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The Algebra of Polynomial Integro-Differential Operators is a Holonomic Bimodule over the Subalgebra of Polynomial Differential Operators

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Abstract In contrast to its subalgebra $A_n := K\langle x_1, \dots, x_n, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\rangle$ of polynomial differential operators (i.e. the n'th Weyl algebra), the algebra $\mathbb{I}_n := K\langle x_1, \dots, x_n, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \int_1, \dots, \int_n\rangle$ of polynomial integro-differential operators is neither left nor right Noetherian algebra; moreover it contains infinite direct sums of nonzero left and right ideals. It is proved that $\mathbb{I}_n$ is a left (right) coherent algebra iff $n = 1$; the algebra $\mathbb{I}_n$ is a *holonomic A_n-bimodule* of length 3^n and has multiplicity 3^n with respect to the filtration of Bernstein, and all 3^n simple factors of $\mathbb{I}_n$ are pairwise non-isomorphic A_n-bimodules. The socle length of the A_n-bimodule $\mathbb{I}_n$ is $n+1$, the socle filtration is found, and the m'th term of the socle filtration has length $\binom{n}{m}2^{n-m}$. This fact gives a new canonical form for each polynomial integro-differential operator. It is proved that the algebra $\mathbb{I}_n$ is the maximal left (resp. right) order in the largest left (resp. right) quotient ring of the algebra $\mathbb{I}_n$.

Keywords The algebra of polynomial integro-differential operators · The Weyl algebra · The socle · The socle length

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1 Introduction

Throughout, ring means an associative ring with one; module means a left module; $\mathbb{N} := \{0, 1, \dots\}$ is the set of natural numbers; K is a field of characteristic zero and K^* is its group of units; $P_n := K[x_1, \dots, x_n]$ is a polynomial algebra over

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K; $\partial_1 := \frac{\partial}{\partial x_1}, \dots, \partial_n := \frac{\partial}{\partial x_n}$ are the partial derivatives (K-linear derivations) of P_n; $\mathrm{End}_K(P_n)$ is the algebra of all K-linear maps from P_n to P_n; the subalgebras $A_n := K\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n\rangle$ and $\mathbb{I}_n := K\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n, \int_1, \dots, \int_n\rangle$ of the algebra $\mathrm{End}_K(P_n)$ are called the n'th *Weyl* algebra and the *algebra of polynomial integro-differential operators* respectively.

The Weyl algebras A_n are Noetherian algebras and domains. The algebras $\mathbb{I}_n$ are neither left nor right Noetherian and not domains. Moreover, they contain infinite direct sums of nonzero left and right ideals [1, 2]. The algebra A_n is isomorphic to its opposite algebra A_n^{op} via the K-algebra *involution*:

$$A_n \to A_n, \quad x_i \mapsto \partial_i, \quad \partial_i \mapsto x_i, \quad i = 1, \dots, n.$$

Therefore, every A_n-bimodule is a left A_{2n}-module and vice versa. *Inequality of Bernstein* [6] states that *each nonzero finitely generated A_n-module has Gelfand-Kirillov dimension which is greater or equal to n*. A finitely generated A_n-module is *holonomic* if it has Gelfand-Kirillov dimension n. The holonomic A_n-modules share many pleasant properties. In particular, all holonomic modules have finite length, each nonzero submodule and factor module of a holonomic module is holonomic. The aim of the paper is to prove Theorem 2.5. In particular, to show that the algebra $\mathbb{I}_n$ is a holonomic A_n-bimodule of length 3^n and has multiplicity 3^n, i.e. a holonomic left A_{2n}-module of length 3^n and has multiplicity 3^n with respect to the filtration of Bernstein. All 3^n simple factors of $\mathbb{I}_n$ are pairwise non-isomorphic A_n-bimodules. We also found the socle filtration of the A_{2n}-module $\mathbb{I}_n$. It turns out that the socle length of the A_{2n}-module is $n+1$, and the length, as an A_{2n}-module, of the m'th socle factor is $\binom{n}{m}2^{n-m}$ (Theorem 2.5.(4)) where $m = 0, 1, \dots, n$. A new K-basis for the algebra $\mathbb{I}_n$ is found which gives a new canonical form for each polynomial integro-differential operator, see Eq. 16. By the very definition,

$$\mathbb{I}_n := \bigotimes_{i=1}^{n} \mathbb{I}_1(i) \simeq \mathbb{I}_1^{\otimes n} \quad \text{where} \quad \mathbb{I}_1(i) := K\left\langle x_i, \partial_i, \int_i \right\rangle;$$

$$A_n := \bigotimes_{i=1}^{n} A_1(i) = A_1^{\otimes n} \quad \text{where} \quad A_1(i) := K\langle x_i, \partial_i\rangle.$$

So, the properties of the algebras $\mathbb{I}_n$ and A_n are 'determined' by the properties of the algebras $\mathbb{I}_1$ and A_1.

At the beginning of Section 2 we collect necessary facts on the algebras $\mathbb{I}_n$. Then we prove Theorem 2.5 in the case when $n = 1$ and prove some necessary results that are used in the proof of Theorem 2.5 (in the general case) which is given at the end of the section.

In Section 3, it is proved that the algebra $\mathbb{I}_n$ is left (right) coherent iff $n = 1$ (Theorem 3.1).

In Section 4, it is proved that the algebra $\mathbb{I}_n$ is the maximal left (resp. right) order in its largest left (resp. right) quotient ring (Theorem 4.3).

The referee of the present paper pointed out in his report that "this paper provides an approach for studying the Belov-Kontsevich Conjecture and correspondence between holonomic D-modules and Lagrangian varieties," see [5, 7] for details.

2 Proof of Theorem 2.5

At the beginning of this section, we collect necessary (mostly elementary) facts on the algebra $\mathbb{I}_1$ from [1] that are used later in the paper.

The algebra $\mathbb{I}_1$ is generated by the elements ∂, $H := \partial x$ and $\int$ (since $x = \int H$) that satisfy the defining relations (Proposition 2.2, [1]):

$$\partial \int = 1, \quad \left[H, \int\right] = \int, \quad [H, \partial] = -\partial, \quad H\left(1 - \int \partial\right) = \left(1 - \int \partial\right) H = 1 - \int \partial, \tag{1}$$

where $[a, b] := ab - ba$ is the *commutator* of elements a and b. The elements of the algebra $\mathbb{I}_1$,

$$e_{ij} := \int^i \partial^j - \int^{i+1} \partial^{j+1}, \quad i, j \in \mathbb{N}, \tag{2}$$

satisfy the relations $e_{ij} e_{kl} = \delta_{jk} e_{il}$ where δ_{jk} is the Kronecker delta function and $\mathbb{N} := \{0, 1, \ldots\}$ is the set of natural numbers. Notice that $e_{ij} = \int^i e_{00} \partial^j$. The matrices of the linear maps $e_{ij} \in \mathrm{End}_K(K[x])$ with respect to the basis $\{x^{[s]} := \frac{x^s}{s!}\}_{s \in \mathbb{N}}$ of the polynomial algebra $K[x]$ are the elementary matrices, i.e.

$$e_{ij} * x^{[s]} = \begin{cases} x^{[i]} & \text{if } j = s, \\ 0 & \text{if } j \neq s. \end{cases}$$

Let $E_{ij} \in \mathrm{End}_K(K[x])$ be the usual matrix units, i.e. $E_{ij} * x^s = \delta_{js} x^i$ for all $i, j, s \in \mathbb{N}$. Then

$$e_{ij} = \frac{j!}{i!} E_{ij}, \tag{3}$$

$Ke_{ij} = KE_{ij}$, and

$$F := \bigoplus_{i,j \geq 0} Ke_{ij} = \bigoplus_{i,j \geq 0} KE_{ij} \simeq M_\infty(K),$$

the algebra (without 1) of infinite dimensional matrices. F is the only proper ideal (i.e. $\neq 0, \mathbb{I}_1$) of the algebra $\mathbb{I}_1$ [1].

$\mathbb{Z}$-grading on the algebra $\mathbb{I}_1$ and the canonical form of an integro-differential operator [1, 3] The algebra $\mathbb{I}_1 = \bigoplus_{i \in \mathbb{Z}} \mathbb{I}_{1,i}$ is a $\mathbb{Z}$-graded algebra ($\mathbb{I}_{1,i} \mathbb{I}_{1,j} \subseteq \mathbb{I}_{1,i+j}$ for all $i, j \in \mathbb{Z}$) where

$$\mathbb{I}_{1,i} = \begin{cases} D_1 \int^i = \int^i D_1 & \text{if } i > 0, \\ D_1 & \text{if } i = 0, \\ \partial^{|i|} D_1 = D_1 \partial^{|i|} & \text{if } i < 0, \end{cases}$$

the algebra $D_1 := K[H] \bigoplus \bigoplus_{i \in \mathbb{N}} Ke_{ii}$ is a *commutative non-Noetherian* subalgebra of $\mathbb{I}_1$, $He_{ii} = e_{ii} H = (i+1) e_{ii}$ for $i \in \mathbb{N}$ (and so $\bigoplus_{i \in \mathbb{N}} Ke_{ii}$ is the direct sum of non-zero ideals Ke_{ii} of the algebra D_1); $(\int^i D_1)_{D_1} \simeq D_1$, $\int^i d \mapsto d$; ${}_{D_1}(D_1 \partial^i) \simeq D_1$, $d\partial^i \mapsto d$, for all $i \geq 0$ since $\partial^i \int^i = 1$. Notice that the maps $\cdot \int^i : D_1 \to D_1 \int^i$, $d \mapsto d \int^i$, and $\partial^i \cdot : D_1 \to \partial^i D_1$, $d \mapsto \partial^i d$, have the same kernel $\bigoplus_{j=0}^{i-1} Ke_{jj}$.

Each element a of the algebra $\mathbb{I}_1$ is a *unique* finite sum

$$a = \sum_{i>0} a_{-i}\partial^i + a_0 + \sum_{i>0} \int^i a_i + \sum_{i,j\in\mathbb{N}} \lambda_{ij}e_{ij} \tag{4}$$

where $a_k \in K[H]$ and $\lambda_{ij} \in K$. This is the *canonical form* of the polynomial integro-differential operator [1].

Definition Let $a \in \mathbb{I}_1$ be as in Eq. 4 and let $a_F := \sum \lambda_{ij}e_{ij}$. Suppose that $a_F \neq 0$ then

$$\deg_F(a) := \min \left\{ n \in \mathbb{N} \mid a_F \in \bigoplus_{i,j=0}^{n} Ke_{ij} \right\} \tag{5}$$

is called the *F-degree* of the element a; $\deg_F(0) := -1$.

Let

$$v_i := \begin{cases} \int^i & \text{if } i > 0, \\ 1 & \text{if } i = 0, \\ \partial^{|i|} & \text{if } i < 0. \end{cases}$$

Then $\mathbb{I}_{1,i} = D_1 v_i = v_i D_1$ and an element $a \in \mathbb{I}_1$ is the unique finite sum

$$a = \sum_{i\in\mathbb{Z}} b_i v_i + \sum_{i,j\in\mathbb{N}} \lambda_{ij}e_{ij} \tag{6}$$

where $b_i \in K[H]$ and $\lambda_{ij} \in K$. So, the set $\{H^j\partial^i, H^j, \int^i H^j, e_{st} \mid i \geq 1;\ j, s, t \geq 0\}$ is a K-basis for the algebra $\mathbb{I}_1$. The multiplication in the algebra $\mathbb{I}_1$ is given by the rule:

$$\int H = (H-1)\int, \quad H\partial = \partial(H-1), \quad \int e_{ij} = e_{i+1,j}, \quad e_{ij}\int = e_{i,j-1},$$
$$\partial e_{ij} = e_{i-1,j} \quad e_{ij}\partial = \partial e_{i,j+1}.$$

$$He_{ii} = e_{ii}H = (i+1)e_{ii}, \quad i \in \mathbb{N},$$

where $e_{-1,j} := 0$ and $e_{i,-1} := 0$.

The factor algebra $B_1 := \mathbb{I}_1/F$ is the simple Laurent skew polynomial algebra $K[H][\partial, \partial^{-1}; \tau]$ where the automorphism $\tau \in \mathrm{Aut}_{K-\mathrm{alg}}(K[H])$ is defined by the rule $\tau(H) = H + 1$, [1]. Let

$$\pi : \mathbb{I}_1 \to B_1, \quad a \mapsto \overline{a} : a + F, \tag{7}$$

be the canonical epimorphism.

The Weyl algebra A_2 is equipped with the, so-called, *filtration of Bernstein*, $A_2 = \bigcup_{i\geq 0} A_{2,\leq i}$ where

$$A_{2,\leq i} := \bigoplus \left\{ Kx_1^{\alpha_1}x_2^{\alpha_2}\partial_1^{\beta_1}\partial_2^{\beta_2} \mid \alpha_1 + \alpha_2 + \beta_1 + \beta_2 \leq i \right\}.$$

The polynomial algebra $P_2 := K[x_1, x_2] \simeq A_2/(A_2\partial_1 + A_2\partial_2)$ is a simple left A_2-module with $\mathrm{End}_{A_2}(P_2) = \ker_{P_2}(\partial_1) \cap \ker_{P_2}(\partial_2) = K$. The standard filtration $\{A_{2,\leq i} \cdot 1\}_{i\in\mathbb{N}}$ of the A_2-module P_2 coincides with the filtration $\{P_{2,\leq i} :=$

$\sum_{\alpha_1,\alpha_2\geq 0}\{Kx_1^{\alpha_1}x_2^{\alpha_2}\,|\,\alpha_1+\alpha_2\leq i\}_{i\in\mathbb{N}}$ on the polynomial algebra P_2 by the total degree, i.e. $P_{2,\leq i}=A_{2,\leq i}\cdot 1$ for all $i\geq 0$, and so $\dim_K(A_{2,\leq i})=\frac{(i+1)(i+2)}{2}$. Therefore, P_2 is a holonomic A_2-module with multiplicity $e(P_2)=1$ and $\mathrm{End}_{A_2}(P_2)\simeq K$. The Weyl algebra A_1 admits the K-isomorphism:

$$\xi : A_1,\to A_1,\quad x\mapsto \partial,\quad \partial\mapsto -x. \tag{8}$$

Then $1\otimes\xi$ is an automorphism of the Weyl algebra A_2. The twisted by the automorphism $1\otimes\xi$ A_2-module P_2,

$$P_2^{1\otimes\xi}\simeq K[x_1,\partial_2]\simeq A_2/(A_2\partial_1+A_2x_2) \tag{9}$$

is a simple holonomic A_2-module with multiplicity 1 and $\mathrm{End}_{A_2}(P_2^{1\otimes\xi})\simeq K$.

The Weyl algebra A_1 is isomorphic to its opposite algebra A_1^{op} via

$$A_1\to A_1^{op},\quad x\mapsto\partial,\quad \partial\mapsto x. \tag{10}$$

In particular, each A_1-bimodule ${}_{A_1}M_{A_1}$ is a left A_2-module: ${}_{A_1}M_{A_1}={}_{A_1\otimes A_1^{op}}M\simeq {}_{A_1\otimes A_1}M={}_{A_2}M$.

Lemma 2.1

1. ${}_{A_1}F_{A_1}=A_1e_{00}A_1\simeq {}_{A_1}(A_1/A_1\partial\otimes A_1/xA_1)_{A_1}$.
2. ${}_{A_2}F\simeq A_2/(A_2\partial_1+A_2\partial_2)\simeq K[x_1,x_2]$ *is a simple holonomic A_2-module with multiplicity 1 with respect to the filtration of Bernstein of the algebra A_2 and* $End_{A_2}(F)\simeq K$.

Proof ${}_{A_1}(A_1/A_1\partial\otimes A_1/xA_1)_{A_1}\overset{(10)}{\simeq}{}_{A_1\otimes A_1}(A_1/A_1\partial\otimes A_1/A_1\partial)\simeq A_2/(A_2\partial_1+A_2\partial_2)\simeq K[x_1,x_2]$ is a simple holonomic A_2-module with multiplicity 1 with respect to the filtration of Bernstein of the algebra A_2 and $\mathrm{End}_{A_2}(F)\simeq K$. Clearly, ${}_{A_1}F_{A_1}=A_1e_{00}A_1$ and the A_1-bimodule homomorphism

$$A_1/A_1\partial\otimes A_1/xA_1\to A_1e_{00}A_1,\quad (1+A_1\partial_1)\otimes(1+xA_1)\mapsto e_{00},$$

is an epimorphism. Therefore, it is an isomorphism by the simplicity of the first A_1-bimodule. □

Proposition 2.2

1. ${}_{A_1}(\mathbb{I}_1/(A_1+F))_{A_1}\simeq A_1/A_1\partial\otimes A_1/\partial A_1$.
2. ${}_{A_2}(\mathbb{I}_1/(A_1+F))\simeq A_1/A_1\partial\otimes A_1/A_1x\simeq A_2/(A_2\partial_1+A_2x_2)\simeq K[x_1,\partial_2]$ *is a simple holonomic A_2-module with multiplicity 1 with respect to the filtration of Bernstein and* $\mathrm{End}_{A_2}(K[x_1,\partial_2])\simeq K$.
3. ${}_{A_1}(\mathbb{I}_1/(A_1+F))\simeq(A_1/A_1\partial)^{(\mathbb{N})}\simeq K[x]^{(\mathbb{N})}$ *is a semi-simple left A_1-module and* $(\mathbb{I}_1/(A_1+F))_{A_1}\simeq(A_1/\partial A_1)^{(\mathbb{N})}\simeq K[x]^{(\mathbb{N})}$ *is a semi-simple right A_1-module.*

Proof

1 and 2. Notice that ${}_{A_2}(A_1/A_1\partial\otimes A_1/A_1x)\simeq {}_{A_2}(A_2/(A_2\partial_1+A_2x_2))\simeq K[x_1,\partial_2]$ is a simple holonomic A_2-module with multiplicity 1 with respect to the filtration of Bernstein and $\mathrm{End}_{A_2}(K[x_1,\partial_2])\simeq K$. The natural filtration of the polynomial algebra $Q':=K[x_1,\partial_2]=\bigcup_{i\geq 0}Q'_{\leq i}$ by the total degree

of the variables, i.e. $Q'_{\leq i} := \bigoplus_{s+t\leq i} Kx_1^s\partial_2^t$, is a standard filtration for the A_2-module $Q' = A_2 \cdot 1$ since $Q'_{\leq i} = A_{2,\leq i}\cdot 1$ for all $i \geq 0$. In particular, $\dim_K(Q'_{\leq i}) = \frac{(i+1)(i+2)}{2}$ for all $i \geq 0$. By Eq. 4, the A_1-bimodule $Q := \mathbb{I}_1/(A_1+F)$ is the direct sum

$$Q = \bigoplus_{i\geq 1} Q_i \tag{11}$$

of its vector subspaces

$$(Q_i)_{K[H]} \simeq \int^i K[H]/x^i K[H] \simeq \int^i K[H]/\int^i (H(H+1)\cdots (H+i-1)) \simeq K[H]/(H(H+1)\cdots(H+i-1)) \tag{12}$$

(since $x^i = (\int H)^i = \int^i H(H+1)\cdots(H+i-1)$ and $\partial^i\int^i = 1$) such that $xQ_i \subseteq Q_{i+1}$, $Q_i x \subseteq Q_{i+1}$, $\partial Q_i \subseteq Q_{i-1}$ and $Q_i\partial \subseteq Q_{i-1}$ for all all $i \geq 1$ where $Q_0 := 0$. Then A_2-module Q has the finite dimensional ascending filtration $Q = \bigcup_{i\geq 0} Q_{\leq i}$ where $Q_{\leq i} := \bigoplus_{1\leq j\leq i+1} Q_j$ and

$$\dim_K(Q_{\leq i}) = \sum_{j=0}^{i}(j+1) = \frac{(i+1)(i+2)}{2} \quad \text{for all } i \geq 0.$$

Since $\partial Q_1 = Q_1\partial = 0$, the simple filtered A_2-module (treated as A_1-bimodule)

$${}_{A_1}Q'_{A_1} = A_1/A_1\partial \otimes A_1/\partial A_1$$

can be seen as a filtered A_2-submodule of Q via $(1+A_1\partial)\otimes(1+\partial A_1) \mapsto \int + A_1 + F$. In particular, for all $i \geq 0$, we have the inclusions $Q'_i \subseteq Q_i$ which are, in fact, equalities since $\dim_K(Q'_i) = \dim_K(Q_i)$. Then,

$$\begin{aligned} {}_{A_1}(\mathbb{I}_1/(A_1+F))_{A_1} &\simeq {}_{A_1}(A_1/A_1\partial \otimes A_1/\partial A_1)_{A_1} \\ &\simeq {}_{A_2}(A_1/A_1\partial \otimes A_1/A_1 x) \simeq K[x_1, \partial_2]. \end{aligned}$$

It is obvious that the A_2-module $K[x_1,\partial_2]$ is a simple A_2-module with multiplicity 1 and $\mathrm{End}_{A_2}(K[x_1,\partial_2]) \simeq K$.

3. Statement 3 follows from statement 1. □

A linear map φ acting in a vector space V is called a *locally nilpotent map* if $V = \bigcup_{i\geq 1}\ker(\varphi^i)$, i.e. for each element $v \in V$ there exists a natural number i such that $\varphi^i(v) = 0$.

It follows from Proposition 2.2 and Eq. 12 that

$$\ker_{\mathbb{I}_1/(A_1+F)}(\partial\cdot)\bigcap \ker_{\mathbb{I}_1/(A_1+F)}(\cdot\partial) = K\left(\int + A_1 + F\right), \tag{13}$$

and that the maps $\partial\cdot : \mathbb{I}_1/(A_1+F) \to \mathbb{I}_1/(A_1+F)$, $u \mapsto \partial u$, and $\cdot\partial : \mathbb{I}_1/(A_1+F) \to \mathbb{I}_1/(A_1+F)$, $u \mapsto u\partial$, are locally nilpotent since

$$\partial * x_1^i x_2^j = i x_1^{i-1} x_2^j, \quad x_1^i x_2^j * \partial = -j x_1^i x_2^{j-1}. \tag{14}$$

Recall that the *socle* $\mathrm{soc}_A(M)$ of a module M over a ring A is the sum of all the simple submodules of M, if they exist, and zero, otherwise.

Theorem 2.3

1. *The* A_1*-bimodule* $\mathbb{I}_1$ *is a holonomic* A_2*-module of length 3 with simple non-isomorphic factors* $F \simeq {}_{A_2}K[x_1, x_2]$, ${}_{A_1}A_{1\,A_1}$ *and* ${}_{A_2}K[x_1, \partial_2]$. *Each factor is a simple holonomic* A_2*-module with multiplicity 1 and its* A_2*-module endomorphism algebra is* K.
2. $\mathrm{soc}_{A_2}(\mathbb{I}_1) = A_1 \bigoplus F$.
3. *The short exact sequence of* A_2*-modules*

$$0 \to A_1 \bigoplus F \to \mathbb{I}_1 \to \mathbb{I}_1/(A_1 + F) \to 0 \tag{15}$$

is non-split.

Proof

1. Statement 1 follows from Lemma 2.1, Proposition 2.2 and Eq. 15.
2. Suppose that the short exact sequence of A_1-bimodules splits, we seek a contradiction. Then, by Proposition 2.2.(1) and Eq. 13, there is a nonzero element, say

$$u = \int + a + f \in \mathbb{I}_1 \ \text{ with } \ a \in A_1 \ \text{ and } \ f \in F$$

such that $\partial u = 0$ and $u\partial = 0$. The first equation implies $1 + \partial a = -\partial f \in A_1 \cap F = 0$, and so $\partial a = -1$ in A_1, a contradiction.
3. Statement 3 follows from statement 2. □

New basis for the algebra $\mathbb{I}_n$ It follows from Eqs. 11, 12 and 15 that

$$\mathbb{I}_1 = \bigoplus_{i,j\geq 0} Kx^i\partial^j \oplus \bigoplus_{k,l\geq 0} Ke_{kl} \oplus \bigoplus \left\{ K\int^s H^t \,|\, s \geq 1, t = 0, 1, \ldots, s-1 \right\}. \tag{16}$$

This gives a new K-basis for the algebra $\mathbb{I}_1$:

$$\left\{ x^i\partial^j, e_{kl}, \int^s H^t \,|\, i, j, k, l \geq 0;\ s \geq 1;\ t = 0, 1, \ldots, s-1 \right\}.$$

By taking n'th tensor product of this basis we obtain a new K-basis for the algebra $\mathbb{I}_n = \mathbb{I}_1^{\otimes n}$.

Lemma 2.4

1. *The* A_1*-bimodule* $\mathbb{I}_1/A_1$ *is a holonomic* A_2*-module of length 2 with simple non-isomorphic factors* $F \simeq {}_{A_2}K[x_1, x_2]$ *and* ${}_{A_2}K[x_1, \partial_2]$. *Each factor is a simple holonomic* A_2*-module with multiplicity 1 and its* A_2*-module endomorphism algebra is* K.
2. $\mathrm{soc}_{A_2}(\mathbb{I}_1/A_1) = F$.

3. *The short exact sequence of A_2-modules*

$$0 \to F \to \mathbb{I}_1/A_1 \to \mathbb{I}_1/(A_1+F) \to 0 \tag{17}$$

is non-split.
4. *The short exact sequence of left A_1-modules Eq. 17 splits and so ${}_{A_1}(\mathbb{I}_1/A_1) \simeq K[x]^{(\mathbb{N})}$ is a semi-simple left A_1-module.*
5. *The short exact sequence of right A_1-modules Eq. 17 does not split, and so $(\mathbb{I}_1/A_1)_{A_1}$ is not a semi-simple right A_1-module.*

Proof

1. Statement 1 follows from Theorem 2.3.(1).
2. Suppose that the short exact sequence of A_1-bimodules Eq. 17 splits, we seek a contradiction. Then, by Proposition 2.2.(1) and Eq. 13, there is a nonzero element, say $u = \int + f + A_1 \in \mathbb{I}_1/A_1$ with $f \in F$ such that $0 = \partial u = 1 + \partial f$ and $0 = u\partial = 1 - e_{00} + f\partial$ in $\mathbb{I}_1/A_1$. The first equality gives $\partial f = 0$ in $\mathbb{I}_1/A_1$, and so $f = \sum_{i\geq 0} \lambda_i e_{0i}$ for some $\lambda_i \in K$. Then the second equality gives $e_{00} = f\partial = \sum_{i\geq 0} \lambda_i e_{0i}\partial = \sum_{i\geq 0} \lambda_i e_{0,i+1}$, a contradiction.
3. Statement 3 follows from statement 2.
4. Let L be the last sum in the decomposition Eq. 16, i.e.

$$\mathbb{I}_1 = A_1 \bigoplus F \bigoplus L. \tag{18}$$

Then $A_1 \bigoplus L$ is a left A_1-submodule of ${}_{A_1}\mathbb{I}_1$ since $\partial \int = 1$, $x = \int H$ and $\int H = (H-1)\int$. Notice that $A_1 \bigoplus L$ is not a right A_1-submodule of $\mathbb{I}_1$ since $\int \partial = 1 - e_{00} \notin A_1 \bigoplus L$. By Eq. 18,

$${}_{A_1}(\mathbb{I}_1/A_1) \simeq F \bigoplus (A_1 + L)/A_1$$

is a direct sum of left A_1-submodules such that ${}_{A_1}F \simeq K[x]^{(\mathbb{N})}$ (Lemma 2.1.(1)) and ${}_{A_1}((A_1+L)/A_1) \simeq \mathbb{I}_1/(A_1+F) \simeq K[x]^{(\mathbb{N})}$ (Proposition 2.2.(3)). Therefore, ${}_{A_1}(\mathbb{I}_1/A_1)$ is a semi-simple module. Therefore, the short exact sequence of left A_1-modules Eq. 17 splits and ${}_{A_1}(\mathbb{I}_1/A_1) \simeq K[x]^{(\mathbb{N})} \bigoplus K[x]^{(\mathbb{N})} \simeq K[x]^{(\mathbb{N})}$.
5. By Proposition 2.2.(1), $(\mathbb{I}_1/(A_1+F))_{A_1} \simeq (A_1/\partial A_1)^{(\mathbb{N})}$. Suppose that the short exact sequence of right A_1-modules Eq. 17 splits, we seek a contradiction. In the factor module $\mathbb{I}_1/(A_1+F)$, $(\int + A_1 + F)\partial = 0$ since $\int \partial = 1 - e_{00} \in A_1 + F$. Then the splitness implies that

$$\left(\int + f + A_1\right)\partial = 0$$

in $\mathbb{I}_1/A_1$ for some element $f \in F$, equivalently, $-e_{00} + f\partial \in A_1 \cap F = 0$ in $\mathbb{I}_1$, i.e. $f\partial = e_{00}$, this is obviously impossible (since $e_{i,j}\partial = e_{i,j+1}$), a contradiction. □

Let M be a module over a ring R. The socle $\mathrm{soc}_R(M)$, if nonzero, is the largest semi-simple submodule of M. The socle of M, if nonzero, is the only essential semi-simple submodule. The *socle chain* of the module M is the ascending chain of its submodules:

$$\mathrm{soc}^0_R(M) := \mathrm{soc}_R(M) \subseteq \mathrm{soc}^1_R(M) \subseteq \cdots \subseteq \mathrm{soc}^i_R(M) \subseteq \cdots$$

where $\mathrm{soc}^i_R(M) := \varphi^{-1}_{i-1}(\mathrm{soc}_R(M/\mathrm{soc}^{i-1}_R(M)))$ where

$$\varphi_{i-1} : M \to M/\mathrm{soc}^{i-1}(M), \quad m \mapsto m + \mathrm{soc}^{i-1}_R(M).$$

Let $\mathrm{soc}^\infty_R(M) := \bigcup_{i\geq 0} \mathrm{soc}^i_R(M)$. If $M = \mathrm{soc}^\infty_R(M)$ then

$$\mathrm{l.soc}_R(M) = 1 + \min\left\{i \geq 0 \mid M = \mathrm{soc}^i_R(M)\right\}$$

is called the *socle length* of the R-module M. So, a nonzero module is semi-simple iff its socle length is 1.

Theorem 2.5

1. *The A_n-bimodule $\mathbb{I}_n$ is a holonomic A_{2n}-module of length 3^n with pairwise non-isomorphic simple factors and each of them is the tensor product $\bigotimes_{i=1}^n M_i$ of simple $A_2(i)$-modules M_i as in Theorem 2.3 for $i = 1, \dots, n$. Each simple factor $\bigotimes_{i=1}^n M_i$ is a simple holonomic A_{2n}-module and has multiplicity 1 (with respect to the filtration of Bernstein on the algebra A_{2n}) and its A_{2n}-module endomorphism algebra is K.*
2. $\mathrm{soc}_{A_{2n}}(\mathbb{I}_n) = \bigotimes_{i=1}^n \mathrm{soc}_{A_2(i)}(\mathbb{I}_1(i)) = \bigotimes_{i=1}^n (A_1(i) \bigoplus F(i))$.
3. *The socle length of the A_{2n}-module $\mathbb{I}_n$ is $n+1$. For each number $m = 0, 1, \dots, n$,*

$$\mathrm{soc}^m_{A_{2n}}(\mathbb{I}_n) = \sum_{i_1+\dots+i_n=m} \bigotimes_{s=1}^n \mathrm{soc}^{i_s}_{A_2(i)}(\mathbb{I}_1(i))$$

   where all $i_s \in \{0, 1\}$ and $\mathrm{soc}^j_{A_2(i)} = \begin{cases} A_1(i) \bigoplus F(i) & \text{if } j = 0, \\ \mathbb{I}_1(i) & \text{if } j = 1. \end{cases}$
4. *For each number $m = 0, 1, \dots, n$,*

$$\mathrm{soc}^m_{A_{2n}}(\mathbb{I}_n)/\mathrm{soc}^{m-1}_{A_{2n}}(\mathbb{I}_n) = \bigoplus_{i_1+\dots+i_n=m} \bigotimes_{s=1}^n \mathrm{soc}^{i_s}_{A_2(i)}(\mathbb{I}_1(i))/\mathrm{soc}^{i_s-1}_{A_2(i)}(\mathbb{I}_1(i))$$

   and its length (as an A_{2n}-module) is $\binom{n}{m}2^{n-m}$ where all $i_s \in \{0, 1\}$ and $\mathrm{soc}^{-1} := 0$.
5. *The left A_{2n}-module $\mathbb{I}_n$ has multiplicity 3^n with respect to the filtration of Bernstein of the Weyl algebra A_{2n}.*

Remark The sum of lengths of all the factors in statement 4 is 3^n as

$$3^n = (1+2)^n = \sum_{m=0}^n \binom{n}{m} 2^{n-m}.$$

Proof

1. By Theorem 2.3.(1), each of the tensor multiples $\mathbb{I}_1(i)$ in $\mathbb{I}_n = \bigotimes_{i=1}^n \mathbb{I}_1(i)$ has the $A_2(i)$-module (i.e. the $A_1(i)$-bimodule) filtration of length 3 with factors M_i as in Theorem 2.3.(1). By considering the tensor product of these filtrations, the A_{2n}-module $\mathbb{I}_n = \bigotimes_{i=1}^n \mathbb{I}_1(i)$ (i.e. the A_n-bimodule) has a filtration (of length 3^n) with factors $\bigotimes_{i=1}^n M_i$. It is obvious that each A_{2n}-module $\bigotimes_{i=1}^n M_i$ is isomorphic to a

twisted A_{2n}-module ${}^{\sigma}P_{2n}$ by an automorphism σ of the Weyl algebra A_{2n} that preserves the filtration of Bernstein on the algebra A_{2n} where

$$P_{2n} = K[x_1, \dots, x_{2n}] \simeq A_{2n}/\sum_{i=1}^{2n} A_{2n}\partial_i.$$

This statement is obvious for $n = 1$, then the general case follows at once. Since the A_{2n}-module P_{2n} is simple, holonomic with multiplicity 1 and $\mathrm{End}_{A_{2n}}(P_{2n}) \simeq K$, then so are all the A_{2n}-modules $\bigotimes_{i=1}^{n} M_i$ (since $e({}^{\sigma}P_{2n}) = e(P_{2n}) = 1$). This finishes the proof of statement 1.

2. Statement 2 follows from statement 3.
3. To prove statement 3 we use induction on n. The initial step when $n = 1$ is true due to Theorem 2.3.(1). Suppose that $n > 1$ and the statement holds for all $n' < n$. Let

$$\left\{s^0 = A_1 \bigoplus F, s^1 = \mathbb{I}_1\right\}$$

be the socle filtration for ${}_{A_1}\mathbb{I}_1{}_{A_1}$ and let $\{s^0, s^1, \dots, s^{n-1}\}$ be the socle filtration for ${}_{A_{n-1}}\mathbb{I}_{n-1}{}_{A_{n-1}}$. We are going to prove that

$$\{s'^0 := s^0 \otimes s^0, s'^1 := s^0 \otimes s^1 + s^1 \otimes s^0, \dots,$$
$$s'^{n-1} := s^0 \otimes s^{n-1} + s^1 \otimes s^{n-2}, s'^n := s^1 \otimes s^{n-1}\}$$

is the socle filtration for ${}_{A_n}\mathbb{I}_n{}_{A_n}$. Notice that $A_n = A_1 \otimes A_{n-1}$, $\mathbb{I}_n = \mathbb{I}_1 \otimes \mathbb{I}_{n-1}$ and $\{s^0 \otimes s^i\}_{i=0}^{n-1}$ is the socle filtration for ${}_{A_{n-1}}(s^0 \otimes \mathbb{I}_{n-1})_{A_{n-1}} = s^0 \otimes ({}_{A_{n-1}}\mathbb{I}_{n-1}{}_{A_{n-1}})$ since the $\mathbb{I}_{n-1}$-bimodules $s^0 \otimes s^i/s^{i-1}$ are semi-simple. Since, for each number $m = 0, 1, \dots, n$, the A_n-subbimodule

$$\overline{s}'^m := s'^m/s'^{m-1} = s^0 \otimes (s^m/s^{m-1}) \bigoplus (s^1/s^0) \otimes (s^{m-1}/s^{m-2}) \quad (\text{where } \overline{s}'^0 = s^0 \otimes s^0)$$

of $\mathbb{I}'_n/s'^{m-1}$ is semi-simple, in order to finish the proof of statement 3 it suffices to show that $\overline{s}'^m$ is an essential A_n-subbimodule of $\mathbb{I}_n/s'^{m-1}$. Let a be a nonzero element of the A_n-bimodule $\mathbb{I}_n/s'^{m-1}$. We have to show that

$$A_n a A_n \cap \overline{s}'^m \neq 0.$$

If $a \in s^0 \otimes \mathbb{I}_{n-1} + s'^{m-1}$ then

$$0 \neq \mathbb{I}_{n-1} a \mathbb{I}_{n-1} \cap s^0 \otimes (s^m/s^{m-1}) \subseteq \overline{s}'^m$$

(since $\{s^0 \otimes s^i\}_{i=0}^{n-1}$ is the socle filtration for ${}_{A_{n-1}}(s^0 \otimes \mathbb{I}_{n-1})_{A_{n-1}}$).
If $a \notin s^0 \otimes \mathbb{I}_{n-1} + s'^{m-1}$ then using the explicit basis $\{x_1^i x_2^j\}_{i,j\geq 0}$ for the A_1-bimodule s^1/s^0 (Proposition 2.2.(1)) and the action of the element ∂ on it (see Eq. 14), we can find natural numbers, say k and l, such that, by Eq. 13, the element

$$a' := \partial^k a \partial^l = \int \otimes u_1 + v_2 \otimes u_2 + \dots + v_s \otimes u_s,$$

is such that $0 \neq u_1 \in \mathbb{I}_{n-1}/s^{m-1}$ (in particular, a' is a nonzero element of $\mathbb{I}_n/s'^{m-1}$); $u_2, \dots, u_s$ are linearly independent elements of $\mathbb{I}_{n-1}$; $v_2, \dots, v_s$ are linearly independent elements of s^0. If the elements $u_1, u_2, \dots, u_s$ are linearly independent then

$$a'' := \partial a' = 1 \otimes u_1 + (\partial v_2) \otimes u_2 + \cdots + (\partial v_s) \otimes u_s$$

is a nonzero element of $s^0 \otimes \mathbb{I}_{n-1}$, and so, by the previous case $\mathbb{I}_n a \mathbb{I}_n \cap \bar{s}'^m \neq 0$. If the elements $u_1, u_2, \dots, u_s$ are linearly dependent then $u_1 = \sum_{i=2}^{s} \lambda_i u_i$ for some elements $\lambda_i \in K$ not all of which are zero ones, say $\lambda_2 \neq 0$. The element a' can be written as $a' = (\lambda_2 \int + v_2) \otimes u_2 + \cdots + (\lambda_s \int + v_s) \otimes u_s$. Then

$$a'' := \partial a' = (\lambda_2 + \partial v_2) \otimes u_2 + \cdots + (\lambda_s + \partial v_s) \otimes u_s.$$

We claim that $a'' \neq 0$. Suppose that $a'' = 0$, we seek a contradiction. Then

$$\lambda_2 + \partial v_2 = 0, \dots, \lambda_s + \partial v_s = 0$$

in $A_1 \bigoplus F$ (since the elements $u_2, \dots, u_s$ are linearly independent). The first equality yields $0 \neq \lambda_2 = \partial b$ in the Weyl algebra A_1 for some element $b \in A_1$. This is clearly impossible. Therefore, a'' is a nonzero element of $s^0 \otimes \mathbb{I}_{n-1}$, and so, by the previous case, $\mathbb{I}_n a \mathbb{I}_n \cap \bar{s}'^m \neq 0$.

4. The equality follows from statement 3. To prove the claim about the length note that $\binom{n}{m}$ is the number of vectors $(i_1, \dots, i_n) \in \{0,1\}^n$ with $i_1 + \cdots + i_n = m$; and for each choice of $(i_1, \dots, i_n)$ the length of the A_{2n}-module $\bigotimes_{s=1}^{n} \mathrm{soc}^{i_s}_{A_2(i)}(\mathbb{I}_1(i))/\mathrm{soc}^{i_s-1}_{A_2(i)}(\mathbb{I}_1(i))$ is 2^{n-m}. Therefore, the length of the A_{2n}-module $\mathrm{soc}^m_{A_{2n}}(\mathbb{I}_n)/\mathrm{soc}^{m-1}_{A_{2n}}(\mathbb{I}_n)$ is $\binom{n}{m}2^{n-m}$.
5. Statement 5 follows from statement 1 and the additivity of the multiplicity on the holonomic modules. □

3 The Algebra $\mathbb{I}_n$ is Coherent iff $n = 1$

The aim of this section is to prove Theorem 3.1.

A module M over a ring R is *finitely presented* if there is an exact sequence of modules

$$R^m \to R^n \to M \to 0.$$

A finitely generated module is a *coherent* module if every finitely generated submodule is finitely presented. A ring R is a *left* (resp. *right*) *coherent ring* if the module ${}_R R$ (resp. R_R) is coherent. *A ring R is a left coherent ring iff, for each element $r \in R$, $\ker_R(\cdot r)$ is a finitely generated left R-module and the intersection of two finitely generated left ideals is finitely generated*, Proposition 13.3, [8]. Each left Noetherian ring is left coherent but not vice versa.

Theorem 3.1 *The algebra $\mathbb{I}_n$ is a left coherent algebra iff the algebra $\mathbb{I}_n$ is a right coherent algebra iff $n = 1$.*

Proof The first 'iff' is obvious since the algebra $\mathbb{I}_n$ is self-dual [1], i.e. is isomorphic to its opposite algebra $\mathbb{I}_n^{op}$. If $n = 1$ the algebra is a left coherent algebra [3]. If $n \geq 2$ then the algebra $\mathbb{I}_2$ is not a left coherent algebra since, by Lemma 3.2,

$$\ker_{\mathbb{I}_n}(\cdot(H_1 - H_2)) = \ker_{\mathbb{I}_2}(\cdot(H_1 - H_2)) \otimes \mathbb{I}_{n-2} \simeq {}_{\mathbb{I}_n}(P_2 \otimes \mathbb{I}_{n-2})^{(\mathbb{N})}$$

is an infinite direct sum of nonzero $\mathbb{I}_n$-modules, hence it is not finitely generated. Therefore, the algebra $\mathbb{I}_n$ is not a left coherent algebra, by Proposition 13.3, [8]. □

Lemma 3.2 $\ker_{\mathbb{I}_2}(\cdot(H_1-H_2))=\ker_{F_2}(\cdot(H_1-H_2))=\bigoplus_{i,j,k\in\mathbb{N}} Ke_{ij}(1)e_{kj}(2)\simeq({}_{\mathbb{I}_2}P_2)^{(\mathbb{N})}$.

Proof The algebra $B_2=\mathbb{I}_2/\mathfrak{a}_2$ is a domain, see [1], where $\mathfrak{a}_2:=F(1)\otimes\mathbb{I}_1(2)+\mathbb{I}_1(1)\otimes F(2)$ and $H_1-H_2\notin\mathfrak{a}_2$. Therefore, $\mathcal{K}:=\ker_{\mathbb{I}_2}(\cdot(H_1-H_2))=\ker_{\mathfrak{a}_2}(\cdot(H_1-H_2))$. Let $F_2:=F(1)\otimes F(2)$. Notice that

$${}_{\mathbb{I}_2}(\mathfrak{a}_2/F_2)_{\mathbb{I}_2}\simeq F(1)\otimes B_1(2)\bigoplus B_1(1)\otimes F(2)$$

is a direct sum of two $\mathbb{I}_2$-bimodules. It follows from the presentation

$$F(1)\otimes B_1(2)=\bigoplus_{i,j\in\mathbb{N},k\in\mathbb{Z}} e_{ij}(1)\otimes\partial_2^k K[H_2]$$

that $\ker_{F(1)\otimes B_1(2)}(\cdot(H_1-H_2))=0$. Similarly, $\ker_{B_1(1)\otimes F(2)}(\cdot(H_1-H_2))=0$ (or use the $(1,2)$-symmetry). Therefore,

$$\mathcal{K}=\ker_{F_2}(\cdot(H_1-H_2))=\bigoplus_{i,j,k\in\mathbb{N}} Ke_{ij}(1)e_{kj}(2)$$

$$=\bigoplus_{j\in\mathbb{N}}(\bigoplus_{i,k\in\mathbb{N}} Ke_{ij}(1)e_{kj}(2))\simeq\bigoplus_{j\in\mathbb{N}}({}_{\mathbb{I}_2}P_2)\simeq{}_{\mathbb{I}_2}(P_2)^{(\mathbb{N})}.$$

□

4 The Algebra $\mathbb{I}_n$ is a Maximal Order

The aim of this section is to prove Theorem 4.3.

Let R be a ring. An element $x\in R$ is *right regular* if $xr=0$ implies $r=0$ for $r\in R$. Similarly, a *left regular element* is defined. A left and right regular element is called a *regular element*. The sets of regular/left regular/right regular elements of a ring R are denoted respectively by $\mathcal{C}_R(0)$, ${}'\mathcal{C}_R(0)$ and $\mathcal{C}'_R(0)$. For an arbitrary ring R there exists the *largest* (w.r.t. inclusion) *left regular denominator set* $S_{l,0}=S_{l,0}(R)$ in the ring R (regular means that $S_{l,0}(R)\subseteq\mathcal{C}_R(0)$), and so $Q_l(R):=S_{l,0}^{-1}R$ *is the largest left quotient ring of* R (Theorem 2.1, [4]). Similarly, for an arbitrary ring R there exists the *largest right regular denominator set* $S_{r,0}=S_{r,0}(R)$ in R, and so $Q_r(R):=RS_{r,0}^{-1}$ is the *largest right quotient ring* of R. The rings $Q_l(R)$ and $Q_r(R)$ were introduced and studied in [4].

Let $\mathrm{End}_K(K[x])$ be the algebra of all linear maps from the vector space $K[x]$ to itself and $\mathrm{Aut}_K(K[x])$ be its group of units (i.e. the group of all invertible linear maps from $K[x]$ to itself). The algebra $\mathbb{I}_1$ is a subalgebra of $\mathrm{End}_K(K[x])$. Theorem 5.6.(1), [4], states that

$$S_{r,0}(\mathbb{I}_1)=\mathbb{I}_1\cap\mathrm{Aut}_K(K[x]),$$

it is the set of all elements of the algebra $\mathbb{I}_1$ that are invertible linear maps in $K[x]$. The set $S_{r,0}(\mathbb{I}_1)$ is huge compared to the group of units $\mathbb{I}_1^*$ of the algebra $\mathbb{I}_1$ which is obviously a subset of $S_{r,0}(\mathbb{I}_1)$.

Let R be a ring. A subring S (not necessarily with 1) of the largest right quotient ring $Q_r(R)$ of the ring R is called a *right order* in $Q_r(R)$ if each element $q \in Q_r(R)$ has the form rs^{-1} for some elements $r, s \in S$. A subring S (not necessarily with 1) of the largest left quotient ring $Q_l(R)$ of the ring R is called a *left order* in $Q_l(R)$ if each element $q \in Q_l(R)$ has the form $s^{-1}r$ for some elements $r, s \in S$.

Let R_1 and R_2 be right orders in $Q_r(\mathbb{I}_n)$. We say that the right orders R_1 and R_2 are *equivalent*, $R_1 \sim R_2$, if there are units $a_1, a_2, b_1, b_2 \in Q_r(\mathbb{I}_n)$ such that

$$a_1 R_1 b_1 \subseteq R_2 \text{ and } a_2 R_2 b_2 \subseteq R_1.$$

Clearly, $\sim$ is an equivalent relation on the set of right orders in $Q_r(\mathbb{I}_n)$. A right order in $Q_r(\mathbb{I}_n)$ is called a *maximal right order* if it is maximal (w.r.t. $\subseteq$) within its equivalence class.

Lemma 4.1 *Let $Q_r(\mathbb{I}_n)$ be the right quotient ring of $\mathbb{I}_n$ and R, S be equivalent right orders in $Q_r(\mathbb{I}_n)$ such that $R \subseteq S$. Then there are equivalent right orders T and T' in $Q_r(\mathbb{I}_n)$ with $R \subseteq T \subseteq S$, $R \subseteq T' \subseteq S$ and units r_1, r_2 of $Q_r(\mathbb{I}_n)$ contained in R such that $r_1 S \subseteq T$, $Tr_2 \subseteq R$ and $Sr_2 \subseteq T'$, $r_1 T' \subseteq R$. In particular, $r_1 S r_2 \subseteq R$.*

Proof By definition, $aSb \subseteq R$ for some units a, b of $Q_r(\mathbb{I}_n)$. Then $a = r_1 s_1^{-1}$ and $b = r_2 s_2^{-1}$ with $r_i, s_i \in R$, and r_i, and s_i are units in $Q_r(\mathbb{I}_n)$. Then $r_1 S r_2 \subseteq r_1 s_1^{-1} S r_2 \subseteq R s_2 \subseteq R$. It is readily checked that $T = R + r_1 S + R r_1 S$ and $T' = R + S r_2 + S r_2 R$ are as claimed. □

Lemma 4.2

1. $\mathcal{C}_{\mathbb{I}_n}(0) \cap \mathfrak{a}_n = \emptyset$, $'\mathcal{C}_{\mathbb{I}_n}(0) \cap \mathfrak{a}_n = \emptyset$ *and* $\mathcal{C}'_{\mathbb{I}_n}(0) \cap \mathfrak{a}_n = \emptyset$.
2. $S_{l,0}(\mathbb{I}_n) \cap \mathfrak{a}_n = \emptyset$ *and* $S_{r,0}(\mathbb{I}_n) \cap \mathfrak{a}_n = \emptyset$.
3. *For all elements* $a \in S_{l,0}(\mathbb{I}_n) \cup S_{r,0}(\mathbb{I}_n)$, $\mathbb{I}_n a \mathbb{I}_n = \mathbb{I}_n$.

Proof

1. Trivial (since every element of the ideal $\mathfrak{a}_n$ is a left and right zero divisor in $\mathbb{I}_n$).
2. Statement 2 follows from statement 1 and the inclusions $S_{l,0}(\mathbb{I}_n), S_{r,0}(\mathbb{I}_n) \subseteq \mathcal{C}_{\mathbb{I}_n}(0)$.
3. If $\mathbb{I}_n a \mathbb{I}_n \neq \mathbb{I}_n$ for some element $a \in S_{l,0}(\mathbb{I}_n) \cup S_{r,0}(\mathbb{I}_n)$ then $a \in \mathbb{I}_n a \mathbb{I}_n \subseteq \mathfrak{a}_n$ (as $\mathfrak{a}_n$ is the only maximal ideal of the algebra $\mathbb{I}_n$). This contradicts statement 2. □

Theorem 4.3 *The algebra $\mathbb{I}_n$ is a maximal left order in $Q_l(\mathbb{I}_n)$ and a maximal right order in $Q_r(\mathbb{I}_n)$.*

Proof Suppose that $\mathbb{I}_n \subseteq S$ and $S \sim \mathbb{I}_n$ for some right order S in $Q_r(\mathbb{I}_n)$. Then $aSb \subseteq \mathbb{I}_n$ for some elements $a, b \in \mathbb{I}_n \cap Q_r(\mathbb{I}_n)^*$, by Lemma 4.1, where $Q_r(\mathbb{I}_n)^*$ is the group of units of the algebra $Q_r(\mathbb{I}_n)$. By Theorem 2.8, [4],

$$\mathbb{I}_n \cap Q_r(\mathbb{I}_n)^* = S_{r,0}(\mathbb{I}_n).$$

Then, by Corollary 4.2.(3),

$$\mathbb{I}_n \supseteq \mathbb{I}_n a S b \mathbb{I}_n = (\mathbb{I}_n a \mathbb{I}_n) S (\mathbb{I}_n b \mathbb{I}_n) = \mathbb{I}_n S \mathbb{I}_n = S,$$

i.e. $\mathbb{I}_n = S$. Then the algebra $\mathbb{I}_n$ is a maximal right order in $Q_r(\mathbb{I}_n)$. Since the algebra $\mathbb{I}_n$ admits an involution [1], the algebra $\mathbb{I}_n$ is also a maximal left order in $Q_l(\mathbb{I}_n)$. □

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