exact sequence $0 = \mathsf{Ext}^{i-1}_\Lambda(H^i, \Lambda) \to \mathsf{Ext}^{i-1}_\Lambda(Z^i, \Lambda) \to \mathsf{Ext}^{i-1}_\Lambda(B^i, \Lambda)$ and an isomorphism $\mathsf{Ext}^{i-1}_\Lambda(Z^i, \Lambda) \cong \mathsf{Ext}^i_\Lambda(B^{i+1}, \Lambda)$ since C^i is a projective module. Therefore we have an injection $\mathsf{Ext}^i_\Lambda(B^{i+1}, \Lambda) \hookrightarrow \mathsf{Ext}^{i-1}_\Lambda(B^i, \Lambda)$. Thus we obtain a chain

$$\mathsf{Ext}^{n-2}_\Lambda(B^{n-1}, \Lambda) \hookrightarrow \cdots \hookrightarrow \mathsf{Ext}^2(B^3, \Lambda) \hookrightarrow \mathsf{Ext}^1_\Lambda(B^2, \Lambda)$$

of injections. Since $\mathsf{Ext}^1_\Lambda(B^2, \Lambda)$ vanishes, so does $\mathsf{Ext}^{n-2}_\Lambda(B^{n-1}, \Lambda)$. The exact sequence $0 \to B^{n-1} \to C^{n-1} \to H^{n-1} \to 0$ and the projectivity of the module C^{n-1} imply $\mathsf{Ext}^{n-1}_\Lambda(H^{n-1}, \Lambda) = 0$. Now we conclude $\mathsf{grade}_\Lambda H^{n-1} \geq n$. □

Remark 3.4 Theorem 3.3 is regarded as a strong version of the implication (1) ⇒ (2) in Theorem 1.2. In fact, one can deduce this implication from Theorem 3.3, as follows. We use induction on n; the case $n = 0$ is trivial. Let $n \geq 1$, and assume that $\Omega^i(\mathsf{mod}\,\Lambda)$ is extension closed for all $1 \leq i \leq n$. The induction hypothesis yields $\mathsf{grade}_\Lambda \mathsf{Ext}^i_{\Lambda^{\mathsf{op}}}(M, \Lambda) \geq i$ for each $1 \leq i \leq n-1$ and each $M \in \mathsf{mod}\,\Lambda^{\mathsf{op}}$. We have $\mathsf{grade}_\Lambda \mathsf{Ext}^n_{\Lambda^{\mathsf{op}}}(M, \Lambda) \geq n-1$: this is trivial for $n = 1$, and for $n \geq 2$ the isomorphism $\mathsf{Ext}^n_{\Lambda^{\mathsf{op}}}(M, \Lambda) \cong \mathsf{Ext}^{n-1}_{\Lambda^{\mathsf{op}}}(\Omega M, \Lambda)$ implies it. By virtue of Theorem 3.3, we obtain $\mathsf{grade}_\Lambda \mathsf{Ext}^n_{\Lambda^{\mathsf{op}}}(M, \Lambda) \geq n$. Thus $\mathsf{grade}_\Lambda \mathsf{Ext}^i_{\Lambda^{\mathsf{op}}}(M, \Lambda) \geq i$ for all $1 \leq i \leq n$ and all $M \in \mathsf{mod}\,\Lambda^{\mathsf{op}}$.

The following result is a consequence of Theorem 3.3, which is used in the proof of Theorem 4.4 stated later.

Corollary 3.5 *Let n be a nonnegative integer. Let R be a commutative noetherian ring satisfying Serre's condition (S_{n-1}). If $\Omega^n(\mathsf{mod}\,R)$ is extension closed, then the local ring $R_\mathfrak{p}$ is Gorenstein for all prime ideals $\mathfrak{p}$ of R with height $n-1$.*

Proof Let $\mathfrak{p}$ be a prime ideal of R with $\mathsf{ht}\,\mathfrak{p} = n-1$. We have

$$\mathsf{grade}_R \mathsf{Ext}^i_R(R/\mathfrak{p}, R) = \mathsf{grade}(\mathsf{ann}_R \mathsf{Ext}^i_R(R/\mathfrak{p}, R)) \geq \mathsf{grade}\,\mathfrak{p} = n-1 \geq i-1$$

for each $1 \leq i \leq n$. Here, the first inequality is shown by the fact that the ideal $\mathsf{ann}_R \mathsf{Ext}^i_R(R/\mathfrak{p}, R)$ contains $\mathfrak{p}$, and the second equality follows from the assumption that R satisfies (S_{n-1}). Applying Theorem 3.3, we obtain $\mathsf{grade}_R \mathsf{Ext}^n_R(R/\mathfrak{p}, R) \geq n$, and therefore $\mathsf{Ext}^{n-1}_R(\mathsf{Ext}^n_R(R/\mathfrak{p}, R), R) = 0$. Localization at $\mathfrak{p}$ yields $\mathsf{Ext}^{n-1}_{R_\mathfrak{p}}(\mathsf{Ext}^n_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), R_\mathfrak{p}), R_\mathfrak{p}) = 0$. Suppose that $\mathsf{Ext}^n_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), R_\mathfrak{p})$ is nonzero. Then it contains $\kappa(\mathfrak{p})$ as a direct summand, and $\mathsf{Ext}^{n-1}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), R_\mathfrak{p})$ is a direct summand of $\mathsf{Ext}^{n-1}_{R_\mathfrak{p}}(\mathsf{Ext}^n_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), R_\mathfrak{p}), R_\mathfrak{p})$, which is zero. If follows that $\mathsf{Ext}^{n-1}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), R_\mathfrak{p}) = 0$, but this contradicts the fact that $R_\mathfrak{p}$ has depth $n-1$. (In fact, $R_\mathfrak{p}$ is a Cohen–Macaulay local ring of dimension $n-1$.) Therefore $\mathsf{Ext}^n_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), R_\mathfrak{p}) = 0$, which implies that $R_\mathfrak{p}$ is Gorenstein; see [8, Theorem (1.1)]. □

We close this section by posing a naive question.

Question 3.6 Under the assumption of Corollary 3.5, is $R_\mathfrak{p}$ a Gorenstein local ring for all prime ideals $\mathfrak{p}$ with height *less than* (or equal to) $n-1$?

4 Local Gorensteinness of Commutative Rings

We begin with proving the following theorem.

Theorem 4.1 *Let $n > t > 0$ be integers. Let R be a commutative noetherian local ring of dimension $d \geq t$ satisfying Serre's condition (S_n). Then the following are equivalent.*

(1) *$R_\mathfrak{p}$ is Gorenstein for all $\mathfrak{p} \in \operatorname{Spec} R$ with $\operatorname{ht} \mathfrak{p} = t$.*
(2) *$R_\mathfrak{p}$ is Gorenstein for all $\mathfrak{p} \in \operatorname{Spec} R$ with $\operatorname{ht} \mathfrak{p} \leq t$.*

Proof It suffices to prove that (1) implies (2). It is enough to show that $R_\mathfrak{p}$ is Gorenstein for all $\mathfrak{p} \in \operatorname{Spec} R$ with $\operatorname{ht} \mathfrak{p} = t - 1$. Suppose that this statement is not true, and consider the counterexample where $d = \dim R$ is minimal. There exists a prime ideal $\mathfrak{q}$ with height $t - 1$ such that $R_\mathfrak{q}$ is not Gorenstein. If $t > 1$, then $\operatorname{grade} \mathfrak{q} = t - 1 > 0$ and there is an R-regular element x in $\mathfrak{q}$. We have $n - 1 > t - 1 > 0$ and $d - 1 \geq t - 1$. The residue ring R/xR is a $(d - 1)$-dimensional local ring that satisfies (S_{n-1}) and is locally Gorenstein in height $t - 1$. The prime ideal $\mathfrak{q}/xR$ of R/xR has height $(t - 1) - 1$, and $(R/xR)_{\mathfrak{q}/xR}$ is non-Gorenstein. This contradicts the minimality of d, and we must have $t = 1$.

Let $0 = \bigcap_{\mathfrak{p} \in \operatorname{Ass} R} L(\mathfrak{p})$ be a primary decomposition of the zero ideal 0 of R. Let A be the set of associated primes $\mathfrak{p}$ such that $R_\mathfrak{p}$ is not Gorenstein, and set $B = \operatorname{Ass} R \setminus A$. As $\mathfrak{q}$ has height $t - 1 = 0$, it belongs to A. Take a prime ideal $\mathfrak{r}$ of height t. The assumption (1) shows that $R_\mathfrak{r}$ is Gorenstein. Choosing a minimal prime $\mathfrak{s}$ contained in $\mathfrak{r}$, we see that $\mathfrak{s}$ belongs to B. Thus both A and B are nonempty. Put $I = \bigcap_{\mathfrak{p} \in A} L(\mathfrak{p})$ and $J = \bigcap_{\mathfrak{p} \in B} L(\mathfrak{p})$.

We claim that the ideal $I + J$ is $\mathfrak{m}$-primary, where $\mathfrak{m}$ stands for the maximal ideal of R. Indeed, assume that there exists a nonmaximal prime ideal P containing $I + J$. Then P contains some prime ideals $P_1 \in A$ and $P_2 \in B$. Set $e := \dim R_P$. We have $e < d$, and the fact that $P_1 \neq P_2$ implies $e \geq 1 = t$. The local ring R_P satisfies (S_n) and is locally Gorenstein in height t. Also, $(R_P)_{P_1 R_P} = R_{P_1}$ is non-Gorenstein, and $\operatorname{ht} P_1 R_P = \operatorname{ht} P_1 = 0 = t - 1$, since R satisfies (S_1) and the equality $\operatorname{Ass} R = \operatorname{Min} R$ holds. We thus get a contradiction to the minimality of d, and the claim follows.

Let $X = \operatorname{Spec} R \setminus \{\mathfrak{m}\}$ be the punctured spectrum, and let $V_1 = \mathsf{V}(I) \cap X$ and $V_2 = \mathsf{V}(J) \cap X$ be closed subsets of X. For each $i = 1, 2$ the set V_i is nonempty since it contains P_i, while $V_1 \cap V_2$ is empty. Thus X is disconnected, and Hartshorne's connectedness theorem [9, Proposition 2.1] implies that R has depth at most 1. Since R satisfies (S_1) and $d \geq t > 0$, it is a Cohen–Macaulay local ring of dimension 1. Our assumption (1) implies that $R = R_\mathfrak{m}$ is Gorenstein, and so is $R_\mathfrak{q}$, which is a contradiction. □

As is seen in the following example, the conclusion of Theorem 4.1 does not necessarily hold if one removes the assumption that R is local.

Example 4.2 Let A and B be commutative noetherian local rings with $\dim A \geq 1$ and $\dim B = 0$. Assume that A is locally Gorenstein in height one and that B is non-Gorenstein. Let $R = A \times B$ be a product ring. Then R is locally Gorenstein in height one, but not so in height zero.

Indeed, let $\mathfrak{p}$ be a prime ideal of R with height one. Then $\mathfrak{p} = P \times B$ for some prime ideal P of A with height one. Hence $R_\mathfrak{p} = A_P$ is Gorenstein. Set $\mathfrak{q} = A \times Q$, where Q is the maximal ideal of B. Then $\mathfrak{q}$ is a minimal prime of R, and $R_\mathfrak{q} = B$ is not Gorenstein.

For a commutative noetherian ring R of Krull dimension d we denote by $\operatorname{Assh} R$ the set of prime ideals $\mathfrak{p}$ of R such that $\dim R/\mathfrak{p} = d$. Note that one has inclusions $\operatorname{Assh} R \subseteq \operatorname{Min} R \subseteq \operatorname{Ass} R$. The next example says that the assertion of Theorem 4.1 is not necessarily true if $n = t$.

Example 4.3 Let S be a regular local ring of dimension 3 with regular system of parameters x, y, z. Then $R = S/(x) \cap (y, z)^2$ is a local ring of dimension 2 satisfying (S_1). The ring R is locally Gorenstein in height 1, but not so in height 0.

In fact, set $\mathfrak{p} = xS$ and $\mathfrak{q} = (y, z)S$. We have $\mathsf{Ass}\, R = \mathsf{Min}\, R = \{\mathfrak{p}R, \mathfrak{q}R\} \supsetneq \{\mathfrak{p}R\} = \mathsf{Assh}\, R$. Hence R has dimension 2, depth 1 and satisfies (S_1). Let P be a prime ideal of R with height 1. Write $P = Q/\mathfrak{p} \cap \mathfrak{q}^2$ for some prime ideal Q of S containing $\mathfrak{p} \cap \mathfrak{q}^2$. If Q contains $\mathfrak{q}$, then $(y, z)S \subsetneq Q \subsetneq (x, y, z)S$, which gives a contradiction. Thus Q does not contain $\mathfrak{q}$ but contains $\mathfrak{p}$, and the ring $R_P = S_Q/xS_Q$ is regular, whence Gorenstein. On the other hand, $R_{\mathfrak{q}R} = S_{\mathfrak{q}}/\mathfrak{q}^2 S_{\mathfrak{q}}$ is an artinian local ring of type 2, whence non-Gorenstein.

Now we prove the following theorem, which is the main result of this section. We should remark that this theorem extends Theorem 1.1 on Cohen–Macaulay local rings with canonical module to arbitrary commutative noetherian local rings. Compare with Corollary 3.5 the implication (1) $\Rightarrow$ (3) in this theorem.

Theorem 4.4 (= Theorem B) *Let n be a nonnegative integer. Let R be a commutative noetherian local ring satisfying Serre's condition (S_n). The following are equivalent.*

(1) *The subcategory $\Omega^n(\mathsf{mod}\, R)$ of $\mathsf{mod}\, R$ is extension closed.*
(2) *The equality $\Omega^n(\mathsf{mod}\, R) = \mathsf{S}_n(R)$ holds.*
(3) *The local ring $R_{\mathfrak{p}}$ is Gorenstein for all prime ideals $\mathfrak{p}$ of R with height less than n.*

Proof It is shown in [7, Theorem 3.8] (see also [10, Lemma 1.3]) that (3) implies (2). It is straightforward to check that (2) implies (1). Let us show that (1) implies (3). Thanks to Corollary 3.5, R is locally Gorenstein in height $n-1$. We may assume $n > 1$. Applying Theorem 4.1 to $t := n-1$ yields that R is locally Gorenstein in height at most $n-1$. □

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Compliance with Ethical Standards

Conflict of interests The authors declare that they have no conflict of interest.

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