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Retractions and Gorenstein Homological Properties

Xiao-Wu Chen · Yu Ye

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Abstract We associate to a localizable module a left retraction of algebras; it is a homological ring epimorphism that preserves singularity categories. We study the behavior of left retractions with respect to Gorenstein homological properties (for example, being Gorenstein algebras or CM-free algebras). We apply the results to Nakayama algebras. It turns out that for a connected Nakayama algebra A, there exists a connected self-injective Nakayama algebra A' such that there is a sequence of left retractions linking A to A'; in particular, the singularity category of A is triangle equivalent to the stable category of A'. We classify connected Nakayama algebras with at most three simple modules according to Gorenstein homological properties.

Keywords Retractions · Gorenstein algebras · Gorenstein projective modules · Nakayama algebras · CM-free algebras

Mathematics Subject Classifications (2010) 16G20, 16E50, 16D90

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X.-W. Chen (✉) · Y. Ye
School of Mathematical Sciences, University of Science and Technology of China,
Hefei 230026, Anhui, People's Republic of China
e-mail: xwchen@mail.ustc.edu.cn

Y. Ye
e-mail: yeyu@ustc.edu.cn

X.-W. Chen · Y. Ye
Wu Wen-Tsun Key Laboratory of Mathematics, USTC, Chinese Academy of Sciences,
Hefei 230026, Anhui, People's Republic of China
URL: http://staff.ustc.edu.cn/~yeyu, http://home.ustc.edu.cn/~xwchen

1 Introduction

Let A be an artin algebra. The modules we consider here are finitely generated left modules. The study of Gorenstein projective modules, that extend projective modules, goes back to Auslander and Bridger [2], and it relates to singularity categories of algebras via the work of Buchweitz, (Maximal Cohen-Macaulay Modules and Tate-Cohomology over Gorenstein Rings, Unpublished); also see [5, 15, 16, 22]. Gorenstein projective modules are also known as modules of G-dimension zero [2], totally reflexive modules [4] or (maximal) Cohen-Macaulay modules [5, (Buchweitz R. O, Maximal Cohen-Macaulay Modules and Tate-Cohomology over Gorenstein Rings, Unpublished)].

Gorenstein projective modules play a central role in Gorenstein homological algebra. For example, resolutions by Gorenstein projective modules give rise to the notion of Gorenstein projective dimension for modules; see [2, 12]. Recall from [15, (Buchweitz R. O, Maximal Cohen-Macaulay Modules and Tate-Cohomology over Gorenstein Rings, Unpublished)] that A is Gorenstein if the regular module A has finite injective dimension on both sides. For example, algebras with finite global dimension and self-injective algebras are Gorenstein. Note that A is Gorenstein if and only if each A-module has finite Gorenstein projective dimension; in other words, Gorenstein algebras in Gorenstein homological algebra play a similar role as algebras of finite global dimension in classical homological algebra. For Gorenstein algebras, Gorenstein projective modules behave quite nicely.

An algebra A is called CM-free [9] provided that any Gorenstein projective module is projective. For example, algebras with finite global dimension are CM-free; indeed, an algebra has finite global dimension if and only if it is Gorenstein and CM-free. We are interested in non-Gorenstein CM-free algebras, or equivalently, CM-free algebras with infinite global dimension.

We have the following obvious trichotomy from the point view of Gorenstein homological algebra: any algebra A is either Gorenstein, or non-Gorenstein CM-free, or non-Gorenstein but not CM-free. We are interested in classifying algebras according to this trichotomy. Recall from [9] that a connected algebra with radical square zero is either Gorenstein or non-Gorenstein CM-free; compare [26].

In this paper, we show that left retractions of algebras are related to these Gorenstein properties; see Proposition 2.6 and Theorem 2.8. Here, we follow [11, 19] to associate a left retraction of algebras to a localizable module; left retractions are homological ring epimorphisms and preserve singularity categories. We mention that related results are obtained in [21] via a completely different method. We apply there results to Nakayama algebras, where similar ideas might trace back to [30] and [7]. It turns out that for a connected Nakayama algebra A, there is a connected self-injective Nakayama algebra A' such that there is a sequence of left retractions linking A to A'; indeed, if A has infinite global dimension, such an algebra A' is unique up to isomorphism; see Theorem 3.8. In particular, the singularity category of A is triangle equivalent to the stable category of A'; see Corollary 3.11. Finally, we classify Nakayama algebras with at most three simple modules according to the above trichotomy. It turns out that there exists a class of such algebras, that are non-Gorenstein but not CM-free; see Proposition 3.14. For the classification, we use the notion of regular element for Nakayama algebras introduced in [13].

Throughout, A is an artin algebra over a commutative artinian ring R. We denote by A-mod the category of finitely generated left A-modules. We denote by $n(A)$ the number of pairwise non-isomorphic simple A-modules. For a module M, $l(M)$ denotes its composition length. We refer to [3] for artin algebras.

2 Left Retractions of Algebras

In this section, we recall left retractions of algebras and their basic properties. We study the behavior of left retractions with respect to Gorenstein homological properties, such as being Gorenstein algebras or CM-free algebras. We point out that left retractions induce triangle equivalences between singularity categories.

2.1 Left Retractions of Categories

Let A be an artin algebra. Recall that a simple A-module S is called *localizable* provided that $\text{proj.dim}_A\, S \leq 1$ and $\text{Ext}^1_A(S, S) = 0$. Here, for each A-module X, $\text{proj.dim}_A\, X$ denotes its projective dimension.

For a localizable A-module S, we consider the *perpendicular subcategory*

$$S^{\perp} = \{X \in A\text{-mod} \mid \text{Hom}_A(S, X) = 0 = \text{Ext}^1_A(S, X)\}.$$

It is an *exact abelian* subcategory of A-mod, that is, the category $S^{\perp}$ is abelian and the inclusion functor $i\colon S^{\perp} \to A$-mod is exact. A nontrivial fact is that the functor i admits a right adjoint i_{λ} which is exact. For details, see [19, Proposition 3.2].

The inclusion functor $i\colon S^{\perp} \to A$-mod is called a *left expansion* of categories, while the right adjoint $i_{\lambda}\colon A\text{-mod} \to S^{\perp}$ is called a *left retraction*; see [11].

Denote by add S the full subcategory of A-mod consisting of finite direct sums of the localizable module S. It is a *Serre subcategory*, that is, it is closed under extensions, submodules and quotient modules. Consider the quotient abelian category A-mod/add S in the sense of Gabriel [17]. Then the functor i_{λ} induces an equivalence $A\text{-mod}/\text{add}\, S \simeq S^{\perp}$, in other words, the functor i_{λ} identifies with the quotient functor $q\colon A\text{-mod} \to A\text{-mod}/\text{add}\, S$; see [11, Lemma 3.1.2].

Set $\Delta = \text{End}_A(S)^{\text{op}}$ to be the opposite algebra of the endomorphism algebra of S; it is a division algebra. There is an equivalence $\text{Hom}_A(S, -)\colon \text{add}\, S \xrightarrow{\sim} \Delta\text{-mod}$ of categories.

Let us recall from [19, Section 3] the construction of the functor $i_{\lambda}\colon A\text{-mod} \to S^{\perp}$. For an A-module X, we take an exact sequence $0 \to X \to X' \to S^{\oplus m_1} \to 0$ such that $\text{Ext}^1_A(S, X') = 0$ and $m_1 = \dim_{\Delta} \text{Ext}^1_A(S, X)$. Then we take an exact sequence $0 \to S^{\oplus m_2} \to X' \to X'' \to 0$ such that $\text{Hom}_A(S, S^{\oplus m_2}) \to \text{Hom}_A(S, X')$ is an isomorphism. It follows that X'' lies in $S^{\perp}$. The composite $X \to X' \to X''$ is the universal morphism of X to $S^{\perp}$, that is, $i_{\lambda}(X) = X''$. We conclude with an exact sequence

$$0 \longrightarrow t(X) \longrightarrow X \xrightarrow{\eta_X} ii_{\lambda}(X) \longrightarrow t'(X) \longrightarrow 0. \tag{2.1}$$

Here, both $t(X)$ and $t'(X)$ lie in add S. Moreover, this yields a functor $t\colon A\text{-mod} \to \text{add}\, S$, which is right adjoint to the inclusion functor $\text{inc}\colon \text{add}\, S \to A\text{-mod}$.

We observe that η is the unit of the adjoint pair (i_λ, i). Consider the endofunctor $L = ii_\lambda\colon A\text{-mod} \to A\text{-mod}$. Since the functor i is fully faithful, the counit $i_\lambda i \to \mathrm{Id}_{S^\perp}$ is an isomorphism. It follows that for each A-module X, $L(\eta_X) = \eta_{L(X)}\colon L(X) \to L^2(X)$ is an isomorphism. In other words, the pair $(L, \eta\colon \mathrm{Id}_{A\text{-mod}} \to L)$ is a *localization functor* in the sense of [6, Section 3].

In summary, for a localizable A-module S, we have the following diagram of functors

$$\Delta\text{-mod} \simeq \mathrm{add}\, S \underset{t}{\overset{\mathrm{inc}}{\rightleftarrows}} A\text{-mod} \underset{i}{\overset{i_\lambda}{\rightleftarrows}} S^\perp. \tag{2.2}$$

2.2 Left Retractions of Algebras

Let (L, η) be the localization functor associated to a localizable module S. Consider the A-module $L(A)$. Recall the isomorphism $L(\eta_A) = \eta_{L(A)}\colon L(A) \to L^2(A)$. For each element $x \in L(A)$, denote by $r_x\colon A \to L(A)$ the morphism given by $r_x(a) = a.x$; here, the dot "." denotes the A-module action. For $x, y \in L(A)$, we define $x \star y = L(\eta_A)^{-1}(L(r_y)(x)) \in L(A)$. This gives rise to an algebra structure on $L(A)$. Observe that $\eta_A(a) \star y = a.y$ for $a \in A$. In particular, $\eta_A\colon A \to L(A)$ is a homomorphism of algebras and the left A-module structure on $L(A)$ coincides with the one induced from η_A.

We call the algebra homomorphism $\eta_A\colon A \to L(A)$ the *left retraction* associated to the localizable module S.

We observe that the left retraction η_A is surjective if and only if $t'(A) = 0$, or equivalently, $\mathrm{Ext}^1_A(S, A) = 0$. This happens if the simple module S is projective, in which case the algebra A is Morita equivalent to a one-point (co-)extension of $L(A)$; compare [3, III. 2].

The following result is well known; compare [19, Corollary 3.9].

Lemma 2.1 *Keep the notation as above. Then we have the following statements:*

(1) *there is an algebra isomorphism* $\Phi\colon L(A) \xrightarrow{\sim} \mathrm{End}_{S^\perp}(i_\lambda(A))^{\mathrm{op}}$ *such that* $i(\Phi(x)) = L(\eta_A)^{-1} \circ L(r_x)$*;*
(2) *there is an equivalence* $S^\perp \simeq L(A)\text{-mod}$ *of categories; moreover, the functor* i *identifies with* $\mathrm{Hom}_{L(A)}(L(A), -)$*, and* i_λ *with* $L(A) \otimes_A -$.

The fully-faithfulness of the functor $i \simeq \mathrm{Hom}_{L(A)}(L(A), -)\colon L(A)\text{-mod} \to A\text{-mod}$ implies that $\eta_A\colon A \to L(A)$ is a ring epimorphism. The exactness of the functor $i_\lambda \simeq L(A) \otimes_A -$ implies that the right A-module $L(A)$ is flat and then projective. In particular, the left retraction $\eta_A\colon A \to L(A)$ is a *left localization* in the sense of [27]. It follows that η_A is a homological ring epimorphism; see [19, Corollary 4.7]. We observe from the exact sequence 2.1 that $\mathrm{proj.dim}_A\, L(A) \le 2$.

We mention that there exists an idempotent e in A such that add S is the kernel of the *Schur functor* $S_e = eA \otimes_A -\colon A\text{-mod} \to eAe\text{-mod}$. Then S_e induces an equivalence $A\text{-mod}/\mathrm{add}\, S \simeq eAe\text{-mod}$, which is further equivalent to $S^\perp$. Then Lemma 2.1(2) implies that $L(A)$ and eAe are Morita equivalent.

From the above discussion, we may identify the Schur functor S_e with the quotient functor q, and thus with i_λ. In particular, the right adjoint of S_e, that is $\mathrm{Hom}_{eAe}(eA, -)$, is exact. Hence, the left eAe-module eA is projective.

Proof For (1), we observe that $L(A) \simeq \mathrm{Hom}_A(A, ii_\lambda(A)) \simeq \mathrm{Hom}_{S^\perp}(i_\lambda(A), i_\lambda(A))$, where the second isomorphism is by the adjoint pair (i_λ, i). The composition is Φ; one checks directly that it is a homomorphism of algebras.

For (2), we observe that $i_\lambda(A)$ is projective in $S^\perp$, since i is exact; see [29, Proposition 2.3.10]. Identifying i_λ with the quotient functor $q\colon A\text{-mod} \to A\text{-mod}/\mathrm{add}\, S$, we infer that each object in $S^\perp$ is a quotient of finite direct sums of copies of $i_\lambda(A)$. Hence, $i_\lambda(A)$ is a projective generator of $S^\perp$. Then using (1), we deduce the desired equivalence of categories. □

2.3 Simple Modules

Let $n(A) = n$ be the number of pairwise non-isomorphic simple A-modules, and let $\{S_1, S_2, \cdots, S_{n-1}, S_n = S\}$ be a complete set of representatives of pairwise non-isomorphic simple A-modules. Set $\Delta_j = \mathrm{End}_A(S_j)^{\mathrm{op}}$; they are division algebras.

We assume that $S_n = S$ is localizable. Let $P_j = P(S_j)$ and $I_j = I(S_j)$ be the projective cover and the injective envelop of S_j, respectively. Since S_n is localizable, its minimal projective presentation takes the following form

$$0 \longrightarrow \bigoplus_{j=1}^{n-1} P_j^{\oplus d_j} \longrightarrow P_n \longrightarrow S_n \longrightarrow 0. \tag{2.3}$$

Here, each $d_j \geq 0$.

Recall that the *Cartan matrix* $C_A = (c_{jk})$ of A is an $n \times n$ matrix such that c_{jk} is the multiplicity of S_j in a composition series of P_k. Consider the homomorphism $C_A\colon \mathbb{Z}^n \to \mathbb{Z}^n$ between free abelian groups induced by multiplication of C_A from the left; we view elements in $\mathbb{Z}^n$ as column vectors. Then Cok C_A is a finitely generated abelian group such that $\mathrm{rk}(\mathrm{Cok}\, C_A) = n - \mathrm{rank}\, C_A$, where we denote by "rk" the rank of an abelian group.

Proposition 2.2 *Consider the left retraction $\eta_A\colon A \to L(A)$ associated to $S_n = S$. Identify $S^\perp$ with $L(A)$-mod. Then we have the following statements:*

(1) *the set $\{i_\lambda(S_1), i_\lambda(S_2), \cdots, i_\lambda(S_{n-1})\}$ is a complete set of representatives of pairwise non-isomorphic simple $L(A)$-modules;*
(2) *there is an isomorphism $\Delta_j \simeq \mathrm{End}_{L(A)}(i_\lambda(S_j))^{\mathrm{op}}$ induced by i_λ for each $1 \leq j \leq n-1$;*
(3) *the $L(A)$-module $i_\lambda(P_j)$ is the projective cover of $i_\lambda(S_j)$ for $1 \leq j \leq n-1$;*
(4) *the $L(A)$-module $i_\lambda(I_j)$ is the injective envelop of $i_\lambda(S_j)$ for $1 \leq j \leq n-1$;*
(5) *the Cartan matrix $C_{L(A)}$ of $L(A)$ is the minor of C_A by deleting the n-th row and n-th column; consequently, we have that* $\det C_{L(A)} = \det C_A$, $\mathrm{rank}\, C_{L(A)} = \mathrm{rank}\, C_A - 1$ *and an isomorphism* $\mathrm{Cok}\, C_{L(A)} \simeq \mathrm{Cok}\, C_A$ *of abelian groups.*

Proof Identify $i_\lambda\colon A\text{-mod} \to S^\perp \simeq L(A)\text{-mod}$ with the quotient functor $q\colon A\text{-mod} \to A\text{-mod}/\mathrm{add}\, S$. We observe that each $L(A)$-module has a submodule series with factors $i_\lambda(S_j)$ for $1 \leq j \leq n-1$. Then (1) and (2) follow from a general fact: given any abelian category $\mathcal{A}$, a Serre subcategory $\mathcal{C}$ and two simple objects S, S' of $\mathcal{A}$ that are not in $\mathcal{C}$, the quotient functor $q\colon \mathcal{A} \to \mathcal{A}/\mathcal{C}$ sends S and S' to simple objects, and induces an isomorphism $\mathrm{Hom}_{\mathcal{A}}(S, S') \to \mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(q(S), q(S'))$.

For (3), we recall that each P_j has a unique maximal submodule R_j such that $P_j/R_j \simeq S_j$. It follows that $i_\lambda(R_j)$ is the unique maximal submodule of $i_\lambda(P_j)$. Here, we use another general fact: for an object X in $\mathcal{A}$, each subobject of $q(X)$ is induced by a subobject of X. Then (3) follows from the facts that $i_\lambda(P_j)$ is projective and that $i_\lambda(P_j)/i_\lambda(R_j) \simeq i_\lambda(S_j)$.

For (4), we observe that each I_j lies in $S^\perp$, and it is injective and indecomposable in $S^\perp$. This implies that $i_\lambda(I_j) = I_j$. Consider the embedding $S_j \to I_j$. We infer that the induced embedding $i_\lambda(S_j) \to i_\lambda(I_j)$ is an injective hull.

For (5), it suffices to observe that the multiplicity of $i_\lambda(S_j)$ in a composition series of $i_\lambda(P_k)$ is the same of the multiplicity of S_j in a composition series of P_k. For the equality of determinants, we observe from Eq. 2.3 that the last column of C_A is the sum of $e_n = (0, 0, \cdots, 0, 1)^t$ and a linear combination of its j-th columns for $1 \leq j \leq n-1$; here, "t" denotes the transpose. Then all the statements follow immediately. □

Recall the *valued quiver* Q_A of an algebra A. Let $\{S_1, S_2, \cdots, S_{n-1}, S_n\}$ and $\{\Delta_1, \Delta_2, \cdots, \Delta_{n-1}, \Delta_n\}$ be as above. Observe that $\mathrm{Ext}^1_A(S_j, S_k)$ has a natural Δ_j-Δ_k-bimodule structure. The quiver Q_A has the vertex set $\{S_1, S_2, \cdots, S_{n-1}, S_n\}$, and there is an arrow from S_j to S_k whenever $\mathrm{Ext}^1_A(S_j, S_k) \neq 0$; this arrow is endowed with a valuation $(\dim_{\Delta_k{}^{\mathrm{op}}} \mathrm{Ext}^1_A(S_j, S_k), \dim_{\Delta_j} \mathrm{Ext}^1_A(S_j, S_k))$. The valuation of Q_A is *trivial* if all the valuations of arrows are $(1, 1)$.

Consider the left localization $\eta_A\colon A \to L(A)$. By Proposition 2.2 the vertex set of the valued quiver $Q_{L(A)}$ of $L(A)$ is obtained by deleting S_n from the vertex set of Q_A. The following result implies the quiver obtained from Q_A by deleting S_n is *smaller* than $Q_{L(A)}$ in the sense of [3, p.244]; their difference only appears in the neighborhood of S_n.

Lemma 2.3 *For $1 \leq j, k \leq n-1$, there is an exact sequence*

$$0 \longrightarrow \mathrm{Ext}^1_A(S_j, S_k) \stackrel{\xi}{\longrightarrow} \mathrm{Ext}^1_{L(A)}(i_\lambda(S_j), i_\lambda(S_k)) \longrightarrow \mathrm{Ext}^1_A(S_j, t'(S_k))$$

where the module $t'(S_k)$ is defined in Eq. 2.1, and ξ is induced by i_λ.

In particular, the map ξ is an isomorphism provided that in Q_A, there is no arrow from S_j to S_n, or no arrow from S_n to S_k.

Proof Recall $S_n = S$ and that in the adjoint pair (i_λ, i) both functors are exact. Then we have $\mathrm{Ext}^1_{L(A)}(i_\lambda(S_j), i_\lambda(S_k)) \simeq \mathrm{Ext}^1_A(S_j, ii_\lambda(S_k))$; see [11, Lemma 2.3.1]. The exact sequence 2.1 for S_k takes the form $0 \to S_k \to ii_\lambda(S_k) \to t'(S_k) \to 0$, since $t(S_k) = 0$. Applying the functor $\mathrm{Hom}_A(S_j, -)$ to this sequence, we are done. Here, one might notice that $\mathrm{Hom}_A(S_j, t'(S_k)) = 0$, since $t'(S_k) \in \mathrm{add}\ S$. For the last statement, we note that $t'(S_k) = 0$ if and only if $\mathrm{Ext}^1_A(S, S_k) = 0$. □

2.4 Homological Properties

For an artin algebra A, we denote by gl.dim A its global dimension. Recall its *finitistic dimension* $\text{fin.dim } A = \sup\{\text{proj.dim}_A\, X \mid X \in A\text{-mod with proj.dim}_A\, X < \infty\}$. For an algebra A with finite global dimension, we have gl.dim A = fin.dim A. We have the following result; compare [7, Lemma 4].

Lemma 2.4 *Let $\eta_A : A \to L(A)$ be the left retraction associated to $S_n = S$. Identify $S^{\perp}$ with $L(A)$-mod. Let $X \in A$-mod. Then the following statements hold:*

(1) $\text{proj.dim}_{L(A)}\, i_\lambda(X) \leq \text{proj.dim}_A\, X \leq \text{proj.dim}_{L(A)}\, i_\lambda(X) + 2$;
(2) $\text{gl.dim}\, L(A) \leq \text{gl.dim}\, A \leq \text{gl.dim}\, L(A) + 2$;
(3) $\text{fin.dim}\, L(A) \leq \text{fin.dim}\, A \leq \text{fin.dim}\, L(A) + 2$.

Proof It suffices to show (1). Since i_λ sends projective A-modules to projective $L(A)$-modules, the left inequality follows. For the right one, recall that $i(L(A)) = L(A)$, viewed as a left A-module, satisfies that $\text{proj.dim}_A\, L(A) \leq 2$. Then for each projective $L(A)$-module Q, $\text{proj.dim}_A\, i(Q) \leq 2$. This implies $\text{proj.dim}_A\, ii_\lambda(X) \leq \text{proj.dim}_{L(A)}\, i_\lambda(X) + 2$. Using the fact that $\text{proj.dim}_A\, S \leq 1$, the result follows from the exact sequence Eq. 2.1. □

Let $P^\bullet = \cdots \to P^{-1} \overset{d^{-1}}{\to} P^0 \overset{d^0}{\to} P^1 \overset{d^1}{\to} P^2 \to \cdots$ be an unbounded complex of projective A-modules. Recall that $Z^1(P^\bullet) = \text{Ker}\, d^1$ is the first cocycle. Following [4], the complex $P^\bullet$ is *totally acyclic* provided that it is acyclic and the dual complex $(P^\bullet)^* = \text{Hom}_A(P^\bullet, A)$ is also acyclic. An A-module X is *Gorenstein projective* provided that there exists a totally acyclic complex $P^\bullet$ such that $Z^1(P^\bullet) \simeq X$; such a complex $P^\bullet$ is called a *complete resolution* of X; see [12].

We denote by A-Gproj the full subcategory consisting of Gorenstein projective A-modules. Any projective module P is *Gorenstein projective*, since its complete resolution can be taken as $\cdots \to 0 \to P \overset{\text{Id}_P}{\to} P \to 0 \to \cdots$. The subcategory A-Gproj is closed under extensions, and thus carries a natural exact structure in the sense of Quillen [23]. As an exact category, A-Gproj is Frobenius, whose projective-injective objects are precisely projective A-modules. By [14, Theorem I.2.8], the *stable category* A-$\underline{\text{Gproj}}$ modulo projective modules is a triangulated category: the translation functor is induced by a quasi-inverse of the syzygy functor, and triangles are induced by short exact sequences with terms in A-Gproj. If A is self-injective, we have that A-Gproj $= A$-mod, and thus the resulting triangulated category A-$\underline{\text{Gproj}}$ coincides with the stable category A-$\underline{\text{mod}}$.

The following fact is well known.

Lemma 2.5 *Let X be an indecomposable Gorenstein projective A-module which is non-projective. Then there exists an exact sequence $0 \to X \to P \to X' \to 0$ such that P is projective and X' is indecomposable Gorenstein projective which is non-projective. Moreover, the morphism $P \to X'$ is a projective cover of X'.*

Proof From the definition, there is an exact sequence $0 \to X \to P \to X' \to 0$ with P projective and X' Gorenstein projective. We may take the sequence such that P has the minimal length. This implies that X' has no projective direct summands and then the indecomposable module X is the first syzygy of X'. So we have that X' is indecomposable and that $P \to X'$ is a projective cover. □

Recall from [9] that an artin algebra A is *CM-free* provided that each Gorenstein projective A-module is projective.

Proposition 2.6 *Let $\eta_A\colon A \to L(A)$ be the left retraction associated to $S_n = S$. Identify $S^\perp$ with $L(A)$-mod. Then for any $X \in A$-Gproj, we have $i_\lambda(X) \in L(A)$-Gproj. In particular, if $L(A)$ is CM-free, so is A.*

Proof The last statement follows from the first one: if $i_\lambda(X)$ is projective, the A-module X has finite projective dimension; see Lemma 2.4(1); then it suffices to recall from [12, Proposition 10.2.3] that a Gorenstein projective module of finite projective dimension is necessarily projective.

For the first statement, it suffices to show that for a totally acyclic complex $P^\bullet$ of A-modules, the complex $i_\lambda(P^\bullet)$ is totally acyclic. Observe that the complex $i_\lambda(P^\bullet)$ consists of projective $L(A)$-modules and is acyclic. From the adjoint pair (i_λ, i), we have the isomorphism $\mathrm{Hom}_{L(A)}(i_\lambda(P^\bullet), L(A)) \simeq \mathrm{Hom}_A(P^\bullet, L(A))$ of complexes. Since the left A-module $L(A)$ has projective dimension at most two and $P^\bullet$ is totally acyclic, by [4, Lemma 2.4(iii)] the complex $\mathrm{Hom}_A(P^\bullet, L(A))$ is acyclic. It follows that the complex $i_\lambda(P^\bullet)$ of $L(A)$-modules is totally acyclic. □

2.5 Gorenstein Algebras

Recall from [15, (Buchweitz R. O, Maximal Cohen-Macaulay Modules and Tate-Cohomology over Gorenstein Rings, Unpublished)] that an artin algebra A is *Gorenstein* provided that the regular A-module has finite injective dimension on both sides. In this case, both injective dimensions are the same, which are called the *virtual dimension* of A and denoted by v.dim A. Observe that for a Gorenstein algebra A, a module has finite injective dimension if and only if it has finite projective dimension; moreover, we have v.dim $A =$ fin.dim A.

An artin algebra A is self-injective, if and only if it is Gorenstein with virtual dimension zero, if and only if A-Gproj $= A$-mod. On the other hand, A has finite global dimension if and only if it is Gorenstein and CM-free; compare [5, Theorem 6.9(ι)].

The following result is well known; see [5, Theorem 6.9(13)] and compare [4, Theorem 3.2].

Lemma 2.7 *Let A be an artin algebra and let $d \geq 0$. Then A is Gorenstein with* v.dim $A \leq d$ *if and only if each A-module X fits into an exact sequence $0 \to G \to P^{-(d-1)} \to \cdots \to P^{-1} \to P^0 \to X \to 0$ with each P^{-j} projective and $G \in A$-Gproj.*

The next result relates the Gorensteinness of A and $L(A)$ in a left retraction. We mention that this result is related to [21, Proposition 6(3)]; consult the remarks after Lemma 2.1.

Theorem 2.8 *Let $\eta_A\colon A \to L(A)$ be the left retraction associated to the localizable module $S_n = S$. Identify $S^\perp$ with $L(A)$-mod. Denote by I_n the injective envelop of S_n. Then A is Gorenstein if and only if $L(A)$ is Gorenstein and* $\mathrm{proj.dim}_{L(A)}\, i_\lambda(I_n) < \infty$.

Proof For the "only if" part, let A be Gorenstein with v.dim $A = d$. Recall that each $L(A)$-module Y is of the form $i_\lambda(X)$ for an A-module X. Take an exact sequence $0 \to G \to P^{-(d-1)} \to \cdots \to P^{-1} \to P^0 \to X \to 0$ of A-modules with each P^{-j} projective and $G \in A$-Gproj. We apply the functor i_λ to this sequence. By

Proposition 2.6 and Lemma 2.7, we get an exact sequence for Y, which implies that $L(A)$ is Gorenstein with v.dim $L(A) \leq d$. Observe that I_n has finite projective dimension, and then by Lemma 2.4(1) we have $\text{proj.dim}_{L(A)}\, i_\lambda(I_n) < \infty$.

For the "if" part, assume that $L(A)$ is Gorenstein and $\text{proj.dim}_{L(A)}\, i_\lambda(I_n) < \infty$. For the Gorensteinness of A, it suffices to show that the regular A-module ${}_AA$ has finite injective dimension and each injective A-module I_j has finite projective dimension.

We observe that $i_\lambda(I_n)$ has finite injective dimension and so does the A-module $ii_\lambda(I_n)$. Here, we recall that i sends injective $L(A)$-modules to injective A-modules. We observe that the exact sequence 2.1 for I_n has the form

$$0 \longrightarrow S_n \longrightarrow I_n \longrightarrow ii_\lambda(I_n) \longrightarrow 0.$$

Here, we use that $t(I_n) \simeq S_n$ and $t'(I_n) = 0$. It follows that S_n has finite injective dimension. Consider the regular A-module ${}_AA$. Since $i_\lambda(A) = L(A)$ has finite injective dimension, the A-module $ii_\lambda(A)$ has finite injective dimension. Then the exact sequence 2.1 for ${}_AA$ implies that ${}_AA$ has finite injective dimension.

By Lemma 2.4(1) the A-module I_n has finite projective dimension. It remains to prove that for each $1 \leq j \leq n-1$, the A-module I_j has finite projective dimension. Proposition 2.2(4) implies that the $L(A)$-module $i_\lambda(I_j)$ is injective. Since $L(A)$ is Gorenstein, $i_\lambda(I_j)$ has finite projective dimension. Then applying Lemma 2.4(1), we are done. □

We observe the following consequence.

Corollary 2.9 *Let S and S' be two non-isomorphic localizable A-modules. Denote by $\eta_A\colon A \to L(A)$ and $\eta'_A\colon A \to L'(A)$ the corresponding left retractions. Then A is Gorenstein if and only if both $L(A)$ and $L'(A)$ are Gorenstein.*

Proof The "only if" part follows from Theorem 2.8.

For the "if" part, assume that $S_n = S$ and $S_1 = S'$. Consider the injective envelop I_n for S. Applying Proposition 2.2(4) to S', we have that the $L'(A)$-module $i'_\lambda(I_n)$ is injective and then has finite projective dimension. By Lemma 2.4(1) for S', the A-module I_n has finite projective dimension. It follows from Lemma 2.4(1) for S that $i_\lambda(I_n)$ has finite projective dimension. Applying Theorem 2.8 for $S_n = S$, we are done. □

2.6 Recollements and Singular Equivalences

We will show that a localizable module induces a recollement [1] of derived categories and a triangle equivalence between singularity categories.

For an artin algebra A, denote by $\mathbf{D}^b(A\text{-mod})$ the bounded derived category of the module category A-mod. Recall that A-mod embeds into $\mathbf{D}^b(A\text{-mod})$ by identifying an A-module with the stalk complex concentrated on degree zero. For an exact functor $F\colon A\text{-mod} \to A'\text{-mod}$ between module categories, we denote by $F^*\colon \mathbf{D}^b(A\text{-mod}) \to \mathbf{D}^b(A'\text{-mod})$ its natural extension on complexes.

Let S be a localizable A-module. Consider the left localization $\eta_A\colon A \to L(A)$. By identifying $S^\perp$ with $L(A)$-mod, the functor $i\colon A\text{-mod} \to S^\perp$ identifies with

$\mathrm{Hom}_{L(A)}(L(A), -)$, which is further isomorphic to $L(A) \otimes_{L(A)} -$. It follows that i admits a right adjoint $i_\rho = \mathrm{Hom}_A(L(A), -)$. Recall the equivalence add $S \simeq \Delta$-mod with $\Delta = \mathrm{End}_A(S)^{\mathrm{op}}$.

Lemma 2.10 *Keep the notation as above. Then we have a recollement of derived categories*

$$\mathbf{D}^b(L(A)\text{-mod}) \underset{\mathbf{R}^b i_\rho}{\overset{i^*_\lambda}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{i^*}}}}} \mathbf{D}^b(A\text{-mod}) \underset{}{\overset{\mathrm{inc}^*}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\mathbf{R}^b t}}}}} \mathbf{D}^b(\Delta\text{-mod})$$

We mention that both functors $i_\rho\colon A\text{-mod} \to L(A)\text{-mod}$ and $t\colon A\text{-mod} \to \text{add } S \simeq \Delta\text{-mod}$ are left exact. The notation $\mathbf{R}^b$ means the right (bounded) derived functor.

Proof The proof is similar to [11, Proposition 3.3.2]. Here, it suffices to note that $\mathbf{R}^b i_\rho$ is well defined, since the left A-module $L(A)$ has finite projective dimension; moreover, $\mathbf{R}^b i_\rho$ is right adjoint to i^*. Similar remarks hold for t, since $t \simeq \mathrm{Hom}_A(S, -)$. □

For an artin algebra A, the *singularity category* $\mathbf{D}_{\mathrm{sg}}(A)$ is the Verdier quotient category of $\mathbf{D}^b(A\text{-mod})$ by the triangulated subcategory perf(A) formed by perfect complexes; see [22, (Buchweitz R. O, Maximal Cohen-Macaulay Modules and Tate-Cohomology over Gorenstein Rings, Unpublished)]. Here, a bounded complex of A-modules is *perfect* provided that it is isomorphic to a bounded complex of projective A-modules in $\mathbf{D}^b(A\text{-mod})$. The triangulated subcategory perf(A) is *thick*, that is, it is closed under taking direct summands.

Consider the following composite of functors

$$G_A\colon A\text{-Gproj} \hookrightarrow A\text{-mod} \longrightarrow \mathbf{D}^b(A\text{-mod}) \longrightarrow \mathbf{D}_{\mathrm{sg}}(A),$$

where from the left side, the first functor is inclusion, the second identifies modules with stalk complexes concentrated on degree zero, and the last is the quotient functor. Observe that the additive functor G_A vanishes on projective modules and then induces uniquely an additive functor $A\text{-}\underline{\mathrm{Gproj}} \to \mathbf{D}_{\mathrm{sg}}(A)$, which is still denoted by G_A.

We recall the following fundamental result.

Lemma 2.11 *The functor $G_A\colon A\text{-}\underline{\mathrm{Gproj}} \to \mathbf{D}_{\mathrm{sg}}(A)$ is a fully faithful triangle functor. Moreover, the algebra A is Gorenstein if and only if G_A is dense and thus a triangle equivalence. In particular, if A is self-injective, we have a triangle equivalence $\mathbf{D}_{\mathrm{sg}}(A) \simeq A\text{-}\underline{\mathrm{mod}}$.*

Proof The result is due to Buchweitz, (Maximal Cohen-Macaulay Modules and Tate-Cohomology over Gorenstein Rings, Unpublished), Theorem 4.4.1 and independently due to Happel [15, Theorem 4.6]. We mention that the "if" part of the second statement follows from [5, Theorem 6.9(8)].

For a self-injective algebra A, we have $A\text{-Gproj} = A\text{-mod}$. Then the final statement, which is also due to [24, Theorem 2.1], follows. □

Recall that a nontrivial triangulated category $\mathcal{T}$ is *minimal* provided that it has no nontrivial thick subcategories. Lemma 2.11 implies that $A\text{-}\underline{\text{Gproj}}$ is a thick subcategory of $\mathbf{D}_{\text{sg}}(A)$. Then we have the following consequence.

Corollary 2.12 *Let A be a non-Gorenstein algebra such that $\mathbf{D}_{\text{sg}}(A)$ is minimal. Then A is CM-free.* □

Recall from [10] that a *singular equivalence* between two algebras A and A' means a triangle equivalence between $\mathbf{D}_{\text{sg}}(A)$ and $\mathbf{D}_{\text{sg}}(A')$. We observe that a left retraction induces a singular equivalence. The equivalence might be viewed as an enhancement of the isomorphism in Proposition 2.2(5). Here, we recall that for an artin algebra A, the Grothendieck group $K_0(\mathbf{D}_{\text{sg}}(A))$ of its singularity category $\mathbf{D}_{\text{sg}}(A)$ is isomorphic to Cok C_A; see [15, 4.1]. In view of the remarks after Lemma 2.1, the following result might be deduced from [8, Theorem 2.1].

Proposition 2.13 *Let $\eta_A\colon A \to L(A)$ be a left retraction associated to a localizable A-module S. Identify $S^{\perp}$ with $L(A)$-mod. Then the functors i_λ^* and i^* induce mutually inverse triangle equivalences between $\mathbf{D}_{\text{sg}}(A)$ and $\mathbf{D}_{\text{sg}}(L(A))$.*

Proof Consider the triangle functor $i_\lambda^*\colon \mathbf{D}^b(A\text{-mod}) \to \mathbf{D}^b(L(A)\text{-mod})$. Denote by thick$\langle S\rangle$ the smallest thick subcategory of $\mathbf{D}^b(A\text{-mod})$ containing S. Observe that thick$\langle S\rangle \subseteq$ perf(A), since proj.dim$_A$ $S \leq 1$. By Lemma 2.4(1), we infer that $i_\lambda^*(\text{perf}(A)) = \text{perf}(L(A))$.

By the recollement in Lemma 2.10, we infer that i_λ^* induces a triangle equivalence $\mathbf{D}^b(A\text{-mod})/\text{thick}\langle S\rangle \simeq \mathbf{D}^b(L(A)\text{-mod})$, which restricts to an equivalence $\text{perf}(A)/\text{thick}\langle S\rangle \simeq \text{perf}(L(A))$. Then we are done with a canonical triangle equivalence in [28, Chaptre I, §2, 4–3 Corollaire]. □

3 Nakayama Algebras

In this section, we recall from [13] some homological properties of Nakayama algebras, and introduce the notion of θ-perfect element that relates to Gorenstein projective modules. We apply the results in the previous section to prove that for a connected Nakayama algebra A, there is a connected self-injective Nakayama algebra A' such that there is a sequence of left retractions linking A to A'. Consequently, the singularity category of A is triangle equivalent to the stable category of A'. We classify Nakayama algebras with at most three simple modules according to the trichotomy: Gorenstein, non-Gorenstein CM-free, non-Gorenstein but not CM-free. It turns out that there is a class of such algebras, that are non-Gorenstein but not CM-free.

3.1 Nakayama Algebras and Homological Properties

Let A be an artin algebra. An A-module is uniserial provided that it has a unique composition series. Recall that A is *Nakayama* provided that all indecomposable projective and all indecomposable injective A-modules are uniserial, or equivalently, all indecomposable A-modules are uniserial.

Assume that A is connected, that is, it does not admit a decomposition as a direct sum of two proper ideals. Then A is Nakayama if and only if its valued quiver is a linear quiver with trivial valuation

$$S_1 \longrightarrow S_2 \longrightarrow \cdots \longrightarrow S_n$$

or an oriented cycle with trivial valuation

$$S_1 \rightleftarrows S_2 \longrightarrow \cdots \longrightarrow S_n.$$

Here, $\{S_1, S_2, \cdots, S_n\}$ is a complete set of representatives of pairwise non-isomorphic simple A-modules, and $n = n(A)$ is the number of pairwise non-isomorphic simple A-modules. In the first case, A is called a *line algebra*; in the second case, A is a *cycle algebra*; compare [20, 30]. Observe that a line algebra A has finite global dimension; indeed, gl.dim $A \le n(A)$.

By an *admissible sequence* of length n we mean a sequence $\mathbf{c} = (c_1, c_2, \cdots, c_n)$ of n positive integers subject to the conditions $2 \le c_j \le c_{j+1} + 1$ for $j = 1, 2, \cdots, n-1$ and $c_n \le c_1 + 1$. If $c_n \ge 2$, all cyclic permutations of $\mathbf{c}$ are admissible. An admissible sequence $\mathbf{c}$ is *normalized* provided that $c_n = 1$, or $c_1 = c_2 = \cdots = c_n$, or c_1 is minimal among all c_j's and $c_n = c_1 + 1$.

For a connected Nakayama algebra A, we order its simple modules as above. Denote by P_j the projective cover of S_j. Then we have exact sequences $P_{j+1} \to P_j \to S_j \to 0$ for $j = 1, 2, \cdots, n-1$. For a line algebra, we have $P_n \simeq S_n$, while for a cycle algebra we have an exact sequence $P_1 \to P_n \to S_n \to 0$. Hence, the Nakayama algebra A corresponds to an admissible sequence $\mathbf{c}(A) = (l(P_1), l(P_2), \cdots, l(P_n))$. Moreover, by cyclic permutations, we may always get a normalized admissible sequence. Recall that A is self-injective if and only if $l(P_1) = l(P_2) = \cdots = l(P_n)$.

An indecomposable A-module X is uniquely determined by its top top(X) and its composition length $l = l(X)$. Here, the top top$(X) = X/\text{rad } X = S$ is simple, and this unique module X is denoted by $S^{[l]}$. Then a complete set of representatives of pairwise non-isomorphic indecomposable A-modules is given by $\{S_j^{[l]} \mid 1 \le j \le n,\ l \le c_j\}$; moreover, the Auslander-Reiten quiver of A is described in [3, VI. 2].

In what follows, we recall the minimal projective resolution of an indecomposable module. Recall that $\mathbf{c}(A) = (c_1, c_2, \cdots, c_n)$ is the admissible sequence of A. Following [13], we introduce a map $\theta\colon \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$ such that $\theta(j) = \phi_n(j + c_j)$. Here, for each positive integer x, $\phi_n(x)$ is uniquely determined by the conditions that $1 \le \phi_n(x) \le n$ and n divides $x - \phi_n(x)$; compare [20, Section 3]. Then we have a descending chain $\{1, 2, \cdots, n\} = \text{Im } \theta^0 \supseteq \text{Im } \theta \supseteq \text{Im } \theta^2 \supseteq \text{Im } \theta^3 \supseteq \cdots$. There is a minimal integer $d(A)$ such that $\text{Im } \theta^{d(A)} = \text{Im } \theta^{d(A)+1}$. Elements in $\text{Im } \theta^{d(A)}$ are called θ*-regular elements*. Observe that $0 \le d(A) \le n(A) - 1$, and that $d(A) = 0$ if and only if A is self-injective.

For an A-module X, denote by soc(X) the socle of X.

Lemma 3.1 [13] *Keep the notation as above. Then we have the following statements:*

(1) soc$(P_j) = S_{\theta(j)-1}$ *for each j, where we identify* 0 *with n;*
(2) *for a nonzero homomorphism $f\colon P_j \to P_k$, we have that $\theta(j) = \theta(k)$ if f is mono,* top(Ker f) $= S_{\theta(k)}$ *otherwise.*

Proof We use the fact that each indecomposable projective module P_j is uniserial, and that from the top, the m-th composition factor in its composition series is $S_{\phi_n(j+m-1)}$. □

We have the following immediate consequence on minimal projective resolutions of indecomposable modules over a Nakayama algebra.

Corollary 3.2 [13, (5)] *Let X be an indecomposable non-projective A-module. Assume that $X = S_j^{[l]}$ and that $k = \phi_n(j+l)$. Then we have the following statements:*

(1) *if* $\text{proj.dim}_A\, X = 2m$ *for* $m \geq 1$*, then there is an exact sequence*

$$0 \to P_{\theta^{m-1}(k)} \to P_{\theta^{m-1}(j)} \to \cdots \to P_{\theta(k)} \to P_{\theta(j)} \to P_k \to P_j \to X \to 0;$$

(2) *if* $\text{proj.dim}_A\, X = 2m+1$ *for* $m \geq 0$*, then there is an exact sequence*

$$0 \to P_{\theta^m(j)} \to P_{\theta^{m-1}(k)} \to P_{\theta^{m-1}(j)} \to \cdots \to P_{\theta(k)} \to P_{\theta(j)} \to P_k \to P_j \to X \to 0;$$

(3) *if* $\text{proj.dim}_A\, X = \infty$*, then there is an exact sequence*

$$\cdots \to P_{\theta^m(k)} \to P_{\theta^m(j)} \to \cdots \to P_{\theta(k)} \to P_{\theta(j)} \to P_k \to P_j \to X \to 0.$$

Proof Take an epimorphism $P_j \xrightarrow{f} X$. We observe that $\text{top}(\text{Ker } f) = S_k$, and thus we have an exact sequence $P_k \to P_j \to X \to 0$. Then we infer the existence of these sequences by applying Lemma 3.1(2) repeatedly. □

The following result is essentially contained in the proof of [13, Theorem (i)].

Corollary 3.3 *Let A be a connected Nakayama algebra with $d(A)$ defined as above. Then we have* $\text{fin.dim}\, A \leq 2d(A) \leq 2n(A) - 2$.

Proof We assume the converse, and take X to be an indecomposable A-module with $2d(A) < \text{proj.dim}_A\, X < \infty$. We use the same notation as in Corollary 3.2. If $\text{proj.dim}_A\, X = 2m$ with $m \geq d(A) + 1$, then both $\theta^{m-1}(k)$ and $\theta^{m-1}(j)$ are θ-regular. By Lemma 3.1(2) we have $\theta^m(k) = \theta^m(j)$. This implies that $\theta^{m-1}(k) = \theta^{m-1}(j)$, since θ induces a bijection on the set of θ-regular elements. This equality is absurd, since we have a proper monomorphism from $P_{\theta^{m-1}(k)}$ to $P_{\theta^{m-1}(j)}$. A similar argument works for the case $\text{proj.dim}_A\, X = 2m + 1$. □

For a nonzero morphism $f\colon P_j \to P_k$ between two indecomposable projective A-modules, we defines its *valuation* $\nu(f)$ as follows. Take two idempotents e_j and e_k in A such that $P_j \simeq Ae_j$ and $P_k \simeq Ae_k$. Then we have a natural isomorphism $\text{Hom}_A(P_j, P_k) \simeq e_j A e_k$, which sends f to $f(e_j)$. There is a unique integer p such that $f(e_j) \in \text{rad}^p\, A$ and $f(e_j) \notin \text{rad}^{p+1}\, A$. We define $p = \nu(f)$.

Lemma 3.4 *Let $f\colon P_j \to P_k$ be a nonzero morphism as above and let $P_{k'}$ be an indecomposable projective A-module. Then we have the following statements:*

(1) $\nu(f) = \nu(f^*)$*, where* $(-)^* = \text{Hom}_A(-, A)$*;*
(2) $l(\text{Cok } f) = \nu(f)$*,* $l(\text{Im } f) = l(P_k) - \nu(f)$ *and* $l(\text{Ker } f) = l(P_j) - l(P_k) + \nu(f)$*;*

(3) *for any nonzero homomorphism* $g\colon P_k \to P_{k'}$, *we have that* $\xi\colon P_j \xrightarrow{f} P_k \xrightarrow{g} P_{k'}$ *is exact if and only if* $l(P_{k'}) = \nu(f) + \nu(g)$;
(4) *if both the above sequence* ξ *and its dual* ξ^* *are exact, then* $l(P_{k'}) = l(P_j^*)$.

Proof Recall that $P_j^* \simeq e_jA$ and $P_k^* \simeq e_kA$. Hence, we have the isomorphism $\mathrm{Hom}_{A^{\mathrm{op}}}(P_k^*, P_j^*) \simeq e_jAe_k$ which sends f^* to $f(e_j)$. Then (1) follows.

For (2), we observe that $\mathrm{Im}\ f = \mathrm{rad}^{\nu(f)}\ P_k$. It follows that $l(\mathrm{Cok}\ f) = \nu(f)$, and then we have the remaining equalities.

For (3), we note that the sequence is exact if and only if $l(\mathrm{Im}\ f) = l(\mathrm{Ker}\ g)$. Applying (2) to f and g, we have the result.

By (3), the exactness of ξ^* implies that $l(P_j^*) = \nu(g^*) + \nu(f^*)$. Using $\nu(g^*) = \nu(g)$ and $\nu(f^*) = \nu(f)$, we are done. □

The following notion is related to Gorenstein projective modules over a Nakayama algebra. A θ-regular element j is called *θ-perfect* provided that $l(P_{\theta^m(j)}) = l(P^*_{\theta^{m+1}(j)})$ for all integers m. Here, we recall that θ induces a bijection on the set of θ-regular elements, on which θ^{-1} is well defined. For a θ-regular element j, $\theta^m(j)$ is θ-perfect for any integer m.

Proposition 3.5 *Let X be an indecomposable Gorenstein projective A-module which is non-projective. Assume that $X = S_j^{[l]}$ and that $k = \phi_n(j+l)$. Then the following statements hold:*

(1) *both j and k are θ-perfect;*
(2) *there is a complete resolution of X as follows*

$$\cdots \to P_{\theta(k)} \to P_{\theta(j)} \to P_k \to P_j \to P_{\theta^{-1}(k)} \to P_{\theta^{-1}(j)} \to \cdots .$$

Proof We have the projective resolution of X as in Corollary 3.2(3). By Lemma 2.5 there is an exact sequence $0 \to X \to P^1 \to X^1 \to 0$ such that P^1 is projective and that X^1 is an indecomposable Gorenstein projective module which is non-projective; moreover, the morphism $P^1 \to X^1$ is a projective cover, and this implies that P^1 is indecomposable. Repeating this argument, we obtain a long exact sequence $0 \to X \to P^1 \xrightarrow{d^1} P^2 \xrightarrow{d^2} P^3 \to \cdots$ such that each P^j is indecomposable projective and each $X^j = \mathrm{Im}\ d^j$ is Gorenstein projective. Then we have the complete resolution of X

$$\cdots \to P_{\theta(k)} \to P_{\theta(j)} \to P_k \to P_j \to P^1 \to P^2 \to \cdots .$$

Here, we recall that an acyclic complex of projective modules is totally acyclic if and only if all its cocycles are Gorenstein projective.

In the above complete resolution, we have the minimal projective resolution of each X^j. In view of Corollary 3.2(3) for X^j's, we have that both j and k are θ-regular, moreover, $P^{2m-1} = P_{\theta^{-m}(k)}$ and $P^{2m} = P_{\theta^{-m}(j)}$ for $m \geq 1$, and then we have (2). The dual complex of the above resolution is acyclic. Then (1) follows from Lemma 3.4(4). □

We observe the following immediate consequence of Proposition 3.5.

Corollary 3.6 *Let A be a connected Nakayama algebra without θ-perfect elements. Then A is CM-free.* □

3.2 Left Retraction Sequence

Let A be a connected Nakayama algebra, which is not self-injective. Let $\mathbf{c}(A) = (c_1, c_2, \cdots, c_n)$ be its normalized admissible sequence. If $c_n = 1$, the simple A-module S_n is projective. Otherwise, $c_n = c_1 + 1$ and then S_n is localizable with projective dimension one, since we have an exact sequence $0 \to P_1 \to P_n \to S_n \to 0$. In both cases, we have that S_n is localizable. Consider the left retraction $\eta_A : A \to L(A)$ associated to S_n. We have the following result, a part of which is similar to [21, Lemmas 7 and 8]; compare [7, Section 2].

For an admissible sequence $\mathbf{c}$ of length n, set $\mathbf{c}' = (c'_1, c'_2, \cdots, c'_{n-1})$ such that $c'_j = c_j - [\frac{c_j+j-1}{n}]$. Here, for a real number x, $[x]$ denotes the largest integer that is not strictly larger than x. Then $\mathbf{c}'$ is an admissible sequence of length $n-1$.

Lemma 3.7 *Keep the notation as above. Then we have the following statements:*

(1) *the functor $i_\lambda : A$-mod $\to L(A)$-mod sends indecomposable modules, that are not isomorphic to S_n, to indecomposable modules; more precisely, we have that $i_\lambda(S_j^{[l]}) = (i_\lambda(S_j))^{[l']}$ with $1 \leq j \leq n-1$ and $l' = l - [\frac{l+j-1}{n}]$, and that $i_\lambda(S_n^{[l]}) = (i_\lambda(S_1))^{[l']}$ with $l' = l - 1 - [\frac{l-1}{n}]$;*
(2) *the algebra $L(A)$ is connected Nakayama with $\mathbf{c}(L(A)) = \mathbf{c}(A)'$;*
(3) *if the algebra A is cyclic, we have $d(L(A)) \leq d(A) \leq d(L(A)) + 1$; here, $d(A)$ is defined as in Section 3.1.*
(4) *$L(A)$ is a line algebra if and only if $c_{n-1} = 2$, that is equivalent to the fact that A is a line algebra or $\mathbf{c}(A) = (2, 2, \cdots, 2, 3)$.*

Proof Set $S = S_n$. We identify $S^\perp$ with $L(A)$-mod, and the functor i_λ with the quotient functor $q : A$-mod $\to A$-mod/add S; see Lemma 2.1.

For (1), it suffices to recall a general fact: given any abelian category $\mathcal{A}$ and a Serre subcategory $\mathcal{C}$, the corresponding quotient functor $q : \mathcal{A} \to \mathcal{A}/\mathcal{C}$ sends a uniserial object to a uniserial object. For any A-module X, we observe that the length of $i_\lambda(X)$ equals the length of X minus the multiplicity of S in a composition series of X.

Any indecomposable $L(A)$-module is isomorphic to the image under i_λ of some indecomposable A-module. Then it follows from (1) that the algebra $L(A)$ is Nakayama. By Lemma 2.3, the valued quiver of $L(A)$ is connected and then $L(A)$ is connected. The statement about admissible sequences in (2) follows from (1).

For (3), set $d = d(A)$ and $d' = d(L(A))$. Consider the map $\theta' : \{1, 2, \cdots, n-1\} \to \{1, 2, \cdots, n-1\}$ given by $\theta'(j) = \phi_{n-1}(j + c'_j)$. Define a surjective map $\pi : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n-1\}$ such that $\pi(n) = 1$ and $\pi(j) = j$ for $j < n$. We claim that $\pi \circ \theta = \theta' \circ \pi$. This follows from (2) and the following fact: for an integer m, we have $\phi_n(m) = \phi_{n-1}(m - [\frac{m-1}{n}])$, if $\phi_n(m) < n$; otherwise, we have $\phi_{n-1}(m - [\frac{m-1}{n}]) = 1$.

We infer from the claim that $\operatorname{Im}\theta'^m = \pi(\operatorname{Im}\theta^m)$ for any $m \geq 0$. By the definition of $d = d(A)$, we have $\operatorname{Im}\theta^d = \operatorname{Im}\theta^{d+1}$, and thus $\operatorname{Im}\theta'^d = \operatorname{Im}\theta'^{d+1}$. It follows that $d' \leq d$. By $\operatorname{Im}\theta'^{d'} = \operatorname{Im}\theta'^{d'+1}$, we infer that $\operatorname{Im}\theta^{d'}$ and $\operatorname{Im}\theta^{d'+1}$ have the same image under π. Then there are three possibilities: $\operatorname{Im}\theta^{d'} = \operatorname{Im}\theta^{d'+1}$, $\operatorname{Im}\theta^{d'} = \operatorname{Im}\theta^{d'+1} \cup \{n\}$ or $\operatorname{Im}\theta^{d'} = \operatorname{Im}\theta^{d'+1} \cup \{1\}$. In each case, we have that $\operatorname{Im}\theta^{d'+1} = \operatorname{Im}\theta^{d'+2}$. Hence, $d \leq d' + 1$. This proves (3).

For (4), we observe that $L(A)$ is a line algebra if and only if $c'_{n-1} = 1$, that is equivalent to $c_{n-1} = 2$. Recall by assumption that $\mathbf{c}(A)$ is normalized. Then we are done. □

The following result associates to any connected Nakayama algebra a self-injective one, via a sequence of left retractions.

Theorem 3.8 *Let A be a connected Nakayama algebra with $n(A)$ the number of pairwise non-isomorphic simple A-modules and C_A the Cartan matrix. Recall the notation $d(A)$ in Section 3.1. Then there is a sequence of homomorphisms between connected Nakayama algebras*

$$A = A_0 \xrightarrow{\eta_0} A_1 \xrightarrow{\eta_1} A_2 \longrightarrow \cdots \xrightarrow{\eta_{r-1}} A_r \tag{3.1}$$

such that $d(A) \leq r \leq n(A) - 1$, each η_i is a left retraction associated to some localizable module, and that A_r is self-injective. Moreover, we have

(1) *the algebra A_r is simple if and only if* gl.dim $A < \infty$*; in this case, $r = n(A) - 1$;*
(2) *if* gl.dim $A = \infty$*, the composition $\eta_{r-1} \circ \cdots \circ \eta_1 \circ \eta_0 \colon A \to A_r$ is uniquely determined by A, and $r = \sharp\{1 \leq j \leq n(A) \mid \operatorname{proj.dim}_A S_j < \infty\}$.*

We will denote by $r(A)$ the unique r of the above sequence 3.1. Hence, we have $d(A) \leq r(A) \leq n(A) - 1$. For A with infinite global dimension, $r(A)$ equals the number of Ψ-regular simple modules; see [20, Corollary 3.6].

Proof Denote by $\{S_1, S_2, \cdots, S_n\}$ a complete set of representatives of pairwise non-isomorphic simple A-modules. If A is self-injective, we set $r = 0$ and $A = A_r$. If A is not self-injective, we apply Lemma 3.7 repeatedly to obtain such a sequence. We apply Proposition 2.2(1) repeatedly to get $n(A_r) = n(A) - r \geq 1$, and then $r \leq n(A) - 1$. Since A_r is self-injective, we have $d(A_r) = 0$. Then applying Lemma 3.7(3) we have $d(A) \leq r$.

Applying Lemma 2.4(2) repeatedly, we have that gl.dim $A < \infty$ if and only if gl.dim $A_r < \infty$; this is equivalent to that A_r is simple, since it is connected self-injective. In this case, $n(A_r) = 1$, and thus $r = n(A) - 1$.

Assume that gl.dim $A = \infty$. Then each simple A_r-module has infinite projective dimension. Consider the adjoint pair (F, G), where $F = A_r \otimes_A - \colon A\text{-mod} \to A_r\text{-mod}$ and $G = \operatorname{Hom}_{A_r}(A_r, -)\colon A_r\text{-mod} \to A\text{-mod}$. By Lemma 2.1 and the remarks afterward, the right A-module A_r is projective and then F is exact; moreover, F identifies with the quotient functor $q\colon A\text{-mod} \to A\text{-mod}/\operatorname{Ker} F$, and the functor G is fully faithful. Here, Ker F is the Serre subcategory formed by A-modules on which F vanishes. It follows from [19, Proposition 2.2] that the image of the functor G is $(\operatorname{Ker} F)^{\perp} = \{X \in A\text{-mod} \mid \operatorname{Hom}_A(M, X) = 0 = \operatorname{Ext}^1_A(M, X) \text{ for all } M \in \operatorname{Ker} F\}$.

We claim that Ker $F = \langle S_j \mid \text{proj.dim}_A\ S_j < \infty, j = 1, 2, \cdots, n(A)\rangle$. Here, for a class $\mathcal{S}$ of A-modules, we denote by $\langle \mathcal{S}\rangle$ the smallest Serre subcategory containing $\mathcal{S}$. Observe that Ker $F = \langle S_j \mid S_j \in \text{Ker}\ F\rangle$. We identify F with the quotient functor q. Then for any simple A-module S_j, $F(S_j)$ is either zero or a simple A_r-module. Applying Lemma 2.4(1) repeatedly, $\text{proj.dim}_A\ S_j < \infty$ if and only if $\text{proj.dim}_{A_r}\ F(S_j) < \infty$, which is equivalent to $F(S_j) = 0$. Then we are done with the claim. We observe from the proof that $n(A_r) = n(A) - \sharp\{1 \leq j \leq n(A) \mid \text{proj.dim}_A\ S_j < \infty\}$, and thus $r = \sharp\{1 \leq j \leq n(A) \mid \text{proj.dim}_A\ S_j < \infty\}$; compare Proposition 2.2(1).

We conclude that the image of the fully faithful functor $G = \text{Hom}_A(A_r, -)$ is uniquely determined by A. Then the uniqueness of the composite homomorphism $A \to A_r$ follows from Lemma 3.9. □

Lemma 3.9 *Let $\phi\colon A \to B$ and $\phi'\colon A \to B'$ be two algebra homomorphisms of artin algebras. Assume that both functors* $\text{Hom}_A(B, -)$ *and* $\text{Hom}_A(B', -)$ *are fully faithful with the same image in A-mod. Then there is an isomorphism $\psi\colon B \to B'$ of algebras such that $\psi \circ \phi = \phi'$*

Proof This is analogous to the bijection in [18, Theorem 1.2]. □

We draw some consequences of Theorem 3.8. In the following result, the first statement is contained in [7, Theorem 6], and the second is due to [13]. By max $\mathbf{c}(A)$ we mean the maximum of $c_j = l(P_j)$ for $1 \leq j \leq n(A)$; it is the Lowey length of the algebra A.

Corollary 3.10 *Let A be a connected Nakayama algebra. Then we have the following statements:*

(1) $\text{gl.dim}\ A < \infty$ *if and only if* $\det C_A = 1$*;*
(2) *if* $\text{gl.dim}\ A < \infty$*, then* $\text{gl.dim}\ A \leq 2n(A) - 2$ *and* $\max \mathbf{c}(A) \leq 2n(A) - 1$.

Proof Consider the admissible sequence $\mathbf{c}(A_r) = (c, c, \cdots, c)$ with $n(A_r)$ copies of c. Then in the Cartan matrix C_{A_r}, the sum of all entries in each column equals c. This implies that c divides $\det C_{A_r}$. It follows that $\det C_{A_r} = 1$ if and only if $\mathbf{c}(A_r) = (1)$, that is, A_r is simple. By Proposition 2.2(5), we have $\det C_A = \det C_{A_r}$. Then (1) follows from Theorem 3.8(1).

The first half in (2) follows from Corollary 3.3. For the second half, we may assume that A is a cycle algebra with its normalized admissible sequence $\mathbf{c}(A)$. Using induction, we assume that $\max \mathbf{c}(A_1) \leq 2n(A) - 3$. Then by Lemma 3.7(2), we have that each $c_1 < 2n(A) - 1$ and $c_j \leq 2n(A) - 1$ for $2 \leq j \leq n(A) - 1$. Since $c_n = c_1 + 1$, we have $c_n \leq 2n(A) - 1$. □

Recall that A_r is self-injective. Then the stable module category A_r-$\underline{\text{mod}}$ has a canonical triangulated structure.

Corollary 3.11 *Let A be a connected Nakayama algebra. Then the composite homomorphism in Eq. 3.1 induces a triangle equivalence*

$$\mathbf{D}_{\text{sg}}(A) \simeq A_r\text{-}\underline{\text{mod}}.$$

It follows that the singularity category $\mathbf{D}_{sg}(A)$ is Krull-Schmidt; its Auslander-Reiten quiver is a truncated tube of rank $n(A) - r(A)$. Moreover, it is a homogeneous tube, that is, a tube of rank one, if and only if $n(A_r) = 1$ and A_r is not simple.

Proof Recall from Lemma 2.11 the triangle equivalence $\mathbf{D}_{sg}(A_r) \simeq A_r$-$\underline{\text{mod}}$. Then the above triangle equivalence follows from Proposition 2.13. □

3.3 Nakayama Algebras with at Most Three Simples

Let A be a connected Nakayama algebra with two simple modules, that is, $n(A) = 2$. We may assume that A is a cycle algebra which is not self-injective. This implies that its normalized admissible sequence is given by $\mathbf{c}(A) = (c, c+1)$ for $c \geq 2$.

Lemma 3.12 *Keep the assumption as above. Then there are two cases:*

(1) *if* $\mathbf{c}(A) = (2k, 2k+1)$ *for* $k \geq 1$*, then* A *is Gorenstein with* v.dim $A = 2$*;* A *has finite global dimension if and only if* $k = 1$*;*
(2) *if* $\mathbf{c}(A) = (2k+1, 2k+2)$ *for* $k \geq 1$*, then* A *is non-Gorenstein CM-free with* fin.dim $A = 1$.

Proof We consider the retraction $\eta\colon A \to L(A)$ associated to S_2. Observe that $L(A)$ has a unique simple module with $\mathbf{c}(L(A)) = ([\frac{c}{2}])$, and that $L(A)$ is self-injective; moreover, the stable category $L(A)$-$\underline{\text{mod}}$, as a triangulated category, is minimal. By Corollary 3.11, the singularity category $\mathbf{D}_{sg}(A)$ is minimal; here $A_r = L(A)$. Then A has finite global dimension if and only if $L(A)$ is simple, that is, $[\frac{c}{2}] = 1$. This is equivalent to that $\mathbf{c}(A) = (2, 3)$.

For (1), we have that the length of the indecomposable $L(A)$-module $i_\lambda(I_2)$ equals $k = [\frac{c}{2}]$, and it is projective. By Theorem 2.8 A is Gorenstein. The same reasoning yields that the algebra A in (2) is non-Gorenstein. In this case, by the minimality of $\mathbf{D}_{sg}(A)$, we infer that A is CM-free; see Corollary 2.12. □

In what follows, we assume that A is a cycle algebra with $n(A) = 3$ which is not self-injective. Then we may assume that the normalized admissible sequence is $\mathbf{c}(A) = (c, c+j, c+1)$ with $c \geq 2$ and $j = 0, 1, 2$.

Corollary 3.13 *Keep the assumption as above. Then* A *has finite global dimension if and only if its normalized admissible sequence* $\mathbf{c}(A)$ *is* $(2, 2, 3)$*,* $(2, 4, 3)$*,* $(3, 4, 4)$ *or* $(3, 5, 4)$. *In this case, the global dimension of* A *equals* 3*,* 2*,* 4 *and* 2*, respectively.*

Proof Recall that A has finite global dimension if and only if so does $L(A)$. This is equivalent to that $\mathbf{c}(L(A))$ is $(2, 1)$ or $(2, 3)$ up to cyclic permutations; see Lemma 3.12(1). Applying $\mathbf{c}(L(A)) = \mathbf{c}(A)'$ in Lemma 3.7(2), we infer the result immediately. □

The following result classifies connected Nakayama algebras with three simple modules according to the trichotomy: Gorenstein, non-Gorenstein CM-free, non-Gorenstein but not CM-free. We mention that this result is obtained independently by Ringel with a different method; see [25, Section 6].

Proposition 3.14 *Let A be a cycle algebra with $n(A) = 3$ which is not self-injective. Denote by $\mathbf{c}(A)$ its normalized admissible sequence. Then we have the following statements:*

(1) *the algebra A is Gorenstein if and only if $\mathbf{c}(A) = (2, 2, 3)$, $(2, 4, 3)$, $(3k, 3k, 3k+1)$, $(3k, 3k+1, 3k+1)$, $(3k, 3k+2, 3k+1)$ or $(3k+1, 3k+2, 3k+2)$ for $k \geq 1$;*
(2) *the algebra A is non-Gorenstein CM-free if and only if $\mathbf{c}(A) = (2, 3, 3)$, $(3k+1, 3k+1, 3k+2)$, $(3k+1, 3k+3, 3k+2)$, $(3k+2, 3k+2, 3k+3)$ or $(3k+2, 3k+4, 3k+3)$ for $k \geq 1$;*
(3) *the algebra A is non-Gorenstein, but not CM-free if and only if $\mathbf{c}(A) = (3k+2, 3k+3, 3k+3)$ for $k \geq 1$; in this case, all indecomposable non-projective Gorenstein projective A-modules are given by $S_2^{[3m]}$ for $1 \leq m \leq k$.*

In case (1), the virtual dimension v.dim A *equals* 3, 2, 2, 4, 2 *and* 2*, respectively.*

Proof It suffices to prove the "if" part of all the statements. For (1), by Corollary 3.13 if $\mathbf{c}(A) = (2, 2, 3)$ or $(2, 4, 3)$, the algebra A has finite global dimension, and thus it is Gorenstein. We consider the case $\mathbf{c}(A) = (3k, 3k, 3k+1)$ for $k \geq 1$. Then the left retraction $L(A)$ of A with respect to S_3 satisfies that $\mathbf{c}(L(A)) = (2k, 2k)$, and thus $L(A)$ is self-injective. Observe that I_3 is projective. Then the $L(A)$-module $i_\lambda(I_3)$ is projective. By Theorem 2.8 the algebra A is Gorenstein. Similar argument works for the remaining three cases.

Recall that a CM-free algebra is non-Gorenstein if and only if it has infinite global dimension. For (2), assume that $\mathbf{c}(A) = (3k+1, 3k+3, 3k+2)$ with $k \geq 1$. Then we have $\mathbf{c}(L(A)) = (2k+1, 2k+2)$ and by Lemma 3.12, $L(A)$ is CM-free of infinite global dimension. Thus by Propositions 2.6 and Lemma 2.4(2), A is CM-free of infinite global dimension. Similar argument works for the case $\mathbf{c}(A) = (3k+2, 3k+2, 3k+3)$ or $(3k+2, 3k+4, 3k+3)$.

We consider the case $\mathbf{c}(A) = (3k+1, 3k+1, 3k+2)$ for $k \geq 1$. Then we have $\mathbf{c}(L(A)) = (2k+1, 2k+1)$, and thus $L(A)$ is self-injective and has infinite global dimension. So by Lemma 2.4(2), the algebra A has infinite global dimension. Observe that the set of θ-regular elements is $\{2, 3\}$, and θ sends 2 to 3, and 3 to 2; moreover, $l(P_2^*) = 3k+1$ and $l(P_3^*) = 3k+1$, where $(-)^* = \mathrm{Hom}_A(-, A)$. Then we infer that there are no θ-perfect elements. By Corollary 3.6 the algebra A is CM-free. The only remaining case in (2) is $(2, 3, 3)$, which follows from the following argument for (3) (take k to be zero).

For (3), we assume that $\mathbf{c}(A) = (3k+2, 3k+3, 3k+3)$. Then we have $\mathbf{c}(L(A)) = (2k+2, 2k+2)$, and thus $L(A)$ is self-injective. Observe that $I_3 = S_2^{[3k+2]}$. It follows that the length of $i_\lambda(I_3)$ is $2k+1$; see Lemma 3.7(1). Hence, the $L(A)$-module $i_\lambda(I_3)$ is not projective and thus has infinite projective dimension. It follows from Theorem 3.8 that A is non-Gorenstein. Observe that the modules $S_2^{[3m]}$ are Gorenstein projective, whose complete resolution is periodic as follows

$$\cdots \to P^2 \xrightarrow{f} P^2 \xrightarrow{g} P^2 \xrightarrow{f} P^2 \xrightarrow{g} P^2 \to \cdots .$$

Here, we have that $\nu(f) = 3(k-m+1)$ and $\nu(g) = 3m$.

We claim that any indecomposable Gorenstein projective A-module X, that is not projective, is of the form $S_2^{[3m]}$. Indeed, the set of θ-regular elements are $\{2, 3\}$, on which θ acts as the identity. Observe that $l(P_2^*) = 3k+3$ and $l(P_3^*) = 3k+2$. Then the only θ-perfect element is 2. It follows from Proposition 3.5 that X fits into an exact sequence $P^2 \to P^2 \to X \to 0$. This implies that X is isomorphic to $S_2^{[3m]}$ for $1 \leq m \leq k$. Then we are done. □

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