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Tilting Theoretical Approach to Moduli Spaces Over Preprojective Algebras

Yuhi Sekiya · Kota Yamaura

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Abstract We apply tilting theory over preprojective algebras Λ to the study of moduli spaces of Λ-modules. We define the categories of semistable modules and give equivalences, so-called reflection functors, between them by using tilting modules over Λ. Moreover we prove that the equivalence induces an isomorphism of K-schemes between moduli spaces. In particular, we study the case when the moduli spaces are related to Kleinian singularities, and generalize some results of Crawley-Boevey (Am J Math 122:1027–1037, 2000).

Keywords Moduli spaces · Preprojective algebras · Tilting theory · McKay correspondence · Kleinian singularities

Mathematics Subject Classifications (2010) Primary 14D22 · 16E30 · 16G20; Secondary 14J10 · 14E16

1 Introduction

In this paper we study moduli spaces of modules over preprojective algebras by using tilting theory. In particular, we focus on the minimal resolutions of Kleinian singularities.

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Y. Sekiya (✉) · K. Yamaura
Graduate School of Mathematics, Nagoya University, Chikusa-ku Nagoya 464-8602, Japan
e-mail: yuhi-sekiya@math.nagoya-u.ac.jp

K. Yamaura
e-mail: m07052d@math.nagoya-u.ac.jp

Moduli spaces of modules are an important tool in a study of resolutions of quotient singularities $\mathbb{A}^n/G$ where G is a finite subgroup of $GL(n, K)$. Let Λ be a K-algebra associated to the McKay quiver Q of G with the relations given by [7]. Let $\mathbf{d}$ be the dimension vector determined by the dimensions of irreducible representations of G and θ a parameter θ satisfying $\theta(\mathbf{d}) = 0$. Then by King [26], we can construct a moduli space $\mathcal{M}_{\theta,\mathbf{d}}(\Lambda)$ of θ-semistable modules of dimension vector $\mathbf{d}$.

If G is contained in $SL(2, K)$, the quotient singularity $\mathbb{A}^2/G$ is called a Kleinian singularity. Due to Kronheimer [27] and Cassens-Slodowy [17], the moduli space $\mathcal{M}_{\theta,\mathbf{d}}(\Lambda)$ is the minimal resolution of $\mathbb{A}^2/G$ for every generic parameter θ. If G is contained in $SL(3, K)$, Bridgeland-King-Reid [10] proved that the moduli space $\mathcal{M}_{\theta,\mathbf{d}}(\Lambda)$ is a crepant resolution of $\mathbb{A}^3/G$ for every generic parameter θ. In this case, since crepant resolutions are not unique, it is natural to consider the problem whether every (projective) crepant resolution can be realized as a moduli space of Λ-modules. Craw-Ishii [18] solved this problem for the case that G is abelian by observing the variation of moduli spaces by changing the parameter θ.

Our interest is to generalize the above Craw-Ishii's result to the case that G is non-abelian. However there are many difficulties to deal with a non-abelian case of dimension three. Thus in this paper we try to develop a new method using homological algebra and apply it to the case that G is *any* finite subgroup of $SL(2, K)$. Although we deal with the two dimensional case in this paper, our method is useful and can be applied to the higher dimensional case (e.g. [33]).

For a finite subgroup G of $SL(2, K)$, the minimal resolution of $\mathbb{A}^2/G$ is unique. So all moduli spaces $\mathcal{M}_{\theta,\mathbf{d}}(\Lambda)$ for generic parameter θ are isomorphic. However they parametrize different θ-semistable Λ-modules. Therefore it makes sense to consider variation of moduli space as variation of θ-semistable Λ-modules. Since we want to deal with θ-semistable Λ-modules categorically, we give the following definition.

Definition 1.1 Let Λ be a K-algebra associated to a finite quiver with relations. For any parameter $\theta \in \Theta$, we define the full subcategory $\mathcal{S}_\theta(\Lambda)$ of ModΛ consisting of θ-semistable Λ-modules. Moreover we denote by $\mathcal{S}_{\theta,\alpha}(\Lambda)$ the full subcategory of $\mathcal{S}_\theta(\Lambda)$ consisting of θ-semistable Λ-modules of dimension vector α if $\mathcal{S}_{\theta,\alpha}(\Lambda)$ is not empty.

The K-algebra Λ associated to a finite subgroup $G \subset SL(2, K)$ is called the preprojective algebra (cf. [2, 3, 7, 34]). To study variation of moduli space, we apply representation theory of preprojective algebras, in particular tilting theory.

Tilting theory is a theory to deal with equivalences of derived categories of modules over algebras (cf. [1, 21]). In tilting theory, tilting modules play an important role. If a tilting module T is given, then the derived category $\mathcal{D}(\text{Mod}\Lambda)$ becomes equivalent to the derived category $\mathcal{D}(\text{ModEnd}_\Lambda(T))$ (cf. [35]).

The notion of preprojective algebra was introduced by Gelfand and Ponomarev [20] and has been studied by many researchers (for example [5, 6, 11, 24]). For a finite quiver Q, it is defined as a K-algebra associated to the double quiver of Q with preprojective relations (in detail see Section 2). Buan-Iyama-Reiten-Scott [11] constructed a set of tilting modules over completions of preprojective algebras of non-Dynkin quivers with no loops as follows. For each vertex $i \in Q_0$, they defined the two sided ideal I_i and proved that any products of these ideals are tilting modules whose endomorphism algebra is isomorphic to Λ. We denote by $\mathcal{I}(\Lambda)$ the set of such tilting modules. Moreover they gave a bijection from the Coxeter group W_Q associated to Q to $\mathcal{I}(\Lambda)$ by $w \longmapsto I_w$ where I_w is well-defined as the product $I_{i_1} \cdots I_{i_\ell}$ for any reduced expression $w = s_{i_1} \cdots s_{i_\ell}$.

In the following, we explain our ideas and state main results in this paper. We are interested in singularity theory, namely the case that Q is an extended Dynkin quiver. But our proof works in a more general setting, that is, Q is a non-Dynkin quiver with no loops. So we prove almost all results for such a setting.

First we prove the above results shown in [11] for non-completed preprojective algebras of non-Dynkin quivers in Section 2 since we want to work globally.

Next we give a relation between the categories $\mathcal{S}_\theta(\Lambda)$ in Section 3. The Coxeter group W_Q acts on both the set of dimension vectors and the parameter space Θ. So the variation of parameter θ is described by an action of W_Q on Θ. Corresponding to $w \in W_Q$, a tilting module I_w is constructed. Then we have the triangle auto-equivalence

$$\mathcal{D}(\mathrm{Mod}\Lambda) \underset{-\overset{\mathbb{L}}{\otimes}_\Lambda I_w}{\overset{\mathbb{R}\mathrm{Hom}_\Lambda(I_w,-)}{\rightleftarrows}} \mathcal{D}(\mathrm{Mod}\Lambda).$$

Our first main result is the following.

Theorem 1.2 (Theorems 3.5 and 3.12) *For any preprojective algebra Λ and any $\theta \in \Theta$ with $\theta_i > 0$, there is an equivalence*

$$\mathcal{S}_\theta(\Lambda) \underset{-\otimes_\Lambda I_i}{\overset{\mathrm{Hom}_\Lambda(I_i,-)}{\rightleftarrows}} \mathcal{S}_{s_i\theta}(\Lambda)$$

which preserves S-equivalence classes. Moreover for any element $w \in W_Q$ and any sufficiently general parameter $\theta \in \Theta$, a composition of the above equivalences gives an equivalence

$$\mathcal{S}_\theta(\Lambda) \underset{-\otimes_\Lambda I_w}{\overset{\mathrm{Hom}_\Lambda(I_w,-)}{\rightleftarrows}} \mathcal{S}_{w\theta}(\Lambda)$$

which preserves S-equivalence classes. Furthermore it restricts to an equivalence between $\mathcal{S}_{\theta,\alpha}(\Lambda)$ and $\mathcal{S}_{w\theta,w\alpha}(\Lambda)$.

We call these functors the reflection functors. A reflection functor induces a bijection on sets of closed points of $\mathcal{M}_{\theta,\alpha}(\Lambda)$ and $\mathcal{M}_{w\theta,w\alpha}(\Lambda)$. It is natural to hope that this bijection is extended to an isomorphism of K-schemes. Our second main result as below says that it is actually true. It is proved by using functors of points in Section 4. The benefit of this paper is that the results obtained in there are very general due to the homological nature of the results, and can be applied to much more complicated situations, for example higher dimensions.

Theorem 1.3 (Theorem 4.20) *Let Λ be a K-algebra and T a tilting Λ-module. Put $\Gamma = \mathrm{End}_\Lambda(T)$. Assume that Λ and Γ are given by finite quivers with relations. If a derived equivalence*

$$\mathcal{D}(\mathrm{Mod}\Lambda) \underset{-\overset{\mathbb{L}}{\otimes}_\Gamma T}{\overset{\mathbb{R}\mathrm{Hom}_\Lambda(T,-)}{\rightleftarrows}} \mathcal{D}(\mathrm{Mod}\Gamma).$$

induces an equivalence

$$\mathcal{S}_{\theta,\alpha}(\Lambda) \underset{-\otimes_\Gamma T}{\overset{\mathrm{Hom}_\Lambda(T,-)}{\rightleftarrows}} \mathcal{S}_{\eta,\beta}(\Gamma)$$

preserving S-equivalence classes, then it induces an isomorphism $\mathcal{M}_{\theta,\alpha}(\Lambda) \xrightarrow{\sim} \mathcal{M}_{\eta,\beta}(\Gamma)$ *of K-schemes where* α, β *are dimension vectors and* θ, η *are parameters.*

We remark that Nakajima [32] has already considered similar isomorphisms between quiver varieties, which is a space of representations over a deformed preprojective algebra. But it is defined as a hyper-Kähler quotient and his method is differential geometric. Maffei [29] also studied a similar thing by using geometric invariant theory. So our result is more or less known. However we obtain a new realization of isomorphisms, namely these induced by the above equivalences between derived categories. Furthermore since our reflection functor is realized as an equivalence of categories, it gives the correspondence between not only the objects but also morphisms between them.

In Section 5, we focus on the Kleinian singularity case. Let G be a finite subgroup of $SL(2, K)$ and Λ the preprojective algebra of the extended Dynkin quiver Q corresponding to G. We denote the vertexes of Q by $0, 1, \ldots, n$ where 0 corresponds to the trivial representation of G. Let S_i be the simple Λ-module associated to a vertex $i \in Q_0$.

For a parameter θ such that $\theta_i > 0$ for any $i \neq 0$, every module in $\mathcal{S}_{\theta,\mathbf{d}}(\Lambda)$ is characterized by the property that it is generated by the vertex 0. Moreover the moduli space $\mathcal{M}_{\theta,\mathbf{d}}(\Lambda)$ is naturally identified with the G-Hilbert scheme [16]. There is a concrete correspondence between exceptional curves on G-Hilbert scheme and irreducible representations of G, which is known as a McKay correspondence [23]. In terms of Λ-modules, a θ-stable Λ-module belongs to exceptional curve E_i if and only if it contains S_i as a submodule [16].

We generalize these characterizations of modules in $\mathcal{S}_{\theta,\mathbf{d}}(\Lambda)$ (Theorem 5.15) and modules on each exceptional curve (Theorem 5.19) for every generic parameter θ as an application of Theorems 1.2 and 1.3. An important idea is considering $\mathbb{R}\mathrm{Hom}_\Lambda(I_w, S_i)$ instead of S_i in the above special case.

2 Tilting Theory on Preprojective Algebras

In this section, we study the representation theory of preprojective algebras of non-Dynkin quivers. The results stated in this section play an important role in our study of moduli spaces of modules over preprojective algebras, especially resolutions of Kleinian singularities, in the latter sections.

In the first subsection we recall tilting modules and equivalences induced from them. In the second subsection we define preprojective algebras and recall their basic properties. In the third subsection we recall the construction of a set of tilting modules over preprojective algebras (Theorem 2.20) shown in [11]. We give a proof of this result again because the setting in this paper is different from that of [11]. Namely they dealt with completed preprojective algebras, but we deal with non-completed ones. We study properties of these tilting modules in the fourth and fifth subsections. In the sixth subsection we study the category of finite dimensional modules over the preprojective algebra of an extended Dynkin quiver.

We give conventions. For a K-algebra Λ, we denote by $\mathrm{Mod}\Lambda$ the category of right Λ-modules, $\mathrm{fd}\Lambda$ the abelian full subcategory of $\mathrm{Mod}\Lambda$ whose objects consist of finite dimensional Λ-modules. We write $D := \mathrm{Hom}_K(-, K)$ the K-dual. The category $\mathrm{fd}\Lambda$ has the duality

$$D : \mathrm{fd}\Lambda \longrightarrow \mathrm{fd}\Lambda^{\mathrm{op}}$$

such that $D \circ D$ is isomorphic to the identity functor.

2.1 Tilting Modules

This subsection is devoted to recalling tilting modules and their applications for the readers convenience.

Definition 2.1 Let Λ be a K-algebra. A Λ-module T is called a *tilting module of projective dimension at most d* if it satisfies the following conditions.

(1) There exists an exact sequence

$$0 \longrightarrow P_d \longrightarrow \cdots\cdots \longrightarrow P_0 \longrightarrow T \longrightarrow 0 \tag{2.1}$$

where each P_i is a finitely generated projective Λ-module.

(2) $\mathrm{Ext}^i_\Lambda(T, T) = 0$ for any $1 \le i \le d$.

(3) There exists an exact sequence

$$0 \longrightarrow \Lambda_\Lambda \longrightarrow T_0 \longrightarrow \cdots\cdots \longrightarrow T_d \longrightarrow 0 \tag{2.2}$$

where each T_i is in $\mathrm{add}T$.

In particular, tilting modules of projective dimension at most one are called classical tilting modules.

Tilting is a self-dual notion in the following sense.

Lemma 2.2 [4] *Let Λ be a K-algebra and T a tilting Λ-module of projective dimension d. We put $\Gamma := \mathrm{End}_\Lambda(T)$. Then T is a tilting Γ^{op}-module of projective dimension d such that the map*

$$\Lambda^{\mathrm{op}} \ni a \longmapsto (t \mapsto ta) \in \mathrm{End}_{\Gamma^{\mathrm{op}}}(T)$$

is a K-algebra isomorphism.

The notion of tilting modules is important since they give derived equivalences between algebras. For an algebra Λ, we denote by $\mathcal{D}(\mathrm{Mod}\Lambda)$ the derived category of $\mathrm{Mod}\Lambda$, and $\mathcal{D}_{\mathrm{fd}}(\mathrm{Mod}\Lambda)$ the triangulated full subcategory of $\mathcal{D}(\mathrm{Mod}\Lambda)$ which consists of complexes whose total homology is in $\mathrm{fd}\Lambda$.

Theorem 2.3 [35, 36] *Let Λ be a K-algebra and T a tilting Λ-module of projective dimension d. We put $\Gamma := \mathrm{End}_\Lambda(T)$. Then there exist triangle equivalences*

$$\mathcal{D}(\mathrm{Mod}\Lambda) \underset{-\overset{\mathbb{L}}{\otimes}_\Gamma T}{\overset{\mathbb{R}\mathrm{Hom}_\Lambda(T,-)}{\rightleftarrows}} \mathcal{D}(\mathrm{Mod}\Gamma),$$

$$\mathcal{D}_{\mathrm{fd}}(\mathrm{Mod}\Lambda) \underset{-\overset{\mathbb{L}}{\otimes}_\Gamma T}{\overset{\mathbb{R}\mathrm{Hom}_\Lambda(T,-)}{\rightleftarrows}} \mathcal{D}_{\mathrm{fd}}(\mathrm{Mod}\Gamma) \quad .$$

To show the second equivalence, we need the following lemma.

Lemma 2.4 *Let X be a bounded complex in $\mathcal{D}_{\mathrm{fd}}(\mathrm{Mod}\Lambda)$. We take integers $k < \ell$ such that $X^i = 0$ for $i < k$ and $i > \ell$. Then there exist triangles in $\mathcal{D}_{\mathrm{fd}}(\mathrm{Mod}\Lambda)$*

$$\begin{array}{c} X_1 \to X \to Y_\ell \to X_1[1] \\ X_2 \to X_1 \to Y_{\ell-1} \to X_2[1] \\ \vdots \\ X_{\ell-k} \to X_{\ell-k+1} \to Y_k \to X_{\ell-k}[1] \\ Y_k \to X_{\ell-k} \to Y_{k+1} \to Y_k[1] \end{array}$$

such that $Y_i \simeq \mathrm{H}^i(X)[-i]$ in $\mathcal{D}_{\mathrm{fd}}(\mathrm{Mod}\Lambda)$ for $k \le i \le \ell$.

Proof Straightforward. □

Proof of Theorem 2.3 The first triangle equivalence is given by [35]. We show that the restriction of the first equivalences induce the second one.

By the condition (1) in the Definition 2.1, for any finite dimensional Λ-module M, $\mathrm{Hom}_\Lambda(T, M)$ and $\mathrm{Ext}^i_\Lambda(T, M)$ are finite dimensional for any $i > 0$. Thus we have $\mathbb{R}\mathrm{Hom}_\Lambda(T, M)$ is in $\mathcal{D}_{\mathrm{fd}}(\mathrm{Mod}\Lambda)$. Since $\mathbb{R}\mathrm{Hom}_\Lambda(T, -)$ is a triangle functor and by Lemma 2.4, $\mathbb{R}\mathrm{Hom}_\Lambda(T, X)$ is in $\mathcal{D}_{\mathrm{fd}}(\mathrm{Mod}\Lambda)$ for any $X \in \mathcal{D}_{\mathrm{fd}}(\mathrm{Mod}\Lambda)$.

Since T is a tilting Λ^{op}-module and $-\overset{\mathbb{L}}{\otimes}_\Lambda T$ is a triangle functor, we can show that $X \overset{\mathbb{L}}{\otimes}_\Lambda T$ is in $\mathcal{D}_{\mathrm{fd}}(\mathrm{Mod}\Lambda)$ for any $X \in \mathcal{D}_{\mathrm{fd}}(\mathrm{Mod}\Lambda)$ by the similar argument as above. □

Next we focus into pairs $(\mathcal{T}, \mathcal{F})$ of full subcategories of categories of finite dimensional modules, which are called torsion pairs.

Definition 2.5 Let Λ be a K-algebra. A pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\mathrm{fd}\Lambda$ is called a *torsion pair* if it satisfies the following conditions.

(1) $\mathrm{Hom}_\Lambda(\mathcal{T}, \mathcal{F}) = 0$.
(2) For any $M \in \mathrm{fd}\Lambda$, there exists an exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

such that $L \in \mathcal{T}$ and $N \in \mathcal{F}$.

We show that a classical tilting module induces two torsion pairs. Let Λ be a K-algebra. For a Λ-module T, we define a pair of full subcategories $\mathcal{T}(T)$ and $\mathcal{F}(T)$ of $\mathrm{fd}\Lambda$ by

$$\mathcal{T}(T) := \{M \in \mathrm{fd}\Lambda \mid \mathrm{Ext}^1_\Lambda(T, M) = 0\}$$

$$\mathcal{F}(T) := \{M \in \mathrm{fd}\Lambda \mid \mathrm{Hom}_\Lambda(T, M) = 0\}.$$

We put $\Gamma := \mathrm{End}_\Lambda(T)$. We also define a pair of full subcategories $\mathcal{X}(T)$ and $\mathcal{Y}(T)$ of $\mathrm{fd}\Gamma$ by

$$\mathcal{X}(T) := \{M \in \mathrm{fd}\Gamma \mid M \otimes_\Gamma T = 0\}$$

$$\mathcal{Y}(T) := \{M \in \mathrm{fd}\Gamma \mid \mathrm{Tor}^\Gamma_1(M, T) = 0\}.$$

By the definition, the above subcategories are closed under extensions and finite direct sums. It is obvious that $\mathcal{X}(T)$ is closed under images and $\mathcal{F}(T)$ is closed under submodules.

If T is a classical tilting Λ-module, then the above pairs form torsion pairs in each category of finite dimensional modules. To show this, we need the following well-known isomorphism.

Lemma 2.6 [13, Chapter VI Proposition 5.1] *Let Λ and Γ be K-algebras. We have a homomorphism*

$$\mathrm{Ext}^i_\Lambda(L, \mathrm{Hom}_\Gamma(M, N)) \longrightarrow \mathrm{Hom}_\Gamma(\mathrm{Tor}^\Lambda_i(L, M), N)$$

for any $L \in \mathrm{Mod}\Lambda$, $M \in \mathrm{Mod}(\Lambda^{\mathrm{op}} \otimes_K \Gamma)$, $N \in \mathrm{Mod}\Gamma$ and $i \in \mathbb{N} \cup \{0\}$. If N is an injective Γ-module, then this is an isomorphism.

Proposition 2.7 [4] *Let Λ be a K-algebra and T a classical tilting Λ-module. We put $\Gamma := \mathrm{End}_\Lambda(T)$. Then the following assertions hold.*

(1) *$\mathcal{T}(T)$ is closed under images.*
(2) *$\mathcal{Y}(T)$ is closed under submodules.*
(3) *$(\mathcal{T}(T), \mathcal{F}(T))$ forms a torsion pair in* $\mathrm{fd}\Lambda$.
(4) *$(\mathcal{X}(T), \mathcal{Y}(T))$ forms a torsion pair in* $\mathrm{fd}\Gamma$.

Proof

(1) Since T is a Λ-module of projective dimension at most one, $\mathcal{T}(T)$ is closed under images.
(2) By Lemma 2.2, T is a Γ^{op}-module of projective dimension at most one. Thus $\mathcal{Y}(T)$ is closed under submodules.
(3) First we show $(\mathcal{T}(T), \mathcal{F}(T))$ satisfies the first condition in Definition 2.5. Since $\mathcal{T}(T)$ is closed under images and $\mathcal{F}(T)$ is closed under submodules, it is enough to show $\mathcal{T}(T) \cap \mathcal{F}(T) = 0$. We take $M \in \mathcal{T}(T) \cap \mathcal{F}(T)$. Then by applying $\mathrm{Hom}_\Lambda(-, M)$ to the exact sequence (2.2) in Definition 2.1, we have an exact sequence

$$0 = \mathrm{Hom}_\Lambda(T_0, M) \longrightarrow \mathrm{Hom}_\Lambda(\Lambda, M) \longrightarrow \mathrm{Ext}^1_\Lambda(T_1, M) = 0.$$

Thus we have $M = 0$.

Next we show that $(\mathcal{T}(T), \mathcal{F}(T))$ satisfies the second condition in Definition 2.5. Since $\mathcal{T}(T)$ is closed under images and finite direct sums, there exists a unique maximal submodule L of M such that $L \in \mathcal{T}(T)$. We put $N := M/L$ and show $N \in \mathcal{F}(T)$. Let $a \in \mathrm{Hom}_\Lambda(T, N)$. By taking a pullback, we consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & E & \longrightarrow & T & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle a} & & \\ 0 & \longrightarrow & L & \xrightarrow[f]{} & M & \xrightarrow[g]{} & N & \longrightarrow & 0 \end{array}$$

where f is the inclusion. Since $\mathcal{T}(T)$ is closed under images and extensions, we have $\mathrm{Im} b \in \mathcal{T}(T)$. Thus by the definition of L, b factors through f. Consequently we have $a = 0$, and hence $\mathrm{Hom}_\Lambda(T, N) = 0$.

(4) Since T is a tilting Γ^{op}-module, a pair $(\mathcal{T}({}_\Gamma T), \mathcal{F}({}_\Gamma T))$ of full subcategories of $\mathrm{fd}\Gamma^{\mathrm{op}}$ forms a torsion pair by (3). By Lemma 2.6, we have

$$(\mathcal{X}(T), \mathcal{Y}(T)) = (D\mathcal{F}({}_\Gamma T), D\mathcal{T}({}_\Gamma T)).$$

Thus we have the assertion by these facts and the fact that D is a duality on $\mathrm{fd}\Lambda$. □

By Proposition 2.7, we have the following lemma. We need this to show Theorem 2.20.

Lemma 2.8 *Let* Λ *be a* K*-algebra and* T *a classical tilting* Λ*-module. We put* $\Gamma := \mathrm{End}_\Lambda(T)$. *For a finite dimensional simple* Λ^{op}*-module* S*, exactly one of the statements* $T \otimes_\Lambda S = 0$ *and* $\mathrm{Tor}_1^\Lambda(T, S) = 0$ *holds.*

Proof By Lemma 2.2, T is a tilting Γ^{op}-module with $\Lambda^{\mathrm{op}} \simeq \mathrm{End}_{\Gamma^{\mathrm{op}}}(T)$. Thus by Proposition 2.7, $(\mathcal{X}({}_\Gamma T), \mathcal{Y}({}_\Gamma T))$ is a torsion pair in $\mathrm{fd}\Lambda^{\mathrm{op}}$. For any finite dimensional simple Λ^{op}-module M, exactly one of the statements $S \in \mathcal{X}({}_\Gamma T)$ and $S \in \mathcal{Y}({}_\Gamma T))$ holds by the conditions of Definition 2.5. Thus the assertion follows. □

Classical tilting modules not only induce two torsion pairs, but also induce the following categorical equivalences.

Lemma 2.9 *Let* Λ *be a* K*-algebra and* T *a tilting* Λ*-module. We put* $\Gamma := \mathrm{End}_\Lambda(T)$. *Then there exist categorical equivalences*

$$\mathcal{T}(T) \underset{-\otimes_\Gamma T}{\overset{\mathrm{Hom}_\Lambda(T,-)}{\rightleftarrows}} \mathcal{Y}(T),$$

$$\mathcal{F}(T) \underset{\mathrm{Tor}_1^\Gamma(-,T)}{\overset{\mathrm{Ext}_\Lambda^1(T,-)}{\rightleftarrows}} \mathcal{X}(T) \;.$$

Proof The assertions immediately follow from Theorem 2.3. □

2.2 Preprojective Algebras

In this subsection, we recall basic facts about preprojective algebras of non-Dynkin quivers. We start with basic definitions.

A *quiver* is a quadruple $Q=(Q_0, Q_1, s, t)$ which consists of a vertex set Q_0, an arrow set Q_1 and maps $s, t : Q_1 \to Q_0$ which associate to each arrow $a \in Q_1$ its source $sa := s(a) \in Q_0$ and its target $ta := t(a) \in Q_0$, respectively. We call $a \in Q_1$ a *loop* if $sa = ta$. A quiver is called non-Dynkin if its underlying graph is not a Dykin graph.

Definition 2.10 Let Q be a finite connected quiver. We define the double quiver $\overline{Q}$ of Q by

$$\overline{Q}_0 := Q_0$$

and

$$\overline{Q}_1 := Q_1 \bigsqcup \left\{ j \xrightarrow{\alpha^*} i \mid i \xrightarrow{\alpha} j \in Q_1 \right\}.$$

Then we have a bijection $(-)^* : \overline{Q}_1 \longrightarrow \overline{Q}_1$ which is defined by

$$\alpha^* := \begin{cases} \alpha^* & (\alpha \in Q_1), \\ \beta & (\alpha = \beta^* \text{ for some } \beta \in Q_1). \end{cases}$$

We define a relation ρ_i for any $i \in \overline{Q}_0$ by

$$\rho_i := \sum_{i \xrightarrow{\alpha} j \in \overline{Q}_1} \epsilon_\alpha \alpha \alpha^*$$

where

$$\epsilon_\alpha := \begin{cases} 1 & (\alpha \in Q_1), \\ -1 & (\alpha^* \in Q_1). \end{cases}$$

A relation $\sum_{i \in \overline{Q}_0} \rho_i$ is called the *preprojective relation*. We call the algebra

$$K\overline{Q}/\langle \rho_i \mid i \in \overline{Q}_0 \rangle$$

the preprojective algebra of Q.

Remark 2.11 We give three remarks.

(1) Let Q and Q' be quivers which have the same underlying graph. Then the preprojective algebra of Q and that of Q' are isomorphic to each other as K-algebras.
(2) The preprojective algebra of a quiver Q is finite dimensional if and only if Q is a Dynkin quiver.
(3) The preprojective algebra of an extended Dynkin quiver is noetherian (see [5]).

In the rest of this section, let Λ be the preprojective algebra of a finite connected non-Dynkin quiver Q which has no loops with the vertex set $Q_0 = \{0, 1, \ldots, n\}$. We define several notations.

Definition 2.12 We define I as the two-sided ideal of Λ which is generated by all arrows of $\overline{Q}$.

We denote by e_i the primitive idempotent of Λ which corresponds to a vertex $i \in \overline{Q}_0$. We define a two-sided ideal I_i of Λ by

$$I_i := \Lambda(1 - e_i)\Lambda.$$

We denote by S_i the simple $\Lambda^{\mathrm{op}} \otimes_K \Lambda$-module which corresponds to a vertex $i \in \overline{Q}_0$. We sometimes regard S_i as a simple Λ-module and a simple Λ^{op}-module.

We give an easy observation. It is obvious that

$$\Lambda/I = K\overline{Q}_0,$$
$$S_i = e_i(\Lambda/I) = (\Lambda/I)e_i.$$

Since Q has no loops, so does $\overline{Q}$. This implies the equations

$$e_j I_i = \begin{cases} e_i I & \text{if } i = j, \\ e_j \Lambda & \text{if } i \neq j, \end{cases}$$
$$\Lambda/I_i = S_i$$

for $i, j \in \overline{Q}_0$.

We prove easy lemmas which are used many times in this paper.

Lemma 2.13 *For $i \in \overline{Q}_0$, we have* $\mathrm{Hom}_\Lambda(I_i, S_i) = 0$.

Proof We take $f \in \mathrm{Hom}_\Lambda(I_i, S_i)$. For $x(1 - e_i)y \in \Lambda(1 - e_i)\Lambda = I_i$, we have $f(x(1 - e_i)y) = f(x(1 - e_i))(1 - e_i)y = 0$. Thus we have $f = 0$. □

We define nilpotent Λ-modules as follows.

Definition 2.14 A Λ-module M is called *nilpotent* if there exists an $m \in \mathbb{N}$ such that $MI^m = 0$. Since $\Lambda/I = K\overline{Q}_0$, a Λ-module M is finite dimensional nilpotent if and only if M has finite length and its composition factors consist of $S_0, S_1, \ldots, S_n$. We denote by $\mathrm{nilp}\Lambda$ the abelian full subcategory of $\mathrm{fd}\Lambda$ which consists of finite dimensional nilpotent Λ-modules.

Lemma 2.15 *Let $M \in \mathrm{nilp}\Lambda$. We put $a_i = \dim \mathrm{Hom}_\Lambda(M, S_i)$ and $b_i = \dim \mathrm{Hom}_\Lambda(S_i, M)$. Then the following assertions hold.*

(1) $M/MI \simeq S_0^{a_0} \oplus S_1^{a_1} \oplus \cdots \oplus S_n^{a_n}$ *as Λ-modules.*
(2) $\mathrm{Soc} M \simeq S_0^{b_0} \oplus S_1^{b_1} \oplus \cdots \oplus S_n^{b_n}$ *as Λ-modules.*
(3) $\mathrm{Hom}_\Lambda(M, S_i) \simeq S_i^{a_i}$ *as Λ^{op}-modules.*
(4) $\mathrm{Hom}_\Lambda(S_i, M) \simeq S_i^{b_i}$ *as Λ-modules.*

Proof We only prove (1) and (3) since the others are proved by the same argument.

(1) Since M/MI is a $K\overline{Q}_0$-module, we can write $M/MI \simeq S_0^{c_0} \oplus S_1^{c_1} \oplus \cdots \oplus S_n^{c_n}$. Since S_i is annihilated by I, we have

$$\mathrm{Hom}_\Lambda(M, S_i) \simeq \mathrm{Hom}_\Lambda(M/MI, S_i) \simeq \bigoplus_{j=0}^{n} \mathrm{Hom}_\Lambda(S_j^{c_j}, S_i) = \mathrm{Hom}_\Lambda(S_i^{c_i}, S_i).$$

These isomorphisms imply that $c_i = a_i$.

(3) The left Λ-action on S_i induces a left Λ-action on $\mathrm{Hom}_\Lambda(M, S_i)$. Thus $\mathrm{Hom}_\Lambda(M, S_i)$ is a Λ^{op}-module. Moreover since $\mathrm{Hom}_\Lambda(M, S_i)$ is annihilated by I_i, it is a $(\Lambda/I_i)^{\mathrm{op}}$-module. Thus we have $\mathrm{Hom}_\Lambda(M, S_i) \simeq S_i^c$ as Λ^{op}-modules for some c. By the calculation in the proof of (1), we have $c = a_i$. □

For finite dimensional nilpotent Λ-modules, the following analogue of Nakayama's lemma holds.

Lemma 2.16 *Let $M \in \mathrm{nilp}\Lambda$. Then the following are equivalent.*

(1) $M = 0$.
(2) $M = MI$.
(3) $\mathrm{Hom}_\Lambda(M, S_i) = 0$ *for any* $i \in \overline{Q_0}$.
(4) $\mathrm{Hom}_\Lambda(S_i, M) = 0$ *for any* $i \in \overline{Q_0}$.

In the rest of this subsection, we recall properties of Λ which are shown by using the assumption that Λ is the preprojective algebra of a non-Dynkin quiver essentially. First we give explicit projective resolutions of simple Λ-modules $S_0, S_1, \ldots, S_n$ which play a crucial role in the representation theory of Λ.

Proposition 2.17 [8, Section 4.1] *For any $i \in \overline{Q_0}$, the following hold.*

(1) *A complex*

$$0 \longrightarrow e_i\Lambda \xrightarrow{(\epsilon_\alpha \alpha^*)_{i \xrightarrow{\alpha} j \in \overline{Q}_1}} \bigoplus_{i \xrightarrow{\alpha} j \in \overline{Q}_1} e_j\Lambda \xrightarrow{(\alpha)_{i \xrightarrow{\alpha} j \in \overline{Q}_1}} e_i\Lambda \longrightarrow S_i \longrightarrow 0 \qquad (2.3)$$

is a projective resolution of the right Λ-module S_i.

(2) *A complex*

$$0 \longrightarrow \Lambda e_i \xrightarrow{(\epsilon_{\alpha^*} \alpha^*)_{j \xrightarrow{\alpha} i \in \overline{Q}_1}} \bigoplus_{j \xrightarrow{\alpha} i \in \overline{Q}_1} \Lambda e_j \xrightarrow{(\alpha)_{j \xrightarrow{\alpha} i \in \overline{Q}_1}} \Lambda e_i \longrightarrow S_i \longrightarrow 0 \qquad (2.4)$$

is a projective resolution of the left Λ-module S_i.

The following property is called the 2-Calabi-Yau property.

Lemma 2.18 *There exists a functorial isomorphism*

$$\mathrm{Hom}_{\mathcal{D}(\mathrm{Mod}\Lambda)}(M, N) \simeq D\mathrm{Hom}_{\mathcal{D}(\mathrm{Mod}\Lambda)}(N, M[2])$$

for any $M \in \mathcal{D}_{\mathrm{fd}}(\mathrm{Mod}\Lambda)$ and any $N \in \mathcal{D}(\mathrm{Mod}\Lambda)$.

Proof See [9, Theorem 9.2] and [25, Lemma 4.1]. □

Now we recall a useful lemma. The dimension vector $\underline{\dim} M$ of a finite dimensional Λ-module M is defined by

$$\underline{\dim} M :=^t (\dim(Me_0), \dim(Me_1), \ldots, \dim(Me_n)) \in \mathbb{Z}^{\overline{Q}_0}.$$

If M is nilpotent, then $\dim(Me_i)$ coincides with the multiplicity of S_i appearing composition factor of M.

Let $(-,-)$ be a symmetric bilinear form on $\mathbb{Z}^{\overline{Q}_0}$ defined by

$$(\alpha,\beta)=\sum_{i\in\overline{Q}_0}2\alpha_i\beta_i-\sum_{a\in\overline{Q}_1}\alpha_{sa}\beta_{ta}.$$

We define $(M,N):=(\underline{\dim}M,\underline{\dim}N)$ for any finite dimensional Λ-modules M, N.

Lemma 2.19 [16, Lemma 1] *Let M and N be finite dimensional Λ-modules. Then the following holds.*

$$\begin{aligned}(M,N)&=\dim\mathrm{Hom}_\Lambda(M,N)-\dim\mathrm{Ext}^1_\Lambda(M,N)+\dim\mathrm{Ext}^2_\Lambda(M,N)\\&=\dim\mathrm{Hom}_\Lambda(M,N)-\dim\mathrm{Ext}^1_\Lambda(M,N)+\dim\mathrm{Hom}_\Lambda(N,M).\end{aligned}$$

2.3 Construction of Classical Tilting Modules Over Preprojective Algebras

In this subsection, we recall the construction of classical tilting modules over preprojective algebras of non-Dynkin quivers which was given by [11]. We keep the notations in the previous section.

In Definition 2.12, we defined the two-sided ideal I_i of Λ by

$$I_i:=\Lambda(1-e_i)\Lambda$$

for any $i\in\overline{Q}_0$. As we observed, there is an exact sequence

$$0\longrightarrow I_i\longrightarrow\Lambda\longrightarrow S_i\longrightarrow 0 \tag{2.5}$$

of $\Lambda^{\mathrm{op}}\otimes_K\Lambda$-modules for any $i\in\overline{Q}_0$ since $\overline{Q}$ has no loops. Now we consider a set

$$\mathcal{I}(\Lambda):=\{I_{i_1}I_{i_2}\cdots I_{i_\ell}\mid l\in\mathbb{N}\cup\{0\},\ i_1,i_2,\ldots,i_\ell\in Q_0\}$$

where $I_{i_1}I_{i_2}\cdots I_{i_\ell}$ is a two-sided ideal which is obtained by the product of ideals $I_{i_1},I_{i_2},\ldots,I_{i_\ell}$. This set of two-sided ideals gives a large class of classical tilting Λ-modules.

Theorem 2.20 [11, Proposition III.1.4. Theorem III.1.6] *Any $T\in\mathcal{I}(\Lambda)$ is a classical tilting Λ-module with* $\mathrm{End}_\Lambda(T)\simeq\Lambda$.

Proof First we show that I_i is a classical tilting Λ-module for any i. We remark that $I_i=(\bigoplus_{j\neq i}e_j\Lambda)\oplus e_iI_i$. By Proposition 2.17, there exists an exact sequence

$$0\longrightarrow e_i\Lambda\longrightarrow\bigoplus_{j\to i}e_j\Lambda\longrightarrow e_iI_i\longrightarrow 0.$$

Thus we have exact sequences of the forms (2.1) and (2.2) in Definition 2.1. We show $\mathrm{Ext}^1_\Lambda(I_i,I_i)=0$. By applying $\mathrm{Hom}_\Lambda(-,I_i)$ to the exact sequence (2.5), we have an exact sequece

$$0=\mathrm{Ext}^1_\Lambda(\Lambda,I_i)\longrightarrow\mathrm{Ext}^1_\Lambda(I_i,I_i)\longrightarrow\mathrm{Ext}^2_\Lambda(S_i,I_i)\longrightarrow\mathrm{Ext}^2_\Lambda(\Lambda,I_i)=0.$$

By this and Lemmas 2.18 and 2.13, we have

$$\mathrm{Ext}^1_\Lambda(I_i, I_i) \simeq \mathrm{Ext}^2_\Lambda(S_i, I_i) \simeq D\mathrm{Hom}_\Lambda(I_i, S_i) = 0.$$

Thus I_i is a classical tilting Λ-module.

Next we show $\mathrm{End}_\Lambda(I_i) \simeq \Lambda$. By applying $\mathrm{Hom}_\Lambda(-, \Lambda)$ to the exact sequence (2.5), we have an exact sequece

$$0 \longrightarrow \mathrm{Hom}_\Lambda(S_i, \Lambda) \longrightarrow \mathrm{Hom}_\Lambda(\Lambda, \Lambda) \longrightarrow \mathrm{Hom}_\Lambda(I_i, \Lambda) \longrightarrow \mathrm{Ext}^1_\Lambda(S_i, \Lambda).$$

Since $\mathrm{Ext}^j_\Lambda(S_i, \Lambda) \simeq D\mathrm{Ext}^{2-j}_\Lambda(\Lambda, S_i) = 0$ for $j = 0, 1$ by Lemma 2.18, we have $\mathrm{Hom}_\Lambda(\Lambda, \Lambda) \simeq \mathrm{Hom}_\Lambda(I_i, \Lambda)$. By applying $\mathrm{Hom}_\Lambda(I_i, -)$ to the exact sequence (2.5), we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}_\Lambda(I_i, I_i) \longrightarrow \mathrm{Hom}_\Lambda(I_i, \Lambda) \longrightarrow \mathrm{Hom}_\Lambda(I_i, S_i).$$

By Lemma 2.13, we have $\mathrm{Hom}_\Lambda(I_i, I_i) \simeq \mathrm{Hom}_\Lambda(I_i, \Lambda)$, and so we have

$$\mathrm{Hom}_\Lambda(\Lambda, \Lambda) \simeq \mathrm{Hom}_\Lambda(I_i, \Lambda) \simeq \mathrm{Hom}_\Lambda(I_i, I_i).$$

Since the above isomorphism is given by

$$\mathrm{Hom}_\Lambda(\Lambda, \Lambda) \ni a\cdot \longmapsto a\cdot \in \mathrm{Hom}_\Lambda(I_i, I_i),$$

we have a K-algebra isomorphism $\Lambda \simeq \mathrm{End}_\Lambda(I_i)$.

Finally we show that $I_{i_\ell} \cdots I_{i_2} I_{i_1}$ is a classical tilting Λ-module with $\mathrm{End}_\Lambda(I_{i_\ell} \cdots I_{i_2} I_{i_1}) = \Lambda$ for any $\ell \in \mathbb{N} \cup \{0\}$ and $i_1, i_2, \ldots, i_\ell \in Q_0$ by induction on ℓ. If $I_{i_\ell} \cdots I_{i_2} I_{i_1} = I_{i_\ell} \cdots I_{i_2}$, it is a classical tilting Λ-module with $\mathrm{End}_\Lambda(I_{i_\ell} \cdots I_{i_2} I_{i_1}) = \Lambda$ by inductive hypothesis.

We assume $I_{i_\ell} \cdots I_{i_2} I_{i_1} \neq I_{i_\ell} \cdots I_{i_2}$. By [37, Corollary 1.7.(3)], $I_{i_\ell} \cdots I_{i_2} \overset{\mathbb{L}}{\otimes}_\Lambda I_{i_1}$ is a tilting complex in $\mathcal{D}(\mathrm{Mod}\Lambda)$, so we have

$$\mathrm{Hom}_{\mathcal{D}(\mathrm{Mod}\Lambda)}(I_{i_\ell} \cdots I_{i_2} \overset{\mathbb{L}}{\otimes}_\Lambda I_{i_1}, I_{i_\ell} \cdots I_{i_2} \overset{\mathbb{L}}{\otimes}_\Lambda I_{i_1}) \simeq \mathrm{Hom}_\Lambda(I_{i_\ell} \cdots I_{i_2}, I_{i_\ell} \cdots I_{i_2}) \simeq \Lambda.$$

Hence it is enough to show that

$$(I_{i_\ell} \cdots I_{i_2}) \overset{\mathbb{L}}{\otimes}_\Lambda I_{i_1} = (I_{i_\ell} \cdots I_{i_2}) \otimes_\Lambda I_{i_1} = I_{i_\ell} \cdots I_{i_2} I_{i_1}$$

and $\mathrm{pd}(I_{i_\ell} \cdots I_{i_2} I_{i_1}) \leq 1$.

Since $\mathrm{pd} S_{i_1} = 2$, we have $\mathrm{Tor}^\Lambda_j(I_{i_\ell} \cdots I_{i_2}, I_{i_1}) \simeq \mathrm{Tor}^\Lambda_{j+2}(\Lambda/(I_{i_\ell} \cdots I_{i_2}), S_{i_1}) = 0$ for any $j \neq 0$. Thus we have $(I_{i_\ell} \cdots I_{i_2}) \overset{\mathbb{L}}{\otimes}_\Lambda I_{i_1} = (I_{i_\ell} \cdots I_{i_2}) \otimes_\Lambda I_{i_1}$.

Now we show $(I_{i_\ell} \cdots I_{i_2}) \otimes_\Lambda I_{i_1} = I_{i_\ell} \cdots I_{i_2} I_{i_1}$. By applying $I_{i_\ell} \cdots I_{i_2} \otimes_\Lambda -$ to an exact sequence

$$0 \longrightarrow I_{i_1} \longrightarrow \Lambda \longrightarrow S_{i_1} \longrightarrow 0,$$

we have a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Tor}^\Lambda_1(I_{i_\ell} \cdots I_{i_2}, S_{i_1}) & \longrightarrow & (I_{i_\ell} \cdots I_{i_2}) \otimes_\Lambda I_{i_1} & \longrightarrow & (I_{i_\ell} \cdots I_{i_2}) \otimes_\Lambda \Lambda & \longrightarrow & (I_{i_\ell} \cdots I_{i_2}) \otimes_\Lambda S_{i_1} \\
 & & \downarrow f & & \downarrow g & & \\
 & & I_{i_\ell} \cdots I_{i_2} I_{i_1} & \longrightarrow & I_{i_\ell} \cdots I_{i_2} & &
\end{array}$$

such that the first row is exact, f is an epimorphism and g is an isomorphism. If $I_{i_\ell} \cdots I_{i_2} \otimes_\Lambda S_{i_1} = 0$, we have $I_{i_\ell} \cdots I_{i_2} I_{i_1} = I_{i_\ell} \cdots I_{i_2}$. This is a contradiction, hence we have $I_{i_\ell} \cdots I_{i_2} \otimes_\Lambda S_{i_1} \neq 0$. By Lemma 2.8 and the induction hypothesis, $\mathrm{Tor}_1^\Lambda(I_{i_\ell} \cdots I_{i_2}, S_{i_1}) = 0$ holds. Thus f is a monomorphism, hence we have $(I_{i_\ell} \cdots I_{i_2}) \otimes_\Lambda I_{i_1} = I_{i_\ell} \cdots I_{i_2} I_{i_1}$. Since $\mathrm{pd}(I_{i_\ell} \cdots I_{i_2}) \leq 1$ and $\mathrm{pd}((I_{i_\ell} \cdots I_{i_2})/(I_{i_\ell} \cdots I_{i_2} I_{i_1})) \leq 2$, we have $\mathrm{pd}(I_{i_\ell} \cdots I_{i_2} I_{i_1}) \leq 1$. The assertion follows. □

In the above proof, the following have been shown.

Lemma 2.21 *Let $i \in \overline{Q}_0$ and $T \in \mathcal{I}(\Lambda)$. If $T \neq TI_i$, then*

$$T \overset{\mathbb{L}}{\otimes}_\Lambda I_i = T \otimes_\Lambda I_i = TI_i.$$

For $T \in \mathcal{I}(\Lambda)$, we have triangle equivalences

$$\mathcal{D}(\mathrm{Mod}\Lambda) \underset{-\overset{\mathbb{L}}{\otimes}_\Lambda T}{\overset{\mathbb{R}\mathrm{Hom}_\Lambda(T,-)}{\rightleftarrows}} \mathcal{D}(\mathrm{Mod}\Lambda),$$

$$\mathcal{D}_{\mathrm{fd}}(\mathrm{Mod}\Lambda) \underset{-\overset{\mathbb{L}}{\otimes}_\Lambda T}{\overset{\mathbb{R}\mathrm{Hom}_\Lambda(T,-)}{\rightleftarrows}} \mathcal{D}_{\mathrm{fd}}(\mathrm{Mod}\Lambda)$$

by Theorems 2.3 and 2.20.

Next we consider the triangulated full subcategory $\mathcal{D}_{\mathrm{nilp}}(\mathrm{Mod}\Lambda)$ of $\mathcal{D}_{\mathrm{fd}}(\mathrm{Mod}\Lambda)$ which consists of complexes whose homologies are in $\mathrm{nilp}\Lambda$. We restrict the above triangle equivalences to this triangulated full subcategory.

Lemma 2.22 *Let T be a classical tilting Λ-module which is in $\mathcal{I}(\Lambda)$. Then there exist triangle equivalences*

$$\mathcal{D}_{\mathrm{nilp}}(\mathrm{Mod}\Lambda) \underset{-\overset{\mathbb{L}}{\otimes}_\Lambda T}{\overset{\mathbb{R}\mathrm{Hom}_\Lambda(T,-)}{\rightleftarrows}} \mathcal{D}_{\mathrm{nilp}}(\mathrm{Mod}\Lambda)\ .$$

Proof By the condition (1) in the Definition 2.1, for any finite dimensional nilpotent Λ-module M, $\mathrm{Hom}_\Lambda(T, M)$ and $\mathrm{Ext}^1_\Lambda(T, M)$ are finite dimensional nilpotent. Thus we have $\mathbb{R}\mathrm{Hom}_\Lambda(T, M)$ is in $\mathcal{D}_{\mathrm{nilp}}(\mathrm{Mod}\Lambda)$. Since $\mathbb{R}\mathrm{Hom}_\Lambda(T, -)$ is a triangle functor and by Lemma 2.4, $\mathbb{R}\mathrm{Hom}_\Lambda(T, X)$ is in $\mathcal{D}_{\mathrm{nilp}}(\mathrm{Mod}\Lambda)$ for any $X \in \mathcal{D}_{\mathrm{nilp}}(\mathrm{Mod}\Lambda)$.

Since T is a classical tilting Λ^{op}-module and $- \overset{\mathbb{L}}{\otimes}_\Lambda T$ is a triangle functor, we can show that $X \overset{\mathbb{L}}{\otimes}_\Lambda T$ is in $\mathcal{D}_{\mathrm{nilp}}(\mathrm{Mod}\Lambda)$ for any $X \in \mathcal{D}_{\mathrm{nilp}}(\mathrm{Mod}\Lambda)$ by the similar argument as above. □

For $T \in \mathcal{I}(\Lambda)$, we have two torsion pairs $(\mathcal{T}(T), \mathcal{F}(T))$ and $(\mathcal{X}(T), \mathcal{Y}(T))$ in $\mathrm{fd}\Lambda$ by Proposition 2.7 and Theorem 2.20. In the case $T = I_i$, we have an explicit description of these torsion pairs.

Lemma 2.23 *The following hold.*

(1) $\mathcal{T}(I_i) = \{M \in \mathrm{fd}\Lambda \mid S_i$ *is not a factor of* $M\}$.
(2) $\mathcal{F}(I_i) = \mathrm{add}S_i$.
(3) $\mathcal{X}(I_i) = \mathrm{add}S_i$.
(4) $\mathcal{Y}(I_i) = \{M \in \mathrm{fd}\Lambda \mid S_i$ *is not a submodule of* $M\}$.

Proof

(1) Let M be a finite dimensional Λ-module. By applying $\mathrm{Hom}_\Lambda(-, M)$ to the exact sequence (2.5), we have $\mathrm{Ext}^1_\Lambda(I_i, M) \simeq \mathrm{Ext}^2_\Lambda(S_i, M)$, and by Lemma 2.18, we have $\mathrm{Ext}^1_\Lambda(I_i, M) \simeq D\mathrm{Hom}_\Lambda(M, S_i)$. Thus by Lemma 2.15 the assertion follows.
(2) Let M be a finite dimensional Λ-module. If M lies in $\mathrm{add}S_i$, we have $\mathrm{Hom}_\Lambda(I_i, M) = 0$ by Lemma 2.13. Conversely we assume $\mathrm{Hom}_\Lambda(I_i, M) = 0$. Then by applying $\mathrm{Hom}_\Lambda(-, M)$ to the exact sequence (2.5), we have $M \simeq \mathrm{Hom}_\Lambda(S_i, M) \in \mathrm{add}S_i$ by Lemma 2.15.

(3) and (4) hold since $(\mathcal{X}(I_i), \mathcal{Y}(I_i)) = (D\mathcal{F}({}_\Lambda I_i), D\mathcal{T}({}_\Lambda I_i))$ as shown in the proof of Proposition 2.7, and by the opposite version of (1) and (2). □

2.4 Description of $\mathcal{I}(\Lambda)$ via the Coxeter Group

In the previous subsection we constructed the set $\mathcal{I}(\Lambda)$ of classical tilting Λ-modules. However elements in $\mathcal{I}(\Lambda)$ has many expressions (e.g. $I_i = I_i^2$). In this subsection, we recall description of elements in $\mathcal{I}(\Lambda)$ by using the Coxeter group associated to Q, which was given in [11]. We keep the notations in the previous subsections.

First we recall the definition of the Coxeter group associated to a finite quiver.

Definition 2.24 For any finite connected quiver Q with no loops (not necessarily non-Dynkin), the *Coxeter group* W_Q associated to Q is defined as a group whose generators are $s_0, \ldots, s_n$ with the relations

$$\begin{cases} s_i^2 = 1, \\ s_i s_j = s_j s_i \text{ if there is no arrows between } i \text{ and } j \text{ in } Q, \\ s_i s_j s_i = s_j s_i s_j \text{ if there is precisely one arrow between } i \text{ and } j \text{ in } Q. \end{cases}$$

In particular, if Q is a Dynkin quiver, we call W_Q the *(finite) Weyl group*, and if Q is an extended Dynkin quiver, we call W_Q the *affine Weyl group*.

Define the *length* $\ell(w)$ of w to be the smallest r for which an expression $w = s_{i_r} \cdots s_{i_1}$ exists and call the expression *reduced*. By convention $\ell(1) = 0$. Clearly $\ell(w) = 1$ if and only if $w = s_i$ for some $i \in Q_0$.

Now we define a correspondence between W_Q and $\mathcal{I}(\Lambda)$. For $w \in W_Q$, we take a reduced expression $w = s_{i_\ell} \cdots s_{i_2} s_{i_1}$. Then we put

$$I_w := I_{i_\ell} \cdots I_{i_2} I_{i_1} \in \mathcal{I}(\Lambda).$$

This gives a correspondence

$$W_Q \ni w \longmapsto I_w \in \mathcal{I}(\Lambda).$$

The following results imply that the correspondence is actually well-defined and bijective. We omit proofs because they are shown by the quite same arguments as in [11].

Proposition 2.25 ([11] Proposition III.1.8) *The following hold.*

(1) $I_i^2 = I_i$.
(2) $I_i I_j = I_j I_i$ *if there are no arrows between i and j in Q.*
(3) $I_i I_j I_i = I_j I_i I_j$ *if there is precisely one arrow between i and j in Q.*

Theorem 2.26 ([11] Theorem III.1.9) *The correspondence* $W_Q \ni w \longmapsto I_w \in \mathcal{I}(\Lambda)$ *is a bijection.*

For any $w \in W_Q$, $\mathbb{R}\mathrm{Hom}_\Lambda(I_w, -)$ and $- \overset{\mathbb{L}}{\otimes}_\Lambda I_w$ are decomposed by using reduced expression of w as follows.

Proposition 2.27 *Let w be an element of W_Q. We take a reduced expression $w = s_{i_\ell} \cdots s_{i_2} s_{i_1}$. Then the following hold.*

(1) $\mathbb{R}\mathrm{Hom}_\Lambda(I_w, -) = \mathbb{R}\mathrm{Hom}_\Lambda(I_{i_\ell}, -) \circ \cdots \circ \mathbb{R}\mathrm{Hom}_\Lambda(I_{i_2}, -) \circ \mathbb{R}\mathrm{Hom}_\Lambda(I_{i_1}, -)$.
(2) $- \overset{\mathbb{L}}{\otimes}_\Lambda I_w = - \overset{\mathbb{L}}{\otimes}_\Lambda I_{i_\ell} \overset{\mathbb{L}}{\otimes}_\Lambda \cdots \overset{\mathbb{L}}{\otimes}_\Lambda I_{i_2} \overset{\mathbb{L}}{\otimes}_\Lambda I_{i_1}$.

Proof (2) Since $w = s_{i_\ell} \cdots s_{i_2} s_{i_1}$ is a reduced expression and [11, Proposition III.1.10.], we have a strict descending chain of tilting ideals

$$I_{i_\ell} \cdots I_{i_2} I_{i_1} \subset I_{i_\ell} \cdots I_{i_2} \subset \cdots\cdots \subset I_{i_\ell}.$$

Thus by Lemma 2.21, we have

$$I_{i_\ell} \overset{\mathbb{L}}{\otimes}_\Lambda \cdots \overset{\mathbb{L}}{\otimes}_\Lambda I_{i_2} \overset{\mathbb{L}}{\otimes}_\Lambda I_{i_1} = I_{i_\ell} \cdots I_{i_2} I_{i_1} = I_w.$$

The assertion follows.

(1) There is an adjoint isomorphism

$$\mathbb{R}\mathrm{Hom}_\Lambda(L \overset{\mathbb{L}}{\otimes}_\Lambda M, N) \simeq \mathbb{R}\mathrm{Hom}_\Lambda(L, \mathbb{R}\mathrm{Hom}_\Lambda(M, N))$$

for any $L, N \in \mathcal{D}(\mathrm{Mod}\Lambda)$ and $M \in \mathcal{D}(\mathrm{Mod}(\Lambda^{\mathrm{op}} \otimes_K \Lambda))$. By (2) and the above isomorphism, we have the assertion. □

2.5 Change of Dimension Vectors

In the Section 2.3, we had triangle equivalences

$$\mathcal{D}_{\mathrm{fd}}(\mathrm{Mod}\Lambda) \underset{- \overset{\mathbb{L}}{\otimes}_\Lambda I_i}{\overset{\mathbb{R}\mathrm{Hom}_\Lambda(I_i, -)}{\rightleftarrows}} \mathcal{D}_{\mathrm{fd}}(\mathrm{Mod}\Lambda) \quad .$$

In this subsection, we study change of dimension vectors via these triangle equivalences. We keep the notations in the previous sections.

First we define a map

$$[-] : \mathcal{D}_{\mathrm{fd}}(\mathrm{Mod}\Lambda) \longrightarrow \mathbb{Z}^{\overline{Q}_0}$$

by

$$[X] := \sum_{i \in \mathbb{Z}} (-1)^i \underline{\dim} \mathrm{H}^i(X).$$

We call $[X]$ *the dimension vector* of X since $[M] = \underline{\dim} M$ holds for $M \in \mathrm{fd}\Lambda$.

One can check that for a triangle $X \to Y \to Z \to X[1]$ in $\mathcal{D}_{\mathrm{fd}}(\mathrm{Mod}\Lambda)$, the equation

$$[Y] = [X] + [Z]$$

holds.

Next we define an action of W_Q on $\mathbb{Z}^{\overline{Q}_0}$ by

$$s_i(\mathbf{x}) := \mathbf{x} - (\mathbf{x}, \mathbf{e}_i)\mathbf{e}_i.$$

Then the following result holds.

Theorem 2.28 *The following diagrams commute.*

$$\begin{array}{ccc} \mathcal{D}_{\mathrm{fd}}(\mathrm{Mod}\Lambda) & \xrightarrow{\mathbb{R}\mathrm{Hom}_\Lambda(I_i, -)} & \mathcal{D}_{\mathrm{fd}}(\mathrm{Mod}\Lambda) \\ \downarrow{\scriptstyle [-]} & & \downarrow{\scriptstyle [-]} \\ \mathbb{Z}^{\overline{Q}_0} & \xrightarrow{s_i} & \mathbb{Z}^{\overline{Q}_0} \end{array} \qquad \begin{array}{ccc} \mathcal{D}_{\mathrm{fd}}(\mathrm{Mod}\Lambda) & \xrightarrow{- \overset{\mathbb{L}}{\otimes}_\Lambda I_i} & \mathcal{D}_{\mathrm{fd}}(\mathrm{Mod}\Lambda) \\ \downarrow{\scriptstyle [-]} & & \downarrow{\scriptstyle [-]} \\ \mathbb{Z}^{\overline{Q}_0} & \xrightarrow{s_i} & \mathbb{Z}^{\overline{Q}_0} \end{array}$$

Proof First we show that the commutativity of the first diagram. We claim that $s_i[M] = [\mathbb{R}\mathrm{Hom}_\Lambda(I_i, M)]$ holds for any finite dimensional Λ-module M. Let M be a finite dimensional Λ-module, and put $\alpha = (\alpha_i) := \underline{\dim} M$.

We have

$$\mathrm{Hom}_\Lambda(I_i, M)e_j = \begin{cases} \mathrm{Hom}_\Lambda(e_i I_i, M) & (i = j) \\ \mathrm{Hom}_\Lambda(e_j \Lambda, M) \simeq M e_j & (i \neq j) \end{cases}$$

and

$$\mathrm{Ext}^1_\Lambda(I_i, M)e_j = \begin{cases} \mathrm{Ext}^1_\Lambda(e_i I_i, M) & (i = j) \\ \mathrm{Ext}^1_\Lambda(e_j \Lambda, M) = 0 & (i \neq j). \end{cases}$$

By applying $\mathrm{Hom}_\Lambda(-, M)$ to the exact sequence

$$0 \longrightarrow e_i \Lambda \longrightarrow \bigoplus_{j \to i} e_j \Lambda \longrightarrow e_i I_i \longrightarrow 0$$

obtained from Proposition 2.17, we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}_\Lambda(e_i I_i, M) \longrightarrow \bigoplus_{j \to i} \mathrm{Hom}_\Lambda(e_j \Lambda, M) \longrightarrow \mathrm{Hom}_\Lambda(e_i \Lambda, M) \longrightarrow \mathrm{Ext}^1_\Lambda(e_i I_i, M) \longrightarrow 0.$$

Thus we have

$$\begin{aligned}&\dim \mathrm{Hom}_\Lambda(e_iI_i, M) - \dim \mathrm{Ext}^1_\Lambda(e_iI_i, M)\\&= \sum_{j\to i} \dim \mathrm{Hom}_\Lambda(e_j\Lambda, M) - \dim \mathrm{Hom}_\Lambda(e_i\Lambda, M)\\&= \sum_{ta=i} \alpha_{sa} - \alpha_i.\end{aligned}$$

Therefore we have

$$\begin{aligned}[\mathbb{R}\mathrm{Hom}_\Lambda(I_i, M)] &= \underline{\dim}\mathrm{Hom}_\Lambda(I_i, M) - \underline{\dim}\mathrm{Ext}^1_\Lambda(I_i, M)\\&= \sum_{j\neq i} \alpha_j \mathbf{e}_j + (\dim \mathrm{Hom}_\Lambda(e_iI_i, M) - \dim \mathrm{Ext}^1_\Lambda(e_iI_i, M))\mathbf{e}_i\\&= \sum_{j\neq i} \alpha_j \mathbf{e}_j + \left(\sum_{ta=i} \alpha_{sa} - \alpha_i\right)\mathbf{e}_i\\&= \alpha - \left(2\alpha_i - \sum_{ta=i} \alpha_{sa}\right)\mathbf{e}_i\\&= \alpha - (\alpha, \mathbf{e}_i)\mathbf{e}_i = s_i\alpha = s_i[M].\end{aligned}$$

Next let $X \in \mathcal{D}_{\mathrm{fd}}(\mathrm{Mod}\Lambda)$. By Lemma 2.4 and since $\mathbb{R}\mathrm{Hom}_\Lambda(I_i, -)$ is a triangle functor, we have

$$[\mathbb{R}\mathrm{Hom}_\Lambda(I_i, X)] = \sum_{j\in\mathbb{Z}} (-1)^j [\mathbb{R}\mathrm{Hom}_\Lambda(I_i, \mathrm{H}^j(X))].$$

Consequently we have

$$\begin{aligned}[\mathbb{R}\mathrm{Hom}_\Lambda(I_i, X)] &= \sum_{j\in\mathbb{Z}} (-1)^j [\mathbb{R}\mathrm{Hom}_\Lambda(I_i, \mathrm{H}^j(X))]\\&= \sum_{j\in\mathbb{Z}} (-1)^j s_i[H^j(X)]\\&= s_i[X].\end{aligned}$$

This equation shows the commutativity of the first diagram.

For the second diagram, since $s_i^2 = 1$ and $[\mathbb{R}\mathrm{Hom}_\Lambda(I_i, X \overset{\mathbb{L}}{\otimes}_\Lambda I_i)] = [X]$, we have $[X \overset{\mathbb{L}}{\otimes}_\Lambda I_i] = s_i[X]$ by the commutativity of the first diagram. □

In particular, the following follows from Theorem 2.28.

Corollary 2.29 *For a finite dimensional Λ-module M, the following hold.*

(1) *If $M \in \mathcal{T}(I_i)$, then* $\underline{\dim}\mathrm{Hom}_\Lambda(I_i, M) = [\mathbb{R}\mathrm{Hom}_\Lambda(I_i, M)] = s_i(\underline{\dim} M)$.
(2) *If $M \in \mathcal{F}(I_i)$, then* $\underline{\dim}\mathrm{Ext}^1_\Lambda(I_i, M) = -[\mathbb{R}\mathrm{Hom}_\Lambda(I_i, M)] = -s_i(\underline{\dim} M)$.
(3) *If $M \in \mathcal{X}(I_i)$, then* $\underline{\dim}\mathrm{Tor}_1^\Lambda(M, I_i) = -[M \overset{\mathbb{L}}{\otimes}_\Lambda I_i] = -s_i(\underline{\dim} M)$.

(4) *If* $M \in \mathcal{Y}(I_i)$, *then* $\underline{\dim} M \otimes_\Lambda I_i = [M \overset{\mathbb{L}}{\otimes}_\Lambda I_i] = s_i(\underline{\dim} M)$.

2.6 The Categories of Finite Dimensional Modules Over Preprojective Algebras of Extended Dynkin Quivers

In this subsection, we focus the case in which Q is an extended Dynkin quiver. Then the double quiver $\overline{Q}$ of Q is one of quivers appeared in Fig. 1, which is known as the McKay quiver of a finite subgroup G of $SL(2, K)$. The associated preprojective algebra Λ is important from the viewpoint of geometry. The moduli spaces of Λ-modules of the dimension vector $\mathbf{d}$ in Fig. 1 are isomorphic to the minimal resolution of the Kleinian singularity $\mathbb{A}^2/G$.

The purpose of this subsection is to show properties of simple Λ-modules. Moreover as an application of it, we represent fdΛ as a direct sum of its full subcategories. We denote by $\mathcal{S}$ the set of isomorphism classes of finite dimensional simple Λ-modules which are not isomorphic to $S_0, S_1, \ldots, S_n$.

Let $\mathbf{d}$ be the dimension vector shown in Fig. 1. For the symmetric bilinear form $(-, -)$ defined in Section 2.2, the following hold.

Lemma 2.30 *For* $\alpha \in \mathbb{Z}^{\overline{Q}_0}$, *a function*

$$(\alpha, -) : \mathbb{Z}^{\overline{Q}_0} \longrightarrow \mathbb{Z}$$

is zero if and only if $\alpha = m\mathbf{d}$ *for some* $m \in \mathbb{Z}$.

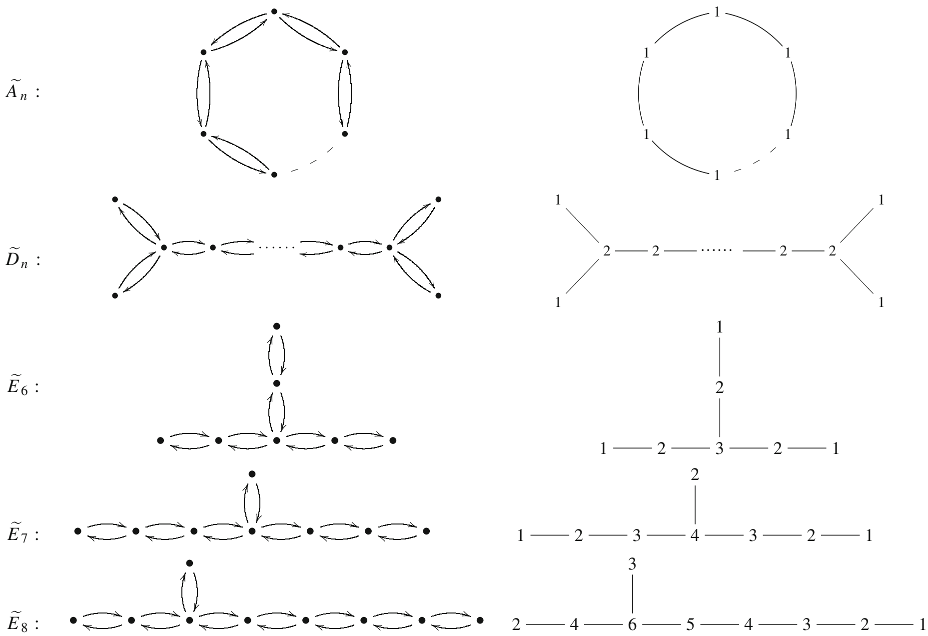


Fig. 1 The double $\overline{Q}$ of extended Dynkin quivers and dimension vectors $\mathbf{d}$

Proof The assertion is verified by easy calculations. □

We state properties of simple Λ-modules.

Lemma 2.31 *The following assertions hold.*

(1) *The dimension vector of a simple Λ-module in $\mathcal{S}$ is $\mathbf{d}$.*
(2) *If a finite dimensional Λ-module M satisfies $\underline{\dim} M = \mathbf{d}$, then M is either nilpotent or simple.*
(3) *Let M and N be simple Λ-modules. If M is in $\mathcal{S}$ and is not isomorphic to N, then we have $\mathrm{Ext}^1_\Lambda(M, N) = \mathrm{Ext}^1_\Lambda(N, M) = 0$.*

Proof

(1) See [15, 3.9].
(2) The assertion follows from (1).
(3) The assertion follows from Lemma 2.19 and (1) and Lemma 2.30. □

For any $S \in \mathcal{S}$, we define the full subcategory $\mathrm{fd}_S\Lambda$ of $\mathrm{fd}\Lambda$ which consists of finite dimensional Λ-modules whose composition factors consist only of S.

Proposition 2.32 *The following assertion holds.*

$$\mathrm{fd}\Lambda = \left(\bigoplus_{S\in\mathcal{S}} \mathrm{fd}_S\Lambda\right) \bigoplus \mathrm{nilp}\Lambda.$$

Proof The assertion follows from Lemma 2.31 (3). □

3 Reflection Functors

In this section, we study variation of θ-semistable modules over preprojective algebras of non-Dynkin quivers by changing parameters. To do this, we apply classical tilting modules and their properties given in the previous section.

3.1 Moduli Space of Modules

We prepare notations about moduli spaces of modules. Let $\Lambda = KQ/\langle R\rangle$ be a K-algebra associated to a finite connected quiver (Q, R) with relations. Denote by $Q_0 = \{0, \ldots, n\}$ the vertex set of Q. We identify Λ-modules and representations of (Q, R) since there is a categorical equivalence between the category of Λ-modules and the category of representations of (Q, R) (cf. [4, Chapter 3]). Let $\mathbf{e}_0, \ldots, \mathbf{e}_n$ be the canonical basis of $\mathbb{Z}^{Q_0}$. Also we denote by $(\mathbb{Z}^{Q_0})^*$ the dual lattice of $\mathbb{Z}^{Q_0}$ with the dual basis $\mathbf{e}_0^*, \ldots, \mathbf{e}_n^*$. We define the parameter space Θ by

$$\Theta := (\mathbb{Z}^{Q_0})^* \otimes_{\mathbb{Z}} \mathbb{Q}.$$

By the canonical pairing, we define $\theta(M) := \langle \theta, \underline{\dim} M\rangle = \sum_{i\in Q_0} \theta_i \dim_K(Me_i)$ for any $\theta = (\theta_i)_{i\in Q_0} \in \Theta$ and any finite dimensional KQ-module M. King provided the notion of stability for modules.

Definition 3.1 [26] For any $\theta \in \Theta$, a finite dimensional Λ-module M is called θ-*semistable* (resp. θ-*stable*) if $\theta(M) = 0$ and, for any non-zero proper submodule N of M, $\theta(N) \geq 0$ (resp. > 0). Two θ-semistable modules are called *S-equivalent* if they have filtrations by θ-stable modules with the same associated graded modules. Moreover for a given indivisible vector α, θ is called *generic* if all θ-semistable modules of dimension vector α are θ-stable.

Remark 3.2 In [26], θ-semistability is defined for a finite dimensional KQ-module. However we see that a finite dimensional Λ-module M is θ-semistable (resp. θ-stable) as an element of fdΛ if and only if the corresponding representation of Q is θ-semistable (resp. θ-stable) as an element of fdKQ (see [26, Theorem 4.1]).

For any $\theta \in \Theta$ and any dimension vector α, we denote by $\mathcal{M}_{\theta,\alpha}(\Lambda)$ the moduli space of θ-semistable Λ-modules of dimension vector α. In fact, it is a coarse moduli space parametrizing S-equivalence classes of θ-semistable Λ-modules of dimension vector α. For an indivisible vector α, if θ is generic, then $\mathcal{M}_{\theta,\alpha}(\Lambda)$ becomes a fine moduli space. In the case, S-equivalence classes are just isomorphism classes.

Since we want to deal with θ-semistable modules categorically, we introduce the following notation.

Definition 3.3 For any parameter $\theta \in \Theta$, we define the full subcategory $\mathcal{S}_\theta(\Lambda)$ of ModΛ consisting of θ-semistable Λ-modules. Moreover we denote by $\mathcal{S}_{\theta,\alpha}(\Lambda)$ the full subcategory of $\mathcal{S}_\theta(\Lambda)$ consisting of θ-semistable Λ-modules of dimension vector α if $\mathcal{S}_{\theta,\alpha}(\Lambda)$ is not empty.

Note that the category $\mathcal{S}_\theta(\Lambda)$ is closed under extensions and direct summands. There is a bijection between the set of S-equivalence classes in $\mathcal{S}_{\theta,\alpha}(\Lambda)$ and the set of closed points on $\mathcal{M}_{\theta,\alpha}(\Lambda)$.

3.2 Simple Reflection Case $w = s_i$

In the rest of this section, let Λ be the preprojective algebra associated to a finite connected non-Dynkin quiver Q with no loops.

Recall that the function $(-,-) : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \to \mathbb{Z}$ denotes the symmetric bilinear form defined in Section 2.2. We define actions of the Coxeter group W_Q associated to Q on $\mathbb{Z}^{Q_0}$ and $(\mathbb{Z}^{Q_0})^*$ as follows. For any simple reflection s_i, any $\alpha \in \mathbb{Z}^{Q_0}$ and any $\theta \in (\mathbb{Z}^{Q_0})^*$, we put

$$s_i\alpha := \alpha - (\alpha, \mathbf{e}_i)\mathbf{e}_i,$$

$$s_i\theta := \theta - \theta_i \sum_{j=1}^{n} (\mathbf{e}_i, \mathbf{e}_j)\mathbf{e}_j^*.$$

Note that the action on defined $\mathbb{Z}^{Q_0}$ is the same action as defined in Section 2.5. And the action on $(\mathbb{Z}^{Q_0})^*$ is extended to Θ linearly. It is easy to see that $s_i\theta(\alpha) = \theta(s_i\alpha)$ for any $\alpha \in \mathbb{Z}^{Q_0}$ and $\theta \in \Theta$.

We want to give an equivalence between $\mathcal{S}_\theta(\Lambda)$ and $\mathcal{S}_{w\theta}(\Lambda)$ for any Coxeter element $w \in W_Q$ by using classical tilting Λ-modules studied in Section 2.

First of all we consider the case when $w \in W_Q$ is a simple reflection s_i since the general case is obtained by a composition of the simple case. The required equivalence is given by the tilting module I_i, which gives the derived equivalence

$$\mathbb{R}\mathrm{Hom}_\Lambda(I_i, -) : \mathcal{D}(\mathrm{Mod}\Lambda) \to \mathcal{D}(\mathrm{Mod}\Lambda).$$

We show that this equivalence induces an equivalence between $\mathcal{S}_\theta(\Lambda)$ and $\mathcal{S}_{s_i\theta}(\Lambda)$. Recall that $\mathcal{T}(T)$ and $\mathcal{Y}(T)$ are full subcategories of fdΛ defined in Section 2.1.

Lemma 3.4 *For any $\theta \in \Theta$ the following hold.*

(1) *If $\theta_i > 0$, then $\mathcal{S}_\theta(\Lambda) \subset \mathcal{T}(I_i)$.*
(2) *If $\theta_i < 0$, then $\mathcal{S}_\theta(\Lambda) \subset \mathcal{Y}(I_i)$.*

Proof

(1) Take any $M \in \mathcal{S}_\theta(\Lambda)$. By Lemma 2.23 we need to show that S_i is not a factor of M. If S_i is a factor of M, then there is an exact sequence $0 \to X \to M \to S_i \to 0$. So we have $\theta_i = \theta(S_i) = \theta(M) - \theta(X) \leq 0$. This contradicts the assumption.
(2) Take any $M \in \mathcal{S}_\theta(\Lambda)$. By Lemma 2.23 we need to show that S_i is not a submodule of M. If S_i is a submodule of M, then we have $\theta_i = \theta(S_i) \geq 0$ since M is θ-semistable. This contradicts the assumption. □

Now we state one of the main results in this paper.

Theorem 3.5 *For any $\theta \in \Theta$ with $\theta_i > 0$, there is a categorical equivalence*

$$\mathcal{S}_\theta(\Lambda) \underset{-\otimes_\Lambda I_i}{\overset{\mathrm{Hom}_\Lambda(I_i,-)}{\rightleftarrows}} \mathcal{S}_{s_i\theta}(\Lambda) \quad .$$

Under this equivalence S-equivalence classes are preserved and θ-stable modules correspond to $s_i\theta$-stable modules. In particular it induces an equivalence between $\mathcal{S}_{\theta,\alpha}(\Lambda)$ and $\mathcal{S}_{s_i\theta,s_i\alpha}(\Lambda)$ for each dimension vector α.

Proof By Lemma 3.4, we have functors $\mathrm{Hom}_\Lambda(I_i, -) : \mathcal{S}_\theta(\Lambda) \to \mathcal{Y}(I_i)$ and $- \otimes_\Lambda I_i : \mathcal{S}_{s_i\theta}(\Lambda) \to \mathcal{T}(I_i)$. Since $\mathrm{Hom}_\Lambda(I_i, -)$ and $- \otimes_\Lambda I_i$ give the equivalence between $\mathcal{T}(I_i)$ and $\mathcal{Y}(I_i)$ by Lemma 2.9 and Theorem 2.20, it is sufficient to show that $\mathrm{Hom}_\Lambda(I_i, M) \in \mathcal{S}_{s_i\theta}(\Lambda)$ for any $M \in \mathcal{S}_\theta(\Lambda)$ and $M \otimes_\Lambda I_i \in \mathcal{S}_\theta(\Lambda)$ for any $M \in \mathcal{S}_{s_i\theta}(\Lambda)$.

Take any $M \in \mathcal{S}_\theta(\Lambda)$. We show that $M' := \mathrm{Hom}_\Lambda(I_i, M)$ is $s_i\theta$-semistable. Since $M \in \mathcal{T}(I_i)$, by Corollary 2.29 we have

$$(s_i\theta)(M') = (s_i\theta)(s_i(\underline{\dim} M)) = \theta(M) = 0.$$

Take any non-zero proper submodule N' of M'. Since $\mathcal{Y}(I_i)$ is closed under submodules and M' belongs to $\mathcal{Y}(I_i)$, so does N'. By applying $- \otimes_\Lambda I_i$ to the exact sequence

$$0 \longrightarrow N' \longrightarrow M' \longrightarrow M'/N' \longrightarrow 0, \tag{3.1}$$

we have an exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^\Lambda(M'/N', I_i) \longrightarrow N' \otimes_\Lambda I_i \xrightarrow{f} M.$$

Since $X := \mathrm{Im} f$ is a submodule of M, we have $\theta(X) \geq 0$. We have $\mathrm{Tor}_1^{\Lambda}(M'/N', I_i) \simeq \mathrm{Tor}_2^{\Lambda}(M'/N', S_i) \overset{2.6}{\simeq} D\mathrm{Ext}^2_{\Lambda}(M'/N', S_i) \overset{2\text{-CY}}{\simeq} \mathrm{Hom}_{\Lambda}(S_i, M'/N') \overset{2.15}{\simeq} S_i^m$ for some integer m where the first isomorphism is given by applying $M'/N' \otimes_{\Lambda} -$ to the exact sequence $0 \to I_i \to \Lambda \to S_i \to 0$. Therefore by Corollary 2.29 we have

$$\begin{aligned}(s_i\theta)(N') = \theta(s_i(\underline{\dim} N')) &= \theta(N' \otimes_{\Lambda} I_i)\\ &= \theta(\mathrm{Tor}_1^{\Lambda}(M'/N', I_i)) + \theta(X)\\ &= \theta(S_i^m) + \theta(X)\\ &= m\theta_i + \theta(X) \geq 0.\end{aligned}$$

Thus M' is $s_i\theta$-semistable. Furthermore if M is θ-stable, then $\theta(X) > 0$, so $(s_i\theta)(N') > 0$. Hence M' is $s_i\theta$-stable.

Conversely we take any $M \in \mathcal{S}_{s_i\theta}(\Lambda)$. We show that $M' := M \otimes_{\Lambda} I_i$ is θ-semistable. Since $M \in \mathcal{Y}(I_i)$, by Corollary 2.29 we have

$$\theta(M') = \theta(s_i(\underline{\dim} M)) = (s_i\theta)(M) = 0.$$

Take any non-zero proper submodule N' of M'. Since $\mathcal{T}(I_i)$ is closed under images and M' belongs to $\mathcal{T}(I_i)$, so does M'/N'. Consider an exact sequence

$$0 \longrightarrow N' \longrightarrow M' \longrightarrow M'/N' \longrightarrow 0.$$

By applying $\mathrm{Hom}_{\Lambda}(I_i, -)$ to the above exact sequence, we have an exact sequence

$$M \overset{g}{\to} \mathrm{Hom}_{\Lambda}(I_i, M'/N') \to \mathrm{Ext}^1_{\Lambda}(I_i, N') \to 0.$$

Since $X := \mathrm{Im} g$ is a factor module of M, we have $s_i\theta(X) \leq 0$. We have $\mathrm{Ext}^1_{\Lambda}(I_i, N') \simeq \mathrm{Ext}^2_{\Lambda}(S_i, N') \overset{2\text{-CY}}{\simeq} D\mathrm{Hom}_{\Lambda}(N', S_i) \overset{2.15}{\simeq} D(S_i^m) \simeq S_i^m$ for some integer m where the first isomorphism is given by applying $\mathrm{Hom}_{\Lambda}(-, N')$ to the exact sequence $0 \to I_i \to \Lambda \to S_i \to 0$. So by Corollary 2.29 we have

$$\begin{aligned}\theta(N') = -\theta(M'/N') &= -(s_i\theta)(s_i(\underline{\dim} M'/N'))\\ &= -(s_i\theta)(\underline{\dim}\mathrm{Hom}_{\Lambda}(I_i, M'/N')\\ &= -(s_i\theta)(\mathrm{Ext}^1_{\Lambda}(I_i, N')) - (s_i\theta)(X)\\ &= -(s_i\theta)(S_i^m) - (s_i\theta)(X)\\ &= m\theta_i - (s_i\theta)(X) \geq 0.\end{aligned}$$

Hence M' is θ-semistable. Furthermore if M is $s_i\theta$-stable, then $(s_i\theta)(X) < 0$, so $\theta(N') > 0$. Hence M' is θ-stable.

Since the functors $\mathrm{Hom}_{\Lambda}(I_i, -)$ and $- \otimes_{\Lambda} I_i$ are exact on $\mathcal{S}_{\theta}(\Lambda)$ and $\mathcal{S}_{s_i\theta}(\Lambda)$ respectively by Lemma 3.4, it is trivial to see that they preserve S-equivalence classes. Finally, the functors induce an equivalence between $\mathcal{S}_{\theta,\alpha}(\Lambda)$ and $\mathcal{S}_{s_i\theta, s_i\alpha}(\Lambda)$ by Corollary 2.29. □

Definition 3.6 The functors $\mathrm{Hom}_{\Lambda}(I_i, -)$ and $- \otimes_{\Lambda} I_i$ in Theorem 3.5 are called *simple reflection functors*. We introduce the notation $\mathbf{s}_i$ for any $i \in Q_0$ as follows:

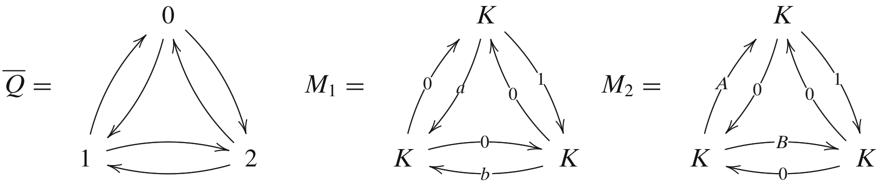
- If $\theta_i > 0$, then $\mathbf{s}_i := \mathrm{Hom}_{\Lambda}(I_i, -) : \mathcal{S}_{\theta}(\Lambda) \to \mathcal{S}_{s_i\theta}(\Lambda)$,

- If $\theta_i < 0$, then $\mathbf{s}_i := - \otimes_\Lambda I_i : \mathcal{S}_\theta(\Lambda) \to \mathcal{S}_{s_i\theta}(\Lambda)$.

Remark 3.7 In the above theorem, we assumed $\theta_i > 0$. The same result holds for the case $\theta_i < 0$. Indeed if $\theta_i < 0$ then apply Theorem 3.5 for $s_i\theta$ since $(s_i\theta)_i > 0$ and $s_i^2 = 1$.

It is natural to hope that the above bijection is extended to an isomorphism of K-schemes. In fact, it is proved in the next section.

Example 3.8 Let Q be an extended Dynkin quiver of type $\widetilde{A}_2$. Put $\mathbf{d} = (d_0, d_1, d_2) = (1, 1, 1)$. For the parameter $\theta = (\theta_0, \theta_1, \theta_2)$ with $\theta(\mathbf{d}) = \theta_1 d_1 + \theta_2 d_2 + \theta_3 d_3 = 0, \theta_1 = 0$ and $\theta_2 > 0$, the following two representations are contained in $\mathcal{S}_{\theta,\mathbf{d}}(\Lambda)$ and give the same S-equivalent class.



By Lemma 2.23, it follows that $M_1 \in \mathcal{T}(I_1)$, but $M_2 \notin \mathcal{T}(I_1)$. So $\mathcal{S}_{\theta,\mathbf{d}}(\Lambda) \not\subset \mathcal{T}(I_1)$, hence $\mathrm{Hom}_\Lambda(I_1, -)$ does not give an equivalence between $\mathcal{S}_{\theta,\mathbf{d}}(\Lambda)$ and $\mathcal{S}_{s_1\theta,\mathbf{d}}(\Lambda)$. This means that the assumption that $\theta_1 \neq 0$ in the above result is essential.

Before proceeding to the general case, we show the following result.

Proposition 3.9 *Let $\theta \in \Theta$ with $\theta_i \neq 0$ and $M \in \mathcal{S}_\theta(\Lambda)$.*

(1) *If $\theta_i > 0$, $M \simeq \mathbf{s}_i(M)$ if and only if S_i is not a submodule of M and $(M, S_i) = 0$.*
(2) *If $\theta_i < 0$, $M \simeq \mathbf{s}_i(M)$ if and only if S_i is not a factor of M and $(M, S_i) = 0$.*

Proof Take any $M \in \mathcal{S}_\theta(\Lambda)$. We only show (1) because (2) is similar. We assume that $M \simeq \mathbf{s}_i(M) = \mathrm{Hom}_\Lambda(I_i, M)$. Since M belongs to $\mathcal{T}(I_i)$ by Lemma 3.4, we have $\mathrm{Hom}_\Lambda(I_i, M) \in \mathcal{Y}(I_i)$. So we have $M \in \mathcal{Y}(I_i)$. Thus S_i is not a submodule of M by Lemma 2.23. Moreover since $M \simeq \mathrm{Hom}_\Lambda(I_i, M)$ and $M \in \mathcal{T}(I_i)$, we have $\underline{\dim} M = \underline{\dim} \mathrm{Hom}_\Lambda(I_i, M) = s_i(\underline{\dim} M)$ by Corollary 2.29. This implies that $(M, S_i) = 0$ holds.

Conversely we assume that S_i is not a submodule of M and $(M, S_i) = 0$. By applying $\mathrm{Hom}_\Lambda(-, M)$ to the exact sequence $0 \to I_i \to \Lambda \to S_i \to 0$, we have an exact sequence

$$\mathrm{Hom}_\Lambda(S_i, M) \longrightarrow \mathrm{Hom}_\Lambda(\Lambda, M) \longrightarrow \mathrm{Hom}_\Lambda(I_i, M) \longrightarrow \mathrm{Ext}^1_\Lambda(S_i, M).$$

Thus it is enough to show that $\mathrm{Hom}_\Lambda(S_i, M) = 0$ and $\mathrm{Ext}^1_\Lambda(S_i, M) = 0$. By the assumption, we have $\mathrm{Hom}_\Lambda(S_i, M) = 0$. On the other hand, since M belongs to $\mathcal{T}(I_i)$, S_i is not a factor of M by Lemma 2.23. Hence by Lemma 2.19, we have

$$\dim \mathrm{Ext}^1_\Lambda(S_i, M) = \dim \mathrm{Hom}_\Lambda(S_i, M) + \dim \mathrm{Hom}_\Lambda(M, S_i) - (S_i, M) = 0.$$

Therefore the assertion holds. □

3.3 General Case $w = s_{i_\ell} \cdots s_{i_1}$

Let w be an element of the Coxeter group W_Q. For an expression $w = s_{i_\ell} \cdots s_{i_1}$, we denote by $\mathbf{s}_{i_\ell} \cdots \mathbf{s}_{i_1}$ the composition of the simple reflection functors $\mathbf{s}_{i_1}, \ldots, \mathbf{s}_{i_\ell}$. We remark that $\mathbf{s}_{i_{k+1}}$ is defined with respect to $s_{i_k} \cdots s_{i_1}\theta$ and not with respect to θ. A composition $\mathbf{s}_{i_\ell} \cdots \mathbf{s}_{i_1}$ of the simple reflection functors gives an equivalence between $\mathcal{S}_\theta(\Lambda)$ and $\mathcal{S}_{w\theta}(\Lambda)$ for a general $\theta \in \Theta$. The next result implies that it does not depend on the choice of the expression of w up to isomorphisms.

Proposition 3.10 *The functors $\mathbf{s}_i$ satisfy the Coxeter relation, namely*

(1) $\mathbf{s}_i\mathbf{s}_i \simeq \mathrm{id}$,
(2) $\mathbf{s}_j\mathbf{s}_i \simeq \mathbf{s}_i\mathbf{s}_j$ *if there are no arrows between i and j in Q,*
(3) $\mathbf{s}_i\mathbf{s}_j\mathbf{s}_i \simeq \mathbf{s}_j\mathbf{s}_i\mathbf{s}_j$ *if there is precisely one arrow between i and j in Q.*

where $\theta_i \neq 0$, $\theta_j \neq 0$ and $\theta_i + \theta_j \neq 0$.

Proof (1) is trivial. (2) There are four cases: (i) $\theta_i > 0$, $(s_i\theta)_j > 0$, (ii) $\theta_i > 0$, $(s_i\theta)_j < 0$, (iii) $\theta_i < 0$, $(s_i\theta)_j > 0$ and (iv) $\theta_i < 0$, $(s_i\theta)_j < 0$. Since j is not connected to i, these four cases are same as the next four cases: (i) $\theta_i > 0$, $\theta_j > 0$, (ii) $\theta_i > 0$, $\theta_j < 0$, (iii) $\theta_i < 0$, $\theta_j > 0$ and (iv) $\theta_i < 0$, $\theta_j < 0$. For example, if $\theta_i > 0$ and $\theta_j < 0$, then for a θ-semistable Λ-module M, we have

$$\mathbf{s}_j\mathbf{s}_i(M) = \mathrm{Hom}(I_i, M) \otimes I_j \overset{3.5}{\simeq} \mathrm{Hom}(I_i, \mathrm{Hom}(I_j, M \otimes I_j)) \otimes I_j \overset{2.27}{\simeq} \mathrm{Hom}(I_i I_j, M \otimes I_j) \otimes I_j$$

$$\overset{2.25}{\simeq} \mathrm{Hom}(I_j I_i, M \otimes I_j)) \otimes I_j \overset{2.27}{\simeq} \mathrm{Hom}(I_j, \mathrm{Hom}(I_i, M \otimes I_j)) \otimes I_j$$

$$\overset{3.5}{\simeq} \mathrm{Hom}(I_i, M \otimes I_j)) = \mathbf{s}_i\mathbf{s}_j(M).$$

The others are similar. (3) There are six cases and the proof is similar to (2). □

Definition 3.11 We denote by $\mathbf{w}$ the composition $\mathbf{s}_{i_\ell} \cdots \mathbf{s}_{i_1}$ of simple reflection functors for an expression $s_{i_\ell} \cdots s_{i_1}$ of w. We call $\mathbf{w}$ the reflection functor.

In summary, we have the following result.

Theorem 3.12 *For any $w \in W$, if $\theta \in \Theta$ is sufficiently general, then there is a categorical equivalence*

$$\mathcal{S}_\theta(\Lambda) \underset{\mathbf{w}^{-1}}{\overset{\mathbf{w}}{\rightleftarrows}} \mathcal{S}_{w\theta}(\Lambda).$$

Under this equivalence S-equivalence classes are preserved and θ-stable modules correspond to $w\theta$-stable modules. In particular it induces the equivalence between $\mathcal{S}_{\theta,\alpha}(\Lambda)$ and $\mathcal{S}_{w\theta,w\alpha}(\Lambda)$.

In particular, it is an interesting problem when $\mathbf{w}$ is described as a hom functor or a tensor functor. If we impose some assumption on θ, we have an explicit description of $\mathbf{w}$ as follows.

Proposition 3.13 *For any $w \in W_Q$, take a reduced expression $w = s_{i_\ell} \cdots s_{i_1}$. Put $w_j = s_{i_j} \cdots s_{i_1}$ for $j = 1, \ldots, \ell$ and $w_0 = 1$. For any $\theta \in \Theta$, the following hold.*

(1) *If $(w_{j-1}\theta)_{i_j} > 0$ holds for any $j = 1, \ldots, \ell$, then we have $\mathbf{w} = \mathrm{Hom}_\Lambda(I_w, -)$ and $\mathcal{S}_\theta(\Lambda) \subset \mathcal{T}(I_w)$, $\mathcal{S}_{w\theta}(\Lambda) \subset \mathcal{Y}(I_w)$.*
(2) *If $(w_{j-1}\theta)_{i_j} < 0$ holds for any $j = 1, \ldots \ell$, then we have $\mathbf{w} = - \otimes_\Lambda I_w$ and $\mathcal{S}_\theta(\Lambda) \subset \mathcal{Y}(I_w)$, $\mathcal{S}_{w\theta}(\Lambda) \subset \mathcal{T}(I_w)$.*

Proof

(1) By the assumption, we have $\mathbf{w} = \mathbf{s}_{i_\ell} \cdots \mathbf{s}_{i_1} = \mathrm{Hom}_\Lambda(I_{i_\ell}, -) \circ \cdots \circ \mathrm{Hom}_\Lambda(I_{i_1}, -)$. Since $s_{i_\ell} \cdots s_{i_1}$ is a reduced expression, $\mathbf{s}_{i_\ell} \cdots \mathbf{s}_{i_1} = \mathrm{Hom}_\Lambda(I_w, -)$ by Lemma 3.4 and Proposition 2.27. Hence we have $\mathcal{S}_\theta(\Lambda) \subset \mathcal{T}(I_w)$, $\mathcal{S}_{w\theta}(\Lambda) \subset \mathcal{Y}(I_w)$.
(2) is similar. □

4 Functors Inducing Morphisms

In the previous section, we obtained an equivalence between $\mathcal{S}_{\theta,\alpha}(\Lambda)$ and $\mathcal{S}_{w\theta\Lambda}(w\alpha)$, and hence a bijection between the closed points on moduli spaces $\mathcal{M}_{\theta,\alpha}(\Lambda)$ and $\mathcal{M}_{w\theta,w\alpha}(\Lambda)$. In this section, we prove that this bijection naturally lifts up to an isomorphism of K-schemes. However expecting applications to more various and complicated situations (e.g. higher dimensions [33]), we observe in a possibly general setting.

Throughout this section, let Λ, Γ be K-algebras which are associated to finite quivers with relations.

4.1 Moduli Spaces of Modules and Functors of Points

We recall the definition of moduli spaces of modules and functors of points. Let $\mathfrak{Sch}$ be the category of K-schemes, $\mathfrak{R}$ the category of finitely generated commutative K-algebras and $\mathfrak{Set}$ the category of sets. By Yoneda's lemma (cf. [19, Proposition VI-2]), $\mathfrak{Sch}$ is equivalent to the full subcategory of the category of covariant functors from $\mathfrak{R}$ to $\mathfrak{Set}$. So a K-scheme X is regarded as a covariant functor $h_X : \mathfrak{R} \to \mathfrak{Set}$ called functor of points.

For simplicity, for any Λ-module M and any $R \in \mathfrak{R}$, Λ^R and M^R stands for $\Lambda \otimes_K R$ and $M \otimes_K R$ respectively. For any $R \in \mathfrak{R}$, $\mathrm{Max}(R)$ denotes the set of maximal ideals of R. For any $\mathfrak{m} \in \mathrm{Max}(R)$, we put $k(\mathfrak{m}) = R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}$.

We introduce the notion of families of Λ-modules of θ-semistable modules. Let (Q, ρ) be a finite quiver with relations and $\Lambda = KQ/\langle\rho\rangle$. We write $\Theta = (\mathbb{Z}^{Q_0})^* \otimes_\mathbb{Z} \mathbb{Q}$ the parameter space.

Definition 4.1 For $\theta \in \Theta$, a Λ^R-module M is called θ-semistable if $M \otimes_R k(\mathfrak{m})$ is θ-semistable for any $\mathfrak{m} \in \mathrm{Max}(R)$. We say that M has dimension vector α if the dimension vector of $M \otimes_R k(\mathfrak{m})$ is α as a Λ-module for any $\mathfrak{m} \in \mathrm{Max}(R)$. We say that two θ-semistable Λ^R-modules M and N are S-equivalent if $M \otimes_R k(\mathfrak{m})$ is S-equivalent to $N \otimes_R k(\mathfrak{m})$ for any $\mathfrak{m} \in \mathrm{Max}(R)$.

Let $\mathcal{S}^R_{\theta,\alpha}(\Lambda)$ denote the category of θ-semistable Λ^R-modules of dimension vector α which are finitely generated and flat over R.

We define the assignment $\mathcal{F}_{\theta,\alpha,\Lambda} : \mathfrak{R} \to \mathfrak{Set}$ by

$$\mathcal{F}_{\theta,\alpha,\Lambda}(R) := \left\{ \begin{array}{l} \text{S-equivalence classes of } \theta\text{-semistable } \Lambda^R\text{-modules of dimension} \\ \text{vector } \alpha \text{ which are finitely generated and flat over } R \end{array} \right\}$$

for any $R \in \mathfrak{R}$. By the following lemma, $\mathcal{F}_{\theta,\alpha,\Lambda}$ is a covariant functor, which is called the moduli functor with respect to θ-semistable Λ-modules of dimension vector α.

Lemma 4.2 *For a morphism $\varphi : R \to S$ in $\mathfrak{R}$, the functor $- \otimes_R S$ takes $\mathcal{S}^R_{\theta,\alpha}(\Lambda)$ to $\mathcal{S}^S_{\theta,\alpha}(\Lambda)$ preserving S-equivalence classes.*

Proof For a Λ^R-module M, we have $M \otimes_R S \otimes_S k(\mathfrak{n}) \simeq M \otimes_R k(\varphi^{-1}(\mathfrak{n}))$ as R-modules for every $\mathfrak{n} \in \mathrm{Max}(S)$, and $\varphi^{-1}(\mathfrak{n}) \in \mathrm{Max}(R)$ since K is algebraically closed. So if M is θ-semistable of dimension α, so does $M \otimes_R S$. We see that $- \otimes_R S$ preserves S-equivalence classes.

If M is finitely generated and flat over R, it is immediate that $M \otimes_R S$ is finitely generated over S. Since M is projective, and $- \otimes_R S$ takes free modules to free modules and preserves direct summands, $M \otimes_R S$ is projective, so is flat over S. □

Remark 4.3 For flat modules, the following is standard. Since $R \in \mathfrak{R}$ is finitely generated commutative K-algebra, it is noetherian. Therefore a finitely generated R-module is flat if and only if it is projective if and only if it is locally free (cf. [31, §7], [12, Section II.5.2]).

Remark 4.4 We can prove the following. For $R \in \mathfrak{R}$ the following hold.

(1) $\Lambda^R \simeq RQ/\langle\rho\rangle$ as K-algebras.
(2) $\mathrm{Mod}\Lambda^R \simeq \mathrm{Rep}_R(Q, \rho)$.
(3) The category of Λ^R-modules which are finitely generated and locally free over R is equivalent to the category of representations in the category of finitely generated and locally free R-modules.

So our definition of families coincides with King's definition [26]. Hence for every $\theta \in \Theta$ and every dimension vector α, a coarse moduli space $\mathcal{M}_{\theta,\alpha}(\Lambda)$ for $\mathcal{F}_{\theta,\alpha,\Lambda}$ exists. We recall the definition of coarse moduli spaces. There is a morphism

$$\Phi_{\theta,\alpha,\Lambda} : \mathcal{F}_{\theta,\alpha,\Lambda} \longrightarrow h_{\mathcal{M}_{\theta,\alpha}(\Lambda)}$$

such that $\Phi_{\theta,\alpha,\Lambda}(K)$ is bijective and, for any K-scheme X and any morphism $\Psi : \mathcal{F}_{\theta,\alpha,\Lambda} \to h_X$, there is a unique morphism $\Omega : h_{\mathcal{M}_{\theta,\alpha}(\Lambda)} \to h_X$ such that $\Psi = \Omega \circ \Phi_{\theta,\alpha,\Lambda}$. In particular if $\Phi_{\theta,\alpha,\Lambda}$ is an isomorphism, then $\mathcal{M}_{\theta,\alpha}(\Lambda)$ is called a fine moduli space.

4.2 Motivation

Let θ, η be stability conditions and α, β dimension vectors with respect to Λ and Γ respectively.

In the study of moduli spaces of modules, we often encounter a situation that there is a functor (resp. an equivalence)

$$F : \mathcal{S}_{\theta,\alpha}(\Lambda) \longrightarrow \mathcal{S}_{\eta,\beta}(\Gamma),$$

which preserves S-equivalence classes. This means that there is a set theoretical map (resp. bijection)

$$f : \mathcal{M}_{\theta,\alpha}(\Lambda) \longrightarrow \mathcal{M}_{\eta,\beta}(\Gamma)$$

between the closed points on the course moduli spaces. In such a case, it is natural to consider the next question:

Question When does the map f lift up to a morphism (resp. an isomorphism) of K-schemes.

Of course it is not expected in full generality. However we prove it in special cases in which the functor is given by a hom functor or a tensor functor (resp. given by a tilting module) under suitable assumptions in the following subsections.

The next proposition gives a condition that f lifts up to a morphism of K-schemes.

Proposition 4.5 *If there is a family* $\{F^R : \mathrm{Mod}\Lambda^R \to \mathrm{Mod}\Gamma^R \mid R \in \mathfrak{R}\}$ *of functors such that* $F^K = F$ *and it satisfies the following conditions:*

(M1) *For any* $R \in \mathfrak{R}$, *the functor* F^R *induces* $\mathcal{S}^R_{\theta,\alpha}(\Lambda) \to \mathcal{S}^R_{\eta,\beta}(\Gamma)$ *preserving S-equivalence classes.*

(M2) *For any ring homomorphism* $R \to S$ *in* $\mathfrak{R}$, *the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{S}^R_{\theta,\alpha}(\Lambda) & \xrightarrow{F^R} & \mathcal{S}^R_{\eta,\beta}(\Gamma) \\ {\scriptstyle -\otimes_R S}\downarrow & & \downarrow{\scriptstyle -\otimes_R S} \\ \mathcal{S}^S_{\theta,\alpha}(\Lambda) & \xrightarrow[F^S]{} & \mathcal{S}^S_{\eta,\beta}(\Gamma) \end{array}$$

then F *induces a morphism* $f : \mathcal{M}_{\theta,\alpha}(\Lambda) \to \mathcal{M}_{\eta,\beta}(\Gamma)$ *of K-schemes.*

Proof By the conditions (M1) and (M2), $\{F^R\}$ determines a natural transformation $\overline{F} : \mathcal{F}_{\theta,\alpha,\Lambda} \to \mathcal{F}_{\eta,\beta,\Gamma}$. By the definition of coarse moduli spaces, for a composition $\Phi_{\eta,\beta,\Gamma} \circ \overline{F}$, there exists a morphism $f : h_{\mathcal{M}_{\theta,\alpha}(\Lambda)} \to h_{\mathcal{M}_{\eta,\beta}(\Gamma)}$ which makes the following diagram commutative:

$$\begin{array}{ccc} \mathcal{F}_{\theta,\alpha,\Lambda} & \xrightarrow{\Phi_{\theta,\alpha,\Lambda}} & h_{\mathcal{M}_{\theta,\alpha}(\Lambda)} \\ {\scriptstyle \overline{F}}\downarrow & & \vdots\downarrow{\scriptstyle f} \\ \mathcal{F}_{\eta,\beta,\Gamma} & \xrightarrow{\Phi_{\eta,\beta,\Gamma}} & h_{\mathcal{M}_{\eta,\beta}(\Gamma)}. \end{array}$$

By Yoneda's lemma, there exists a corresponding morphism $f : \mathcal{M}_{\theta,\alpha}(\Lambda) \to \mathcal{M}_{\eta,\beta}(\Gamma)$ of K-schemes. □

By the above result, we only need to check that a given functor F satisfies the conditions (M1) and (M2). In the following, we concretely work case-by-case.

4.3 Hom Functors Inducing Morphisms

In this subsection, we consider the case where the functor F is given as a hom functor. Let T be a $\Gamma^{\mathrm{op}} \otimes_K \Lambda$-module. Then we have a family of functors

$$\{\mathrm{Hom}_{\Lambda^R}(T^R, -) : \mathrm{Mod}\Lambda^R \to \mathrm{Mod}\Gamma^R \mid R \in \mathfrak{R}\}$$

such that $\mathrm{Hom}_{\Lambda^K}(T^K, -) = \mathrm{Hom}_{\Lambda}(T, -)$. We give a sufficient condition for the family of functors to satisfies the conditions (M1) and (M2).

Theorem 4.6 *Assume that the following conditions hold.*

(i) *There exists an exact sequence*

$$0 \to P_d \to P_{d-1} \to \cdots\cdots P_1 \to P_0 \to T \to 0 \tag{4.1}$$

in $\mathrm{Mod}\Lambda$ *where* P_i *is a finitely generated projective* Λ*-module for any* i*.*

(ii) $\mathrm{Hom}_{\Lambda}(T, -) : \mathrm{Mod}\Lambda \to \mathrm{Mod}\Gamma$ *induces* $\mathcal{S}_{\theta,\alpha}(\Lambda) \to \mathcal{S}_{\eta,\beta}(\Gamma)$ *preserving S-equivalence classes.*

(iii) $\mathrm{Ext}^i_{\Lambda}(T, M) = 0$ *for any* $M \in \mathcal{S}_{\theta,\alpha}(\Lambda)$ *and any* $i > 0$*.*

Then the functor $\mathrm{Hom}_{\Lambda}(T, -)$ *induces a morphism of K-schemes* $\mathcal{M}_{\theta,\alpha}(\Lambda) \to \mathcal{M}_{\eta,\beta}(\Gamma)$*.*

In the following, we give a proof of Theorem 4.6. We divide the proof into a few steps. The following facts are basic and very useful.

Lemma 4.7 *Let* $R \in \mathfrak{R}$*. Let P be a* Λ*-module, M a* Λ^R*-module and N an R-module. Then there exists a morphism*

$$\mathrm{Hom}_{\Lambda^R}(P^R, M) \otimes_R N \longrightarrow \mathrm{Hom}_{\Lambda}(P, M \otimes_R N)$$

given by $\varphi \otimes n \longmapsto (x \longmapsto \varphi(x \otimes 1) \otimes n)$*, which are functorial in P, M and N. Moreover if P is finitely generated projective, then it is an isomorphism.*

Proof The assertion is proved straightforward. □

Lemma 4.8 *Let* $R \in \mathfrak{R}$*. Let P be a finitely generated projective* Λ^R*-module.*

(1) *If a* Λ^R*-module M is finitely generated over R, then so is* $\mathrm{Hom}_{\Lambda^R}(P, M)$*.*
(2) *If* Λ^R*-module M is flat over R, then so is* $\mathrm{Hom}_{\Lambda^R}(P, M)$*.*

Proof

(1) Since P is a direct summand of $(\Lambda^R)^n$ for some integer n, $\mathrm{Hom}_{\Lambda^R}(P, M)$ is a direct summand of $\mathrm{Hom}_{\Lambda^R}((\Lambda^R)^n, M) \simeq M^n$. Hence if M is finitely generated over R, then M^n is also finitely generated and so is $\mathrm{Hom}_{\Lambda^R}(P, M)$.
(2) is similar. □

Proposition 4.9 *Let T be a Λ-module satisfying the condition* (i) *in Theorem* 4.6*. For $R \in \mathfrak{R}$, let M be a Λ^R-module satisfying the following conditions.*

(ii) *M is finitely generated and flat over R.*
(ii) *For any $\mathfrak{m} \in \operatorname{Max}(R)$ and $i > 0$, $\operatorname{Ext}^i_\Lambda(T, k(\mathfrak{m}) \otimes_R M) = 0$ holds.*

Then the following hold.

(1) *For any $i > 0$, $\operatorname{Ext}^i_{\Lambda^R}(T^R, M) = 0$ holds.*
(2) *$\operatorname{Hom}_{\Lambda^R}(T^R, M)$ is a finitely generated flat R-module.*
(3) *For any ring homomorphism $R \to S$ in $\mathfrak{R}$, $\operatorname{Hom}_{\Lambda^R}(T^R, M) \otimes_R S \simeq \operatorname{Hom}_{\Lambda^S}(T^S, M \otimes_R S)$ holds.*

Proof

(1) It is sufficient to show that $\operatorname{Ext}^i_{\Lambda^R}(T^R, M) \otimes_R k(\mathfrak{m}) = 0$ for any $\mathfrak{m} \in \operatorname{Max}(R)$. By the assumption, there is a projective resolution

$$0 \longrightarrow P_d \xrightarrow{f_d} P_{d-1} \xrightarrow{f_{d-1}} P_{d-2} \xrightarrow{f_{d-2}} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} T \to 0 \qquad (4.2)$$

of T where P_i is finitely generated over Λ for any $i = 0, \dots, d$. For any $i = 0, \dots, d$, by putting $C_i = \operatorname{Im} f_i$ we have short exact sequences

$$0 \longrightarrow C_{i+1} \longrightarrow P_i \longrightarrow C_i \longrightarrow 0. \qquad (4.3)$$

Note that $C_0 = T$ and $C_d = P_d$. By applying $\operatorname{Hom}_{\Lambda^R}(- \otimes_K R, M) = (\operatorname{Hom}_{\Lambda^R}(-, M)) \circ (- \otimes_K R)$ to (4.3), we obtain

$$\operatorname{Ext}^j_{\Lambda^R}(C^R_{i+1}, M) \simeq \operatorname{Ext}^{j+1}_{\Lambda^R}(C^R_i, M),$$

for any $j \geq 1$, so we have $\operatorname{Ext}^1_{\Lambda^R}(C^R_i, M) \simeq \operatorname{Ext}^{i+1}_{\Lambda^R}(T^R, M)$ for any $i = 0, \dots, d-1$. Hence it is enough to show that $\operatorname{Ext}^1_{\Lambda^R}(C^R_i, M) \otimes_R k(\mathfrak{m}) = 0$ for any $i = 0, \dots, d-1$ and any $\mathfrak{m} \in \operatorname{Max}(R)$. For any $\mathfrak{m} \in \operatorname{Max}(R)$, by applying $(- \otimes_R k(\mathfrak{m})) \circ (\operatorname{Hom}_{\Lambda^R}(- \otimes_K R, M))$ and $\operatorname{Hom}_\Lambda(-, M \otimes_R k(\mathfrak{m}))$ to the exact sequence (4.3), by Lemma 4.7 we have a commutative diagram, and call it (Y),

$$\begin{array}{ccccccc}
\operatorname{Hom}_{\Lambda^R}(P^R_i, M) \otimes_R k(\mathfrak{m}) & \xrightarrow{a_{i+1}} & \operatorname{Hom}_{\Lambda^R}(C^R_{i+1}, M) \otimes_R k(\mathfrak{m}) & \longrightarrow & \operatorname{Ext}^1_{\Lambda^R}(C^R_i, M) \otimes_R k(\mathfrak{m}) & \longrightarrow & 0 \\
\downarrow g_i & & \downarrow h_{i+1} & & & & \\
\operatorname{Hom}_\Lambda(P_i, M \otimes_R k(\mathfrak{m})) & \xrightarrow{a'_{i+1}} & \operatorname{Hom}_\Lambda(C_{i+1}, M \otimes_R k(\mathfrak{m})) & \longrightarrow & \operatorname{Ext}^1_\Lambda(C_i, M \otimes_R k(\mathfrak{m})) & \longrightarrow & 0
\end{array}$$

with exact rows where g_i is an isomorphism, and

$$\operatorname{Ext}^j_\Lambda(C_{i+1}, M \otimes_R k(\mathfrak{m})) \simeq \operatorname{Ext}^{j+1}_\Lambda(C_i, M \otimes_R k(\mathfrak{m}))$$

for any $j \geq 1$. So, since $C_0 = T$ and $M \otimes_R k(\mathfrak{m}) \in \mathcal{S}_{\theta,\alpha}(\Lambda)$, by the assumption we have $\operatorname{Ext}^1_\Lambda(C_i, M \otimes_R k(\mathfrak{m})) \simeq \operatorname{Ext}^{i+1}_\Lambda(T, M \otimes_R k(\mathfrak{m})) = 0$ for any $i = 0, \dots, d-1$.
Now we use an induction on i to prove $\operatorname{Ext}^1_{\Lambda^R}(C^R_i, M) \otimes_R k(\mathfrak{m}) = 0$. We assume that $\operatorname{Ext}^1_{\Lambda^R}(C^R_j, M) \otimes_R k(\mathfrak{m}) = 0$ holds for any $j = i+1, \dots, d-1$ and h_{i+1} is an isomorphism. Then $\operatorname{Ext}^1_\Lambda(C_i, M \otimes_R k(\mathfrak{m})) = 0$ and h_{i+1} is an isomorphism implies

that $\mathrm{Ext}^1_{\Lambda^R}(C_i^R, M)\otimes_R k(\mathfrak{m})=0$ by (Y). Furthermore, by the induction hypothesis, by applying $\mathrm{Hom}_{\Lambda^R}(-\otimes_K R, M)$ to Eq. 4.3 we have an exact sequence

$$0\to \mathrm{Hom}_{\Lambda^R}(C_j^R, M)\to \mathrm{Hom}_{\Lambda^R}(P_j^R, M)\to \mathrm{Hom}_{\Lambda^R}(C_{j+1}^R, M)\to 0 \quad (4.4)$$

for any $j=i,\dots,d-1$. By applying $-\otimes_R k(\mathfrak{m})$ to Eq. 4.4, since $\mathrm{Hom}_{\Lambda^R}(P_j, M)$ are flat over R by Lemma 4.8, we have $\mathrm{Tor}_\ell^R(\mathrm{Hom}_{\Lambda^R}(C_j^R, M), k(\mathfrak{m}))\simeq \mathrm{Tor}_{\ell+1}^R(\mathrm{Hom}_{\Lambda^R}(C_{j+1}^R, M), k(\mathfrak{m}))$ for any $j=i,\dots,d-1$ and $\ell\geq 1$. So, since $C_d=P_d$, we have $\mathrm{Tor}_1^R(\mathrm{Hom}_{\Lambda^R}(C_{i+1}^R, M), k(\mathfrak{m}))\simeq \mathrm{Tor}_{d-i}^R(\mathrm{Hom}_{\Lambda^R}(P_d^R, M), k(\mathfrak{m}))=0$. Thus by Lemma 4.7 we have a commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Hom}_{\Lambda^R}(C_i^R, M)\otimes_R k(\mathfrak{m}) & \xrightarrow{b_i} & \mathrm{Hom}_{\Lambda^R}(P_i^R, M)\otimes_R k(\mathfrak{m}) & \xrightarrow{a_{i+1}} & \mathrm{Hom}_{\Lambda^R}(C_{i+1}^R, M)\otimes_R k(\mathfrak{m}) \\
& \downarrow h_i & & \downarrow g_i & & \downarrow h_{i+1} \\
0 \longrightarrow & \mathrm{Hom}_{\Lambda}(C_i, M\otimes_R k(\mathfrak{m})) & \xrightarrow{b'_i} & \mathrm{Hom}_{\Lambda}(P_i, M\otimes_R k(\mathfrak{m})) & \xrightarrow{a'_{i+1}} & \mathrm{Hom}_{\Lambda}(C_{i+1}, M\otimes_R k(\mathfrak{m}))
\end{array}$$

where each row is exact. Since g_i, h_{i+1} are isomorphisms, h_i is also an isomorphism.
In the case $i=d-1$, since $C_d=P_d$, we have $\mathrm{Ext}^1_{\Lambda^R}(C_d^R, M)\otimes_R k(\mathfrak{m})=0$, and h_d is an isomorphism. Consequently it follows that $\mathrm{Ext}^i_{\Lambda^R}(T^R, M)=0$ for any $i>0$.

(2) Since $\mathrm{Ext}^i_{\Lambda^R}(T^R, M)=0$ for any $i>0$ by (1), by applying $\mathrm{Hom}_{\Lambda^R}(-\otimes_K R, M)$ to the exact sequence (4.1), we have an exact sequene

$$0\to \mathrm{Hom}_{\Lambda^R}(T^R, M)\to \mathrm{Hom}_{\Lambda^R}(P_0^R, M)\to\cdots\to \mathrm{Hom}_{\Lambda^R}(P_d^R, M)\to 0. \quad (4.5)$$

Now R is noetherian. So, since $\mathrm{Hom}_{\Lambda^R}(P_0^R, M)$ and $\mathrm{Hom}_{\Lambda^R}(P_1^R, M)$ are finitely generated over R by Lemma 4.8, $\mathrm{Hom}_{\Lambda^R}(T^R, M)$ is also finitely generated over R.
Moreover, since M is flat over R, $\mathrm{Hom}_{\Lambda^R}(P_i, M)$ are flat over R for all $i=0,\dots,d$ by Lemma 4.8, so $\mathrm{Hom}_{\Lambda^R}(T^R, M)$ is also flat over R.

(3) By applying $-\otimes_R S$ to the sequence (4.5), since $\mathrm{Hom}_{\Lambda^R}(P_i^R, M)$ is flat over R for any $i=0,\dots,d$ by Lemma 4.8, we have an exact sequence

$$0\to \mathrm{Hom}_{\Lambda^R}(T^R, M)\otimes_R S\to \mathrm{Hom}_{\Lambda^R}(P_0^R, M)\otimes_R S\to \mathrm{Hom}_{\Lambda^R}(P_1^R, M)\otimes_R S.$$

So by applying $\mathrm{Hom}_{\Lambda^S}(-\otimes_K S, M\otimes_R S)$ to the exact sequence (4.1), we have a commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Hom}_{\Lambda^R}(T^R, M)\otimes_R S & \longrightarrow & \mathrm{Hom}_{\Lambda^R}(P_0^R, M)\otimes_R S & \longrightarrow & \mathrm{Hom}_{\Lambda^R}(P_1^R, M)\otimes_R S \\
& & & \downarrow \phi & & \downarrow \psi \\
0 \longrightarrow & \mathrm{Hom}_{\Lambda^S}(T^S, M\otimes_R S) & \longrightarrow & \mathrm{Hom}_{\Lambda^S}(P_0^S, M\otimes_R S) & \longrightarrow & \mathrm{Hom}_{\Lambda^S}(P_1^S, M\otimes_R S)
\end{array}$$

where each row is exact. By Lemma 4.7, ϕ and ψ are isomorphisms, therefore we have $\mathrm{Hom}_{\Lambda^R}(T^R, M)\otimes_R S\simeq \mathrm{Hom}_{\Lambda^S}(T^S, M\otimes_R S)$. □

Proposition 4.10 *Under the assumption in Theorem 4.6, for any* $R \in \mathfrak{R}$*, the functor* $\mathrm{Hom}_{\Lambda^R}(T^R, -) : \mathrm{Mod}\Lambda^R \to \mathrm{Mod}\Gamma^R$ *induces a functor* $\mathcal{S}^R_{\theta,\alpha}(\Lambda) \to \mathcal{S}^R_{\eta,\beta}(\Gamma)$ *preserving S-equivalence classes.*

Proof Let $R \in \mathfrak{R}$ and $M \in \mathcal{S}^R_{\theta,\alpha}(\Lambda)$. Note that by the assumption (i), (ii) and (iii) in Theorem 4.6, we can apply Proposition 4.9 to M.

For any $\mathfrak{m} \in \mathrm{Max}(R)$, we have a natural ring homomorphism $R \to k(\mathfrak{m}) \simeq K$ in $\mathfrak{R}$. So by Proposition 4.9 (3) we have an isomorphism

$$\mathrm{Hom}_{\Lambda^R}(T^R, M) \otimes_R k(\mathfrak{m}) \simeq \mathrm{Hom}_\Lambda(T, M \otimes_R k(\mathfrak{m})).$$

Since $M \otimes_R k(\mathfrak{m}) \in \mathcal{S}_{\theta,\alpha}(\Lambda)$, we have $\mathrm{Hom}_\Lambda(T, M \otimes_R k(\mathfrak{m})) \in \mathcal{S}_{\eta,\beta}(\Gamma)$ by the assumption (ii), so $\mathrm{Hom}_{\Lambda^R}(T^R, M) \otimes_R k(\mathfrak{m}) \in \mathcal{S}_{\eta,\beta}(\Gamma)$. Moreover by Proposition 4.9 (2) $\mathrm{Hom}_{\Lambda^R}(T^R, M)$ is a finitely generated flat R-module. Therefore we have $\mathrm{Hom}_{\Lambda^R}(T^R, M) \in \mathcal{S}^R_{\eta,\beta}(\Gamma)$. In addition, since $\mathrm{Hom}_\Lambda(T, -)$ preserves S-equivalence classes, $\mathrm{Hom}_{\Lambda^R}(T^R, -)$ preserves S-equivalence classes. □

Now we are ready to prove Theorem 4.6.

Proof of Theorem 4.6 For every $R \in \mathfrak{R}$, put $F^R := \mathrm{Hom}_{\Lambda^R}(T^R, -)$. By Proposition 4.10 and 4.9 (3), the family $\{F^R\}$ satisfies the conditions (M1) and (M2). Therefore the assertion holds by Proposition 4.5. □

4.4 Tensor Functors Inducing Morphisms

In this subsection, we consider the case where the functor F is given as a tensor functor. Let T be a $\Lambda^{\mathrm{op}} \otimes_K \Gamma$-module. Then we have a family of functors

$$\{- \otimes_{\Lambda^R} T^R : \mathrm{Mod}\Lambda^R \to \mathrm{Mod}\Gamma^R \mid R \in \mathfrak{R}\}$$

such that $- \otimes_{\Lambda^K} T^K = - \otimes_\Lambda T$. We give a sufficient condition for the family of functors to satisfy the conditions (M1) and (M2).

Theorem 4.11 *Assume the following conditions.*

(i) *There exists an exact sequence*

$$\cdots\cdots \to P_d \to P_{d-1} \to \cdots\cdots \to P_1 \to P_0 \to T \to 0 \tag{4.6}$$

in $\mathrm{Mod}\Lambda^{\mathrm{op}}$ *where* P_i *is a finitely generated projective* Λ^{op}*-module for any* i*.*

(ii) $- \otimes_\Lambda T : \mathrm{Mod}\Lambda \to \mathrm{Mod}\Gamma$ *induces* $\mathcal{S}_{\theta,\alpha}(\Lambda) \to \mathcal{S}_{\eta,\beta}(\Gamma)$ *preserving S-equivalence classes.*

(iii) $\mathrm{Tor}^\Lambda_i(M, T) = 0$ *for any* $M \in \mathcal{S}_{\theta,\alpha}(\Lambda)$ *and any* $i > 0$*.*

Then the functor $- \otimes_\Lambda T$ *induces a morphism of* K*-schemes* $\mathcal{M}_{\theta,\alpha}(\Lambda) \to \mathcal{M}_{\eta,\beta}(\Gamma)$*.*

In the following, we give a proof of Theorem 4.11. We divide the proof into a few steps. In the previous subsection, Theorem 4.6 is proved by using only basic facts of homological algebra. However to prove Theorem 4.11, we need some technical observations.

We use the notation $E(M)$ which stands for the injective hull of a module M over a ring R.

Lemma 4.12 *Let R be a noetherian commutative ring. For a finitely generated R-module M, the following conditions are equivalent.*

(1) $M = 0$.
(2) $\mathrm{Hom}_R(M, R/\mathfrak{m}) = 0$ *for any* $\mathfrak{m} \in \mathrm{Max}(R)$.
(3) $\mathrm{Hom}_R(M, E(R/\mathfrak{m})) = 0$ *for any* $\mathfrak{m} \in \mathrm{Max}(R)$.

Proof of Theorem 4.6 The assertion (1) ⇒ (3) ⇒ (2) is obvious. We show (2) ⇒ (1). By the assumption (2), we have

$$0 = \mathrm{Hom}_R(M, R/\mathfrak{m}) \simeq \mathrm{Hom}_R(M, \mathrm{Hom}_{R/\mathfrak{m}}(R/\mathfrak{m}, R/\mathfrak{m})) \stackrel{2.6}{\simeq} \mathrm{Hom}_{R/\mathfrak{m}}(M \otimes_R R/\mathfrak{m}, R/\mathfrak{m})$$

for any $\mathfrak{m} \in \mathrm{Max}(R)$. So since $R/\mathfrak{m}$ is a field, we have $M \otimes_R R/\mathfrak{m} = 0$ for any $\mathfrak{m} \in \mathrm{Max}(R)$. By [30, Theorem 4.8], we have $M = 0$. □

Lemma 4.13 *Let $R \in \mathfrak{R}$.*

(1) *Let T be a Λ^{op}-module satisfying the condition (i) in Theorem 4.11, M a Λ^R-module which is finitely generated over R, and $\mathfrak{m} \in \mathrm{Max}(R)$. If $\mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(T, \mathrm{Hom}_R(M, X)) = 0$ holds for any R-module X of finite length and any $i > 0$, then $\mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(T, \mathrm{Hom}_R(M, E(R/\mathfrak{m}))) = 0$ holds for any $i > 0$.*
(2) *Let M be a finitely generated R-module, and $\mathfrak{m} \in \mathrm{Max}(R)$. If $\mathrm{Ext}^i_R(M, R/\mathfrak{m}) = 0$ holds for any $i > 0$, then $\mathrm{Ext}^i_R(M, \mathrm{Hom}_R(X, E(R/\mathfrak{m}))) = 0$ holds for any finitely generated R-module X and any $i > 0$.*
(3) *Let T be a finitely generated Λ^{op}-module and M a Λ^R-module which is finitely generated over R. If $\mathrm{Tor}^R_i(M \otimes_\Lambda T, R/\mathfrak{m}) = 0$ holds for any $\mathfrak{m} \in \mathrm{Max}(R)$ and any $i > 0$, then $\mathrm{Tor}^R_i(M \otimes_\Lambda T, X) = 0$ holds for any finitely generated R-module X and any $i > 0$.*

Proof

(1) It is enough to show the case $i = 1$. There exists an exact sequence

$$0 \longrightarrow K \stackrel{f}{\longrightarrow} P \longrightarrow T \longrightarrow 0$$

where P is a finitely generated projective Λ^{op}-module and K is a finitely generated Λ^{op}-module. By applying $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(-, \mathrm{Hom}_R(M, E(R/\mathfrak{m})))$ to the above exact sequence, we have an exact sequence

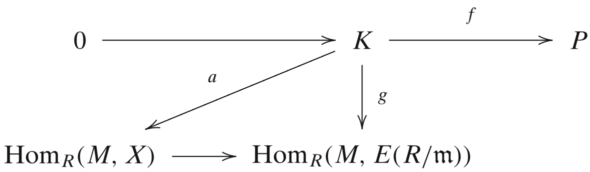
$$\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(P, \mathrm{Hom}_R(M, E(R/\mathfrak{m}))) \stackrel{(\circ f)}{\longrightarrow} \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(K, \mathrm{Hom}_R(M, E(R/\mathfrak{m}))) \longrightarrow \mathrm{Ext}^1_{\Lambda^{\mathrm{op}}}(T, \mathrm{Hom}_R(M, E(R/\mathfrak{m}))) \longrightarrow 0.$$

We show that $(\circ f)$ is a surjection.
We take $g \in \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(K, \mathrm{Hom}_R(M, E(R/\mathfrak{m})))$.

Claim There exists a submodule X of $E(R/\mathfrak{m})$ which has finite length such that $\mathrm{Im}h \subset X$ for any $h \in \mathrm{Im}g$.

Since K is a finitely generated Λ^{op}-module, there are $h_1, \ldots, h_t \in \mathrm{Hom}_R(M, E(R/\mathfrak{m}))$ such that $\mathrm{Im}g = \Lambda h_1 + \cdots + \Lambda h_t$. We put $X := \sum_{i=1}^{t} \mathrm{Im}h_i$. Then since M is a finitely generated R-module and by [28, Corollary 3.85], X has finite length. Moreover since $\mathrm{Im}(ah_i) \subset \mathrm{Im}h_i$ for any $a \in \Lambda$, we have $\mathrm{Im}h \subset X$ for any $h \in \mathrm{Im}g$. The claim has been proved.

Now by the above claim, $\mathrm{Im}g$ is contained in a submodule $\mathrm{Hom}_R(M, X)$ of the Λ^{op}-module $\mathrm{Hom}_R(M, E(R/\mathfrak{m}))$. So there is a map $a \in \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(K, \mathrm{Hom}_R(M, X))$ such that the following diagram commutes.



Here the map of the lower row is a map induced by an inclusion $X \to E(R/\mathfrak{m})$. Since we have $\mathrm{Ext}^1_{\Lambda^{\mathrm{op}}}(T, \mathrm{Hom}_R(M, X)) = 0$ by the assumption, a factors through f. Thus g factors through f. Hence $(\circ f)$ is surjective.

(2) It is enough to show the case $i = 1$. There exists an exact sequence

$$0 \longrightarrow K \xrightarrow{f} P \longrightarrow M \longrightarrow 0$$

where P is a finitely generated projective R-module and K is a finitely generated R-module. By applying $\mathrm{Hom}_R(-, \mathrm{Hom}_R(X, E(R/\mathfrak{m})))$ to the above exact sequence, we have an exact sequence

$$\mathrm{Hom}_R(P, \mathrm{Hom}_R(X, E(R/\mathfrak{m}))) \xrightarrow{(\circ f)} \mathrm{Hom}_R(K, \mathrm{Hom}_R(X, E(R/\mathfrak{m})))$$

$$\longrightarrow \mathrm{Ext}^1_R(M, \mathrm{Hom}_R(X, E(R/\mathfrak{m}))) \longrightarrow 0.$$

We show that $(\circ f)$ is a surjection.

We take $g \in \mathrm{Hom}_R(K, \mathrm{Hom}_R(X, E(R/\mathfrak{m})))$.

Claim There exists a submodule Y of X such that X/Y has finite length and $Y \subset \mathrm{Ker}h$ for any $h \in \mathrm{Im}g$.

Since K is a finitely generated R-module, there are $h_1, \ldots, h_t \in \mathrm{Hom}_R(X, E(R/\mathfrak{m}))$ such that $\mathrm{Im}g = Rh_1 + \cdots + Rh_t$. Since $X/\mathrm{Ker}h_i \simeq \mathrm{Im}h_i$ has finite length by [28, Corollary 3.85], there are maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_s$ of R such that $XI \subset \mathrm{Ker}h_i$ where $I = \mathfrak{m}_1 \cdots \mathfrak{m}_s$. We put $Y := XI$. Then since X is a finitely generated R-module, X/Y has finite length. Moreover since $\mathrm{Ker}h_i \subset \mathrm{Ker}(ah_i)$ for any $a \in R$, we have $Y \subset \mathrm{Ker}h$ for any $h \in \mathrm{Im}g$. The claim have been proved.

Now we put $Z := X/Y$. By the above claim, $\mathrm{Im}\, g$ is contained in a submodule $\mathrm{Hom}_R(Z, E(R/\mathfrak{m}))$ of $\mathrm{Hom}_R(X, E(R/\mathfrak{m}))$. So there is a map $a \in \mathrm{Hom}_R(K, \mathrm{Hom}_R(Z, E(R/\mathfrak{m})))$ such that the following diagram commutes.

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \xrightarrow{f} & P \\ & \swarrow^{a} & \downarrow{\scriptstyle g} & & \\ \mathrm{Hom}_R(Z, E(R/\mathfrak{m})) & \longrightarrow & \mathrm{Hom}_R(X, E(R/\mathfrak{m})) & & \end{array}$$

Here the map of the lower row is a map induced by a natural surjection $X \to Z$. Since we have

$$\begin{aligned} \mathrm{Ext}^1_R(M, \mathrm{Hom}_R(R/\mathfrak{m}, E(R/\mathfrak{m}))) &\simeq \mathrm{Ext}^1_R(M, \mathrm{Hom}_R(R/\mathfrak{m}, R/\mathfrak{m})) \\ &\simeq \mathrm{Ext}^1_R(M, R/\mathfrak{m}) = 0 \end{aligned}$$

by the assumption and

$$\mathrm{Ext}^1_R(M, \mathrm{Hom}_R(R/\mathfrak{n}, E(R/\mathfrak{m}))) = 0$$

for any $\mathfrak{n} \in \mathrm{Max}(R)\backslash\{\mathfrak{m}\}$, we have $\mathrm{Ext}^1_R(M, \mathrm{Hom}_R(Z, E(R/\mathfrak{m}))) = 0$. Therefore a factors through f. Thus g factors through f. Hence $(\circ f)$ is surjective.

(3) First we note that $M \otimes_\Lambda T$ is a finitely generated R-module since T is a finitely generated Λ^{op}-module and M is a finitely generated R-module.
Next for any $\mathfrak{m} \in \mathrm{Max}(R)$, we have

$$\begin{aligned} \mathrm{Ext}^i_R(M \otimes_\Lambda T, R/\mathfrak{m})) &\simeq \mathrm{Ext}^i_R(M \otimes_\Lambda T, \mathrm{Hom}_R(R/\mathfrak{m}, E(R/\mathfrak{m}))) \\ &\overset{2.6}{\simeq} \mathrm{Hom}_R(\mathrm{Tor}^R_i(M \otimes_\Lambda T, R/\mathfrak{m}), E(R/\mathfrak{m})), \end{aligned}$$

and this vanishes by the assumption.
Now let X be a finitely generated R-module. By the above arguments and (2), we have $\mathrm{Ext}^i_R(M \otimes_\Lambda T, \mathrm{Hom}_R(X, E(R/\mathfrak{m}))) = 0$ for any $\mathfrak{m} \in \mathrm{Max}(R)$. So we have

$$\mathrm{Hom}_R(\mathrm{Tor}^R_i(M \otimes_\Lambda T, X), E(R/\mathfrak{m})) \overset{2.6}{\simeq} \mathrm{Ext}^i_R(M \otimes_\Lambda T, \mathrm{Hom}_R(X, E(R/\mathfrak{m}))) = 0$$

for any $\mathfrak{m} \in \mathrm{Max}(R)$ and for any $i > 0$. Thus we have $\mathrm{Tor}^R_i(M \otimes_\Lambda T, X) = 0$ for any $i > 0$ by Lemma 4.12. □

Lemma 4.14 *Let $R \in \mathfrak{R}$.*

(1) *For any Λ^{op}-module L, Λ^R-module M and ring homomorphism $R \to S$ in $\mathfrak{R}$, there exists an isomorphism which is functorial in L and M*

$$\mathrm{Tor}^{\Lambda^R}_i(M, L^S) \simeq \mathrm{Tor}^\Lambda_i(S \otimes_R M, L)$$

for any $i \geq 0$. In particular if we consider the identity map $R \to R$, then we have $\mathrm{Tor}^{\Lambda^R}_i(M, L^R) \simeq \mathrm{Tor}^\Lambda_i(M, L)$.

(2) *Let T be a Λ^{op}-module, X an R-module, and M a Λ^R-module. We assume that M is a flat R-module and $\mathrm{Tor}_i^{\Lambda}(M, T) = 0$ for any $i > 0$. Then there exists an isomorphism*

$$\mathrm{Tor}_i^R(M \otimes_\Lambda T, X) \simeq \mathrm{Tor}_i^\Lambda(M \otimes_R X, T)$$

for any $i > 0$.

Proof

(1) The assertion for the case $i = 0$ is straightforward. In the following, we show the assertion for the case $i > 0$. We take a projective resolution

$$\cdots\cdots \to P_d \to P_{d-1} \to \cdots\cdots \to P_1 \to P_0 \to L \to 0$$

of L. By applying $(S \otimes_R M) \otimes_\Lambda -$ to the above exact sequence, we have a complex

$$\cdots\cdots \to (S \otimes_R M) \otimes_\Lambda P_d \to (S \otimes_R M) \otimes_\Lambda P_{d-1} \to \cdots\cdots$$
$$\to (S \otimes_R M) \otimes_\Lambda P_1 \to (S \otimes_R M) \otimes_\Lambda P_0.$$

On the other hand by applying $(M \otimes_{\Lambda^R} (- \otimes_K S))$ to the projective resolution of L, we have a complex

$$\cdots\cdots \to M \otimes_{\Lambda^R} P_d^S \to M \otimes_{\Lambda^R} P_{d-1}^S \to \cdots\cdots \to M \otimes_{\Lambda^R} P_1^S \to M \otimes_{\Lambda^R} P_0^S.$$

Since the above two complexes are isomorphic by the second assertion for the case $i = 0$, we have an isomorphism

$$\mathrm{Tor}_i^\Lambda(S \otimes_R M, L) \simeq \mathrm{Tor}_i^{\Lambda^R}(M, L^S)$$

for any $i > 0$.

(2) We take a projective resolution

$$\cdots\cdots \to P_d \to P_{d-1} \to \cdots\cdots \to P_1 \to P_0 \to T \to 0$$

of T. By applying $M \otimes_\Lambda -$ to the above exact sequence, we have a complex

$$\cdots\cdots \to M \otimes_\Lambda P_d \to M \otimes_\Lambda P_{d-1} \to \cdots\cdots$$
$$\to M \otimes_\Lambda P_1 \to M \otimes_\Lambda P_0 \to M \otimes_\Lambda T \to 0.$$

This is a flat resolution of $M \otimes_\Lambda T$ since this is exact by our assumption and $M \otimes_\Lambda P_i$ is a flat R-module for any $i \geq 0$. By applying $- \otimes_R X$ to the above exact sequence, we have a complex

$$\cdots\cdots \to (M \otimes_\Lambda P_d) \otimes_R X \to (M \otimes_\Lambda P_{d-1}) \otimes_R X \to \cdots\cdots$$
$$\to (M \otimes_\Lambda P_1) \otimes_R X \to (M \otimes_\Lambda P_0) \otimes_R X$$

whose i-th homology is $\mathrm{Tor}_i^R(M \otimes_\Lambda T, X)$.
On the other hand by applying $(M \otimes_R X) \otimes_\Lambda -$ to the projective resolution of T, we have a complex

$$\cdots\cdots \to (M \otimes_R X) \otimes_\Lambda P_d \to (M \otimes_R X) \otimes_\Lambda P_{d-1} \to \cdots\cdots$$
$$\to (M \otimes_R X) \otimes_\Lambda P_1 \to (M \otimes_R X) \otimes_\Lambda P_0.$$

Since the above complexes coincide, we have the desired isomorphism. □

Proposition 4.15 *Let T be a Λ^{op}-module satisfying the condition (i) in Theorem 4.11. Let $R \in \mathfrak{R}$, and M a Λ^R-module satisfying the following conditions.*

(i) *M is finitely generated and flat over R.*
(ii) *For any $\mathfrak{m} \in \mathrm{Max}(R)$ and any $i > 0$, $\mathrm{Tor}_i^{\Lambda}(M \otimes_R k(\mathfrak{m}), T) = 0$ holds.*

Then the following hold.

(1) *For any $i > 0$, $\mathrm{Tor}_i^{\Lambda^R}(M, T^R) = 0 = \mathrm{Tor}_i^{\Lambda}(M, T)$ hold.*
(2) *$M \otimes_{\Lambda^R} T^R$ is a finitely generated flat R-module.*

Proof

(1) By Lemma 4.14 (1), it is enough to show that $\mathrm{Tor}_i^{\Lambda}(M, T) = 0$ for any $i > 0$. By Lemma 4.12, we show that $\mathrm{Hom}_R(\mathrm{Tor}_i^{\Lambda}(M, T), E(R/\mathfrak{m})) = 0$ for any $\mathfrak{m} \in \mathrm{Max}(R)$. By a variation of Lemma 2.6, we have $\mathrm{Hom}_R(\mathrm{Tor}_i^{\Lambda}(M, T), E(R/\mathfrak{m})) \simeq \mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(T, \mathrm{Hom}_R(M, E(R/\mathfrak{m})))$. So in the following, we show that $\mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(T, \mathrm{Hom}_R(M, E(R/\mathfrak{m}))) = 0$ for any $\mathfrak{m} \in \mathrm{Max}(R)$.
By Lemma 4.13 (1), it is enough to show that $\mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(T, \mathrm{Hom}_R(M, X)) = 0$ for any R-module X of finite length. In addition, we have $\mathrm{Ext}^1_R(M, R/\mathfrak{m}) = 0$ for any $\mathfrak{m} \in \mathrm{Max}(R)$ since M is a finitely generated flat R-module and Remark 4.3. Thus $\mathrm{Hom}_R(M, X)$ is filtered by R-modules of the form $\mathrm{Hom}_R(M, R/\mathfrak{m})$. This fact implies that it is enough to show that $\mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(T, \mathrm{Hom}_R(M, R/\mathfrak{m})) = 0$ for any $\mathfrak{m} \in \mathrm{Max}(R)$. However it holds by the isomorphisms
$$\begin{aligned}\mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(T, \mathrm{Hom}_R(M, R/\mathfrak{m})) &\simeq \mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(T, \mathrm{Hom}_{R/\mathfrak{m}}(M \otimes_R (R/\mathfrak{m}), R/\mathfrak{m})) \\ &\simeq \mathrm{Hom}_{R/\mathfrak{m}}(\mathrm{Tor}_i^{\Lambda}(M \otimes_R (R/\mathfrak{m}), T), R/\mathfrak{m})\end{aligned}$$
and the assumption (ii). Thus we have the assertion.
(2) By Lemma 4.14 (1), we have $M \otimes_{\Lambda^R} T^R \simeq M \otimes_{\Lambda} T$. So we show that $M \otimes_{\Lambda} T$ is a finitely generated flat R-module. First $M \otimes_{\Lambda} T$ is a finitely generated R-module since T is a finitely generated Λ^{op}-module and M is a finitely generated R-module.
Next we show that $M \otimes_{\Lambda} T$ is a flat R-module by proving $\mathrm{Tor}_i^R(M \otimes_{\Lambda} T, X) = 0$ for any finitely generated R-module X and any $i > 0$. By Lemma 4.13 (3), it is enough to show that $\mathrm{Tor}_i^R(M \otimes_{\Lambda} T, R/\mathfrak{m}) = 0$ for any $\mathfrak{m} \in \mathrm{Max}(R)$ and any $i > 0$. By the assumption (i), the statement (1) and Lemma 4.14 (2), we have $\mathrm{Tor}_i^R(M \otimes_{\Lambda} T, R/\mathfrak{m}) \simeq \mathrm{Tor}_i^{\Lambda}(M \otimes_R (R/\mathfrak{m}), T)$ for any $\mathfrak{m} \in \mathrm{Max}(R)$. This vanishes by the assumption (ii). □

In the Hom functor case, we needed some assumptions to show the commutativity of the functors in Proposition 4.9 (3). In the tensor functor case, however, we need no assumption.

Proposition 4.16 *For any ring homomorphism $R \to S$ in $\mathfrak{R}$, $(M \otimes_{\Lambda^R} T^R) \otimes_R S \simeq (M \otimes_R S) \otimes_{\Lambda^S} T^S$ holds.*

Proof By Lemma 4.14 (1), we have
$$(M \otimes_{\Lambda^R} T^R) \otimes_R S \simeq (M \otimes_{\Lambda} T) \otimes_R S \simeq (M \otimes_R S) \otimes_{\Lambda} T \simeq (M \otimes_R S) \otimes_{\Lambda^S} T^S.$$
□

Proposition 4.17 *Under the assumption in Theorem 4.11, for any $R \in \mathfrak{R}$, the functor $- \otimes_{\Lambda^R} T^R : \mathrm{Mod}\Lambda^R \to \mathrm{Mod}\Gamma^R$ induces a functor $\mathcal{S}^R_{\theta,\alpha}(\Lambda) \to \mathcal{S}^R_{\eta,\beta}(\Gamma)$ preserving S-equivalence classes.*

Proof Let $R \in \mathfrak{R}$ and $M \in \mathcal{S}^R_{\theta,\alpha}(\Lambda)$. For any $\mathfrak{m} \in \mathrm{Max}(R)$, we have a natural ring homomorphism $R \to k(\mathfrak{m}) \simeq K$ in $\mathfrak{R}$. So by Proposition 4.16, $(M \otimes_{\Lambda^R} T^R) \otimes_R k(\mathfrak{m}) \simeq (M \otimes_R k(\mathfrak{m})) \otimes_\Lambda T$ holds. Since $M \otimes_R k(\mathfrak{m}) \in \mathcal{S}_{\theta,\alpha}(\Lambda)$, we have $(M \otimes_{\Lambda^R} T^R) \otimes_R k(\mathfrak{m}) \in \mathcal{S}_{\eta,\beta}(\Gamma)$ by the assumption (ii). Moreover by the assumption (i) (ii) and (iii), we can apply Proposition 4.15 to M. Thus $M \otimes_{\Lambda^R} T^R$ is a finitely generated flat R-module. Therefore we have $M \otimes_{\Lambda^R} T^R \in \mathcal{S}^R_{\eta,\beta}(\Gamma)$. In addition, since $- \otimes_\Lambda T$ preserves S-equivalence classes, $- \otimes_{\Lambda^R} T^R$ preserves S-equivalence classes. □

Now we are ready to prove Theorem 4.11.

Proof of Theorem 4.11 For every $R \in \mathfrak{R}$, put $F^R := - \otimes_{\Lambda^R} T^R$. By Proposition 4.17 and 4.16, the family $\{F^R\}$ satisfies the conditions (M1) and (M2). Thus by Proposition 4.5 the assertion holds. □

4.5 Tiltings Inducing Isomorphisms

In this subsection we consider the case where the equivalence $F : \mathcal{S}_{\theta,\alpha}(\Lambda) \to \mathcal{S}_{\eta,\beta}(\Gamma)$ is given by a tilting Λ-module. To show that F induces an isomorphism of K-schemes $\mathcal{M}_{\theta,\alpha}(\Lambda) \to \mathcal{M}_{\eta,\beta}(\Gamma)$, we first show that a tilting module is preserved under base change.

Proposition 4.18 *Let R be a commutative K-algebra and T a tilting Λ-module of projective dimension at most d. Then $T^R = T \otimes_K R$ is a tilting Λ^R-module of projective dimension d whose endomorphism ring is isomorphic to $\mathrm{End}_\Lambda(T) \otimes_K R$.*

To prove the above we need the following result.

Lemma 4.19 *Let $R \in \mathfrak{R}$ and X and Y be Λ-modules. Then there exists an isomorphism*

$$\mathrm{Ext}^i_{\Lambda^R}(X^R, Y^R) \simeq \mathrm{Ext}^i_\Lambda(X, Y) \otimes_K R$$

for any $i \geq 0$.

Proof We take a projective resolution

$$\cdots\cdots \longrightarrow P_d \longrightarrow P_{d-1} \longrightarrow \cdots\cdots \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

of X. By applying $- \otimes_K R$ to this exact sequence, we have a projective resolution

$$\cdots\cdots \longrightarrow P^R_d \longrightarrow P^R_{d-1} \longrightarrow \cdots\cdots \longrightarrow P^R_0 \longrightarrow X^R \longrightarrow 0$$

of X^R as a Λ^R-module. By applying $\mathrm{Hom}_{\Lambda^R}(-, Y^R)$ to this exact sequence, we have a complex

$$\mathrm{Hom}_{\Lambda^R}(P^R_0, Y^R) \longrightarrow \cdots\cdots \longrightarrow \mathrm{Hom}_{\Lambda^R}(P^R_{d-1}, Y^R) \longrightarrow \mathrm{Hom}_{\Lambda^R}(P^R_d, Y^R) \longrightarrow \cdots\cdots$$

On the other hand by applying $\mathrm{Hom}_\Lambda(-, Y) \otimes_K R$ to the projective resolution of X, we have a complex

$$\mathrm{Hom}_\Lambda(P_0, Y) \otimes_K R \longrightarrow \cdots\cdots \longrightarrow \mathrm{Hom}_\Lambda(P_{d-1}, Y) \otimes_K R$$
$$\longrightarrow \mathrm{Hom}_\Lambda(P_d, Y) \otimes_K R \longrightarrow \cdots\cdots$$

whose i-th homology is $\mathrm{Ext}^i_\Lambda(X, Y) \otimes_K R$. By Lemma 4.7, the above two complexes are isomorphic. Thus we have $\mathrm{Ext}^i_{\Lambda^R}(X^R, Y^R) \simeq \mathrm{Ext}^i_\Lambda(X, Y) \otimes_K R$. □

Proof of Proposition 4.18 Since T is a tilting Λ-module of projective dimension at most d, there exist exact sequences

$$0 \longrightarrow P_d \longrightarrow \cdots\cdots \longrightarrow P_0 \longrightarrow T \longrightarrow 0$$

where each P_i is a finitely generated projective Λ-module and

$$0 \longrightarrow \Lambda_\Lambda \longrightarrow T_0 \longrightarrow \cdots\cdots \longrightarrow T_d \longrightarrow 0$$

where each T_i is in $\mathrm{add}T$.

First we can show that T^R satisfies the conditions (1) and (3) in Definition 2.1 by applying $- \otimes_K R$ to the above exact sequences. Next since T is a tilting Λ-module and by Lemma 4.19, we have $\mathrm{Ext}^i_{\Lambda^R}(T^R, T^R) \simeq \mathrm{Ext}^i_\Lambda(T, T) \otimes_K R = 0$. So T^R satisfies the condition (2) in Definition 2.1. Thus T^R is a tilting Λ^R-module. Finally by Lemma 4.19, we have $\mathrm{End}_{\Lambda^R}(T^R) \simeq \mathrm{End}_\Lambda(T) \otimes_K R$. □

Now we state the main result in this subsection.

Theorem 4.20 *Let T be a tilting Λ-module and $\Gamma := \mathrm{End}_\Lambda(T)$. Assume that the following conditions hold.*

(i) $\mathrm{Hom}_\Lambda(T, -) : \mathrm{Mod}\Lambda \to \mathrm{Mod}\Gamma$ *induces* $\mathcal{S}_{\theta,\alpha}(\Lambda) \to \mathcal{S}_{\eta,\beta}(\Gamma)$ *preserving S-equivalence classes.*
(i) $\mathrm{Ext}^i_\Lambda(T, M) = 0$ *for any* $M \in \mathcal{S}_{\theta,\alpha}(\Lambda)$ *and any* $i > 0$.
(iii) $- \otimes_\Gamma T : \mathrm{Mod}\Gamma \to \mathrm{Mod}\Lambda$ *induces* $\mathcal{S}_{\eta,\beta}(\Gamma) \to \mathcal{S}_{\theta,\alpha}(\Lambda)$ *preserving S-equivalence classes.*
(iv) $\mathrm{Tor}^\Gamma_i(M, T) = 0$ *for any* $M \in \mathcal{S}_{\eta,\beta}(\Gamma)$ *and any* $i > 0$.

Equivalently triangle equivalences

$$\mathcal{D}(\mathrm{Mod}\Lambda) \underset{-\overset{\mathbb{L}}{\otimes}_\Gamma T}{\overset{\mathbb{R}\mathrm{Hom}_\Lambda(T,-)}{\rightleftarrows}} \mathcal{D}(\mathrm{Mod}\Gamma)$$

induce equivalences

$$\mathcal{S}_{\theta,\alpha}(\Lambda) \underset{-\otimes_\Gamma T}{\overset{\mathrm{Hom}_\Lambda(T,-)}{\rightleftarrows}} \mathcal{S}_{\eta,\beta}(\Gamma)$$

preserving S-equivalence classes. Then the functors $\mathrm{Hom}_\Lambda(T, -)$ *and* $- \otimes_\Lambda T$ *induce an isomorphism* $\mathcal{M}_{\theta,\alpha}(\Lambda) \cong \mathcal{M}_{\eta,\beta}(\Gamma)$ *of K-schemes.*

Proof For every $R \in \mathfrak{R}$, put $F^R := \mathrm{Hom}_{\Lambda^R}(T^R, -)$ and $G^R := - \otimes_{\Lambda^R} T^R$. By Proposition 4.18 we have a triangle equivalences

$$\mathcal{D}(\mathrm{Mod}\Lambda) \underset{- \overset{\mathbb{L}}{\otimes}_{\Gamma^R} T^R}{\overset{\mathbb{R}\mathrm{Hom}_{\Lambda^R}(T^R,-)}{\rightleftarrows}} \mathcal{D}(\mathrm{Mod}\Gamma).$$

By Proposition 4.10 and 4.17, these induce functors

$$\mathcal{S}^R_{\theta,\alpha}(\Lambda) \underset{G^R}{\overset{F^R}{\rightleftarrows}} \mathcal{S}^R_{\eta,\beta}(\Gamma)$$

preserving S-equivalence classes. However, by Proposition 4.9 (1) and 4.15 (1), these are actually equivalences. So these induce bijections

$$\mathcal{F}_{\theta,\alpha,\Lambda}(R) \underset{G^R}{\overset{F^R}{\rightleftarrows}} \mathcal{F}_{\eta,\beta,\Gamma}(R).$$

So by Proposition 4.9 (3) and 4.16, we have natural isomorphisms

$$\mathcal{F}_{\theta,\alpha,\Lambda} \underset{\overline{G}}{\overset{\overline{F}}{\rightleftarrows}} \mathcal{F}_{\eta,\beta,\Gamma}$$

defined by $\overline{F}(R) := F^R$ and $\overline{G}(R) := G^R$. Since $h_{\mathcal{M}_{\theta,\alpha}(\Lambda)}$ and $h_{\mathcal{M}_{\eta,\beta}(\Gamma)}$ are coarse moduli spaces of $\mathcal{F}_{\theta,\alpha,\Lambda}$ and $\mathcal{F}_{\eta,\beta,\Gamma}$ respectively, by uniqueness $h_{\mathcal{M}_{\theta,\alpha}(\Lambda)} \simeq h_{\mathcal{M}_{\eta,\beta}(\Gamma)}$. Hence $\mathcal{M}_{\theta,\alpha}(\Lambda) \cong \mathcal{M}_{\eta,\beta}(\Gamma)$ as K-schemes. □

Now we return to the setting in the previous section. We show that the categorical equivalence shown in Theorem 3.12 induces an isomorphism of K-schemes between moduli spaces.

Corollary 4.21 *For any preprojective algebra $\Lambda = K\overline{Q}/\langle R\rangle$ of a non-Dynkin quiver Q, any element w of the Coxeter group W_Q and any sufficiently general parameter $\theta \in \Theta$, the equivalence* **w** *induces an isomorphism of K-schemes*

$$\mathcal{M}_{\theta,\alpha}(\Lambda) \simeq \mathcal{M}_{w\theta,w\alpha}(\Lambda).$$

Proof It is enough to show the assertion for the setting in Theorem 3.5. Let $\theta \in \Theta$ such that $\theta_i > 0$. Then by Theorem 3.5, we have categorical equivalences

$$\mathcal{S}_{\theta,\alpha}(\Lambda) \underset{-\otimes_\Lambda I_i}{\overset{\mathrm{Hom}_\Lambda(I_i,-)}{\rightleftarrows}} \mathcal{S}_{s_i\theta,s_i\alpha}(\Lambda)$$

where I_i is a tilting Λ-module whose endomorphism algebra is isomorphic to Λ (see Section 2.3). By Lemma 3.4 and Theorem 3.5, we can apply Theorem 4.20 to the above functors. Thus the above functors induce an isomorphisms $\mathcal{M}_{\theta,\alpha}(\Lambda) \simeq \mathcal{M}_{s_i\theta,s_i\alpha}(\Lambda)$ of K-schemes. □

5 Kleinian Singularity Case

In this section, we investigate Kleinian singularities. We assume K is an algebraically closed field of characteristic 0. Let G be a finite subgroup of $SL(2, K)$ of type Γ, that is, $\Gamma = A_m (m \geq 1)$, $D_m (m \geq 4)$, E_6, E_7 or E_8. We denote by $\widetilde{\Gamma}$ the type of the extended Dynkin diagram of Γ. We denote by Λ the preprojective algebra associated to the extended Dynkin quiver of type $\widetilde{\Gamma}$. Note that the double $\overline{Q}$ of Q is the so-called McKay quiver of G (see Fig. 1). The vertex set of $\overline{Q}$ is denoted by $\overline{Q}_0 = \{0, 1, \ldots, n\}$ where 0 corresponds to the trivial representation of G. Let $\Theta = (\mathbb{Z}^{Q_0})^* \otimes_{\mathbb{Z}} \mathbb{Q}$ be a parameter space and W the Weyl group of type Γ (see Sections 2 and 3.1).

5.1 Moduli Space and Parameter Space

First we recall the relation between moduli spaces of Λ-module and the Kleinian singularity $\mathbb{A}^2/G$, and the chamber structure of the parameter space Θ and the Weyl group.

Let $\mathbf{d}$ be the dimension vector whose entries are the dimensions of irreducible representations of G (see Fig. 1). Since we are especially interested in moduli spaces of Λ-modules of dimension vector $\mathbf{d}$, we define a subset $\Theta_{\mathbf{d}}$ of Θ as follows:

$$\Theta_{\mathbf{d}} = \{\theta \in \Theta \mid \theta(\mathbf{d}) = 0\}.$$

Then $\theta \in \Theta_{\mathbf{d}}$ is called *generic* if any θ-semistable module is θ-stable. It is known that, for any $\theta \in \Theta_{\mathbf{d}}$, the moduli space $\mathcal{M}_{\theta,\mathbf{d}}(\Lambda)$ gives a partial resolution of the Kleinian singularity $\mathbb{A}^2/G$. Moreover if θ is generic, the next result is well-known.

Theorem 5.1 [10, 17, 27] *If $\theta \in \Theta_{\mathbf{d}}$ is generic, then $\mathcal{M}_{\theta,\mathbf{d}}(\Lambda)$ is isomorphic to the minimal resolution of the Kleinian singularity $\mathbb{A}^2/G$ via the natural projective morphism*

$$\mathcal{M}_{\theta,\mathbf{d}}(\Lambda) \to \mathcal{M}_{\mathbf{0},\mathbf{d}}(\Lambda) \cong \mathbb{A}^2/G.$$

Next we recall the chamber structure of $\Theta_{\mathbf{d}}$ given in [17, 27]. We prepare some notations about root systems (cf. [22]). We write $\widetilde{X}_* = \mathbb{Z}^{Q_0}$ which is regarded as the affine root lattice of type $\widetilde{\Gamma}$ with a symmetric bilinear form $(-,-)$ defined in Section 2.2. In this case we have

$$(\mathbf{e}_i, \mathbf{e}_j) = \begin{cases} 2 & i = j \\ -1 & i \text{ and } j \text{ are adjacent vertices in } Q \\ 0 & i \text{ and } j \text{ are adjacent vertices in } Q \end{cases}.$$

Since $\mathbf{d} \in \widetilde{X}_*$ is a minimal imaginary root of $\widetilde{X}_*$ (namely $(\alpha, \mathbf{d}) = 0$ holds for all $\alpha \in \widetilde{X}_*$, and further any $\mathbf{d}'$ with such a property is written as $\mathbf{d}' = m\mathbf{d}$ for some $m \in \mathbb{Z}$), the quotient lattice

$$X_* := \widetilde{X}_*/\mathbb{Z}\mathbf{d}$$

becomes the finite root lattice of type Γ with the induced bilinear form, again we denote it by $(-,-)$. We denote the image of $\alpha \in \widetilde{X}_*$ by $\overline{\alpha}$. Then $\overline{\mathbf{e}}_1, \ldots, \overline{\mathbf{e}}_n$ form a basis of X_* since $\overline{\mathbf{e}}_0 = -d_1\overline{\mathbf{e}}_1 - \cdots - d_n\overline{\mathbf{e}}_n$ holds. The dual of X_* is given as the sublattice

$$X^* := \{\theta \in \widetilde{X}^* \mid \theta(\mathbf{d}) = 0\}$$

of $\widetilde{X}^*$. For any $\theta \in X^*$, since $\theta(\mathbf{d}) = 0$, we can define $\theta(\overline{\alpha}) := \theta(\alpha)$ for any $\overline{\alpha} \in X_*$. For the finite root lattice X_*, let Φ be the finite root system, $\Delta = \{\overline{\mathbf{e}}_1, \ldots, \overline{\mathbf{e}}_n\} \subset \Phi$ a simple root system of Φ and Π (resp. $-\Pi$) the positive (resp. negative) root system corresponding to Δ i.e. $\Pi = \Phi \cap \mathbb{Z}_{\geq 0}\Delta$. Let W be the finite Weyl group associated to the finite root system Φ, which is a finite group generated by simple reflections $s_1, \ldots, s_n$ where s_i is defined by $s_i(\alpha) = \alpha - (\alpha, \overline{\mathbf{e}}_i)\overline{\mathbf{e}}_i$ for $\alpha \in \Phi$. Note that W is regarded as a subgroup of the Coxeter group W_Q. For any element $w \in W$, $w\Delta := \{w\overline{\mathbf{e}}_1, \ldots, w\overline{\mathbf{e}}_n\}$ is also a simple root system of Φ.

Now we describe a chamber structure of $\Theta_{\mathbf{d}}$. Let Θ^{gen} be a subset of $\Theta_{\mathbf{d}}$ consisting of generic parameters. Each connected component in $\Theta_{\mathbf{d}}^{\mathrm{gen}}$ is called a GIT chamber. θ and θ' are contained in the same GIT chamber if and only if $\mathcal{S}_{\theta,\mathbf{d}}(\Lambda) = \mathcal{S}_{\theta',\mathbf{d}}(\Lambda)$ holds. On the other hand, for any element $w \in W$ the subset

$$C(w) = \{\theta \in \Theta_{\mathbf{d}} \mid \theta(\alpha) > 0 \text{ for any } \alpha \in w\Delta\}$$

of $\Theta_{\mathbf{d}}$ is called a Weyl chamber.

Proposition 5.2 [17, 27] *For any $\theta \in \Theta_{\mathbf{d}}$, θ is generic if and only if $\theta(\alpha) \neq 0$ for any real root $\alpha \in \widetilde{X}^*$ strictly between 0 and $\mathbf{d}$, equivalently if and only if $\theta(\alpha) \neq 0$ for any $\alpha \in \Phi$.*

Thus GIT chambers coincide with Weyl chambers:

$$\Theta_{\mathbf{d}}^{\mathrm{gen}} = \coprod_{w \in W} C(w).$$

We just call them chambers. It is known that W acts on the set of chambers simply transitively [22].

In the rest of this section, we only consider generic parameters. Then the categories $\mathcal{S}_{\theta,\mathbf{d}}(\Lambda)$ and the moduli spaces $\mathcal{M}_{\theta,\mathbf{d}}(\Lambda)$ are classified by the elements of W, thus we give the following definition.

Definition 5.3 For any $\theta \in C(w)$, we write $\mathcal{S}_w = \mathcal{S}_{\theta,\mathbf{d}}(\Lambda)$ and $\mathcal{M}_w = \mathcal{M}_{\theta,\mathbf{d}}(\Lambda)$.

The next fact is used in the following.

Lemma 5.4 *Let $w \in W$ and $i \in \{1, \ldots, n\}$. Then the following are equivalent;*

(1) $\ell(s_i w) > \ell(w)$,
(2) $\overline{\mathbf{e}}_i \in w\Pi$, *equivalently* $-\overline{\mathbf{e}}_i \in s_i w\Pi$,
(3) $\theta_i > 0$ *for any* $\theta \in C(w)$,
(4) $\theta_i < 0$ *for any* $\theta \in C(s_i w)$.

Proof (1) $\Leftrightarrow$ (2) follows from [22, 1.6]. By the definition of chambers the rest is clear. □

5.2 Description of the Reflection Functor

Next we revisit the reflection functor in the Kleinian singularity case. We describe any reflection functor concretely as a composition of simple reflection functors. First we consider the simple reflection case.

Proposition 5.5 *For any $w \in W$, if $\ell(w) < \ell(s_i w)$, then we have $\mathcal{S}_w \subset \mathcal{T}(I_i)$ and $\mathcal{S}_{s_i w} \subset \mathcal{Y}(I_i)$, and there is a categorical equivalence*

$$\mathcal{S}_w \underset{-\otimes_\Lambda I_i}{\overset{\mathrm{Hom}_\Lambda(I_i,-)}{\rightleftarrows}} \mathcal{S}_{s_i w}.$$

Proof If we take a $\theta \in C(w)$, then by Lemma 5.4 we have $\theta_i > 0$. So the assertion follows from Lemma 3.4 and Theorem 3.5. □

By virtue of Proposition 3.10, for any $w \in W$, the corresponding reflection functor $\mathbf{w} = \mathbf{s}_{i_\ell} \cdots \mathbf{s}_{i_1}$ does not depend on an expression $s_{i_\ell} \cdots s_{i_1}$ of w up to isomorphisms. First we observe the relation between $\mathcal{S}_1$ and $\mathcal{S}_w$ for any $w \in W$.

Proposition 5.6 *For any $w \in W$, we have $\mathcal{S}_1 \subset \mathcal{T}(I_w)$ and $\mathcal{S}_w \subset \mathcal{Y}(I_w)$, and there is a categorical equivalence*

$$\mathcal{S}_1 \underset{\mathbf{w}^{-1} \simeq -\otimes_\Lambda I_w}{\overset{\mathbf{w} \simeq \mathrm{Hom}_\Lambda(I_w,-)}{\rightleftarrows}} \mathcal{S}_w.$$

Proof Let $\theta \in C(1)$. Take any reduced expression $s_{i_\ell} \cdots s_{i_1}$ of w. If we write $w_j = s_{i_j} \cdots s_{i_1}$ for any $j = 1, \ldots, \ell$ and $w_0 = 1$, then $\ell(s_{j+1} w_j) > \ell(w_j)$ holds for any $j = 0, \ldots, \ell - 1$. So by Lemma 5.4, we have $(w_j\theta)_{i_{j+1}} > 0$ for any $j = 0, \ldots, \ell - 1$. Hence the assertion follows from Proposition 3.13. □

Theorem 5.7 *For any $w_1, w_2 \in W$, there is a categorical equivalence*

$$\mathcal{S}_{w_1} \underset{\mathbf{w}_1\mathbf{w}_2^{-1} \simeq \mathrm{Hom}_\Lambda(I_{w_1}, -\otimes_\Lambda I_{w_2})}{\overset{\mathbf{w}_2\mathbf{w}_1^{-1} \simeq \mathrm{Hom}_\Lambda(I_{w_2}, -\otimes_\Lambda I_{w_1})}{\rightleftarrows}} \mathcal{S}_{w_2}.$$

Proof By Proposition 5.6, we have categorical equivalenecs

$$\mathcal{S}_{w_1} \underset{\mathrm{Hom}_\Lambda(I_{w_1},-)}{\overset{-\otimes_\Lambda I_{w_1}}{\rightleftarrows}} \mathcal{S}_1 \underset{-\otimes_\Lambda I_{w_2}}{\overset{\mathrm{Hom}_\Lambda(I_{w_2},-)}{\rightleftarrows}} \mathcal{S}_{w_2}.$$

Therefore the assertion follows. □

Corollary 5.8 *For any $w_1, w_2 \in W$, $\mathcal{M}_{w_1}$ is isomorphic to $\mathcal{M}_{w_2}$ as a variety, which is induced by reflection functors $\mathbf{w}_2\mathbf{w}_1^{-1}$ and $\mathbf{w}_1\mathbf{w}_2^{-1}$.*

Proof This follows from Corollary 4.21. □

Remark 5.9 We only considered the case when $\theta \in \Theta_{\mathbf{d}}$ is generic. However we can prove similar results for any $\theta \in \Theta$ and any dimension vector α if we impose some assumptions on θ to use Theorem 3.5.

5.3 Properties of $S_i^w = \mathbb{R}\mathrm{Hom}_\Lambda(I_w, S_i)$

In the rest of this section, we study properties of modules M contained in $\mathcal{S}_w$ by using our previous results. We consider the following two problems.

(1) Is there a homological characterization of the condition $M \in \mathcal{S}_w$?
(2) Is there a characterization of the exceptional curves in $\mathcal{M}_w$?

The complexes defined below will play an important role to solve the above problems. Recall that S_i denotes the simple Λ-module corresponding to a vertex $i \in Q_0$ and $\mathbf{e}_i = \underline{\dim} S_i$.

Definition 5.10 For any $w \in W$ and $i \in Q_0$, we denote by $S_i^w := \mathbb{R}\mathrm{Hom}_\Lambda(I_w, S_i)$. We mention that

$$S_i^w = \begin{cases} \mathrm{Hom}_\Lambda(I_w, S_i) & \text{if } S_i \in \mathcal{T}(I_w), \\ \mathrm{Ext}^1_\Lambda(I_w, S_i)[-1] & \text{if } S_i \in \mathcal{F}(I_w) \end{cases}.$$

Lemma 5.11 *Precisely either $S_i \in \mathcal{T}(I_w)$ or $S_i \in \mathcal{F}(I_w)$ holds. Moreover*

(1) *$S_i \in \mathcal{T}(I_w)$ if and only if $w\mathbf{e}_i \in \Pi$. In this case, we have $S_i^w = \mathrm{Hom}_\Lambda(I_w, S_i)$ and $\underline{\dim}\mathrm{Hom}_\Lambda(I_w, S_i) = w\mathbf{e}_i$.*
(2) *$S_i \in \mathcal{F}(I_w)$ if and only if $w\mathbf{e}_i \in -\Pi$. In this case we have $S_i^w[1] = \mathrm{Ext}^1_\Lambda(I_w, S_i)$ and $\underline{\dim}\mathrm{Ext}^1_\Lambda(I_w, S_i) = -w\mathbf{e}_i$.*

Proof Since $(\mathcal{T}(I_w), \mathcal{F}(I_w))$ is a torsion theory and S_i is simple, by considering the canonical exact sequence $0 \to T \to S_i \to F \to 0$ with $T \in \mathcal{T}(I_w)$ and $F \in \mathcal{F}(I_w)$, precisely either $S_i \in \mathcal{T}(I_w)$ or $S_i \in \mathcal{F}(I_w)$ holds. The rest follows from the equality

$$w\mathbf{e}_i = [S_i^w] = \underline{\dim}(\mathrm{Hom}_\Lambda(I_w, S_i)) - \underline{\dim}(\mathrm{Ext}^1_\Lambda(I_w, S_i))$$

which is given by Theorem 2.28. □

Remark 5.12 For any $w \in W$, the dimension vectors $[S_1^w], \ldots, [S_n^w]$ induce the simple root system $w\Delta = \{w\overline{\mathbf{e}}_1, \ldots, w\overline{\mathbf{e}}_n\}$.

5.4 Homological Interpretation of the Stability Condition

We give a homological characterization of the condition $M \in \mathcal{S}_w$ for any $w \in W$. Fix a vertex $v \in Q_0$. Then a Λ-module M is said to be v-generated if the dimension of Me_v is 1 and M is generated by an element of Me_v. We give characterizations of this notion for finite dimensional nilpotent Λ-modules.

Lemma 5.13 *Let M be a non-zero finite dimensional nilpotent Λ-module, and $v \in Q_0$. Then the following are equivalent.*

(1) *M is v-generated.*
(2) $M/MI \simeq S_v$.
(3) $\dim \operatorname{Hom}_\Lambda(M, S_v) = \begin{cases} 1 & (i = v), \\ 0 & (i \neq v). \end{cases}$

Proof

(1) $\Rightarrow$ (2) We take $m \in Me_v$. Then we can define Λ-homomorphism

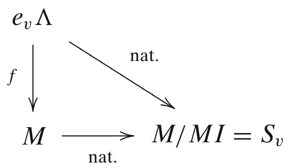
$$f : e_v\Lambda \ni e_v a \longmapsto ma \in M.$$

By the assumption (1), this is a surjection. Thus by applying $- \otimes_\Lambda (\Lambda/I)$ to f, we have a surjection

$$f \otimes_\Lambda (\Lambda/I) : e_v\Lambda/e_v I \longrightarrow M/MI.$$

By Lemma 2.16, M/MI is not zero. Moreover since $e_v\Lambda/e_v I = S_v$, M satisfies the condition (2).

(2) $\Rightarrow$ (1) Since $e_v\Lambda$ is projective, there is $f \in \operatorname{Hom}_\Lambda(e_v\Lambda, M)$ such that the following diagram commutes.



Since $f(e_v) \in Me_v$, it is enough to show that f is a surjection. We put $N := \operatorname{Im} f$. By applying $- \otimes_\Lambda (\Lambda/I)$ to the exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0,$$

we have the exact sequence

$$N/(NI) \longrightarrow M/(MI) \longrightarrow (M/N)/(M/N)I \longrightarrow 0.$$

Since the map $N/(NI) \to M/(MI) = S_v$ is not zero by the the above commutative diagram, we have $(M/N)/(M/N)I = 0$. Thus by Lemma 2.16, we have $M/N = 0$. Consequently f is a surjection.

(2) $\Leftrightarrow$ (3) This equivalence follows from Lemma 2.15 (1). □

As an application of the above, the following result follows.

Lemma 5.14 *For any Λ-module M of dimension vector* **d***, the following are equivalent.*

(1) $M \in \mathcal{S}_1$.
(2) *M is 0-generated.*
(3) $\operatorname{Hom}_\Lambda(M, S_i) = 0$ *holds for all $i = 1, \ldots, n$.*

Proof (1) ⇔ (2) is a well known result (see [14, Exercise 4.12]). We show (2) ⇔ (3). By Lemma 2.31 (2), M is either simple or nilpotent. If M is simple, then M satisfies both of (2) and (3). If M is nilpotent, then the assertion follows from Lemma 5.13 and $d_0 = 1$. □

Next we consider the general case. The next is the main result in this subsection.

Theorem 5.15 *For any $w \in W$ and any Λ-module M of dimension vector* $\mathbf{d}$, *the following are equivalent.*

(1) $M \in \mathcal{S}_w$.
(2) *For all $i = 1, \dots, n$, the following hold.*

$$\begin{cases} \mathrm{Hom}_\Lambda(M, S_i^w) = 0 & \text{if } w\mathbf{e}_i \in \Pi, \\ \mathrm{Hom}_\Lambda(S_i^w[1], M) = 0 & \text{if } w\mathbf{e}_i \in -\Pi. \end{cases}$$

To prove Theorem 5.15, we need the next technical lemma.

Lemma 5.16 *For $w \in W \backslash \{1\}$, we take a reduced expression $w = s_{i_\ell} \cdots s_{i_1}$. Then there exists $i \in \{1, \dots, n\}$ such that $w\mathbf{e}_i \in -\Pi$ and $\mathrm{Hom}_\Lambda(S_i^w[1], S_{i_\ell}) \neq 0$. In particular $S_i^w[1]/S_i^w[1]I$ contains S_{i_ℓ} as a direct summand.*

Proof First we show that $\mathrm{Tor}_1^\Lambda(S_{i_\ell}, I_w)$ is in $\mathcal{F}(I_w)$. By Proposition 2.27, we have $I_w = I_{i_\ell} \otimes_\Lambda I_{s_{i_\ell} w}$. By Lemma 2.23, $S_{i_\ell} \in \mathcal{X}(I_{i_\ell})$ holds, so we have $S_{i_\ell} \otimes_\Lambda I_{i_\ell} = 0$. Thus we have $S_{i_\ell} \otimes_\Lambda I_w = S_{i_\ell} \otimes_\Lambda I_{i_\ell} \otimes_\Lambda I_{s_{i_\ell} w} = 0$. This means that $S_{i_\ell} \in \mathcal{X}(I_w)$, and so $\mathrm{Tor}_1^\Lambda(S_{i_\ell}, I_w) \in \mathcal{F}(I_w)$ by Theorem 2.9.

Next $\mathrm{Tor}_1^\Lambda(S_{i_\ell}, I_w)$ is in $\mathrm{nilp}\Lambda$ and is non-zero by Lemma 2.22. Moreover we show that S_0 does not appear in composition factors of $\mathrm{Tor}_1^\Lambda(S_{i_\ell}, I_w)$. By Corollary 2.29, we have $\underline{\dim}\mathrm{Tor}_1^\Lambda(S_{i_\ell}, I_w) = -[S_{i_\ell} \overset{\mathbb{L}}{\otimes}_\Lambda I_w] = -w\mathbf{e}_{i_\ell}$. Since s_0 does not appear in a reduced expression of w, one can see that $(w\mathbf{e}_{i_\ell})_0 = 0$. By this fact, S_0 does not appear in composition factors of $\mathrm{Tor}_1^\Lambda(S_{i_\ell}, I_w)$.

Next by the above argument and Lemma 2.16, there exists $i \in \{1, \dots, n\}$ such that $\mathrm{Hom}_\Lambda(S_i, \mathrm{Tor}_1^\Lambda(S_{i_\ell}, I_w)) \neq 0$. This implies S_i is a submodule of $\mathrm{Tor}_1^\Lambda(S_{i_\ell}, I_w)$. Since $\mathcal{F}(I_w)$ is closed under submodules, $S_i \in \mathcal{F}(I_w)$. By Lemma 5.11, we have $w\mathbf{e}_i \in -\Pi$. Moreover we have

$$\begin{aligned} \mathrm{Hom}_\Lambda(S_i^w[1], S_{i_\ell}) &\simeq \mathrm{Hom}_{\mathcal{D}(\mathrm{Mod}\Lambda)}(\mathbb{R}\mathrm{Hom}_\Lambda(I_w, S_i[1]), S_{i_\ell}) \\ &\simeq \mathrm{Hom}_{\mathcal{D}(\mathrm{Mod}\Lambda)}(S_i[1], \mathrm{Tor}_1^\Lambda(I_w, S_{i_\ell})[1]) \\ &\simeq \mathrm{Hom}_\Lambda(S_i, \mathrm{Tor}_1^\Lambda(I_w, S_{i_\ell})) \neq 0. \end{aligned}$$

□

Proof of Theorem 5.15 We take a Λ-module M of dimension vector $\mathbf{d}$. Then by Lemma 2.31 (2), M is either nilpotent or simple Λ-module.

First we assume that M is simple, and show that it satisfies the both conditions (1) and (2). It satisfies the condition (1) since any simple Λ-module of dimension vector $\mathbf{d}$ is in $\mathcal{S}_w$. By Lemma 2.22, S_i^w (resp. $S_i^w[1]$) is nilpotent for $w\mathbf{e}_i \in \Pi$ (resp. $w\mathbf{e}_i \in -\Pi$). Thus M satisfies the condition (2) by Proposition 2.32.

Next we assume that M is nilpotent. Let $\ell = \ell(w)$. We prove the assertion by induction on ℓ. In the case $\ell = 0$, namely the case $w = 1$, the assertion follows from Lemma 5.14.

We assume that $\ell > 0$ and the assertion holds for any w' with $\ell(w') < \ell$. We take a reduced expression $w = s_{i_\ell} \cdots s_{i_1}$, and define $w' := s_{i_{\ell-1}} \cdots s_{i_1}$. Then we have $w = s_{i_\ell} w'$, and $\ell > \ell(w')$ holds.

We show (1) $\Rightarrow$ (2). Assume that $M \in \mathcal{S}_w$. Then by Proposition 5.5, we have $M \in \mathcal{Y}(I_{i_\ell})$ and $N := M \otimes_\Lambda I_{i_\ell} \in \mathcal{S}_{w'}$. So by the induction hypothesis, for all $i = 1, \ldots, n$, N satisfies

$$\begin{cases} \mathrm{Hom}_\Lambda(N, S_i^{w'}) = 0 & \text{if } w'\mathbf{e}_i \in \Pi, \\ \mathrm{Hom}_\Lambda(S_i^{w'}[1], N) = 0 & \text{if } w'\mathbf{e}_i \in -\Pi. \end{cases}$$

Now we divide the proof into three cases.

(i) The case $w'\mathbf{e}_i \in \Pi$ and $w'\mathbf{e}_i \neq \mathbf{e}_{i_\ell}$. Then we have $S_i \in \mathcal{T}(I_{w'})$ by Lemma 5.11. Moreover in this case $w\mathbf{e}_i = s_{i_\ell} w'\mathbf{e}_i \in \Pi$ holds. So by Lemma 5.11, we have $S_i \in \mathcal{T}(I_w)$. Thus by Proposition 2.27, we have $\mathrm{Hom}_\Lambda(M, S_i^w) \simeq \mathrm{Hom}_{\mathcal{D}(\mathrm{Mod}\Lambda)}(M, \mathbb{R}\mathrm{Hom}_\Lambda(I_{i_\ell}, S_i^{w'})) \simeq \mathrm{Hom}_{\mathcal{D}(\mathrm{Mod}\Lambda)}(M \overset{\mathbb{L}}{\otimes}_\Lambda I_{i_\ell}, S_i^{w'}) \simeq \mathrm{Hom}_\Lambda(N, S_i^{w'}) = 0$.

(ii) The case $w'\mathbf{e}_i \in -\Pi$. By Lemma 5.11, we have $S_i \in \mathcal{F}(I_{w'})$. Moreover in this case $w\mathbf{e}_i = s_{i_\ell} w'\mathbf{e}_i \in -\Pi$ holds. So we have $S_i \in \mathcal{F}(I_w)$ by Lemma 5.11. Thus by Proposition 2.27, we have $\mathrm{Hom}_\Lambda(S_i^w[1], M) \simeq \mathrm{Hom}_{\mathcal{D}(\mathrm{Mod}\Lambda)}(\mathbb{R}\mathrm{Hom}_\Lambda(I_{i_\ell}, S_i^{w'})[1], M) \simeq \mathrm{Hom}_{\mathcal{D}(\mathrm{Mod}\Lambda)}(S_i^{w'}[1], M \overset{\mathbb{L}}{\otimes}_\Lambda I_{i_\ell}) \simeq \mathrm{Hom}_\Lambda(S_i^{w'}[1], N) = 0$.

(iii) In the case $w'\mathbf{e}_i = \mathbf{e}_{i_\ell}$. In this case $w\mathbf{e}_i = s_{i_\ell} w'\mathbf{e}_i = -\mathbf{e}_{i_\ell}$. So by Lemma 5.11, we have $S_i^w[1] \simeq S_{i_\ell}$. By Lemma 2.23, $M \in \mathcal{Y}(I_{i_\ell})$ implies that $\mathrm{Hom}_\Lambda(S_{i_\ell}, M) = 0$. Thus we have $\mathrm{Hom}_\Lambda(S_i^w[1], M) = 0$.

Since (i), (ii) and (iii) cover all cases, M satisfies the condition (2).

Next we show (2) $\Rightarrow$ (1). Assume that M satisfies the condition (2). By Lemma 5.16, there exists $i \in \{1, \ldots, n\}$ such that $we_i \in -\Pi$ and S_{i_ℓ} is a factor of $S_i^w[1]$. The assumption $\mathrm{Hom}_\Lambda(S_i^w[1], M) = 0$ implies $\mathrm{Hom}_\Lambda(S_{i_\ell}, M) = 0$. Hence S_{i_ℓ} is not a submodule of M. Thus by Lemma 2.23, we have $M \in \mathcal{Y}(I_{i_\ell})$.

By the similar argument to the proof of (1) $\Rightarrow$ (2), we see that $N := M \otimes_\Lambda I_{i_\ell}$ satisfies the condition (2). So by the induction hypothesis, we have $N \in \mathcal{S}_{w'}$. Consequently by Proposition 5.5, we have $M \in \mathcal{S}_w$. □

5.5 Analogues of the McKay Correspondence

We give a characterization of the exceptional curves on $\mathcal{M}_w$ for any $w \in W$.

Crawley-Boevey [15] observed that $\mathcal{M}_1$ is identified with the G-Hilbert scheme via the Morita equivalence between Λ and the skew group ring. Ito and Nakamura [23] explained the McKay correspondence, which is a one-to-one correspondence between the set of exceptional curves on the minimal resolution of $\mathbb{A}^2/G$ and the set of non-trivial irreducible representation of G, by using the G-Hilbert scheme. Crawley-Boevey [15] reformulated it in terms of Λ-modules as follows.

Theorem 5.17 [15, Theorem 2] *Let $N \in \mathcal{S}_1$. Then the socle of N has at most two simple summands, and if two, they are not isomorphic. If $i \in \{1, \ldots, n\}$, then*

$$E_i := \{N \in \mathcal{S}_1 \mid S_i \text{ is a submodule of } N\}/\simeq$$

is a closed subset of $\mathcal{M}_1$ isomorphic to $\mathbb{P}^1_K$. Moreover E_i meets E_j if and only if i and j are adjacent in Q, and in this case they meet at only one point.

We generalize Theorem 5.17 for any $w \in W$. For all $i \in \{1, \ldots, n\}$, we define a subset E_i^w of $\mathcal{M}_w$ by

$$E_i^w := \{\mathrm{Hom}_\Lambda(I_w, M) \mid M \in E_i\}/\simeq .$$

Proposition 5.18 *If $i \in \{1, \ldots, n\}$, then E_i^w is a closed subset of $\mathcal{M}_w$ isomorphic to $\mathbb{P}^1_K$. Moreover E_i^w meets E_j^w if and only if i and j are adjacent in Q, and in this case they meet at only one point.*

Proof It follows immediately from Theorem 5.6, 5.17 and Corollary 4.21. □

Now we state a main result in this subsection. Although each exceptional curve E_i is characterized by a simple module S_i, each exceptional curve E_i^w is characterized by S_i^w.

Theorem 5.19 *We take any $w \in W$ and $i \in \{1, \ldots, n\}$. For any $M \in \mathcal{S}_w$, the following hold.*

(1) *If $w\mathbf{e}_i \in \Pi$, then $M \in E_i^w$ if and only if S_i^w is a submodule of M.*
(2) *If $w\mathbf{e}_i \in -\Pi$, then $M \in E_i^w$ if and only if $S_i^w[1]$ is a factor module of M.*

In the rest we prove Theorem 5.19.

Lemma 5.20 *Let $i \in \{1, \ldots, n\}$. For any $N \in E_i$, there exist non-split exact sequences*

$$0 \longrightarrow S_i \longrightarrow L_i^+ \longrightarrow N \longrightarrow 0, \tag{5.1}$$

$$0 \longrightarrow S_i \longrightarrow N \longrightarrow L_i^- \longrightarrow 0 \tag{5.2}$$

such that L_i^+ and L_i^- are 0-generated nilpotent Λ-modules of dimension vector $\mathbf{d} + \mathbf{e}_i$ and $\mathbf{d} - \mathbf{e}_i$ respectively.

Proof First we make the exact sequence (5.1). By Theorem 5.17 we have $\dim \mathrm{Hom}_\Lambda(S_i, N) = 1$. Since N is 0-generated, we have $\mathrm{Hom}_\Lambda(N, S_i) = 0$ by Lemma 5.13. Since $\underline{\dim} N = \mathbf{d}$, we have $(N, S_i) = 0$ by Lemma 2.30. Thus by Lemma 2.19 we have $\dim \mathrm{Ext}^1_\Lambda(N, S_i) = 1$. We can take the exact sequence (5.1) as the non-split exact sequence corresponding to a non-zero element in $\mathrm{Ext}^1_\Lambda(N, S_i)$.

Next we make the exact sequence (5.2). Since S_i is a submodule of N, we have the non-split exact sequence (5.2) naturally.

Then it is obvious that $\underline{\dim} L_i^+ = \mathbf{d} + \mathbf{e}_i$ and $\underline{\dim} L_i^- = \mathbf{d} - \mathbf{e}_i$. Moreover L_i^+ and L_i^- are non-zero nilpotent since so does N.

Finally we show that L_i^+ and L_i^- are 0-generated. We prove that L_i^+ satisfies the third condition of Lemma 5.13. For $j \in \{1, \dots, n\}$ by applying $\mathrm{Hom}_\Lambda(-, S_j)$ to the exact sequence (5.1), we have an exact sequence

$$\mathrm{Hom}_\Lambda(N, S_j) \longrightarrow \mathrm{Hom}_\Lambda(L_i^+, S_j) \longrightarrow \mathrm{Hom}_\Lambda(S_i, S_j) \longrightarrow \mathrm{Ext}^1_\Lambda(N, S_j).$$

Since N is in $\mathcal{S}_1$, we have $\mathrm{Hom}_\Lambda(N, S_j) = 0$ by Lemma 5.14. If $i \neq j$, we have $\mathrm{Hom}_\Lambda(S_i, S_j) = 0$, and so $\mathrm{Hom}_\Lambda(L_i^+, S_j) = 0$. If $i = j$, then the map $\mathrm{Hom}_\Lambda(S_i, S_i) \longrightarrow \mathrm{Ext}^1_\Lambda(N, S_i)$ is non-zero since the exact sequence (5.1) is not split. By this fact and $\dim \mathrm{Hom}_\Lambda(S_i, S_i) = 1$, we have $\mathrm{Hom}_\Lambda(L_i^+, S_i) = 0$. Moreover we have $\dim \mathrm{Hom}_\Lambda(L_i^+, S_0) = 1$ since $(\underline{\dim} L_i^+)_0 = 1$ and Lemma 2.16.

We prove that L_i^- satisfies the third condition of Lemma 5.13. For $j \in \{1, \dots, n\}$ by applying $\mathrm{Hom}_\Lambda(-, S_j)$ to the exact sequence (5.2), we have a injective map

$$0 \to \mathrm{Hom}_\Lambda(L_i^-, S_j) \longrightarrow \mathrm{Hom}_\Lambda(N, S_j).$$

Since N is in $\mathcal{S}_1$, we have $\mathrm{Hom}_\Lambda(N, S_j) = 0$ by Lemma 5.14. Thus we have $\mathrm{Hom}_\Lambda(L_i^-, S_j) = 0$. Moreover we have $\dim \mathrm{Hom}_\Lambda(L_i^-, S_0) = 1$ since $(\underline{\dim} L_i^-)_0 = 1$ and Lemma 2.16. □

Lemma 5.21 *For any 0-generated finite dimensional Λ-module M is contained in $\mathcal{T}(I_w)$ for any $w \in W$. In particular, $L_i^+, L_i^- \in \mathcal{T}(I_w)$ for any $w \in W$ and $i \in \{1, \dots, n\}$.*

Proof For any 0-generated finite dimensional Λ-module M, $\mathrm{Ext}^1_\Lambda(I_w, M) \simeq \mathrm{Ext}^2_\Lambda(\Lambda/I_w, M) \simeq D\mathrm{Hom}_\Lambda(M, \Lambda/I_w)$ holds. Therefore it is enough to show that $\mathrm{Hom}_\Lambda(M, \Lambda/I_w) = 0$.

We may assume that M is indecomposable. If M is not nilpotent, then $\mathrm{Hom}_\Lambda(M, \Lambda/I_w) = 0$ since Λ/I_w is in $\mathrm{nilp}\Lambda$ and by Proposition 2.32.

We assume that M is nilpotent. By Lemma 5.13, $M/MI \simeq S_0$. Thus if there is a non-zero Λ-homomorphism from M to Λ/I_w, then S_0 should be a composition factor of Λ/I_w. However S_0 does not appear in composition factors of Λ/I_w. By these arguments, we have $\mathrm{Hom}_\Lambda(M, \Lambda/I_w) = 0$. □

For any $i = 1, \dots, n$, we put $(L_i^+)^w := \mathrm{Hom}_\Lambda(I_w, L_i^+)$ and $(L_i^-)^w := \mathrm{Hom}_\Lambda(I_w, L_i^-)$.

Lemma 5.22 *For any $M \in E_i^w$, there exist non-split exact sequences:*

(1) *If $w\mathbf{e}_i \in \Pi$,*

$$0 \longrightarrow S_i^w \longrightarrow M \longrightarrow (L_i^-)^w \longrightarrow 0.$$

(2) *If $w\mathbf{e}_i \in -\Pi$,*

$$0 \longrightarrow (L_i^+)^w \longrightarrow M \longrightarrow S_i^w[1] \longrightarrow 0.$$

Proof By the definition of E_i^w, there exists $N \in E_i$ such that $M \simeq \mathrm{Hom}_\Lambda(I_w, N)$. By Lemma 5.20, there is the exact sequences (5.1) and (5.2).

(1) If $w\mathbf{e}_i \in \Pi$, then we have $S_i \in \mathcal{T}(I_w)$ by Lemma 5.11. So we have $\mathrm{Ext}^1_\Lambda(I_w, S_i) = 0$. Thus by applying $\mathrm{Hom}_\Lambda(I_w, -)$ to the exact sequence (5.2), we have

$$0 \to \mathrm{Hom}_\Lambda(I_w, S_i) \to \mathrm{Hom}_\Lambda(I_w, N) \to \mathrm{Hom}_\Lambda(I_w, L_i^-) \to 0.$$

(2) If $w\mathbf{e}_i \in -\Pi$, then we have $S_i \in \mathcal{F}(I_w)$ by Lemma 5.11. So we have $\mathrm{Hom}_\Lambda(I_w, S_i) = 0$. Moreover by Lemma 5.21, we have $\mathrm{Ext}^1_\Lambda(I_w, L_i^+) = 0$. Thus by applying $\mathrm{Hom}_\Lambda(I_w, -)$ to the exact sequence (5.1), we have

$$0 \to \mathrm{Hom}_\Lambda(I_w, L_i^+) \to \mathrm{Hom}_\Lambda(I_w, N) \to \mathrm{Ext}^1_\Lambda(I_w, S_i) \to 0.$$

□

Proof of Theorem 5.19 We take any $M \in \mathcal{S}_w$. Then by Proposition 5.6, there exists $N \in \mathcal{S}_1$ such that $M = \mathrm{Hom}_\Lambda(I_w, N)$. We note that $M \in \mathcal{Y}(I_w)$ and $N \in \mathcal{T}(I_w)$. Let $i \in \{1, \ldots, n\}$.

First we consider the case $w\mathbf{e}_i \in \Pi$. If $M \in E_i^w$, then S_i^w is a submodule of M by Lemma 5.22. Conversely we assume that S_i^w is a submodule of M. By the assumption $w\mathbf{e}_i \in \Pi$ and Lemma 5.11, we have $S_i \in \mathcal{T}(I_w)$. Thus $\mathrm{Hom}_\Lambda(S_i, N) \simeq \mathrm{Hom}_\Lambda(S_i^w, M) \neq 0$ by Lemma 2.9. Hence S_i is a submodule of N, and so $N \in E_i$. Consequently we have $M \in E_i^w$.

Next we consider the case $w\mathbf{e}_i \in -\Pi$. If $M \in E_i^w$, then $S_i^w[1]$ is a factor module of M by Lemma 5.22. Conversely we assume that $S_i^w[1]$ is a factor module of M. Since N is a 0-generated nilpotent Λ-module of dimension vector $\mathbf{d}$, we have $\dim \mathrm{Hom}_\Lambda(S_i, N) = \dim \mathrm{Ext}^1_\Lambda(N, S_i)$ by Lemmas 2.19, 2.30 and 5.13. Thus if $\mathrm{Ext}^1_\Lambda(N, S_i) \neq 0$, we have $M \in E_i^w$ by the same argument as the last two sentences in the above proof.

In the following, we show $\mathrm{Ext}^1_\Lambda(N, S_i) \neq 0$. We show that the canonical exact sequence

$$0 \to X \to M \to S_i^w[1] \to 0 \tag{5.3}$$

is not split. If it is split, then we have $M \simeq S_i^w[1]$ since M is indecomposable. By the assumption $w\mathbf{e}_i \in -\Pi$, we have $S_i \in \mathcal{F}(I_w)$ by Lemma 5.11, and so $S_i^w[1] \in \mathcal{X}(I_w)$ by Lemma 2.9. Thus M is in $\mathcal{X}(I_w) \cap \mathcal{Y}(I_w)$, and so $M = 0$ by Proposition 2.7 (4). This contradicts to $M \neq 0$.

By applying $- \otimes_\Lambda I_w$ to the exact sequence (5.3), we have an exact sequence

$$\mathrm{Tor}_1^\Lambda(M, I_w) \to \mathrm{Tor}_1^\Lambda(S_i^w[1], I_w) \to X \otimes_\Lambda I_w \to M \otimes_\Lambda I_w \to S_i^w[1] \otimes_\Lambda I_w.$$

We have $\mathrm{Tor}_1^\Lambda(M, I_w) = 0$ since $M \in \mathcal{Y}(I_w)$, and have $\mathrm{Tor}_1^\Lambda(S_i^w[1], I_w) \simeq S_i$, $S_i^w[1] \otimes_\Lambda I_w = 0$ since $S_i^w[1] \in \mathcal{X}(I_w)$. Thus we have an exact sequence

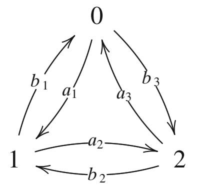
$$0 \to S_i \to X \otimes_\Lambda I_w \to N \to 0. \tag{5.4}$$

This is not split. Indeed since X is a submodule of M, it is in $\mathcal{Y}(I_w)$ by Proposition 2.7 (2) , and so $X \otimes_\Lambda I_w$ is in $\mathcal{T}(I_w)$ by Lemma 2.9. Thus since $X \otimes_\Lambda I_w, N \in \mathcal{T}(I_w)$ and $S_i \in \mathcal{F}(I_i)$ and by Lemma 2.9, the exact sequence (5.4) returns to (5.3) by applying $\mathrm{Hom}_\Lambda(I_w, -)$.

Now we obtain the non-split exact sequence (5.4). This implies that $\mathrm{Ext}^1_\Lambda(N, S_i) \neq 0$. □

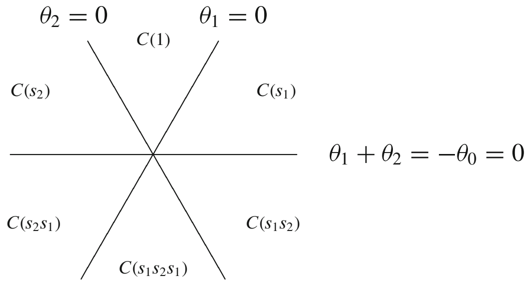
6 Example

Let G be a finite subgroup of $SL(2, K)$ of order three which is generated by $\sigma = \mathrm{diag}(\epsilon, \epsilon^2)$ where ϵ is a primitive third root of unity. Then the McKay quiver Q of G, a preprojective relation R and the dimension vector $\mathbf{d}$ of the irreducible representations are given by

$Q =$ 

$$R = \left\{ \begin{array}{c} a_1b_1 - b_3a_3, \\ a_2b_2 - b_1a_1, \\ a_3b_3 - b_2a_2 \end{array} \right\} \quad \mathbf{d} = \begin{array}{ccc} & 1 & \\ 1 & & 1 \end{array}.$$

The chamber structure of the parameter space $\Theta \in \mathbb{Q}^2$ is as follows.



First we consider $\mathcal{M}_1$. Then the exceptional set $E_1 \cup E_2$ is a chain of two $\mathbb{P}^1$'s and by Theorem 5.17, these are given as follows.

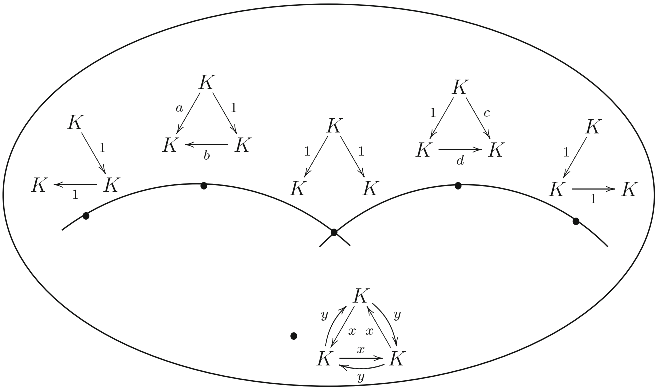
$$E_1 = \{M \in \mathcal{S}_{\theta,\mathbf{d}}(\Lambda) \mid \mathrm{Hom}_\Lambda(S_1, M) \neq 0\} = \left\{ \begin{array}{ccc} & K & \\ {\scriptstyle a} \swarrow & & \searrow {\scriptstyle 1} \\ K & \underset{b}{\leftarrow} & K \end{array} \;\middle|\; (a, b) \in \mathbb{P}^1_K \right\} / \simeq,$$

$$E_2 = \{M \in \mathcal{S}_{\theta,\mathbf{d}}(\Lambda) \mid \mathrm{Hom}_\Lambda(S_2, M) \neq 0\} = \left\{ \begin{array}{ccc} & K & \\ {\scriptstyle 1} \swarrow & {\scriptstyle d} & \searrow {\scriptstyle c} \\ K & \rightarrow & K \end{array} \;\middle|\; (c, d) \in \mathbb{P}^1_K \right\} / \simeq,$$

and the intersection of E_1 and E_2 is

$$E_1 \cap E_2 = \left\{ \begin{array}{ccc} & K & \\ {\scriptstyle 1} \swarrow & & \searrow {\scriptstyle 1} \\ K & & K \end{array} \right\}.$$

Note that we omit to write zero maps in each representations and actually consider isomorphism classes of them. Pictorially $\mathcal{M}_1$ is described as follows where $(x, y) \neq (0, 0)$ is a point in $\mathbb{A}^2$.



Next we observe the relation between $\mathcal{M}_1$ and $\mathcal{M}_{s_1}$.

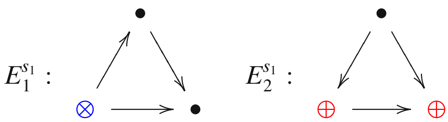
$$E_1^{s_1} = \left\{ \begin{array}{ccc} & K & \\ {\scriptstyle a'}\nearrow & & \searrow{\scriptstyle 1} \\ K & \xrightarrow[b']{} & K \end{array} \;\middle|\; (a', b') \in \mathbb{P}^1_K \right\},$$

$$E_2^{s_1} = E_2 \setminus \left\{ \begin{array}{ccc} & K & \\ {\scriptstyle 1}\swarrow & & \searrow{\scriptstyle 1} \\ K & & K \end{array} \right\} \cup \left\{ \begin{array}{ccc} & K & \\ {\scriptstyle 1}\swarrow & & \\ K & \xrightarrow[]{1} & K \end{array} \right\} = \left\{ \begin{array}{ccc} & K & \\ {\scriptstyle d'}\swarrow & & \searrow{\scriptstyle c'} \\ K & \xrightarrow[]{1} & K \end{array} \;\middle|\; (c', d') \in \mathbb{P}^1_K \right\},$$

and the intersection of $E_1^{s_1}$ and $E_2^{s_1}$ is

$$E_1^{s_1} \cap E_2^{s_1} = \left\{ \begin{array}{ccc} & K & \\ & & \searrow{\scriptstyle 1} \\ K & \xrightarrow[]{1} & K \end{array} \right\}.$$

Now $s_1\Delta = \{-\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2\}$, and the dimension of $\mathrm{Ext}^1_\Lambda(I_1, S_1) \simeq S_1$ is $\mathbf{e}_1$.
We express the exceptional curves on $\mathcal{M}_{s_1}$ by



where $\otimes$ implies the quotient $\mathrm{Ext}^1_\Lambda(I_1, S_1)$ and $\oplus \longrightarrow \oplus$ the submodule $\mathrm{Hom}_\Lambda(I_1, S_2)$.

For all chambers, if we draw the exceptional curves by using the above expression, then the result is described in Fig. 2.

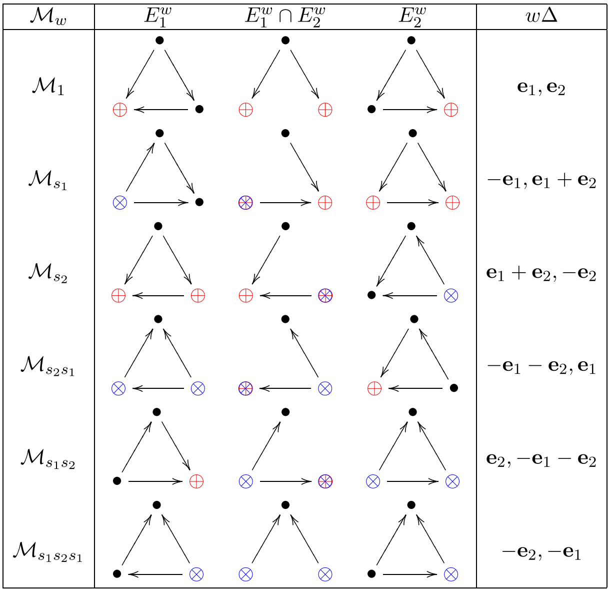


Fig. 2 Characterization of exceptional curves

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