Algebr Represent Theor (2015) 18:1623–1647
DOI 10.1007/s10468-015-9554-7

Schrödinger Representations from the Viewpoint of Tensor Categories

Kenichi Shimizu[1] · **Michihisa Wakui**[2]

Received: 11 December 2014 / Accepted: 19 June 2015 / Published online: 17 July 2015

Abstract The Drinfel'd double $D(A)$ of a finite-dimensional Hopf algebra A is a Hopf algebraic counterpart of the monoidal center construction. Majid introduced an important representation of $D(A)$, which he called the Schrödinger representation. We study this representation from the viewpoint of the theory of tensor categories. One of our main results is as follows: If two finite-dimensional Hopf algebras A and B over a field $\boldsymbol{k}$ are monoidally Morita equivalent, *i.e.*, there exists an equivalence $F : {}_A\mathsf{M} \to {}_B\mathsf{M}$ of $\boldsymbol{k}$-linear monoidal categories, then the equivalence ${}_{D(A)}\mathsf{M} \approx {}_{D(B)}\mathsf{M}$ induced by F preserves the Schrödinger representation. Here, ${}_A\mathsf{M}$ for an algebra A means the category of left A-modules. As an application, we construct a family of invariants of finite-dimensional Hopf algebras under the monoidal Morita equivalence. This family is parameterized by braids. The invariant associated to a braid $\boldsymbol{b}$ is, roughly speaking, defined by "coloring" the closure of $\boldsymbol{b}$ by the Schrödinger representation. We investigate what algebraic properties this family have and, in particular, show that the invariant associated to a certain braid closely relates to the number of irreducible representations.

Presented by Susan Montgomery.

For this research the first author (K.S.) is supported by Grant-in-Aid for JSPS Fellows (24·3606), and the second author (M.W.) is partially supported by Grant-in-Aid for Scientific Research (No. 22540058), JSPS.

✉ Michihisa Wakui
wakui@kansai-u.ac.jp

Kenichi Shimizu
x12005i@math.nagoya-u.ac.jp

[1] Graduate School of Mathematics, Nagoya University, Furo-cho, Chikusa-ku, Nagoya 464-8602, Japan

[2] Department of Mathematics, Faculty of Engineering Science, Kansai University, Suita-shi, Osaka 564-8680, Japan

Keywords Hopf algebra · Monoidal category · Drinfel'd double · Schrödinger representation · Monoidal Morita invariant · Braiding

Mathematics Subject Classification (2010) 16D90 · 16T05 · 18D10

1 Introduction

Drinfel'd doubles of Hopf algebras [4] are one of the most important objects in not only Hopf algebra theory, but also in other areas including category theory and low-dimensional topology. Let A be a finite-dimensional Hopf algebra over a field $\boldsymbol{k}$, and $(D(A), \mathcal{R})$ be its Drinfel'd double. Due to Majid [21], it is known that there is a canonical representation of $D(A)$ on A, which is called the *Schrödinger representation* (or the *Schrödinger module*). This representation is an extension of the adjoint representation of A, and originates from quantum mechanics as explained in Majid's book [21, Examples 6.1.4 & 7.1.8] (see Section 2 for the precise definition of the Schrödinger representation). The Schrödinger module is also addressed by Fang [9] as an algebra in the Yetter-Drinfel'd category ${}^{A}_{A}\mathcal{YD}$ via the Miyashita-Ulbrich action.

In this paper, we study the Schrödinger module over the Drinfel'd double from the viewpoint of the theory of tensor categories. We say that two finite-dimensional Hopf algebras A and B over the same field $\boldsymbol{k}$ are *monoidally Morita equivalent* if ${}_{A}\mathbf{M}$ and ${}_{B}\mathbf{M}$ are equivalent as $\boldsymbol{k}$-linear monoidal categories, where ${}_{H}\mathbf{M}$ for an algebra H is the category of H-modules. One of our main results is that the Schrödinger module is an invariant under the monoidal Morita equivalence in the following sense: If $F : {}_{A}\mathbf{M} \to {}_{B}\mathbf{M}$ is an equivalence of $\boldsymbol{k}$-linear monoidal categories, then the equivalence ${}_{D(A)}\mathbf{M} \approx {}_{D(B)}\mathbf{M}$ induced by F preserves the Schrödinger modules. To prove this result, we introduce the notion of the *Schrödinger object* for a monoidal category by using the monoidal center construction. It turns out that the Schrödinger module over $D(A)$ is characterized as the Schrödinger object for ${}_{A}\mathbf{M}$. Once such a characterization is establishe d, the above result easily follows from general arguments.

As an application of the above category-theoretical understanding of the Schrödinger module, we construct a new family of *monoidal Morita invariants*, *i.e.*, invariants of finite-dimensional Hopf algebras under the monoidal Morita equivalence. Some monoidal Morita invariants have been discovered and studied; see, *e.g.*, [6–8, 15, 24, 25, 32, 34]. Our family of invariants is parametrized by braids. Roughly speaking, the invariant associated with a braid $\boldsymbol{b}$ is defined by "coloring" the closure of $\boldsymbol{b}$ by the Schrödinger module. Since the categorical dimension (in the sense of Majid [21]) of the Schrödinger module is a special case of our invariants, we call the invariant associated with $\boldsymbol{b}$ the *braided dimension* of the Schrödinger module associated with $\boldsymbol{b}$. We investigate what algebraic properties this family of invariants have and, in particular, show that the invariant associated to a certain braid closely relates to the number of irreducible representations.

This paper organizes as follows: In Section 2, we recall the definition of the Schrödinger module over the Drinfel'd double $D(A)$ of a finite-dimensional Hopf algebra A. We also describe the definition of another Schrödinger representation of $D(A)$ on $A^{*\mathrm{cop}}$, which is introduced by Majid, and is called co-Schrödinger representation [21, Proposition 6.2.7]. We refer the left $D(A)$-module as the *co-Schrödinger module*. Following [12] we also describe the definition of Radford's induction functors and those properties. It is shown that the Schrödinger module and the co-Schödinger module are isomorphic to the images of the trivial left A-module and the trivial right A-comodule under Radford's induction functors, respectively. Furthermore, we examine the relationship between the Schrödinger module over $D(A^*)$ and the co-Schrödinger module over $D(A)$.

In Section 3, we study the categorical aspects of the Schrödinger module and the co-Schrödinger module. We introduce a Schrödinger object for a monoidal category $\mathcal{C}$ as the object of $\mathcal{Z}(\mathcal{C})$ representing the functor $\mathrm{Hom}_{\mathcal{C}}(\Pi(-), \mathbb{I})$, where $\mathcal{Z}(\mathcal{C})$ is the monoidal center of $\mathcal{C}$, $\Pi : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ is the forgetful functor, and $\mathbb{I}$ is the unit object of $\mathcal{C}$. By using the properties of Radford's induction functor, we show that the Schrödinger module over $D(A)$ is a Schrödinger object for ${}_A\mathbf{M}$ under the identification $\mathcal{Z}({}_A\mathbf{M}) \approx {}_{D(A)}\mathbf{M}$. Once this characterization is obtained, the invariance of the Schrödinger module (stated above) is easily proved. A similar result for the co-Schrödinger module is also proved.

In Section 4, based on our category-theoretical understanding of the Schrödinger module, we introduce a family of monoidal Morita invariants parameterized by braids. We give formulas for the invariants associated with a certain series of braids and give some applications. Note that some monoidal Morita invariants, such as ones introduced in [6] and [32], factor through the Drinfel'd double construction. Our invariants have an advantage that they do not factor through that. On the other hand, our invariants have a disadvantage in the non-semisimple situation: For any braid $\boldsymbol{b}$, the braided dimension of the Schrödinger module of $D(A)$ associated with $\boldsymbol{b}$ is zero unless A is cosemisimple. From this result, we could say that our invariants are not interesting as monoidal Morita invariants for non-cosemisimple Hopf algebras. However, endomorphisms on the Schödinger module induced by braids are not generally zero and thus may have some information about A. To demonstrate, we give an example of a morphism induced by a braid, which turns out to be closely related to the unimodularlity of A.

2 Preliminaries

2.1 Hopf Algebras

For the basic theory of Hopf algebras, we refer the reader to Abe [1], Montgomery [23] and Sweedler [33]. We first fix some notations: Throughout this paper, $\boldsymbol{k}$ is an arbitrary field and all vector spaces, algebras, coalgebras, etc., are assumed to be over $\boldsymbol{k}$. For vector spaces X and Y (over $\boldsymbol{k}$), their tensor product over $\boldsymbol{k}$ is denoted by $X \otimes Y$. There is a natural isomorphism

$$T_{X,Y} : X \otimes Y \longrightarrow Y \otimes X, \quad x \otimes y \mapsto y \otimes x \quad (x \in X, y \in Y).$$

Given a coalgebra C, we denote the comultiplication and the counit of C respectively by Δ_C and ε_C (or simply by Δ and ε if there is no confusion). To express the comultiplication, we use the Sweedler notation such as $\Delta(c) = c_{(1)} \otimes c_{(2)}$. Recall that the opposite algebra A^{op} of an algebra A is obtained by reversing the order of the multiplication of A. The opposite coalgebra C^{cop} is obtained from C by replacing its comultiplication with $\Delta^{\mathrm{cop}} := T_{C,C} \circ \Delta$. These notations are also used for a bialgebra. The antipode of a bialgebra B is denoted by S_B (or by S) if it exists (*i.e.*, if B is a Hopf algebra).

The dual space of X is denoted by $X^* = \mathrm{Hom}_{\boldsymbol{k}}(X, \boldsymbol{k})$. For $f \in X^*$ and $x \in X$, we often write $f(x)$ as $\langle f, x \rangle$. If C is a coalgebra, then C^* is an algebra with the multiplication defined by $\langle pq, c \rangle = \langle p, c_{(1)} \rangle \langle q, c_{(2)} \rangle$ for $p, q \in C^*$ and $c \in C$. If A is a finite-dimensional algebra, then A^* is a coalgebra with the comultiplication determined by $\langle p_{(1)}, a \rangle \langle p_{(2)}, b \rangle = \langle p, ab \rangle$ for $p \in A^*$ and $a, b \in A$. We note that if B is a finite-dimensional bialgebra, then B^* is. If, moreover, B is a Hopf algebra, then B^* is also a Hopf algebra with antipode S_B^*.

2.2 Monoidal Categories

For the basic theory of monoidal categories, we refer the reader to Mac Lane [20], Kassel [17] and Joyal and Street [14]. Here we fix related notations: Given a (strict) monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{I})$ with tensor product $\otimes$ and unit object $\mathbb{I}$, we denote by $\mathcal{C}^{\text{rev}} = (\mathcal{C}, \otimes^{\text{rev}}, \mathbb{I})$ the category $\mathcal{C}$ with the 'reversed' tensor product given by $X \otimes^{\text{rev}} Y = Y \otimes X$. By a *left dual* of $X \in \mathcal{C}$, we mean a triple (X^*, e_X, n_X) consisting of an object $X^* \in \mathcal{C}$ and morphisms

$$e_X : X^* \otimes X \longrightarrow \mathbb{I} \quad \text{(the evaluation)} \quad \text{and} \quad n_X : \mathbb{I} \longrightarrow X \otimes X^* \quad \text{(the coevaluation)}$$

such that $(\mathrm{id}_X \otimes e_X) \circ (n_X \otimes \mathrm{id}_X)$ and $(e_X \otimes \mathrm{id}_{X^*}) \circ (\mathrm{id}_{X^*} \otimes n_X)$ are the identities. The monoidal category $\mathcal{C}$ is said to be *left rigid* if every object of $\mathcal{C}$ has a left dual. A *rigid monoidal category* is a monoidal category $\mathcal{C}$ such that both $\mathcal{C}$ and $\mathcal{C}^{\text{rev}}$ are left rigid.

Given a braided monoidal category $\mathcal{B}$ with braiding c, we denote by $\mathcal{B}^{\text{mir}}$ the braided monoidal category obtained from $\mathcal{B}$ by replacing its braiding with the 'mirror' braiding c^{mir} defined by $c^{\text{mir}}_{X,Y} = c^{-1}_{Y,X}$. We always regard the monoidal category $\mathcal{B}^{\text{rev}}$ as a braided monoidal category with braiding c^{rev} given by

$$c^{\text{rev}}_{X,Y} : X \otimes^{\text{rev}} Y = Y \otimes X \xrightarrow{c_{Y,X}} X \otimes Y = Y \otimes^{\text{rev}} X \quad (X, Y \in \mathcal{B}).$$

Now let B be a bialgebra. Then the category ${}_B\mathbf{M}$ of left B-modules and the category $\mathbf{M}^B$ of right B-comodules are $\boldsymbol{k}$-linear monoidal categories (here, a $\boldsymbol{k}$-linear monoidal category means a monoidal category $\mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a vector space for all objects $X, Y \in \mathcal{C}$, and the composition and the tensor product of morphisms in $\mathcal{C}$ are linear in each variable). We note that there is an isomorphism of monoidal categories

$${}_{B^{\text{cop}}}\mathbf{M} \cong ({}_B\mathbf{M})^{\text{rev}}. \tag{2.1}$$

If B is finite-dimensional, there are also isomorphisms of monoidal categories

$${}_B\mathbf{M} \cong \mathbf{M}^{B^*} \quad \text{and} \quad {}_{B^*}\mathbf{M} \cong \mathbf{M}^B. \tag{2.2}$$

If B is a quasitriangular bialgebra with universal R-matrix $R \in B \otimes B$, then ${}_B\mathbf{M}$ is a braided monoidal category with the braiding given by

$$c^R_{M,N} : M \otimes N \to N \otimes M, \quad m \otimes n \mapsto T_{M,N}(R \cdot (m \otimes n)) \quad (m \in M, n \in N) \tag{2.3}$$

for $M, N \in {}_B\mathbf{M}$. We always regard B^{cop} and B^{op} as quasitriangular bialgebras with universal R-matrix $R_{21} := T_{B,B}(R)$. By our convention, the category isomorphism (2.1) preserves the braidings. Also, the antipode of B (if it exists) induces an isomorphism $S_B : B^{\text{op}} \to B^{\text{cop}}$ of quasitriangular Hopf algebras (note that the antipode of a quasitriangular Hopf algebra is always bijective [28]).

2.3 The Center Construction

The *center* of a (strict) monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{I})$, denoted by $\mathcal{Z}(\mathcal{C})$, is a braided monoidal category defined as follows: An object of $\mathcal{Z}(\mathcal{C})$ is a pair $(V, c_{-,V})$ consisting of an object $V \in \mathcal{C}$ and a natural isomorphism $c_{-,V} : (-) \otimes V \longrightarrow V \otimes (-)$ such that

$$c_{X \otimes Y,V} = (c_{X,V} \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes c_{Y,V})$$

for all $X, Y \in \mathcal{C}$. A morphism $f : (V, c_{-,V}) \to (W, c_{-,W})$ in $\mathcal{Z}(\mathcal{C})$ is a morphism $f : V \to W$ in $\mathcal{C}$ compatible with $c_{-,V}$ and $c_{-,W}$, and the composition of morphisms in $\mathcal{Z}(\mathcal{C})$ is defined by the composition of morphisms in $\mathcal{C}$. Thus there is a functor

$$\Pi_{\mathcal{C}} : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}, \quad (V, c_{-,V}) \mapsto V,$$

called the *forgetful functor*. The category $\mathcal{Z}(\mathcal{C})$ is a braided monoidal category such that $\Pi_{\mathcal{C}}$ is a strict monoidal functor; see [17, XIII.4] for details.

Since the center construction is described purely in terms of monoidal categories, it is natural to expect that equivalent monoidal categories have equivalent centers. We omit to give a proof of the following well-known fact, since the proof is easy but quite long.

Lemma 2.1 *For a monoidal equivalence $F : \mathcal{C} \longrightarrow \mathcal{D}$ between monoidal categories $\mathcal{C}$ and $\mathcal{D}$, there exists a braided monoidal equivalence $\mathcal{Z}(F) : \mathcal{Z}(\mathcal{C}) \longrightarrow \mathcal{Z}(\mathcal{D})$ such that $\Pi_{\mathcal{D}} \circ \mathcal{Z}(F) = F \circ \Pi_{\mathcal{C}}$ as monoidal functors.*

We note that if $\mathcal{C}$ is a $\boldsymbol{k}$-linear monoidal category, then so is $\mathcal{Z}(\mathcal{C})$ in such a way that the functor $\Pi_{\mathcal{C}}$ is a $\boldsymbol{k}$-linear functor. If $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a $\boldsymbol{k}$-linear monoidal equivalence between $\boldsymbol{k}$-linear monoidal categories, then the functor $\mathcal{Z}(F)$ is $\boldsymbol{k}$-linear.

2.4 The Yetter-Drinfel'd Category

Let A be a Hopf algebra. A (left-right) *Yetter-Drinfel'd A-module* is a triple $(M, \cdot, \rho)$ such that $(M, \cdot)$ is a left A-module, (M, ρ) is a right A-comodule, and the *Yetter-Drinfel'd condition*

$$(a_{(1)} \cdot m_{(0)}) \otimes (a_{(2)} m_{(1)}) = (a_{(2)} \cdot m)_{(0)} \otimes (a_{(2)} \cdot m)_{(1)} a_{(1)} \tag{2.4}$$

is satisfied for all $a \in A$ and $m \in M$ (where the coaction ρ is denoted by $\rho(m) = m_{(0)} \otimes m_{(1)}$ in Sweedler's notation). The *Yetter-Drinfel'd category* over A, denoted by ${}_A\mathcal{YD}^A$, is the category whose objects are the Yetter-Drinfel'd A-modules and whose morphisms are the A-linear and A-colinear maps between them. For $M, N \in {}_A\mathcal{YD}^A$, their tensor product $M \otimes N \in {}_A\mathcal{YD}^A$ is defined to be the tensor product A-module $M \otimes N$ with coaction

$$m \otimes n \mapsto m_{(0)} \otimes n_{(0)} \otimes n_{(1)} m_{(1)} \quad (m \in M, n \in N).$$

The category ${}_A\mathcal{YD}^A$ is a $\boldsymbol{k}$-linear monoidal category with this operation. Moreover, it has the braiding c given by

$$c_{M,N}(m \otimes n) = n_{(0)} \otimes n_{(1)} m \quad (M, N \in {}_A\mathcal{YD}^A, m \in M, n \in N).$$

It is known that there is an isomorphism of $\boldsymbol{k}$-linear braided monoidal categories

$${}_A\mathcal{YD}^A \cong \mathcal{Z}({}_A\mathbf{M}). \tag{2.5}$$

To describe this isomorphism, we let $R_A : {}_A\mathcal{YD}^A \to {}_A\mathbf{M}$ denote the functor forgetting the comodule structure. Then (2.5) is given by $M \longmapsto \left(R_A(M), c'_{-,M}\right)$, where $c'_{X,M}(x \otimes m) = m_{(0)} \otimes m_{(1)} x$ $(m \in M, x \in X \in {}_A\mathbf{M})$.

There is also an isomorphism of $\boldsymbol{k}$-linear braided monoidal categories

$${}_A\mathcal{YD}^A \cong \mathcal{Z}\left(\mathbf{M}^A\right)^{\mathrm{rev}} \tag{2.6}$$

given as follows. Let $R^A : {}_A\mathcal{YD}^A \to \mathbf{M}^A$ be the functor forgetting the A-module structure (we remark that R^A is not a monoidal functor since the tensor product of ${}_A\mathcal{YD}^A$ extends that of $\mathbf{M}^{A^{\mathrm{op}}}$). The isomorphism (2.6), say $F : {}_A\mathcal{YD}^A \longrightarrow \mathcal{Z}\left(\mathbf{M}^A\right)^{\mathrm{rev}}$, is then given by

$$F(M) = \left(R^A(M),\ c''_{-,M}\right),$$

where $c''_{X,M}(x \otimes m) = x_{(1)}m \otimes x_{(0)}$ $\left(m \in M, x \in X \in \mathbf{M}^A\right)$. The monoidal structure $F(M) \otimes^{\mathrm{rev}} F(N) \cong F(M \otimes N)$ is given by $T_{N,M}$.

3 The Schrödinger Module of the Drinfel'd Double

3.1 The Drinfel'd Double

Let A be a finite-dimensional Hopf algebra (thus S_A is invertible [27]). The *Drinfel'd double* $D(A)$ of A [4] is a Hopf algebra such that $D(A) = A^{*\mathrm{cop}} \otimes A$ as a coalgebra. For $p \in A^*$ and $a \in A$, we write $p \otimes a \in D(A)$ as $p \bowtie a$. Then the unit, the multiplication and the antipode of $D(A)$ are given respectively by $1_{D(A)} = \varepsilon_A \bowtie 1_A$,

$$\begin{aligned}(p \bowtie a) \cdot \left(p' \bowtie a'\right) &= \left\langle p'_{(1)}, S_A^{-1}(a_{(3)})\right\rangle \left\langle p'_{(3)}, a_{(1)}\right\rangle pp'_{(2)} \bowtie a_{(2)}a',\\ S_{D(A)}(p \bowtie a) &= \left\langle p_{(1)}, a_{(3)}\right\rangle \left\langle S_{A^*}^{-1}(p_{(3)}), a_{(1)}\right\rangle S_{A^*}^{-1}\left(p_{(2)}\right) \bowtie S_A\left(a_{(2)}\right)\end{aligned}$$

for all $p, p' \in A^*$ and $a, a' \in A$. The Hopf algebra $D(A)$ is quasitriangular with the universal R-matrix given by $\mathcal{R} = \sum_{i=1}^d (\varepsilon_A \bowtie e_i) \otimes \left(e_i^* \bowtie 1_A\right) \in D(A) \otimes D(A)$, where $\{e_i\}_{i=1}^d$ is a basis of A and $\left\{e_i^*\right\}_{i=1}^d$ is its dual basis.

For a finite-dimensional vector space V, we denote by $\iota_V : V \longrightarrow V^{**}$ the canonical isomorphism. As shown by Radford [29, Theorem 3], there is an isomorphism

$$\tau : D(A) \longrightarrow D(A^{\mathrm{op\,cop}*})^{\mathrm{op}}, \quad p \bowtie a \mapsto \iota_A(a) \bowtie p \qquad (p \in A^*,\ a \in A)$$

of quasitriangular Hopf algebras. Note that an isomorphism $f : A \longrightarrow B$ of finite-dimensional Hopf algebras induces an isomorphism

$$D(f) := \left(f^{-1}\right)^* \otimes f : D(A) \longrightarrow D(B)$$

of quasitriangular Hopf algebras. Now we define ϕ_A to be the composition

$$\phi_A : D(A) \xrightarrow{\ \tau\ } D(A^{\mathrm{op\,cop}*})^{\mathrm{op}} \xrightarrow{\ D\left(S_{A^*}^{-1}\right)^{\mathrm{op}}\ } D(A^*)^{\mathrm{op}} \xrightarrow{\ S_{D(A^*)}\ } D(A^*)^{\mathrm{cop}}. \quad (3.1)$$

By definition, $\phi_A(p \bowtie a) = (\iota_A(1_A) \bowtie p) \cdot (\iota_A(a) \bowtie \varepsilon_A)$ for $p \in A^*$ and $a \in A$. Since each arrow in Eq. 3.1 is an isomorphism of quasitriangular Hopf algebras, so is ϕ_A. We note the following property of ϕ_A:

Lemma 3.1 $\phi_{A^*} \circ \phi_A = D(\iota_A)$.

Proof As is well-known, $\iota_{A^*} = \left(\iota_A^{-1}\right)^*$. Hence, for all $a \in A$ and $p \in A^*$,

$$\begin{aligned}(\phi_{A^*} \circ \phi_A)(p \bowtie a) &= \phi_{A^*}(\iota_A(1_A) \bowtie p) \cdot \phi_{A^*}(\iota_A(a) \bowtie \varepsilon_A)\\ &= (\iota_{A^*}(p) \bowtie \iota_A(1_A)) \cdot (\iota_{A^*}(\varepsilon_A) \bowtie \iota_A(a)) = D(\iota_A)(p \bowtie a).\end{aligned}$$

□

The category ${}_{D(A)}\mathbf{M}$ is isomorphic to the Yetter-Drinfel'd category ${}_A\mathcal{YD}^A$ as a $\boldsymbol{k}$-linear braided monoidal category; see, *e.g.*, [17, IX.5]. To describe the isomorphism, we note that A and $A^{*\mathrm{cop}}$ can be regarded as Hopf subalgebras of $D(A)$ by

$$A \longrightarrow D(A), \quad a \mapsto a \bowtie \varepsilon \quad (a \in A) \quad \text{and} \quad A^{*\mathrm{cop}} \longrightarrow D(A), \quad p \mapsto 1 \bowtie p \quad (p \in A^*),$$

respectively. Since $D(A)$ is generated by the subalgebras A and $A^{*\mathrm{cop}}$ $(= A^{\mathrm{op}*})$, we can view a left $D(A)$-module M as a left A-module M endowed with a left $A^{\mathrm{op}*}$-module structure (with some compatibility conditions). The isomorphism ${}_{D(A)}\mathbf{M} \cong {}_A\mathcal{YD}^A$ is the functor that leaves the underlying A-modules fixed and translates the left $A^{\mathrm{op}*}$-module structure into a right A^{op}-comodule structure via (2.2).

In what follows, we identify ${}_{D(A)}\mathbf{M}$ with ${}_A\mathcal{YD}^A$. Then the category-theoretical meaning of the isomorphism ϕ_A is explained as follows: We consider the composition

$$ {}_A\mathcal{YD}^A \xrightarrow{(2.6)} \mathcal{Z}\left(\mathbf{M}^A\right)^{\mathrm{rev}} \xrightarrow{(2.2)} \mathcal{Z}\left({}_{A^*}\mathbf{M}\right)^{\mathrm{rev}} \xrightarrow{(2.5)} \left({}_{A^*}\mathcal{YD}^{A^*}\right)^{\mathrm{rev}} \tag{3.2}$$

of isomorphisms of $\boldsymbol{k}$-linear braided monoidal categories. The isomorphism (3.2) only translates the left A-module structure and the right A-comodule structure of an object of ${}_A\mathcal{YD}^A$ into a right A^*-comodule structure and a left A^*-module structure, respectively.

Lemma 3.2 *The isomorphism* $\phi_A^\sharp : {}_{D(A)^{*\mathrm{cop}}}\mathbf{M} \to {}_{D(A)}\mathbf{M}$ *induced by* ϕ_A *coincides with the following composition of isomorphisms of* $\boldsymbol{k}$*-linear braided monoidal categories:*

$$ {}_{D(A^*)^{\mathrm{cop}}}\mathbf{M} \xrightarrow{(2.1)} \left({}_{D(A^*)}\mathbf{M}\right)^{\mathrm{rev}} = \left({}_{A^*}\mathcal{YD}^{A^*}\right)^{\mathrm{rev}} \xrightarrow{(3.2)} {}_A\mathcal{YD}^A = {}_{D(A)}\mathbf{M}. \tag{3.3}$$

Proof Since $\phi_A^\sharp$ and Eq. 3.3 are strict monoidal functors being the identity on morphisms, it is sufficient to check the claim on the level of objects. Let M be a left $D(A^*)^{\mathrm{cop}}$-module. We denote by $\star$ and $m \mapsto m_{[0]} \otimes m_{[1]}$ the action and the coaction of A^* on the Yetter-Drinfel'd A^*-module corresponding to M, respectively. The left $D(A)$-module corresponding to M via (3.3) is the vector space M with the action given by

$$(p \bowtie a) \cdot m = p \star m_{[0]} \langle m_{[1]}, a \rangle \quad (p \in A^*, a \in A, m \in M).$$

On the other hand, the action of $D(A^*)$ on M is expressed by

$$(\xi \bowtie p) \cdot m = (p \star m)_{[0]} \langle \xi, (p \star m)_{[1]} \rangle \quad (\xi \in A^{**}, p \in A^*, m \in M).$$

Hence, for $p \bowtie a \in D(A)$ and $m \in M$, we compute

$$\phi_A(p \bowtie a) \cdot m = (\iota_A(1_A) \bowtie p) \cdot (\iota_A(a) \bowtie \varepsilon_A) \cdot m = p \star m_{[0]} \langle m_{[1]}, a \rangle.$$

This means that the left $D(A)$-module $\phi_A^\sharp(M)$ coincides with the left $D(A)$-module corresponding to M via the isomorphism (3.3). □

3.2 Schrödinger Modules

Let A be a finite-dimensional Hopf algebra. The Drinfel'd double $D(A)$ has two canonical representations, which are called the (co)-Schrödinger representation as described in Majid's book [21, Examples 6.1.4, 7.1.8, Proposition 6.2.7]. The Schrödinger representation is obtained by unifying the left adjoint action of A and the right A^*-action $\leftharpoonup$, and the co-Schrödinger representation is formally obtained from the Schrödinger representation by replacing A with A^* and 'left' with 'right', respectively (See below (3.8) and (3.9) for precise definition). It is generalized to quasi-Hopf case by Bulacu and Torrecillas [2, Section 3], and to the Drinfel'd double of a generalized Hopf pairing by Fang [9, Section 2]. Specializing in our setting, we will describe these representations below.

There are four actions defined as follows.

$$a \blacktriangleright c = a_{(1)}cS(a_{(2)}) \qquad (a, c \in A), \tag{3.4}$$

$$a \leftharpoondown p = \langle a_{(1)}, p\rangle a_{(2)} \qquad (p \in A^*,\ a \in A), \tag{3.5}$$

$$q \blacktriangleleft p = S(p_{(1)})qp_{(2)} \qquad (p, q \in A^*), \tag{3.6}$$

$$a \rightharpoonup q = q_{(1)}\langle a, q_{(2)}\rangle \qquad (q \in A^*,\ a \in A). \tag{3.7}$$

By using these actions two left actions $\bullet$ of $D(A)$ on A and A^* can be defined by

$$(p \bowtie a) \bullet b = (a \blacktriangleright b) \leftharpoondown S^{-1}(p), \tag{3.8}$$

$$(p \bowtie a) \bullet q = (a \rightharpoonup q) \blacktriangleleft S^{-1}(p) \tag{3.9}$$

for all $a, b \in A^*$ and $p, q \in A^*$. We call $\mathrm{Sch}_A := (A, \bullet)$ and $\mathrm{Sch}^A := (A^*, \bullet)$ the *Schrödinger module* and the *co-Schrödinger module* of $D(A)$, respectively.

Recall that we have identified ${}_{D(A)}\mathbf{M}$ with ${}_A\mathcal{YD}^A$. The Yetter-Drinfel'd module corresponding to Sch_A is the vector space A with the action $\blacktriangleright$ and the coaction

$$\mathrm{Sch}_A \to \mathrm{Sch}_A \otimes A, \quad a \mapsto a_{(2)} \otimes S^{-1}\left(a_{(1)}\right) \quad (a \in \mathrm{Sch}_A).$$

The Yetter-Drinfel'd module corresponding to the co-Schrödinger module Sch^A is the vector space A^* with the action $\rightharpoonup$ and the coaction $q \mapsto q_{(0)} \otimes q_{(1)}$ determined by

$$\left\langle q_{(0)}, a\right\rangle q_{(1)} = \left\langle q, a_{(2)}\right\rangle S^{-1}\left(a_{(3)}\right) a_{(1)} \quad \left(q \in \mathrm{Sch}^A, a \in A\right).$$

Indeed, for all $p \in A^*$, $q \in \mathrm{Sch}^A$ and $a \in A$, we have

$$\langle q_{(0)}, a\rangle\langle p, q_{(1)}\rangle = \langle q, a_{(2)}\rangle \left\langle p_{(1)}, S^{-1}(a_{(3)})\right\rangle \langle p_{(2)}, a_{(1)}\rangle = \left\langle p_{(2)}qS^{-1}(p_{(1)}), a\right\rangle = \left\langle q \blacktriangleleft S^{-1}(p), a\right\rangle.$$

Note that $\mathrm{Sch}_A \in {}_A\mathcal{YD}^A$ can be defined for a not necessarily finite-dimensional Hopf algebra A with bijective antipode. Thus the same symbol will be used for such a Hopf algebra. On the other hand, the finiteness of A seems to be needed to define Sch^A.

3.3 Radford's Induction Functors and Schrödinger Modules

Let A be a bialgebra over $\boldsymbol{k}$ such that A^{op} has antipode $\overline{S}$. For a left A-module $L \in {}_A\mathbf{M}$, the vector space $I_A(L) := L \otimes A$ is a Yetter-Drinfel'd module with the action and the coaction given respectively by

$$h \cdot (l \otimes a) = \left(h_{(2)} \cdot l\right) \otimes h_{(3)}a\overline{S}(h_{(1)}), \quad \text{and} \quad \rho(l \otimes a) = \left(l \otimes a_{(1)}\right) \otimes a_{(2)}$$

for $h, a \in A$ and $l \in L$. The assignment $L \mapsto I_A(L)$ is a right adjoint functor of the forgetful functor $R_A : {}_A\mathcal{YD}^A \longrightarrow {}_A\mathbf{M}$ (Radford [30, Proposition 2], Hu and Zhang [12, Lemma 2.1]). The adjunction is given by

$$\begin{aligned} \mathrm{Hom}_{{}_A\mathbf{M}}(R_A(M), V) \ni f \longmapsto \varphi(f) \in \mathrm{Hom}_{{}_A\mathcal{YD}^A}(M, I_A(V)),& \\ \bigl(\varphi(f)\bigr)(m) = f(m_{(0)}) \otimes m_{(1)} \quad (m \in M)& \end{aligned} \tag{3.10}$$

for $V \in {}_A\mathbf{M}$ and $M \in {}_A\mathcal{YD}^A$.

Remark 3.3 Let $F : \mathcal{C} \to \mathcal{D}$ and $\widetilde{F} : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{D}}$ be equivalences of categories, and let U and U' be functors such that the following diagram commutes up to isomorphism:

$$\begin{array}{ccc} \widetilde{\mathcal{C}} & \xrightarrow{\widetilde{F}} & \widetilde{\mathcal{D}} \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U'} \\ \mathcal{C} & \xrightarrow[F]{} & \mathcal{D} \end{array}$$

It is easy to see that if U has a left (right) adjoint I, then the functor $I' := \widetilde{F} \circ I \circ \overline{F}$, where $\overline{F}$ is a quasi-inverse of F, is a left (right) adjoint of U'. Now let A be a finite-dimensional Hopf algebra. Applying the above observation to the forgetful functor $U = R_A$ and the restriction functor $U' = \mathrm{Res}_A^{D(A)}$, we may say that the functor I_A is right adjoint to $\mathrm{Res}_A^{D(A)}$ under the identification ${}_{D(A)}\mathsf{M} \cong {}_A\mathcal{YD}^A$.

The functor I_A will be referred to as *Radford's induction functor*. The following proposition expresses the Yetter-Drinfel'd module Sch_A by Radford's induction functor:

Proposition 3.4 *Let A be a Hopf algebra over $\boldsymbol{k}$ with bijective antipode. Then, the $\boldsymbol{k}$-linear map $\Phi : \mathrm{Sch}_A \longrightarrow I_A(\boldsymbol{k})$ defined by $\Phi(a) = 1 \otimes S^{-1}(a)$ for all $a \in A$ is an isomorphism of Yetter-Drinfel'd A-modules. Here, $\boldsymbol{k}$ in $I_A(\boldsymbol{k})$ means the trivial left A-module.*

Proof We identify $I_A(\boldsymbol{k})$ with A as a vector space via $1 \otimes a \leftrightarrow a$. Then the Yetter-Drinfel'd module $I_A(\boldsymbol{k})$ is the vector space A with the left action $h \triangleright a = h_{(2)} a S^{-1}(h_{(1)})$ $(h, a \in A)$ and the right coaction $a \mapsto a_{(1)} \otimes a_{(2)}$ $(a \in A)$. The claim follows from the fact that S is an anti-algebra and anti-coalgebra map. □

There is also a comodule-version of Radford's induction functor: Let A be a bialgebra over $\boldsymbol{k}$, and suppose that A^{op} has an antipode $\overline{S}$. For a right A-comodule N, the vector space $I^A(N) = A \otimes N$ is a Yetter-Drinfel'd A-module by

$$h \cdot (a \otimes n) = ha \otimes n \quad \text{and} \quad \rho(h \otimes n) = (h_{(2)} \otimes n_{(0)}) \otimes h_{(3)} n_{(1)} \overline{S}(h_{(1)})$$

for all $h, a \in A$ and $n \in N$. The functor defined by $N \mapsto I^A(N)$ is in fact a left adjoint functor of the forgetful functor ${}_A\mathcal{YD}^A \to \mathsf{M}^{A^{\mathrm{op}}}$ (Radford [30, Proposition 1], Hu and Zhang [12, Remark 2.2]), and thus we may say that I^A is left adjoint to $\mathrm{Res}_{A^{*\mathrm{cop}}}^{D(A)}$ under the identification ${}_A\mathcal{YD}^A \cong {}_{D(A)}\mathsf{M}$ if A is finite-dimensional (*cf.* Remark 3.3). Moreover, since $\mathsf{M}^A = \mathsf{M}^{A^{\mathrm{op}}}$ as categories, we may view I^A as a left adjoint of $R^A : {}_A\mathcal{YD}^A \longrightarrow \mathsf{M}^A$ of Section 2.4. We also call I^A *Radford's induction functor*. The co-Schrödinger module is expressed by using I^A as follows:

Proposition 3.5 *Let A be a finite-dimensional Hopf algebra over $\boldsymbol{k}$. Then, the $\boldsymbol{k}$-linear map $\Phi : \mathrm{Sch}^A \longrightarrow (I^A(\boldsymbol{k}))^*$ defined by $\Phi(q) = S^{-1}(q) \otimes 1$ for all $q \in \mathrm{Sch}^A$ is an isomorphism of Yetter-Drinfel'd A-modules. Here, $\boldsymbol{k}$ in $I^A(\boldsymbol{k})$ means the trivial right A-comodule.*

Proof Let M be a finite-dimensional Yetter-Drinfel'd A-module. Then its left dual module M^* is the dual vector space of M with the action $\rightharpoonup$ and the coaction determined by

$$\langle a \rightharpoonup p, m \rangle = \langle p, S(a) m \rangle \quad \text{and} \quad \langle p_{(0)}, m \rangle p_{(1)} = \langle p, m_{(0)} \rangle S^{-1}(m_{(1)}),$$

respectively, for $p \in M^*$, $a \in A$ and $m \in M$ (since $M \otimes N$ for $M, N \in {}_A\mathcal{YD}^A$ is the tensor product right A^{op}-comodule, we need to use $S_{A^{\mathrm{op}}} = S_A^{-1}$ to define the coaction of A on M^*). Now we identify $I^A(\boldsymbol{k})$ with A as a vector space. Then $(I^A(\boldsymbol{k}))^*$ is the vector space A^* with the action $\rightharpoonup$ and the coaction $q \mapsto q_{(0)} \otimes q_{(1)}$ determined by

$$\langle a \rightharpoonup p, b\rangle = \langle p, S(a)b\rangle \quad \text{and} \quad \langle q_{(0)}, b\rangle q_{(1)} = \langle p, b_{(2)}\rangle S^{-2}(b_{(1)})S^{-1}(b_{(3)})$$

for $a, b \in A$ and $p \in \left(I^A(\boldsymbol{k})\right)^*$. As in Proposition 3.4, the proof now follows from the fact that the antipode is an anti-algebra and anti-coalgebra map. □

3.4 The Tensor Product of Schrödinger Modules

In this subsection, we compute the tensor product of Schrödinger modules by using Propositions 3.4 and 3.5. We first note the following properties of Radford's induction:

Lemma 3.6 *Let A be a Hopf algebra over $\boldsymbol{k}$ with bijective antipode. Then:*

(1) There is a natural isomorphism of Yetter-Drinfel'd modules

$$\Phi : I_A(V) \otimes M \longrightarrow I_A(V \otimes R_A(M)) \quad \left(V \in {}_A\mathbf{M}, M \in {}_A\mathcal{YD}^A\right)$$

given by $\Phi(v \otimes a \otimes m) = v \otimes m_{(0)} \otimes m_{(1)}a$ for $v \in V$, $a \in A$ and $m \in M$.

(2) There is a natural isomorphism of Yetter-Drinfel'd modules

$$\Psi : I^A(R^A(M) \otimes V) \longrightarrow I^A(V) \otimes M \quad \left(V \in \mathbf{M}^A, M \in {}_A\mathcal{YD}^A\right)$$

given by $\Psi(a \otimes m \otimes v) = a_{(1)} \otimes v \otimes a_{(2)}m$ for $v \in V$, $a \in A$ and $m \in N$.

Proof It is routine to check that the map Φ is a morphism in ${}_A\mathcal{YD}^A$. Instead of doing the computation, it can be also confirmed by observing that Φ arises as the composition

$$I_A(V) \otimes M \xrightarrow{\eta} I_A R_A(I_A(V) \otimes M) = I_A(R_A I_A(V) \otimes R_A(M)) \xrightarrow{I_A(\epsilon \otimes \mathrm{id})} I_A(V \otimes R_A(M)),$$

where η and ϵ are the unit and the counit of the adjunction (3.10), respectively. It is easy to see that the inverse of Φ is given by $\Phi^{-1}(v \otimes m \otimes a) = v \otimes S(m_{(1)})a \otimes m_{(0)}$ for $a \in A$, $v \in V$ and $m \in M$. Hence (1) is proved. Part (2) can be proved in a similar way. □

For a Hopf algebra A, we denote by Adj_A the adjoint representation of A, *i.e.*, the vector space A endowed with the left A-action $\blacktriangleright$ given by Eq. 3.4.

Proposition 3.7 *Let A be a Hopf algebra over $\boldsymbol{k}$ with bijective antipode, and let $n \geq 1$ be an integer. Then there is an isomorphism $\mathrm{Sch}_A^{\otimes n} \cong I_A\left(\mathrm{Adj}_A^{\otimes(n-1)}\right)$ of Yetter-Drinfel'd A-modules.*

Proof By Proposition 3.4 and Lemma 3.6,

$$\mathrm{Sch}_A^{\otimes n} \cong I_A(\boldsymbol{k}) \otimes \mathrm{Sch}_A^{\otimes(n-1)} \cong I_A\left(\boldsymbol{k} \otimes R_A(\mathrm{Sch}_A)^{\otimes(n-1)}\right) \cong I_A\left(\mathrm{Adj}_A^{\otimes(n-1)}\right).$$

□

The following result is a non-semisimple generalization of a part of [3, Proposition 4].

Proposition 3.8 *If A is a finite-dimensional Hopf algebra over $\boldsymbol{k}$, then there is an isomorphism of left $D(A)$-modules* $\mathrm{Sch}_A \otimes \mathrm{Sch}^A \cong D(A)$.

Proof Let, in general, H be a finite-dimensional Hopf algebra. For a left H-module X, we denote by X_0 the vector space X with left H-action $h \cdot x = \varepsilon(h)x$ ($h \in H$, $x \in X$). Then there are natural isomorphisms $H \otimes X \cong H \otimes X_0 \cong X \otimes H$ of left H-modules. It is also known that the left H-module $H \in {}_H\mathbf{M}$ is self-dual. Using these facts, we obtain natural isomorphisms

$$\mathrm{Hom}_H(X \otimes H, Y) \cong \mathrm{Hom}_{\boldsymbol{k}}(X, Y) \cong \mathrm{Hom}_H(X, Y \otimes H) \quad (X, Y \in {}_H\mathbf{M}) \tag{3.11}$$

of vector spaces. Now we give a proof of the claim: Since $(R_A \circ I^A)(\boldsymbol{k}) \cong A$ as left A-modules, we have natural isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{D(A)}\left(X, \mathrm{Sch}_A \otimes \mathrm{Sch}^A\right) &\cong \mathrm{Hom}_{D(A)}\left(X, I_A(\boldsymbol{k}) \otimes I^A(\boldsymbol{k})^*\right)\\
&\cong \mathrm{Hom}_{D(A)}\left(X \otimes I^A(\boldsymbol{k}), I_A(\boldsymbol{k})\right)\\
&\cong \mathrm{Hom}_A\left(R_A(X \otimes I^A(\boldsymbol{k})), \boldsymbol{k}\right)\\
&\cong \mathrm{Hom}_A(R_A(X) \otimes A, \boldsymbol{k})\\
&\cong \mathrm{Hom}_{\boldsymbol{k}}(X, \boldsymbol{k}) \quad \text{(by (3.11))}
\end{aligned}$$

for $X \in {}_{D(A)}\mathbf{M}$. On the other hand, $\mathrm{Hom}_{D(A)}(X, D(A)) \cong \mathrm{Hom}_{\boldsymbol{k}}(X, \boldsymbol{k})$ again by Eq. 3.11. Hence the result follows from Yoneda's lemma. □

3.5 Categorical Aspects of the Schrödinger Module

Let $\mathcal{C}$ be a monoidal category. A *Schrödinger object* for $\mathcal{C}$ is an object $\mathbf{S} \in \mathcal{Z}(\mathcal{C})$ such that there exists a natural isomorphism $\mathrm{Hom}_{\mathcal{C}}\left(\Pi_{\mathcal{C}}(\mathbf{X}), \mathbb{I}_{\mathcal{Z}(\mathcal{C})}\right) \cong \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}(\mathbf{X}, \mathbf{S})$ for $\mathbf{X} \in \mathcal{Z}(\mathcal{C})$. Note that such an object is unique up to isomorphism by Yoneda's lemma (if it exists). The following lemma is obvious from the definition:

Lemma 3.9 *Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories.*

(1) *Suppose that $\Pi_{\mathcal{C}} : \mathcal{Z}(\mathcal{C}) \longrightarrow \mathcal{C}$ has a right adjoint functor $I_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{Z}(\mathcal{C})$. Then the object $I_{\mathcal{C}}(\mathbb{I})$ is a Schrödinger object for $\mathcal{C}$.*

(2) *Suppose that $\mathbf{S}_{\mathcal{C}}$ is a Schrödinger object for $\mathcal{C}$, and there exists an equivalence $F : \mathcal{C} \longrightarrow \mathcal{D}$ of monoidal categories. Then $\mathcal{Z}(F)(\mathbf{S}_{\mathcal{C}})$ is a Schrödinger object for $\mathcal{D}$.*

Recall that there is an isomorphism ${}_A\mathcal{YD}^A \cong \mathcal{Z}({}_A\mathbf{M})$. By interpreting Lemmas 2.1 and 3.9 in terms of the Yetter-Drinfel'd category through this isomorphism, we obtain:

Theorem 3.10 *Let A and B be Hopf algebras over $\boldsymbol{k}$ with bijective antipodes. Suppose that there is an equivalence $F : {}_A\mathbf{M} \longrightarrow {}_B\mathbf{M}$ of $\boldsymbol{k}$-linear monoidal categories. Then:*

(1) *There exists an equivalence $\widetilde{F} : {}_A\mathcal{YD}^A \longrightarrow {}_B\mathcal{YD}^B$ of $\boldsymbol{k}$-linear braided monoidal categories such that $R_B \circ \widetilde{F} = F \circ R_A$ as monoidal functors.*

(2) *The equivalence $\widetilde{F}$ satisfies $I_B \circ F \cong \widetilde{F} \circ I_A$.*

(3) $\mathrm{Sch}_A \in {}_A\mathcal{YD}^A$ *is a Schrödinger object for ${}_A\mathbf{M}$ under $\mathcal{Z}({}_A\mathbf{M}) \cong {}_A\mathcal{YD}^A$.*

(4) *The equivalence $\widetilde{F}$ preserves the Schrödinger modules, i.e., $\widetilde{F}(\mathrm{Sch}_A) \cong \mathrm{Sch}_B$.*

Proof The following functor $\widetilde{F}$ satisfies the conditions required in Part (1):

$$\widetilde{F} : {}_A\mathcal{YD}^A \xrightarrow{(2.5)} \mathcal{Z}({}_A\mathbf{M}) \xrightarrow{\mathcal{Z}(F)} \mathcal{Z}({}_B\mathbf{M}) \xrightarrow{(2.5)} {}_B\mathcal{YD}^B.$$

Part (2) follows from Remark 3.3. Parts (3) and (4) are obtained by translating Lemma 3.9 through the isomorphism (2.5). □

Remark 3.11 An equivalence ${}_A\mathcal{YD}^A \approx {}_B\mathcal{YD}^B$ of $\boldsymbol{k}$-linear braided monoidal categories does not preserve the Schrödinger module in general.

Remark 3.12 As Masuoka pointed out to us, the above theorem can be also derived from the point of view of cocycle deformations by using the action given in [22, Proposition 5.1].

3.6 Categorical Aspects of the Co-Schrödinger Module

We have explored categorical aspects of the Schrödinger module. There is a similar result for the co-Schrödinger module:

Theorem 3.13 *Let A and B be Hopf algebras over $\boldsymbol{k}$ with bijective antipodes. Suppose that there is an equivalence $F : \mathbf{M}^A \longrightarrow \mathbf{M}^B$ of $\boldsymbol{k}$-linear monoidal categories. Then:*

(1) There exists an equivalence $\widetilde{F} : {}_A\mathcal{YD}^A \longrightarrow {}_B\mathcal{YD}^B$ of $\boldsymbol{k}$-linear braided monoidal categories such that $R^B \circ \widetilde{F} = F \circ R^A$ as monoidal functors.
(2) The equivalence $\widetilde{F}$ satisfies $I^B \circ F \cong \widetilde{F} \circ I^A$.
(3) Suppose that A and B are finite-dimensional. Then the equivalence $\widetilde{F}$ preserves the co-Schrödinger module, i.e., $\widetilde{F}\left(\mathrm{Sch}^A\right) \cong \mathrm{Sch}^B$.

Proof The following functor $\widetilde{F}$ satisfies the conditions required in Part (1):

$$\widetilde{F} : {}_A\mathcal{YD}^A \xrightarrow{(2.6)} \mathcal{Z}(\mathbf{M}^A)^{\mathrm{rev}} \xrightarrow{\mathcal{Z}(F)^{\mathrm{rev}}} \mathcal{Z}(\mathbf{M}^B)^{\mathrm{rev}} \xrightarrow{(2.6)} {}_B\mathcal{YD}^B.$$

Part (2) follows from Remark 3.3. Part (3) follows from Proposition 3.5 and the fact that an equivalence of monoidal categories preserves the left duals. □

We have introduced the notion of a Schrödinger object to explain categorical nature of the Schrödinger module. There is a bit technical way to understand the co-Schrödinger object in terms of a Schrödinger object:

Theorem 3.14 *Let $\Phi_A : \left({}_A\mathcal{YD}^A\right)^{\mathrm{rev}} \longrightarrow \mathcal{Z}(\mathbf{M}^A)$ be the isomorphism of $\boldsymbol{k}$-linear braided monoidal categories given in (2.6). If A is finite-dimensional, then $\Phi_A\left(\mathrm{Sch}^A\right) \in \mathcal{Z}\left(\mathbf{M}^A\right)$ is a Schrödinger object for $\mathbf{M}^A$.*

Proof For $X \in {}_A\mathcal{YD}^A$, we have

$$\begin{aligned}
\mathrm{Hom}_{{}_A\mathcal{YD}^A}\left(X, \mathrm{Sch}^A\right) &\cong \mathrm{Hom}_{{}_A\mathcal{YD}^A}\left(X \otimes I^A(\boldsymbol{k}), \boldsymbol{k}\right) && \text{(by Proposition 3.5)}\\
&\cong \mathrm{Hom}_{{}_A\mathcal{YD}^A}\left(I^A R^A(X), \boldsymbol{k}\right) && \text{(by Lemma 3.6)}\\
&\cong \mathrm{Hom}_{\mathbf{M}^A}\left(R^A(X), \boldsymbol{k}\right).
\end{aligned}$$

Since $\Pi_{\mathbf{M}^A} \circ \Phi_A = R^A$ as functors, $\mathrm{Hom}_{\mathcal{Z}(\mathbf{M}^A)}\left(X, \Phi_A(\mathrm{Sch}^A)\right) \cong \mathrm{Hom}_{\mathbf{M}^A}\left(\Pi_{\mathbf{M}^A}(X), \boldsymbol{k}\right)$, that is $\Phi_A\left(\mathrm{Sch}^A\right)$ is a Schrödinger object for $\mathbf{M}^A$. □

Recall from Section 3.1 that there is an isomorphism $\phi_A : D(A) \to D(A^*)^{\mathrm{cop}}$ of quasi-triangular Hopf algebras. The following theorem may be proved in a more direct way, but we prefer to prove it by emphasizing the role of the notion of the Schrödinger object:

Theorem 3.15 *There are isomorphisms of left $D(A)$-modules*

$$\phi_A^\sharp(\mathrm{Sch}_{A^*}) \cong \mathrm{Sch}^A \quad \textit{and} \quad \phi_A^\sharp\left(\mathrm{Sch}^{A^*}\right) \cong \mathrm{Sch}_A$$

where $\phi_A^\sharp : {}_{D(A^*)^{\mathrm{cop}}}\mathbf{M} \to {}_{D(A)}\mathbf{M}$ is the functor induced by ϕ_A.

Proof By Lemma 3.2, we have the following commutative diagram:

$$\begin{array}{ccccccc}
{}_{D(A^*)^{\mathrm{cop}}}\mathbf{M} & = & {}_{A^*}\mathcal{YD}^{A^*} & \xrightarrow{(2.5)} & \mathcal{Z}({}_{A^*}\mathbf{M}) & \xrightarrow{\Pi} & {}_{A^*}\mathbf{M} \\
\phi_A^\sharp \downarrow & & & & \approx \downarrow \text{by (2.2)} & & \approx \downarrow \text{by (2.2)} \\
{}_{D(A)}\mathbf{M} & = & {}_{A}\mathcal{YD}^{A} & \xrightarrow[(2.6)]{} & \mathcal{Z}(\mathbf{M}^A) & \xrightarrow[\Pi]{} & \mathbf{M}^A
\end{array}$$

(we omit 'rev' since the monoidal structure is not needed here). We chase $\mathrm{Sch}_{A^*} \in {}_{D(A^*)^{\mathrm{cop}}}\mathbf{M}$ around this diagram. By Lemma 3.9 and Theorem 3.10, the object in $\mathcal{Z}\left(\mathbf{M}^A\right)$ corresponding to Sch_{A^*} is a Schrödinger object for $\mathbf{M}^A$. On the other hand, by Theorem 3.14, the object in $\mathcal{Z}(\mathbf{M}^A)$ corresponding to $\mathrm{Sch}^A \in {}_{D(A)}\mathbf{M}$ is also the Schrödinger object for $\mathbf{M}^A$. Since the Schrödinger object is unique up to isomorphism, we have an isomorphism

$$\phi_A^\sharp(\mathrm{Sch}_{A^*}) \cong \mathrm{Sch}^A.$$

Applying this result to A^*, we obtain $\phi_{A^*}^\sharp(\mathrm{Sch}_{A^{**}}) \cong \mathrm{Sch}^{A^*}$. Now let $\psi_A^\sharp : {}_{D(A^{**})}\mathbf{M} \to {}_{D(A)}\mathbf{M}$ be the isomorphism induced by the isomorphism $\psi_A = D(\iota_A)$ of quasitriangular Hopf algebras appeared in Lemma 3.1. By that lemma, we have

$$\phi_A^\sharp\left(\mathrm{Sch}^{A^*}\right) \cong \phi_A^\sharp\phi_{A^*}^\sharp(\mathrm{Sch}_{A^{**}}) \cong \psi_A^\sharp(\mathrm{Sch}_{A^{**}}) \cong \mathrm{Sch}_A.$$

□

4 Applications

Motivated by the construction of quantum representations of the n-strand braid group B_n due to Reshetikhin and Turaev [31], a family of monoidal Morita invariants of a finite-dimensional Hopf algebra, which is indexed by braids, can be obtained from the Schrödinger module.

Let A be a finite-dimensional Hopf algebra. It turns out that the invariant associated with the identity element $\mathbf{1} \in B_1$ is equal to the categorical dimension of the Schrödinger module Sch_A in the sense of Majid [21], and thus equal to $\mathrm{Tr}\left(S^2\right)$ by [21, Example 9.3.8] (see Bulacu and Torrecillas [2] for the case of quasi-Hopf algebras). As is well-known, $\mathrm{Tr}\left(S^2\right)$ has the following representation-theoretic meaning: $\mathrm{Tr}\left(S^2\right) \neq 0$ if and only if A

is semisimple and cosemisimple [26]. In this section, we show that the invariants derived from other braids, like , involve further interesting results connecting with representation theory.

The invariant associated with a braid $\boldsymbol{b}$ is, roughly speaking, defined by "coloring" the closure of $\boldsymbol{b}$ by the Schrödinger module as if we were computing the quantum invariant of a (framed) link. Such an operation is not allowed in general since ${}_{D(A)}\mathbf{M}$ may not be a ribbon category. So we will use the (partial) braided trace, introduced below, to define invariants.

4.1 Partial Traces in Braided Monoidal Categories

From now on, all monoidal categories are assumed to be strict although almost all definitions and results do not need this assumption.

Let $\mathcal{B}$ be a left rigid braided monoidal category with braiding c. We choose a left dual (X^*, e_X, n_X) for each object $X \in \mathcal{B}$. Let $f : X \otimes Y \longrightarrow X \otimes Z$ be a morphism in $\mathcal{B}$. Then the following composition $\underline{\mathrm{Tr}}_{\mathcal{B}}^{l,X}(f) : Y \longrightarrow Z$ can be defined:

$$Y = \mathbb{I} \otimes Y \xrightarrow{n_X \otimes \mathrm{id}_Y} X \otimes X^* \otimes Y \xrightarrow{c^{-1}_{X^*,X} \otimes \mathrm{id}_Y} X^* \otimes X \otimes Y \xrightarrow{\mathrm{id} \otimes f} X^* \otimes X \otimes Z \xrightarrow{e_X \otimes \mathrm{id}_Z} \mathbb{I} \otimes Z = Z.$$

We frequently write $\underline{\mathrm{Tr}}^{l,X}(f)$ for simplicity. The morphism $\underline{\mathrm{Tr}}^{l,X}(f) : Y \longrightarrow Z$ is said to be the *left partial braided trace* of f on X. Similarly, for a morphism $f : Y \otimes X \longrightarrow Z \otimes X$, the *right partial braided trace* $\underline{\mathrm{Tr}}^{r,X}(f)$ $\big(= \underline{\mathrm{Tr}}_{\mathcal{B}}^{r,X}(f)\big)$ is defined by

$$Y = Y \otimes \mathbb{I} \xrightarrow{\mathrm{id}_Y \otimes n_X} Y \otimes X \otimes X^* \xrightarrow{f \otimes \mathrm{id}} Z \otimes X \otimes X^* \xrightarrow{\mathrm{id} \otimes c_{X,X^*}} Z \otimes X^* \otimes X \xrightarrow{\mathrm{id}_Z \otimes e_X} Z \otimes \mathbb{I} = Z,$$

see Fig. 1.

The left and right partial braided traces on X do not depend on the choice of left duals of X, and they have the following properties. For morphisms $f : X \otimes Y \longrightarrow X \otimes Z$, $g : Y' \longrightarrow Y$, $h : Z \longrightarrow Z'$,

$$\underline{\mathrm{Tr}}^{l,X}\big((\mathrm{id}_X \otimes h) \circ f \circ (\mathrm{id}_X \otimes g)\big) = h \circ \underline{\mathrm{Tr}}^{l,X}(f) \circ g : Y \longrightarrow Z. \tag{4.1}$$

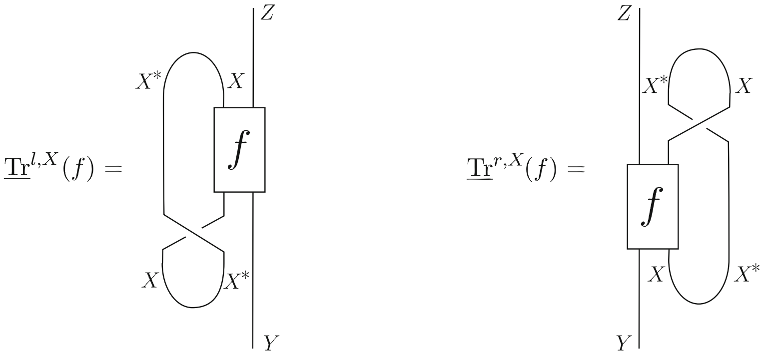

Fig. 1 The left and right partial traces (the diagrams are read upwards)

For an endomorphism $f : X \longrightarrow X$ in $\mathcal{B}$, the left braided trace $\underline{\mathrm{Tr}}^{l,X}(f)$ and the right braided trace $\underline{\mathrm{Tr}}^{r,X}(f)$ are defined by $\underline{\mathrm{Tr}}^{l,X}(f) := \underline{\mathrm{Tr}}^{l,X}(f \otimes \mathrm{id}_{\mathbb{I}})$ and $\underline{\mathrm{Tr}}^{r,X}(f) := \underline{\mathrm{Tr}}^{r,X}(\mathrm{id}_{\mathbb{I}} \otimes f)$. They coincide with the following compositions, respectively.

$$\begin{aligned} \underline{\mathrm{Tr}}^{l,X}(f) \quad &: \mathbb{I} \xrightarrow{n_X} X \otimes X^* \xrightarrow{c^{-1}_{X^*,X}} X^* \otimes X \xrightarrow{\mathrm{id}\otimes f} X^* \otimes X \xrightarrow{e_X} \mathbb{I}, \\ \underline{\mathrm{Tr}}^{r,X}(f) \quad &: \mathbb{I} \xrightarrow{n_X} X \otimes X^* \xrightarrow{f\otimes \mathrm{id}} X \otimes X^* \xrightarrow{c_{X,X^*}} X^* \otimes X \xrightarrow{e_X} \mathbb{I}. \end{aligned}$$

The left and right partial traces are related as follows. Let $\mathcal{B}$ be a left rigid braided monoidal category chosen left duals (X^*, e_X, n_X) for all objects X in $\mathcal{B}$. Then, for two objects X, Y in $\mathcal{B}$ there is a natural isomorphism $j_{X,Y} : Y^* \otimes X^* \longrightarrow (X \otimes Y)^*$ such that $e_{X\otimes Y} \circ (j_{X,Y} \circ \mathrm{id}_{X\otimes Y}) = e_Y \circ (\mathrm{id}_{Y^*} \otimes e_X \otimes \mathrm{id}_Y)$ [11]. For any morphism $f : X \longrightarrow Y$ in $\mathcal{B}$, there is a unique morphism ${}^t f : Y^* \longrightarrow X^*$ in $\mathcal{B}$, which is characterized by $e_X \circ ({}^t f \otimes \mathrm{id}_X) = e_Y \circ (\mathrm{id}_{Y^*} \otimes f)$. Then:

Lemma 4.1 *For any morphism $f : X \otimes Y \longrightarrow X \otimes Z$ in $\mathcal{B}$,*

$$\underline{Tr}^{r,X^*}_{\mathcal{B}}\left(j^{-1}_{X,Y} \circ {}^t f \circ j_{X,Z}\right) = {}^t\!\left(\underline{Tr}^{l,X}_{\mathcal{B}^{mir}}(f)\right).$$

Proof The equation of the lemma is obtained from a graphical calculus depicted as in Fig. 2. □

Lemma 4.2 *Let $\mathcal{B}$ and $\mathcal{B}'$ be left rigid braided monoidal categories, and $(F, \phi, \omega) : \mathcal{B} \longrightarrow \mathcal{B}'$ be a braided monoidal functor. Then for any morphism $f : X \otimes Y \longrightarrow X \otimes Z$ in $\mathcal{B}$*

$$F\left(\underline{Tr}^{l,X}(f)\right) = \underline{Tr}^{l,F(X)}\left(\phi^{-1}_{X,Z} \circ F(f) \circ \phi_{X,Y}\right). \tag{4.2}$$

Proof For each $X \in \mathcal{B}$ we choose a left dual (X^*, e_X, n_X). Then $\left(F(X^*), e'_{F(X)}, n'_{F(X)}\right)$ is a left dual of $F(X)$, where

$$\begin{aligned} e'_{F(X)} \quad &:= \omega^{-1} \circ F(e_X) \circ \phi_{X^*,X} : F(X^*) \otimes F(X) \longrightarrow \mathbb{I}', \\ n'_{F(X)} \quad &:= \phi^{-1}_{X,X^*} \circ F(n_X) \circ \omega : \mathbb{I}' \longrightarrow F(X) \otimes F(X^*). \end{aligned}$$

By using this left dual of $F(X)$ and computing the partial braided trace $\underline{\mathrm{Tr}}^{l,F(X)}\big(\phi^{-1}_{X,Z} \circ F(f) \circ \phi_{X,Y}\big)$, we have the desired Eq. 4.2. □

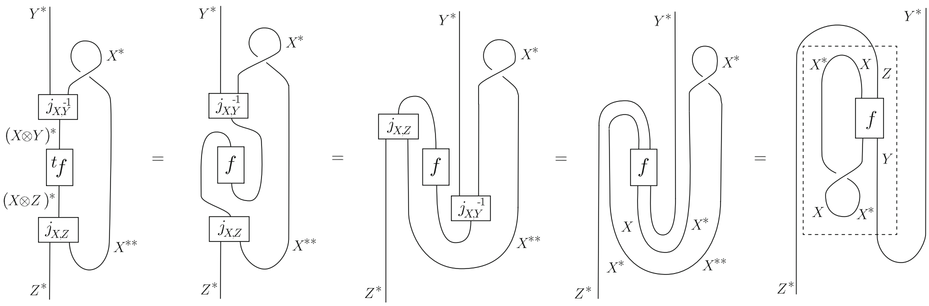

Fig. 2 A graphical calculus for the proof of Lemma 4.1

Let M be an object in $\mathcal{B}$. For each endomorphism $f \in \mathrm{End}(M^{\otimes n})$ and each positive integer k $(1 \leq k \leq n)$, we set $\underline{\mathrm{Tr}}^{l,k}(f) := \underline{\mathrm{Tr}}^{l,M^{\otimes k}}(f)$, $\underline{\mathrm{Tr}}^{r,k}(f) := \underline{\mathrm{Tr}}^{r,M^{\otimes k}}(f)$, and

$$\underline{\widetilde{\mathrm{Tr}}}^{l}(f) := \Big(\overbrace{\underline{\mathrm{Tr}}^{l,1} \circ \cdots \circ \underline{\mathrm{Tr}}^{l,1}}^{n}\Big)(f), \quad \underline{\widetilde{\mathrm{Tr}}}^{r}(f) := \Big(\overbrace{\underline{\mathrm{Tr}}^{r,1} \circ \cdots \circ \underline{\mathrm{Tr}}^{r,1}}^{n}\Big)(f). \tag{4.3}$$

The modified traces (4.3) are preserved by a braided monoidal functor. More precisely:

Proposition 4.3 *Let $\mathcal{B}$ and $\mathcal{B}'$ be left rigid braided monoidal categories, and $(F, \phi, \omega) : \mathcal{B} \longrightarrow \mathcal{B}'$ be a braided monoidal functor. Let M be an object in $\mathcal{B}$, and k be a positive integer, and define the isomorphism $\phi^{(k)} : F(M)^{\otimes k} \longrightarrow F\left(M^{\otimes k}\right)$ in $\mathcal{B}'$ by*

$$\phi^{(1)} := id_{F(M)}, \quad \phi^{(k)} := \phi_{M,M^{\otimes(k-1)}} \circ \left(id_{F(M)} \otimes \phi^{(k-1)}\right) \qquad (k \geq 2).$$

Then for an endomorphism f on $M^{\otimes n}$ in $\mathcal{B}$, the following equations hold.

$$\underline{\widetilde{Tr}}^{l}\left((\phi^{(n)})^{-1} \circ F(f) \circ \phi^{(n)}\right) = \omega^{-1} \circ \left(F\left(\underline{\widetilde{Tr}}^{l}(f)\right)\right) \circ \omega, \tag{4.4}$$

$$\underline{\widetilde{Tr}}^{r}\left((\phi^{(n)})^{-1} \circ F(f) \circ \phi^{(n)}\right) = \omega^{-1} \circ \left(F\left(\underline{\widetilde{Tr}}^{r}(f)\right)\right) \circ \omega. \tag{4.5}$$

Proof We set $g := \left(\phi^{(n)}\right)^{-1} \circ F(f) \circ \phi^{(n)}$. By Eq. 4.1 and Lemma 4.2 we have

$$\begin{aligned}\underline{\mathrm{Tr}}^{l,1}(g) &= \left(\phi^{(n-1)}\right)^{-1} \circ \left(\underline{\mathrm{Tr}}^{l,1}\left(\phi^{-1}_{M,M^{\otimes(n-1)}} \circ F(f) \circ \phi_{M,M^{\otimes(n-1)}}\right)\right) \circ \phi^{(n-1)} \\ &= \left(\phi^{(n-1)}\right)^{-1} \circ F\left(\underline{\mathrm{Tr}}^{l,1}(f)\right) \circ \phi^{(n-1)}.\end{aligned}$$

The same arguments for $f_1 := \underline{\mathrm{Tr}}^{l,1}(f)$ and $g_1 := \underline{\mathrm{Tr}}^{l,1}(g)$ provide the equation

$$\underline{\mathrm{Tr}}^{l,1}\left(\underline{\mathrm{Tr}}^{l,1}(g)\right) = \left(\phi^{(n-2)}\right)^{-1} \circ F\left(\underline{\mathrm{Tr}}^{l,1}\left(\underline{\mathrm{Tr}}^{l,1}(f)\right)\right) \circ \phi^{(n-2)}.$$

By repeating the same arguments, the equation

$$\Big(\overbrace{\underline{\mathrm{Tr}}^{l,1} \circ \cdots \circ \underline{\mathrm{Tr}}^{l,1}}^{n-1}\Big)(g) = F\Big(\big(\overbrace{\underline{\mathrm{Tr}}^{l,1} \circ \cdots \circ \underline{\mathrm{Tr}}^{l,1}}^{n-1}\big)(f)\Big) \tag{4.6}$$

is obtained. Setting $f_{n-1} := \Big(\overbrace{\underline{\mathrm{Tr}}^{l,1} \circ \cdots \circ \underline{\mathrm{Tr}}^{l,1}}^{n-1}\Big)(f)$ and applying $\underline{\mathrm{Tr}}^{l,1}$ to the Eq. 4.6, we have the desired equation

$$\underline{\widetilde{\mathrm{Tr}}}^{l}(g) = \underline{\mathrm{Tr}}^{l,1}\Big(F\big(f_{n-1}\big)\Big) = \omega^{-1} \circ F\left(\underline{\mathrm{Tr}}^{l,1}(f_{n-1})\right) \circ \omega = \omega^{-1} \circ F\left(\underline{\widetilde{\mathrm{Tr}}}^{l}(f)\right) \circ \omega.$$

As in a similar way, the Eq. 4.5 can be shown by using $\phi^{(k)} = \phi_{M^{\otimes(k-1)},M} \circ (\phi^{(k-1)} \otimes \mathrm{id}_{F(M)})$. □

As the same manner of the proof of the above proposition with help from Lemma 4.1 we have:

Proposition 4.4 *Let $\mathcal{B}$ be a left rigid braided monoidal category. Let M be an object in $\mathcal{B}$, and k be a positive integer, and define the isomorphism $j^{(k)} : (M^*)^{\otimes k} \longrightarrow \left(M^{\otimes k}\right)^*$ in $\mathcal{B}$ by*

$$j^{(1)} := id_{M^*}, \quad j^{(k)} := j_{M^{\otimes(k-1)},M} \circ \left(id_{M^*} \otimes j^{(k-1)}\right) \qquad (k \geq 2).$$

Then for an endomorphism f on $M^{\otimes n}$ in $\mathcal{B}$, the following equation holds

$$\underline{\widetilde{\mathrm{Tr}}}^{r}_{\mathcal{B}}\left((j^{(n)})^{-1}\circ {}^{t}f\circ j^{(n)}\right)=\underline{\widetilde{\mathrm{Tr}}}^{l}_{\mathcal{B}^{mir}}(f). \tag{4.7}$$

4.2 Construction of Monoidal Morita Invariants

In this subsection we introduce a family of monoidal Morita invariants of a finite-dimensional Hopf algebra by using partial braided traces.

Let $\mathcal{B}$ be a left rigid braided monoidal category with braiding c, and M be an object in $\mathcal{B}$. Then there is a representation $\rho_M : B_n \longrightarrow \mathrm{Aut}(M^{\otimes n})$ of the n-strand braid group B_n such that each positive crossing and negative crossing correspond to $c_{M,M}$ and $c^{-1}_{M,M}$, respectively [31]. For each $\boldsymbol{b}\in B_n$ we set

$$\underline{\boldsymbol{b}\text{-}\dim}^{l}_{\mathcal{B}}(M):=\underline{\widetilde{\mathrm{Tr}}}^{l}_{\mathcal{B}}\big(\rho_M(\boldsymbol{b})\big),\quad \underline{\boldsymbol{b}\text{-}\dim}^{r}_{\mathcal{B}}(M):=\underline{\widetilde{\mathrm{Tr}}}^{r}_{\mathcal{B}}\big(\rho_M(\boldsymbol{b})\big).$$

For simplicity we write $\underline{\boldsymbol{b}\text{-}\dim}$ instead of $\underline{\boldsymbol{b}\text{-}\dim}_{\mathcal{B}}$. If $\boldsymbol{b}$ is the identity element $\mathbf{1}\in B_1$, then

$$\underline{\mathbf{1}\text{-}\dim}^{r}(M)=(\text{the categorical dimension of } M) \tag{4.8}$$

in the sense of [21, Subsection 9.3].

Lemma 4.5 *Let M and N be two objects in $\mathcal{B}$.*

(1) *If M and N are isomorphic, then $\underline{\boldsymbol{b}\text{-}\dim}^{l}(M)=\underline{\boldsymbol{b}\text{-}\dim}^{l}(N)$, $\underline{\boldsymbol{b}\text{-}\dim}^{r}(M)=\underline{\boldsymbol{b}\text{-}\dim}^{r}(N)$.*

(1) $\underline{\boldsymbol{b}\text{-}\dim}^{r}_{\mathcal{B}}(M^*)=\underline{\boldsymbol{b}\text{-}\dim}^{l}_{\mathcal{B}^{mir}}(M)$.

Proof (1) Let $\varphi : M \longrightarrow N$ be an isomorphism. The map $\varphi^{\otimes n} : M^{\otimes n} \longrightarrow N^{\otimes n}$ is also an isomorphism. Let $\rho_M : B_n \longrightarrow \mathrm{Aut}(M^{\otimes n})$ and $\rho_N : B_n \longrightarrow \mathrm{Aut}(N^{\otimes n})$ be the representations induced from the braiding c. Since $c_{N,N}\circ(\varphi\otimes\varphi)=(\varphi\otimes\varphi)\circ c_{M,M}$ from naturality of c, the endomorphisms $f:=\rho_M(\boldsymbol{b})$ and $g:=\rho_N(\boldsymbol{b})$ satisfy $g\circ\varphi^{\otimes n}=\varphi^{\otimes n}\circ f$. Thus, g is expressed as $g=(\varphi^{\otimes n})\circ f\circ(\varphi^{\otimes n})^{-1}$, and it follows from Proposition 4.3 that $\underline{\boldsymbol{b}\text{-}\dim}^{l}(N)=\underline{\widetilde{\mathrm{Tr}}}^{l}(g)=\underline{\widetilde{\mathrm{Tr}}}^{l}(f)=\underline{\boldsymbol{b}\text{-}\dim}^{l}(M)$. The equation $\underline{\boldsymbol{b}\text{-}\dim}^{r}(M)=\underline{\boldsymbol{b}\text{-}\dim}^{r}(N)$ is also shown by the same argument.

(2) By the definition of the natural isomorphism $j_{M,N} : N^*\otimes M^* \longrightarrow (M\otimes N)^*$, it is easy to see that $j_{M,N}\circ c_{M^*,N^*}={}^{t}(c_{M,N})\circ j_{N,M}$. It follows that the representation $\rho_{M^*} : B_n \longrightarrow \mathrm{Aut}((M^*)^{\otimes n})$ induced from the braiding c satisfies $j^{(n)}\circ\rho_{M^*}(\boldsymbol{b})={}^{t}\big(\rho_M(\boldsymbol{b})\big)\circ j^{(n)}$, where $j^{(n)}$ is the isomorphism defined in Proposition 4.4. Thus we have $\underline{\boldsymbol{b}\text{-}\dim}^{r}_{\mathcal{B}}(M^*)=\underline{\widetilde{\mathrm{Tr}}}^{r}_{\mathcal{B}}\left((j^{(n)})^{-1}\circ{}^{t}\big(\rho_M(\boldsymbol{b})\big)\circ j^{(n)}\right)=\underline{\widetilde{\mathrm{Tr}}}^{l}_{\mathcal{B}^{\mathrm{mir}}}\big(\rho_M(\boldsymbol{b})\big)=\underline{\boldsymbol{b}\text{-}\dim}^{l}_{\mathcal{B}^{\mathrm{mir}}}(M)$. □

In what follows, we only consider $\boldsymbol{k}$-linear (braided) monoidal categories such that $\mathrm{End}(\mathbb{I})\cong\boldsymbol{k}$. In this case the braided traces of endomorphisms can be regarded as elements in $\boldsymbol{k}$. By Proposition 4.3, if $(F,\phi,\omega) : \mathcal{B} \longrightarrow \mathcal{B}'$ is a $\boldsymbol{k}$-linear braided monoidal functor between left rigid braided monoidal categories, then for an endomorphism f on $M^{\otimes n}$ in $\mathcal{B}$,

$$\underline{\widetilde{\mathrm{Tr}}}^{l}\big((\phi^{(n)})^{-1}\circ F(f)\circ\phi^{(n)}\big)=\underline{\widetilde{\mathrm{Tr}}}^{l}(f),\quad \underline{\widetilde{\mathrm{Tr}}}^{r}\left((\phi^{(n)})^{-1}\circ F(f)\circ\phi^{(n)}\right)=\underline{\widetilde{\mathrm{Tr}}}^{r}(f) \tag{4.9}$$

as elements in $\boldsymbol{k}$.

Let $\mathcal{C}$ be a $\boldsymbol{k}$-linear monoidal category such that $\mathrm{End}(\mathbb{I}) \cong \boldsymbol{k}$. Suppose that it has a Schrödinger object $\mathbf{S} \in \mathcal{Z}(\mathcal{C})$, and there is a left dual of $\mathbf{S}$. Then, one can define $\underline{\boldsymbol{b}\text{-Sdim}}^l(\mathcal{C})$, $\underline{\boldsymbol{b}\text{-Sdim}}^r(\mathcal{C}) \in \boldsymbol{k}$ by

$$\underline{\boldsymbol{b}\text{-Sdim}}^l(\mathcal{C}) = \underline{\boldsymbol{b}\text{-dim}}^l_{\mathcal{Z}(\mathcal{C})}(\mathbf{S}), \qquad \underline{\boldsymbol{b}\text{-Sdim}}^r(\mathcal{C}) = \underline{\boldsymbol{b}\text{-dim}}^r_{\mathcal{Z}(\mathcal{C})}(\mathbf{S}). \tag{4.10}$$

Theorem 4.6 *$\underline{\boldsymbol{b}\text{-Sdim}}^l(\mathcal{C})$ and $\underline{\boldsymbol{b}\text{-Sdim}}^r(\mathcal{C})$ are invariant under $\boldsymbol{k}$-linear monoidal equivalences.*

Proof Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a $\boldsymbol{k}$-linear monoidal equivalence, and $\mathbf{S}_{\mathcal{C}}$ and $\mathbf{S}_{\mathcal{D}}$ are Schrödinger objects for $\mathcal{C}$ and $\mathcal{D}$, respectively. By Lemma 3.9, $\big(\mathcal{Z}(F)\big)(\mathbf{S}_{\mathcal{C}}) \cong \mathbf{S}_{\mathcal{D}}$ as objects in $\mathcal{Z}(\mathcal{D})$. It follows from Eq. 4.9 and Lemma 4.5 that $\underline{\boldsymbol{b}\text{-Sdim}}^l(\mathcal{C}) = \underline{\boldsymbol{b}\text{-dim}}^l_{\mathcal{Z}(\mathcal{C})}(\mathbf{S}_{\mathcal{C}}) = \underline{\boldsymbol{b}\text{-dim}}^l_{\mathcal{Z}(\mathcal{C})}(\mathbf{S}_{\mathcal{D}}) = \underline{\boldsymbol{b}\text{-Sdim}}^l(\mathcal{D})$. The right version can be proved by the same argument. □

Applying the above theorem to the module category over a finite-dimensional Hopf algebra, we have:

Corollary 4.7 *Let A and B be finite-dimensional Hopf algebras over $\boldsymbol{k}$. If ${}_A\mathbf{M}$ and ${}_B\mathbf{M}$ are equivalent as $\boldsymbol{k}$-linear monoidal categories, then $\underline{\boldsymbol{b}\text{-dim}}^l(\mathrm{Sch}_A) = \underline{\boldsymbol{b}\text{-dim}}^l(\mathrm{Sch}_B)$ for all $\boldsymbol{b} \in B_n$. The same statement holds for $\underline{\boldsymbol{b}\text{-dim}}^r$.*

By using Theorem 3.15 we see that the monoidal Morita invariants $\underline{\boldsymbol{b}\text{-dim}}^l$ and $\underline{\boldsymbol{b}\text{-dim}}^r$ of the co-Schrödinger modules Sch^A and Sch^{A^*} are computable from the monoidal Morita invariants of the Schrödinger modules Sch_{A^*} and Sch_A, respectively.

Proposition 4.8 *Let A be a finite-dimensional Hopf algebra over $\boldsymbol{k}$. For any $\boldsymbol{b} \in B_n$, the following equations hold.*

(1) $\underline{\boldsymbol{b}\text{-dim}}^l\left(\mathrm{Sch}^{A^*}\right) = \underline{\boldsymbol{b}\text{-dim}}^l(\mathrm{Sch}_A)$, $\underline{\boldsymbol{b}\text{-dim}}^r\left(\mathrm{Sch}^{A^*}\right) = \underline{\boldsymbol{b}\text{-dim}}^r(\mathrm{Sch}_A)$.

(2) $\underline{\boldsymbol{b}\text{-dim}}^l(\mathrm{Sch}_{A^*}) = \underline{\boldsymbol{b}\text{-dim}}^l\left(\mathrm{Sch}^A\right)$, $\underline{\boldsymbol{b}\text{-dim}}^r(\mathrm{Sch}_{A^*}) = \underline{\boldsymbol{b}\text{-dim}}^r\left(\mathrm{Sch}^A\right)$.

Proof (1) Let $\phi^\sharp_A : \left({}_{D(A^*)}\mathbf{M}\right)^{\mathrm{rev}} \longrightarrow {}_{D(A)}\mathbf{M}$ be the equivalence of braided monoidal categories defined in Theorem 3.15. Setting $M := \mathrm{Sch}^{A^*}$, we have $\underline{\boldsymbol{b}\text{-dim}}^l_{\mathcal{R}}\left(\phi^\sharp_A(M)\right) = \underline{\boldsymbol{b}\text{-dim}}^l_{\mathcal{R}'}(M)$ from the proof of Theorem 4.6. Since $\phi^\sharp_A(M)$ and Sch_A are isomorphic as left $D(A)$-modules by Theorem 3.10(4), it follows from Lemma 4.5(1) that $\underline{\boldsymbol{b}\text{-dim}}^l_{\mathcal{R}}\left(\phi^\sharp_A(M)\right) = \underline{\boldsymbol{b}\text{-dim}}^l_{\mathcal{R}}(\mathrm{Sch}_A)$. Thus, the first equation is obtained. Similarly, the rest equations of Parts (1) and (2) can be also proved. □

4.3 Examples

In this subsection we show several applications and examples of our invariants.

Given a quasitriangular Hopf algebra (A, R), we use the notation $\underline{\widetilde{\mathrm{Tr}}}_R$ for the modified braided trace $\underline{\widetilde{\mathrm{Tr}}}$ in the left rigid braided category $\left({}_A\mathbf{M}^{\mathrm{fd}}, c^R\right)$, where ${}_A\mathbf{M}^{\mathrm{fd}}$ is the full subcategory of ${}_A\mathbf{M}$ whose objects are finite-dimensional. For computation the following example is useful.

Example 4.9 Let A be a quasitriangular Hopf algebra with universal R-matrix $R = \sum_j \alpha_j \otimes \beta_j$, and let $u = \sum_j S(\beta_j)\alpha_j$ be the Drinfel'd element of it. As is well-known,

$$\sum_j S(\alpha_j) \otimes \beta_j = R^{-1} = \sum_j \alpha_j \otimes S^{-1}(\beta_j), \quad (S \otimes S)(R) = R, \quad u^{-1} = \sum_j \beta_j S^2(\alpha_j), \tag{4.11}$$

and $S^2(a) = uau^{-1}$ for all $a \in A$ [5, 28].

Let M be a finite-dimensional left A-module. For any $a \in A$ the action of a on M is denoted by $\underline{a}_M$. Then for any A-module endomorphism f on $M^{\otimes n}$ the following formulas hold:

$$\widetilde{\underline{\mathrm{Tr}}}^l_R(f) = \mathrm{Tr}\left(\left(\underline{u}_M^{-1} \otimes \cdots \otimes \underline{u}_M^{-1}\right) \circ f\right), \tag{4.12}$$

$$\widetilde{\underline{\mathrm{Tr}}}^r_R(f) = \mathrm{Tr}\left(\left(\underline{u}_M \otimes \cdots \otimes \underline{u}_M\right) \circ f\right), \tag{4.13}$$

where Tr in the right-hand side stands for the usual trace on linear transformations.

Proof Here, we only prove the first equation since the second equation can be proved by the same argument. The Eq. 4.12 can be shown by induction on n as follows.

Let $\{e_i\}_{i=1}^d$ be a basis for M. For any $a \in A$, $a \cdot e_i$ is expressed as $a \cdot e_i = \sum_{i'=1}^d M_{i',i}(a)\, e_{i'}$ for some $M_{i',i}(a) \in \boldsymbol{k}$. Then $\widetilde{\underline{\mathrm{Tr}}}^l(f) = \underline{\mathrm{Tr}}^{l,1}(f) = \sum M_{i',i}(\beta_j) M_{k,i'}(f) M_{i,k}\left(S^2(\alpha_j)\right) = \sum M_{i',i'}\left(\underline{u}_M^{-1} \circ f\right) = \mathrm{Tr}\left(\underline{u}_M^{-1} \circ f\right)$.

Next, assume that the equation $\widetilde{\underline{\mathrm{Tr}}}^l(g) = \mathrm{Tr}\big((\underline{u}_M^{-1})^{\otimes(n-1)} \circ g\big)$ holds for any A-module endomorphism g on $M^{\otimes(n-1)}$. Let f be an A-module endomorphism on $M^{\otimes n}$. Then $g := \underline{\mathrm{Tr}}^{l,1}(f)$ is an A-module endomorphism on $M^{\otimes(n-1)}$. Applying the induction hypothesis, we have $\widetilde{\underline{\mathrm{Tr}}}^l(f) = \widetilde{\underline{\mathrm{Tr}}}^l(g) = \mathrm{Tr}\big((\underline{u}_M^{-1})^{\otimes(n-1)} \circ g\big) = \mathrm{Tr}\big((\underline{u}_M^{-1})^{\otimes n} \circ f\big)$. □

Let A be a finite-dimensional Hopf algebra. In view of Example 4.9, it is important to know the action of the Drinfel'd element $u \in D(A)$ on a given $D(A)$-module M to compute the braided dimension of M. Below we give formulas for the actions of u and $S(u)$ on the Schrödinger module Sch_A.

Recall that a *left integral* in A is an element $\Lambda \in A$ such that $a\Lambda = \varepsilon(a)\Lambda$ for all $a \in A$. A *right integral* in A is a left integral in A^{op}. It is known that a non-zero left integral $\Lambda \in A$ always exists (under our assumption that A is finite-dimensional), and is unique up to a scalar multiple. Hence one can define $\alpha \in A^*$ by $\Lambda a = \langle \alpha, a \rangle \Lambda$ for $a \in A$. The map α is in fact an algebra map, and does not depend on the choice of Λ. We call α the *distinguished grouplike element* of A^*. The Hopf algebra A is said to be *unimodular* if the distinguished grouplike element $\alpha \in A^*$ is the counit of A, or, equivalently, $\Lambda \in A$ is central.

Lemma 4.10 *With the above notations, we have*

$$u \bullet a = S^2(a_{(1)})\big\langle \alpha^{-1}, a_{(2)} \big\rangle \quad \textit{and} \quad S(u) \bullet a = S^{-2}(a)$$

for all $a \in \mathrm{Sch}_A$, where $\alpha^{-1} = \alpha \circ S$.

Proof Let $\{e_i\}$ be a basis of A, and let $\{e_i^*\}$ be the dual basis. Recall that the universal R-matrix of $D(A)$ is given by $\mathcal{R} = \sum_i \alpha_i \otimes \beta_i$, where $\alpha_i = \varepsilon \bowtie e_i$ and $\beta_i = e_i^* \bowtie 1$. We first compute the action of $S(u)$. By Eq. 4.11, we have

$$S(u) = \sum_i S(S(\beta_i)\alpha_i) = \sum_i S(\alpha_i) S^2(\beta_i) = \sum_i \alpha_i S(\beta_i).$$

Hence, for all $a \in \mathrm{Sch}_A$, we have

$$\begin{aligned} S(u) \bullet a &= \sum_i (\varepsilon \bowtie e_i) \bullet S_{D(A)}(e_i^* \bowtie 1) \bullet a \\ &= \sum_i e_i \blacktriangleright a_{(2)} \left\langle e_i^*, S^{-2}(a_{(1)}) \right\rangle \\ &= S^{-2}\left(a_{(1)}\right) \blacktriangleright a_{(2)} \qquad \left(\text{by} \sum_i e_i \langle e_i^*, x\rangle = x\right) \\ &= S^{-2}\left(a_{(1)}\right) a_{(3)} S\left(S^{-2}(a_{(2)})\right) = S^{-2}(a). \end{aligned}$$

Next, we compute the action of u. Fix a non-zero right integral $\lambda \in A^*$, and define $g \in A$ to be the unique element such that $p\lambda = \langle p, g\rangle\lambda$ for all $p \in A^*$ (*i.e.*, the distinguished grouplike element of $(A^{\mathrm{cop}})^{**} = (A^{*\,\mathrm{op}})^*$ regarded as an element of A). Radford showed that $D(A)$ is unimodular and $\alpha \bowtie g$ is the distinguished grouplike element [29, Theorem 4(a) and Corollary 7]. Hence, by [28, Theorem 2], we have $u = S(u) \cdot (\alpha \bowtie g)$ in $D(A)$. Using this formula and Radford's formula of the fourth power of the antipode [27], we compute, for all $a \in \mathrm{Sch}_A$,

$$\begin{aligned} u \bullet a &= S(u) \bullet ((\alpha \bowtie g) \bullet a) \\ &= S^{-2}\Big(\langle \alpha^{-1}, g a_{(1)} g^{-1}\rangle g a_{(2)} g^{-1}\Big) = S^{-2}\Big(S^4(\alpha^{-1} \rightharpoonup a)\Big) = S^2\left(a_{(1)}\right)\left\langle \alpha^{-1}, a_{(2)}\right\rangle. \end{aligned}$$

□

Combining Example 4.9 and Lemma 4.10, we obtain the following proposition:

Proposition 4.11 *Let A be a finite-dimensional Hopf algebra over* $\mathbf{k}$. *If A is involutory (i.e. the square of the antipode is the identity) and unimodular, then we have*

$$\underline{\boldsymbol{b}\text{-}\dim}^l(\mathrm{Sch}_A) = \underline{\boldsymbol{b}\text{-}\dim}^r(\mathrm{Sch}_A) = \mathrm{Tr}(\rho(\boldsymbol{b}))$$

for all $\boldsymbol{b} \in B_n$, *where* $\rho : B_n \longrightarrow Aut((\mathrm{Sch}_A)^{\otimes n})$ *is the braid group action.*

We denote by $\sigma_i \in B_n$ $(i = 1, \ldots, n-1)$ the braid of n strands with only one positive crossing between the i-th and the $(i+1)$-st strands. For integers p and q with $p \geq 2$, the braid

$$\boldsymbol{t}_{p,q} := (\sigma_1 \sigma_2 \cdots \sigma_{p-1})^q \in B_p$$

is called the (p, q)-torus braid, as its closure is the (p, q)-torus link. The below is an example of the computation of the braided dimension associated with $\boldsymbol{b} = \boldsymbol{t}_{2,q}$.

Lemma 4.12 *Let* (A, R) *be a quasitriangular Hopf algebra over* $\mathbf{k}$, *and u be the Drinfel'd element of it. For each non-negative integer m and finite-dimensional left A-module X,*

$$\underline{\boldsymbol{t}_{2,q}\text{-}\dim}^l X = \begin{cases} Tr\big(\underline{u^{m-1}(u^{-m})_{(1)}}_X\big) Tr\left(\underline{u^{m-1}(u^{-m})_{(2)}}_X\right) & \text{if } q = 2m, \\ Tr\big(\underline{\big(u^{m-1} \otimes u^{m-1}\big)\Delta(u^{-m})R_{21}}_{X\otimes X} \circ T_{X,X}\big) & \text{if } q = 2m+1, \end{cases}$$

$$\underline{\boldsymbol{t}_{2,q}\text{-}\dim}^r X = \begin{cases} Tr\big(\underline{u^{m+1}(u^{-m})_{(1)}}_X\big) Tr\left(\underline{u^{m+1}(u^{-m})_{(2)}}_X\right) & \text{if } q = 2m, \\ Tr\big(\underline{(u^{m+1} \otimes u^{m+1})\Delta(u^{-m})R_{21}}_{X\otimes X} \circ T_{X,X}\big) & \text{if } q = 2m+1. \end{cases}$$

Here, for elements $a, b \in A$ the notation $\underline{a \otimes b}_{X \otimes X}$ stands for the left action on $X \otimes X$ defined by $x \otimes y \longmapsto (a \cdot x) \otimes (b \cdot y)$ for all $x, y \in X$, and $\underline{\boldsymbol{t}_{2,q}\text{-}\dim^l} X$, $\underline{\boldsymbol{t}_{2,q}\text{-}\dim^r} X$ are the braided dimensions in the case of $\mathcal{B} = ({}_A\mathbf{M}, c^R)$.

Proof The formula for $\underline{\boldsymbol{t}_{2,q}\text{-}\dim^l} X$ can be obtained as follows. Let $\{e_s\}_{s=1}^d$ be a basis for X, and $\{e_s^*\}_{s=1}^d$ be its dual basis. Let $R^{(q)}$ be the element in $A \otimes A$ defined by

$$R^{(q)} = \begin{cases} (R_{21}R)^m & \text{if } q = 2m, \\ (R_{21}R)^m R_{21} & \text{if } q = 2m+1. \end{cases}$$

By Example 4.9, we see that

$$\underline{\boldsymbol{t}_{2,q}\text{-}\dim^l} X = \begin{cases} \mathrm{Tr}\left(\underline{(u^{-1} \otimes u^{-1})R^{(q)}}_{X \otimes X}\right) & \text{if } q \text{ is even}, \\ \mathrm{Tr}\left(\underline{(u^{-1} \otimes u^{-1})R^{(q)}}_{X \otimes X} \circ T_{X,X}\right) & \text{if } q \text{ is odd}. \end{cases} \tag{4.14}$$

Since $R_{21}R = \Delta(u^{-1})(u \otimes u) = (u \otimes u)\Delta(u^{-1})$ [5], it follows that $(R_{21}R)^m = (u^m \otimes u^m)\Delta(u^{-m})$. Substituting this equation to Eq. 4.14 we obtain the formula for $\underline{\boldsymbol{t}_{2,q}\text{-}\dim^l} X$ in the lemma. By a similar consideration, the formula for $\underline{\boldsymbol{t}_{2,q}\text{-}\dim^r} X$ can be obtained. □

In the case where A is semisimple, the braided dimension of the Schrödinger module associated with $\boldsymbol{t}_{2,2}$ has the following representation-theoretic meaning:

Theorem 4.13 *Suppose that $\boldsymbol{k}$ is an algebraically closed field of characteristic zero. If A is a finite-dimensional semisimple Hopf algebra over $\boldsymbol{k}$, then*

$$\underline{\boldsymbol{t}_{2,2}\text{-}\dim^l}(\mathrm{Sch}_A) = \underline{\boldsymbol{t}_{2,2}\text{-}\dim^r}(\mathrm{Sch}_A) = \dim(A)\,\sharp Irr(A),$$

where $\sharp Irr(A)$ is the number of isomorphism classes of irreducible A-modules.

Proof It is sufficient to show $\underline{\boldsymbol{t}_{2,2}\text{-}\dim^l}(\mathrm{Sch}_A) = \dim(A)\,\sharp\mathrm{Irr}(A)$ in view of Proposition 4.11. By the assumption, Radford's induction functor $I_A : {}_A\mathbf{M} \longrightarrow {}_{D(A)}\mathbf{M}$ is isomorphic to the functor $D(A) \otimes_A (-)$ by [13, Lemma 2.3]. Combining this fact with Proposition 3.6, we have

$$\mathrm{Sch}_A \otimes \mathrm{Sch}_A \cong I_A(\mathrm{Adj}_A) \cong D(A) \otimes_A \mathrm{Adj}_A.$$

Hence, by Lemma 4.12,

$$\begin{aligned} \underline{\boldsymbol{t}_{2,2}\text{-}\dim^l}(\mathrm{Sch}_A) &= \mathrm{Tr}\left(\underline{(u^{-1})_{(1)}}_{\mathrm{Sch}_A}\right) \mathrm{Tr}\left(\underline{(u^{-1})_{(2)}}_{\mathrm{Sch}_A}\right) \\ &= \mathrm{Tr}\left(\underline{u^{-1}}_{\mathrm{Sch}_A \otimes \mathrm{Sch}_A}\right) = \mathrm{Tr}\left(\underline{u^{-1}}_{D(A) \otimes_A \mathrm{Adj}_A}\right). \end{aligned} \tag{4.15}$$

We use some results on the Frobenius-Schur indicator [19]. Let V be a finite-dimensional left A-module. The "third formula" [16, Section 6.4] of the n-th Frobenius-Schur indicator $\nu_n(V)$ $(n = 1, 2, \ldots)$ expresses $\nu_n(V)$ by using the Drinfel'd element, as

$$\nu_n(V) = \frac{1}{\dim(A)} \mathrm{Tr}\left(\underline{u^n}_{D(A) \otimes_A V}\right).$$

Since u is of finite order [6], $\dim(A)\,\nu_n(V) \in \mathbb{Z}[\xi]$ $(\subset \boldsymbol{k})$, where $\xi \in \boldsymbol{k}$ is a root of unity of the same order as u. Hence, if we denote by $z \mapsto \overline{z}$ the ring automorphism of $\mathbb{Z}[\xi]$ defined by $\xi \mapsto \xi^{-1}$, then we have

$$\mathrm{Tr}\left(\underline{u^{-n}}_{D(A) \otimes_A V}\right) = \overline{\dim(A)\,\nu_n(V)}.$$

On the other hand, the "first formula" [16, Section 2.3] yields $\nu_1(V) = \dim\big(\mathrm{Hom}_A(\boldsymbol{k}, V)\big)$. Considering the case where V is the adjoint representation Adj_A, we obtain

$$\mathrm{Tr}\left(\underline{u^{-1}}_{D(A)\otimes_A \mathrm{Adj}_A}\right) = \overline{\dim(A)\cdot \nu_1(\mathrm{Adj}_A)} = \overline{\dim(A)\cdot \dim\big(\mathrm{Hom}(\boldsymbol{k}, \mathrm{Adj}_A)\big)} = \dim(A)\,\sharp\mathrm{Irr}(A).$$

Now the result follows Eq. 4.15. □

As this theorem suggests, the Schrödinger module Sch_A has much information about the category of A-modules, at least, in the semisimple case. However, the computation of the braided dimension is not easy in general. Fortunately, if A is a group algebra, then the braided dimension of Sch_A closely relates to the link group of the closure of the braid, and can be computed in the following way:

Theorem 4.14 *Let $\boldsymbol{b} \in B_n$. If $A = \boldsymbol{k}[G]$ is the group algebra of a finite group G, then*

$$\underline{\boldsymbol{b}\text{-}\dim}^l(\mathrm{Sch}_A) = \underline{\boldsymbol{b}\text{-}\dim}^r(\mathrm{Sch}_A) = \sharp Hom(\pi_1(\mathbb{R}^3 \setminus \widehat{\boldsymbol{b}}), G)$$

in $\boldsymbol{k}$, where $\widehat{\boldsymbol{b}}$ is the link obtained by closing the braid $\boldsymbol{b}$, and π_1 means the fundamental group.

Proof Set $X = \mathrm{Sch}_A$ for simplicity. Then the braiding $c_{X,X}$ is given by

$$c_{X,X}(g\otimes h) = h \otimes \left(h^{-1} \blacktriangleright g\right) = h\otimes h^{-1}gh \quad (g, h \in G).$$

Let B_n act on G^n by

$$\varrho(\sigma_i)(g_1, \dots, g_n) = \left(g_1, \dots, g_{i-1}, g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, g_{i+2}, \dots, g_n\right) \quad (g_1, \dots, g_n \in G).$$

By Proposition 4.11, $\underline{\boldsymbol{b}\text{-}\dim}^l(\mathrm{Sch}_A)$ and $\underline{\boldsymbol{b}\text{-}\dim}^r(\mathrm{Sch}_A)$ are equal to the number of fixed points of $\varrho(\boldsymbol{b})$ regarded as an element of $\boldsymbol{k}$. On the other hand, the number of fixed points of $\varrho(\boldsymbol{b})$ has been studied by Freyd and Yetter [10] in relation with link invariants arising from crossed G-sets. The claim of this theorem follows from [10, Proposition 4.2.5]. □

Example 4.15 We consider the case where $A = \boldsymbol{k}[G]$ is the group algebra of a finite group G. If $\boldsymbol{k}$ is an algebraically closed field of characteristic zero, then we obtain

$$\begin{aligned}\underline{\boldsymbol{t}_{2,2}\text{-}\dim}^l(\mathrm{Sch}_A) &= \underline{\boldsymbol{t}_{2,2}\text{-}\dim}^r(\mathrm{Sch}_A) = |G|\cdot\sharp\mathrm{Conj}(G),\\ \underline{\boldsymbol{t}_{2,2}\text{-}\dim}^l(\mathrm{Sch}_{A^*}) &= \underline{\boldsymbol{t}_{2,2}\text{-}\dim}^r(\mathrm{Sch}_{A^*}) = |G|^2\end{aligned} \tag{4.16}$$

by Theorem 4.13, where $\mathrm{Conj}(G)$ is the set of conjugacy classes of G. In particular,

$$\underline{\boldsymbol{t}_{2,2}\text{-}\dim}^l(\mathrm{Sch}_A) \neq \underline{\boldsymbol{t}_{2,2}\text{-}\dim}^l(\mathrm{Sch}_{A^*})$$

whenever G is non-abelian. This result is interesting from the viewpoint that some other monoidal Morita invariants, such as ones introduced in [6] and [32], are in fact invariants of the braided monoidal category of the representations of the Drinfel'd double.

In topology, the link $\widehat{\boldsymbol{t}}_{2,2}$ is known as the Hopf link. Since $\pi_1\left(\mathbb{R}^3\setminus\widehat{\boldsymbol{t}}_{2,2}\right)$ is the free abelian group of rank two, we have

$$\underline{\boldsymbol{t}_{2,2}\text{-}\dim}^l(\mathrm{Sch}_A) = \underline{\boldsymbol{t}_{2,2}\text{-}\dim}^r(\mathrm{Sch}_A) = \sharp\mathrm{Comm}(G) \tag{4.17}$$

by Theorem 4.14, where $\mathrm{Comm}(G) = \{(x, y) \in G\times G \mid xy = yx\}$. Comparing Eq. 4.16 with Eq. 4.17, we get $|G|\cdot\sharp\mathrm{Conj}(G) = \sharp\mathrm{Comm}(G)$. Although this formula itself is well-known in finite group theory, we expect that some non-trivial formulas for finite groups (or, more generally, for finite-dimensional semisimple Hopf algebras) would be obtained via the investigation of the braided dimension.

By Eq. 4.8 and [21, Example 9.3.8], we have $\underline{\mathbf{1}\text{-}\dim}^r(\mathrm{Sch}_A) = \mathrm{Tr}\left(S_A^2\right)$ (see [2] for the quasi-Hopf case). In particular, $\underline{\mathbf{1}\text{-}\dim}^r(\mathrm{Sch}_A) = 0$ whenever A is not cosemisimple by [18, Theorem 2.5(b)]. More strongly, we have the following theorem:

Theorem 4.16 *Let A be a finite-dimensional Hopf algebra. If A is not cosemisimple, then we have* $\underline{\boldsymbol{b}\text{-}\dim}^l(\mathrm{Sch}_A) = \underline{\boldsymbol{b}\text{-}\dim}^r(\mathrm{Sch}_A) = 0$ *for all braids* $\boldsymbol{b}$.

Proof Let, in general, X be a finite-dimensional Hopf algebra, let $\Lambda \in X \setminus \{0\}$ be a left integral, and let $\lambda \in X^*$ be the right integral such that $\langle \lambda, \Lambda \rangle = 1$. By [26, Proposition 2 (a)],

$$\mathrm{Tr}\Big(X \longrightarrow X;\ x \mapsto S^2\left(x_{(2)}\right)\langle p, x_{(1)}\rangle\Big) = \langle \lambda, 1\rangle\langle p, \Lambda\rangle$$

for all $p \in X^*$. By the Maschke theorem, the right-hand side is identically zero if X is not cosemisimple. Thus, applying the above formula to $X = A^{\mathrm{cop}}$ and $X = A^{\mathrm{op\ cop}}$, we have

$$\mathrm{Tr}\Big(A \longrightarrow A;\ a \mapsto S^{\pm 2}\left(a_{(1)}\right)\langle p, a_{(2)}\rangle\Big) = 0 \tag{4.18}$$

for all $p \in A^*$.

Now, let $\boldsymbol{b} \in B_n$ be a braid. By Eq. 4.6, $\underline{\boldsymbol{b}\text{-}\dim}^l(\mathrm{Sch}_A) = \underline{\mathrm{Tr}}^l\left(\widetilde{f}\right)$, where

$$\widetilde{f} = \Bigg(\overbrace{\underline{\mathrm{Tr}}^{l,1} \circ \cdots \circ \underline{\mathrm{Tr}}^{l,1}}^{n-1}\Bigg)(\rho(b)).$$

Let $f : \mathrm{Adj}_A \longrightarrow \boldsymbol{k}$ be the A-linear map corresponding to $\widetilde{f}$ under the isomorphism

$$\mathrm{End}_{D(A)}(\mathrm{Sch}_A) \xrightarrow{\cong} \mathrm{Hom}_{D(A)}(\mathrm{Sch}_A, I_A(\boldsymbol{k})) \xrightarrow{\cong} \mathrm{Hom}_A(\mathrm{Adj}_A, \boldsymbol{k})$$

given by Eq. 3.10 and Proposition 3.4. Then we have $\widetilde{f}(a) = \langle f, a_{(2)}\rangle\, a_{(1)}$ for all $a \in \mathrm{Sch}_A$, and therefore $\underline{\boldsymbol{b}\text{-}\dim}^l(\mathrm{Sch}_A)$ is equal to the trace of the linear map

$$\mathrm{Sch}_A \longrightarrow \mathrm{Sch}_A; \quad a \mapsto u^{-1} \bullet \widetilde{f}(a) = S^{-2}\left(a_{(1)}\right)\langle \alpha, a_{(2)}\rangle\langle f, a_{(3)}\rangle \quad (a \in \mathrm{Sch}_A)$$

by Lemma 4.10. Hence, $\underline{\boldsymbol{b}\text{-}\dim}^l(\mathrm{Sch}_A) = 0$ by Eq. 4.18. The equation $\underline{\boldsymbol{b}\text{-}\dim}^r(\mathrm{Sch}_A) = 0$ is proved in a similar way. □

By this theorem, we could say that the braided dimension of the Schrödinger module is not interesting as a monoidal Morita invariant for non-cosemisimple Hopf algebras. However, the endomorphism of Sch_A induced by a braid, such as $\widetilde{f}$ in the above proof, is not generally zero, and thus may have some information about A. For example, let us consider the map

$$z_M := \underline{\mathrm{Tr}}^{r,1}(\rho_M(\sigma_1)) : M \longrightarrow M \tag{4.19}$$

for finite-dimensional $M \in {}_{D(A)}\mathbf{M}$, where $\rho_M : B_2 \longrightarrow \mathrm{Aut}\left(M^{\otimes 2}\right)$ is the action of B_2. One can check that z_M is given by the action of $z := uS(u)$ on M. Hence, if $M = \mathrm{Sch}_A$, then

$$z_M(a) = z \bullet a = a_{(1)}\langle \alpha^{-1}, a_{(2)}\rangle \quad (a \in \mathrm{Sch}_A) \tag{4.20}$$

by Lemma 4.10, where $\alpha \in A^*$ is the distinguished grouplike element. Therefore this map has the following information: z_M for $M = \mathrm{Sch}_A$ is the identity if and only if A is unimodular.

Acknowledgments The authors would like to thank Professor Akira Masuoka for showing a direction to solve our problem and for continuous encouragement. They are also thank to the referees for helpful comments.

References

1. Abe, E.: Hopf algebras. Cambridge Univ. Press, Cambridge (1980). (original Japanese version published by Iwanami Shoten, Tokyo, 1977)
2. Bulacu, D., Torrecillas, B.: The representation-theoretic rank of double of quasi-Hopf algebras. J. Pure Appl. Algebra **212**, 919–940 (2008)
3. Burciu, S.: On some representations of the Drinfel'd double. J. Algebra **296**, 480–504 (2006)
4. Drinfel'd, V.G.: Quantum groups. In: Proceedings of the international congress of mathematics, Berkeley, CA, pp. 798–820 (1987)
5. Drinfel'd, V.G.: On almost cocommutative Hopf algebras. Leningrad Math. J. **1**, 321–342 (1990)
6. Etingof, P., Gelaki, S.: On the exponent of finite-dimensional Hopf algebras. Math. Res. Lett. **6**, 131–140 (1999)
7. Etingof, P., Gelaki, S.: Isocategorical groups, international mathematical research notices no. 2 (2001)
8. Etingof, P., Gelaki, S.: On the quasi-exponent of finite-dimensional Hopf algebras. Math. Res. Lett. **9**, 277–287 (2002)
9. Fang, X.: Quantum groups, q-Boson algebras and quantized Weyl algebras. Int. J. Math. **22**, 675–694 (2011)
10. Freyd, P., Yetter, D.N.: Braided compact closed categories with applications to low dimensional topology. Adv. Math. **77**, 156–182 (1989)
11. Freyd, P., Yetter, D.N.: Coherence theorems via knot theory. J. Pure Appl. Algebra **78**, 49–76 (1992)
12. Hu, J., Zhang, Y.: The β-character algebra and a commuting pair in Hopf algebras. Algebra Represent. Theory **10**, 497–516 (2007)
13. Hu, J., Zhang, Y.: Induced modules of semisimple Hopf algebras. Algebra Colloquium **4**, 571–584 (2007)
14. Joyal, A., Street, R.: Braided tensor categories. Adv. Math. **102**, 20–78 (1993)
15. Kashina, Y., Montgomery, S., Ng, S.-H.: On the trace of the antipode and higher indicators. Isr. J. Math. **188**, 57–89 (2012)
16. Kashina, Y., Sommerhäuser, Y., Zhu, Y.: On higher Frobenius-Schur indicators, Mem. Amer. Math. Soc., vol. 181. American Mathematical Society (2006)
17. Kassel, C.: Quantum groups, Graduate Texsts in Math, vol. 155. Springer, New York (1995)
18. Larson, R.G., Radford, D.E.: Finite dimensional cosemisimple Hopf algebras in characteristic 0 are semisimple. J. Algebra **117**, 267–289 (1988)
19. Linchenko, V., Montgomery, S.: A Frobenius-Schur theorem for Hopf algebras. Algebr. Represent. Theory **3**, 347–355 (2000)
20. MacLane, S.: Categories for the working mathematician, graduate texsts in math, vol. 5. Springer, New York (1971)
21. Majid, S.: Foundations of quantum group theory. Cambridge University Press (1995)
22. Masuoka, A.: Construction of quantized enveloping algebras by cocycle deformation. Arab. J. Sci. Eng.–Theme Issues **33**(2C), 387–406 (2008)
23. Montgomery, S.: Hopf algebras and their action on rings, C.B.M.S, vol. 82. American Mathematical Society (1993)
24. Ng, S.-H., Schauenburg, P.: Higher Frobenius-Schur indicators for pivotal categories, Contemp. Math. vol. 441. Amer. Math. Soc., Providence, RI (2007)
25. Ng, S.-H., Schauenburg, P.: Frobenius-Schur indicators and exponents of spherical categories. Adv. Math. **211**, 34–71 (2007)
26. Radford, D.E.: The trace function and Hopf algebras. J. Algebra **163**, 583–622 (1994)
27. Radford, D.E.: The order of the antipode of a finite dimensional Hopf algebra is finite. Amer. J. Math. **98**, 333–355 (1976)
28. Radford, D.E.: On the antipode of a quasitriangular Hopf algebra. J. Algebra **151**, 1–11 (1992)
29. Radford, D.E.: Minimal quasitriangular Hopf algebras. J. Algebra **157**, 285–315 (1993)
30. Radford, D.E.: On oriented quantum algebras derived from representations of the quantum double of a finite-dimensional Hopf algebras. J. Algebra **270**, 670–695 (2003)
31. Reshetikhin, N.Y., Turaev, V.G.: Ribbon graphs and their invariants derived from quantum groups. Comm. Math. Phys. **127**, 1–26 (1990)

32. Shimizu, K.: Monoidal Morita invariants for finite group algebras. J. Algebra **323**, 397–418 (2010)
33. Sweedler, M.E.: Hopf algebras. Benjamin, New York (1969)
34. Wakui, M.: Polynomial invariants for a semisimple and cosemisimple Hopf algebra of finite dimension. J. Pure Appl. Algebra **214**, 701–728 (2010)