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Reducing Homological Conjectures by n-Recollements

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Abstract n-recollements of triangulated categories and n-derived-simple algebras are introduced. The relations between the n-recollements of derived categories of algebras and the Cartan determinants, homological smoothness and Gorensteinness of algebras respectively are clarified. As applications, the Cartan determinant conjecture is reduced to 1-derived-simple algebras, and the Gorenstein symmetry conjecture is reduced to 2-derived-simple algebras.

Keywords n-recollement · n-derived-simple algebra · Cartan determinant · Homologically smooth algebra · Gorenstein algebra

Mathematics Subject Classification (2010) 16G10 · 18E30

1 Introduction

Throughout k is a fixed field and all algebras are associative k-algebras with identity unless stated otherwise. Recollements of triangulated categories were introduced by Beilinson, Bernstein and Deligne [3], and play an important role in algebraic geometry and

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representation theory. Here, we focus on the recollements of derived categories of algebras which are the generalization of derived equivalences and provide a useful reduction technique for some homological properties such as the finiteness of global dimension [1, 21, 32], the finiteness of finitistic dimension [9, 17] and the finiteness of Hochschild dimension [15], some homological invariants such as K-theory [1, 8, 25, 29, 30, 35], Hochschild homology and cyclic homology [19] and Hochschild cohomology [15], and some homological conjectures such as the finitistic dimension conjecture [9, 17] and the Hochschild homology dimension conjecture [14].

In a recollement, two functors in the first layer always preserve compactness, i.e., send compact objects to compact ones, but other functors are not the case in general. If a recollement is *perfect*, i.e., two functors in the second layer also preserve compactness, then the Hochschild homologies, cyclic homologies and K-groups of the middle algebra are the direct sum of those of outer two algebras respectively [1, 8, 19]. Moreover, in this situation, the relations between recollements and the finitistic dimensions of algebras can be displayed very completely [9]. In order to clarify the relations between recollements and the homological smoothness and Gorensteinness of algebras respectively, we need even more layers of functors preserving compactness, which leads to the concept of n-recollement of triangulated categories inspired by that of ladder [4], and further n-derived-simple algebra. In terms of n-recollements, the relations between recollements and the Cartan determinants, homological smoothness and Gorensteinness of algebras respectively are expressed as follows.

Theorem I. *Let A, B and C be finite dimensional algebras, and $\mathcal{D}(\mathrm{Mod}A)$ admit an n-recollement relative to $\mathcal{D}(\mathrm{Mod}B)$ and $\mathcal{D}(\mathrm{Mod}C)$ with $n \geq 2$. Then* $\det C(A) = \det C(B) \cdot \det C(C)$.

Theorem II. *Let A, B and C be algebras, and $\mathcal{D}(\mathrm{Mod}A)$ admit an n-recollement relative to $\mathcal{D}(\mathrm{Mod}B)$ and $\mathcal{D}(\mathrm{Mod}C)$.*

(1) $n = 1$*: if A is homologically smooth then so is B;*
(2) $n = 2$*: if A is homologically smooth then so are B and C;*
(3) $n \geq 3$*: A is homologically smooth if and only if so are B and C.*

Theorem III. *Let A, B and C be finite dimensional algebras, and $\mathcal{D}(\mathrm{Mod}A)$ admit an n-recollement relative to $\mathcal{D}(\mathrm{Mod}B)$ and $\mathcal{D}(\mathrm{Mod}C)$.*

(1) $n = 3$*: if A is Gorenstein then so are B and C;*
(2) $n \geq 4$*: A is Gorenstein if and only if so are B and C.*

As applications of Theorem I and Theorem III, we will show that the Cartan determinant conjecture and the Gorenstein symmetry conjecture can be reduced to 1-derived-simple algebras and 2-derived-simple algebras respectively.

The paper is organized as follows: In Section 2, we will introduce the concepts of n-recollement of triangulated categories and n-derived-simple algebra, and provide some typical examples, constructions and existence criteria of n-recollements of derived categories of algebras. In Section 3, Theorem I is obtained and the Cartan determinant conjecture is reduced to 1-derived-simple algebras. In Section 4, we will prove Theorem II. In Section 5, Theorem III is shown and the Gorenstein symmetry conjecture is reduced to 2-derived-simple algebras.

2 *n*-Recollements and *n*-Derived-Simple Algebras

In this section, we will introduce the concepts of n-recollement of triangulated categories and n-derived-simple algebra, and provide some typical examples, constructions and existence criteria of the n-recollements of derived categories of algebras. As we will see, the language of n-recollements is very convenient for us to observe the relations between recollements and certain homological properties, especially the Gorensteinness of algebras.

2.1 *n*-Recollements of Triangulated Categories

Definition 1 (Beilinson-Bernstein-Deligne [3]) Let $\mathcal{T}_1, \mathcal{T}$ and $\mathcal{T}_2$ be triangulated categories. A *recollement* of $\mathcal{T}$ relative to $\mathcal{T}_1$ and $\mathcal{T}_2$ is given by

$$\mathcal{T}_1 \;\underset{\xleftarrow{\;i^!\;}}{\overset{\xleftarrow{\;i^*\;}}{\xrightarrow{i_*=i_!}}}\; \mathcal{T} \;\underset{\xleftarrow{\;j_*\;}}{\overset{\xleftarrow{\;j_!\;}}{\xrightarrow{j^!=j^*}}}\; \mathcal{T}_2$$

such that

(R1) (i^*, i_*), $(i_!, i^!)$, $(j_!, j^!)$ and (j^*, j_*) are adjoint pairs of triangle functors;
(R2) i_*, $j_!$ and j_* are full embeddings;
(R3) $j^! i_* = 0$ (and thus also $i^! j_* = 0$ and $i^* j_! = 0$);
(R4) for each $X \in \mathcal{T}$, there are triangles

$$\begin{array}{l} j_! j^! X \to X \to i_* i^* X \to \\ i_! i^! X \to X \to j_* j^* X \to \end{array}$$

where the arrows to and from X are the counits and the units of the adjoint pairs respectively.

Definition 2 Let $\mathcal{T}_1$, $\mathcal{T}$ and $\mathcal{T}_2$ be triangulated categories, and n a positive integer. An *n-recollement* of $\mathcal{T}$ relative to $\mathcal{T}_1$ and $\mathcal{T}_2$ is given by $n+2$ layers of triangle functors

$$\mathcal{T}_1 \;\begin{array}{c} \xleftarrow{\;i_1\;} \\ \xrightarrow{\;i_2\;} \\ \xleftarrow{\;i_3\;} \\ \xrightarrow{\;i_4\;} \\ \vdots \end{array}\; \mathcal{T} \;\begin{array}{c} \xleftarrow{\;j_1\;} \\ \xrightarrow{\;j_2\;} \\ \xleftarrow{\;j_3\;} \\ \xrightarrow{\;j_4\;} \\ \vdots \end{array}\; \mathcal{T}_2$$

such that every consecutive three layers form a recollement, and denoted by $(\mathcal{T}_1, \mathcal{T}, \mathcal{T}_2, i_1, i_2, \cdots, i_{n+2}, j_1, j_2, \cdots, j_{n+2})$.

Obviously, a 1-recollement is nothing but a recollement. Moreover, if $\mathcal{T}$ admits an n-recollement relative to $\mathcal{T}_1$ and $\mathcal{T}_2$, then it must admit a p-recollement relative to $\mathcal{T}_1$ and $\mathcal{T}_2$ for all $1 \le p \le n$ and a q-recollement relative to $\mathcal{T}_2$ and $\mathcal{T}_1$ for all $1 \le q \le n-1$.

Remark 1 Let $\mathcal{T}$ be a skeletally small k-linear triangulated category with finite dimensional Hom-sets and split idempotents. If $\mathcal{T}$ has a Serre functor and admits a recollement relative to $\mathcal{T}_1$ and $\mathcal{T}_2$ then it admits an n-recollement relative to $\mathcal{T}_1$ and $\mathcal{T}_2$ (resp. $\mathcal{T}_2$ and $\mathcal{T}_1$) for all $n \in \mathbb{Z}^+$ by [18, Theorem 7].

2.2 *n*-Recollements of Derived Categories of Algebras

Let A be an algebra. Denote by Mod$A$ the category of right A-modules, and by mod$A$, Proj$A$, proj$A$ and inj$A$ its full subcategories consisting of all finitely generated modules, projective modules, finitely generated projective modules and finitely generated injective modules, respectively. For $* \in \{\text{nothing}, -, +, b\}$, denote by $\mathcal{D}^*(\text{Mod}A)$ the derived category of (cochain) complexes of objects in Mod$A$ satisfying the corresponding boundedness condition. Denote by $K^b(\text{proj}A)$ (resp. $K^b(\text{Proj}A)$) the homotopy category of bounded complexes of objects in proj$A$ (resp. Proj$A$). If A is finite dimensional then we denote by $\mathcal{D}^b(\text{mod}A)$ the derived category of bounded complexes of objects in mod$A$ and by $K^b(\text{inj}A)$ the homotopy category of bounded complexes of objects in inj$A$. Up to isomorphism, the objects in $K^b(\text{proj}A)$ are precisely all the compact objects in $\mathcal{D}(\text{Mod}A)$. For convenience, we do not distinguish $K^b(\text{proj}A)$ from the *perfect derived category* $\mathcal{D}_{\text{per}}(A)$ of A, i.e., the full triangulated subcategory of $\mathcal{D}A$ consisting of all compact objects, which will not cause any confusion. Moreover, we also do not distinguish $K^b(\text{inj}A)$, $\mathcal{D}^b(\text{Mod}A)$, $\mathcal{D}^b(\text{mod}A)$, $\mathcal{D}^-(\text{Mod}A)$ and $\mathcal{D}^+(\text{Mod}A)$ from their essential images under the canonical full embeddings into $\mathcal{D}(\text{Mod}A)$. Usually, we just write $\mathcal{D}A$ instead of $\mathcal{D}(\text{Mod}A)$.

In this paper, we focus on the n-recollements of derived categories of algebras, i.e., all three triangulated categories in an n-recollement are the derived categories of algebras. Clearly, in an n-recollement, the upper n layers of functors have right adjoints preserving direct sums, thus they preserve compactness.

Now we provide some typical examples of n-recollements.

Example 1 (1) Stratifying ideals [11]. Let A be an algebra, and e an idempotent of A such that AeA is a *stratifying ideal*, i.e., $Ae \otimes^L_{eAe} eA \cong AeA$ canonically. Then $\mathcal{D}A$ admits a 1-recollement relative to $\mathcal{D}(A/AeA)$ and $\mathcal{D}(eAe)$.

(2) Triangular matrix algebras [1, Example 3.4]. Let B and C be algebras, M a C-B-bimodule, and $A = \begin{bmatrix} B & 0 \\ M & C \end{bmatrix}$. Then $\mathcal{D}A$ admits a 2-recollement relative to $\mathcal{D}B$ and $\mathcal{D}C$. Furthermore, $\mathcal{D}A$ admits a 3-recollement relative to $\mathcal{D}C$ and $\mathcal{D}B$ if ${}_CM \in K^b(\text{proj}C^{\text{op}})$, and $\mathcal{D}A$ admits a 3-recollement relative to $\mathcal{D}B$ and $\mathcal{D}C$ if $M_B \in K^b(\text{proj}B)$. What is more, $\mathcal{D}A$ admits a 4-recollement relative to $\mathcal{D}C$ and $\mathcal{D}B$ if ${}_CM \in K^b(\text{proj}C^{\text{op}})$ and $M_B \in K^b(\text{proj}B)$. Note that the algebras A, B and C here need not be finite dimensional.

(3) Let A be a finite dimensional algebra of finite global dimension and $\mathcal{D}A$ admit a recollement relative to $\mathcal{D}B$ and $\mathcal{D}C$. Then this recollement can be extended to an n-recollement for all $n \in \mathbb{Z}^+$ (Ref. [1, Proposition 3.3]).

(4) A derived equivalence induces a *trivial* n-recollement, i.e., an n-recollement whose left term or right term is zero, for all $n \in \mathbb{Z}^+$.

Next we provide two constructions of n-recollements from a given n-recollement by tensor product algebras and opposite algebras, which generalize [15, Theorem 1 and

Theorem 2]. For this, we need to introduce the concept of standard n-recollement which generalizes that of standard recollement [15, Definition 1].

Definition 3 Let B, A and C be algebras. An n-recollement $(\mathcal{D}B, \mathcal{D}A, \mathcal{D}C, i_1, i_2, \cdots, i_{n+2}, j_1, j_2, \cdots, j_{n+2})$ is said to be *standard* and *defined by* $Y \in \mathcal{D}(A^{\mathrm{op}} \otimes B)$ and $X \in \mathcal{D}(C^{\mathrm{op}} \otimes A)$ if $i_1 \cong - \otimes_A^L Y$ and $j_1 \cong - \otimes_C^L X$.

Remark 2 Let B, A and C be algebras, and $(\mathcal{D}B, \mathcal{D}A, \mathcal{D}C, i_1, i_2, \cdots, i_{n+2}, j_1, j_2, \cdots, j_{n+2})$ a standard n-recollement defined by $Y \in \mathcal{D}(A^{\mathrm{op}} \otimes B)$ and $X \in \mathcal{D}(C^{\mathrm{op}} \otimes A)$. Since the right adjoint functor is unique up to natural isomorphism, we know two functors in the first layer are isomorphic to derived tensor product functors, two functors in the p-th layer are isomorphic to both derived Hom functors and derived tensor product functors for all $2 \leq p \leq n+1$, and two functors in the last layer are isomorphic to derived Hom functors. More precisely, denote $X^{*_A} = \mathrm{RHom}_A(X, A)$, then

$$
\begin{array}{lll}
i_1 \cong - \otimes_A^L Y, & j_1 \cong - \otimes_C^L X, & \\
i_2 \cong - \otimes_B^L Y^{*_B}, & j_2 \cong - \otimes_A^L X^{*_A}, & \\
i_3 \cong - \otimes_A^L Y^{*_B *_A}, & j_3 \cong - \otimes_C^L X^{*_A *_C}, & \\
i_4 \cong - \otimes_B^L Y^{*_B *_A *_B}, & j_4 \cong - \otimes_A^L X^{*_A *_C *_A}, & \\
\vdots & \vdots & \\
i_{n+1} \cong - \otimes_B^L Y^{*_B (*_A *_B)^{\frac{n-1}{2}}}, & j_{n+1} \cong - \otimes_A^L X^{*_A (*_C *_A)^{\frac{n-1}{2}}}, & \text{if } n \text{ is odd,} \\
i_{n+1} \cong - \otimes_A^L Y^{(*_B *_A)^{\frac{n}{2}}}, & j_{n+1} \cong - \otimes_C^L X^{(*_A *_C)^{\frac{n}{2}}}, & \text{if } n \text{ is even,} \\
i_{n+2} \cong \mathrm{RHom}_A(Y^{*_B (*_A *_B)^{\frac{n-1}{2}}}, -), & j_{n+2} \cong \mathrm{RHom}_C(X^{*_A (*_C *_A)^{\frac{n-1}{2}}}, -), & \text{if } n \text{ is odd,} \\
i_{n+2} \cong \mathrm{RHom}_B(Y^{(*_B *_A)^{\frac{n}{2}}}, -), & j_{n+2} \cong \mathrm{RHom}_A(X^{(*_A *_C)^{\frac{n}{2}}}, -), & \text{if } n \text{ is even.}
\end{array}
$$

In particular, every consecutive three layers of a standard n-recollement form a standard recollement.

Proposition 1 *Let A, B and C be algebras. If $\mathcal{D}A$ admits an n-recollement relative to $\mathcal{D}B$ and $\mathcal{D}C$ then it admits a standard n-recollement relative to $\mathcal{D}B$ and $\mathcal{D}C$.*

Proof If $n = 1$ then the proposition is just [15, Proposition 3]. If $n \geq 2$ then we assume that $(\mathcal{D}B, \mathcal{D}A, \mathcal{D}C, i_1, i_2, \cdots, i_{n+2}, j_1, j_2, \cdots, j_{n+2})$ is an n-recollement. It follows from [15, Proposition 3] and its proof that there is a standard recollement $\left(\mathcal{D}B, \mathcal{D}A, \mathcal{D}C, i_1', i_2', i_3', j_1', j_2', j_3'\right)$ such that $i_1' \cong - \otimes_A^L Y$ and $j_1' \cong - \otimes_C^L X$ for some $Y \in \mathcal{D}(A^{\mathrm{op}} \otimes B)$ and $X \in \mathcal{D}(C^{\mathrm{op}} \otimes A)$, and $j_1'C \cong j_1C$. Since (j_1, j_2) and $\left(j_1', j_2'\right)$ are adjoint pairs and $j_1'C \cong j_1C$, we have $H^p\left(j_2'A\right) \cong H^p(j_2A)$ for all $p \in \mathbb{Z}$. By [1, Lemma 2.7], we know $j_2'A, j_2A \in \mathcal{D}^b(\mathrm{Mod}A)$. Hence $j_2'A \cong j_2A$. Since j_2 restricts to $K^b(\mathrm{proj})$, so is j_2'. Thus the standard recollement $\left(\mathcal{D}B, \mathcal{D}A, \mathcal{D}C, i_1', i_2', i_3', j_1', j_2', j_3'\right)$ can be extended one step downwards by [1, Proposition 3.2 (a)], and we obtain a standard 2-recollement $\left(\mathcal{D}B, \mathcal{D}A, \mathcal{D}C, i_1', i_2', i_3', i_4', j_1', j_2', j_3', j_4'\right)$. Inductively, we get a standard n-recollement $\left(\mathcal{D}B, \mathcal{D}A, \mathcal{D}C, i_1', i_2', \cdots i_{n+2}', j_1', j_2', \cdots, j_{n+2}'\right)$. □

Remark 3 Two n-recollements $(\mathcal{T}_1, \mathcal{T}, \mathcal{T}_2, i_1, i_2, \cdots, i_{n+2}, j_1, j_2, \cdots, j_{n+2})$ and $\left(\mathcal{T}_1', \mathcal{T}', \mathcal{T}_2', i_1', i_2', \cdots, i_{n+2}', j_1', j_2', \cdots, j_{n+2}'\right)$ are said to be *equivalent* if

$\mathrm{Im}i_{2p} = \mathrm{Im}i'_{2p}$ and $\mathrm{Im}j_{2p-1} = \mathrm{Im}j'_{2p-1}$ for all p. By [18, Theorem 3.6], we have $(\mathrm{Im}j_1, \mathrm{Im}i_2, \mathrm{Im}j_3, \mathrm{Im}i_4, \cdots) = (\mathrm{Tria}(j_1C), (j_1C)^{\perp}, (j_1C)^{\perp\perp}, (j_1C)^{\perp\perp\perp}, \cdots) = \left(\mathrm{Tria}\left(j'_1C\right), \left(j'_1C\right)^{\perp}, \left(j'_1C\right)^{\perp\perp}, \left(j'_1C\right)^{\perp\perp\perp}, \cdots\right) = (\mathrm{Im}j'_1, \mathrm{Im}i'_2, \mathrm{Im}j'_3, \mathrm{Im}i'_4, \cdots)$ where for a class $\mathcal{X}$ of objects in a triangulated category $\mathcal{T}$ with direct sums, $\mathrm{Tria}\mathcal{X}$ denotes the smallest full triangulated subcategory of $\mathcal{T}$ containing $\mathcal{X}$ and closed under direct sums, and $\mathcal{X}^{\perp}$ denotes the full triangulated subcategory of $\mathcal{T}$ consisting of all objects $T \in \mathcal{T}$ satisfying $\mathrm{Hom}_{\mathcal{T}}(X[p], T) = 0$ for all $X \in \mathcal{X}$ and $p \in \mathbb{Z}$. Namely, the new constructed n-recollement in Proposition 1 is equivalent to the original given one.

Proposition 2 *Let A, B, C and E be algebras, and $Y \in \mathcal{D}(A^{\mathrm{op}} \otimes B)$ and $X \in \mathcal{D}(C^{\mathrm{op}} \otimes A)$ define a standard n-recollement of $\mathcal{D}(A)$ relative to $\mathcal{D}(B)$ and $\mathcal{D}(C)$. Then $E \otimes Y$ and $E \otimes X$ define a standard n-recollement of $\mathcal{D}(E \otimes A)$ relative to $\mathcal{D}(E \otimes B)$ and $\mathcal{D}(E \otimes C)$. Moreover, the triangle functors in both n-recollements are isomorphic to the derived functors of the same forms.*

Proof It follows from Remark 2 and [15, Theorem 1]. □

Proposition 3 *Let A, B and C be algebras, and $Y \in \mathcal{D}(A^{\mathrm{op}} \otimes B)$ and $X \in \mathcal{D}(C^{\mathrm{op}} \otimes A)$ define a standard n-recollement of $\mathcal{D}(A)$ relative to $\mathcal{D}(B)$ and $\mathcal{D}(C)$. If n is odd then $Y^{*_B(*_A*_B)^{\frac{n-1}{2}}}$ and $X^{*_A(*_C*_A)^{\frac{n-1}{2}}}$ define a standard n-recollement $(\mathcal{D}(B^{\mathrm{op}}), \mathcal{D}(A^{\mathrm{op}}), \mathcal{D}(C^{\mathrm{op}}), i_1, i_2, \cdots, i_{n+2}, j_1, j_2, \cdots, j_{n+2})$ with*

$$\begin{array}{ll}
i_1 \cong Y^{*_B(*_A*_B)^{\frac{n-1}{2}}} \otimes^L_A -, & j_1 \cong X^{*_A(*_C*_A)^{\frac{n-1}{2}}} \otimes^L_C -, \\
\vdots & \vdots \\
i_{n-2} \cong Y^{*_B*_A*_B} \otimes^L_A -, & j_{n-2} \cong X^{*_A*_C*_A} \otimes^L_C -, \\
i_{n-1} \cong Y^{*_B*_A} \otimes^L_B -, & j_{n-1} \cong X^{*_A*_C} \otimes^L_A -, \\
i_n \cong Y^{*_B} \otimes^L_A -, & j_n \cong X^{*_A} \otimes^L_C -, \\
i_{n+1} \cong Y \otimes^L_B -, & j_{n+1} \cong X \otimes^L_A -, \\
i_{n+2} \cong \mathrm{RHom}_{A^{\mathrm{op}}}(Y, -), & j_{n+2} \cong \mathrm{RHom}_{C^{\mathrm{op}}}(X, -).
\end{array}$$

If n is even then $X^{(*_A*_C)^{\frac{n}{2}}}$ and $Y^{(*_B*_A)^{\frac{n}{2}}}$ define a standard n-recollement $(\mathcal{D}(C^{\mathrm{op}}), \mathcal{D}(A^{\mathrm{op}}), \mathcal{D}(B^{\mathrm{op}}), i_1, i_2, \cdots, i_{n+2}, j_1, j_2, \cdots, j_{n+2})$ with

$$\begin{array}{ll}
i_1 \cong X^{(*_A*_C)^{\frac{n}{2}}} \otimes^L_A -, & j_1 \cong Y^{(*_B*_A)^{\frac{n}{2}}} \otimes^L_B -, \\
\vdots & \vdots \\
i_{n-2} \cong X^{*_A*_C*_A} \otimes^L_C -, & j_{n-2} \cong Y^{*_B*_A*_B} \otimes^L_A -, \\
i_{n-1} \cong X^{*_A*_C} \otimes^L_A -, & j_{n-1} \cong Y^{*_B*_A} \otimes^L_B -, \\
i_n \cong X^{*_A} \otimes^L_C -, & j_n \cong Y^{*_B} \otimes^L_A -, \\
i_{n+1} \cong X \otimes^L_A -, & j_{n+1} \cong Y \otimes^L_B -, \\
i_{n+2} \cong \mathrm{RHom}_{C^{\mathrm{op}}}(X, -), & j_{n+2} \cong \mathrm{RHom}_{A^{\mathrm{op}}}(Y, -).
\end{array}$$

Proof It follows from Remark 2 and [15, Theorem 2]. □

Usually we pay more attention to the n-recollements of derived categories of finite dimensional algebras. In this situation, we have some useful existence criteria of n-recollements.

Lemma 1 *Let A and B be finite dimensional algebras, and the triangle functor $F : \mathcal{D}A \to \mathcal{D}B$ left adjoint to $G : \mathcal{D}B \to \mathcal{D}A$. Then:*

(1) *F restricts to $K^b(\text{proj})$ if and only if G restricts to $\mathcal{D}^b(\text{mod})$;*
(2) *F restricts to $\mathcal{D}^b(\text{mod})$ if and only if G restricts to $K^b(\text{inj})$.*

Proof (1) follows from [1, Lemma 2.7], and (2) is proved with an argument entirely dual to (1). □

Lemma 2 *Let A, B and C be finite dimensional algebras, and*

$$\begin{array}{ccccc} & \xleftarrow{i^*} & & \xleftarrow{j_!} & \\ \mathcal{D}B & \xrightarrow{i_*} & \mathcal{D}A & \xrightarrow{j^!} & \mathcal{D}C \\ & \xleftarrow{i^!} & & \xleftarrow{j_*} & \end{array}$$

a recollement. Then the following statements hold:

(1) *i^* and $j_!$ restrict to $K^b(\text{proj})$;*
(2) *i_* and $j^!$ restrict to $\mathcal{D}^b(\text{mod})$;*
(3) *$i^!$ and j_* restrict to $K^b(\text{inj})$.*

Proof (1) is clear. (2) and (3) follow from Lemma 1. □

Lemma 3 *Let A, B and C be finite dimensional algebras, and*

$$\begin{array}{ccccc} & \xleftarrow{i^*} & & \xleftarrow{j_!} & \\ \mathcal{D}B & \xrightarrow{i_*} & \mathcal{D}A & \xrightarrow{j^!} & \mathcal{D}C \\ & \xleftarrow{i^!} & & \xleftarrow{j_*} & \end{array} \qquad \text{(R)}$$

a recollement. Then the following statements are equivalent:

(1) *The recollement (R) can be extended one-step downwards;*
(2) *i_* or/and $j^!$ restricts to $K^b(\text{proj})$;*
(2') *$i_*B \in K^b(\text{proj}A)$ or/and $j^!A \in K^b(\text{proj}C)$;*
(3) *$i^!$ or/and j_* restricts to $\mathcal{D}^b(\text{mod})$;*
(4) *The recollement (R) restricts to $\mathcal{D}^-(\text{Mod})$.*

Proof (1) $\Leftrightarrow$ (2): It follows from [1, Proposition 3.2 (a)].
(2) $\Leftrightarrow$ (2'): It follows from [1, Lemma 2.5].

(2) $\Leftrightarrow$ (3): It follows from Lemma 1.
(4) $\Leftrightarrow$ (2'): It follows from [1, Proposition 4.11].

□

Lemma 4 *Let A, B and C be finite dimensional algebras, and*

$$\mathcal{D}B \underset{\overset{\longleftarrow}{i^!}}{\overset{\overset{i^*}{\longleftarrow}}{\xrightarrow{i_*}}} \mathcal{D}A \underset{\overset{\longleftarrow}{j_*}}{\overset{\overset{j_!}{\longleftarrow}}{\xrightarrow{j^!}}} \mathcal{D}C \qquad \text{(R)}$$

a recollement. Then the following statements are equivalent:

(1) *The recollement (R) can be extended one-step upwards;*
(2) *i_* or/and $j^!$ restricts to $K^b(\mathrm{inj})$;*
(2') *$i_*(DB) \in K^b(\mathrm{inj}A)$ or/and $j^!(DA) \in K^b(\mathrm{inj}C)$ where $D = \mathrm{Hom}_k(-, k)$;*
(3) *i^* or/and $j_!$ restricts to $\mathcal{D}^b(\mathrm{mod})$;*
(4) *The recollement (R) restricts to $\mathcal{D}^+(\mathrm{Mod})$.*

Proof This lemma is proved with an argument entirely dual to Lemma 3. □

Proposition 4 *Let A, B and C be finite dimensional algebras. Then the following conditions are equivalent:*

(1) *$\mathcal{D}A$ admits a 2-recollement relative to $\mathcal{D}B$ and $\mathcal{D}C$;*
(2) *$\mathcal{D}A$ admits a recollement relative to $\mathcal{D}B$ and $\mathcal{D}C$ in which two functors in the second layer restrict to $K^b(\mathrm{proj})$;*
(3) *$\mathcal{D}A$ admits a recollement relative to $\mathcal{D}B$ and $\mathcal{D}C$ in which two functors in the third layer restrict to $\mathcal{D}^b(\mathrm{mod})$;*
(4) *$\mathcal{D}^-(\mathrm{Mod}A)$ admits a recollement relative to $\mathcal{D}^-(\mathrm{Mod}B)$ and $\mathcal{D}^-(\mathrm{Mod}C)$;*
(5) *$\mathcal{D}A$ admits a recollement relative to $\mathcal{D}C$ and $\mathcal{D}B$ in which two functors in the first layer restrict to $\mathcal{D}^b(\mathrm{mod})$;*
(6) *$\mathcal{D}A$ admits a recollement relative to $\mathcal{D}C$ and $\mathcal{D}B$ in which two functors in the second layer restrict to $K^b(\mathrm{inj})$;*
(7) *$\mathcal{D}^+(\mathrm{Mod}A)$ admits a recollement relative to $\mathcal{D}^+(\mathrm{Mod}C)$ and $\mathcal{D}^+(\mathrm{Mod}B)$.*

Proof (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4): By [1, Proposition 4.1], any $\mathcal{D}^-(\mathrm{Mod})$-recollement can be lifted to a $\mathcal{D}(\mathrm{Mod})$-recollement. Then it follows from Lemma 3.

(1) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7): Analogous to [1, Proposition 4.1], any $\mathcal{D}^+(\mathrm{Mod})$-recollement can be lifted to a $\mathcal{D}(\mathrm{Mod})$-recollement as well. Then it follows from Lemma 4. □

Proposition 5 *Let A, B and C be finite dimensional algebras. Then the following conditions are equivalent:*

(1) *$\mathcal{D}A$ admits a 3-recollement relative to $\mathcal{D}B$ and $\mathcal{D}C$;*
(2) *$\mathcal{D}A$ admits a recollement relative to $\mathcal{D}B$ and $\mathcal{D}C$ in which all functors restrict to $K^b(\mathrm{proj})$;*

(3) $\mathcal{D}^b(\mathrm{mod}A)$ *admits a recollement relative to* $\mathcal{D}^b(\mathrm{mod}C)$ *and* $\mathcal{D}^b(\mathrm{mod}B)$*;*
(4) $\mathcal{D}A$ *admits a recollement relative to* $\mathcal{D}B$ *and* $\mathcal{D}C$ *in which all functors restrict to* $K^b(\mathrm{inj})$*;*
(5) $\mathcal{D}^b(\mathrm{Mod}A)$ *admits a recollement relative to* $\mathcal{D}^b(\mathrm{Mod}C)$ *and* $\mathcal{D}^b(\mathrm{Mod}B)$.

Proof (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) : It follows from [1, Proposition 4.1] and Lemma 3.
(1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) : It follows from [1, Proposition 4.1] and Lemma 4.
(3) $\Leftrightarrow$ (5) : It follows from [1, Proposition 4.1 and Corollary 4.9]. □

2.3 *n*-Derived-Simple Algebras

For any recollement of derived categories of finite dimensional algebras, the Grothendieck group of the middle algebra is the direct sum of those of the outer two algebras [1, Proposition 6.5]. Thus the process of reducing homological properties, homological invariants and homological conjectures by recollements must terminate after finitely many steps. This leads to derived simple algebras, whose derived categories admit no nontrivial recollements any more. This definition dates from Wiedemann [32], where the author considered the stratifications of bounded derived categories. Later on, recollements of unbounded and bounded above derived categories attract considerable attention, and so do the corresponding derived simple algebras [1]. When we consider the stratifications along n-recollements, n-derived-simple algebras are defined naturally.

Definition 4 A finite dimensional algebra A is said to be *n-derived-simple* if its derived category $\mathcal{D}A$ admits no nontrivial n-recollements.

Clearly, an n-derived-simple algebra must be indecomposable/connected. Note that 1-derived-simple algebras are just the $\mathcal{D}$(Mod)-derived simple algebras. For finite dimensional algebras, by Proposition 4 and Proposition 5, 2 (resp. 3)-derived-simple algebras are exactly $\mathcal{D}^-$(Mod) (resp. $\mathcal{D}^b$(mod))-derived simple algebras in the sense of [1]. Moreover, n-derived-simple algebras must be p-derived-simple for all $p \geq n$, and it is worth noting that for a finite dimensional algebra A of finite global dimension, the n-derived-simplicity of A does not depend on the choice of n.

Although it is difficult to find out all the n-derived-simple algebras, there are still some known examples.

Example 2 (1) Finite dimensional local algebras, blocks of finite group algebras and indecomposable representation-finite symmetric algebras are 1-derived-simple [23, 32];
(2) Some finite dimensional two-point algebras of finite global dimension are n-derived-simple for all $n \in \mathbb{Z}^+$ (Ref. [16, 24]);
(3) Indecomposable symmetric algebras are 2-derived-simple [23];
(4) There exist 2-derived-simple algebras which are not 1-derived-simple [1, Example 5.8], 3-derived-simple algebras which are not 2-derived-simple [1, Example 5.10], and 4-derived-simple algebras which are not 3-derived-simple [1, Example 4.13], respectively.

Let's end this section by listing some known results on reducing homological conjectures via recollements. First, the *finitistic dimension conjecture*, which says that every finite dimensional algebra has finite finitistic dimension, was reduced to 3-derived-simple algebras by Happel [17]. Recently, Chen and Xi extended his result by reducing the finitistic dimension conjecture to 2-derived-simple algebras [9]. Second, it follows from [19, Proposition 2.9(b)] and [1, Proposition 2.14] that the *Hochschild homology dimension conjecture*, which states that the finite dimensional algebras of finite Hochschild homology dimension are of finite global dimension [14], can be reduced to 2-derived-simple algebras. Last but not least, both *vanishing conjecture* and *dual vanishing conjecture* can be reduced to 3-derived-simple algebras [34].

3 *n*-Recollements and Cartan Determinants

In this section, we will observe the relations between n-recollements and the Cartan determinants of algebras, and reduce the Cartan determinant conjecture to 1-derived-simple algebras.

Let $\mathcal{E}$ be a skeletally small exact category, F the free abelian group generated by the isomorphism classes $[X]$ of objects X in $\mathcal{E}$, and F_0 the subgroup of F generated by $[X]-[Y]+[Z]$ for all conflations $0\to X\to Y\to Z\to 0$ in $\mathcal{E}$. The *Grothendieck group* $K_0(\mathcal{E})$ of $\mathcal{E}$ is the factor group F/F_0. The Grothendieck group of a skeletally small triangulated category is defined similarly by replacing conflations with triangles.

Let A be a finite dimensional algebra and $\{P_1,\cdots,P_r\}$ a complete set of pairwise non-isomorphic indecomposable projective A-modules. Then their tops $\{S_1,\cdots,S_r\}$ form a complete set of pairwise non-isomorphic simple A-modules. The map $C_A : K_0(\mathrm{proj}A)\to K_0(\mathrm{mod}A)$, $[P]\mapsto[P]$, is called the *Cartan map* of A, which can be extended to $C_A : K_0(K^b(\mathrm{proj}A))\to K_0(\mathcal{D}^b(\mathrm{mod}A))$, $[X]\mapsto[X]$. The matrix of the Cartan map C_A with respect to the $\mathbb{Z}$-basis $\{[P_1],\cdots,[P_r]\}$ of $K_0(\mathrm{proj}A)$ and the $\mathbb{Z}$-basis $\{[S_1],\cdots,[S_r]\}$ of $K_0(\mathrm{mod}A)$ is called the *Cartan matrix* of A, and denoted by $C(A)$. Namely, $C(A)$ is the $r\times r$ matrix whose (i,j)-th entry c_{ij} is the multiplicity of S_i in P_j. Obviously, c_{ij} equals to the composition length of the $\mathrm{End}_A(P_i)$-module $\mathrm{Hom}_A(P_i,P_j)$, or $\dim_k\mathrm{Hom}_A(P_i,P_j)/\dim_k\mathrm{End}_A(S_i)$.

Now we study the relation between n-recollements and the Cartan determinant of algebras. For convenience, we define $\det C(0)=1$. The following theorem is just Theorem I.

Theorem 1 *Let A', A and A'' be finite dimensional algebras, and $\mathcal{D}A$ admit an n-recollement relative to $\mathcal{D}A'$ and $\mathcal{D}A''$ with $n\geq 2$. Then* $\det C(A)=\det C(A')\cdot\det C(A'')$.

Proof It follows from Proposition 4 and Lemma 2 that $\mathcal{D}A$ admits a recollement

$$\begin{array}{ccccc} & \xleftarrow{\;i^*\;} & & \xleftarrow{\;j_!\;} & \\ \mathcal{D}A' & \xrightarrow{\;i_*\;} & \mathcal{D}A & \xrightarrow{\;j^!\;} & \mathcal{D}A'' \\ & \xleftarrow[\;i^!\;]{} & & \xleftarrow[\;j_*\;]{} & \end{array}$$

such that i^*, i_*, $j_!$ and $j^!$ restrict to $K^b(\mathrm{proj})$, and i_*, $i^!$, $j^!$ and j_* restrict to $\mathcal{D}^b(\mathrm{mod})$.

Let $\{P_1', \cdots, P_{r'}'\}$ (resp. $\{P_1, \cdots, P_r\}$, $\{P_1'', \cdots, P_{r''}''\}$) be a complete set of pairwise non-isomorphic indecomposable projective A'-modules (resp. A-modules, A''-modules). Then their tops $\{S_1', \cdots, S_{r'}'\}$ (resp. $\{S_1, \cdots, S_r\}$, $\{S_1'', \cdots, S_{r''}''\}$) form a complete set of pairwise non-isomorphic simple A'-modules (resp. A-modules, A''-modules). By [8, Theorem 1.1] or [1, Proposition 6.5], we have $r' + r'' = r$.

Consider the triangles $j_!j^!P_u \to P_u \to i_*i^*P_u \to$ for all $1 \leq u \leq r$. Since $P_u \in K^b(\text{proj}A)$, we have $j^!P_u \in K^b(\text{proj}A'') = \text{tria}\left\{P_1'', \cdots, P_{r''}''\right\} \subseteq \mathcal{D}A''$ and $i^*P_u \in K^b(\text{proj}A') = \text{tria}\{P_1', \cdots, P_{r'}'\} \subseteq \mathcal{D}A'$. Here, for a class $\mathcal{X}$ of objects in a triangulated category $\mathcal{T}$, $\text{tria}\mathcal{X}$ denotes the smallest strict full triangulated subcategory of $\mathcal{T}$ containing $\mathcal{X}$. Furthermore, we have $j_!j^!P_u \in \text{tria}\{j_!P_1'', \cdots, j_!P_{r''}''\} \subseteq \mathcal{D}A$ and $i_*i^*P_u \in \text{tria}\{i_*P_1', \cdots, i_*P_{r'}'\} \subseteq \mathcal{D}A$. Hence $P_u \in \text{tria}\{i_*P_1', \cdots, i_*P_{r'}', j_!P_1'', \cdots, j_!P_{r''}''\} \subseteq \mathcal{D}A$, and $K^b(\text{proj}A) = \text{tria}\{P_1, \cdots, P_r\} = \text{tria}\{i_*P_1', \cdots, i_*P_{r'}', j_!P_1'', \cdots, j_!P_{r''}''\} \subseteq \mathcal{D}A$. Therefore, $\mathcal{B}_{\mathcal{P}} := \{[i_*P_1'], \cdots, [i_*P_{r'}'], [j_!P_1''], \cdots, [j_!P_{r''}'']\}$ is a $\mathbb{Z}$-basis of $K_0(K^b(\text{proj}A))$.

Consider the triangles $i_*i^!S_u \to S_u \to j_*j^!S_u \to$ for all $1 \leq u \leq r$. Since $S_u \in \mathcal{D}^b(\text{mod}A)$, we have $i^!S_u \in \mathcal{D}^b(\text{mod}A') = \text{tria}\{S_1', \cdots, S_{r'}'\} \subseteq \mathcal{D}A'$ and $j^!S_u \in \mathcal{D}^b(\text{mod}A'') = \text{tria}\{S_1'', \cdots, S_{r''}''\} \subseteq \mathcal{D}A''$. Furthermore, we have $i_*i^!S_u \in \text{tria}\{i_*S_1', \cdots, i_*S_{r'}'\} \subseteq \mathcal{D}A$ and $j_*j^!S_u \in \text{tria}\{j_*S_1'', \cdots, j_*S_{r''}''\} \subseteq \mathcal{D}A$. Hence $S_u \in \text{tria}\{i_*S_1', \cdots, i_*S_{r'}', j_*S_1'', \cdots, j_*S_{r''}''\} \subseteq \mathcal{D}A$, and $\mathcal{D}^b(\text{mod}A) = \text{tria}\{S_1, \cdots, S_r\} = \text{tria}\{i_*S_1', \cdots, i_*S_{r'}', j_*S_1'', \cdots, j_*S_{r''}''\} \subseteq \mathcal{D}A$. So $\mathcal{B}_{\mathcal{S}} := \{[i_*S_1'], \cdots, [i_*S_{r'}'], [j_*S_1''], \cdots, [j_*S_{r''}'']\}$ is a $\mathbb{Z}$-basis of $K_0(\mathcal{D}^b(\text{mod}A))$.

Let $C(A') = (c_{pq}')$ and $C(A'') = (c_{st}'')$. Then $[P_q'] = \sum_{p=1}^{r'} c_{pq}'[S_p']$ in $K_0(\mathcal{D}^b(\text{mod}A'))$ for all $1 \leq q \leq r'$ and $[P_t''] = \sum_{s=1}^{r''} c_{st}''[S_s'']$ in $K_0(\mathcal{D}^b(\text{mod}A''))$ for all $1 \leq t \leq r''$. Since the functor i_* is triangle, we have $[i_*P_q'] = \sum_{p=1}^{r'} c_{pq}'[i_*S_p']$ for all $1 \leq q \leq r'$. Assume that $[j_!P_t''] = \sum_{p=1}^{r'} x_{p,r'+t}[i_*S_p'] + \sum_{s=1}^{r''} x_{r'+s,r'+t}[j_*S_s'']$ in $K_0(\mathcal{D}^b(\text{mod}A))$ with $x_{p,r'+t}, x_{r'+s,r'+t} \in \mathbb{Z}$ for all $1 \leq t \leq r''$. Since the functor $j^!$ is triangle and $j^!i_* = 0$, we have $[P_t''] = [j^!j_!P_t''] = \sum_{p=1}^{r'} x_{p,r'+t}[j^!i_*S_p'] + \sum_{s=1}^{r''} x_{r'+s,r'+t}[j^!j_*S_s''] = \sum_{s=1}^{r''} x_{r'+s,r'+t}[S_s'']$ in $K_0(\mathcal{D}^b(\text{mod}A''))$. Thus $x_{r'+s,r'+t} = c_{st}''$ for all $1 \leq s, t \leq r''$. Hence the matrix of the Cartan map $C_A : K_0(K^b(\text{proj}A)) \to K_0(\mathcal{D}^b(\text{mod}A))$ with respect to the $\mathbb{Z}$-basis $\mathcal{B}_{\mathcal{P}}$ of $K_0(K^b(\text{proj}A))$ and the $\mathbb{Z}$-basis $\mathcal{B}_{\mathcal{S}}$ of $K_0(\mathcal{D}^b(\text{mod}A))$ is the block-decomposed matrix $\begin{bmatrix} C(A') & * \\ 0 & C(A'') \end{bmatrix}$. Since the matrix of the Cartan map C_A with respect to the $\mathbb{Z}$-basis $\{[P_1], \cdots, [P_r]\}$ of $K_0(K^b(\text{proj}A))$ and the $\mathbb{Z}$-basis $\{[S_1], \cdots, [S_r]\}$ of $K_0(\mathcal{D}^b(\text{mod}A))$ is $C(A)$, there are invertible integer matrices $U, V \in GL_r(\mathbb{Z})$ such that $U \cdot C(A) \cdot V = \begin{bmatrix} C(A') & * \\ 0 & C(A'') \end{bmatrix}$. The determinant of an invertible integer matrix is ± 1, thus $\det C(A) = \pm \det C(A') \cdot \det C(A'')$.

On the other hand, we can define a $\mathbb{Z}$-bilinear form

$$\langle -, - \rangle : K_0(K^b(\text{proj}A)) \times K_0(K^b(\text{proj}A)) \to \mathbb{Z}$$

by

$$\langle [X], [Y] \rangle := \sum_{l \in \mathbb{Z}} (-1)^l \dim_k \text{Hom}_{K^b(\text{proj}A)}(X, Y[l]),$$

for all $X, Y \in K^b(\text{proj}A)$.

Since i_* and $j_!$ are full embeddings and $j^! i_* = 0$, we have

$$\begin{aligned}
\langle \left[i_* P'_p\right], \left[i_* P'_q\right] \rangle &= \dim_k \mathrm{Hom}_{A'}(P'_p, P'_q), \quad p, q = 1, \cdots, r'; \\
\langle \left[j_! P''_s\right], \left[i_* P'_p\right] \rangle &= 0, \qquad\qquad\qquad\qquad s = 1, \cdots, r''; \ p = 1, \cdots, r'; \\
\langle \left[j_! P''_s\right], \left[j_! P''_t\right] \rangle &= \dim_k \mathrm{Hom}_{A''}(P''_s, P''_t), \ s, t = 1, \cdots, r''.
\end{aligned}$$

Thus the matrix of $\langle -, - \rangle$ with respect to the $\mathbb{Z}$-basis $\mathcal{B}_{\mathcal{P}}$ of $K_0(K^b(\mathrm{proj}A))$ is $\begin{bmatrix} D' \cdot C(A') & * \\ 0 & D'' \cdot C(A'') \end{bmatrix}$ where $D' = \mathrm{diag}\{c'_1, \cdots, c'_{r'}\}$ with $c'_p = \dim_k \mathrm{End}_{A'}(S'_p)$ for all $p = 1, \cdots, r'$ and $D'' = \mathrm{diag}\{c''_1, \cdots, c''_{r''}\}$ with $c''_s = \dim_k \mathrm{End}_{A''}(S''_s)$ for all $s = 1, \cdots, r''$.

Let $D = \mathrm{diag}\{c_1, \cdots, c_r\}$ with $c_u = \dim_k \mathrm{End}_A(S_u)$ for all $u = 1, \cdots, r$. Since the matrix of $\langle -, - \rangle$ with respect to the $\mathbb{Z}$-basis $\{[P_1], \cdots, [P_r]\}$ of $K_0(K^b(\mathrm{proj}A))$ is $D \cdot C(A)$, there exists an invertible integer matrix $T \in GL_r(\mathbb{Z})$ such that $D \cdot C(A) = T^t \cdot \begin{bmatrix} D' \cdot C(A') & * \\ 0 & D'' \cdot C(A'') \end{bmatrix} \cdot T$. It follows that $\det C(A)$ and $\det C(A') \cdot \det C(A'')$ have the same sign since $\det D'$, $\det D''$ and $\det D$ are strictly positive integers. Thus $\det C(A) = \det C(A') \cdot \det C(A'')$. □

Next we study the Cartan determinant conjecture. In 1954, Eilenberg showed that if A is a finite dimensional algebra of finite global dimension then $\det C(A) = \pm 1$ (Ref. [12]). After that, the following conjecture was posed:

Cartan Determinant Conjecture. Let A be an artin algebra of finite global dimension. Then $\det C(A) = 1$.

The Cartan determinant conjecture remains open except for some special classes of algebras, such as the algebras of global dimension two [37], the positively graded algebras [33], the Cartan filtered algebras [13], the left serial algebras [7], the quasi-hereditary algebras [6], and the artin algebras admitting a strongly adequate grading by an aperiodic commutative monoid [28].

Applying Theorem 1 to the trivial 2-recollement in Example 4 (4), we can obtain the following corollary which generalizes [5, Proposition 1.5] to an arbitrary base field.

Corollary 1 *If A and B are derived equivalent finite dimensional algebras then* $\det C(A) = \det C(B)$. *In particular, one of them satisfies the Cartan determinant conjecture if and only if so does the other.*

Proposition 6 *Let A', A and A'' be finite dimensional algebras, and $\mathcal{D}A$ admit a recollement relative to $\mathcal{D}A'$ and $\mathcal{D}A''$. If both A' and A'' satisfy the Cartan determinant conjecture, then so does A. In particular, the Cartan determinant conjecture is true for all finite dimensional algebras if and only if it is true for all 1-derived-simple algebras of finite global dimension.*

Proof If A is of finite global dimension then so are A' and A'' by [1, Proposition 2.14]. Thus $\det C(A') = \det C(A'') = 1$ by the assumption and the recollement induces a 2-recollement, see Example 1 (3). By Theorem 1, we have $\det C(A) = \det C(A') \cdot \det C(A'') = 1$.

For any finite dimensional algebra A, by [1, Proposition 6.5], $\mathcal{D}A$ admits a finite stratification of derived categories along recollements with 1-derived-simple factors. Thus the second statement holds. □

Although Proposition 6 provides a reduction technique, the Cartan determinant conjecture seems far from being settled, because it is still a problem to deal with all the 1-derived-simple algebras of finite global dimension. Nonetheless, for the known examples described in Example 2 (1) and (2), the Cartan determinant conjecture holds true [16, 24].

Let's end this section by pointing out that Theorem 1 can be applied to prove the n-derived-simplicity of certain algebras as well.

Remark 4 A finite dimensional two-point algebra A with $\det C(A) \leq 0$ must be 2-derived-simple: Otherwise, $\mathcal{D}A$ admits a non-trivial 2-recollement relative to $\mathcal{D}B$ and $\mathcal{D}C$. Then both B and C are finite dimensional local algebras since A has only two simple modules up to isomorphism. Therefore, $\det C(B) > 0$ and $\det C(C) > 0$. By Theorem 1, we get $\det C(A) > 0$. It is a contradiction. The examples of this kind of 2-derived-simple algebras include:

(1) $1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$, $(\alpha\beta)^n = 0 = (\beta\alpha)^n, n \in \mathbb{Z}^+$;

(2) $\gamma \circlearrowright 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \circlearrowleft \delta$, $\alpha\beta = \beta\alpha = \gamma^2 = \delta^2 = \gamma\alpha - \alpha\delta = \delta\beta - \beta\gamma = 0$;

(3) Let A be one of the algebras in (1) and (2), and B an arbitrary finite dimensional elementary local algebra. Then the tensor product algebra $A \otimes_k B$ is again 2-derived-simple by the same reason.

Remark 5 A representation-finite selfinjective two-point algebra A with $\det C(A) \leq 0$ must be 1-derived-simple. Indeed, for a representation-finite selfinjective algebra A, if $\mathcal{D}A$ admits a recollement relative to $\mathcal{D}B$ and $\mathcal{D}C$, this recollement must be perfect [23, Proposition 4.1]. Therefore, the 2-derived-simplicity of these algebras implies the 1-derived-simplicity. For example, the algebras in Remark 4 (1) are 1-derived-simple.

4 *n*-Recollements and Homological Smoothness

In this section, we will observe the relation between n-recollements and the homological smoothness of algebras.

Let A be an algebra and $A^e := A^{\text{op}} \otimes_k A$ its enveloping algebra. The algebra A is said to be *smooth* if the projective dimension of A as an A^e-module is finite, i.e., A is isomorphic in $\mathcal{D}(A^e)$ to an object in $K^b(\text{Proj} A^e)$ (Refs. [31]). The algebra A is said to be *homologically smooth* if A is compact in $\mathcal{D}(A^e)$, i.e., A is isomorphic in $\mathcal{D}(A^e)$ to an object in $K^b(\text{proj} A^e)$ (Ref. [22]). Clearly, all homologically smooth algebras are smooth. Moreover, if A is a finite dimensional algebra then the concepts of smoothness and homological smoothness coincide. However, they are different in general. For example, the infinite Kronecker algebra is smooth but not homologically smooth [15, Remark 4].

Let A and B be two derived equivalent algebras. Then, by [27, Proposition 2.5], there is a triangle equivalence functor from $\mathcal{D}(A^e)$ to $\mathcal{D}(B^e)$ sending A_{A^e} to B_{B^e}. Since the equivalence functor can restrict to $K^b(\mathrm{Proj})$ and $K^b(\mathrm{proj})$, both the smoothness and the homological smoothness of algebras are invariant under derived equivalences. Moreover, the relations between recollements and the smoothness of algebras have been clarified in [15]:

Proposition 7 *(See* [15, Theorem 3]*) Let A, B and C be algebras, and $\mathcal{D}A$ admit a recollement relative to $\mathcal{D}B$ and $\mathcal{D}C$. Then A is smooth if and only if so are B and C.*

However, Proposition 7 is not correct for homological smoothness any more. Here is an example:

Example 3 (See [15, Remark 4]) Let A be the infinite Kronecker algebra $\begin{bmatrix} k & 0 \\ k^{(\mathbb{N})} & k \end{bmatrix}$. Then by Example 1 (2), $\mathcal{D}A$ admits a 2-recollement relative to $\mathcal{D}k$ and $\mathcal{D}k$, but A is not homologically smooth.

Due to Example 3, even though $\mathcal{D}A$ admits a 2-recollement relative to $\mathcal{D}B$ and $\mathcal{D}C$, the homological smoothness of B and C can not imply the homological smoothness of A. Nonetheless, we have the following theorem which is just Theorem II.

Theorem 2 *Let A, B and C be algebras, and $\mathcal{D}A$ admit an n-recollement relative to $\mathcal{D}B$ and $\mathcal{D}C$.*

(1) $n = 1$*: if A is homologically smooth then so is B;*
(2) $n = 2$*: if A is homologically smooth then so are B and C;*
(3) $n \geq 3$*: A is homologically smooth if and only if so are B and C.*

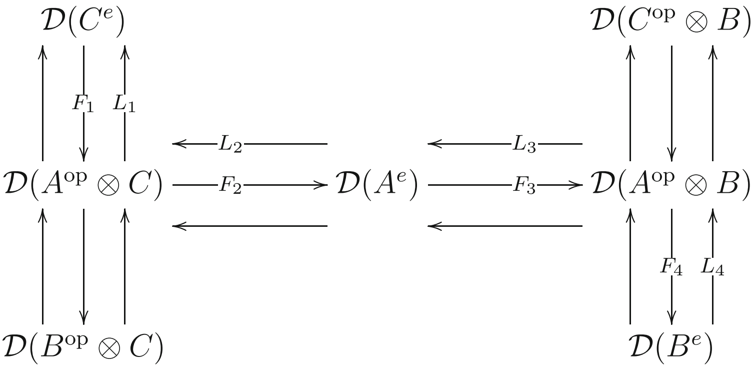
Proof (1) See [20, Proposition 3.10 (c)].

(2) If A is homologically smooth then B is also homologically smooth by (1). Since $n = 2$, we have a recollement of $\mathcal{D}A$ relative to $\mathcal{D}C$ and $\mathcal{D}B$, and thus C is also homologically smooth by (1) again.
(3) Assume $\mathcal{D}A$ admits a 3-recollement relative to $\mathcal{D}B$ and $\mathcal{D}C$. Consider the recollement formed by the middle three layers. By [15, Proposition 3], we may assume that it is of the form

$$\mathcal{D}C \underset{\xleftarrow{\quad i^! \quad}}{\overset{\xleftarrow{i^* \cong - \otimes_A^L Y}}{\xrightarrow{i_* \cong - \otimes_C^L Y^{*C}}}} \mathcal{D}A \underset{\xleftarrow{\quad j_* \quad}}{\overset{\xleftarrow{j_! \cong - \otimes_B^L X}}{\xrightarrow{j^! \cong - \otimes_A^L X^{*A}}}} \mathcal{D}B \qquad (\mathrm{R}')$$

where $X \in \mathcal{D}(B^{\mathrm{op}} \otimes A)$ and $Y \in \mathcal{D}(A^{\mathrm{op}} \otimes C)$. Clearly, ${}_AY$ and Y_A^{*C} are compact since the recollement can be extended one step upwards and one step downwards respectively [1, Proposition 3.2 and Lemma 2.8].

By [15, Theorem 1] and [15, Theorem 2], a recollement of derived categories of algebras induces those of tensor product algebras and opposite algebras respectively. Thus we have the following three recollements induced by the recollement (R′):



where $L_1 \cong Y^{*_C} \otimes_A^L -$, $F_1 \cong Y \otimes_C^L -$, $L_2 \cong - \otimes_A^L Y$, $F_2 \cong - \otimes_C^L Y^{*_C}$, $L_3 \cong - \otimes_B^L X$, $F_3 \cong - \otimes_A^L X^{*_A}$, $L_4 \cong X^{*_A} \otimes_B^L -$ and $F_4 \cong X \otimes_A^L -$. Consider the canonical triangle

$$L_3 F_3 A \longrightarrow A \longrightarrow F_2 L_2 A \longrightarrow \text{ in } D(A^e),$$

and note that $F_2 L_2 A \cong Y \otimes_C^L Y^{*_C} \cong F_2 F_1 C$, $L_3 F_3 A \cong X^{*_A} \otimes_B^L X \cong L_3 L_4 B$. Clearly, the functors L_3 and L_4 preserve compactness, so are F_1 and F_2 since ${}_A Y$ and $Y_A^{*_C}$ are compact. If $B \in K^b(\mathrm{proj} B^e)$ and $C \in K^b(\mathrm{proj} C^e)$ then $F_2 F_1 C$ and $L_3 L_4 B$ are compact, i.e., $F_2 L_2 A$ and $L_3 F_3 A$ are compact. Applying these to the above triangle, we get $A \in K^b(\mathrm{proj} A^e)$. Namely, the homological smoothness of B and C implies that of A. □

According to Example 3 and the statement followed, we see that in Theorem 2 (3), the requirement $n \geq 3$ is optimal.

Applying Theorem 2 to triangular matrix algebras, we get the following corollary which provides a construction of homologically smooth algebras.

Corollary 2 *Let B and C be algebras, M a C-B-bimodule, and* $A := \begin{bmatrix} B & 0 \\ M & C \end{bmatrix}$.

(1) *If A is homologically smooth, then so are B and C;*
(2) *If B and C are homologically smooth and ${}_C M \in K^b(\mathrm{proj} C^{\mathrm{op}})$ or $M_B \in K^b(\mathrm{proj} B)$, then A is also homologically smooth.*

Proof It follows from Example 1(2) and Theorem 2. □

5 *n*-Recollements and Gorensteinness

In this section, we will observe the relations between n-recollements and the Gorensteinness of algebras, and reduce the Gorenstein symmetry conjecture to 2-derived-simple algebras.

A finite dimensional algebra A is said to be *Gorenstein* if $\mathrm{id}_A A < \infty$ and $\mathrm{id}_{A^{\mathrm{op}}} A < \infty$. Clearly, a finite dimensional algebra A is Gorenstein if and only if $K^b(\mathrm{proj}A) = K^b(\mathrm{inj}A)$ as strict full triangulated subcategories of $\mathcal{D}A$. It is well-known that every derived equivalent functor between the derived categories of finite dimensional algebras restricts to $K^b(\mathrm{proj})$, $K^b(\mathrm{inj})$ and $\mathcal{D}^b(\mathrm{mod})$. Thus the Gorensteinness of algebras is invariant under derived equivalences. It is natural to consider the relation between recollements and the Gorensteinness of algebras. In [26], Pan proved that the Gorensteinness of A implies the Gorensteinness of B and C if there exists a recollement of $\mathcal{D}^b(\mathrm{mod}A)$ relative to $\mathcal{D}^b(\mathrm{mod}B)$ and $\mathcal{D}^b(\mathrm{mod}C)$. Now we complete it using the language of n-recollements. The following theorem is just Theorem III.

Theorem 3 *Let A, B and C be finite dimensional algebras, and $\mathcal{D}A$ admit an n-recollement relative to $\mathcal{D}B$ and $\mathcal{D}C$.*

(1) $n = 3$*: if A is Gorenstein then so are B and C;*
(2) $n \geq 4$*: A is Gorenstein if and only if so are B and C.*

Proof (1) It follows from Proposition 5 that $\mathcal{D}^b(\mathrm{mod}A)$ admits a recollement relative to $\mathcal{D}^b(\mathrm{mod}C)$ and $\mathcal{D}^b(\mathrm{mod}B)$. Therefore, the statement follows from Pan [26]. Here we provide another proof. Consider the following recollement consisting of the middle three layers of functors of the 3-recollement:

$$\mathcal{D}(C)\ \begin{array}{c}\xleftarrow{\quad i^* \quad}\\ \xrightarrow{\quad i_* \quad}\\ \xleftarrow[\quad i^! \quad]{}\end{array}\ \mathcal{D}(A)\ \begin{array}{c}\xleftarrow{\quad j_! \quad}\\ \xrightarrow{\quad j^! \quad}\\ \xleftarrow[\quad j_* \quad]{}\end{array}\ \mathcal{D}(B)\ .$$

By Lemma 2, i^*, i_*, $j_!$ and $j^!$ restrict to $K^b(\mathrm{proj})$, and i_*, $i^!$, $j^!$ and j_* restrict to $K^b(\mathrm{inj})$. If A is Gorenstein then $K^b(\mathrm{proj}A) = K^b(\mathrm{inj}A)$.

Note that $DC := \mathrm{Hom}_k(C, k)$. Thus $DC \cong i^*i_*(DC) \in i^*i_*(K^b(\mathrm{inj}C)) \subseteq i^*(K^b(\mathrm{inj}A)) = i^*(K^b(\mathrm{proj}A)) \subseteq K^b(\mathrm{proj}C)$. Thus, $\mathrm{pd}_C(DC) < \infty$, equivalently, $\mathrm{id}_{C^{\mathrm{op}}} C < \infty$. On the other hand, $C \cong i^!i_*C \in i^!i_*K^b(\mathrm{proj}C) \subseteq i^!K^b(\mathrm{proj}A) = i^!K^b(\mathrm{inj}A) \subseteq K^b(\mathrm{inj}C)$. Thus, $\mathrm{id}_C C < \infty$. Therefore, C is Gorenstein.

Similarly, $DB \cong j^!j_*(DB) \in j^!j_*(K^b(\mathrm{inj}B)) \subseteq j^!(K^b(\mathrm{inj}A)) = j^!(K^b(\mathrm{proj}A)) \subseteq K^b(\mathrm{proj}B)$. Thus, $\mathrm{pd}_B(DB) < \infty$, equivalently, $\mathrm{id}_{B^{\mathrm{op}}} B < \infty$. On the other hand, $B \cong j^!j_!B \in j^!j_!K^b(\mathrm{proj}B) \subseteq j^!K^b(\mathrm{proj}A) = j^!K^b(\mathrm{inj}A) \subseteq K^b(\mathrm{inj}B)$. Thus, $\mathrm{id}_B B < \infty$. Therefore, B is Gorenstein.

(2) Let

$$\mathcal{D}B\ \begin{array}{c}\xleftarrow{\qquad\quad}\\ \xrightarrow{\qquad\quad}\\ \xleftarrow{\quad i^* \quad}\\ \xrightarrow[\quad i_* \quad]{}\\ \xleftarrow{\qquad\quad}\\ \xrightarrow{\qquad\quad}\end{array}\ \mathcal{D}A\ \begin{array}{c}\xleftarrow{\qquad\quad}\\ \xrightarrow{\qquad\quad}\\ \xleftarrow{\quad j_! \quad}\\ \xrightarrow[\quad j^! \quad]{}\\ \xleftarrow{\qquad\quad}\\ \xrightarrow{\qquad\quad}\end{array}\ \mathcal{D}C$$

be a 4-recollement. By Lemma 2, i^*, $j_!$, i_* and $j^!$ restrict to both $K^b(\mathrm{proj})$ and $K^b(\mathrm{inj})$. If both B and C are Gorenstein, then $K^b(\mathrm{proj}B) = K^b(\mathrm{inj}B)$ and $K^b(\mathrm{proj}C) = K^b(\mathrm{inj}C)$.

Consider the triangle $j_!j^!(DA) \to DA \to i_*i^*(DA) \to$. We have $j_!j^!(DA) \in j_!j^!K^b(\mathrm{inj}A) \subseteq j_!K^b(\mathrm{inj}C) = j_!K^b(\mathrm{proj}C) \subseteq K^b(\mathrm{proj}A)$ and $i_*i^*(DA) \in$

$i_*i^*K^b(\text{inj}A) \subseteq i_*K^b(\text{inj}B) = i_*K^b(\text{proj}B) \subseteq K^b(\text{proj}A)$. Thus, $DA \in K^b(\text{proj}A)$, i.e., $\text{pd}_A(DA) < \infty$. Hence, $\text{id}_{A^{\text{op}}}A < \infty$.

Similarly, consider the triangle $j_!j^!A \to A \to i_*i^*A \to$. We have $j_!j^!A \in j_!j^!K^b(\text{proj}A) \subseteq j_!K^b(\text{proj}C) = j_!K^b(\text{inj}C) \subseteq K^b(\text{inj}A)$ and $i_*i^*A \in i_*i^*K^b(\text{proj}A) \subseteq i_*K^b(\text{proj}B) = i_*K^b(\text{inj}B) \subseteq K^b(\text{inj}A)$. Thus, $A \in K^b(\text{inj}A)$, i.e., $\text{id}_A A < \infty$. Therefore, A is Gorenstein. □

Applying Theorem 3 to triangular matrix algebras, we get the following corollary (Ref. [10, Theorem 3.3] and [36, Theorem 2.2 (iii)]), which imply the condition $n \geq 4$ in Theorem 3 (2) is optimal.

Corollary 3 *Let B and C be finite dimensional algebras, M a finitely generated C-B-bimodule, and $A = \begin{bmatrix} B & 0 \\ M & C \end{bmatrix}$. If two of the following conditions hold then so does the other one:*

(1) *A is Gorenstein;*
(2) *B and C are Gorenstein;*
(3) $\text{pd}_{C^{\text{op}}}M < \infty$ *and* $\text{pd}_B M < \infty$.

Proof "(1)+(2) ⇒ (3)": Assume that A is Gorenstein. Set $e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. By [1, Example 3.4], there is a 2-recollement of the form

$$\begin{array}{ccccc} & \xleftarrow{\quad i^* \quad} & & \xleftarrow{\quad j_! = -\otimes^L_C e_2A \quad} & \\ \mathcal{D}B & \xrightarrow{\quad i_* = -\otimes^L_B e_1A \quad} & \mathcal{D}A & \xrightarrow{\quad j^! \quad} & \mathcal{D}C\ . \\ & \xleftarrow{\quad i^! = -\otimes^L_A Ae_1 \quad} & & \xleftarrow{\quad j_* \quad} & \\ & \xrightarrow{\qquad\qquad} & & \xrightarrow{\qquad\qquad} & \end{array}$$

It follows from Lemma 2 that $i^!$ restricts to $K^b(\text{inj})$, and further restricts to $K^b(\text{proj})$ by the Gorensteinness of A and B. Thus, $i^!A = B \oplus M_B \in K^b(\text{proj}B)$. Hence, $\text{pd}_B M < \infty$. Similarly, it follows from Lemma 2 that $j^!$ restricts to $K^b(\text{proj})$, and further restricts to $K^b(\text{inj})$ by the Gorensteinness of A and C. By Lemma 1, $j_!$ restricts to $\mathcal{D}^b(\text{mod})$. Since $j_! = -\otimes^L_C e_2A$, this is equivalent to ${}_C(e_2A) \in K^b(\text{proj}C^{\text{op}})$ (Ref. [1, Lemma 2.8]). Note that ${}_C(e_2A) = C \oplus {}_CM$, thus $\text{pd}_{C^{\text{op}}}M < \infty$.

"(2)+(3) ⇒ (1)" and "(1)+(3) ⇒ (2)": Assume that $\text{pd}_{C^{\text{op}}}M < \infty$ and $\text{pd}_B M < \infty$, then by Example 1 (2), the above 2-recollement can be extended one step upwards and one step downwards to a 4-recollement. By Theorem 3, A is Gorenstein if and only if B and C are Gorenstein. □

Next we study the Gorenstein symmetry conjecture.

Gorenstein symmetry conjecture *Let A be an artin algebra. Then* $\text{id}_A A < \infty$ *if and only if* $\text{id}_{A^{\text{op}}}A < \infty$.

This conjecture is listed in Auslander-Reiten-Smalø's book [2, p.410, Conjecture (13)], and it closely connects with other homological conjectures. For example, it is known that the finitistic dimension conjecture implies the Gorenstein symmetry conjecture. But so far all these conjectures are still open. As mentioned before, the finitistic dimension conjecture can be reduced to 2-derived-simple algebras. Now, let us utilize Theorem 3 to reduce the Gorenstein symmetry conjecture to 2-derived-simple algebras.

Proposition 8 *Let A, B and C be finite dimensional algebras, and $\mathcal{D}A$ admit a 2-recollement relative to $\mathcal{D}B$ and $\mathcal{D}C$. If both B and C satisfy the Gorenstein symmetry conjecture, then so does A. In particular, the Gorenstein symmetry conjecture is true for all finite dimensional algebras if and only if it is true for all 2-derived-simple algebras.*

Proof Assume that

$$\mathcal{D}B \underset{\longleftarrow}{\overset{\longleftarrow}{\underset{\longleftarrow i^! \longrightarrow}{\overset{\longrightarrow i_* \longrightarrow}{}}}} \mathcal{D}A \underset{\longleftarrow}{\overset{\longleftarrow}{\underset{\longleftarrow j_* \longrightarrow}{\overset{\longrightarrow j^! \longrightarrow}{}}}} \mathcal{D}C \qquad (\mathrm{R}'')$$

is a 2-recollement, and both B and C satisfy the Gorenstein symmetry conjecture.

If $\mathrm{id}_A A < \infty$, then $K^b(\mathrm{proj}A) \subseteq K^b(\mathrm{inj}A)$. By Lemma 2, we have $B \cong i^! i_* B \in i^! i_*(K^b(\mathrm{proj}B)) \subseteq i^!(K^b(\mathrm{proj}A)) \subseteq i^!(K^b(\mathrm{inj}A)) \subseteq K^b(\mathrm{inj}B)$, i.e., $\mathrm{id}_B B < \infty$. Since B satisfies the Gorenstein symmetry conjecture, we obtain that B is Gorenstein. By Lemma 2 again, we have $i^! A \in i^!(K^b(\mathrm{proj}A)) \subseteq i^!(K^b(\mathrm{inj}A)) \subseteq K^b(\mathrm{inj}B) = K^b(\mathrm{proj}B)$. Due to Lemma 3, $i^! A \in K^b(\mathrm{proj}B)$ implies that the 2-recollement (R$''$) can be extended one step downwards. Therefore, we get a 2-recollement of $\mathcal{D}A$ relative to $\mathcal{D}C$ and $\mathcal{D}B$. Analogous to the above proof, we obtain that C is Gorenstein and the 2-recollement (R$''$) can be extended two steps downwards to a 4-recollement of $\mathcal{D}A$ relative to $\mathcal{D}B$ and $\mathcal{D}C$. By Theorem 3, A is Gorenstein. Thus $\mathrm{id}_{A^{\mathrm{op}}} A < \infty$.

Now we have two ways to prove that $\mathrm{id}_{A^{\mathrm{op}}} A < \infty$ implies $\mathrm{id}_A A < \infty$. One is to mimic the paragraph above. The other is as follows: By Proposition 1 and Proposition 3, $\mathcal{D}A^{\mathrm{op}}$ admit a 2-recollement relative to $\mathcal{D}C^{\mathrm{op}}$ and $\mathcal{D}B^{\mathrm{op}}$. Since both B and C satisfy the Gorenstein symmetry conjecture, so do B^{op} and C^{op}. By the conclusion of the paragraph above, we know that $\mathrm{id}_{A^{\mathrm{op}}} A^{\mathrm{op}} < \infty$ implies $\mathrm{id}_A A^{\mathrm{op}} < \infty$, equivalently $\mathrm{id}_{A^{\mathrm{op}}} A < \infty$ implies $\mathrm{id}_A A < \infty$. □

Remark 6 Note from Proposition 8 that the class of finite dimensional algebras which satisfy the Gorenstein symmetry conjecture is invariant under derived equivalences.

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