

Signature Characters of Highest-Weight Representations of $U_q(\mathfrak{gl}_n)$

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Abstract We consider $U_q(\mathfrak{gl}_n)$, the quantum group of type A for $|q| = 1$, q generic. We provide formulas for signature characters of irreducible finite-dimensional highest weight modules and Verma modules. In both cases, the technique involves combinatorics of the Gelfand-Tsetlin bases. As an application, we obtain information about unitarity of finite-dimensional irreducible representations for arbitrary q : we classify the continuous spectrum of the unitarity locus. We also recover some known results in the classical limit $q \rightarrow 1$ that were obtained by different means. Finally, we provide several explicit examples of signature characters.

Keywords Quantum groups · Combinatorics

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1 Introduction

Let G be a group and V an irreducible complex representation of G . Determining (1) if V admits a non-degenerate invariant Hermitian form and (2) whether this form is positive-definite (i.e., the representation V is *unitary*) is an important problem in representation theory. The complete classification of unitary representations remains open in many cases (for example, noncompact Lie groups).

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A refinement of unitarity may be found in the context of signature characters. More precisely, suppose F is a finite-dimensional vector space equipped with a non-degenerate invariant Hermitian form $\langle \cdot, \cdot \rangle$ and an orthogonal basis $B = \{e_i\}$ with respect to the form. We define the signature of F to be

$$s(F) = \sum_{e_i \in B} \text{sgn}(\langle e_i, e_i \rangle),$$

that is, $s(F)$ is the number of basis elements with positive norm minus the number of basis elements with negative norm (this does not depend on the choice of orthogonal basis). Now let V be a (possibly infinite-dimensional) irreducible representation, with invariant Hermitian form $\langle \cdot, \cdot \rangle$, orthogonal basis B , and weight space decomposition $V = \bigoplus V_\mu$ with V_μ finite-dimensional. Suppose also that B is compatible with the weight space decomposition. Then the signature character is

$$ch_s(V) = \sum_{\mu} s(V_\mu) e^\mu,$$

where the sum is over degrees μ . Note that if V is unitary, $ch_s(V)$ is the usual character with respect to the grading. Thus, the signature character encodes information about failure of unitarity.

In a previous work [5], we have computed signature characters for rational Cherednik algebras and Hecke algebras of type A . In [1] and [4], classifications of unitary representations were provided for these algebras, respectively. In this paper, we consider the case of $U_q(\mathfrak{gl}_n)$, the quantum group of type A with $|q| = 1$ and q sufficiently generic (so, in particular, it is not a root of unity such that representations we study are reducible). We use the Gelfand-Tsetlin basis of (1) finite-dimensional, irreducible highest weight modules and (2) Verma modules, along with a combinatorial algorithm, to compute the signature of the norm of an arbitrary basis element. This allows us to compute the signature character (for both the finite and infinite-dimensional representations) in terms of q -binomial coefficients. As a byproduct of our method, the count $s(V_\mu)$ for a particular weight space can be determined in terms of data coming from the appropriate Gelfand-Tsetlin patterns.

We use our results to obtain information about when the finite-dimensional highest weight representation is unitary for general q : namely, for $n \geq 3$, if the indexing weight λ has distinct parts then the continuous spectrum of the unitarity locus (in the terminology of [1]) is an arc around $q = 1$ (with endpoints determined by λ). As a consequence, in the limit $q \rightarrow 1$ the form on finite-dimensional representations is positive-definite, agreeing with the fact that these representations of \mathfrak{gl}_n are unitary. For the Verma modules, if λ is a negative weight, we recover Wallach’s formula [6] in the limit $q \rightarrow 1$. We also provide a signature character formula in the limit $q \rightarrow 1$ for arbitrary λ for the Verma modules. Finally, we provide explicit formulas for signature characters for small values of n , and particular representations.

We mention that Wai-Ling Yee [7, 8] studied this problem in the classical limit $q \rightarrow 1$ for any Lie algebra. She used a wall-crossing technique (starting with Wallach’s formula [6] for λ in a particular region) to compute formulas for the signature character. However, our method is different from hers, since our approach relies on the combinatorics of the Gelfand-Tsetlin basis.

The outline of the paper is as follows. In the first section, we set up some notation and recall some preliminary facts about representation theory of $U_q(\mathfrak{gl}_n)$. In the second section, we compute signature characters for the finite-dimensional representations. In the third section, we compute signature characters for the infinite-dimensional representations. In both

of these sections, we include the applications mentioned above (unitarity and analysis of the classical $q \rightarrow 1$ limit). Finally, in the last section, we calculate signature characters in some particular cases.

2 Preliminaries

We set up some notation that will be used throughout the article and recall some standard facts and results from the literature; we will follow [3] and [2].

Let $q = e^{\pi is}$ for $0 < s < 2$ be on the unit circle and for any $k \in \mathbb{Z}$

$$[k] = [k]_q = \frac{q^k - q^{-k}}{q - q^{-1}} = \frac{\sin(\pi sk)}{\sin(\pi s)} \in \mathbb{R},$$

and $\{\cdot\} : \mathbb{R} \setminus 0 \rightarrow \{\pm 1\}$ be the sign map. As mentioned in the introduction, we will assume q is not a root of unity. Note the symmetries $[-k] = (-1)[k]$ and $[k]_q = [k]_{q^{-1}}$. Since norm factors only involve $[k]$, we may in fact restrict to $0 < s < 1$, in particular $\sin(\pi s)$ is positive. We will use the following computation throughout the paper

$$\{[k]_q\} = \left\{ \frac{e^{k\pi is} - e^{-k\pi is}}{e^{\pi is} - e^{-\pi is}} \right\} = \left\{ \frac{\sin(k\pi s)}{\sin(\pi s)} \right\} = \{\sin(k\pi s)\} = (-1)^{\lfloor ks \rfloor}. \tag{1}$$

We also note that $\lim_{q \rightarrow 1} [k] = k$.

For $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0}$, let $L(\lambda)$ denote the finite-dimensional $U_q(\mathfrak{gl}_n)$ -module with highest weight λ . Recall that this module has a basis $\{\zeta_\Lambda\}$ indexed by Gelfand-Tsetlin patterns $\Lambda = (\lambda_{i,j})$ for $1 \leq j \leq i \leq n$ with $\lambda_{n,i} = \lambda_i$ for $1 \leq i \leq n$, which comes from the multiplicity-one decomposition associated with the chain of subalgebras

$$U_q(\mathfrak{gl}_1) \subset U_q(\mathfrak{gl}_2) \subset \dots \subset U_q(\mathfrak{gl}_n).$$

Recall that Gelfand-Tsetlin patterns satisfy the *interlacing condition*

$$\lambda_{m,i} \geq \lambda_{m-1,i} \geq \lambda_{m,i+1} \tag{2}$$

for all m, i . These arrays may be visualized as follows

$$\begin{array}{cccccc} \lambda_1 & & \lambda_2 & & \dots & & \lambda_{n-1} & & \lambda_n \\ & \lambda_{n-1,1} & & \lambda_{n-1,2} & \dots & \lambda_{n-1,n-2} & & \lambda_{n-1,n-1} & \\ & & \ddots & & \dots & & \ddots & & \\ & & & \lambda_{2,1} & & \lambda_{2,2} & & & \\ & & & & & \lambda_{1,1} & & & \end{array}$$

The highest weight element is ζ_{Λ_0} , where Λ_0 is the pattern with $\lambda_{m,i} = \lambda_i$ for all m, i . We will write $GT(\lambda)$ to denote the Gelfand-Tsetlin basis of $L(\lambda)$.

For $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_i - \lambda_{i+1} \in \mathbb{R}$ (i.e., generic differences), let $M(\lambda)$ denote the Verma module with highest weight λ . Recall that this module has a basis $\{\zeta_\Lambda\}$ indexed by arrays $\Lambda = (\lambda_{i,j})$ for $1 \leq j \leq i \leq n$ with $\lambda_{n,i} = \lambda_i$ for $1 \leq i \leq n$. Such arrays must satisfy the condition

$$\lambda_{m,i} - \lambda_{m-1,i} \in \mathbb{Z}_{\geq 0}$$

for all m, i (crucially, the interlacing condition need not hold).

The highest weight element is ζ_{Λ_0} where Λ_0 is the array with $\lambda_{m,i} = \lambda_i$ for all m, i . We will write $A(\lambda)$ to denote this basis of $M(\lambda)$.

For $\zeta_\Lambda \in L(\lambda)$ or $M(\lambda)$ we define the weight function by

$$\text{wt}(\zeta_\Lambda) = (\lambda_{11}, \lambda_{21} + \lambda_{22} - \lambda_{11}, \dots, \sum_{i=1}^{n-1} \lambda_{n-1,i} - \sum_{i=1}^{n-2} \lambda_{n-2,i}, \sum_{i=1}^n \lambda_i - \sum_{i=1}^{n-1} \lambda_{n-1,i}) \in \mathbb{R}^n. \tag{3}$$

Note that $\lambda - \text{wt}(\zeta_\Lambda)$ is a vector in Λ_r^+ , the positive root lattice. So $\text{wt}(\zeta_\Lambda) = \lambda - \mu$, for $\mu \in \Lambda_r^+$.

We will write (\cdot, \cdot) to denote the standard Hermitian form on $L(\lambda)$ and $M(\lambda)$. We recall that bases $GT(\lambda)$ and $A(\lambda)$ are orthogonal with respect to this form, and that the operators $e_m, f_m \in U_q(\mathfrak{gl}_n)$ are adjoint with respect to this form. We will compute $ch_s(L(\lambda))$ and $ch_s(M(\lambda))$ by computing the signs of the norms $(\zeta_\Lambda, \zeta_\Lambda)$ for basis elements ζ_Λ . We will use a combinatorial algorithm that describes $(\zeta_\Lambda, \zeta_\Lambda)$ in terms of $(\zeta_{\Lambda_0}, \zeta_{\Lambda_0})$.

We define the coefficients

$$v_{m,i} = i - \lambda_{m,i} - 1$$

and

$$v_i = i - \lambda_i - 1.$$

We will write $v_{m,i}^\Lambda$ to indicate the underlying array Λ if it is not clear from context (for example, if we are comparing different arrays and wish to emphasize which array the coefficients are coming from).

We also define the following coefficients, which are expressed in terms of $[k]$:

$$\beta_{mi\Lambda} = \prod_{j=1}^i \frac{[v_{mi} - v_{m+1,j} + 1]}{[v_{mi} - v_j + 1]} \prod_{j=1}^{i-1} \frac{[v_{mi} - v_{m-1,j}]}{[v_{mi} - v_j]}$$

and

$$\gamma_{mi\Lambda} = \prod_{j=1}^i [v_{mi} - v_j] \prod_{j=1}^{i-1} [v_{mi} - v_j - 1] \prod_{j=i+1}^{m+1} [v_{m+1,j} - v_{mi}] \prod_{j=i}^{m-1} [v_{m-1,j} - v_{mi} + 1]$$

and

$$\tau_{mi\Lambda} = \prod_{\substack{j=1 \\ j \neq i}}^m \frac{1}{[v_{mi} - v_{mj}]}.$$

We also write, for $X \in \mathbb{Z}_{>0}$

$$[X]! = [X][X - 1] \cdots [2][1]$$

and for $X \in \mathbb{R}, k \in \mathbb{Z}_+$

$$[X]_k! = [X][X - 1] \cdots [X - k]$$

and

$$(X)_k! = X(X - 1) \cdots (X - k) = \lim_{q \rightarrow 1} [X]_k!.$$

We note that (assuming $X \in \mathbb{R} \setminus \mathbb{Z}$ and $k \in \mathbb{Z}_+$)

$$\begin{aligned} \{(X)_k!\} &= \begin{cases} 1, & \text{if } X - k > 0 \\ (-1)^{k+1}, & \text{if } X < 0 \\ (-1)^{\lfloor X-k \rfloor}, & \text{if } X > 0 \text{ and } X - k < 0 \end{cases} \\ &= (-1)^{\min\{0, \lfloor X \rfloor + 1\}} (-1)^{\min\{0, \lfloor X-k \rfloor\}}. \end{aligned} \tag{4}$$

We will need the following result which describes how the quantum group operators act on basis elements in terms of Gelfand-Tsetlin patterns (or arrays in the case of Verma modules).

Theorem 2.1 ([3]) *Let $\zeta_\Lambda \in GT(\lambda)$ and $m < n$. The operators e_m and f_m (for $m = 1, \dots, n - 1$) act on the module $L(\lambda)$ (or $M(\lambda)$) as follows*

$$e_m \cdot \zeta_\Lambda = \sum_{i \leq m} \gamma_{mi\Lambda} \tau_{mi\Lambda} \zeta_{\Lambda_{m,i}^+}$$

and

$$f_m \cdot \zeta_\Lambda = \sum_{i \leq m} \beta_{mi\Lambda} \tau_{mi\Lambda} \zeta_{\Lambda_{m,i}^-},$$

where $\Lambda_{m,i}^+$ and $\Lambda_{m,i}^-$ are the Gelfand-Tsetlin patterns (arrays) obtained from Λ by increasing and decreasing (resp.) the (m, i) -entry by 1.

3 Signature Characters of Finite-Dimensional Modules

Let $L(\lambda)$ be the irreducible finite-dimensional representation with highest weight λ . Let Λ be a Gelfand-Tsetlin pattern with first row equal to λ ; recall that it satisfies the interlacing condition (2). We will compute the sign $\{(\zeta_\Lambda, \zeta_\Lambda)\}$ through a series of lemmas that will illustrate the combinatorial technique mentioned in Section 1.

Lemma 3.1 *Let $\Lambda = (\lambda_{i,j})$ be a Gelfand-Tsetlin pattern with first row $\lambda = (\lambda_1, \dots, \lambda_n)$. Let X be the array with entry (m, i) of Λ increased by 1 (for $1 \leq i \leq m < n$), and all other entries remain the same. We have*

$$(\zeta_X, \zeta_X) = \frac{\beta_{miX} \tau_{miX}}{\gamma_{mi\Lambda} \tau_{mi\Lambda}} (\zeta_\Lambda, \zeta_\Lambda).$$

Proof Fix m, i . Let $X = \Lambda_{m,i}^+$, then $\Lambda = X_{m,i}^-$. So by Theorem 2.1 and orthogonality, we have

$$(\zeta_X, e_m \zeta_\Lambda) = \gamma_{mi\Lambda} \tau_{mi\Lambda} (\zeta_X, \zeta_X).$$

On the other hand, $(\zeta_X, e_m \zeta_\Lambda) = (f_m \zeta_X, \zeta_\Lambda)$ by adjointness, and again by Theorem 2.1 and orthogonality

$$(f_m \zeta_X, \zeta_\Lambda) = \beta_{miX} \tau_{miX} (\zeta_\Lambda, \zeta_\Lambda).$$

So we have

$$(\zeta_X, \zeta_X) = \frac{\beta_{miX} \tau_{miX}}{\gamma_{mi\Lambda} \tau_{mi\Lambda}} (\zeta_\Lambda, \zeta_\Lambda).$$

□

We will now measure the sign change between the two norms.

Lemma 3.2 *Let $\Lambda = (\lambda_{i,j})$ be a Gelfand-Tsetlin pattern with first row $\lambda = (\lambda_1, \dots, \lambda_n)$. Let X be the array with entry (m, i) of Λ increased by 1, and all other entries the same. We have*

$$\{(\zeta_X, \zeta_X)\} = s_{mi\Lambda} \{(\zeta_\Lambda, \zeta_\Lambda)\},$$

where $s_{mi\Lambda}$ is equal to

$$(-1) \left\{ \prod_{j=1, j \neq i}^m [v_{mi}^\Lambda - v_{mj}^\Lambda] [v_{mi}^\Lambda - v_{mj}^\Lambda - 1] \right\} \left\{ \prod_{j=1}^{m+1} [v_{mi}^\Lambda - v_{m+1,j}^\Lambda] \right\} \left\{ \prod_{j=1}^{m-1} [v_{mi}^\Lambda - 1 - v_{m-1,j}^\Lambda] \right\}.$$

Proof By Lemma 3.1, we have

$$\begin{aligned}
 (\zeta_X, \zeta_X) &= \frac{\beta_{miX} \tau_{miX}}{\gamma_{mi\Delta} \tau_{mi\Delta}} (\zeta_\Delta, \zeta_\Delta) = \prod_{j=1, j \neq i}^m \frac{[v_{mi}^\Delta - 1 - v_{mj}^\Delta]^{-1}}{[v_{mi}^\Delta - v_{mj}^\Delta]^{-1}} \prod_{j=1}^i \frac{[v_{mi}^\Delta - v_{m+1,j}^\Delta]}{[v_{mi}^\Delta - v_j^\Delta]} \prod_{j=1}^{i-1} \frac{[v_{mi}^\Delta - 1 - v_{m-1,j}^\Delta]}{[v_{mi}^\Delta - 1 - v_j^\Delta]} \\
 &\times \prod_{j=1}^i \frac{1}{[v_{mi}^\Delta - v_j^\Delta]} \prod_{j=1}^{i-1} \frac{1}{[v_{mi}^\Delta - v_{j-1}^\Delta]} \prod_{j=i+1}^{m+1} \frac{1}{[v_{m+1,j}^\Delta - v_{mi}^\Delta]} \prod_{j=i}^{m-1} \frac{1}{[v_{m-1,j}^\Delta - v_{mi}^\Delta + 1]} (\zeta_\Delta, \zeta_\Delta),
 \end{aligned}$$

where we have used $v_{m,i}^X = i - \lambda_{m,i}^X - 1 = i - (\lambda_{m,i}^\Delta + 1) - 1 = v_{m,i}^\Delta - 1$ and $\lambda_{m,i}^X$ denotes the (m, i) entry of X . Using $[k] = (-1)[-k]$ and looking at signs only, we have

$$\begin{aligned}
 \{(\zeta_X, \zeta_X)\} &= \{(\zeta_\Delta, \zeta_\Delta)\} \left\{ \prod_{j=1, j \neq i}^m [v_{mi}^\Delta - v_{mj}^\Delta] [v_{mi}^\Delta - v_{mj}^\Delta - 1] \right\} (-1)^{m-i+1} \\
 &\times \left\{ \prod_{j=1}^{m+1} [v_{mi}^\Delta - v_{m+1,j}^\Delta] \right\} (-1)^{m-i} \left\{ \prod_{j=1}^{m-1} [v_{mi}^\Delta - 1 - v_{m-1,j}^\Delta] \right\} \\
 &= \{(\zeta_\Delta, \zeta_\Delta)\} (-1) \left\{ \prod_{j=1, j \neq i}^m [v_{mi}^\Delta - v_{mj}^\Delta] [v_{mi}^\Delta - v_{mj}^\Delta - 1] \right\} \left\{ \prod_{j=1}^{m+1} [v_{mi}^\Delta - v_{m+1,j}^\Delta] \right\} \\
 &\times \left\{ \prod_{j=1}^{m-1} [v_{mi}^\Delta - 1 - v_{m-1,j}^\Delta] \right\},
 \end{aligned}$$

as desired. □

Lemma 3.3 *Let $\Delta = (\lambda_{i,j})$ be a Gelfand-Tsetlin array with first row $\lambda = (\lambda_1, \dots, \lambda_n)$. Suppose that increasing the (m, i) entry of Δ by k (and keeping the other entries fixed) results in a Gelfand-Tsetlin pattern X . We have*

$$\{(\zeta_X, \zeta_X)\} = s_{mi\Delta}^{(k)} \{(\zeta_\Delta, \zeta_\Delta)\},$$

where

$$\begin{aligned}
 s_{mi\Delta}^{(k)} &= \left\{ \prod_{j=1, j \neq i}^m [v_{mi}^\Delta - v_{mj}^\Delta] [v_{mi}^\Delta - k - v_{mj}^\Delta] \right\} (-1)^k \\
 &\times \left\{ \prod_{j=1}^{m+1} [v_{mi}^\Delta - v_{m+1,j}^\Delta] [v_{mi}^\Delta - v_{m+1,j}^\Delta - 1] \cdots [v_{mi}^\Delta - v_{m+1,j}^\Delta - (k-1)] \right\} \\
 &\times \left\{ \prod_{j=1}^{m-1} [v_{mi}^\Delta - v_{m-1,j}^\Delta - 1] [v_{mi}^\Delta - v_{m-1,j}^\Delta - 2] \cdots [v_{mi}^\Delta - v_{m-1,j}^\Delta - k] \right\}.
 \end{aligned}$$

Proof We iterate the result of Lemma 3.2 k -times, and take into account signs, to obtain the formula. □

Lemma 3.4 *Let $1 \leq i \leq m < n$ be fixed. Let Δ be a Gelfand-Tsetlin pattern with first row $\lambda = (\lambda_1, \dots, \lambda_n)$. Suppose it has entries $\lambda_{i,j}$ such that $\lambda_{l,j} = \lambda_j$ for $j < i$ and any l and*

$\lambda_{l,i} = \lambda_i$ for $l > m$. Let X be the Gelfand-Tsetlin pattern that agrees with Λ in all entries except (m, i) , where it is equal to $\lambda_{m,i} + (\lambda_i - \lambda_{m,i}) = \lambda_i$. We have

$$\{(\zeta_X, \zeta_X)\} = s_{mi\Lambda}^{(\lambda_i - \lambda_{m,i})} \{(\zeta_\Lambda, \zeta_\Lambda)\},$$

where

$$s_{mi\Lambda}^{(\lambda_i - \lambda_{m,i})} = \left\{ \prod_{i < j \leq m} [v_{mi}^\Lambda - v_{mj}^\Lambda][v_i - v_{mj}^\Lambda] \right\} (-1)^{v_{mi}^\Lambda - v_i} \left\{ [v_{mi}^\Lambda - v_i]_{(\lambda_i - \lambda_{m,i} - 1)}! \right\} \\ \times \left\{ \prod_{i < j \leq m+1} [v_{mi}^\Lambda - v_{m+1,j}^\Lambda]_{(\lambda_i - \lambda_{m,i} - 1)}! \right\} \left\{ \prod_{i \leq j \leq m-1} [v_{mi}^\Lambda - v_{m-1,j-1}^\Lambda]_{(\lambda_i - \lambda_{m,i} - 1)}! \right\}.$$

Proof We first note that, because of the specified parts of Λ , we have

$$v_{mj}^\Lambda = v_j \text{ for } j < i \\ v_{m+1,i}^\Lambda = v_i \\ v_{m-1,j}^\Lambda = v_j \text{ for } j < i \\ v_{m+1,j}^\Lambda = v_j \text{ for } j < i.$$

We then use Lemma 3.3 for this particular choice of Λ and cancel terms according to whether they appear an even or odd number of times. □

Theorem 3.1 *Let Λ be a Gelfand-Tsetlin pattern with first row $\lambda = (\lambda_1, \dots, \lambda_n)$. Then*

$$\{(\zeta_\Lambda, \zeta_\Lambda)\} = \prod_{1 \leq i \leq m < n} s_{mi\Lambda}^{(\lambda_i - \lambda_{m,i})} \\ = \prod_{1 \leq i \leq m < n} \left\{ \prod_{i < j \leq m} [v_{mi}^\Lambda - v_{mj}^\Lambda][v_i - v_{mj}^\Lambda] \right\} (-1)^{v_{mi}^\Lambda - v_i} \left\{ [v_{mi}^\Lambda - v_i]_{(\lambda_i - \lambda_{m,i} - 1)}! \right\} \\ \times \left\{ \prod_{i < j \leq m+1} [v_{mi}^\Lambda - v_{m+1,j}^\Lambda]_{(\lambda_i - \lambda_{m,i} - 1)}! \right\} \left\{ \prod_{i \leq j \leq m-1} [v_{mi}^\Lambda - v_{m-1,j-1}^\Lambda]_{(\lambda_i - \lambda_{m,i} - 1)}! \right\}.$$

Proof Obtained from Lemma 3.4, starting with Λ and applying the required series of steps in order to produce Λ_0 . □

Definition 1 *For $\zeta_\Lambda \in GT(\lambda)$ let*

$$s_q(\zeta_\Lambda) = \prod_{1 \leq i \leq m < n} \left\{ \prod_{i < j \leq m} [v_{mi}^\Lambda - v_{mj}^\Lambda][v_i - v_{mj}^\Lambda] \right\} (-1)^{v_{mi}^\Lambda - v_i} \left\{ [v_{mi}^\Lambda - v_i]_{(\lambda_i - \lambda_{m,i} - 1)}! \right\} \\ \times \left\{ \prod_{i < j \leq m+1} [v_{mi}^\Lambda - v_{m+1,j}^\Lambda]_{(\lambda_i - \lambda_{m,i} - 1)}! \right\} \left\{ \prod_{i \leq j \leq m-1} [v_{mi}^\Lambda - v_{m-1,j-1}^\Lambda]_{(\lambda_i - \lambda_{m,i} - 1)}! \right\},$$

which is equal to ± 1 , depending on Λ and q .

As a result of the previous computation of the sign of the norm for any basis element and Definition 1, we obtain a formula for the signature character.

Theorem 3.2 *We have*

$$ch_s(L(\lambda)) = \sum_{\zeta_\Lambda \in GT(\lambda)} s_q(\zeta_\Lambda) e^{wt(\zeta_\Lambda)}.$$

Recall that $wt(\cdot)$ was defined in Eq. 3.

We will provide an alternate formula for $s_q(\zeta_\Lambda)$ by cancelling off signs. This will be easier to work with.

Lemma 3.5 *Let Λ be a Gelfand-Tsetlin pattern with first row $\lambda = (\lambda_1, \dots, \lambda_n)$. We have*

$$\begin{aligned} s_q(\zeta_\Lambda) &= \prod_{1 \leq i \leq m < n} \left\{ \prod_{i < j \leq m} [v_{mj}^\Lambda - v_{mi}^\Lambda] [v_{mj}^\Lambda - v_i] \right\} \left\{ [v_{mi}^\Lambda - v_i]_{(\lambda_i - \lambda_{m,i} - 1)!} \right\} \\ &\times \left\{ \prod_{i < j \leq m+1} [v_{m+1,j}^\Lambda - v_{mi}^\Lambda + (\lambda_i - \lambda_{m,i} - 1)]_{(\lambda_i - \lambda_{m,i} - 1)!} \right\} \\ &\times \left\{ \prod_{i \leq j \leq m-1} [v_{m-1,j}^\Lambda - v_{mi}^\Lambda + (\lambda_i - \lambda_{m,i})]_{(\lambda_i - \lambda_{m,i} - 1)!} \right\} \\ &= \prod_{1 \leq i \leq m < n} \left\{ \prod_{i < j \leq m} [v_{mj}^\Lambda - v_{mi}^\Lambda] [v_{mj}^\Lambda - v_i] \right\} \left\{ [v_{mi}^\Lambda - v_i]_{(\lambda_i - \lambda_{m,i} - 1)!} \right\} \\ &\times \left\{ \prod_{i < j \leq m+1} [v_{m+1,j}^\Lambda - v_i - 1]_{(\lambda_i - \lambda_{m,i} - 1)!} \right\} \\ &\times \left\{ \prod_{i \leq j \leq m-1} [v_{m-1,j}^\Lambda - v_i]_{(\lambda_i - \lambda_{m,i} - 1)!} \right\} \end{aligned} \tag{5}$$

where now all q -integers $[n]$ appearing above satisfy $n \geq 0$.

Proof The formulas are obtained from algebraic manipulations applied to the formula for $s_q(\zeta_\Lambda)$ in Definition 1, noting that $[m]_q = (-1)[-m]_q$ for all $m \in \mathbb{Z}$.

To show that every q -integer $[n]$ appearing in the formula satisfies $n \geq 0$, we use the interlacing condition. For $i < j \leq m$ we have

$$\begin{aligned} v_{mi}^\Lambda - v_{mj}^\Lambda &= (i - \lambda_{m,i} - 1) - (j - \lambda_{m,j} - 1) = (i - j) + (\lambda_{m,j} - \lambda_{m,i}) \leq 0 \\ v_i - v_{mj}^\Lambda &= (i - \lambda_i - 1) - (j - \lambda_{m,j} - 1) = (i - j) + (\lambda_{m,j} - \lambda_i) \leq 0. \end{aligned}$$

We also have

$$v_{mi}^\Lambda - v_i = (i - \lambda_{m,i} - 1) - (i - \lambda_i - 1) = \lambda_i - \lambda_{m,i} \geq 0.$$

For $i < j \leq m + 1$, we have

$$v_{mi}^\Lambda - v_{m+1,j}^\Lambda = (i - \lambda_{m,i} - 1) - (j - \lambda_{m+1,j} - 1) = (i - j) + (\lambda_{m+1,j} - \lambda_{m,i}) \leq 0.$$

Finally, for $i \leq j \leq m - 1$, we have

$$\begin{aligned} v_{mi}^\Lambda - v_{m-1,j}^\Lambda - 1 &= (i - \lambda_{m,i} - 1) - (j - \lambda_{m-1,j} - 1) - 1 \\ &= (i - j) + (\lambda_{m-1,j} - \lambda_{m,i}) - 1 < 0. \end{aligned}$$

Putting the negative signed contributions together yields

$$(-1)^{v_{mi}^\Lambda - v_i} (-1)^{(\lambda_i - \lambda_{m,i})(m+1-i)} (-1)^{(\lambda_i - \lambda_{m,i})(m-i)} = (-1)^{v_{mi}^\Lambda - v_i + \lambda_i - \lambda_{m,i}} = 1$$

as desired. □

We now use Theorem 3.2 to obtain some results about unitarity of $L(\lambda)$ for particular values of s (recall that $q = e^{\pi is}$), when λ has distinct parts. In particular, for $n = 2$, we will give a characterization of such s , and for $n \geq 3$ we will show that the only interval is the one around $q = 1$.

Lemma 3.6 *Let $n = 2$ and $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_1 - \lambda_2 > 0$. Then $L(\lambda)$ is unitary if and only if the signs*

$$\{[d]\} = \{[\lambda_1 - \lambda_2 - d + 1]\},$$

for all $1 \leq d \leq \lambda_1 - \lambda_2$.

Proof We let Λ be the Gelfand-Tsetlin pattern with λ_1, λ_2 in the first row and $\lambda_{11} = \lambda_1 - d$ in the second row for $1 \leq d \leq \lambda_1 - \lambda_2$. We use Eq. 5 to compute

$$s_q(\zeta_\Lambda) = \{[d][d - 1] \cdots [2][1]\} \times \{[\lambda_1 - \lambda_2][\lambda_1 - \lambda_2 - 1] \cdots [\lambda_1 - \lambda_2 - d + 1]\}.$$

So, $L(\lambda)$ is unitary if and only if the ratio

$$\frac{\{[d][d - 1] \cdots [2][1]\} \times \{[\lambda_1 - \lambda_2][\lambda_1 - \lambda_2 - 1] \cdots [\lambda_1 - \lambda_2 - d + 1]\}}{\{[d - 1] \cdots [2][1]\} \times \{[\lambda_1 - \lambda_2][\lambda_1 - \lambda_2 - 1] \cdots [\lambda_1 - \lambda_2 - d + 2]\}} = \{[d][\lambda_1 - \lambda_2 - d + 1]\}$$

is equal to 1, which gives the result. □

Lemma 3.7 *Let $\lambda = (\lambda_1, \lambda_2)$ and $\lambda' = (\lambda_1 + 1, \lambda_2)$. Then if $L(\lambda)$ and $L(\lambda')$ are both unitary, the signs $\{[1]\}, \{[2]\}, \dots, \{[\lambda_1 - \lambda_2]\}, \{[\lambda_1 - \lambda_2 + 1]\}$ are all equal to 1, or equivalently, $s < \frac{1}{\lambda_1 - \lambda_2 + 1}$.*

Proof Using Lemma 3.6 for $L(\lambda)$ gives that $[1]$ and $[\lambda_1 - \lambda_2]$ have the same sign. Similarly, applying it to $L(\lambda')$ gives that $[1]$ and $[\lambda_1 - \lambda_2 + 1]$ have the same sign, as do $[2]$ and $[\lambda_1 - \lambda_2]$. By iterating this argument and using $\{[1]\} = 1$, we get the result. The equivalent condition is obtained by using Eq. 1. □

Theorem 3.3 *Let $n \geq 3$ and $q = e^{\pi is}$. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition with distinct parts, $\lambda_i - \lambda_j > 0$ for $1 \leq i < j \leq n$. Then the module $L(\lambda)$ is unitary if and only if $0 < s < 1/M_\lambda$, where*

$$M_\lambda = (\lambda_1 - \lambda_n) + (n - 2).$$

Proof First we will show that $L(\lambda)$ is unitary for $0 < s < \frac{1}{M_\lambda}$, but it is not unitary for $\frac{1}{M_\lambda} < s < \frac{1}{\lambda_1 - \lambda_n + (n-3)}$. Recall by Eq. 1, we have

$$\{[k]_q\} = (-1)^{\lfloor ks \rfloor}.$$

Let $0 < s < \frac{1}{M_\lambda}$ and $\zeta_\Lambda \in GT(\lambda)$ be arbitrary. Recall the formula for $s_q(\zeta_\Lambda)$ in Eq. 5. Using the interlacing condition, one can check that each term $[k]$ appearing in that formula satisfies $0 < k \leq v_n^\Lambda - v_1^\Lambda - 1$, so that

$$0 < ks \leq (v_n^\Lambda - v_1^\Lambda - 1)s < \frac{v_n^\Lambda - v_1^\Lambda - 1}{M_\lambda} = 1.$$

Thus, $[ks] = 0$, so that each factor appearing in the formula is positive. Thus, $s_q(\zeta_\Lambda) = 1$ for all ζ_Λ , so the representation $L(\lambda)$ is unitary for s in the interval $(0, 1/M_\lambda)$.

Now let $\frac{1}{M_\lambda} < s < \frac{1}{\lambda_1 - \lambda_n + (n-3)}$. Consider the right-justified GT pattern Λ , with $\Lambda_{ij} = \lambda_{j+1}$ for $1 \leq i < n$. Inspecting Eq. 5, we find a single term $[k]$ with $k = v_{n-1, n-1}^\Lambda - v_1^\Lambda = \lambda_1 - \lambda_n + (n - 2)$, so

$$1 < ks < 2,$$

since $\frac{1}{\lambda_1 - \lambda_n + (n-2)} < s < \frac{2}{\lambda_1 - \lambda_n + (n-2)}$, and hence $(-1)^{[ks]} = (-1)$. On the other hand, one can see that all other terms $[k']$ that appear in Eq. 5 satisfy $k' < \lambda_1 - \lambda_n + (n - 2)$, and since $s < \frac{1}{\lambda_1 - \lambda_n + (n-3)}$, we have $0 < sk' < 1$. Thus $(-1)^{[sk']} = 1$, and therefore $s_q(\zeta_\Lambda) = -1$ and $L(\lambda)$ is not unitary.

Now we will show that if $L(\lambda)$ is unitary, we must have the above bounds on s . We will induct on n . For the base case $n = 3$, let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ and assume $L(\lambda)$ is unitary. Then the restriction of $L(\lambda)$ as a $U_q(\mathfrak{gl}_2)$ -module must decompose as a sum of unitary submodules, and these are indexed by μ interlacing λ . Thus $L(\mu')$ with $\mu' = (\lambda_1, \lambda_3)$ is unitary. Similarly $L(\mu)$ with $\mu = (\lambda_1 - 1, \lambda_3)$ is also unitary. By Lemma 3.7, this implies $s < \frac{1}{\lambda_1 - \lambda_3}$. But in the first part of the proof, we proved that $L(\lambda)$ is not unitary for $\frac{1}{\lambda_1 - \lambda_3 + 1} < s < \frac{1}{\lambda_1 - \lambda_3}$. Thus, we must have $0 < s < \frac{1}{\lambda_1 - \lambda_3 + 1}$, which establishes the base case. We assume the result for $n - 1 \geq 3$, and will show it holds for n . Let $\lambda = (\lambda_1, \dots, \lambda_n)$ and suppose $L(\lambda)$ is unitary. As in the argument for $n = 3$, the restriction of $L(\lambda)$ over $U_q(\mathfrak{gl}_{n-1})$ must decompose as a sum of unitary submodules, indexed by μ interlacing λ . In particular, we may take μ with distinct parts satisfying $\mu_1 = \lambda_1$ and $\mu_{n-1} = \lambda_n$. Thus, by the induction hypothesis, we must have $s < \frac{1}{\lambda_1 - \lambda_n + (n-3)}$. But in the first part of the proof, we showed that $L(\lambda)$ is not unitary for $\frac{1}{\lambda_1 - \lambda_n + (n-2)} < s < \frac{1}{\lambda_1 - \lambda_n + (n-3)}$. Thus $0 < s < \frac{1}{\lambda_1 - \lambda_n + (n-2)}$ as desired. □

Remark Note that the previous result implies that $L(\lambda)$ is unitary in the classical limit $q = 1$, which is known from representation theory of \mathfrak{gl}_n . (This is also immediately seen by inspection of Eq. 5 since $\lim_{q \rightarrow 1} [k] = k$ and in that formula all appearing k are positive.)

We note that at $n = 2$, there are several intervals of s for which $L(\lambda)$ is unitary. We use Lemma 3.6 to compute some examples. Let $\lambda = (\lambda_1, \lambda_2)$. We obtain the following intervals for s depending on $\lambda_1 - \lambda_2$ where $L(\lambda)$ is unitary:

- For $\lambda_1 - \lambda_2 = 3$, we get $0 < s < \frac{1}{3}$ and $\frac{2}{3} < s < 1$.
- For $\lambda_1 - \lambda_2 = 4$, we get $0 < s < \frac{1}{4}$ and $\frac{1}{2} < s < \frac{2}{3}$.
- For $\lambda_1 - \lambda_2 = 5$, we get $0 < s < \frac{1}{5}$ and $\frac{4}{5} < s < 1$.
- For $\lambda_1 - \lambda_2 = 6$, we get $0 < s < \frac{1}{6}$ and $\frac{2}{5} < s < \frac{1}{2}$ and $\frac{2}{3} < s < \frac{3}{4}$.
- For $\lambda_1 - \lambda_2 = 7$, we get $0 < s < \frac{1}{7}$, $\frac{1}{3} < s < \frac{2}{5}$, $\frac{3}{5} < s < \frac{2}{3}$, and $\frac{6}{7} < s < 1$.

4 Signature Characters for Verma Modules

In this section, we compute signature characters for the modules $M(\lambda)$. The Gelfand-Tsetlin formulae in Theorem 2.1 still hold, but the basis elements are indexed by arrays, not patterns (recall the discussion in the first section).

Definition 2 For $\zeta_\Lambda \in A(\lambda)$, let

$$\begin{aligned} \tilde{s}_q(\zeta_\Lambda) = & \prod_{1 \leq i \leq m < n} \left\{ \prod_{i < j \leq m} [v_{mj}^\Lambda - v_{mi}^\Lambda] [v_{mj}^\Lambda - v_i^\Lambda] \right\} \left\{ [v_{mi}^\Lambda - v_i^\Lambda]_{(\lambda_i - \lambda_{m,i} - 1)}! \right\} \\ & \times \left\{ \prod_{i < j \leq m+1} [v_{m+1,j}^\Lambda - v_i^\Lambda - 1]_{(\lambda_i - \lambda_{m,i} - 1)}! \right\} \\ & \times \left\{ \prod_{i \leq j \leq m-1} [v_{m-1,j}^\Lambda - v_i^\Lambda]_{(\lambda_i - \lambda_{m,i} - 1)}! \right\}, \end{aligned}$$

it is equal to ± 1 depending on Λ and q .

Note that the q -integers $[n]_q$ appearing above do not necessarily satisfy $n \geq 0$ if Λ is a Gelfand-Tsetlin array, unlike for Gelfand-Tsetlin patterns (see Lemma 3.5).

Theorem 4.1 We have

$$ch_s(M(\lambda)) = \sum_{\zeta_\Lambda \in A(\lambda)} \tilde{s}_q(\zeta_\Lambda) e^{wt(\zeta_\Lambda)}.$$

Recall that $wt(\cdot)$ was defined in Eq. 3.

Proof The proof of Theorem 3.2 in the previous section relied only on algebraic manipulations applied to the formulas from Theorem 2.1, and did not use the interlacing condition. The proof therefore carries over directly to the case of Verma modules and Gelfand-Tsetlin arrays, which gives the stated result. \square

Definition 3 For $\zeta_\Lambda \in A(\lambda)$, let

$$\begin{aligned} \tilde{c}(\zeta_\Lambda) = & \prod_{1 \leq i \leq m < n} \left\{ \prod_{i < j \leq m} (v_{mj}^\Lambda - v_{mi}^\Lambda)(v_{mj}^\Lambda - v_i^\Lambda) \right\} \\ & \times \left\{ \prod_{i < j \leq m+1} (v_{m+1,j}^\Lambda - v_i^\Lambda - 1)(v_{m+1,j}^\Lambda - v_i^\Lambda - 2) \cdots (v_{m+1,j}^\Lambda - v_i^\Lambda - 1 - (\lambda_i - \lambda_{m,i} - 1)) \right\} \\ & \times \left\{ \prod_{i < j \leq m-1} (v_{m-1,j}^\Lambda - v_i^\Lambda)(v_{m-1,j}^\Lambda - v_i^\Lambda - 1) \cdots (v_{m-1,j}^\Lambda - v_i^\Lambda - (\lambda_i - \lambda_{m,i} - 1)) \right\}. \end{aligned}$$

Proposition 4.1 For $\zeta_\Lambda \in A(\lambda)$, we have

$$\lim_{q \rightarrow 1} \tilde{s}_q(\zeta_\Lambda) = \tilde{c}(\zeta_\Lambda).$$

Proof Follows since

$$(v_{mi}^\Lambda - v_i^\Lambda) (v_{mi}^\Lambda - v_i^\Lambda - 1) \cdots (v_{mi}^\Lambda - v_i^\Lambda - (\lambda_i - \lambda_{m,i} - 1))$$

is positive, as is

$$(v_{m-1,i}^\Lambda - v_i^\Lambda) (v_{m-1,i}^\Lambda - v_i^\Lambda - 1) \cdots (v_{m-1,i}^\Lambda - v_i^\Lambda - (\lambda_i - \lambda_{m,i} - 1)).$$

□

We use this to provide a formula for the signature character of the representation $M(\lambda)$ of \mathfrak{gl}_n .

Corollary 4.1 *We have*

$$\lim_{q \rightarrow 1} ch_s(M(\lambda)) = \sum_{\zeta_\Lambda \in A(\lambda)} \tilde{c}(\zeta_\Lambda) e^{wt(\zeta_\Lambda)}.$$

We now study the case where λ is in a particular region, termed the *Wallach region* (as in [7]):

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n$$

with $\lambda_{i+1} - \lambda_i \geq 1$. In this case, the signature character has an explicit factorized form. We note that this was first computed by Wallach (for all Lie algebras, not just \mathfrak{gl}_n), and then used by Yee in determining signature characters for λ in any region [6–8].

Proposition 4.2 *For λ in the Wallach region, we have*

$$\tilde{c}(\zeta_\Lambda) = \prod_{1 \leq i \leq n-1} (-1)^{\lambda_i - \lambda_{ii}}.$$

Proof We use the formula (from the previous section, although one can use either)

$$\begin{aligned} \tilde{s}_q(\zeta_\Lambda) = & \prod_{1 \leq i \leq m < n} \left\{ \prod_{i < j \leq m} [v_{mi}^\Lambda - v_{mj}^\Lambda] [v_i - v_{mj}^\Lambda] \right\} (-1)^{v_{mi}^\Lambda - v_i} \left\{ [v_{mi}^\Lambda - v_i]_{(\lambda_i - \lambda_{m,i} - 1)!} \right\} \\ & \times \left\{ \prod_{i < j \leq m+1} [v_{mi}^\Lambda - v_{m+1,j}^\Lambda]_{(\lambda_i - \lambda_{m,i} - 1)!} \right\} \left\{ \prod_{i \leq j \leq m-1} [v_{mi}^\Lambda - v_{m-1,j}^\Lambda - 1]_{(\lambda_i - \lambda_{m,i} - 1)!} \right\}. \end{aligned}$$

First note that the signature of an arbitrary weight space is constant over this region, since the region does not intersect any degeneracy hyperplanes. So to compute the signature of a fixed weight space, we can take the differences $\lambda_{i+1} - \lambda_i$ to be very large. We check for fixed $1 \leq i \leq m < n$:

$$\begin{aligned} v_{mi}^\Lambda - v_{mj}^\Lambda &> 0, \quad v_i^\Lambda - v_{mj}^\Lambda > 0 \\ v_{mi}^\Lambda - v_i^\Lambda &> 0 \\ v_{mi}^\Lambda - v_{m+1,j}^\Lambda &> 0 \\ v_{mi}^\Lambda - v_{m-1,j}^\Lambda - 1 &> 0, \text{ unless } j = i \text{ in which case } < 0. \end{aligned}$$

So for $i \neq m$, we get $\text{sign}(-1)^{v_{mi}^\Lambda - v_i^\Lambda} (-1)^{\lambda_i - \lambda_{m,i}} = +1$ and for $i = m$, we get $(-1)^{v_{ii}^\Lambda - v_i^\Lambda}$. So the total is

$$\prod_{1 \leq i \leq n-1} (-1)^{\lambda_i - \lambda_{ii}},$$

as desired.

Note that one could instead use Lemma 3.2 (i.e., just increasing the node (m, i) by one) and iterate to prove the result. The sign of that step is -1 if $i = m$ and 1 otherwise. \square

Corollary 4.2 *The formula for the signature character in the Wallach region is*

$$\frac{e^\lambda}{\prod_{1 \leq i \leq m < n} \left(1 + \frac{x_m}{x_i}\right)}.$$

Proof We consider moves corresponding to (m, i) that increase entries in (r, i) for $r \leq m$ by one unit. By Proposition 4.2, such a move induces a sign change of (-1) . We also have

$$\frac{1}{1 + \frac{x_m}{x_i}} = 1 - (x_m/x_i) + (x_m/x_i)^2 - (x_m/x_i)^3 + \dots.$$

On the other hand, such moves (varying over all $1 \leq i \leq m < n$) generate all possible weights. This provides the desired bijection between the two quantities. \square

5 Examples

In this section, we use Theorem 3.2 and Theorem 4.1 to compute some examples of signature characters for small values of n . We will also use Corollary 4.1 to compute signature characters of Verma modules in the classical limit $q \rightarrow 1$ for small values of n .

1. $n = 2$, **formula for $ch_s(L(\lambda))$** . We let $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_1 - \lambda_2 \in \mathbb{Z}_+$. We index Gelfand-Tsetlin patterns by Λ_i with λ_1, λ_2 in the first row and $\lambda_{11} = \lambda_1 - i$ in the second row for $i = 0, 1, \dots, (\lambda_1 - \lambda_2)$. We use Definition 1 to compute

$$\begin{aligned} s(\zeta_{\Lambda_i}) &= (-1)^{v_{11}-v_1} \{[v_{11} - v_1]_{(\lambda_1-\lambda_{11}-1)}!\} \{[v_{11} - v_{22}]_{(\lambda_1-\lambda_{11}-1)}!\} \\ &= (-1)^i \{[i]_{(i-1)}!\} \{[(\lambda_2 - \lambda_1) + i - 1]_{(i-1)}!\}. \end{aligned}$$

So we have

$$ch_s(L(\lambda)) = \sum_{0 \leq i \leq (\lambda_1 - \lambda_2)} \{[i]!\} \{[\lambda_1 - \lambda_2]_{(i-1)}!\} e^{(\lambda_1 - i, \lambda_2 + i)}.$$

2. $n = 3$, **formula for $ch_s(L(\lambda))$** . We let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, with $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0}$. We let $\Lambda = (\lambda_1, \lambda_2, \lambda_3; \lambda_1 - d_1, \lambda_2 - d_2; \lambda_1 - d_1 - d_3)$ denote an arbitrary Gelfand-Tsetlin pattern indexing the basis. Note that we must have $0 \leq d_1 \leq (\lambda_1 - \lambda_2)$ and $0 \leq d_2 \leq (\lambda_2 - \lambda_3)$ and $0 \leq d_3 \leq (\lambda_1 - d_1) - (\lambda_2 - d_2)$ to satisfy the interlacing condition. We use Eq. 5 to compute

$$\begin{aligned} s_q(d_1, d_2, d_3) &= s_q(\zeta_\Lambda) \\ &= \{[d_1]![d_2]![d_1 + d_3]![d_2 - d_1 + v_2 - v_1][d_2 + v_2 - v_1] \\ &\quad \times [v_2 - v_1 - 1]_{(d_1-1)}![v_3 - v_1 - 1]_{(d_1-1)}!\} \\ &\quad \times \{[d_1 + d_3]_{(d_1-1)}![v_3 - v_2 - 1]_{(d_2-1)}![d_2 + v_2 - v_1]_{(d_1+d_3-1)}!\}. \end{aligned}$$

So we have

$$\sum_{\substack{0 \leq d_1 \leq (\lambda_1 - \lambda_2) \\ 0 \leq d_2 \leq (\lambda_2 - \lambda_3) \\ 0 \leq d_3 \leq (\lambda_1 - d_1) - (\lambda_2 - d_2)}} s_q(d_1, d_2, d_3) e^{(\lambda_1 - d_1 - d_3, \lambda_2 - d_2 + d_3, \lambda_3 + d_1 + d_2)},$$

where $s_q(d_1, d_2, d_3)$ is as computed in Eq. 6.

- $n = 2$, **formula for $ch_s(M(\lambda))$** . We let $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_1 - \lambda_2 \in \mathbb{R}$. We have the same formula as $ch_s(L(\lambda))$, except the basis is parametrized differently:

$$ch_s(M(\lambda)) = \sum_{0 \leq i} (-1)^i \{[i]_{i-1}!\} \{[(\lambda_2 - \lambda_1) + i - 1]_{i-1}!\} e^{(\lambda_1 - i, \lambda_2 + i)}.$$

- $n = 3$, **formula for $ch_s(M(\lambda))$** . We let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_i - \lambda_{i+1} \in \mathbb{R}$. We have the same formula as $ch_s(L(\lambda))$, except the basis is parametrized differently:

$$ch_s(M(\lambda)) = \sum_{d_1, d_2, d_3 \geq 0} s_q(d_1, d_2, d_3) e^{(\lambda_1 - d_1 - d_3, \lambda_2 - d_2 + d_3, \lambda_3 + d_1 + d_2)},$$

where $s_q(d_1, d_2, d_3)$ is computed in Eq. 6.

- $n = 2$, **formula for $\lim_{q \rightarrow 1} ch_s(M(\lambda))$** . Using (3) and Eq. 4, we have

$$\begin{aligned} \lim_{q \rightarrow 1} ch_s(M(\lambda)) &= \sum_{0 \leq i} (-1)^i \{(i)(i-1) \cdots (1)\} \{(\lambda_2 - \lambda_1 + i - 1) \\ &\quad \times (\lambda_2 - \lambda_1 + i - 2) \cdots (\lambda_2 - \lambda_1)\} e^{(\lambda_1 - i, \lambda_2 + i)} \\ &= \sum_{0 \leq i} (-1)^i (-1)^{\min\{0, \lfloor \lambda_2 - \lambda_1 + i - 1 \rfloor + 1\}} (-1)^{\min\{0, \lfloor \lambda_2 - \lambda_1 \rfloor\}} e^{(\lambda_1 - i, \lambda_2 + i)}. \end{aligned}$$

- $n = 3$, **formula for $\lim_{q \rightarrow 1} ch_s(M(\lambda))$** . We note that in Eq. 6, the terms $d_1, d_2, d_1 + d_3$ are all positive, so we have

$$\begin{aligned} \lim_{q \rightarrow 1} s_q(d_1, d_2, d_3) &= \{(d_2 - d_1 + v_2 - v_1)(d_2 + v_2 - v_1)(v_2 - v_1 - 1)_{(d_1-1)}! \\ &\quad \times (v_3 - v_1 - 1)_{(d_1-1)}!(v_3 - v_2 - 1)_{(d_2-1)}!(d_2 + v_2 - v_1)_{(d_1+d_3-1)}!\}. \end{aligned}$$

Thus,

$$\begin{aligned} \lim_{q \rightarrow 1} ch_s(M(\lambda)) &= (-1)^{\min\{0, \lfloor v_2 - v_1 - 1 \rfloor + 1\}} (-1)^{\min\{0, \lfloor v_3 - v_2 - 1 \rfloor + 1\}} (-1)^{\min\{0, \lfloor v_3 - v_1 - 1 \rfloor + 1\}} \\ &\quad \times \sum_{d_1, d_2, d_3 \geq 0} \left(\{d_2 - d_1 + v_2 - v_1\} \{d_2 + v_2 - v_1\} (-1)^{\min\{0, \lfloor v_2 - v_1 - 1 - (d_1 - 1) \rfloor\}} \right. \\ &\quad \times (-1)^{\min\{0, \lfloor v_3 - v_1 - 1 - (d_1 - 1) \rfloor\}} (-1)^{\min\{0, \lfloor v_3 - v_2 - 1 - (d_2 - 1) \rfloor\}} (-1)^{\min\{0, \lfloor d_2 + v_2 - v_1 \rfloor + 1\}} \\ &\quad \left. \times (-1)^{\min\{0, \lfloor d_2 + v_2 - v_1 - (d_1 + d_3 - 1) \rfloor\}} \right) e^{(\lambda_1 - d_1 - d_3, \lambda_2 - d_2 + d_3, \lambda_3 + d_1 + d_2)}; \end{aligned}$$

recall that $v_i = i - \lambda_i - 1$.

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