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Modules Over Endomorphism Rings

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Abstract For a finite dimensional K-algebra Λ over an algebraically closed field K and for a basic Λ-module M, we study M with its natural structure as a module over the endomorphism ring $\mathrm{End}_\Lambda(M)$.

Keywords Modules · Endomorphism rings · Quiver representations · Finite dimensional K-algebras

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1 Introduction

Artin algebras have the property, which distinguishes them from artin rings, that endomorphism rings of finitely generated modules are again artin algebras. These examples of artin algebras are very important, since many algebras that are studied in representation theory are described as endomorphism rings of appropriate modules. For example, Auslander algebras, tilted algebras, cluster tilted algebras, among others.

Also, endomorphism rings are used in processes of induction on the number of pairwise non-isomorphic simple modules of the algebra. In fact, if Λ has l pairwise non-isomorphic simple modules and P is the sum of the projective covers of $l-1$ of them, then the endomorphism ring of P has $l-1$ pairwise non-isomorphic simple modules.

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In this paper, for an artin algebra Λ we denote by $\mathrm{mod}(\Lambda)$ the category of finitely generated left Λ-modules. Moreover, if M is a Λ-module then M is a $\Gamma = \mathrm{End}_\Lambda(M)$-module when we define $f \cdot m = f(m)$ for $m \in M$ and $f \in \Gamma$, and therefore M is a right Γ^{op}-module. The categories $\mathrm{mod}(\Lambda)$ and $\mathrm{mod}(\Gamma^{op})$ can be compared through the pair of adjoint functors, $F = \mathrm{Hom}_\Lambda(M, -) : \mathrm{mod}(\Lambda) \to \mathrm{mod}(\Gamma^{op})$ and $G = {}_\Lambda M_{\Gamma^{op}} \otimes_{\Gamma^{op}} - : \mathrm{mod}(\Gamma^{op}) \to \mathrm{mod}(\Lambda)$. These functors induce equivalences between appropriate subcategories of $\mathrm{mod}(\Lambda)$ and $\mathrm{mod}(\Gamma^{op})$. For example, $\mathrm{add}(M)$ and $\mathrm{proj}(\Gamma^{op})$, where $\mathrm{add}(M)$ is the full subcategory of $\mathrm{mod}(\Lambda)$ consisting of the direct summands of direct sums of copies of M, and $\mathrm{proj}(\Gamma^{op})$ is the full subcategory of $\mathrm{mod}(\Gamma^{op})$ consisting of the finitely generated projective Γ^{op}-modules. If M is a tilting module then F and G induce inverse equivalences between $\tau({}_\Lambda M)$ and $D\tau(M_{\Gamma^{op}})$, where $\tau({}_\Lambda M)$ is the torsion class induced by M. In [1] M. Auslander considered the full subcategory C_1^M of $\mathrm{mod}(\Lambda)$ consisting of the modules X having a presentation $M_1 \to M_0 \to X \to 0$ with $M_i \in \mathrm{add}(M)$, and such that the induced sequence $F(M_1) \to F(M_0) \to F(X) \to 0$ is exact in $\mathrm{mod}(\Gamma^{op})$. Then F induces an equivalence between C_1^M and the image of the restriction of the functor F to C_1^M.

Thus, for a Λ-module M it is interesting to study and describe the Γ-module M, and answer some elementary questions, at least in some cases. For example, how to describe the composition factors of ${}_\Gamma M$, or the indecomposable summands of ${}_\Gamma M$. We will assume that Λ is given as a factor of the path algebra of a quiver Q modulo an admissible ideal I. This is, $\Lambda \simeq KQ/I$.

If M is a basic module in $\mathrm{mod}(\Lambda)$, we begin by explaining how to get the ordinary quiver of $\Gamma = \mathrm{End}_\Lambda(M)$ and its relations, from the knowledge of the Auslander-Reiten quiver of Λ. Then, with this data, and given the representation associated to the Λ-module M, we obtain the representation associated to M as a module over Γ (Theorem 1).

We describe a family of l summands M_k' of M in $\mathrm{mod}(\Gamma)$, where l is the number of pairwise non-isomorphic simple modules of Λ. We study conditions for these summands to be all non-zero and for them to be indecomposable and pairwise non-isomorphic modules. For example, we prove that M has the first property if and only if it is sincere, and it has the second one provided ${}_\Lambda M_{\Gamma^{op}}$ is a faithfully balanced bimodule. Thus, when M is a basic tilting module, we prove that the M_k' are precisely the indecomposable summands of ${}_\Gamma M$. Also we describe the summands of ${}_\Gamma M$ in the case in that M is a $\star$-module (in the sense defined by Colpi in [2]), and when M is a generator or a cogenerator of $\mathrm{mod}(\Lambda)$. In particular, if M is simultaneously a generator and a cogenerator of $\mathrm{mod}(\Lambda)$ then the summands M_k' of ${}_\Gamma M$ are a complete set of pairwise non-isomorphic indecomposable projective injective modules of $\mathrm{mod}(\Gamma)$.

Finally, in the last section of this work we consider the functors $\overline{F} = \mathrm{Hom}_\Lambda(-, M) : \mathrm{mod}(\Lambda) \to \mathrm{mod}(\Gamma)$ and $\overline{G} = \mathrm{Hom}_\Gamma(-, M) : \mathrm{mod}(\Gamma) \to \mathrm{mod}(\Lambda)$, and we describe the representations associated to $F(X)$ and $\overline{F}(X)$, for a Λ-module X, and the representations associated to $G(Y)$ and $\overline{G}(Y)$, for a Γ-module Y.

Throughout this paper algebra means *finite-dimensional K-algebra*, where K is an algebraically closed field. When Λ is an algebra the term 'Λ-module' will mean *finitely generated left Λ-module*. The full subcategory of finitely generated projective Λ-modules is denoted by $\mathrm{proj}(\Lambda)$.

2 The Γ-module M

Let $M = \bigoplus_{i=1}^n M_i$, where the M_i's are pairwise non-isomorphic indecomposable Λ-modules. We start by showing how to obtain the ordinary quiver of $\Gamma^{op} = \mathrm{End}_\Lambda(M)^{op}$ and its relations, from the knowledge of the Auslander-Reiten quiver of Λ (Proposition 1 and Remark 4). Let $\mathbf{i}$ be the vertex corresponding to the projective module $P_{\mathbf{i}} = \mathrm{Hom}_\Lambda(M, M_i)$. We will see that the arrows from the vertex $\mathbf{i}$ to the vertex $\mathbf{j}$ in the ordinary quiver of Γ^{op} are in bijective correspondence with a basis of 'the irreducible morphisms in add(M)' from M_j to M_i, in the sense that will be explained in Remark 2.

We start the section by using the well known fact that a map between indecomposable projective modules $h : P \to Q$ such that $\mathrm{Im}(h) \subseteq rQ$, satisfies that $\mathrm{Im}(h) \nsubseteq r^2Q$ if and only it is 'irreducible in add(Λ)'. We state this more precisely in the following lemma, where $\mathrm{rad}_\Lambda(X, Y)$ denotes the radical of $\mathrm{Hom}_\Lambda(X, Y)$ (see [3], Chapter V, Section 7).

Lemma 1 *Let $h : P \to rQ$ be a morphism of indecomposable projective Λ-modules. Then $\mathrm{Im}(h) \nsubseteq r^2Q$ if and only if the induced morphism $h : P \to Q$ satisfies $h \neq g \circ f$ for all $f \in \mathrm{rad}_\Lambda(P, Q')$, $g \in \mathrm{rad}_\Lambda(Q', Q)$ and Q' projective.*

Lemma 2 *Let Λ be a basic finite dimensional K-algebra and $1 = e_1 + e_2 + \ldots + e_l$ a decomposition of 1 into a sum of primitive orthogonal idempotents. Let $P_i = \Lambda e_i$ and $S_i = P_i/rP_i$ for $i = 1, \ldots, l$. Let Q_Λ be the ordinary quiver of Λ and let i be the vertex of Q_Λ corresponding to the simple Λ-module S_i. Then the following conditions are equivalent.*

(a) *There exist at least t arrows α from the vertex i to the vertex j in Q_Λ.*
(b) *There exist $\omega_1, \ldots, \omega_t$ in $\mathrm{Hom}_\Lambda(P_j, P_i)$ which are not isomorphisms, and such that the relation $\sum_{s=1}^t a_s\omega_s = g \circ f$ with $a_1, \ldots, a_t \in K$, $f \in \mathrm{rad}_\Lambda(P_j, Q)$, $g \in \mathrm{rad}_\Lambda(Q, P_i)$ and Q projective, implies $a_s = 0$ for $s = 1, \ldots, t$.*

Proof We know that the number of arrows α from the vertex i to the vertex j coincides with $\dim_K(\mathrm{Ext}^1_\Lambda(S_i, S_j))$, and this number is equal to $\dim_K(\mathrm{Hom}_\Lambda(P_j, rP_i/r^2P_i))$ (see [3], Chapter III, Proposition 1.14).

If $\omega : P_j \to P_i$ is not an isomorphism, then $\mathrm{Im}(\omega) \subseteq rP_i$. In this case we denote by $\overline{\omega}$ the morphism induced $P_j \to rP_i/r^2P_i$. The equivalence between (a) and (b) follows from the next remarks.

Let $\varphi_1, \ldots, \varphi_t \in \mathrm{Hom}_\Lambda(P_j, rP_i/r^2P_i)$ and $\omega_1, \ldots, \omega_t \in \mathrm{Hom}_\Lambda(P_j, P_i)$ such that $\overline{\omega}_s = \varphi_s$ for all $1 \leq s \leq t$. If $a_1, \ldots, a_t \in K$, then $a_1\varphi_1 + \ldots + a_t\varphi_t = 0$ in $\mathrm{Hom}_\Lambda(P_j, rP_i/r^2P_i)$, if and only if $\mathrm{Im}(a_1\omega_1 + \ldots + a_t\omega_t) \subseteq r^2P_i$. By Lemma 1 we know that this last condition is equivalent to saying that there exist morphisms $f \in \mathrm{rad}_\Lambda(P_j, Q)$, $g \in \mathrm{rad}_\Lambda(Q, P_i)$ with Q projective such that $a_1\omega_1 + \ldots + a_t\omega_t = g \circ f$. □

Remark 1 Let A and B be Λ-modules such that $B = \amalg_{j=1}^m B_j$, and $f = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} : A \to \amalg_{j=1}^m B_j$. We recall that $f \in \mathrm{rad}_\Lambda(A, B)$ if and only if $f_j \in \mathrm{rad}_\Lambda(A, B_j)$ for all j. Moreover, if A and B_j are indecomposable for all j then $f \in \mathrm{rad}_\Lambda(A, B)$ if and only if f_j is not an isomorphism for all j.

Proposition 1 *Let Λ be a basic finite dimensional K-algebra, l the number of pairwise non-isomorphic simple Λ-modules and $M = \bigoplus_{i=1}^n M_i$, where the M_i's are pairwise non-isomorphic indecomposable Λ-modules. Let $\Gamma^{op} = \mathrm{End}_\Lambda(M)^{op}$ and let $S_{\mathbf{i}}$ be the simple Γ^{op}-module corresponding to the projective module $P_{\mathbf{i}} = \mathrm{Hom}_\Lambda(M, M_i)$. Let $Q_{\Gamma^{op}}$ be the ordinary quiver of Γ^{op} and $\mathbf{i}$ the vertex of $Q_{\Gamma^{op}}$ corresponding to the simple Γ^{op}-module $S_{\mathbf{i}}$. The following conditions are equivalent.*

(a) *There exist t arrows α from the vertex* $\mathbf{i}$ *to the vertex* $\mathbf{j}$ *in* $Q_{\Gamma^{op}}$.
(b) *There exist* $f_1, \ldots, f_t \in \mathrm{Hom}_\Lambda(M_j, M_i)$ *such that, if* $a_1, \ldots, a_t \in K$ *are non-zero then* $\sum_{s=1}^t a_s f_s \neq g \circ h$, *with* $h = \begin{pmatrix} h_1 \\ \vdots \\ h_r \end{pmatrix} : M_j \to \bigoplus_{k=1}^r M_{i_k}$, $g = (g_1 \ \cdots \ g_r) : \bigoplus_{k=1}^r M_{i_k} \to M_i$ *where* $M_{i_k} \in \{M_1, \ldots, M_n\}$ *and* h_k, g_k *are non-isomorphisms for* $k = 1, \ldots, r$.

Proof We apply Lemma 2 to the projective Γ^{op}-modules $P_\mathbf{i} = \mathrm{Hom}_\Lambda(M, M_i)$. The proposition follows from Remark 1 and the equivalence of categories $\mathrm{Hom}_\Lambda(M, -) : \mathrm{add}(M) \to \mathrm{proj}(\Gamma^{op})$. □

Remark 2 From the above proposition we obtain that there is a bijection between the set of arrows from the vertex $\mathbf{i}$ to the vertex $\mathbf{j}$ in $Q_{\Gamma^{op}}$, and a maximal set of morphisms from M_j to M_i satisfying (b) in Proposition 1. Equivalently, morphisms that define a basis of the space of 'the irreducible morphisms in $\mathrm{add}(M)$', that is, a basis of $\mathrm{Hom}_\Lambda(M_j, M_i)$ modulo the morphisms $M_j \to M_i$ which can be written as a composition $M_j \xrightarrow{g} M' \xrightarrow{h} M_i$ with $M' \in \mathrm{add}(M)$, where g is not a split epimorphism and h is not a split monomorphism.

Remark 3 Given a presentation (Q_Γ, I_Γ) of $\Gamma = \mathrm{End}_\Lambda(M)$, we are interested in describing the representation of (Q_Γ, I_Γ) associated to ${}_\Gamma M$.

Note that, from Remark 2 and Proposition 1 applied to the algebra Γ, we get that there is a bijection between the set of arrows from the vertex $\mathbf{i}$ to the vertex $\mathbf{j}$ in Q_Γ and a maximal set of morphisms from M_i to M_j satisfying (b) in Proposition 1.

In the sequel, f_α denotes the morphism associated to the arrow $\alpha : \mathbf{i} \to \mathbf{j}$ under this bijection. We regard $f_\alpha : M_i \to M_j$ as an element of Γ in the natural way: we identify f_α with the composition $M \xrightarrow{\pi_i} M_i \xrightarrow{f_\alpha} M_j \xrightarrow{\iota_j} M$, where π_i and ι_j are the canonical projection and inclusion, respectively. In this way, the family $\{\overline{f}_\alpha : \alpha \in Q_\Gamma\}$ is a basis of $\mathrm{rad}\Gamma/\mathrm{rad}^2\Gamma$. Let $e_\mathbf{i}$ be the composition $M \xrightarrow{\pi_i} M_i \xrightarrow{\iota_i} M$. Then, by Theorem 1.9 in Chapter III in [3], the f_α's together with a complete set $e_\mathbf{1}, e_\mathbf{2}, \ldots, e_\mathbf{n}$ of primitive ortogonal idempotents generate the algebra Γ.

From now on, we consider the corresponding presentation (Q_Γ, I_Γ) of Γ. If $X = (X(i), \varphi_\alpha)_{i=1}^l$ is a representation of (Q_Γ, I_Γ), we write ${}_\Gamma X$ for the associated Γ-module. That is, ${}_\Gamma X = \bigoplus_{i=1}^l X(i)$ with the structure of Γ-module given by

$$f_\alpha \star x_i = \varphi_\alpha(x_i), \ e_\mathbf{t} \star x_i = \delta_{\mathbf{t}_i} x_i$$

for $x_i \in X(i)$, $\alpha : \mathbf{i} \to \mathbf{j}$ in Q_Γ, and $1 \leq \mathbf{t} \leq \mathbf{n}$, where $\delta_{\mathbf{t}_i}$ is the Kronecker delta.

Remark 4 For a path $\gamma = \alpha_r ... \alpha_1$ in Q_Γ, we define $f_\gamma = f_{\alpha_r} \circ ... \circ f_{\alpha_1}$. Let $a_1, ..., a_t \in K$ and $\gamma_1, ..., \gamma_t$ in Q_Γ. From Proposition 1 we get that $\sum_{s=1}^t a_s \gamma_s \in I_\Gamma$ if and only if $\sum_{s=1}^t a_s \mathrm{Hom}_\Lambda(M, f_{\gamma_s}) = 0$ in Γ. Moreover, this is the case if and only if $\sum_{s=1}^t a_s f_{\gamma_s} = 0$, as we see using the equivalence of categories $\mathrm{Hom}_\Lambda(M, -) : \mathrm{add}(M) \to \mathrm{proj}(\Gamma^{op})$.

Our next theorem describes a family of l direct summands M'_k of the Γ-module M, where l is the number of pairwise non-isomorphic simple modules of Λ. For $\alpha : \mathbf{i} \to \mathbf{j}$ in Q_Γ, we have the family of morphisms $\{f_\alpha(k) : M_i(k) \to M_j(k)\}_{k=1}^l$ associated to the morphism $f_\alpha : M_i \to M_j$. Here $M_i(k) = \mathrm{Hom}_\Lambda(P_k, M_i)$ and $M_j(k) = \mathrm{Hom}_\Lambda(P_k, M_j)$

are the vector spaces corresponding to the vertex k in the representations associated to M_i and M_j, respectively, and $f_\alpha(k)(\varphi) = f_\alpha \circ \varphi$ for each $\varphi \in M_i(k)$.

Theorem 1 *Let Λ be a basic finite dimensional K-algebra, ${}_\Lambda M = \bigoplus_{i=1}^n M_i$, where the M_i's are pairwise non-isomorphic indecomposable Λ-modules. Let (Q_Γ, I_Γ) be the presentation of $\Gamma = \mathrm{End}_\Lambda(M)$ defined in Remark 3 and let $M_i(k) = \mathrm{Hom}_\Lambda(P_k, M_i)$ be the vector space corresponding to the vertex k in the representation associated to ${}_\Lambda M_i$. Then the representation of (Q_Γ, I_Γ) associated to M is $(M_i, f_\alpha)_{i=1}^n$. Moreover, $M'_k = (M_i(k), f_\alpha(k))_{i=1}^n$ is a representation of (Q_Γ, I_Γ) and ${}_\Gamma M \simeq \bigoplus_{k=1}^l {}_\Gamma M'_k$.*

Proof We know that the operation $\cdot$ that defines M as a Γ-module is given by $f \cdot m = f(m)$ for $f \in \Gamma$ and $m \in M$. On the other hand, the Γ-module associated to the representation $(M_i, f_\alpha)_{i=1}^n$ is the abelian group $M = \bigoplus_{i=1}^n M_i$ with the operation $\star$ defined in Remark 3. So we only need to show that $f(m) = f \star m$ for each $m \in M$ and each $f \in \Gamma$. Since ${}_\Lambda M = \bigoplus_{i=1}^n M_i$, it is sufficient to show that $f(m_s) = f \star m_s$ for each $m_s \in M_s$ and each $f \in \Gamma$.

As we observed in Remark 3, the f_α's together with the complete set $e_{\mathbf{1}}, e_{\mathbf{2}}, \ldots, e_{\mathbf{n}}$ of primitive ortogonal idempotents generate the algebra Γ. Then, it only remains to prove that $e_{\mathbf{i}}(m_s) = e_{\mathbf{i}} \star m_s$ and, for $\alpha : \mathbf{s} \to \mathbf{r}$ in Q_Γ, that $f_\alpha(m_s) = f_\alpha \star m_s$ for each $m_s \in M_s$ and each $f_\alpha : M_s \to M_r$. This holds due to Remark 3.

Now, $M'_k = (M_i(k), f_\alpha(k))_{i=1}^n$ is a representation of Q_Γ. The fact that the linear maps $f_\alpha(k)$ satisfy the relations in I_Γ is a direct consequence of the fact that the f_α's do so, as we observed in Remark 4. Then M'_k is a representation of (Q_Γ, I_Γ). Moreover, we have that

$$\bigoplus_{k=1}^{l} M'_k = \left(\bigoplus_{k=1}^{l} M_i(k), \bigoplus_{k=1}^{l} f_\alpha(k) \right)_{i=1}^{n} = (M_i, f_\alpha)_{i=1}^n .$$

So the modules $\bigoplus_{k=1}^l {}_\Gamma M'_k$ and ${}_\Gamma M$ are isomorphic. This completes the proof of the theorem. □

Keeping the above notations, we obtain that in the diagram

$$\begin{array}{ccc} M_i(k) & \xrightarrow{M_i(\beta)} & M_i(t) \\ {\scriptstyle f_\alpha(k)}\downarrow & & \downarrow{\scriptstyle f_\alpha(t)} \\ M_j(k) & \xrightarrow{M_j(\beta)} & M_j(t) \end{array}$$

the rows describe the summands of ${}_\Lambda M$, and the columns describe the summands M'_k of ${}_\Gamma M$ given in the previous theorem, for any arrow $\beta : k \to t$ in Q_Λ.

Remark 5 Let M be a Λ-module such that ${}_\Gamma M \simeq \bigoplus_{k=1}^l {}_\Gamma M'_k$, with the notations used in Theorem 1. Then ${}_\Gamma M'_k \simeq \mathrm{Hom}_\Lambda(P_k, M) \simeq D\mathrm{Hom}_\Lambda(M, I_k)$. In fact, the first isomorphism holds because

$${}_\Gamma M'_k \simeq \oplus_{i=1}^n M_i(k) \simeq \oplus_{i=1}^n \mathrm{Hom}_\Lambda(P_k, M_i) \simeq \mathrm{Hom}_\Lambda(P_k, M) \simeq e_k M.$$

The second isomorphism is a well known fact (see [4], Chapter III, Lemma 2.11).

The next example illustrates Theorem 1.

Example 1 Let Λ be the path algebra of the quiver

$$Q_\Lambda : \quad \underset{1}{\circ} \longrightarrow \underset{2}{\circ} \longrightarrow \underset{3}{\circ}.$$

Consider the Λ-module $M = \bigoplus_{i=1}^{3} M_i$, with $M_1 = 3$, $M_2 = 1$ and $M_3 = \begin{smallmatrix}1\\2\\3\end{smallmatrix}$. In this case, the algebra $\Gamma = \mathrm{End}_\Lambda(M)$ is given by the quiver

$$Q_\Gamma : \quad \underset{1}{\circ} \xrightarrow{\varepsilon} \underset{3}{\circ} \xrightarrow{\mu} \underset{2}{\circ}$$

with the relation $\mu\varepsilon = 0$.

We want to give a description of M as Γ-module. In the next diagram the rows represent the summands of ${}_\Lambda M$, and the vertical arrows correspond to the morphisms between the M_i's associated to the arrows ε and μ in the quiver of Γ.

$$\begin{array}{lccccc}
{}_\Lambda M_1 : & 0 & \longrightarrow & 0 & \longrightarrow & K \\
& \downarrow & & \downarrow & & \downarrow{\scriptstyle 1} \\
{}_\Lambda M_3 : & K & \xrightarrow{1} & K & \xrightarrow{1} & K \\
& \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow \\
{}_\Lambda M_2 : & K & \longrightarrow & 0 & \longrightarrow & 0\,.
\end{array}$$

Here, the columns represent the summands of ${}_\Gamma M$. That is,

$${}_\Gamma M = \begin{smallmatrix}3\\2\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}1\\3\end{smallmatrix}.$$

In this case, these summands are indecomposable.

In the following example we show that the summands that appear in the description of M as Γ-module are not always indecomposable.

Example 2 Let Λ be the path algebra of the quiver

$$Q_\Lambda : \quad \underset{1}{\circ} \rightrightarrows \underset{2}{\circ}$$

and consider the Λ-module $M = M_1 \oplus M_2$, where $M_1 = 2$ and $M_2 = {}_2{}^1{}_2{}^1{}_2$. In this case, the algebra $\Gamma = \mathrm{End}_\Lambda(M)$ is defined by the quiver

$$Q_\Gamma : \quad \underset{1}{\circ} \overset{\longrightarrow}{\rightrightarrows} \underset{2}{\circ} \qquad .$$

In what follows, we describe M as Γ-module.

In the next diagram the rows represent the summands of ${}_\Lambda M$, and the vertical arrows correspond to the morphism between the M_i's which determine the arrows in the quiver of Γ.

$$\begin{array}{lccc}
{}_\Lambda M_1 : & 0 & \rightrightarrows & K \\
& \downarrow\downarrow\downarrow & & \downarrow{\scriptstyle \iota_1}\downarrow{\scriptstyle \iota_2}\downarrow{\scriptstyle \iota_3} \\
{}_\Lambda M_2 : & K^2 & \overset{h_1}{\underset{h_2}{\rightrightarrows}} & K^3
\end{array}$$

where $h_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$, $h_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ and ι_j is the inclusion map in the j-th coordinate for $j = 1, 2, 3$. Here, the columns represent the summands of ${}_\Gamma M$. That is

$${}_\Gamma M = (2 \oplus 2) \oplus {}_{2}\,{}^{1}_{2}\,{}_{2}.$$

In this case, we obtain that the first summand ${}_\Gamma M_1 = 2 \oplus 2$ is not indecomposable.

Diverse questions arise from Theorem 1, among them: when are the M'_k's all indecomposable modules?, when are they all non-zero?, when are they pairwise non-isomorphic? We next remind the reader of some necessary concepts to answer these questions (see [5] and [4]).

Let S and T be rings. For a left S-module M we have a canonical map

$$\lambda : S \to \mathrm{End}_{T^{op}}(M)$$

such that for $s \in S$ and $m \in M$, $\lambda(s) : m \mapsto sm$. And for a right T-module M we have a canonical map

$$\rho : T \to \mathrm{End}_S(M)$$

such that for $m \in M$ and $t \in T$, $\rho(t) : m \mapsto mt$.

The module ${}_S M$ (respectively, M_T) is **faithful** if and only if λ (respectively, ρ) is injective.

For a bimodule ${}_S M_T$ the maps λ and ρ are ring homomorphisms. Then ${}_S M_T$ is said to be a **balanced bimodule**, if both λ and ρ are surjective. If λ and ρ are isomorphisms then ${}_S M_T$ is called a **faithfully balanced bimodule**.

Also we recall that a Λ-module M is **sincere** if every simple Λ-module is a composition factor of M. This is the case if and only if $\mathrm{Hom}_\Lambda(P, M) \neq 0$ for all projective Λ-modules $P \neq 0$.

Proposition 2 *Let ${}_\Lambda M = \bigoplus_{i=1}^{n} M_i$, where the M_i's are pairwise non-isomorphic indecomposable Λ-modules, and let l be the number of pairwise non-isomorphic simple modules of Λ. Let $\Gamma = \mathrm{End}_\Lambda(M)$ and let ${}_\Gamma M \simeq \bigoplus_{k=1}^{l} M'_k$ be the decomposition given in Theorem 1. Then:*

(a) *${}_\Lambda M$ is sincere if and only if ${}_\Gamma M'_k \neq 0$ for all $k = 1, ..., l$.*
(b) *If ${}_\Lambda M_{\Gamma^{op}}$ is a faithfully balanced bimodule, then the Γ-modules M'_k are pairwise non-isomorphic and indecomposable.*

Proof We recall that $M'_k = (M_i(k), f_\alpha(k))_{i=1}^{n}$.

(a) Suppose that M is sincere and that there exists $s \in \{1, ..., l\}$ such that $M'_s = 0$. Then $M_i(s) = 0$ for all $i = 1, ..., n$. That is, $0 = e_s M = \mathrm{Hom}_\Lambda(P_s, M)$, which contradicts the sincerity of M. Thus, ${}_\Gamma M'_k \neq 0$ for all $k = 1, ..., l$.

Now, suppose that ${}_\Gamma M'_k \neq 0$ for all $k = 1, ..., l$. Then, for each $k = 1, ..., l$, there exists $i \in \{1, ..., n\}$ such that $M_i(k) \neq 0$. That is, for each $k = 1, ..., l$, there exists $i \in \{1, ..., n\}$ such that $\mathrm{Hom}_\Lambda(P_k, M_i) \neq 0$. Hence, $\mathrm{Hom}_\Lambda(P_k, M) \neq 0$ for all $k = 1, ..., l$, which proves that M is sincere.

(b) Next, we assume that ${}_\Lambda M_{\Gamma^{op}}$ is a faithfully balanced bimodule. In particular, ${}_\Lambda M$ is faithful. Then, M is a sincere Λ-module and then, by (a), it follows that ${}_\Gamma M'_k \neq 0$ for all $k = 1, ..., l$. Since we assume that ${}_\Lambda M_{\Gamma^{op}}$ is a faithfully balanced bimodule, we have an algebra isomorphism $\Lambda \simeq \mathrm{End}_{\Gamma}({}_{\Gamma}M) = \Omega$. So the number of pairwise non-isomorphic simple modules of Ω is l. Since Λ is basic, Ω is also basic. Thus the l projective summands $\mathrm{Hom}_{\Gamma}(M'_1, M), \mathrm{Hom}_{\Gamma}(M'_2, M), ..., \mathrm{Hom}_{\Gamma}(M'_l, M)$ of Ω are pairwise non-isomorphic. Then these are all the indecomposable projective modules. Finally, from the duality between $\mathrm{add}({}_{\Gamma}M)$ and $\mathrm{proj}(\Omega)$ we get that $M'_1, M'_2, ..., M'_l$ are pairwise non-isomorphic and indecomposable.

□

Example 3 The next example shows that the fact that a Λ-module is faithful does not guarantee that in its decomposition in direct sum of Γ-modules, the summands are pairwise non-isomorphic.

Let Λ be the path algebra of the quiver

$$\underset{1}{\circ} \longrightarrow \underset{2}{\circ} \longrightarrow \cdots \longrightarrow \underset{l}{\circ}.$$

and ${}_{\Lambda}M = {}_{\Lambda}P_1$. Then M is a faithful Λ-module and, since $\Gamma = \mathrm{End}_{\Lambda}(M) \simeq K$, we have that ${}_{\Gamma}M \simeq K^l$.

Example 4 Let Λ be the path algebra of the quiver

with the relation $\alpha^3 = 0$, and let us consider the Λ-module $M = \begin{smallmatrix}1\\1\end{smallmatrix}$. Here, the algebra $\Gamma = \mathrm{End}_{\Lambda}(M)$ is given by the quiver

with the relation $\mu^2 = 0$. It is easy to check that ${}_{\Gamma}M = \begin{smallmatrix}1\\1\end{smallmatrix}$. In this case $\Lambda \not\simeq \mathrm{End}({}_{\Gamma}M)$. That is, ${}_{\Lambda}M$ is not faithfully balanced. From this, it may be conclude that the converse of Proposition 2 (b) does not hold.

Let T be a basic tilting module. Then it is well known that ${}_{\Lambda}T_{\Gamma^{op}}$ is a faithfully balanced bimodule. From Proposition 2 we have the following result.

Corollary 1 *If ${}_{\Lambda}T = \bigoplus_{i=1}^{n} T_i$ is a basic tilting module, then in the decomposition ${}_{\Gamma}T = \bigoplus_{k=1}^{n} T'_k$ given in Theorem 1, the T'_k are pairwise non-isomorphic indecomposable modules.*

In [4] (Chapter VI, Section 6) the authors find the composition factors of the indecomposable summands of ${}_{\Gamma}T$ with different techniques.

For a module ${}_{\Lambda}X$ we denote by $Gen(X)$ the full subcategory of all modules Y in $\mathrm{mod}(\Lambda)$ generated by X, that is, the modules Y such that there exists an integer $d \geq 0$ and an epimorphism $X^d \to Y$ of Λ-modules. $Cogen(X)$ is defined dually. We notice that X is a generator of $\text{mod}(\Lambda)$ if and only if $\Lambda \in \text{add}(X)$, and X is a cogenerator of $\text{mod}(\Lambda)$ if and only if $D(\Lambda_\Lambda) \in \text{add}(X)$. We recall that a Λ-module X is a $\star$-module (as defined by Colpi in [2]) if $\text{Hom}_\Lambda(X, -) : \text{Gen}({}_\Lambda X) \to \text{Cogen}(D\, {}_\Gamma X)$ is an equivalence of categories. We next describe ${}_\Gamma M$ when ${}_\Lambda M$ is a $\star$-module.

Corollary 2 *Let ${}_\Lambda M$ be a $\star$-module and let ${}_\Gamma M \simeq \bigoplus_{k=1}^{l} M'_k$ be the decomposition given in Theorem 1. Then:*

(a) ${}_\Gamma M'_k \neq 0$ *if and only if* $Ann({}_\Lambda M).S_k = 0$, *where* $S_k = \Lambda e_k / r\Lambda e_k$.
(b) *The Γ-modules M'_k that are non-zero, are pairwise non-isomorphic and indecomposable.*

Proof Let $\Phi = \Lambda/\text{Ann}({}_\Lambda M)$. We know that $\Gamma = \text{End}_\Lambda(M) = \text{End}_\Phi(M)$. Since ${}_\Lambda M$ is a $\star$-module, ${}_\Phi M$ is tilting (see [6], Corollary 2). Then, by Corollary 1, ${}_\Phi M$ has m pairwise non-isomorphic indecomposable summands, where m is the number of pairwise non-isomorphic simple Φ-modules. Then (a) and (b) follow from the fact that the simple Φ-modules are the simple Λ-modules S such that $\text{Ann}({}_\Lambda M).S = 0$. □

We next give a description of the projective injective Γ^{op}-modules when all the injective Λ-modules are in $\text{add}(M)$.

Proposition 3 *Let M be a cogenerator of* $\text{mod}(\Lambda)$ *and let* $\Gamma^{op} = \text{End}_\Lambda(M)^{op}$. *If P is a projective injective Γ^{op}-module then $P \simeq \text{Hom}_\Lambda(M, I)$ for some injective Λ-module I.*

Proof Since P is projective over Γ^{op}, there exists a module X in $\text{add}(M)$ such that $P \simeq \text{Hom}_\Lambda(M, X)$. Let $j : X \to I$ be an injective envelope of X. Then the induced monomorphism $\text{Hom}_\Lambda(M, j) : \text{Hom}_\Lambda(M, X) \to \text{Hom}_\Lambda(M, I)$ splits, because we assume that the module $\text{Hom}_\Lambda(M, X) \simeq P$ is injective. That is, there exists $t : \text{Hom}_\Lambda(M, I) \to \text{Hom}_\Lambda(M, X)$ such that $t \circ \text{Hom}_\Lambda(M, j) = id_{\text{Hom}_\Lambda(M,X)}$. We chose $X \in \text{add}(M)$ and, since M is a cogenerator of $\text{mod}(\Lambda)$, we know that $I \in \text{add}(M)$. Using that the functor $\text{Hom}_\Lambda(M, -)|_{\text{add}(M)}$ is full we find $h : I \to X$ such that $t = \text{Hom}_\Lambda(M, h)$, and using that it is a faithful functor we conclude that $id_X = h \circ j$. From this, the essential monomorphism j splits and is thus an isomorphism. This proves that $P \simeq \text{Hom}_\Lambda(M, I)$, as desired. □

Remark 6 In the particular case when Λ is an artin algebra of finite representation type and $\text{add}(M) = \text{mod}(\Lambda)$, Proposition 3 and its converse are proven in [3] (Chapter VI, Lemma 5.3).

The converse in the above proposition is not true. If I is an injective Λ-module, $\text{Hom}_\Lambda(M, I)$ is not always an injective Γ^{op}-module, even assuming that I is in $\text{add}(M)$. In fact, let us consider ${}_\Lambda M = D(\Lambda)$ and l the number of pairwise non-isomorphic simple modules of Λ. We know that, for all $k = 1, ..., l$, the modules $\text{Hom}_\Lambda(D(\Lambda), I_k)$ are projective over $\Gamma^{op} = \text{End}(D(\Lambda))^{op} \simeq \Lambda$. If Λ is not selfinjective then not all of them are injective.

Proposition 4 *Let M be a Λ-module and let ${}_\Gamma M \simeq \bigoplus_{k=1}^{l} {}_\Gamma M'_k$ be the decomposition given in Theorem 1.*

(a) *If M is a generator of* mod(Λ) *then $M'_1, \ldots, M'_l$ are pairwise non-isomorphic projective indecomposable Γ-modules. In particular, ${}_{\Gamma}M$ is projective.*
(b) *If M is a cogenerator of* mod(Λ) *then $M'_1, \ldots, M'_l$ are pairwise non-isomorphic injective indecomposable Γ-modules. In particular, ${}_{\Gamma}M$ is injective.*

Proof By Remark 5 we know that ${}_{\Gamma}M'_k \simeq \mathrm{Hom}_{\Lambda}(P_k, M) \simeq D\mathrm{Hom}_{\Lambda}(M, I_k)$, for all $k = 1, \ldots, l$.

Suppose that M is a generator of mod(Λ). That is, all the projective Λ-modules are in add(M). Then, using the duality between add(M) and proj(Γ), we get that the Γ-modules $M'_k \simeq \mathrm{Hom}_{\Lambda}(P_k, M)$ are projective indecomposable pairwise non-isomorphic.

Assume now that M is a cogenerator of mod(Λ). That is, all the injective Λ-modules are in add(M). Then, from the equivalence between add(M) and proj(Γ^{op}) it follows that the Γ^{op}-modules $\mathrm{Hom}_{\Lambda}(M, I_k)$ are projective indecomposable pairwise non-isomorphic. That is, the Γ-modules $M'_k \simeq D\mathrm{Hom}_{\Lambda}(M, I_k)$ are injective indecomposable pairwise non-isomorphic. □

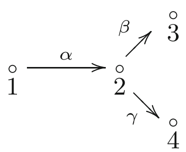
When ${}_{\Lambda}M$ is a generator and a cogenerator of mod(Λ) we obtain a stronger result: not only ${}_{\Gamma}M$ is projective injective, but also any projective injective Γ-module is in add(${}_{\Gamma}M$), as we state in the following corollary.

Corollary 3 *Assume M is a generator and a cogenerator of* mod(Λ). *Then the Γ-modules M'_k in the decomposition ${}_{\Gamma}M = \bigoplus_{k=1}^{l} M'_k$ given in Theorem 1 are a complete set of pairwise non-isomorphic indecomposable projective injective modules of* mod(Γ).

Proof Since $D(\Lambda_{\Lambda}) \in \mathrm{add}(M)$ and $\Lambda \in \mathrm{add}(M)$, from Proposition 4 we get that the Γ-modules M'_k are projective injective indecomposable pairwise non-isomorphic. Now let us see that these are all. Suppose that N is a projective injective indecomposable Γ-module. Thus $D(N)$ is a projective injective indecomposable Γ^{op}-module and, by Proposition 3, $D(N) \simeq \mathrm{Hom}_{\Lambda}(M, I)$ for some injective indecomposable Λ-module I. Hence, using Remark 5, we get $N \simeq D\mathrm{Hom}_{\Lambda}(M, I) \simeq M'_k$ for some $1 \leq k \leq l$, which completes the proof of the corollary. □

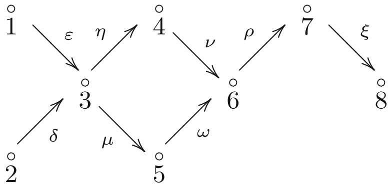
We observe that, if we assume that Λ is a nonsemisimple artin algebra of finite representation type and M is an additive generator for mod(Λ), then $\Gamma_M^{op} = \mathrm{End}({}_{\Lambda}M)^{op}$ is an Auslander algebra (see [3], Proposition 5.4 in Chapter VI), and the previous corollary applies in this case.

Example 5 Let Λ be the path algebra of the quiver



with the relations $\beta\alpha = 0$ and $\gamma\alpha = 0$. We consider M as the direct sum of all the indecomposable Λ-modules. That is, $M = \bigoplus_{i=1}^{8} M_i$, where $M_1 = 3$, $M_2 = 4$, $M_3 = {}_3{}^{2}{}_4$, $M_4 = \begin{smallmatrix}2\\4\end{smallmatrix}$, $M_5 = \begin{smallmatrix}2\\3\end{smallmatrix}$, $M_6 = 2$, $M_7 = \begin{smallmatrix}1\\2\end{smallmatrix}$ and $M_8 = 1$. Then M is a additive generator for mod(Λ).

Here, the algebra $\Gamma = \mathrm{End}_\Lambda(M)$ is the opposite of the Auslander algebra of Λ, and is given by the quiver



with the relations $\eta\varepsilon = 0$, $\mu\delta = 0$, $\xi\rho = 0$ and $\nu\eta = \omega\mu$ (which coincides with the AR-quiver of Λ). The indecomposable projective injective modules over Γ are:

$$P_7 = I_8 = \begin{smallmatrix} 7 \\ 8 \end{smallmatrix},\ P_3 = I_7 = \begin{smallmatrix} & 3 & \\ 4 & & 5 \\ & 6 & \\ & 7 & \end{smallmatrix},\ P_1 = I_5 = \begin{smallmatrix} 1 \\ 3 \\ 5 \end{smallmatrix} \text{ and } P_2 = I_4 = \begin{smallmatrix} 2 \\ 3 \\ 4 \end{smallmatrix}.$$

Then, by Corollary 3,

$$_\Gamma M = \begin{smallmatrix} 7 \\ 8 \end{smallmatrix} \oplus \begin{smallmatrix} & 3 & \\ 4 & & 5 \\ & 6 & \\ & 7 & \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 3 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 \\ 4 \end{smallmatrix}.$$

The following example shows that if we only assume that the Λ-module M is a cogenerator of $\mathrm{mod}(\Lambda)$, then the summands of M as Γ-module, which we know are injective, are not necessarily projective.

Example 6 Let Λ be a non-selfinjective algebra and let $_\Lambda M = D(\Lambda_\Lambda)$. Then $\Gamma = \mathrm{End}_\Lambda(M) \simeq \Lambda^{op}$. In this case, since Λ^{op} is also not a selfinjective algebra, we know that the summands of $_\Gamma M = {}_\Lambda M$ are injective modules but not all of them are projective.

3 Some Particular Representations

Let M be a basic Λ-module. For $\Gamma = \mathrm{End}(_\Lambda M)$, we consider the functors

$$\mathrm{mod}(\Lambda) \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{mod}(\Gamma^{op}),$$

where $F = \mathrm{Hom}_\Lambda(M, -)$ and $G = M \otimes_{\Gamma^{op}} -$, and the functors

$$\mathrm{mod}(\Lambda) \underset{\overline{G}}{\overset{\overline{F}}{\rightleftarrows}} \mathrm{mod}(\Gamma),$$

where $\overline{F} = \mathrm{Hom}_\Lambda(-, M)$ and $\overline{G} = \mathrm{Hom}_\Gamma(-, M)$.

Note that,

$$G = M \otimes_{\Gamma^{op}} - \simeq D\mathrm{Hom}_\Gamma(M, D-) \simeq D\mathrm{Hom}_\Gamma(M, -)D$$

(see [7] p. 120). Then, we consider the functor $H = \mathrm{Hom}_\Gamma(M, -) : \mathrm{mod}(\Gamma) \to \mathrm{mod}(\Lambda^{op})$.

Our aim in this section is to describe the representations associated to the image of a module under each of these functors.

We know that F and G define inverse equivalences between $\mathrm{add}(M)$ and $\mathrm{proj}(\Gamma^{op})$, and $\overline{F}$ and $\overline{G}$ define inverse dualities between $\mathrm{add}(M)$ and $\mathrm{proj}(\Gamma)$. The following lemma extends these results and will be useful in what follows. In the proof we use properties of the trace $\mathrm{tr}_M(X)$ and the reject $\mathrm{Rej}_M(X)$ of M in a module X (see [5], Chapter II).

Lemma 3 *Let $M' \in \text{add}(M)$, $X \in \text{mod}(\Lambda)$ and $Y \in \text{mod}(\Gamma)$. Then:*

(a) $\text{Hom}_{\Gamma^{op}}(F({}_\Lambda M'), F(X)) \simeq \text{Hom}_\Lambda({}_\Lambda M', X)$.
(b) $\text{Hom}_\Lambda(\overline{G}({}_\Gamma M'), \overline{G}(Y)) \simeq \text{Hom}_\Gamma(Y, {}_\Gamma M')$.
(c) $\text{Hom}_\Gamma(\overline{F}({}_\Lambda M'), \overline{F}(X)) \simeq \text{Hom}_\Lambda(X, {}_\Lambda M')$.
(d) $\text{Hom}_{\Lambda^{op}}(H({}_\Gamma M'), H(Y)) \simeq \text{Hom}_\Gamma({}_\Gamma M', Y)$.

Proof Let $X \in \text{mod}(\Lambda)$ and $Y \in \text{mod}(\Gamma)$. Let M' be a Λ-module such that $M' \in \text{add}(M)$. It is well known that

$$\text{tr}_M(X) \in \text{Gen}(M) \text{ and } Y/\text{Rej}_M(Y) \in \text{Cogen}(M);$$

and that

$$F(\text{tr}_M(X)) \simeq F(X) \text{ and } \overline{G}(Y/\text{Rej}_M(Y)) \simeq \overline{G}(Y).$$

Hence, we are reduced to proving (a) and (b) for $X \in \text{Gen}({}_\Lambda M)$ and $Y \in \text{Cogen}({}_\Gamma M)$, respectively.

Let $X \in \text{Gen}({}_\Lambda M)$. We now prove that the morphism $F_{M',X} : \text{Hom}_\Lambda(M', X) \to \text{Hom}_{\Gamma^{op}}(F(M'), F(X))$ induced by F is an isomorphism.

The proof that $F_{M',X}$ is an epimorphism follows using that F induces an equivalence between $\text{add}(M)$ and $\text{proj}(\Gamma^{op})$, and from the well known fact that for $X \in \text{Gen}({}_\Lambda M)$ we can find an epimorphism $f : M^r \to X$ such that $F(f) : F(M^r) \to F(X)$ is also an epimorphism. To prove that $F_{M',X}$ is injective, let $t : M' \to X$ be such that $F_{M',X}(t) = 0$. Then any composition $M^s \stackrel{g}{\to} M' \stackrel{t}{\to} M$ is zero. Since M' is in $\text{add}(M)$ we can choose g to be an epimorphism, and obtain then that $t = 0$.

Similar arguments applied to the case $Y \in \text{Cogen}({}_\Gamma M)$ prove (b).

The proofs of (c) and (d) are analogous to the previous ones using that $X/\text{Rej}_M(X) \in \text{Cogen}(M)$ in the first case, and that $\text{tr}_M(Y) \in \text{Gen}(M)$ in the other case. □

Given a finite dimensional K-algebra A with bounded quiver (Q_A, I_A) and an A-module X, we will denote by $V_X = (V_X(i), h_{X,\alpha})_{i \in Q_{A_0}, \alpha \in Q_{A_1}}$ the representation of the quiver associated to X.

We assume again that l is the number of pairwise non-isomorphic simple modules over the basic finite dimensional K-algebra Λ, and that $M = \bigoplus_{i=1}^n M_i$, where the M_i's are pairwise non-isomorphic indecomposable Λ-modules. Let $\Gamma = \text{End}_\Lambda(M)$. Then $Q_{\Lambda_0} = \{1, ..., l\}$ and $Q_{\Gamma_0^{op}} = \{\mathbf{1}, ..., \mathbf{n}\}$.

We use the notation established in Remark 3: $f_\varepsilon : M_i \to M_j$ is the morphism associated to the arrow $\varepsilon : \mathbf{i} \to \mathbf{j}$ in Q_{Γ_1}.

Let $\alpha : i \to j$ in Q_{Λ_1} and $\alpha^{op} : j \to i$ the corresponding arrow in $Q_{\Lambda_1^{op}}$. We denote by $\alpha\ .$ the left multiplication by α, which coincides with the right multiplication $.\ \alpha^{op}$ by α^{op}.

Proposition 5 *Let $X \in \text{mod}(\Lambda)$, $Y \in \text{mod}(\Gamma)$ and $Z \in \text{mod}(\Gamma^{op})$. Then:*

(a) $V_{F(X)} = (\text{Hom}_\Lambda(M_i, X), \text{Hom}_\Lambda(f_\varepsilon, X))_{i=1}^n$, *where* $\varepsilon \in Q_{\Gamma_1^{op}}$.
(b) $V_{\overline{G}(Y)} = (\text{Hom}_\Gamma(Y, M_i'), \text{Hom}_\Gamma(Y, (\alpha\ .)))_{i=1}^l$, *where* $\alpha \in Q_{\Lambda_1}$.
(c) $V_{\overline{F}(X)} = (\text{Hom}_\Lambda(X, M_i), \text{Hom}_\Lambda(X, f_\varepsilon))_{i=1}^n$, *where* $\varepsilon \in Q_{\Gamma_1}$.
(d) $V_{H(Y)} = (\text{Hom}_\Gamma({}_\Gamma M_i', Y), \text{Hom}_\Gamma((.\ \alpha^{op}), Y))_{i=1}^l$, *where* $\alpha^{op} \in Q_{\Lambda_1^{op}}$.
(e) $V_{G(Z)} = ({}_\Gamma M_i' \otimes_{\Gamma^{op}} Z, (\alpha\ .) \otimes_{\Gamma^{op}} Z)_{i=1}^l$, *where* $\alpha \in Q_{\Lambda_1}$.

Proof (*a*) The vertex of $Q_{\Gamma^{op}}$ associated to $P_{\mathbf{i}} = F(M_i)$ is $\mathbf{i}$. We know that $V_{F(X)}(\mathbf{i}) = e_{\mathbf{i}}F(X)$. Here

$$e_{\mathbf{i}}F(X) \simeq \mathrm{Hom}_{\Gamma^{op}}(\Gamma^{op}e_{\mathbf{i}}, F(X)) \simeq \mathrm{Hom}_{\Gamma^{op}}(P_{\mathbf{i}}, F(X)) = \\ = \mathrm{Hom}_{\Gamma^{op}}(F(M_i), F(X)) \simeq \mathrm{Hom}_{\Lambda}(M_i, X),$$

where the last isomorphism is a consequence of Lemma 3 (a). Hence, $V_{F(X)}(\mathbf{i}) \simeq \mathrm{Hom}_{\Lambda}(M_i, X)$.

On the one hand if $\varepsilon : \mathbf{r} \to \mathbf{s}$ is an arrow of $(Q_{\Gamma^{op}}, I_{\Gamma^{op}})$ we know that $h_{F(X),\varepsilon} : e_{\mathbf{r}}F(X) \to e_{\mathbf{s}}F(X)$ is the map induced by the left multiplication by ε, that is, $h_{F(X),\varepsilon}(t) = \varepsilon.t$ for $t \in e_{\mathbf{r}}F(X) \simeq \mathrm{Hom}_{\Lambda}(M_r, X)$. We prove next that $h_{F(X),\varepsilon}$ coincides with the map $\mathrm{Hom}_{\Lambda}(f_\varepsilon, X) : \mathrm{Hom}_{\Lambda}(M_r, X) \longrightarrow \mathrm{Hom}_{\Lambda}(M_s, X)$. This follows from the fact that $m_s.\varepsilon = f_\varepsilon(m_s)$ for all $m_s \in M_s$ (see Remark 3). In fact, for $t \in \mathrm{Hom}_{\Lambda}(M_r, X)$ and $m_s \in M_s$ we have

$$\mathrm{Hom}_{\Lambda}(f_\varepsilon, X)(t)(m_s) = (t \circ f_\varepsilon)(m_s) = t(f_\varepsilon(m_s)) = t(m_s.\varepsilon) = (\varepsilon.t)(m_s).$$

Then, $\varepsilon.t = t \circ f_\varepsilon$ for all $t \in \mathrm{Hom}_{\Lambda}(M_r, X)$, which shows that $h_{F(X),\varepsilon} = \mathrm{Hom}_{\Lambda}(f_\varepsilon, X)$.

Therefore, $V_{F(X)} = (\mathrm{Hom}_{\Lambda}(M_i, X), \mathrm{Hom}_{\Lambda}(f_\varepsilon, X))_{i=1}^{n}$, where $\varepsilon \in Q_{\Gamma_1^{op}}$.

(*b*) Prior to describing the representation of (Q_Λ, I_Λ) associated to $\overline{G}(Y)$, let us recall that $M'_k = (M_i(k), f_\alpha(k))_{i=1}^{n}$ is a representation of (Q_Γ, I_Γ) and that ${}_\Gamma M \simeq \bigoplus_{k=1}^{l} {}_\Gamma M'_k$ (see Theorem 1).

The vertex of Q_Λ corresponding to the projective Λ-module P_i is i. We know that $V_{\overline{G}(Y)}(i) = e_i\overline{G}(Y)$. Now, ${}_\Gamma M'_i \simeq e_iM$ (see Remark 5), and a straightforward argument proves that $e_i\overline{G}(Y) = e_i\mathrm{Hom}_\Gamma(Y, M) = \mathrm{Hom}_\Gamma(Y, e_iM)$, so that $V_{\overline{G}(Y)}(i) = \mathrm{Hom}_\Gamma(Y, M'_i)$.

Let $\alpha : r \to s$ be an arrow in Q_{Λ_1}. Since $\alpha e_r\overline{G}(Y)) = e_s\alpha\overline{G}(Y) \subset e_s\overline{G}(Y)$, the K-linear map $h_{\overline{G}(Y),\alpha} : e_r\overline{G}(Y) \to e_s\overline{G}(Y)$ is the left multiplication by α. That is, $h_{\overline{G}(Y),\alpha}(t) = \alpha.t = (\alpha\ .) \circ t$ for all $t \in e_r\overline{G}(Y) \simeq \mathrm{Hom}_\Gamma(Y, M'_r)$. Then, $h_{\overline{G}(Y),\alpha} = \mathrm{Hom}_\Gamma(Y, (\alpha\ .))$.

Finally, $V_{\overline{G}(Y)} = (\mathrm{Hom}_\Gamma(Y, M'_i), \mathrm{Hom}_\Gamma(Y, (\alpha\ .)))_{i=1}^{l}$, where $\alpha \in Q_{\Lambda_1}$.

The proof of (*c*) is analogous to (*a*) using (*c*) of Lemma 3.

(*d*) The vertex of Q_Λ corresponding to the projective Λ-module P_i is i. Since ${}_\Gamma M'_i \simeq e_iM$, one can readily verify that $V_{H(Y)}(i) = e_i^{op}H(Y) \simeq H(Y)e_i \simeq \mathrm{Hom}_\Gamma({}_\Gamma M'_i, Y)$.

The arrow $\alpha^{op} : s \to r$ in $Q_{\Lambda_1^{op}}$ correspond to the arrow $\alpha : r \to s$ in Q_{Λ_1}. Then the K-linear map $h_{H(Y),\alpha^{op}} : H(Y)e_s \to H(Y)e_r$ is the right multiplication by α^{op}. That is, $h_{H(Y),\alpha^{op}}(t) = t \circ (\alpha\ .) = t \circ (.\ \alpha^{op})$ for all $t \in H(Y)e_s \simeq \mathrm{Hom}_\Gamma({}_\Gamma M'_s, Y)$. Then, $h_{H(Y),\alpha^{op}} = \mathrm{Hom}_\Gamma((.\ \alpha^{op}), Y)$.

Hence, $V_{H(Y)} = (\mathrm{Hom}_\Gamma({}_\Gamma M'_i, Y), \mathrm{Hom}_\Gamma((.\ \alpha^{op}), Y))_{i=1}^{l}$, where $\alpha^{op} \in Q_{\Lambda_1^{op}}$.

(*e*) Using the item (*d*), we can prove that for a Γ^{op}-module Z the representation of (Q_Λ, I_Λ) associated to the Λ-module $G(Z) \simeq (DHD)(Z)$ is

$$V_{G(Z)} = (D\mathrm{Hom}_\Gamma({}_\Gamma M'_i, DZ), D\mathrm{Hom}_\Gamma((.\ \alpha^{op}), DZ))_{i=1}^{l} = \\ = ({}_\Gamma M'_i \otimes_{\Gamma^{op}} Z, (\alpha\ .) \otimes_{\Gamma^{op}} Z)_{i=1}^{l},$$

where $\alpha \in Q_{\Lambda_1}$. □

To illustrate the results in this section we use the algebra Λ given in Example 1. Then Λ is the path algebra of the quiver

$$Q_\Lambda : \quad \underset{1}{\circ} \xrightarrow{\alpha} \underset{2}{\circ} \xrightarrow{\beta} \underset{3}{\circ},$$

and we consider the Λ-module $M = \bigoplus_{i=1}^{3} M_i$, with $M_1 = 3$, $M_2 = 1$ and $M_3 = \begin{smallmatrix}1\\2\\3\end{smallmatrix}$. We will give two examples. In the first one we choose a Λ-module X and find the representation of $(Q_{\Gamma^{op}}, I_{\Gamma^{op}})$ associated to the Γ^{op}-module $F(X)$. In the second one we calculate the representation of $Q_{\Lambda^{op}}$ associated to the Λ^{op}-module $H(Y)$, for a Γ-module Y.

Example 7 We know that the algebra $\Gamma^{op} = \mathrm{End}_\Lambda(M)^{op}$ is given by the quiver

$$Q_{\Gamma^{op}}: \quad \underset{1}{\circ} \xleftarrow{\ \varepsilon\ } \underset{3}{\circ} \xleftarrow{\ \mu\ } \underset{2}{\circ}$$

with the relation $\varepsilon\mu = 0$.

Next, given the Λ-module $X = S_1$, we want to describe the representation of $(Q_{\Gamma^{op}}, I_{\Gamma^{op}})$ associated to the Γ^{op}-module

$$F(X) = \mathrm{Hom}_\Lambda(M, X) = \mathrm{Hom}_\Lambda(3 \oplus 1 \oplus \begin{smallmatrix}1\\2\\3\end{smallmatrix},\ 1).$$

By (a) of Proposition 5 we know that the representation associated to $F(X)$ is $(\mathrm{Hom}_\Lambda(M_i, X), \mathrm{Hom}_\Lambda(f_\eta, X))_{i=1}^{3}$, where η runs the set of arrows of $Q_{\Gamma^{op}}$. Now, $\mathrm{Hom}_\Lambda(M_1, X) = \mathrm{Hom}_\Lambda(3,\ 1) = 0$, and $\mathrm{Hom}_\Lambda(M_i, X) \simeq K$ for $i = 1, 2$. Then we get a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Hom}_\Lambda(M_1, X) & \xleftarrow{\mathrm{Hom}_\Lambda(f_\varepsilon, X)} & \mathrm{Hom}_\Lambda(M_3, X) & \xleftarrow{\mathrm{Hom}_\Lambda(f_\mu, X)} & \mathrm{Hom}_\Lambda(M_2, X) \\
\simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow \\
0 & \longleftarrow & K & \xleftarrow{\ id\ } & K\,.
\end{array}$$

Thus, $F(X) = \begin{smallmatrix}2\\3\end{smallmatrix}$.

Example 8 The algebra $\Gamma = \mathrm{End}_\Lambda(M)$ is given by the quiver

$$Q_\Gamma: \quad \underset{1}{\circ} \xrightarrow{\ \varepsilon\ } \underset{3}{\circ} \xrightarrow{\ \mu\ } \underset{2}{\circ}$$

with the relation $\mu\varepsilon = 0$. Moreover, ${}_\Gamma M = \bigoplus_{k=1}^{3}\ {}_\Gamma M'_k$, where ${}_\Gamma M'_1 = \begin{smallmatrix}3\\2\end{smallmatrix}$, ${}_\Gamma M'_2 = 3$ and ${}_\Gamma M'_3 = \begin{smallmatrix}1\\3\end{smallmatrix}$ (see Example 1).

Now we consider ${}_\Gamma Y = \begin{smallmatrix}1\\3\end{smallmatrix}$. Our aim is to calculate the representation of $Q_{\Lambda^{op}}$ associated to the Λ^{op}-module

$$H(Y) = \mathrm{Hom}_\Gamma(M, Y) = \mathrm{Hom}_\Gamma\ (\begin{smallmatrix}3\\2\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}1\\3\end{smallmatrix},\ \begin{smallmatrix}1\\3\end{smallmatrix}).$$

We know that it is $(\mathrm{Hom}_\Gamma(\ {}_\Gamma M'_i, Y), \mathrm{Hom}_\Gamma((.\ \gamma), Y))_{i=1}^{3}$, where γ runs the set of arrows of $Q_{\Lambda^{op}}$. Since $\mathrm{Hom}(\ {}_\Gamma M'_i, Y) \simeq K$ for $i = 1, 2, 3$, we obtain the following commutative diagram

$$\begin{array}{ccccc}
\mathrm{Hom}({}_\Gamma M'_1, Y) & \xleftarrow{\mathrm{Hom}((.\ \alpha^{op}), Y)} & \mathrm{Hom}({}_\Gamma M'_2, Y) & \xleftarrow{\mathrm{Hom}((.\ \beta^{op}), Y)} & \mathrm{Hom}({}_\Gamma M'_3, Y) \\
\simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow \\
K & \xleftarrow{\ id\ } & K & \xleftarrow{\ id\ } & K\,.
\end{array}$$

Hence, $H(Y) = \begin{smallmatrix}3\\2\\1\end{smallmatrix}$.

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