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From Leibniz Algebras to Lie 2-algebras

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Abstract In this paper, we construct a Lie 2-algebra associated to every Leibniz algebra via the skew-symmetrization.

Keywords Leibniz algebras · Lie 2-algebras · Omni-Lie algebras · Courant algebroids

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The notion of a Leibniz algebra was introduced by Loday in [4, 5], which is a vector space $\mathfrak{g}$, endowed with a linear map $[\cdot,\cdot]_\mathfrak{g}:\mathfrak{g}\otimes\mathfrak{g}\longrightarrow\mathfrak{g}$ satisfying

$$\big[x,[y,z]_\mathfrak{g}\big]_\mathfrak{g}=\big[[x,y]_\mathfrak{g},z\big]_\mathfrak{g}+\big[y,[x,z]_\mathfrak{g}\big]_\mathfrak{g},\quad \forall\, x,y,z\in\mathfrak{g}. \tag{1}$$

The **left center** is given by

$$\mathrm{Z}(\mathfrak{g})=\big\{x\in\mathfrak{g}|\ [x,y]_\mathfrak{g}=0,\quad \forall y\in\mathfrak{g}\big\}. \tag{2}$$

It is obvious that $\mathrm{Z}(\mathfrak{g})$ is an ideal and the quotient Leibniz algebra $\mathfrak{g}/\mathrm{Z}(\mathfrak{g})$ is actually a Lie algebra since $[x,x]_\mathfrak{g}\in\mathrm{Z}(\mathfrak{g})$, for all $x\in\mathfrak{g}$.

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A Lie 2-algebra is a categorification of a Lie algebra, which is equivalent to a 2-term L_∞-algebra (see [1, 8] for more details).

Definition 1 A **Lie 2-algebra** is a graded vector space $\mathcal{G} = \mathfrak{g}_1 \oplus \mathfrak{g}_0$, together with linear maps $\{l_k : \wedge^k \mathcal{G} \longrightarrow \mathcal{G}, k = 1, 2, 3\}$ of degrees $\deg(l_k) = k - 2$ satisfying the following equalities:

(a) $l_1 l_2(x, a) = l_2(x, l_1(a))$,
(b) $l_2(l_1(a), b) = l_2(a, l_1(b))$,
(c) $l_2(x, l_2(y, z)) + c.p. = l_1 l_3(x, y, z)$,
(d) $l_2(x, l_2(y, a)) + l_2(y, l_2(a, x)) + l_2(a, l_2(x, y)) = l_3(x, y, l_1(a))$,
(e) $l_3(l_2(x, y), z, w) + c.p. = l_2(l_3(x, y, z), w) + c.p.$,

for all $x, y, z, w \in \mathfrak{g}_0,\ a, b \in \mathfrak{g}_1$, where $c.p.$ means cyclic permutations.

Given a Leibniz algebra $\left(\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g}\right)$, introduce the following skew-symmetric bracket on $\mathfrak{g}$:

$$[\![x, y]\!] = \frac{1}{2}\left([x, y]_\mathfrak{g} - [y, x]_\mathfrak{g}\right), \quad \forall x, y \in \mathfrak{g}, \tag{3}$$

and denote by $J_{x,y,z}$ the corresponding Jacobiator, i.e.

$$J_{x,y,z} = [\![x, [\![y, z]\!]]\!] + [\![y, [\![z, x]\!]]\!] + [\![z, [\![x, y]\!]]\!]. \tag{4}$$

Proposition 2 *Let* $(\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g})$ *be a Leibniz algebra.*

(i) *For all* $x, y, z \in \mathfrak{g}$, *we have*

$$J_{x,y,z} = \frac{1}{4}\left([[z, y]_\mathfrak{g}, x]_\mathfrak{g} + [[x, z]_\mathfrak{g}, y]_\mathfrak{g} + [[y, x]_\mathfrak{g}, z]_\mathfrak{g}\right). \tag{5}$$

(ii) $J_{x,y,z} \in \mathrm{Z}(\mathfrak{g})$, *i.e.* $J_{x,y,z}$ *is in the left center of* $(\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g})$.
(iii) *For all* $x, y, z, w \in \mathfrak{g}$, *we have*

$$\begin{aligned} &[\![x, J_{y,z,w}]\!] - [\![y, J_{x,z,w}]\!] + [\![z, J_{x,y,w}]\!] - [\![w, J_{x,y,z}]\!] \\ &- J_{[\![x,y]\!],z,w} + J_{[\![x,z]\!],y,w} - J_{[\![x,w]\!],y,z} - J_{[\![y,z]\!],x,w} + J_{[\![y,w]\!],x,z} - J_{[\![z,w]\!],x,y} = 0. \end{aligned} \tag{6}$$

Proof The first conclusion is obtained by straightforward computations. For any $w \in \mathfrak{g}$, by Eq. 1 and the fact that for all $x \in \mathfrak{g}$, $[x, x]_\mathfrak{g} \in \mathrm{Z}(\mathfrak{g})$, we have

$$\begin{aligned} \left[J_{x,y,z}, w\right]_\mathfrak{g} &= \frac{1}{4}\left([[[z, y]_\mathfrak{g}, x]_\mathfrak{g} + [[x, z]_\mathfrak{g}, y]_\mathfrak{g} + [[y, x]_\mathfrak{g}, z]_\mathfrak{g}, w]_\mathfrak{g}\right) \\ &= \frac{1}{4}\left([[z, [y, x]_\mathfrak{g}]_\mathfrak{g} - [y, [z, x]_\mathfrak{g}]_\mathfrak{g} - [[z, x]_\mathfrak{g}, y]_\mathfrak{g} + [[y, x]_\mathfrak{g}, z]_\mathfrak{g}, w]_\mathfrak{g}\right) \\ &= 0, \end{aligned}$$

which implies that $J_{x,y,z} \in \mathrm{Z}(\mathfrak{g})$. At last, since the bracket $\llbracket\cdot,\cdot\rrbracket$ given by Eq. 4 is skew-symmetric, we have

$$
\begin{aligned}
& \llbracket x, J_{y,z,w}\rrbracket - \llbracket y, J_{x,z,w}\rrbracket + \llbracket z, J_{x,y,w}\rrbracket - \llbracket w, J_{x,y,z}\rrbracket \\
& -J_{\llbracket x,y\rrbracket,z,w} + J_{\llbracket x,z\rrbracket,y,w} - J_{\llbracket x,w\rrbracket,y,z} - J_{\llbracket y,z\rrbracket,x,w} + J_{\llbracket y,w\rrbracket,x,z} - J_{\llbracket z,w\rrbracket,x,y} \\
= & \underline{\llbracket x, J_{y,z,w}\rrbracket} \underbrace{- \llbracket y, J_{x,z,w}\rrbracket}_{} + \underline{\underline{\llbracket z, J_{x,y,w}\rrbracket}} - \underline{\underline{\underline{\llbracket w, J_{x,y,z}\rrbracket}}} \\
& - \llbracket \llbracket x,y\rrbracket, \llbracket z,w\rrbracket\rrbracket - \underline{\underline{\llbracket z, \llbracket w, \llbracket x,y\rrbracket\rrbracket\rrbracket}} - \underline{\underline{\underline{\llbracket w, \llbracket\llbracket x,y\rrbracket, z\rrbracket\rrbracket}}} \\
& + \llbracket \llbracket x,z\rrbracket, \llbracket y,w\rrbracket\rrbracket + \underbrace{\llbracket y, \llbracket w, \llbracket x,z\rrbracket\rrbracket\rrbracket}_{} + \underline{\underline{\underline{\llbracket w, \llbracket\llbracket x,z\rrbracket, y\rrbracket\rrbracket}}} \\
& - \llbracket \llbracket x,w\rrbracket, \llbracket y,z\rrbracket\rrbracket \underbrace{- \llbracket y, \llbracket z, \llbracket x,w\rrbracket\rrbracket\rrbracket}_{} - \underline{\underline{\llbracket z, \llbracket\llbracket x,w\rrbracket, y\rrbracket\rrbracket}} \\
& - \llbracket \llbracket y,z\rrbracket, \llbracket x,w\rrbracket\rrbracket \underline{- \llbracket x, \llbracket w, \llbracket y,z\rrbracket\rrbracket\rrbracket} - \underline{\underline{\underline{\llbracket w, \llbracket\llbracket y,z\rrbracket, x\rrbracket\rrbracket}}} \\
& + \llbracket \llbracket y,w\rrbracket, \llbracket x,z\rrbracket\rrbracket + \underline{\llbracket x, \llbracket z, \llbracket y,w\rrbracket\rrbracket\rrbracket} + \underline{\underline{\llbracket z, \llbracket\llbracket y,w\rrbracket, x\rrbracket\rrbracket}} \\
& - \llbracket \llbracket z,w\rrbracket, \llbracket x,y\rrbracket\rrbracket \underline{- \llbracket x, \llbracket y, \llbracket z,w\rrbracket\rrbracket\rrbracket} \underbrace{- \llbracket y, \llbracket\llbracket z,w\rrbracket, x\rrbracket\rrbracket}_{} \\
= & - \llbracket \llbracket x,y\rrbracket, \llbracket z,w\rrbracket\rrbracket + \llbracket \llbracket x,z\rrbracket, \llbracket y,w\rrbracket\rrbracket - \llbracket \llbracket x,w\rrbracket, \llbracket y,z\rrbracket\rrbracket \\
& - \llbracket \llbracket y,z\rrbracket, \llbracket x,w\rrbracket\rrbracket + \llbracket \llbracket y,w\rrbracket, \llbracket x,z\rrbracket\rrbracket - \llbracket \llbracket z,w\rrbracket, \llbracket x,y\rrbracket\rrbracket \\
= & \ 0.
\end{aligned}
$$

The proof is finished. □

Next, for a Leibniz algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$, we consider the graded vector space $\mathcal{G} = \mathrm{Z}(\mathfrak{g}) \oplus \mathfrak{g}$, where $\mathrm{Z}(\mathfrak{g})$ is of degree 1, $\mathfrak{g}$ is of degree 0. Define a degree -1 differential $l_1 = \mathfrak{i} : \mathrm{Z}(\mathfrak{g}) \longrightarrow \mathfrak{g}$, the inclusion. Define a degree 0 skew-symmetric bilinear map l_2 and a degree 1 totally skew-symmetric trilinear map l_3 on $\mathcal{G}$ by

$$
\begin{cases}
l_2(x,y) = \llbracket x,y\rrbracket = \frac{1}{2}\left([x,y]_{\mathfrak{g}} - [y,x]_{\mathfrak{g}}\right) & \forall\, x,y \in \mathfrak{g}, \\
l_2(x,c) = -l_2(c,x) = \llbracket x,c\rrbracket = \frac{1}{2}[x,c]_{\mathfrak{g}} & \forall\, x \in \mathfrak{g},\ c \in \mathrm{Z}(\mathfrak{g}), \\
l_2(c_1,c_2) = 0 & \forall\, c_1,c_2 \in \mathrm{Z}(\mathfrak{g}), \\
l_3(x,y,z) = J_{x,y,z} & \forall\, x,y,z \in \mathfrak{g}.
\end{cases}
\tag{7}
$$

The following theorem is our main result, which says that one can obtain a Lie 2-algebra via the skew-symmetrization of a Leibniz algebra.

Theorem 3 *Let* $\left(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}\right)$ *be a Leibniz algebra. Then* $(\mathrm{Z}(\mathfrak{g}) \oplus \mathfrak{g}, l_1, l_2, l_3)$ *is a Lie 2-algebra, where* l_i *are given by* Eq. 7.

Proof By the definition of l_1, l_2 and l_3, it is obvious that Conditions (a)–(d) in Definition 1 hold. By (iii) in Proposition 2, Condition (e) also holds. Thus, $(\mathrm{Z}(\mathfrak{g}) \oplus \mathfrak{g}, l_1, l_2, l_3)$ is a Lie 2-algebra. □

Example 4 (**Omni-Lie algebras**) The notion of an omni-Lie algebra was introduced by Weinstein in [10] to study the linearization of the standard Courant algebroid. An **omni-Lie algebra** associated to a vector space V is a triple $(\mathfrak{gl}(V) \oplus V, (\cdot,\cdot)_+, \{\cdot,\cdot\})$, where $(\cdot,\cdot)_+$ is the V-valued pairing given by

$$
(A+u, B+v)_+ = Au + Bv, \quad \forall\, A+u, B+v \in \mathfrak{gl}(V) \oplus V, \tag{8}
$$

and $\{\cdot,\cdot\}$ is the bilinear bracket operation given by

$$\{A+u, B+v\} = [A, B] + Av. \tag{9}$$

It is straightforward to verify that $(\mathfrak{gl}(V)\oplus V, \{\cdot,\cdot\})$ is a Leibniz algebra. Furthermore, if we consider the skew-symmetric bracket $[\![\cdot,\cdot]\!]$, we have

$$[\![A+u, B+v]\!] = \frac{1}{2}\left(\{A+u, B+v\} - \{B+v, A+u\}\right) = [A, B] + \frac{1}{2}(Av - Bu). \tag{10}$$

The factor of $\frac{1}{2}$ in Eq. 10 spoils the Jacobi identity. More precisely, we have

$$\begin{aligned}[\![[\![A+u, B+v]\!], C+w]\!] + c.p. &= \frac{1}{4}\left([A, B]w + [B, C]u + [C, A]v\right)\\ &\triangleq T(A+u, B+v, C+w).\end{aligned}$$

Thus, $[\![\cdot,\cdot]\!]$ is not a Lie bracket. However, we can extend the omni-Lie algebra $\mathfrak{gl}(V)\oplus V$ to the Lie 2-algebra whose degree-0 part is $\mathfrak{gl}(V)\oplus V$,

$$\begin{cases} V \stackrel{0+\mathrm{id}}{\longrightarrow} \mathfrak{gl}(V)\oplus V, & \\ l_2(e_1, e_2) = [\![e_1, e_2]\!], & \text{for } e_1, e_2 \in \mathfrak{gl}(V)\oplus V,\\ l_2(e, f) = [\![e, f]\!], & \text{for } e \in \mathfrak{gl}(V)\oplus V,\ f\in V,\\ l_3(e_1, e_2, e_3) = -T(e_1, e_2, e_3), & \text{for } e_1, e_2, e_3 \in \mathfrak{gl}(V)\oplus V.\end{cases} \tag{11}$$

such that the Jacobiator is measured by a ternary bracket taking value in the degree-1 part V. See [9] for details.

Example 5 (**Courant algebroids**) Courant algebroids were first introduced in [3] to study the double of Lie bialgebroids (see [6] for an alternative definiton). See the review article [2] for more details. The standard Courant algebroid associated to a manifold M is $(\mathcal{T} = TM\oplus T^*M, (\cdot,\cdot)_+, \{\cdot,\cdot\}, \rho)$, where $\rho: \mathcal{T}\longrightarrow TM$ is the projection, the canonical pairing $(\cdot,\cdot)_+$ is given by

$$(X+\xi, Y+\eta) = \frac{1}{2}\left(\xi(Y) + \eta(X)\right), \quad \forall\, X, Y \in \mathfrak{X}(M),\ \xi, \eta\in\Omega^1(M), \tag{12}$$

the bracket $\{\cdot,\cdot\}$ is given by

$$\{X+\xi, Y+\eta\} \triangleq [X, Y] + L_X\eta - i_Y d\xi. \tag{13}$$

It is straightforward to verify that $\left(\mathfrak{X}(M)\oplus\Omega^1(M), \{\cdot,\cdot\}\right)$ is a Leibniz algebra. Furthermore, if we consider the skew-symmetric bracket $[\![\cdot,\cdot]\!]$:

$$[\![X+\xi, Y+\eta]\!] = \frac{1}{2}\left(\{X+\xi, Y+\eta\} - \{Y+\eta, X+\xi\}\right),$$

we have

$$[\![X+\xi, Y+\eta]\!] = [X, Y] + L_X\eta - L_Y\xi + \frac{1}{2}d(\xi(Y) - \eta(X)). \tag{14}$$

However, $(\mathfrak{X}(M)\oplus\Omega^1(M), [\![\cdot,\cdot]\!])$ is not a Lie algebra. Instead, one can construct a Lie 2-algebra. More precisely, we have

$$[\![[\![e_1, e_2]\!], e_3]\!] + c.p. = dT(e_1, e_2, e_3), \quad \forall\, e_1, e_2, e_3 \in \Gamma(\mathcal{T}), \tag{15}$$

where $T(e_1, e_2, e_3) \in C^\infty(M)$ is given by

$$T(e_1, e_2, e_3) = \frac{1}{3}\Big(\big(\llbracket e_1, e_2 \rrbracket, e_3\big)_+ + c.p.\Big). \tag{16}$$

The associated Lie 2-algebra is given by

$$\begin{cases} \Omega^1_{\mathrm{cl}}(M) & \stackrel{0+\mathrm{id}}{\longrightarrow} & \Gamma(\mathcal{T}), & \\ l_2(e_1, e_2) & = & \llbracket e_1, e_2 \rrbracket, & \text{for } e_1, e_2 \in \Gamma(\mathcal{T}), \\ l_2(e, \xi) & = & \llbracket e, \xi \rrbracket, & \text{for } e \in \Gamma(\mathcal{T}),\ \xi \in \Omega^1_{\mathrm{cl}}(M), \\ l_3(e_1, e_2, e_3) & = & -dT(e_1, e_2, e_3), & \text{for } e_1, e_2, e_3 \in \Gamma(\mathcal{T}), \end{cases}$$

where $\Omega^1_{\mathrm{cl}}(M)$ denotes the set of closed 1-forms. See [7] for the general construction of a Lie 2-algebra from a Courant algebroid.

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