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Young's Seminormal Form and Simple Modules for S_n in Characteristic p

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Abstract We realize the integral Specht modules for the symmetric group S_n as induced modules from the subalgebra of the group algebra generated by the Jucys–Murphy elements. We deduce from this that the simple modules for $\mathbb{F}_p S_n$ are generated by reductions modulo p of the corresponding Jucys–Murphy idempotents.

Keywords Symmetric group · Specht modules · Jucys–Murphy elements

Mathematics Subject Classifications (2010) 20C30 · 20C20 · 05E10

1 Introduction

This article is a continuation of the investigation pursued in [24, 25] that seeks to demonstrate the importance of Young's seminormal basis for the modular, that is characteristic p, representation theory of the symmetric group S_n. A main obstacle is here that Young's seminormal basis is defined over the field $\mathbb{Q}$ and indeed there seems to be a general consensus that this obstacle makes Young's seminormal basis a characteristic zero phenomenon, essentially. Still we believe that Young's seminormal basis is a fundamental object for the modular representation theory as well, and we think that the results of our works provide strong evidence in favor of this claim.

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Let Par_n be the set of partitions of n and let $S(\lambda)$ be the integral Specht module for S_n associated with $\lambda \in \mathrm{Par}_n$. Then, as has been known for a long time, the irreducible $\mathbb{Q}S_n$-modules are classified by the $S_{\mathbb{Q}}(\lambda) := S(\lambda) \otimes_{\mathbb{Z}} \mathbb{Q}$ when $\lambda \in \mathrm{Par}_n$, whereas the reduced Specht modules $S(\lambda) \otimes_{\mathbb{Z}} \mathbb{F}_p$ are reducible in general. In fact the irreducible modules $D(\lambda)$ for $\mathbb{F}_p S_n$ are classified by the set of p-restricted partitions $\mathrm{Par}_n^{\mathrm{res}}$ and are obtained as $D(\lambda) = S(\lambda)/\operatorname{rad}(\cdot, \cdot)$ where $(\cdot, \cdot)$ is a certain symmetric bilinear and S_n-invariant form on $S(\lambda)$.

The decomposition numbers $[S(\lambda) : D(\mu)]$ for $\mathbb{F}_p S_n$ have been the topic of much research activity in recent years, but still remain unknown in general and even the dimensions of $D(\mu)$ are not known in general. But using the theory of Young's seminormal form we obtain in this work, as our main Theorem 5, a construction of $D(\mu)$ that may be a good starting point for obtaining combinatorial expressions for $\dim D(\mu)$.

The basic principles behind this construction are parallels of standard methods in the modular representation theory of algebraic groups. Indeed, let $S(\lambda)^{\circledast}$ denote the contragredient dual of $S(\lambda)$. Then $(\cdot, \cdot)$ corresponds to a homomorphism $c_\lambda : S(\lambda) \to S(\lambda)^{\circledast}$. Moreover, for $\lambda \in \mathrm{Par}_n^{\mathrm{res}}$ we have that $D(\lambda) = im\, \overline{c_\lambda}$ where $\overline{c_\lambda}$ is the reduced homomorphism modulo p. Passing to the representation theory of an algebraic group G over an algebraically closed field of characteristic p, the Weyl module $\Delta(\lambda)$, the dual Weyl module $\nabla(\lambda)$ and the simple module $L(\lambda)$ correspond to $S(\lambda)$, $S(\lambda)^{\circledast}$ and $D(\lambda)$ and the bilinear form $(\cdot, \cdot)$ on $S(\lambda)$ corresponds to a form on $\Delta(\lambda)$ that we denote the same way. It induces a G-linear homomorphism $c_\lambda : \Delta(\lambda) \to \nabla(\lambda)$ and the simple module satisfies $L(\lambda) = im\, c_\lambda$. But in the G-module setting, $\nabla(\lambda)$ can also be constructed as the module of global sections of a line bundle on the associated flag manifold, and using this, one obtains a new construction of c_λ without using the bilinear form. The properties of this new construction of c_λ then provide a useful method for obtaining information on $L(\lambda)$, see eg. [1, 12].

Returning to the symmetric group, we then look for a different construction of c_λ. For this we prove in our Theorem 3 that $S(\lambda)$ is induced from a certain subalgebra, denoted GZ_n, of the group algebra, corresponding to the fact that $\nabla(\lambda)$ is induced from a Borel subgroup of G. Given this, our new construction of c_λ is obtained from a Frobenius reciprocity argument.

At the basis of our work are the famous Jucys–Murphy elements L_k, $k = 1, 2, \ldots, n$ that were introduced independently by Jucys and Murphy in [13–15] and [18]. They give rise to idempotents E_t of $\mathbb{Q}S_n$, the Jucys–Murphy idempotents, indexed by λ-tableaux t, that are closely related to Young's seminormal basis of the Specht module $S_{\mathbb{Q}}(\lambda)$. Moreover they commute with each other and therefore generate a commutative subalgebra of the group algebra. This is the algebra GZ_n that was mentioned above, the Gelfand–Zetlin algebra. In the case of the ground field $\mathbb{Q}$ it was considered by Okounkov and Vershik in [22] as a kind of Cartan subalgebra of a semisimple Lie algebra, but for us it is important to work with an integral version of GZ_n, where the analysis of [22] fails.

We have now formulated the main ingredients of our result. The surprisingly simple final result is that $D(\lambda)$ is generated by $a_\lambda E_\lambda$ where $E_\lambda = E_{t^\lambda}$ and a_λ is the least common multiple of the denominators of E_λ. It should be noted that, even though it appears to be a very natural idea to investigate the $\mathbb{Z}S_n$ or $\mathbb{F}_p S_n$-submodule of $S(\lambda)$ generated by $a_\lambda E_\lambda$, the only reference in the literature along these lines is [25], as far as we know.

In an important recent paper [4], Brundan and Kleshchev showed that $\mathbb{F}_p S_n$ is a $\mathbb{Z}$-graded algebra in a nontrivial way by establishing an isomorphism between $\mathbb{F}_p S_n$ and the cyclotomic KLR-algebra, i.e. cyclotomic Khovanov–Lauda–Rouquier algebra, of type A. Their results work in greater generality than $\mathbb{F}_p S_n$ but we shall only consider this case. Hu and Mathas refined in [9] this graded structure on $\mathbb{F}_p S_n$ to a graded cellular algebra structure by constructing an explicit graded cellular basis. A second goal of our paper is to show that key features of their constructions can be carried out entirely within the theory of Young's seminormal form, as developed by Murphy. We hope that this approach to their results, together with our main Theorem 5, may provide a combinatorial expression for dim $D(\lambda)$.

The generators of the cyclotomic KLR-algebra are

$$\left\{ e(\mathbf{i}) \mid \mathbf{i} \in (\mathbb{F}_p)^n \right\} \cup \{y_1, \ldots, y_n\} \cup \{\psi_1, \ldots, \psi_{n-1}\}$$

and Brundan and Kleshchev prove in [4] their Theorem by constructing elements in $\mathbb{F}_p S_n$, denoted the same way, that verify the cyclotomic KLR-algebra relations. The y_i are essentially Jucys–Murphy operators and $e(\mathbf{i})$ are certain idempotents, not necessarily nonzero. In fact they can be identified with the idempotents constructed in [19] by summing Jucys–Murphy idempotents E_t over tableaux classes. The elements ψ_i are the most difficult to handle and the paper [4] takes as starting point for this certain explicitly given intertwining elements ϕ_i. These intertwiners, together with the $e(\mathbf{i})$ and y_i, already satisfy relations that are close to the cyclotomic KLR-relations but still need to be adjusted to get the complete match.

We here give a natural construction of the intertwining elements ϕ_i within Murphy's theory for the seminormal basis. Indeed, we see them as natural analogues of certain elements Ψ_i of the Hecke algebra that appear in [20], although only in the semisimple case. We show that Murphy's ideas, in a suitable sense, can be carried out over $\mathbb{F}_p$ as well. From this we obtain a cellular basis for $\mathbb{F}_p S_n$ using a modification of the constructions done in [9].

Let us sketch the layout of the paper. In Section 2 we fix the basic notation of the paper. It is mostly standard, except possibly for the notion of tableau class which was introduced in [19]. We also review the construction from [19] of the tableau class idempotents. Section 3 contains the construction of the intertwiners $\Psi_{L,i}$. This requires a control of the denominators of the Jucys–Murphy idempotents that are involved in the tableau class idempotents. In Section 4 we construct the cellular basis. In Section 5 we first introduce the Gelfand–Zetlin algebra GZ_n and then set up the induction functor. We then prove that the Specht module is induced up from a "rank one" module of GZ_n. An important ingredient for this is a uniqueness statement, due to James [11], of the integral Specht module. Finally in Section 6 we deduce our main results.

Note that the notation used throughout the paper may vary slightly from the one used in the introduction.

2 Basic Notation and Idempotents in Positive Characteristic

We are concerned with the representation theory of the symmetric group S_n in positive characteristic. Let us first set up the basic notation. Let p be a prime. We use the ground rings R, $\mathbb{Q}$ and $\mathbb{F}_p$, where R is the localization of $\mathbb{Z}$ at the prime ideal (p) and where $\mathbb{F}_p$ is the finite field of p elements. Then R is a local ring with maximal ideal pR and $R/pR = \mathbb{F}_p$. Let n be a positive integer and let S_n be the symmetric group on n letters. An n-composition is a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ of positive integers with sum n. An n-partition is an n-composition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ such that $\lambda_i \geq \lambda_{i+1}$ for all i. The set of n-partitions is denoted Par_n. For $\lambda \in \mathrm{Par}_n$, the associated Young diagram, also denoted λ, is the graphical representation of λ through n empty boxes in the plane. The first λ_1 boxes are placed in the first row, the next λ_2 boxes are placed in the second row, left aligned with respect to the first row, etc. This is the English notation for Young diagrams. The boxes are denoted the *nodes* of λ and are indexed using matrix convention. Thus the node of λ indexed by $[2, 3]$ is the one situated in the second row and the third column of λ. The p-residue diagram of a partition λ is obtained by writing $j - i \bmod p$ in the $[i, j]$'th node of λ. The $[i, j]$'th node is called a k-node of λ if $k = j - i \bmod p$.

Let t be a λ-tableau, i.e. a filling of the nodes of λ using the numbers of $\{1, 2, \dots, n\}$, each once. We write $t[i, j] = k$ if the $[i, j]$'th node of t is filled in with k and $r_t(k) = j - i$ if $t[i, j] = k$. Then $r_t(k)$ is also referred to as the content of t at the node containing k. For $k \in \{1, 2, \dots, n\}$ we define $t(k) := [i, j]$ where $t[i, j] = k$. A tableau t is said to be row standard if $t[i, j] \leq t[i, j+1]$ for all i, j such that the terms are defined. The set of row standard tableaux of all n-partitions is denoted $\mathrm{RStd}(n)$. A tableau t is said to be standard if $t[i, j] \leq t[i, j+1]$ and $t[i, j] \leq t[i+1, j]$ for all i, j such that the terms are defined. For λ an n-partition, we define $\mathit{Shape}(t) := \lambda$ if t is a λ-tableau. The set of standard tableaux of n-partitions is denoted $\mathrm{Std}(n)$ and the set of standard tableaux of shape λ is denoted $\mathrm{Std}(\lambda)$.

If t and s are tableaux we write $t \sim_p s$ if $r_t(k) = r_s(k) \bmod p$ whenever $t[i, j] = s[i_1, j_1] = k$. This defines an equivalence relation on the set of all tableaux which can be restricted to an equivalence relation on the set of standard tableaux. When we refer to a tableau class we always mean a class with respect to the last relation, consisting of standard tableaux. If t is a standard tableau we denote its class by $[t]$. In general we refer to tableau classes using capital letters, like T or S. We denote by $\mathfrak{C}_n$ the set of tableau classes of all n-partitions.

For t a tableau, the multiset of residues $\{r_t(k) \mid k = 1, \dots, n\}$ depends only on $\mathit{Shape}(t)$ and this induces an equivalence relation on Par_n that we also denote $\sim_p$. The blocks of $\mathbb{F}S_n$ are given by this equivalence relation according to the Nakayama conjecture, see for example Wildon's notes [27]. Clearly, for two tableaux s, t we have that $t \sim_p s$ implies $\mathit{Shape}(t) \sim_p \mathit{Shape}(s)$.

We use the convention that S_n acts on the right on $\{1, \dots, n\}$ and hence on tableaux. In other words, we multiply cycles in S_n from the left to the right.

For t a λ-tableau, we define the associated element $d(t) \in S_n$ by

$$t^\lambda d(t) = t$$

where t^λ denotes the highest λ-tableau, having the numbers $\{1, 2, \dots, n\}$ filled in along rows. Highest refers to the dominance order $\trianglelefteq$ on tableaux. It is derived from the dominance order $\trianglelefteq$ on compositions given by

$$\lambda \trianglelefteq \mu \text{ if } \sum_{i=1}^{m} \lambda_i \leq \sum_{i=1}^{m} \mu_i \quad \text{for } m = 1, 2, \dots, \min(k, l)$$

for $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_l)$ by viewing tableaux as series of compositions. Similarly the dominance order can be extended to pairs of tableaux in the following way

$$(s, t) \trianglelefteq (s_1, t_1) \text{ if } s \trianglelefteq s_1 \text{ and } t \trianglelefteq t_1.$$

In [9] this order on pairs of tableaux is called the strong dominance order and is written $\blacktriangleleft$. The dominance orders are all partial.

Let t be a λ-tableau with node (i, j). The (i, j)-hook consists of the nodes to the right and below the (i, j) node, its cardinality is the hook-length $h_{i,j}$. The product of all hook-lengths only depends on λ and is denoted h_λ. The hook-quotient is $\gamma_{t,n} = \prod \frac{h_{i,j}}{h_{i,j}-1}$ with the product taken over all nodes in the row of λ that contains n, omitting hooks of length one. For general i, we define $\gamma_{t,i}$ similarly, by first deleting from t the nodes containing $i+1, i+2, \ldots, n$. We set $\gamma_t = \prod_{i=2}^{n} \gamma_{t,i}$.

In general, when we use λ as a subscript it refers to the tableau t^λ. In this situation we have

$$\gamma_\lambda = \gamma_{t^\lambda} = \prod_i \lambda_i!.$$

For $k = 1, 2, \ldots, n$ the Jucys–Murphy elements $L_k \in \mathbb{Z}S_n$ are defined by

$$L_k := (1, k) + (2, k) + \ldots + (k-1, k)$$

with the convention that $L_1 := 0$. They commute with each other and satisfy the following commutation relations with the simple transpositions

$$\begin{aligned} (k-1, k)L_k &= L_{k-1}(k-1, k) + 1 \\ (k-1, k)L_{k-1} &= L_k(k-1, k) - 1 \\ (k-1, k)L_l &= L_l(k-1, k) \qquad \text{if } l \neq k-1, l \neq k. \end{aligned} \tag{1}$$

These elements are a key ingredient for understanding the representation theory of S_n. Their generalizations appear in many contexts of representation theory, for example as the degenerate affine Hecke algebra, where the L_k are commuting generators that satisfy the above relations with the simple transpositions. In the original works of Jucys and Murphy [13–15, 18], the L_k's were used to construct orthogonal idempotents $E_t \in \mathbb{Q}S_n$, indexed by tableaux t, and to derive Young's seminormal form from them. We denote these idempotents the Jucys–Murphy idempotents. Their construction is as follows

$$E_t := \prod_{\{c \mid -n < c < n\}} \prod_{\{i \mid r_t(i) \neq c\}} \frac{L_i - c}{r_t(i) - c}.$$

For t standard we have $E_t \neq 0$, whereas for t nonstandard either $E_t = 0$, or $E_t = E_s$ for some standard tableau s related to t, see [19], page 260. For example, if t is obtained from the standard tableau s by interchanging $k-1$ and k that occur in the same row or column of s then $E_t = 0$. Running over all standard tableaux, the E_t form a set of primitive and complete idempotents, that is their sum is 1. Moreover, they are eigenvectors for the action of the Jucys–Murphy operators in $\mathbb{Q}S_n$, since

$$(L_k - r_t(k))E_t = 0 \text{ or equivalently } L_k = \sum_{t \in \mathrm{Std}(n)} r_t(k)E_t \tag{2}$$

which is the key formula for deriving Young's seminormal basis from them. In this situation Eq. 1 gives Young's seminormal form for the action of σ_i on the seminormal basis.

Unfortunately, the E_t contain many denominators and hence it is not possible to reduce them modulo p. In order to overcome this obstacle, Murphy introduced in [19] certain elements E_T for each tableau class T. They are defined as follows

$$E_T := \sum_{t \in T} E_t. \tag{3}$$

He showed that the E_T's, with T varying over all classes, give a set of complete orthogonal idempotents in RS_n. The most difficult part of this is to show that $E_T \in RS_n$ since they are clearly orthogonal, idempotent and complete (note at this point that in [19] it is not stated clearly that the sum in Eq. 3, going over the elements of the tableau class T, should only involve standard tableaux). We now present his proof that $E_T \in RS_n$, in our notation. Several of its ingredients will be important for us. See also [17] for a presentation of this and related results from an abstract point of view.

A key point is to consider F_t for t any tableau, given by

$$F_t := \prod_{\{c \,|\, -n<c<n\}} \prod_{\{i \,|\, r_t(i) \neq c \mod p\}} \frac{L_i - c}{r_t(i) - c}. \tag{4}$$

It is clear that $F_t \in RS_n$ and that all F_t's and E_T's commute. The denominator of F_t depends only on the underlying partition $Shape(t) = \lambda$ of t and is denoted w^λ. Although w^λ is not constant on the classes, we have that $w^\lambda = w^\mu$ modulo p if $\lambda \sim_p \mu$. Especially, if $s \sim_p t$ we get that $w^{\mathrm{Shape}(s)} = w^{\mathrm{Shape}(t)}$ modulo p. The numerator of F_t only depends on the class $[t]$ of t and so we have

$$F_s = F_t \quad \text{if } s \sim_p t \text{ and } Shape(s) = Shape(t).$$

Suppose that $t \in T$. Using Eq. 2 we get that

$$F_t E_s = \begin{cases} w^{\mathrm{Shape}(s)}/w^{\mathrm{Shape}(t)}\, E_s & \text{if } s \sim_p t \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

and so we deduce

$$F_t = \frac{1}{w^\lambda} \sum_{s \in T} w^{\mathrm{Shape}(s)}\, E_s. \tag{6}$$

Hence $E_T F_t = F_t$ where we set $T = [t]$. Using this we get for any positive integer m that

$$\begin{aligned}(E_T - F_t)^m &= \textstyle\sum_{i=0}^{m} \binom{m}{i} (-1)^{m-i} E_T^i F_t^{m-i} \\ &= E_T - 1 + 1 + \textstyle\sum_{i=0}^{m-1} \binom{m}{i} (-1)^{m-i} E_T^i F_t^{m-i} \\ &= E_T - 1 + \textstyle\sum_{i=0}^{m} \binom{m}{i} (-1)^{m-i} F_t^{m-i} = E_T - 1 + (1 - F_t)^m.\end{aligned}$$

Combining this with Eq. 6 we arrive at the formula

$$E_T = 1 - (1 - F_t)^m + \sum_{s \in T} \left(1 - \frac{w^{\mathrm{Shape}(s)}}{w^\lambda}\right)^m E_s. \tag{7}$$

Using it, the proof that $E_T \in RS_n$ follows by taking m big enough for

$$\left(1 - \frac{w^{\mathrm{Shape}(s)}}{w^\lambda}\right)^m E_s \in RS_n$$

to hold for all $s \in T$.

3 Commutation Rules

Our first aim is to generalize certain results valid for E_t to E_T. We are especially looking for a generalization for E_T of the elements denoted Ψ_t in [20]. For this we need to work out the commutation relations between E_T and the simple transpositions $\sigma_k = (k-1, k)$.

Assume that t and $t\sigma_k$ are standard tableaux and write $T := [t]$. We first consider the case where $[t\sigma_k] = [t]$, that is $r_t(k-1) = r_t(k) \bmod p$. We prove the following Lemma.

Lemma 1 *In the above situation* $[\sigma_k t] = [t] = T$ *we have*

$$\sigma_k E_T = E_T \sigma_k.$$

Proof We consider the commutator $[\sigma_k, E_T]$. By the previous section it belongs to RS_n. We show that it actually belongs to $p^N RS_n$ for any positive (big) integer N, from which the result follows. Fix therefore such an N. We use formula 7 and first consider the individual terms of that sum. We choose m big enough for $\left(1 - \frac{w^{\mathrm{Shape}(t_1)}}{w^\mu}\right)^m E_{t_1} \in p^N RS_n$ to hold for all $t_1 \in T$. From this we get that

$$\left[\sigma_k, \sum_{t_1 \in T} \left(1 - \frac{w^{\mathrm{Shape}(t_1)}}{w^\mu}\right)^m E_{t_1}\right] \in p^N RS_n$$

and so by Eq. 7 it is enough to prove that σ_k commutes with F_t.

Now by the commutation rules Eq. 1, we have that σ_k commutes with all terms of F_t of the form $L_i - c$ where $i \neq k-1, k$. The remaining terms may be grouped together in pairs of the form

$$(L_{k-1} - c)(L_k - c)$$

since by assumption $r_t(k-1) = r_t(k) \bmod p$. But these expressions are symmetric in L_{k-1} and L_k and therefore commute with σ_k by the commutation rules Eq. 1. The Lemma is proved. □

We next consider the case where s, t are both standard and $s = t\sigma_k \notin [t]$, that is $r_t(k-1) \neq r_t(k) \bmod p$. We set $S := [s]$ and $T := [t]$. In order to work out the commutation rule between σ and E_T in this case, we first consider $E := E_S + E_T$. We need the following auxiliary Lemma.

Lemma 2 *E belongs to RS_n and commutes with σ_k.*

Proof Clearly E belongs to RS_n since E_S and E_T do. For each $s \in S$ we have that either $s\sigma_k \in T$ or $s\sigma_k$ is nonstandard, and conversely for each $t \in T$ we have that either $t\sigma_k \in S$ or $t\sigma_k$ is nonstandard. Accordingly, the sum $E = \sum_{t\in T} E_t + \sum_{s\in S} E_s$ may be split into terms $E_s + E_{s\sigma_k}$ with $s \in S$ and $s\sigma_k \in T$, terms E_s with $s\sigma_k$ nonstandard and terms E_t with $t\sigma_k$ nonstandard.

The first kind of terms are symmetric in L_{k-1} and L_k and therefore commutes with σ_k. For the second kind of terms, $k-1$ and k occur in the same row or column of s, next to each other. Thus, using Theorem 6.4 of [20] together with the formula for x_{tt} of *loc. cit.* appearing eight lines above that Theorem 6.4, we get that $E_s\sigma_k = \pm E_s$, where the sign is positive iff $k-1$ and k are in the same row of s. Applying the antiautomorphism $*$ of RS_n that fixes the simple transpositions we get $\sigma_k E_s = \pm E_s$, with the same sign as before, and so indeed E_s and σ_k commute. Finally, the third kind of terms involving E_s are treated the same way. □

For each tableau class T we choose an arbitrary $t \in T$ and define

$$r_T(i) := r_t(i).$$

Thus $r_T(i) \in \mathbb{Z}$, hence also $r_T(i) \in R$, but it is only well defined modulo p. With this notation we can formulate our next Lemma.

Lemma 3 *Assume that S and T are chosen as above. Then there is a positive integer m_1 such that the following formulas hold for $m \geq m_1$*

$$E_T = \left(\frac{L_k - r_S(k)}{r_T(k) - r_S(k)}\right)^m E, \qquad E_S = \left(\frac{L_{k-1} - r_S(k)}{r_T(k) - r_S(k)}\right)^m E.$$

Proof For tableaux t and s we define $E_{t,s} = E_t + E_s$. Then obviously $E_{t,s}$ is idempotent and by Eq. 2, we have that $L_k = \sum_{u\in \mathrm{Std}(n)} r_u(k) E_u$. Hence for standard tableaux s and t such that $s = \sigma_k t$ we get that

$$E_t = \left(\frac{L_k - r_s(k)}{r_t(k) - r_s(k)}\right) E_{t,s}. \tag{8}$$

Similarly we have that

$$E_s = \left(\frac{L_{k-1} - r_s(k)}{r_t(k) - r_s(k)}\right) E_{t,s}. \tag{9}$$

Note that Eqs. 8 and 9 hold even if only s or t is standard, as long as $s = \sigma_k t$. The formulas are used in Murphy's papers but unfortunately they do not generalize to E_T or E_S since we do *not* have $L_k = \sum_{U\in\mathfrak{C}_n} r_U(k) E_U$ even though $r_u(k) = r_{u_1}(k)$ mod p for $u \sim_p u_1$. The problem is that the individual E_u lie in $\mathbb{Q}S_n$ rather than RS_n.

In order to solve this problem we proceed as follows. Consider first an $a \in pR$. From the binomial expansion we get the following formula in $R[x]$, valid for any positive integer m

$$(x+a)^{p^m} = x^{p^m} \mod p^{m+1} R[x]. \tag{10}$$

We deduce from it the formula

$$\left(\frac{x+a}{c+d}\right)^{p^m} = \left(\frac{x}{c}\right)^{p^m} \mod p^{m+1}R[x] \tag{11}$$

for any $c \in R^\times$ and $a, d \in pR$.

Set $T' := T \cup Ss_k$. Then $T' \setminus T$ consists of nonstandard tableaux and so $E_T = \sum_{t\in T'} E_t$. Using Eq. 8, for m is large enough we get

$$\begin{aligned}E_T &= \sum\nolimits_{t\in T'} E_t = \sum\nolimits_{t\in T', s=t\sigma_k} \left(\frac{L_k - r_s(k)}{r_t(k) - r_s(k)}\right) E_{t,s}\\ &= \sum\nolimits_{t\in T', s=t\sigma_k} \left(\frac{L_k - r_s(k)}{r_t(k) - r_s(k)}\right)^{p^m} E_{t,s} = \sum\nolimits_{t\in T', s=t\sigma_k} \left(\frac{L_k - r_S(k)}{r_T(k) - r_S(k)}\right)^{p^m} E_{t,s}\end{aligned}$$

The last equality follows from Eq. 11, since for any N we may choose m big enough to make the difference of the two sides belong to $p^N RS_n$. From the last expression we then get that E_T is equal to

$$\left(\frac{L_k - r_S(k)}{r_T(k) - r_S(k)}\right)^{p^m} (E_S + E_T) = \left(\frac{L_k - r_S(k)}{r_T(k) - r_S(k)}\right)^{p^m} E$$

as claimed. The other equality is proved the same way. By choosing m even bigger we obtain an m_1 that works for both equations. □

At this stage Murphy constructs in [20], using the formulas 8 and 9, elements Ψ_t and Φ_t of $\mathbb{Q}S_n$ satisfying

$$E_\lambda \Phi_t = \Psi_t E_t. \tag{12}$$

The construction is as follows. Let t be any λ-tableau and let k be an integer between 1 and n. The radial length between the nodes $t[k]$ and $t[k-1]$ is defined as $h_{t,k} = h_k := r_t(k-1) - r_t(k)$. Let $d(t) = \sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_N}$ be a reduced expression of $d(t)$. We associate with it a sequence of tableaux $t_1 = t^\lambda, t_2, \ldots, t_{N+1} = t$ by setting recursively $t_{k+1} := t_k s_{i_k}$. Then Φ_t and Ψ_t are given by the formulas

$$\begin{aligned}\Phi_t &:= \left(\sigma_{i_1} - \frac{1}{h_{t_1,i_1}}\right)\left(\sigma_{i_2} - \frac{1}{h_{t_2,i_2}}\right)\ldots\left(\sigma_{i_N} - \frac{1}{h_{t_N,i_N}}\right)\\ \Psi_t &:= \left(\sigma_{i_1} + \frac{1}{h_{t_1,i_1}}\right)\left(\sigma_{i_2} + \frac{1}{h_{t_2,i_2}}\right)\ldots\left(\sigma_{i_N} + \frac{1}{h_{t_N,i_N}}\right)\end{aligned} \tag{13}$$

As noted in [20], Φ_t and Ψ_t actually do depend on the chosen decomposition of $d(t)$, and not just on $d(t)$, and so the notation is slightly misleading. On the other hand, the key property 12 holds independently of the choice of reduced expression of $d(t)$, and so we just take anyone.

Our aim is to construct similar elements for E_T and E_S. For this we need the following commutation rules between σ_k and the powers L_k^m and $(L_k - a)^m$.

Lemma 4 *For $m \in \mathbb{N}$ and $a \in R$ the following formulas hold:*

(a) $\sigma_k L_k^m = L_{k-1}^m \sigma_k + \sum_{i=0}^{m-1} L_{k-1}^i L_k^{m-i-1}$
(b) $\sigma_k (L_k - a)^m = (L_{k-1} - a)^m \sigma_k + \sum_{i=0}^{m-1} (L_{k-1} - a)^i (L_k - a)^{m-i-1}$.

Proof Formula (a) is proved using a straightforward induction on the commutation rules given in Eq. 1. Formula (b) is proved the same way, since $L_k - a$ satisfies the same commutation rules with σ_k as L_k does. □

We generalize the concept of radial length to tableaux classes by setting

$$h_{T,k} = h_k := r_T(k-1) - r_T(k) \in \mathbb{Z} \subseteq R$$

for k any integer between 1 and n. It depends on the choices of $r_T(k)$ and is therefore only unique modulo p.

We are now going to construct certain elements $\Psi_{L,t}$, verifying a generalization of Eq. 12 for the E_T's. Set first $h_L(k) = h_L := L_{k-1} - L_k$. Modelled on Ψ_t, we shall construct $\Psi_{L,t}$ as products of expressions of the form

$$\sigma_k - \frac{1}{h_L}.$$

On the other hand, for such expressions to make sense in general, one would need to consider an appropriate completion of the group ring, and define $\frac{1}{h_L}$ inside it as a power series. We here take a simpler approach, always considering L_k and L_{k-1} as elements of $\mathrm{End}_{\mathbb{F}_p}(V)$ for V a $\mathbb{F}_p$-vector space such that $L_{k-1} - L_k + \alpha \in \mathrm{End}_{\mathbb{F}_p}(V)$ is nilpotent for some $\alpha \in \mathbb{F}_p^\times$. Under that assumption, $\frac{1}{h_L}$ can be defined as the corresponding geometric series, which is finite. The next Lemma should be seen in this light.

Lemma 5 *Suppose that s, t are standard tableaux with $s = t\sigma_k$ and that $T := [t]$ and $S := [s]$ are different tableaux classes. Let $h := h_{T,k}$. Then $L_{k-1} - L_k - h$ acts nilpotently in $E_T(\mathbb{F}_p S_n)$. Especially, $L_{k-1} - L_k$ is invertible as an element of $\mathrm{End}_{\mathbb{F}_p}(E_T(\mathbb{F}_p S_n))$.*

Proof Notice that since $E_T \in RS_n$, we have that the product $E_T(\mathbb{F}_p S_n)$ is well defined. Consider first $L_{k-1} - L_k - h$ as an element of RS_n. Using formula 2 we have that

$$(L_{k-1} - L_k - h)^N = \sum_u (r_u(k-1) - r_u(k) - r_T(k-1) + r_T(k))^N E_u.$$

Multiplied by E_T it gives the formula

$$(L_{k-1} - L_k - h)^N E_T = \sum_{u \in T} (r_u(k-1) - r_u(k) - r_T(k-1) + r_T(k))^N E_u.$$

Each coefficient of E_u is here a multiple of p. Hence we may take N large enough for $(L_{k-1} - L_k - h)^N E_T$ to belong to $p^m RS_n$. We reduce modulo p and get the statement of the Lemma. □

We can now prove the following Lemma.

Lemma 6 *Let* s, t, S, T, h, m *be as in the previous Lemma and let* $h_L := L_{k-1} - L_k$. *View* $1/h_L$ *as an element of* $\mathrm{Hom}_{\mathbb{F}_p}(E_T(\mathbb{F}_p S_n), \mathbb{F}_p S_n)$ *via the previous Lemma. Then for* $N \in \mathbb{N}$ *and* $a \in \mathbb{F}_p$ *the following formulas hold in* $Hom_{\mathbb{F}_p}(E_T(\mathbb{F}_p S_n), \mathbb{F}_p S_n)$

(a) $\left(\sigma_k - \frac{1}{h_L}\right) L_k^N = L_{k-1}^N \left(\sigma_k - \frac{1}{h_L}\right)$
(b) $\left(\sigma_k - \frac{1}{h_L}\right)(L_k - a)^N = (L_{k-1} - a)^N \left(\sigma_k - \frac{1}{h_L}\right)$.

Proof Using the previous Lemma and the fact that E_T commutes with L_k^r and $(L_k - a)^r$ we first notice that the expressions are well defined transformations of $E_T(\mathbb{F}_p S_n)$. Let us now show (a). Since L_k and L_{k-1} commute it is equivalent to

$$h_L \left(\sigma_k L_k^N - L_{k-1}^N \sigma_k\right) = L_k^N - L_{k-1}^N$$

and hence, using Lemma 4, to the valid expression

$$(L_k - L_{k-1}) \sum_{i=0}^{N-1} L_{k-1}^i L_k^{N-i-1} = L_k^N - L_{k-1}^N.$$

Formula (b) is proved the same way. □

We now obtain the following result.

Lemma 7 *Let the notation be as above. Then we have*

$$\left(\sigma_k - \frac{1}{h_L(k)}\right) E_T = E_S \left(\sigma_k - \frac{1}{h_L(k)}\right)$$

in $\mathrm{Hom}_{\mathbb{F}_p}(E_T(\mathbb{F}_p S_n), \mathbb{F}_p S_n)$.

Proof The proof is obtained by combining Lemmas 2, 3 and 6. □

The Lemma is a generalization of Lemma 6.2 from [20], where L_k, L_{k-1} and hence h_L act semisimply. Note that the second minus sign is there a plus sign, corresponding to the fact that the eigenvalues of h_L on E_s and E_t are equal but with opposite signs.

Set $T^\lambda := [t^\lambda]$. For $d(t) = \sigma_{i_1}\sigma_{i_2} \dots \sigma_{i_N}$ in reduced form we define

$$\Psi_{L,d(t)} := \left(\sigma_{i_1} - \frac{1}{h_L(i_1)}\right)\left(\sigma_{i_2} - \frac{1}{h_L(i_2)}\right) \dots \left(\sigma_{i_N} - \frac{1}{h_L(i_N)}\right) \tag{14}$$

where $\frac{1}{h_L(i_j)}$ is set to 1 when $[t_j] = [t_{j-1}]$. Combining Lemmas 1 and 7 we get the following Theorem.

Theorem 1 $E_{T^\lambda} \Psi_{L,d(t)} = \Psi_{L,d(t)} E_T$.

We view $\sigma_k - \frac{1}{h_L(k)}$ as an analogue of the Khovanov–Lauda generator ψ_i, or more precisely of the element denoted ϕ_i in [4]. These *intertwining* elements are the starting point of their work. In our approach the ϕ-elements have a representation theoretical interpretation coming from the theory of the seminormal basis whereas they appear somewhat pulled out of the sleeve in [4].

4 A Cellular Basis

In this section we use the results from the previous sections to construct a cellular basis for $\mathbb{F}_p S_n$. Our construction is inspired by the one given by Hu and Mathas in [9].

Let us first introduce some notation. For λ a partition of n we let S_λ denote the row stabilizer of t^λ. Let x_λ and y_λ be the elements of RS_n given by

$$x_\lambda = \sum_{\sigma \in S_\lambda} \sigma \quad \text{and } y_\lambda = \sum_{\sigma \in S_\lambda} (-1)^{|\sigma|} \sigma$$

where $|\sigma|$ is the sign of σ. For a pair (s, t) of λ-tableaux we define

$$x_{st} = d(s)^{-1} x_\lambda d(t) \quad \text{and } y_{st} = d(s)^{-1} y_\lambda d(t).$$

If s is a λ-tableau we get that x_{ss} is the sum of the elements of the row-stabilizer of s. A similar comment applies to y_{ss}.

The set $\{x_{st}\}$ with (s, t) running over pairs of standard λ-tableaux and λ over partitions of n gives Murphy's standard basis for RS_n. Similarly $\{y_{st}\}$ gives the dual standard basis. They are cellular bases in the sense of Graham and Lehrer [8], with respect to the dominance order. This implies that $RS_n^{\trianglerighteq\lambda}$ and $RS_n^{\triangleright\lambda}$, defined by

$$RS_n^{\trianglerighteq\lambda} := \text{span}_R\{x_{st} \,|\, (s, t) \text{ pair of } \mu\text{-tableaux with } \mu \trianglerighteq \lambda\}$$
$$RS_n^{\triangleright\lambda} := \text{span}_R\{x_{st} \,|\, (s, t) \text{ pair of } \mu\text{-tableaux with } \mu \triangleright \lambda\}$$

are ideals of RS_n. The associated left cell module is

$$C(\lambda) := R\{x_{s\lambda} \,|\, s \text{ is a } \lambda\text{-tableau}\} \text{ mod } RS_n^{\triangleright\lambda}.$$

In modern terminology, as used for example in [16], it is often referred to as the Specht module, although it rather corresponds to the dual Specht module defined via Young symmetrizers.

Working over the ground fields $\mathbb{F}_p$ and $\mathbb{Q}$, we get ideals $\mathbb{F}_p S_n^{\triangleright\lambda}, \mathbb{F}_p S_n^{\trianglerighteq\lambda}$ and $\mathbb{Q}S_n^{\triangleright\lambda}, \mathbb{Q}S_n^{\trianglerighteq\lambda}$ of $\mathbb{F}_p S_n$ and $\mathbb{Q}S_n$, using constructions similar to the ones for $RS_n^{\triangleright\lambda}$, $RS_n^{\trianglerighteq\lambda}$. Similarly, we get cell modules $\overline{C(\lambda)}$ and $C_\mathbb{Q}(\lambda)$ for $\mathbb{F}_p S_n$ and $\mathbb{Q}S_n$. We use the same notation $x_{s\lambda}$ for the classes of $x_{s\lambda}$ in $C(\lambda)$, $\overline{C(\lambda)}$ or $C_\mathbb{Q}(\lambda)$. They form bases for $C(\lambda)$ over R, for $\overline{C(\lambda)}$ over $\mathbb{F}_p$ and for $C_\mathbb{Q}(\lambda)$ over $\mathbb{Q}$, when $s \in \text{Std}(\lambda)$. Hence we have the base change properties $\overline{C(\lambda)} := C(\lambda) \otimes_R \mathbb{F}_p$ and $C_\mathbb{Q}(\lambda) := C(\lambda) \otimes_R \mathbb{Q}$.

We need to recall another basis for RS_n that was also introduced by Murphy, see [20]. For $\lambda \in \text{Par}_n$, we define

$$\xi_\lambda = \prod_{i=1}^{n} (L_i + \rho_\lambda(i)) \tag{15}$$

where $\rho_\lambda(i) = k$ for $t^\lambda(i) = [k, l]$, that is $\rho_\lambda(i)$ is the row number of the i-node of t^λ. For any pair (s, t) of λ-tableaux we set

$$\xi_{st} := d(s)^{-1} \xi_\lambda d(t).$$

Then $\{\xi_{st}\}$ is a basis for RS_n when (s, t) runs over the same parameter set as above. This follows from Theorem 4.5 of [20], saying that

$$\xi_\lambda = x_\lambda + \sum_{t \in \mathrm{RStd}(n),\, t \trianglerighteq t^\lambda} x_{tt} \tag{16}$$

and Theorem 4.18 of [21], saying that for any two tableaux $u, v \in \mathrm{RStd}(n)$ of the same shape, the element x_{uv} can be expanded in terms of standard basis elements $x_{u'v'}$ where $(u', v') \trianglerighteq (u, v)$. We also get from this and Eq. 16, using that $RS_n^{\rhd\lambda}$ is an ideal in RS_n, that the images of $\{\xi_{s\lambda}\}$ in $C(\lambda)$ coincide with $\{x_{s\lambda}\}$, for s standard λ-tableaux.

Motivated by the construction done by Hu and Mathas in [9] we now introduce for each pair of standard tableaux (s, t) of the same shape λ the following elements of $\mathbb{F}_p S_n$

$$\psi_{st} := \Psi^*_{L,d(s)} \xi_\lambda E_{T^\lambda} \Psi_{L,d(t)} \tag{17}$$

where $\Psi_{L,d(s)}, \Psi_{L,d(t)}$ are as in Eq. 14 and where once again $*$ is the antiautomorphism that fixes the transpositions. Since $\frac{1}{h_L(k)}$ is a polynomial expression of Jucys–Murphy elements, $\Psi^*_{L,d(s)}$ is obtained from $\Psi_{L,d(s)}$ by reversing the factors. Note that in ψ_{st} the two middle factors ξ_λ and E_{T^λ} commute.

We aim at proving that the set of ψ_{st} is a cellular basis for $\mathbb{F}_p S_n$ when (s, t) runs over pairs of standard tableaux of the same shape. We begin with the following preparatory Lemma.

Lemma 8 *For every partition λ of n we have triangular expansions*

$$\xi_\lambda = x_\lambda + \sum_{(u,v) \rhd (s,t)} c_{uv} x_{uv}, \quad \xi_\lambda E_{T^\lambda} = x_\lambda + \sum_{(u,v) \rhd (s,t)} d_{uv} x_{uv}$$

where $c_{uv} \in R$, $d_{uv} \in \mathbb{F}_p$.

Proof The first expansion follows from Eq. 16 and Theorem 4.18 of [21].

In order to obtain the second expansion, we first recall Corollary 2.15 of [10] saying that for any pair of standard tableaux (s, t) of the same shape we have

$$x_{st} L_k = r_t(k) x_{st} + \sum_{(u,v) \rhd (s,t)} c_{uv} x_{uv} \tag{18}$$

where (u, v) runs over pairs of standard tableaux of the same shape and $c_{uv} \in R$. Especially, we get

$$x_\lambda L_k = r_\lambda(k) x_\lambda + \sum_{(u,v) \rhd (t^\lambda, t^\lambda)} c_{uv} x_{uv}.$$

From this we get via the definition of E_t that

$$x_\lambda E_t = \delta_{tt^\lambda} x_\lambda + \sum_{(u,v) \rhd (t^\lambda, t^\lambda)} d_{uv} x_{uv}$$

for certain $d_{uv} \in \mathbb{Q}$ where δ_{tt^λ} is the Kronecker delta. Using the definition of E_T we deduce from this that

$$x_\lambda E_T = x_\lambda + \sum_{(u,v) \rhd (t^\lambda, t^\lambda)} d_{uv} x_{uv}$$

where (u, v) still is a pair of standard tableaux of the same shape but where we now have $d_{uv} \in R$. We then finish the proof of the Lemma by reducing modulo p. □

Remark For $\mu := Shape(\sigma)$ and $\nu := Shape(s)$ we define $(\sigma, \tau) \succeq (s, t)$ if either $(\sigma, \tau) \trianglerighteq (s, t)$ and $\mu = \nu$, or if $\mu \triangleright \nu$. Then the triangularity property 18 appears already in [3] if $\trianglerighteq$ is replaced by $\succeq$.

The next result gives the promised cellularity property for $\{\psi_{st}\}$.

Theorem 2 *For pairs of standard tableaux (s, t) of the same shape we have*

$$\psi_{st} = x_{st} + \sum_{(\sigma,\tau)\succ(s,t)} a_{\sigma\tau} x_{\sigma\tau}, \quad a_{\sigma\tau} \in \mathbb{F}_p.$$

where $\succeq$ is the partial order introduced in the above remark. Moreover, with respect to the dominance order, the set

$$\{\psi_{s,t} \,|\, (s, t) \text{ pair of standard tableaux of the same shape}\}$$

defines a cellular basis for $\mathbb{F}_p S_n$ with cell modules $C(\lambda)$.

Proof We have

$$\psi_{st} := \Psi^*_{L,d(s)} E_{T^\lambda} \xi_\lambda \Psi_{L,d(t)}.$$

For $d(s) = \sigma_{i_1}\sigma_{i_2} \ldots \sigma_{i_N}$ in reduced form, we assume that

$$\left(\sigma_{i_{k-1}} - \frac{1}{h_L(i_{k-1})}\right) \ldots \left(\sigma_{i_1} - \frac{1}{h_L(i_1)}\right) E_{T^\lambda} \xi_\lambda \tag{19}$$

is equal to $x_{\sigma_{i_{k-1}} t^\lambda, \lambda}$ plus a linear combination of terms x_{uv} such that $(u, v) \succ (\sigma_{i_{k-1}} t^\lambda, \lambda)$, and consider the action of $\sigma_{i_k} - \frac{1}{h_L(i_k)}$ on Eq. 19. If $\frac{1}{h_L(i_k)} \neq 1$ then $\frac{1}{h_L(i_k)} = \frac{1}{L_{i_{k-1}} - L_{i_k}}$ is a linear combination of terms of the form

$$(L_{i_{k-1}} - L_{i_k} - r_\lambda(i_{k-1}) + r_\lambda(i_k))^l$$

and so $\frac{1}{h_L(i_k)}$ acts upper triangularily on Eq. 19, even with respect to $\trianglerighteq$. If $\frac{1}{h_L(i_k)} = 1$, the same conclusion holds trivially. We then consider the action of σ_{i_k} on Eq. 19 where we get $x_{\sigma_{i_k} t^\lambda, \lambda}$ plus a linear combination of x_{ab} with $(a, b) \succ (\sigma_{i_k} t^\lambda, \lambda)$. Indeed, for the terms x_{ab} in Eq. 19 satisfying $Shape(a) \triangleright \lambda$ this follows from Theorem 4.18 of [21], whereas for the terms x_{ab} with $Shape(a) = \lambda$ we use the compatibility of the Bruhat order and the dominance order given in Lemma 3.8 of *loc. cit.* The right action of $\Psi_{L,d(t)}$ is treated the same way, and we have then proved the triangularity property of the Theorem.

Finally, since we know that $\{x_{st}\}$ is cellular, an argument similar to the one given in Theorem 5.8 of [9] now gives the cellularity of $\{\psi_{st}\}$ with $*$-involution satisfying $\psi^*_{st} = \psi_{ts}$. The Theorem is proved. □

From the general theory of cellular algebras there is an associated bilinear invariant form on $\overline{C(\lambda)}$, that we denote $\langle \cdot, \cdot \rangle_\lambda$. It it given by

$$\psi_{\lambda s} \psi_{t\lambda} = \langle \psi_{s\lambda}, \psi_{t\lambda} \rangle_\lambda \, \psi_\lambda \bmod \mathbb{F}_p S_n^{\triangleright\lambda}$$

Its radical rad_λ is a submodule of $\overline{C(\lambda)}$ and $\overline{C(\lambda)}/\mathrm{rad}_\lambda$ is either simple or zero. An important point of the theory of cellular algebras is that this gives rise to a classification of the simple modules for $\mathbb{F}_p S_n$. Indeed, every simple module for $\mathbb{F}_p S_n$ is of the form $\overline{C(\lambda)}/\mathrm{rad}_\lambda$ for a unique λ.

Our next Lemma shows that $\langle\cdot,\cdot\rangle_\lambda$ is in block form with respect to our basis. Note that bases that block diagonalise the bilinear form have also been found in [17] and in [5].

Lemma 9 *The basis $\{\psi_{s\lambda} \mid s$ standard λ-tableau$\}$ of $\overline{C(\lambda)}$ is in block form with respect to $\langle\cdot,\cdot\rangle_\lambda$ with blocks given by the tableau classes.*

Proof Suppose that s, t are standard λ-tableaux and that the tableau classes $S := [s]$ and $T := [t]$ are different. Then we have that

$$\psi_{\lambda s}\psi_{t\lambda} = \xi_\lambda E_{T^\lambda}\Psi_{L,d(s)}\Psi^*_{L,d(t)}E_{T^\lambda}\xi_\lambda.$$

Using Theorem 1 and its $*$-version, and noting that $E^*_U = E_U$ for all U since E_U is a sum of products of Jucys–Murphy operators, we get that this is equal to

$$\xi_\lambda\Psi_{L,d(s)}E_S E_T\Psi^*_{Ld(t)}\xi_\lambda.$$

But $E_S E_T = 0$ and the Lemma follows. □

5 Specht Modules and Jucys–Murphy Operators

In this section we give a new realization of the Specht modules, using Jucys–Murphy operators.

An essential ingredient of our construction is the use of what we denote the Gelfand–Zetlin subalgebra of RS_n as a kind of Cartan subalgebra of a semisimple Lie algebra. This is in accordance with ideas promoted by Okounkov and Vershik in the article "A new approach to the representation theory of the symmetric group", [22]. Their approach also applies to a wider class of algebras than the group algebra of the symmetric group, but relies heavily on the algebras being semisimple. Moreover, the Specht modules themselves have in their approach apparently so far only been treated from the "old" point of view. In this section we realize the Specht module as induced modules from the Gelfand–Zetlin subalgebra, at least over R and $\mathbb{Q}$ and thus partially remedy these deficiencies. It would be interesting to investigate to what extent these results hold in positive characteristic.

Define $\mathrm{GZ}_n \subseteq RS_n$ to be the Gelfand–Zetlin algebra, the R-subalgebra of RS_n generated by the Jucys–Murphy operators:

$$\mathrm{GZ}_n := \langle L_i \mid i = 1, \dots, n\rangle.$$

This definition is not quite equivalent to the one used by for example Okounkov and Vershik in [22]. They first of all work over a field of characteristic zero and even in that case, our definition of the Gelfand–Zetlin algebra is actually a Theorem in [22] that characterizes the subalgebra.

GZ_n is a commutative subalgebra of RS_n and it contains the center $Z(RS_n)$ of RS_n—indeed by Theorem 1.9 of [19] we know that $Z(RS_n)$ consists of the symmetric polynomials in the L_k.

We aim at defining an induction functor from GZ_n-modules to RS_n-modules. For this we first need to state a few categorical generalities on R-modules.

For an R-module M we define $M^* := \mathrm{Hom}_R(M, R)$. If M is also a left RS_n-module, M^* becomes a right RS_n-module and vice-versa. Let R-modfg denote the category of finitely generated R-modules and let RS_n-modfg denote the subcategory whose objects are also left RS_n-modules.

Since R is Euclidean, we have for $M \in R$-modfg that

$$M = \mathrm{Fr}(M) \oplus \mathrm{Tor}(M)$$

where $\mathrm{Fr}(M)$ is the free part of M and $\mathrm{Tor}(M)$ the torsion part of M. If $f : M \to N$ is a morphism in R-modfg then clearly $f(\mathrm{Tor}(M)) \subseteq \mathrm{Tor}(N)$ and from this we deduce that $M \mapsto \mathrm{Tor}(M)$ is a left exact functor on R-modfg. On the other hand $M \mapsto \mathrm{Fr}(M)$ is an exact functor. Indeed, we may define it as $\mathrm{Fr}(M) := M^{**}$ which shows that it is a covariant functor in the first place. But for $M \in R$-modfg the canonical map $M \to M^{**}$ induces an isomorphism $M/\mathrm{Tor}(M) \to \mathrm{Fr}(M)$. This gives a natural transformation from the functor $M \mapsto M/\mathrm{Tor}(M)$ to Fr and hence, since $M \mapsto M/\mathrm{Tor}(M)$ is right exact, we get that Fr is right exact as well, whereas left exactness follows directly from the definitions.

From this we get that Fr induces an exact functor on RS_n-modfg. Indeed, if M is a left RS_n-module then also $\mathrm{Fr}(M) = M^{**}$ is a left RS_n-module and exactness follows from exactness at R-modfg level.

We let GZ_n-modfg denote the subcategory of R-modfg whose objects are also GZ_n-modules. Finally, we define R-modfr as the category of finitely generated free R-modules and RS_n-modfr as the subcategory whose objects are also left RS_n-modules.

After these preparations we are in position to define the induction functor. For $M \in \mathrm{GZ}_n$-modfg we define

$$\mathrm{Ind}(M) := \mathrm{Fr}(RS_n \otimes_{\mathrm{GZ}_n} M).$$

Then $\mathrm{Fr}(RS_n \otimes_{\mathrm{GZ}_n} M) \in RS_n$-modfr. Furthermore, by the above considerations we have that $M \mapsto \mathrm{Ind}(M)$ is a right exact functor from GZ_n-modfg to RS_n-modfr.

An important property of Ind is the following Frobenius reciprocity rule

$$\mathrm{Hom}_{\mathrm{GZ}_n}(M, N) \cong \mathrm{Hom}_{RS_n}(\mathrm{Ind}(M), N)$$

for $M \in \mathrm{GZ}_n$-modfg and $N \in \mathrm{GZ}_n$-modfr. It follows from

$$\mathrm{Hom}_R(M, N) \cong \mathrm{Hom}_R(F(M), N)$$

for $M \in R$-modfg, $N \in R$-modfr and the usual Frobenius reciprocity for induction.

For us the most important case of the above construction is the following. Let λ be a partition of n and let I_λ be the ideal of GZ_n generated by $L_i - r_\lambda(i)$ for $i = 1, \ldots, n$. Set

$$1_\lambda := \mathrm{GZ}_n / I_\lambda.$$

Then we may consider 1_λ as a left GZ_n-module. As we point out in the final remarks of this section, it is free of rank one over R with generator 1. The action of L_i on 1 is multiplication by $r_\lambda(i)$. We next define

$$\mathrm{Ind}(\lambda) := \mathrm{Ind}(1_\lambda).$$

We aim at studying $\mathrm{Ind}(\lambda)$ at some depth, our main result being a proof of the isomorphism $\mathrm{Ind}(\lambda) \cong C(\lambda)$. The following Lemma is a first step towards this.

Let t_λ be the lowest λ-tableau having $1, 2 \ldots, n$ filled in along columns and define $s_\lambda := d(t_\lambda) \in S_n$. Set

$$z_\lambda := x_\lambda s_\lambda y_{\lambda'} = x_\lambda s_{\lambda'}^{-1} y_{\lambda'} \in RS_n. \tag{20}$$

Then $x_\lambda RS_n$ is isomorphic to the right permutation module studied in [11] where the isomorphism maps the tabloid $\{t^\lambda\}$ to x_λ. The alternate column sum of t^λ is $\kappa_{t^\lambda} = s_{\lambda'}^{-1} y_{\lambda'} s_{\lambda'} = s_\lambda y_{\lambda'} s_\lambda^{-1}$ and so the isomorphism maps the polytableau e_{t^λ} of [11] to $x_\lambda s_{\lambda'}^{-1} y_{\lambda'} s_{\lambda'}$. In other words $x_\lambda s_\lambda y_{\lambda'} RS_n$ identifies with the Specht module S^λ of [11]. For s a λ-tableau we set

$$z_{\lambda s} := x_\lambda s_\lambda y_{\lambda'} d(s').$$

Then $\{z_{\lambda s} \mid s \in \mathrm{Std}(\lambda)\}$ is a basis for $z_\lambda RS_n$, see eg. Lemma 5.1 and Theorem 5.6 of [7].

Lemma 10 *With the above notation we have*

$$\{x \in RS_n \mid L_i x = r_\lambda(i)\, x \textit{ for all } i\} = z_\lambda RS_n.$$

Proof Let us denote by ${}_\lambda\mathcal{LS}_n$ the left hand side of the Lemma. We first prove that ${}_\lambda\mathcal{LS}_n \supseteq z_\lambda RS_n$. Now ${}_\lambda\mathcal{LS}_n$ certainly is a right submodule of RS_n and it is known, see for example [20], page 498, that

$$x_{ab} s_\lambda y_{\lambda'} = 0 \quad \text{unless } \mu \trianglelefteq \lambda \tag{21}$$

where $\mu = Shape(s) = Shape(b)$. Combining this with the fact that L_i acts upper triangularily on the $\{x_{st}\}$-basis, as is seen by applying $*$ to Eq. 18, we find that z_λ belongs to ${}_\lambda\mathcal{LS}_n$, from which the inclusion $\supseteq$ indeed follows.

In order to show the other inclusion $\subseteq$ we first work in $\mathbb{Q}S_n$, and define

$${}_\lambda\mathcal{LS}_{\mathbb{Q},n} := \{x \in \mathbb{Q}S_n \mid L_i x = r_\lambda(i)\, x \text{ for all } i\}.$$

Setting $t = t_\lambda$ we recall from [20], page 511 that

$$E_\lambda = h_\lambda^{-1} z_{\lambda t} \Psi_t^* \tag{22}$$

and from this we deduce that ${}_\lambda\mathcal{LS}_{\mathbb{Q},n} = z_\lambda \mathbb{Q}S_n$. We then get that

$${}_\lambda\mathcal{LS}_n = z_\lambda \mathbb{Q}S_n \cap RS_n = z_\lambda \mathbb{Q}S_n \cap x_\lambda RS_n.$$

Here the last equality follows from the facts that $\{x_{st}\}$ is an R-basis of RS_n and that $z_\lambda \mathbb{Q}S_n \subseteq x_\lambda \mathbb{Q}S_n$. Finally, since $z_\lambda \mathbb{Q}S_n = S_\mathbb{Q}(\lambda)$ is the Specht module defined over $\mathbb{Q}$, we get from Corollary 8.9 of [11], which is based on the Garnir relations, that

$$z_\lambda \mathbb{Q}S_n \cap x_\lambda RS_n \subseteq z_\lambda RS_n$$

and the Lemma is proved. □

Recall that RS_n is equipped with a symmetric nondegenerate bilinear form $\langle \cdot, \cdot \rangle$, given by

$$\langle a, b \rangle := \mathrm{coeff}_1(ab)$$

where $\operatorname{coeff}_1(x)$ is the coefficient of 1 when $x \in RS_n$ is expanded in the canonical basis of RS_n. It is associative in the following sense

$$\langle ab\,, c\rangle = \langle a, bc\rangle \quad \text{for all } a, b\,, c \in RS_n.$$

The form induces an RS_n-bimodule isomorphism $RS_n \cong RS_n^* = \operatorname{Hom}_R(RS_n, R)$ where the RS_n-bimodule structure on RS_n^* is given as follows

$$af(x)b := f(bxa) \quad \text{for all } a, b\,, x \in RS_n \text{ and } f \in RS_n^*.$$

We can now prove the promised result on $\operatorname{Ind}(\lambda)$.

Theorem 3 *For λ any partition of n there is an isomorphism of RS_n-modules*

$$\operatorname{Ind}(\lambda) \cong C(\lambda).$$

Proof Define $\mathrm{LGZ}_\lambda := \sum_i RS_n(L_i - r_\lambda(i))$. Then LGZ_λ is a left ideal of RS_n and by the definitions we have that

$$\operatorname{Ind}(\lambda) = \operatorname{Fr}(RS_n/\,\mathrm{LGZ}_\lambda) = (RS_n/\,\mathrm{LGZ}_\lambda)^{**}. \tag{23}$$

But $\langle\cdot,\cdot\rangle$ is nondegenerate, and therefore it induces an isomorphism of the right RS_n-modules

$$(RS_n/\,\mathrm{LGZ}_\lambda)^* \cong (\mathrm{LGZ}_\lambda)^\perp$$

where $(\mathrm{LGZ}_\lambda)^\perp := \{x \in RS_n \mid \langle x, \mathrm{LGZ}_\lambda\rangle = 0\}$. On the other hand, using the symmetry, associativity and nondegeneracy of $\langle\cdot,\cdot\rangle$ we find that $x \in (\mathrm{LGZ}_\lambda)^\perp$ iff $(L_i - r_\lambda(i))x = 0$ for all i. We then deduce from the previous Lemma that

$$(\mathrm{LGZ}_\lambda)^\perp = z_\lambda RS_n.$$

Thus, we are reduced to showing that $z_\lambda RS_n^* \cong C(\lambda)$. This is a little variation of a well-known fact, that normally is presented using either two left or two right modules. In our setting, with one left and one right module, the pairing $z_\lambda RS_n \times C(\lambda) \mapsto R$ is given by the rule $(z_{\lambda t}, x_{s\lambda}) \mapsto \operatorname{coeff}_\lambda(z_{\lambda t}x_{s\lambda})$ where for any $u \in RS_n$ we define $\operatorname{coeff}_\lambda(u)$ as the coefficient of x_λ when u is expanded in the x_{st}-basis. □

We now deduce the following universal property of $C(\lambda)$. We consider it analogous to the universal property for the Weyl module of an algebraic group, which is a consequence of the Kempf's vanishing Theorem of the cohomology of the line bundle on the flag manifold given by a dominant weight, see eg. [2, 23].

Theorem 4 *Let $M \in RS_n$-modfr. Let*

$${}_\lambda M := \{m \in M \mid L_i m = r_\lambda(i)m \textit{ for all } i\}.$$

Then $\operatorname{Hom}_{RS_n}(C(\lambda), M) = {}_\lambda M$.

Proof Any $m \in {}_\lambda M$ induces a map in $\operatorname{Hom}_{GZ_n}(1_\lambda, M)$ and then by Frobenius reciprocity a map in $\operatorname{Hom}_{RS_n}(\operatorname{Ind}(\lambda), M) = \operatorname{Hom}_{RS_n}(C(\lambda), M)$. On the other hand, any element of $f \in \operatorname{Hom}_{GZ_n}(1_\lambda, M)$ gives rise to an element of ${}_\lambda M$, namely the image $f(1)$. □

Remark Let $\mathrm{GZ}_{\mathbb{Q}}$ be the original Gelfand–Zetlin algebra introduced in [22], that is the $\mathbb{Q}$-subalgebra of $\mathbb{Q}S_n$ generated by the Jucys–Murphy elements $L_i, i = 1, \dots, n$. We let $I_{\mathbb{Q},\lambda}$ denote the ideal of $\mathrm{GZ}_{\mathbb{Q}}$ generated by the $L_i - r_\lambda(i)$ for $i = 1, \dots, n$ and set $1_{\mathbb{Q},\lambda} := \mathrm{GZ}_{\mathbb{Q}} / I_{\mathbb{Q},\lambda}$ and can then define $\mathrm{Ind}_{\mathbb{Q}}(\lambda)$ as the $\mathbb{Q}S_n$-module induced up from $1_{\mathbb{Q},\lambda}$. Now the same series of arguments as the one used above, even with some simplifications in Lemma 8, leads to the isomorphism

$$\mathrm{Ind}_{\mathbb{Q}}(\lambda) \cong C_{\mathbb{Q}}(\lambda).$$

But in this case the result could actually also have been obtained as follows. From Eq. 2 we have that

$$\mathrm{GZ}_{\mathbb{Q}} = \langle E_t \,|\, t \in \mathrm{Std}(n) \rangle. \tag{24}$$

Here the $\{E_t\}$ even form a $\mathbb{Q}$-basis for $\mathrm{GZ}_{\mathbb{Q}}$ since they are orthogonal idempotents. On the other hand, since $E_\lambda \xi_\lambda = \gamma_\lambda E_\lambda$, as is proved on page 508 of [20], we get that the basis for $\mathbb{Q}S_n$ constructed in the previous section, in this case takes the form

$$\left\{ \Psi_s^* E_\lambda \Phi_t \,|\, s, t \in \mathrm{Std}(\lambda), \lambda \in \mathrm{Par}_n \right\}.$$

Let $ev_\lambda : \mathrm{GZ}_{\mathbb{Q}} \to 1_{\mathbb{Q},\lambda}$ be the quotient map. Then one checks, using the fact that the contents $r_t(i)$ determine t uniquely, that

$$ev_\lambda(E_t) = \begin{cases} 1 & \text{if } t = t^\lambda \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

It now follows from $E_\lambda \Phi_t = \Psi_t E_t$ that $\mathrm{Ind}_{\mathbb{Q}}(\lambda)$ has basis

$$\left\{ \Psi_s^* E_\lambda \,|\, s \in \mathrm{Std}(\lambda), \lambda \in \mathrm{Par}_n \right\}$$

and the claim of the Remark follows from this.

Remark From Eq. 25 we also get that $1_{\mathbb{Q},\lambda}$ is of dimension one over $\mathbb{Q}$ with basis $\{E_\lambda\}$, and from this we conclude that 1_λ is free of rank one over R. Indeed, using $1_\lambda = \mathrm{GZ}_n / I_\lambda$ one checks that 1_λ is cyclic over R, generated by 1. On the other hand, using exactness of the tensor functor $M \to M \otimes_R \mathbb{Q}$ we get that $1_{\mathbb{Q},\lambda} = 1_\lambda \otimes_R Q$. Hence $\dim_{\mathbb{Q}}(1_\lambda \otimes_R Q) = 1$ and so 1_λ is torsion free, and thus free of rank one over R.

Remark In general RS_n is *not* free over GZ_n. Indeed, if RS_n were free over GZ_n then $\mathbb{Q}S_n$ would be free over $\mathrm{GZ}_{\mathbb{Q}}$. Since $\{ E_t \,|\, t \in \mathrm{Std}(n)\}$ is a basis of $\mathrm{GZ}_{\mathbb{Q}}$ we can determine the dimension of $\mathrm{GZ}_{\mathbb{Q}}$. For instance, for $n = 3$ we find $\dim \mathrm{GZ}_{\mathbb{Q}} = 4$ which does not divide $\dim \mathbb{Q}S_n = 6$.

6 Simples

Let G be an algebraic group over an algebraically closed field k of characteristic p. Let B be a Borel subgroup of G with maximal torus $T \subset B$ and let $X(T)$ (resp. $X(T)^+$) be the set of weights (resp. dominant weights) with respect to B and T. For $\lambda \in X(T)^+$ there is an associated Weyl module $\Delta(\lambda)$ with unique simple quotient $L(\lambda)$. It is the reduction modulo p of a $\mathbb{Z}$-form for a module for the corresponding complex group. The finite dimensional simple modules for G are classified by $L(\lambda)$ where $\lambda \in X(T)^+$. We write $\nabla(\lambda) := \Delta(\lambda)^*$ where $*$ is the contravariant duality functor on finite dimensional G-modules. We may realize $\nabla(\lambda)$ as the G-module $H^0(\lambda)$ of global sections of the line bundle on G/B associated with λ.

Let $\langle\cdot,\cdot\rangle_\lambda$ be a nonzero contravariant form on $\Delta(\lambda)$. It induces a G-linear map $c_\lambda : \Delta(\lambda) \to \nabla(\lambda)$. As a matter of fact, since $\langle\cdot,\cdot\rangle_\lambda$ is unique up to multiplication by a nonzero scalar, we have that c_λ generates $\mathrm{Hom}_G(\Delta(\lambda), \nabla(\lambda))$ and that $im\, c_\lambda$ is isomorphic to $L(\lambda)$. In this sense, $\Delta(\lambda)$ and $\nabla(\lambda)$ give rise to a *realization* of $L(\lambda)$.

In this section we try to carry over this realization of the simple G-modules to the case of the symmetric group. As we shall see, the results of the previous section provide a suitable solution to this problem.

Let M be a left RS_n-module. The contragredient dual $M^{\circledast}$ of M is defined to be $M^* := \mathrm{Hom}_R(M, R)$ with RS_n-action given by $(\sigma f)(x) := f(\sigma^{-1}x)$ for $\sigma \in S_n, x \in M$ and $f \in M^*$. It is a left RS_n-module as well.

Using Theorem 5.3 of [21], with a small modification since we are working with left modules, we have that the contragredient dual of $C(\lambda)$ is

$$C(\lambda)^{\circledast} = RS_n\, y_{\lambda'} s_\lambda^{-1} x_\lambda = RS_n\, y_{\lambda'} s_{\lambda'} x_\lambda. \tag{26}$$

This isomorphism is also valid in the specialized situation

$$\overline{C(\lambda)}^{\circledast} = \mathbb{F}_p S_n\, y_{\lambda'} s_{\lambda'} x_\lambda. \tag{27}$$

Let $(\cdot,\cdot)_\lambda$ be the bilinear form on $C(\lambda)$ associated with Murphy's standard basis, following [21] or the general cellular algebra theory, see [8]. It is given by

$$(x_{s\lambda}, x_{t\lambda})_\lambda = \mathrm{coeff}_\lambda(x_{\lambda s}\, x_{t\lambda})$$

where once again $\mathrm{coeff}_\lambda(u)$ is the coefficient of x_λ when u is expanded in the x_{st}-basis. It induces an RS_n-homomorphism $c_\lambda : C(\lambda) \to C(\lambda)^{\circledast}$, or setting $z'_\lambda := y_{\lambda'} s_{\lambda'} x_\lambda$ and using Eq. 27 and Theorem 3

$$c_\lambda : \mathrm{Ind}(\lambda) = \mathrm{Fr}(RS_n \otimes_{\mathrm{GZ}_n} 1_\lambda) \to RS_n z'_\lambda.$$

In general c_λ is injective since $(\cdot,\cdot)_\lambda$ is nondegenerate over R, but not surjective. We can now state and prove our main result.

Theorem 5

(a) *There is $a_\lambda \in \mathbb{Q}$ such that $a_\lambda E_\lambda \in RS_n$ and such that c_λ corresponds to $1 \mapsto a_\lambda E_\lambda$ under Frobenius reciprocity.*

(b) *The simple $\mathbb{F}_p S_n$-module $D(\lambda)$ associated with λ is given by $D(\lambda) = \mathbb{F}_p S_n \overline{a_\lambda E_\lambda}$.*

Proof By Theorems 3 and 4, and the fact that $C(\lambda)^{\circledast}$ is free over R, we have

$$\mathrm{Hom}_{RS_n}\left(\mathrm{Ind}(\lambda), C(\lambda)^{\circledast}\right) = \mathrm{Hom}_{\mathrm{GZ}_n}\left(1_\lambda, C(\lambda)^{\circledast}\right)$$

hence c_λ is given by $1 \mapsto m_\lambda$ where $m_\lambda \in {}_\lambda(RS_n z'_\lambda)$. From Lemma 10 we then have

$$m_\lambda \in RS_n z'_\lambda \cap z_\lambda RS_n = RS_n s_\lambda z'_\lambda \cap z_\lambda s_\lambda^{-1} RS_n.$$

But the Young preidempotent $e := z_\lambda s_\lambda^{-1}$ satisfies $e^2 = \gamma_\lambda \gamma_{\lambda'}\, e$ since it can be rewritten as $e = x_\lambda \kappa_{t^\lambda}$ with κ_{t^λ} as above. Hence we get

$$m_\lambda = \frac{1}{\gamma_\lambda \gamma_{\lambda'}}\, z_\lambda s_\lambda^{-1} m s_\lambda z'_\lambda = \frac{1}{\gamma_\lambda \gamma_{\lambda'}}\, x_\lambda s_\lambda y_{\lambda'} s_\lambda^{-1} m s_\lambda y_{\lambda'} s_\lambda^{-1} x_\lambda$$

for some $m \in RS_n$. On the other hand, it is known that the R-module $x_\lambda RS_n y_{\lambda'}$ is free of rank one, generated by $x_\lambda s_\lambda y_{\lambda'}$, see for example [20], page 498, and so we may rewrite m_λ as follows

$$m_\lambda = a_\lambda x_\lambda s_\lambda y_{\lambda'} s_\lambda^{-1} x_\lambda$$

for some $a_\lambda \in \mathbb{Q}$. We now recall the expression for $z_{\lambda t}$ given on page 511 of loc. cit. which in our notation becomes

$$x_\lambda s_\lambda y_{\lambda'} s_\lambda^{-1} = b_\lambda E_\lambda s_\lambda E_t$$

where $b_\lambda \in \mathbb{Q}$ and t is the lowest λ-tableau. Applying $*$ to it we get

$$s_\lambda y_{\lambda'} s_\lambda^{-1} x_\lambda = b_\lambda E_t s_\lambda^{-1} E_\lambda.$$

Combining these expressions and using that $y_{\lambda'}$ is a preidempotent, we find the following formula for m_λ, up to a scalar in $\mathbb{Q}$

$$m_\lambda = E_\lambda s_\lambda E_t s_\lambda^{-1} E_\lambda.$$

We then finally use the version of Young's seminormal form that is developed on page 152 of [24] and obtain

$$m_\lambda = a_\lambda E_\lambda$$

where a_λ is a (new) scalar in $\in \mathbb{Q}$. This finishes the proof of (a).

We next show (b). From the definitions we have that

$$RS_n / \mathrm{LGZ}_\lambda = RS_n \otimes_{\mathrm{GZ}_n} 1_\lambda = \mathrm{Ind}(\lambda) \oplus \mathrm{Tor}(RS_n \otimes_{\mathrm{GZ}_n} 1_\lambda)$$

and so by (a) we have that $c_\lambda : \mathrm{Ind}(\lambda) \to C(\lambda)^{\circledast}$ is given by $w \in RS_n \mapsto a_\lambda w E_\lambda$ since $C(\lambda)^{\circledast}$ is torsion-free. Reducing c_λ modulo p we get the homomorphism $\overline{c_\lambda}$

$$\overline{C(\lambda)} = \mathrm{Ind}(\lambda) \otimes_R \mathbb{F}_p \xrightarrow{\overline{c_\lambda}} C(\lambda)^{\circledast} \otimes_R \mathbb{F}_p = \overline{C(\lambda)}^{\circledast}$$

given by $w \otimes 1 \mapsto a_\lambda w E_\lambda \otimes 1$ for $w \in RS_n$. We deduce from this that the image of $\overline{c_\lambda}$ is the submodule of $\overline{C(\lambda)}^{\circledast}$ generated by $\overline{a_\lambda E_\lambda} = a_\lambda E_\lambda \otimes_R \mathbb{F}_p$. But from the general principles explained above, this is equal to $D(\lambda)$. The Theorem is proved. □

Remark So far we do not have an exact formula for a_λ. On the other hand, since c_λ is unique up to multiplication by an element of R, and since $\overline{c_\lambda}$ is nonzero iff λ is p-restricted, we may simply choose for a_λ the least common multiple of the denominators of the coefficients of E_λ when expanded in the canonical basis of RS_n. The case where λ is not p-restricted is not relevant for us, of course.

Remark The Theorem gives rise to an algorithm for calculating $\dim D(\lambda)$ that goes as follows. Let $\mathbf{D}(\lambda)$ be the $\dim S(\lambda) \times n!$ matrix over $\mathbb{F}_p$ that has $\overline{a_\lambda E_\lambda}$ in the first row and $\overline{d(t)^{-1} a_\lambda E_\lambda}$ for $t \in \mathrm{Std}(\lambda) \setminus \{t^\lambda\}$ in the other rows. Then $\dim D(\lambda) = \operatorname{rank} \mathbf{D}(\lambda)$. Note that E_λ can be calculated using formula 22. We have implemented this algorithm in the GAP system. We have checked $n < 8$ for all relevant primes and found complete match with the known dimensions for $D(\lambda)$, as given by Mathas's Specht-package.

On the other hand, although the first row of the matrix $\mathbf{D}(\lambda)$ easily determines the other rows, the algorithm cannot be expected to perform better than the usual algorithm for calculating $\dim D(\lambda)$ that goes via the $\dim S(\lambda) \times \dim S(\lambda)$ matrix associated with the bilinear form on $S(\lambda)$ – in general $n!$ is much bigger than $\dim S(\lambda)$, after all.

Remark As was pointed out to us by Mathas, a generator for $D(\lambda)$ is given on page 41 in [11]. In our terminology it is $x_\lambda s_\lambda y_{\lambda'} s_\lambda^{-1} x_\lambda$ and hence, by the arguments of the Theorem, it coincides with our generator. Our final expression of it is somewhat shorter, but still does not permit calculations much beyond the ones already indicated.

Remark It is known from [20] that $\text{coeff}_1(E_\lambda) = \frac{1}{h_\lambda}$ where h_λ is the hook-product as above. In fact since RS_n is a symmetric algebra we get from Proposition 9.17 of [6] that this fact also holds for E_t when $t \in \text{Std}(\lambda)$. Based on GAP calculations we conjecture that the coefficient of any $w \in S_n$ in E_λ is either zero or of the form $\frac{1}{k_w h_\lambda}$ for some nonzero integer k_w. According to our GAP-calculations, a similar statement does not hold for the general E_t.

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