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Unbounded Induced Representations of ∗-Algebras

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Abstract Induced representations of $*$-algebras by unbounded operators in Hilbert space are investigated. Conditional expectations of a $*$-algebra $\mathcal{A}$ onto a unital $*$-subalgebra $\mathcal{B}$ are introduced and used to define inner products on the corresponding induced modules. The main part of the paper is concerned with group graded $*$-algebras $\mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$ for which the $*$-subalgebra $\mathcal{B} := \mathcal{A}_e$ is commutative. Then the canonical projection $p : \mathcal{A} \to \mathcal{B}$ is a conditional expectation and there is a partial action of the group G on the set $\widehat{\mathcal{B}}^+$ of all characters of $\mathcal{B}$ which are nonnegative on the cone $\sum \mathcal{A}^2 \cap \mathcal{B}$. The complete Mackey theory is developed for $*$-representations of $\mathcal{A}$ which are induced from characters of $\widehat{\mathcal{B}}^+$. Systems of imprimitivity are defined and two versions of the Imprimitivity Theorem are proved in this context. A concept of well-behaved $*$-representations of such $*$-algebras $\mathcal{A}$ is introduced and studied. It is shown that well-behaved representations are direct sums of cyclic well-behaved representations and that induced representations of well-behaved representations are again well-behaved. The theory applies to a large variety of examples. For important examples such as the Weyl algebra, enveloping algebras of the Lie algebras $su(2)$, $su(1,1)$, and of the Virasoro algebra, and $*$-algebras generated by dynamical systems our theory is carried out in great detail.

Dedicated to the memory of A.U. Klimyk (14.04.1939–22.07.2008).

Presented by Yuri Drozd.

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1 Introduction

Induced representations are a fundamental tool in representation theory of groups and algebras. They were first defined in 1898 for finite groups by G. Frobenius and in 1955 for algebras by D.G. Higman. If $\mathcal{B}$ is a subalgebra of an algebra $\mathcal{A}$ and V is a left $\mathcal{B}$-module, then the left $\mathcal{A}$-module $\mathcal{A} \otimes_{\mathcal{B}} V$ with action defined by $a_0(a \otimes v) := a_0 a \otimes v$ is called *induced module* of V.

In his seminal paper, M. Rieffel [35] introduced induced representations for C^*-algebras and developed a major part of Mackey's theory in this context. In purely algebraic setting induced representations have been studied in [2]. Another pioneering paper is due to J.M.G. Fell [11]. A detailed treatment of this theory is given in the monograph [12]. An essential step in Rieffel's inducing process is the definition of an inner product on the algebraic tensor product $\mathcal{A} \otimes_{\mathcal{B}} V$. That is, if there exists a conditional expectation p from a C^*-algebra $\mathcal{A}$ onto its C^*-subalgebra $\mathcal{B}$ and if a Hilbert space $(V, \langle \cdot, \cdot \rangle)$ is a Hermitian $\mathcal{B}$-module (that is, $\langle bx, y \rangle = \langle x, b^* y \rangle$ for $x, y \in V$ and $b \in \mathcal{B}$), then there exists a pre-inner product $\langle \cdot, \cdot \rangle_0$ on $\mathcal{A} \otimes_{\mathcal{B}} V$ such that

$$\langle a_1 \otimes v_1, a_2 \otimes v_2 \rangle_0 := \langle p(a_2^* a_1) v_1, v_2 \rangle \tag{1}$$

and the quotient space of $\mathcal{A} \otimes_{\mathcal{B}} V$ by the null space of the form $\langle \cdot, \cdot \rangle_0$ is a Hermitian $\mathcal{A}$-module.

The aim of the present paper is to develop the basics of a theory of *unbounded* induced $*$-representations for complex unital $*$-algebras. In contrast to the case of C^*-algebras there are various notions of positivity for general $*$-algebras that lead to different definitions of conditional expectations. The subtleties of positivity play a central role for our theory. In this respect our notion is different from those defined in [16] and [18]. We shall define (see Definition 4 below) a *conditional expectation* from a unital $*$-algebra $\mathcal{A}$ to a unital $*$-subalgebra $\mathcal{B}$ to be a $\mathcal{B}$-linear projection p of $\mathcal{A}$ onto $\mathcal{B}$ which preserves involution and units and satisfies the following positivity condition:

$$p\left(\sum \mathcal{A}^2\right) \subseteq \mathcal{B} \cap \sum \mathcal{A}^2.$$

Then a cyclic Hermitian $\mathcal{B}$-module V is "inducible" to $\mathcal{A}$ via p if and only if every element of $\mathcal{B} \cap \sum \mathcal{A}^2$ is represented by a positive symmetric operator on V.

Many bounded or unbounded $*$-representations of $*$-algebras $\mathcal{A}$ are induced from appropriate $*$-subalgebras $\mathcal{B}$ in our setting. In Sections 9–11 we shall see that for a number of important $*$-algebras the "nice" irreducible $*$-representations are precisely those representations which are induced from characters which are non-negative on the cone $\mathcal{B} \cap \sum \mathcal{A}^2$. Among them are the $*$-algebras of the quantum group $SU_q(2)$ and of the Podles' spheres which have only bounded representations. This underlines the crucial role of positivity and it shows that our theory might be useful for general countably generated group graded $*$-algebras. It should be emphasized for all our examples neither the theory in [12] nor induction of C^*-algebras applies.

Let us briefly explain the basic idea for the Weyl algebra. We do not carry out all details of proofs, because this is just the special case $f(t) = 1 + t$ of the $*$-algebra treated in Section 10.

Example 1 Let $\mathcal{A}$ be the Weyl algebra $\mathbb{C}\langle a, a^* | aa^* - a^*a = 1\rangle$ and let $\mathcal{B}$ be the unital $*$-subalgebra $\mathbb{C}[N]$ of polynomials in $N := a^*a$. Each element $x \in \mathcal{A}$ can be written as

$$x = \sum_{r=0}^{k} a^r f_r(N) + \sum_{s=1}^{l} a^{*s} f_{-s}(N)$$

with polynomials $f_j \in \mathbb{C}[N]$ uniquely determined by x. Defining $p(x) = f_0(N)$, we obtain a conditional expectation p from $\mathcal{A}$ to $\mathcal{B}$. It can be proved (see [14] or formula (14) below) that an element $f(N) \in \mathbb{C}[N]$ belongs to $\mathcal{B} \cap \sum \mathcal{A}^2$ if and only if there are polynomials $g_0, \ldots, g_k \in \mathbb{C}[N]$ such that

$$f(N) = g_0(N)^* g_0(N) + N g_1(N)^* g_1(N) + \cdots + N(N-1)\cdots(N-k+1) g_k(N)^* g_k(N). \quad (2)$$

For $\lambda \in \mathbb{R}$, let $V_\lambda = \mathbb{C}$ be the one-dimensional $\mathcal{B}$-module given by $N = \lambda$. It is not difficult to show that $f(N) = f(\lambda) \geq 0$ for each polynomial $f(N)$ of the form (2) if and only if $\lambda \in \mathbb{N}_0$.

Now suppose that $\lambda \in \mathbb{N}_0$. Let $\mathcal{H}_\lambda$ denote the Hilbert space obtained from the pre-inner product (1) on $\mathcal{A} \otimes_{\mathcal{B}} V_\lambda$. Clearly, the vectors $a^r \otimes 1$, $a^{*(r+1)} \otimes 1$, where $r \in \mathbb{N}_0$, form a base of the vector space $\mathcal{A} \otimes_{\mathcal{B}} V_\lambda$. From the relation $aa^* - a^*a = 1$ it follows that

$$a^r a^{*r} = (N+1)\ldots(N+r),\ \ a^{*r} a^r = N(N-1)\ldots(N-r+1) \quad (3)$$

for $r \in \mathbb{N}_0$. If $r > \lambda$, then $p(a^{*r}a^r)(\lambda) = 0$, so $a^r \otimes 1$ belongs to the kernel of the form (1). Set

$$e_k := \sqrt{k!\lambda!^{-1}}\, a^{\lambda-k} \otimes 1 \text{ for } k = 0, \ldots, \lambda \text{ and } e_{k+\lambda} := \sqrt{\lambda!(\lambda+k)!^{-1}}\, a^{*k} \otimes 1 \text{ for } k \in \mathbb{N}.$$

From Eqs. 1 and 3 we easily compute that $\langle e_k, e_n\rangle_0 = \delta_{kn}$ for $k, n \in \mathbb{N}_0$. Hence $\{e_k; k \in \mathbb{N}_0\}$ is an orthonormal base of $\mathcal{H}_\lambda$. From the definition of e_k we immediately obtain that

$$a^* e_k = \sqrt{k+1}\, e_{k+1} \text{ and } a e_k = \sqrt{k}\, e_{k-1} \text{ for } k \in \mathbb{N}_0,\ \text{ where } e_{-1} := 0.$$

This shows that for each $\lambda \in \mathbb{N}_0$ the Hermitian $\mathcal{A}$-module induced from the $\mathcal{B}$-module V_λ via p is nothing but the *Bargman–Fock representation* of the Weyl algebra.

If $\lambda \notin \mathbb{N}_0$, the form (1) is not positive semi-definite. Indeed, by Eq. 3 we have $\langle a \otimes 1, a \otimes 1\rangle_0 = \lambda < 0$ if $\lambda < 0$ and $\langle a^{k+1} \otimes 1, a^{k+1} \otimes 1\rangle_0 = \lambda \cdots (\lambda - k + 1)(\lambda - k) < 0$ if $k - 1 < \lambda < k$ for $k \in \mathbb{N}$.

Summarizing, we have shown that the $\mathcal{B}$-module V_λ is inducible to a Hermitian $\mathcal{A}$-module if and only if $f(N) = f(\lambda) \geq 0$ for all $f \in \mathcal{B} \cap \sum \mathcal{A}^2$ or equivalently if $\lambda \in \mathbb{N}_0$.

Our paper is organized in the following way. In Section 2 we study induced $*$-representations defined by rigged modules. We follow mainly the approach given in Chapter XI of [12] with some necessary modifications needed for unbounded representations. As an application we show that the well-behaved representations of $*$-algebras defined in [40] by means of compatible pairs are induced representations coming from certain rigged modules. Section 3 is concerned with conditional expectations of general $*$-algebras. We give various definitions depending on the corresponding positivity conditions and develop a number of examples for these notions. Section 4 is devoted to G-graded $*$-algebras $\mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$ for a discrete group G. If H is a subgroup of G, then there exists a canonical conditional expectation of $\mathcal{A}$ on the $*$-subalgebra $\mathcal{A}_H = \oplus_{h\in H}\ \mathcal{A}_h$. Hence $*$-representations of $\mathcal{A}_H$ can be induced to a $*$-representations of $\mathcal{A}$. From Section 6 on we are dealing with G-graded $*$-algebras $\mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$ for which the $*$-subalgebra $\mathcal{B} := A_e$ is commutative. There is a large variety of G-graded $*$-algebras (Weyl algebra, enveloping algebras of $su(2)$ and $su(1,1)$, quotients of the enveloping algebra of the Virasoro algebra, $*$-algebras associated with dynamical systems, quantum disc algebras, Podles' quantum spheres, quantum algebras, and many others) that have this property. In Section 5 we study systems of imprimitivity and prove our first Imprimitivity Theorem. In Section 6 we show that there is a partial action of the group G on the set $\widehat{\mathcal{B}}^+$ of all characters of the commutative $*$-algebra $\mathcal{B}$ which are nonnegative on the cone $\mathcal{B}\cap\sum\mathcal{A}^2$. This partial action is used for a detailed study of the inducing process from characters of the set $\widehat{\mathcal{B}}^+$. In particular, we characterize irreducible representations and equivalent representations in terms of stabilizer groups of characters.

A fundamental problem in unbounded representation theory is to define and characterize well-behaved representations of a general $*$-algebra. In Section 7 we develop a new concept of well-behaved representations for G-graded $*$-algebras $\mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$ with commutative $*$-subalgebra $\mathcal{A}_e$. Among others we prove that well-behaved representations decompose into direct sums of cyclic well-behaved representations. This theorem is technically rather involved and it is probably the deepest result of our paper. In Section 8 we define well-behaved systems of imprimitivity and prove an Imprimitivity Theorem for well-behaved representations. The next two sections of the paper are devoted to detailed treatments of important examples. In Section 9 we study the enveloping algebras of three Lie algebras. For the real Lie algebras $su(2)$ and $su(1,1)$ we prove that the induced representations from characters of $\widehat{\mathcal{B}}^+$ are precisely the representations dU, where U is an irreducible unitary representation of the Lie group $SU(2)$ resp. of the universal covering group of $SU(1,1)$. For the enveloping algebra of the Virasoro algebra we characterize irreducible $*$-representations with finite-dimensional weight spaces as induced representations from characters of $\widehat{\mathcal{B}}^+$. In Section 10 we investigate $*$-algebras associated with some dynamical systems. For all these examples well-behaved representations according to our definition in Section 7 coincide with distinguished "nice" representations of these $*$-algebras thereby showing the usefulness of our concept of well-behavedness and emphasizing the role of positivity. In Section 11 we mention a number of other examples for which our theory applies.

We close this introduction by collecting some definitions and notations.

By a $*$-*algebra* we mean a complex associative algebra $\mathcal{A}$ equipped with a mapping $a \mapsto a^*$ of $\mathcal{A}$ into itself, called the *involution* of $\mathcal{A}$, such that $(\lambda a + \mu b)^* = \bar{\lambda} a^* +$

$\bar{\mu}b^*$, $(ab)^* = b^*a^*$ and $(a^*)^* = a$ for $a, b \in \mathcal{A}$ and $\lambda, \mu \in \mathbb{C}$. The unit of $\mathcal{A}$ (if it exists) will be denoted by $\mathbf{1}_{\mathcal{A}}$ and the group of all $*$-automorphisms of $\mathcal{A}$ by Aut$\mathcal{A}$. We shall say that a group G *acts as automorphism group on* $\mathcal{A}$ if there is a group homomorphism $g \mapsto \alpha_g$ of G into Aut$\mathcal{A}$. A subset $\mathcal{C}$ of $\mathcal{A}_h := \{a \in \mathcal{A} : a = a^*\}$ is called a *pre-quadratic module* if $\mathcal{C} + \mathcal{C} \subseteq \mathcal{C}$, $\mathbb{R}_+ \cdot \mathcal{C} \subseteq \mathcal{C}$, and $a^*\mathcal{C}a \in \mathcal{C}$ for all $a \in \mathcal{A}$. A *quadratic module* of $\mathcal{A}$ is a pre-quadratic module $\mathcal{C}$ such that $\mathbf{1}_{\mathcal{A}} \in \mathcal{C}$ (see e.g. [42]). The wedge

$$\sum \mathcal{A}^2 := \left\{ \sum_{j=1}^{n} a_j^* a_j;\ a_1, \ldots, a_n \in \mathcal{A}, n \in \mathbb{N} \right\}$$

of all finite sums of squares is obviously the smallest quadratic module of $\mathcal{A}$.

Throughout this paper we use some terminology and results from unbounded representation theory in Hilbert space (see e.g. in [39]). In particular, we shall speak about $*$-representations rather than Hermitian modules. Let us repeat some basic notions and facts.

Let $\mathcal{D}$ be a dense linear subspace of a Hilbert space $\mathcal{H}$ with scalar product $\langle \cdot, \cdot \rangle$. A *$*$-representation* of a $*$-algebra $\mathcal{A}$ on $\mathcal{D}$ is an algebra homomorphism π of $\mathcal{A}$ into the algebra $L(\mathcal{D})$ of linear operators on $\mathcal{D}$ such that $\langle \pi(a)\varphi, \psi \rangle = \langle \varphi, \pi(a^*)\psi \rangle$ for all $\varphi, \psi \in \mathcal{D}$ and $a \in \mathcal{A}$. We call $\mathcal{D}(\pi) := \mathcal{D}$ the *domain* of π and write $\mathcal{H}_\pi := \mathcal{H}$. Two $*$-representations π_1 and π_2 of $\mathcal{A}$ are *(unitarily) equivalent* if there exists an isometric linear mapping U of $\mathcal{D}(\pi_1)$ onto $\mathcal{D}(\pi_2)$ such that $\pi_2(a) = U\pi_1(a)U^{-1}$ for $a \in \mathcal{A}$. The *direct sum representation* $\pi_1 \oplus \pi_2$ acts on the domain $\mathcal{D}(\pi_1) \oplus \mathcal{D}(\pi_2)$ by $(\pi_1 \oplus \pi_2)(a) = \pi_1(a) \oplus \pi_2(a)$, $a \in \mathcal{A}$. A $*$-representation π is called *irreducible* if a direct sum decomposition $\pi = \pi_1 \oplus \pi_2$ is only possible when $\mathcal{D}(\pi_1) = \{0\}$ or $\mathcal{D}(\pi_2) = \{0\}$. If T is a Hilbert space operator, $\mathcal{D}(T)$, Ran$T$, $\overline{T}$ and T^* denote its domain, its range, its closure and its adjoint, respectively.

Suppose that π is a $*$-representation of $\mathcal{A}$. If $\mathcal{C}$ is a pre-quadratic module of $\mathcal{A}$, π is called $\mathcal{C}$-positive if $\langle \pi(c)\varphi, \varphi \rangle \geq 0$ for all $c \in \mathcal{C}$ and $\varphi \in \mathcal{D}(\pi)$. We denote by Res$_{\mathcal{B}}\pi$ the restriction of π to a $*$-subalgebra $\mathcal{B}$. The *graph topology* of π is the locally convex topology on the vector space $\mathcal{D}(\pi)$ defined by the norms $\varphi \mapsto \|\varphi\| + \|\pi(a)\varphi\|$, where $a \in \mathcal{A}$. If $\mathcal{D}(\overline{\pi})$ denotes the completion of $\mathcal{D}(\pi)$ in the graph topology of π, then $\overline{\pi}(a) := \overline{\pi(a)} \upharpoonright \mathcal{D}(\overline{\pi})$, $a \in \mathcal{A}$, defines a $*$-representation of $\mathcal{A}$ with domain $\mathcal{D}(\overline{\pi})$, called the *closure* of π. In particular, π is *closed* if and only if $\mathcal{D}(\pi)$ is complete in the graph topology of π. By a *core for* π we mean a dense linear subspace $\mathcal{D}_0$ of $\mathcal{D}(\pi)$ with respect to the graph topology of π. A $*$-representation π is called *non-degenerate* if $\pi(\mathcal{A})\mathcal{D}(\pi) := \mathrm{Lin}\,\{\pi(a)\varphi; a \in \mathcal{A}, \varphi \in \mathcal{D}(\pi)\}$ is dense in $\mathcal{D}(\pi)$ in the graph topology of π. If $\mathcal{A}$ is unital and π is non-degenerate, then we have $\pi(1_{\mathcal{A}})\varphi = \varphi$ for all $\varphi \in \mathcal{D}(\pi)$. We say that π is *cyclic* if there exists a vector $\varphi \in \mathcal{D}(\pi)$ such that $\pi(\mathcal{A})\varphi$ is dense in $\mathcal{D}(\pi)$ in the graph topology of π. Further, π is called *self-adjoint* if $\mathcal{D}(\pi)$ is the intersection of all domains $\mathcal{D}(\pi(a)^*)$, where $a \in \mathcal{A}$. The *(strong) commutant* $\pi(\mathcal{A})'$ consists of all bounded operators T on $\mathcal{H}_\pi$ such that $T\mathcal{D}(T) \subseteq \mathcal{D}(T)$ and $\pi(a)T\varphi = T\pi(a)\varphi$ for $a \in \mathcal{A}$. If π is self-adjoint, $\pi(\mathcal{A})'$ is a von Neumann algebra. A closed $*$-representation π of a commutative $*$-algebra $\mathcal{B}$ is called *integrable* if $\overline{\pi(b^*)} = \pi(b)^*$ for all $b \in \mathcal{B}$.

2 Rigged Modules and Induced Representations

2.1

Let $\mathcal{B}$ be a $*$-algebra. From [12], p. 1078, we repeat the following

Definition 1 *A right $\mathcal{B}$-rigged module* is a right $\mathcal{B}$-module $\mathfrak{X}$ equipped with a map $\langle\cdot,\cdot\rangle_{\mathcal{B}}:\mathfrak{X}\times\mathfrak{X}\to\mathcal{B}$ which is $\mathbb{C}$-linear in the first variable and $\mathbb{C}$-anti-linear in the second variable and satisfies the following conditions:

(i) $\langle x,y\rangle_{\mathcal{B}}=(\langle y,x\rangle_{\mathcal{B}})^*$ for $x,y\in\mathfrak{X}$,
(ii)$_1$ $\langle xb,y\rangle_{\mathcal{B}}=\langle x,y\rangle_{\mathcal{B}}b$ for $x,y\in\mathfrak{X}$ and $b\in\mathcal{B}$.

Clearly, (i) and (ii)$_1$ are equivalent to the conditions (i) and (ii)$_2$, where

(ii)$_2$ $\langle x,yb\rangle_{\mathcal{B}}=b^*\langle x,y\rangle_{\mathcal{B}}$ for $x,y\in\mathfrak{X}$ and $b\in\mathcal{B}$.

Suppose that $(\mathfrak{X},\langle\cdot,\cdot\rangle)$ is a right $\mathcal{B}$-rigged module. By (ii)$_1$ and (ii)$_2$ we have

(ii) $\langle xb_1,yb_2\rangle_{\mathcal{B}}=b_2^*\langle x,y\rangle_{\mathcal{B}}b_1$ for $x,y\in\mathfrak{X}$ and $b_1,b_2\in\mathcal{B}$.

Suppose that ρ is a $*$-representation of $\mathcal{B}$ on $(\mathcal{D}(\rho),\langle\cdot,\cdot\rangle)$. Let $\mathfrak{X}\otimes_{\mathcal{B}}\mathcal{D}(\rho)$ denote the quotient of the tensor product $\mathfrak{X}\otimes\mathcal{D}(\rho)$ over $\mathbb{C}$ by the subspace

$$\mathcal{N}_\rho=\left\{\sum_{k=1}^r x_kb_k\otimes\varphi_k-\sum_{k=1}^r x_k\otimes\rho(b_k)\varphi_k;\ x_k\in\mathfrak{X},\ b_k\in\mathcal{B},\ \varphi_k\in\mathcal{D}(\rho),\ r\in\mathbb{N}\right\}.$$

Lemma 1

$$\left\langle\sum_k x_k\otimes\varphi_k,\sum_l y_l\otimes\psi_l\right\rangle_0:=\sum_{k,l}\langle\rho(\langle x_k,y_l\rangle_{\mathcal{B}})\varphi_k,\psi_l\rangle,\tag{4}$$

where $x_k,y_l\in\mathfrak{X}$ and $\varphi_k,\psi_l\in\mathcal{D}(\rho)$, is a well-defined Hermitian sesquilinear form $\langle\cdot,\cdot\rangle_0$ on the tensor products $\mathfrak{X}\otimes\mathcal{D}(\rho)$ and $\mathfrak{X}\otimes_{\mathcal{B}}\mathcal{D}(\rho)$.

Proof Obviously, $\langle\cdot,\cdot\rangle_0$ is well-defined on the tensor product $\mathfrak{X}\otimes\mathcal{D}(\rho)$ over $\mathbb{C}$. To prove that $\langle\cdot,\cdot\rangle_0$ is also well-defined on the tensor product $\mathfrak{X}\otimes_{\mathcal{B}}\mathcal{D}(\rho)$ it suffices to show that $\langle\zeta,\eta\rangle_0=0$ and $\langle\eta,\zeta\rangle_0=0$ for arbitrary vectors $\eta=\sum y_j\otimes\psi_j\in\mathfrak{X}\otimes\mathcal{D}(\rho)$ and $\zeta=\sum_k x_kb_k\otimes\varphi_k-\sum_k x_k\otimes\rho(b_k)\varphi_k\in\mathcal{N}_\rho$. From (ii)$_1$ we obtain

$$\sum_{k,l}\langle\rho(\langle x_kb_k,y_l\rangle_{\mathcal{B}})\varphi_k,\psi_l\rangle=\sum_{k,l}\langle\rho(\langle x_k,y_l\rangle_{\mathcal{B}})\rho(b_k)\varphi_k,\psi_l\rangle.$$

Using condition (i) it follows from the latter that $\langle\zeta,\eta\rangle_0=0$. Similarly, (i) and (ii)$_2$ yield $\langle\eta,\zeta\rangle_0=0$. Condition (i) implies that $\langle\cdot,\cdot\rangle_0$ is Hermitian (that is $\langle\zeta,\eta\rangle_0=\overline{\langle\eta,\zeta\rangle_0}$ for all $\zeta,\eta\in\mathfrak{X}\otimes\mathcal{D}(\rho)$ resp. $\zeta,\eta\in\mathfrak{X}\otimes_{\mathcal{B}}\mathcal{D}(\rho)$.) □

Let $\mathcal{C}$ be the set of finite sums of elements $\langle x,x\rangle_{\mathcal{B}}$, where $x\in\mathcal{X}$. Then $\mathcal{C}$ is a pre-quadratic module of the $*$-algebra $\mathcal{B}$. Indeed, condition (ii) implies that $b^*cb\in\mathcal{C}$ for $b\in\mathcal{B}$ and $c\in\mathcal{C}$.

Let $\mathrm{Rep}_c\mathcal{B}$ denote the family of all direct sums of cyclic $*$-representations of $\mathcal{B}$. Note that each cyclic $*$-representation is obviously non-degenerate.

Lemma 2 *If $\rho \in \mathrm{Rep}_c\mathcal{B}$ and ρ is $\mathcal{C}$-positive, then $\langle\cdot,\cdot\rangle_0$ is a nonnegative sesquilinear form on $\mathfrak{X} \otimes_{\mathcal{B}} \mathcal{D}(\rho)$.*

Proof Assume first that ρ is a cyclic representation with a cyclic vector $\xi \in \mathcal{D}(\rho)$. Take $\eta = \sum_{k=1}^{n} x_k \otimes \psi_k \in \mathfrak{X} \otimes_{\mathcal{B}} \mathcal{D}(\rho)$ and fix $\varepsilon > 0$. Since ξ is cyclic, there exist $b_1, \ldots, b_n \in \mathcal{B}$ such that $\|\rho(b_k)\xi - \psi_k\| < \varepsilon$ and $\|\rho(\langle x_k, x_l\rangle_{\mathcal{B}})(\rho(b_k)\xi - \psi_k)\| < \varepsilon$ for all $k, l = 1, \ldots, n$. Then for $k, l = 1, \ldots, n$ we get

$$|\langle\rho(\langle x_k, x_l\rangle_{\mathcal{B}})\psi_k, \psi_l\rangle - \langle\rho(\langle x_k, x_l\rangle_{\mathcal{B}})\rho(b_k)\xi, \rho(b_l)\xi\rangle| \leq$$

$$\leq |\langle\rho(\langle x_k, x_l\rangle_{\mathcal{B}})\psi_k, \psi_l - \rho(b_l)\xi\rangle| + |\langle\rho(\langle x_k, x_l\rangle_{\mathcal{B}})(\rho(b_k)\xi - \psi_k), \rho(b_l)\xi\rangle| \leq$$

$$\leq \|\rho(\langle x_k, x_l\rangle_{\mathcal{B}})\psi_k\|\,\varepsilon + \|\rho(b_l)\xi\|\,\varepsilon \leq \|\rho(\langle x_k, x_l\rangle_{\mathcal{B}})\psi_k\|\,\varepsilon + \|\psi_l\|\,\varepsilon + \varepsilon^2.$$

Therefore $\langle\eta,\eta\rangle_0 = \sum_{k,l=1}^{n}\langle\rho(\langle x_k, x_l\rangle_{\mathcal{B}})\psi_k, \psi_l\rangle$ can be approximated as small as we want by

$$\sum_{k,l=1}^{n}\langle\rho(\langle x_k, x_l\rangle_{\mathcal{B}})\rho(b_k)\xi, \rho(b_l)\xi\rangle = \sum_{k,l=1}^{n}\langle\rho(\langle x_k b_k, x_l b_l\rangle_{\mathcal{B}})\xi, \xi\rangle$$

$$= \left\langle \rho\left(\left\langle \sum_{k=1}^{n} x_k b_k, \sum_{k=1}^{n} x_k b_k \right\rangle_{\mathcal{B}}\right)\xi, \xi\right\rangle,$$

which is nonnegative. This implies that $\langle\eta,\eta\rangle_0$ is also nonnegative.

In the case when ρ is a direct sum of cyclic representations ρ_i use the equality $\mathfrak{X} \otimes_{\mathcal{B}} \mathcal{D}(\rho) = \sum_i \mathfrak{X} \otimes_{\mathcal{B}} \mathcal{D}(\rho_i)$. □

Remark There is a counter-part of Lemma 2 for ∗-representations ρ of $\mathcal{B}$ which are not necessarily direct sums of cyclic ∗-representations. If ρ is *non-degenerate* and *completely positive* with respect to the corresponding matrix ordering (see [39], 11.1 and 11.2, for this concept), then the sesquilinear form $\langle\cdot,\cdot\rangle_0$ is nonnegative on $\mathfrak{X} \otimes \mathcal{D}(\rho)$ resp. $\mathfrak{X} \otimes_{\mathcal{B}} \mathcal{D}(\rho)$.

2.2

Now let $\mathcal{A}$ be another ∗-algebra.

Definition 2 *A right $\mathcal{B}$-rigged left $\mathcal{A}$-module* is a right $\mathcal{B}$-rigged module $(\mathfrak{X}, \langle\cdot,\cdot\rangle_{\mathcal{B}})$ which is a left $\mathcal{A}$-module such that

(iii) $\langle ax, y\rangle_{\mathcal{B}} = \langle x, a^* y\rangle_{\mathcal{B}}$ for $a \in \mathcal{A},\ x, y \in \mathfrak{X}$.

A *right $\mathcal{B}$-rigged $\mathcal{A}$–$\mathcal{B}$-bimodule* is a right $\mathcal{B}$-rigged left $\mathcal{A}$-module satisfying

(iv) $(ax)b = a(xb)$ for $a \in \mathcal{A},\ b \in \mathcal{B},\ x \in \mathfrak{X}$.

Lemma 3 *Suppose $(\mathfrak{X}, \langle\cdot,\cdot\rangle_{\mathcal{B}})$ is a right $\mathcal{B}$-rigged left $\mathcal{A}$-module (resp. $\mathcal{A}-\mathcal{B}$-bimodule). Then*

$$\pi_0(a)\left(\sum_k x_k \otimes \varphi_k\right) = \sum_k a x_k \otimes \varphi_k,\ a \in \mathcal{A}, \tag{5}$$

where $x_k \in \mathfrak{X}$, $\varphi_k \in \mathcal{D}(\rho)$, *is a well-defined homomorphism of* $\mathcal{A}$ *into the linear mappings of the vector space* $\mathfrak{X} \otimes \mathcal{D}(\rho)$ *(resp.* $\mathfrak{X} \otimes_{\mathcal{B}} \mathcal{D}(\rho)$*) such that*

$$\langle \pi_0(a)\zeta, \eta\rangle_0 = \langle \zeta, \pi_0(a^*)\eta\rangle_0 \text{ for } a \in \mathcal{A},\ \zeta, \eta \in \mathfrak{X} \otimes \mathcal{D}(\rho)$$
$$\text{resp. } \zeta, \eta \in \mathfrak{X} \otimes_{\mathcal{B}} \mathcal{D}(\rho). \tag{6}$$

Proof Since $\mathfrak{X}$ is a left $\mathcal{A}$-module, π_0 is an algebra homomorphism into $L(\mathfrak{X} \otimes \mathcal{D}(\rho))$. Equation 6 follows then immediately by combining Eqs. 4, 5 and Definition 2 (iv).

If $\mathfrak{X}$ is an $\mathcal{A} - \mathcal{B}$-bimodule, π_0 is well-defined on $\mathfrak{X} \otimes_{\mathcal{B}} \mathcal{D}(\rho)$, since by (iv) we have

$$\pi_0(a)\left(\sum_k x_k b_k \otimes \varphi_k\right) = \sum_k a(x_k b_k) \otimes \varphi_k = \sum_k (a x_k) b_k \otimes \varphi_k$$
$$= \sum_k a x_k \otimes \rho(b_k)\varphi_k = \pi_0(a)\left(\sum_k x_k \otimes \rho(b_k)\varphi_k\right).$$

□

Lemma 4 *Suppose* $\mathfrak{X}$ *is a right* $\mathcal{B}$*-rigged left* $\mathcal{A}$*-module and* ρ *is a* $*$*-representation of* $\mathcal{B}$ *such that the sesquilinear form* $\langle \cdot, \cdot \rangle_0$ *on* $\mathfrak{X} \otimes_{\mathcal{B}} \mathcal{D}(\rho)$ *is nonnegative. Let* $\langle \cdot, \cdot \rangle$ *be the scalar product on the quotient space* $\mathcal{D}(\pi_0) := (\mathfrak{X} \otimes_{\mathcal{B}} \mathcal{D}(\rho))/\mathcal{K}_\rho$ *defined by* $\langle [\eta], [\zeta] \rangle = \langle \eta, \zeta \rangle_0$*, where* $\mathcal{K}_\rho := \{\eta : \langle \eta, \eta \rangle_0 = 0\}$ *and* $[\eta] := \eta + \mathcal{K}_\rho$*. Then*

$$\pi_0(a)[\eta] = [\pi_0(a)\eta],\ a \in \mathcal{A},\ \eta \in \mathfrak{X} \otimes \mathcal{D}(\rho),$$

defines a $*$*-representation* π_0 *of* $\mathcal{A}$ *on the pre-Hilbert space* $(\mathcal{D}(\pi_0), \langle \cdot, \cdot \rangle)$.

Proof Because of Lemma 3 it suffices to check that $\pi(a)$ is well-defined on $\mathcal{D}(\pi_0)$, that is, $\pi_0(a)\mathcal{K}_\rho \subseteq \mathcal{K}_\rho$. Let $\eta \in \mathcal{K}_\rho$. Using Eq. 6 and the Cauchy–Schwarz inequality for the nonnegative sesquilinear form $\langle \cdot, \cdot \rangle_0$ we obtain

$$\langle \pi_0(a)\eta, \pi_0(a)\eta \rangle_0 = \langle \eta, \pi_0(a^*)\pi_0(a)\eta \rangle_0 = \langle \eta, \pi_0(a^*a)\eta \rangle_0 \leq$$
$$\leq \langle \eta, \eta \rangle_0^{1/2} \langle \pi_0(a^*a)\eta, \pi_0(a^*a)\eta \rangle_0^{1/2} = 0.$$

That is, $\pi_0(a)\eta \in \mathcal{K}_\rho$. □

Let π denote the closure of the $*$-representation π_0 from Lemma 4.

Definition 3 We say the $*$-representation π of $\mathcal{A}$ *is induced from the* $*$*-representation* ρ *of* $\mathcal{B}$ *via the right* $\mathcal{B}$*-rigged left* $\mathcal{A}$*-module* $\mathfrak{X}$ or simply π is *induced from* ρ. A $*$-representation ρ of $\mathcal{B}$ is called *inducible* (from $\mathcal{B}$ to $\mathcal{A}$) if the sesquilinear form (4) is nonnegative.

We denote π by $\mathrm{Ind}_{\mathcal{B}\uparrow\mathcal{A}}\rho$ or simply by $\mathrm{Ind}\rho$ if no confusion can arise. The main assertions of the preceding lemmas are summarized by the following proposition.

Proposition 1 *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are ∗-algebras and $\mathfrak{X}$ is a right $\mathcal{B}$-rigged left $\mathcal{A}$-module. If ρ is a ∗-representation of $\mathcal{B}$ such that the sesquilinear form $\langle \cdot, \cdot \rangle_0$ on $\mathfrak{X} \otimes \mathcal{D}(\rho)$ given by Eq. 4 is nonnegative, then* Indρ *is a closed ∗-representation of $\mathcal{A}$ defined on the core $(\mathfrak{X} \otimes \mathcal{D}(\rho))/\mathcal{K}_\rho$ by*

$$\mathrm{Ind}\rho(a)\left[\sum_k x_k \otimes \varphi_k\right] = \left[\sum_k a x_k \otimes \varphi_k\right], \text{ where } a \in \mathcal{A}, x_k \in \mathfrak{X}, \varphi_k \in \mathcal{D}(\rho).$$

If ρ is a $\mathcal{C}$-positive ∗-representation from $\mathrm{Rep}_c\mathcal{B}$, *then the form $\langle \cdot, \cdot \rangle_0$ is nonnegative and hence the induced representation* Indρ *exists. If $\mathfrak{X}$ is a right $\mathcal{B}$-rigged $\mathcal{A} - \mathcal{B}$–bimodule, then the core $(\mathfrak{X} \otimes \mathcal{D}(\rho))/\mathcal{K}_\rho$ is a quotient of the tensor product $\mathfrak{X} \otimes_{\mathcal{B}} \mathcal{D}(\rho)$.*

For applications the following proposition is convenient.

Proposition 2 *Let $\mathcal{A}$ and $\mathcal{B}$ be ∗-algebras and let $\mathcal{X}$ be a right $\mathcal{B}$-rigged left $\mathcal{A}$-module. Let ρ be a ∗-representation of $\mathcal{B}$. Assume that there exists a Hilbert space $(\mathcal{H}_1, \langle \cdot, \cdot \rangle_1)$ and a (well-defined) linear mapping $\Phi : \mathcal{X} \otimes \mathcal{D}(\rho) \to \mathcal{H}_1$ such that $\mathcal{D}_1 := \Phi(\mathcal{X} \otimes \mathcal{D}(\rho))$ is dense in $\mathcal{H}_1$ and*

$$\langle \Phi(x \otimes \varphi), \Phi(y \otimes \psi) \rangle_1 = \langle \rho(\langle x, y \rangle_{\mathcal{B}})\varphi, \psi \rangle, \quad x, y \in \mathcal{X}, \varphi, \psi \in \mathcal{D}(\rho). \tag{7}$$

Then ρ is inducible and Indρ *is unitarily equivalent to the closure of the ∗-representation π_1 on $\mathcal{D}_1$ defined by $\pi_1(a)(\Phi(x \otimes \varphi)) = \Phi(ax \otimes \varphi)$, where $a \in \mathcal{A}, x \in \mathcal{X}, \varphi \in \mathcal{D}(\rho)$.*

Proof Define a the linear mapping U of $\mathcal{X} \otimes \mathcal{D}(\rho)$ onto $\mathcal{D}_1$ by $U(\eta) = \Phi(\sum_k x_k \otimes \varphi_k)$ for $\eta = \sum_k x_k \otimes \varphi_k$. Comparing Eqs. 4 and 7 we see that the form (4) is nonnegative, so ρ is inducible. Further it follows that $\eta \in \mathcal{K}_\rho$ if and only if $\Phi(\sum_k x_k \otimes \varphi_k) = 0$. Hence U yields an isometric linear mapping, denoted again by U, of the unitary space $((\mathfrak{X} \otimes \mathcal{D}(\rho))/\mathcal{K}_\rho, \langle \cdot, \cdot \rangle)$ onto the unitary space $(\mathcal{D}_1, \langle \cdot, \cdot \rangle_1)$ such that $\pi_1(a) = U\, \mathrm{Ind}\rho(a) U^{-1}, a \in \mathcal{A}$. □

Remark Above we have defined induced representations for a right $\mathcal{B}$-rigged left $\mathcal{A}$-module $\mathcal{X}$. However, except for Example 2 in all applications below $\mathcal{X}$ is even a right $\mathcal{B}$-rigged $\mathcal{A}$–$\mathcal{B}$-bimodule. Moreover, if $\mathcal{X}$ is a right $\mathcal{B}$-rigged left $\mathcal{A}$-module, then using the axioms $(\mathrm{ii})_1$ and (iii) we compute

$$\langle (ax)b - a(xb), y \rangle_{\mathcal{B}} = \langle ax, y \rangle_{\mathcal{B}} b - \langle xb, a^* y \rangle_{\mathcal{B}} = \langle x, a^* y \rangle_{\mathcal{B}} b - \langle x, a^* y \rangle_{\mathcal{B}} b = 0.$$

for arbitrary $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $x, y \in \mathcal{X}$. That is, all elements $(ax)b - a(xb)$ are annihilated by $\mathcal{X}$ with respect to the $\mathcal{B}$-valued form $\langle \cdot, \cdot \rangle_{\mathcal{B}}$. In particular, if this form is nondegenerate, then the right $\mathcal{B}$-rigged left $\mathcal{A}$-module $\mathcal{X}$ is a right $\mathcal{B}$-rigged $\mathcal{A}$–$\mathcal{B}$-bimodule.

The following lemma is needed in Section 7 below.

Lemma 5 *Suppose $\mathfrak{X}$ is a right $\mathcal{B}$-rigged left $\mathcal{A}$-module (resp. $\mathcal{A} - \mathcal{B}$-bimodule) and ρ is an inducible cyclic ∗-representation of $\mathcal{B}$ with cyclic vector $v \in \mathcal{D}(\rho)$. Then the linear subspace of vectors $[x \otimes v]$, where $x \in \mathfrak{X}$, is a core of $\pi =$ Indρ.*

Proof It suffices to show that for arbitrary $\varepsilon > 0$, $a \in \mathcal{A}$, $x \in \mathfrak{X}$, and $w \in \mathcal{D}(\rho)$ there exists $b \in \mathcal{B}$ such that $\|\pi(a)([x \otimes w] - [x \otimes \rho(b)v])\| < \varepsilon$. Since v is cyclic, there is a $b \in \mathcal{B}$ such that $\|\rho(\langle ax, ax\rangle_{\mathcal{B}})(\rho(b)v - w)\| < \varepsilon$ and $\|\rho(b)v - w\| < \varepsilon$. Using the Cauchy–Schwarz inequality we get

$$\|\pi(a)([x \otimes w] - [x \otimes \rho(b)v])\|^2 = \|[ax \otimes (w - \rho(b)v)]\|^2$$

$$= \langle \rho(\langle ax, ax\rangle_{\mathcal{B}})(w - \rho(b)v), (w - \rho(b)v)\rangle_0 < \varepsilon^2.$$

□

The next lemma is a standard fact about induced representations. We omit its simple proof.

Lemma 6 *Suppose $\mathfrak{X}$ is a right $\mathcal{B}$-rigged left $\mathcal{A}$-module (resp. $\mathcal{A} - \mathcal{B}$-bimodule) and ρ is a $*$-representation of $\mathcal{B}$. Assume that ρ is a direct sum of representations ρ_i, $i \in I$. Then ρ is inducible if and only if each ρ_i is inducible. Moreover,* $\mathrm{Ind}\rho = \oplus_{i\in I}\mathrm{Ind}\rho_i$.

We close this section by showing that the considerations of [40] fit nicely into the theory of induced representations.

Example 2 Compatible pairs in the sense of [40].

Let $\mathcal{A}$ and $\mathcal{B}$ be two $*$-algebras. Following [40], we call $(\mathcal{A}, \mathcal{B})$ a *compatible pair* if $\mathcal{B}$ is a left $\mathcal{A}$-module, with a left action denoted by $\triangleright$, such that

$$(a \triangleright b)^* c = b^*(a^* \triangleright c) \text{ for } a \in \mathcal{A} \text{ and } b \in \mathcal{B}. \tag{8}$$

Now let $(\mathcal{A}, \mathcal{B})$ be such a compatible pair. We equip $\mathfrak{X} = \mathcal{B}$ with the $\mathcal{B}$-valued sesquilinear form $\langle b, c\rangle_{\mathcal{B}} := c^* b$, $b, c \in \mathcal{B}$, and with the right $\mathcal{B}$-action given by the multiplication. Then $(\mathfrak{X}, \langle\cdot,\cdot\rangle_{\mathcal{B}})$ is a right $\mathcal{B}$-rigged left $\mathcal{A}$-module. Indeed, axioms (i) and $(\mathrm{ii})_2$ are obvious. Axiom (iii) follows from Eq. 8, since for arbitrary $a \in \mathcal{A}$ and $b, c \in \mathcal{B}$ we have

$$\langle a \triangleright b, c\rangle_{\mathcal{B}} = c^*(a \triangleright b) = (a^* \triangleright c)^* b = \langle b, a^* \triangleright c\rangle_{\mathcal{B}}.$$

Suppose that $\rho \in \mathrm{Rep}_c\, \mathcal{B}$. Since bounded $*$-representations acting on the whole Hilbert space are obviously in $\mathrm{Rep}_c\, \mathcal{B}$, this covers all representations of $\mathcal{B}$ considered in [40]. Since the pre-quadratic module $\mathcal{C}$ for the form $\langle\cdot,\cdot\rangle_{\mathcal{B}}$ is $\sum \mathcal{B}^2$, ρ is $\mathcal{C}$-positive. Therefore, by Proposition 1, ρ induces a $*$-representation $\pi = \mathrm{Ind}\rho$ of $\mathcal{A}$. We shall give a more explicit description of this representation π expressed by formula (9) below.

Clearly, an element $\zeta = \sum b_k \otimes \varphi_k \in \mathfrak{X} \otimes \mathcal{D}(\rho)$ belongs to the kernel $\mathcal{K}_\rho$ of the sesquilinear form $\langle\cdot,\cdot\rangle_0$ if and only if

$$\langle \zeta, \zeta\rangle_0 = \sum_{k,l} \langle \rho(\langle b_k, b_l\rangle_{\mathcal{B}})\varphi_k, \varphi_l\rangle = \left\langle \sum_k \rho(b_k)\varphi_k, \sum_l \rho(b_l)\varphi_l \right\rangle = 0$$

or equivalently if $\sum_k \rho(b_k)\varphi_k = 0$. Hence $\mathcal{K}_\rho$ is the kernel of the mapping

$$\mathcal{B} \otimes \mathcal{D}(\rho) \ni \sum_k b_k \otimes \varphi_k \mapsto \sum_k \rho(b_k)\varphi_k \in \rho(\mathcal{B})\mathcal{D}(\rho),$$

so we have an isomorphism of vector spaces $\mathcal{D}(\pi_0) = (\mathcal{B} \otimes \mathcal{D}(\rho))/\mathcal{K}_\rho$ and $\rho(\mathcal{B})\mathcal{D}(\rho)$. If we identify $\mathcal{D}(\pi_0)$ and $\rho(\mathcal{B})\mathcal{D}(\rho)$ by identifying $b \otimes \varphi$ and $\rho(b)\varphi$, then we have

$$\pi(a)\left(\sum_k \rho(b_k)\varphi_k\right) = \pi_0(a)\left(\sum_k \rho(b_k)\varphi_k\right) = \sum_k \rho(a \triangleright b_k)\varphi_k \tag{9}$$

for $a \in \mathcal{A}$. This formula shows that the $*$-representation π_0 and its closure $\pi = \mathrm{Ind}\rho$ as defined above are precisely the $*$-representations $\tilde{\rho}$ and ρ' as defined in [40], Proposition 1.1. That is, *all well-behaved $*$-representations ρ' of $\mathcal{A}$ associated with the compatible pair $(\mathcal{A}, \mathcal{B})$ in the sense of* [40] *are induced $*$-representations* Ind ρ. Note that the well-behaved $*$-representations in the sense of [40] are closely related to representations constructed from unbounded C^*-seminorms (see [1], Chapter 8, for details).

In [40] a number of examples of compatible pairs are developed. A typical example of a compatible pair $(\mathcal{A}, \mathcal{B})$ is obtained as follows: $\mathcal{B}$ is the $*$-algebra $C_0^\infty(G)$ of a Lie group G with convolution multiplication, $\mathcal{A}$ is the enveloping algebra $\mathcal{U}(g)$ of the Lie algebra $\mathfrak{g}$ of G and $x \triangleright f$ is the action of $x \in \mathcal{U}(\mathfrak{g})$ as a right-invariant differential operator on $f \in C_0^\infty(G)$. Note that as in all other examples of compatible pairs treated in [40] the $*$-algebra $\mathcal{B}$ has no unit.

Moreover, all examples described in [40] are of the following form: $\mathcal{A}$ and $\mathcal{B}$ are $*$-subalgebras of a common unital $*$-algebra $\mathfrak{A}$ and the left action of $a \in \mathcal{A}$ on $b \in \mathcal{B}$ is just the multiplication in the larger algebra $\mathfrak{A}$. In this case it follows at once from the $*$-algebra axioms that condition (8) is valid and that $(\mathfrak{X}, \langle\cdot,\cdot\rangle_{\mathcal{B}})$ is a right $\mathcal{B}$-rigged $\mathcal{A} - \mathcal{B}$–bimodule.

3 Conditional Expectations

In the rest of this paper we assume that $\mathcal{B}$ is a unital $*$-subalgebra of a unital $*$-algebra $\mathcal{A}$.

Most examples of rigged modules are derived from conditional expectations. This is a fundamental concept for this paper. Since positivity will play a crucial role in what follows, we require various versions of this notion.

Definition 4 A linear map $p : \mathcal{A} \to \mathcal{B}$ is called a *conditional expectation* of $\mathcal{A}$ onto $\mathcal{B}$ if

(i) $p(a^*) = p(a)^*$, $p(b_1 a b_2) = b_1 p(a) b_2$ for all $a \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$, $p(\mathbf{1}_\mathcal{A}) = \mathbf{1}_\mathcal{B}$, and p is positive in the sense that
(ii) $p(\sum \mathcal{A}^2) \subseteq \sum \mathcal{A}^2 \cap \mathcal{B}$.

A linear map p satisfying only condition (i) is called a $\mathcal{B}$*-bimodule projection* of $\mathcal{A}$ onto $\mathcal{B}$.

A conditional expectation p will be called a *strong conditional expectation* if

$(\mathrm{ii})_1$ $p(\sum \mathcal{A}^2) \subseteq \sum \mathcal{B}^2$.

Let $\mathcal{C}_\mathcal{A}$ and $\mathcal{C}_\mathcal{B}$ be pre-quadratic modules of $\mathcal{A}$ resp. $\mathcal{B}$. A $\mathcal{B}$-bimodule projection p will be called $(\mathcal{C}_\mathcal{A}, \mathcal{C}_\mathcal{B})$*-conditional expectation* of $\mathcal{A}$ onto $\mathcal{B}$ if

$(\mathrm{ii})_2$ $p(\mathcal{C}_\mathcal{A}) \subseteq \mathcal{C}_\mathcal{B}$.

Note that axiom (i) implies that any *$\mathcal{B}$-bimodule projection* of $\mathcal{A}$ onto $\mathcal{B}$ is indeed a projection of $\mathcal{A}$ onto $\mathcal{B}$.

The bridge of these notions to rigged modules is given by the following simple lemma.

Lemma 7 *Suppose that $p : \mathcal{A} \to \mathcal{B}$ is a $\mathcal{B}$-bimodule projection of $\mathcal{A}$ onto $\mathcal{B}$ and define $\langle b, c\rangle_{\mathcal{B}} := p(c^*b)$ for $b, c \in \mathcal{B}$ and $\mathfrak{X} := \mathcal{A}$. Then $(\mathfrak{X}, \langle\cdot,\cdot\rangle_{\mathcal{B}})$ is a right $\mathcal{B}$-rigged $\mathcal{A} - \mathcal{B}$-bimodule with left and right actions given by the multiplications in $\mathcal{A}$.*

Proof Conditions (i), $(\text{ii})_1$, (iii) and (iv) in Definitions 1 and 2 follow at once from (i) in Definition 4 and the $*$-algebra axioms. For instance, we verify $(\text{ii})_1$. If $x, y \in \mathfrak{X}(= \mathcal{A})$ and $b \in \mathcal{B}$, then using axiom (i) in Definition 4 we have $\langle xb, y\rangle_{\mathcal{B}} = p(y^*xb) = p(y^*x)b = \langle x, y\rangle_{\mathcal{B}} b$. □

Definition 5 A $\mathcal{B}$-bimodule projection p of $\mathcal{A}$ onto $\mathcal{B}$ is called *faithful* if $p(x^*x) = 0$ for some $x \in \mathcal{A}$ implies that $x = 0$.

The next lemma illustrates the importance of this notion.

Lemma 8 *Suppose that p is a faithful $\mathcal{B}$-bimodule projection of $\mathcal{A}$ onto $\mathcal{B}$. Let $\pi_i,\ i \in I$, be a family of inducible $*$-representations of $\mathcal{B}$ which separates the elements of $\mathcal{B}$. Then the family* $\mathrm{Ind}\pi_i,\ i \in I$, *separates the elements of $\mathcal{A}$.*

Proof Let $a \in \mathcal{A}$, $a \neq 0$. Since p is faithful, $p(a^*a) \neq 0$. Since the family $\pi_i, i \in I$, separate the elements of $\mathcal{B}$, there exist a representation $\pi_{i_0}, i_0 \in I$, and a vector $\varphi \in \mathcal{D}(\pi_{i_0})$ such that $\pi_{i_0}(p(a^*a))\varphi \neq 0$. Then we have $\|\mathrm{Ind}\pi_{i_0}(a)[1 \otimes \varphi]\| = \|\pi_{i_0}(p(a^*a))\varphi\| \neq 0$. □

The following simple proposition is taken from [44]. It characterizes a $\mathcal{B}$-bimodule projection in terms of its kernel.

Proposition 3 *There exists a $\mathcal{B}$-bimodule projection from $\mathcal{A}$ onto $\mathcal{B}$ if and only if there exists a $*$-invariant subspace $\mathcal{T} \subseteq \mathcal{A}$ such that $\mathcal{A} = \mathcal{B} \oplus \mathcal{T}$ and*

$$\mathcal{B}\mathcal{T}\mathcal{B} \subseteq \mathcal{T}. \tag{10}$$

If this is true, the $\mathcal{B}$-bimodule projection p is uniquely defined by the requirement $\ker p = \mathcal{T}$ *and we have $p(\sum \mathcal{A}^2) = \sum \mathcal{B}^2 + p(\sum \mathcal{T}^2)$.*

Proof Let p be a $\mathcal{B}$-bimodule projection from $\mathcal{A}$ onto $\mathcal{B}$ and put $\mathcal{T} = \ker p$. For $t \in \mathcal{T}$ and $b_1, b_2 \in \mathcal{B}$ we have $p(b_1tb_2) = b_1p(t)b_2 = 0$ and $p(t^*) = p(t)^* = 0$, so that $\mathcal{T}$ satisfies Eq. 10 and is $*$-invariant. For arbitrary $a \in \mathcal{A}$ we have $p(a) \in \mathcal{B}$ and $a - p(a) \in \mathcal{T}$, so that $\mathcal{A} = \mathcal{B} \oplus \mathcal{T}$.

Conversely, if $\mathcal{T}$ is given, one easily checks that the linear map p defined by $p(b) = b,\ b \in \mathcal{B}$, and $p(t) = 0,\ t \in \mathcal{T}$, is indeed a $\mathcal{B}$-bimodule projection. □

In the remaining part of this section we develop a number of examples. In the first example we use Proposition 3 to show that there is no $\mathcal{B}$-bimodule projection.

Example 3 Let $\mathcal{A}$ be the Weyl algebra from Example 1. As it is well-known, the Hermitian elements $P = \frac{1}{\sqrt{2}} i(a^* - a)$ and $Q = \frac{1}{\sqrt{2}}(a^* + a)$ satisfy the commutation relation $PQ - QP = -i$.

We show that *there is no $\mathcal{B}$-bimodule projection of $\mathcal{A}$ onto $\mathcal{B} := \mathbb{C}[P]$*. Assume to the contrary there is such a projection p and let $\mathcal{T}$ be its kernel. Then, since $\mathcal{A} = \mathcal{B} \oplus \mathcal{T}$, there exists a polynomial $f \in \mathbb{C}[t]$ such that $Q + f(P) \in \mathcal{T}$. By Eq. 10 we have $PQ + Pf(P)$ and $QP + f(P)P \in \mathcal{T}$ which implies that $PQ - QP = -i \in \mathcal{T}$. Hence $\mathbf{1}_{\mathcal{A}} \in \mathcal{T}$ and so $p = 0$ which is a contradiction.

Using Proposition 3 one can check that the map p defined in Example 1 is the unique $\mathcal{B}$-bimodule projection from $\mathcal{A}$ onto $\mathcal{B} := \mathbb{C}[N]$.

Example 4 Let $q_1, \ldots, q_n \in \mathcal{A}$ be a decomposition of unit of the unital $*$-algebra $\mathcal{A}$, that is, $q_1 + \cdots + q_n = 1$ and $q_i = q_i^2 = q_i^*$ for $i = 1, \ldots, n$. It is not difficult to show that $q_i q_j = 0$ for all $i \neq j$ and that the map

$$p : a \mapsto q_1 a q_1 + \cdots + q_n a q_n$$

is a conditional expectation of $\mathcal{A}$ onto the $*$-subalgebra $\mathcal{B} = \{b \in \mathcal{A} : b = p(b)\}$. If $\mathcal{A}$ is an O^*-algebra, then p is faithful.

Example 5 Suppose that G is a discrete group and H is a subgroup of G. Let $\mathcal{A} = \mathbb{C}[G]$ and $\mathcal{B} = \mathbb{C}[H]$ be the group algebras of G and H, respectively. Recall that the group algebra $\mathbb{C}[G]$ of a discrete group G is a unital $*$-algebra with multiplication given by the convolution and involution determined by the inversion of group elements. More precisely, $\mathbb{C}[G]$ is a complex vector space with basis given by the group elements of G and the product of two base element g and h is just the group product gh and g^* is the inverse g^{-1}. Let p be the canonical projection of $\mathbb{C}[G]$ onto $\mathbb{C}[H]$ defined by $p(g) = g$ if $g \in H$ and $p(g) = 0$ if $g \notin H$.

Proposition 4 *p is a faithful strong conditional expectation of $\mathbb{C}[G]$ onto $\mathbb{C}[H]$.*

Proof It is clear from its definition that p satisfies condition (i) of the Definition 4, so p is a $\mathbb{C}[H]$-bimodule projection.

We shall prove that $p(\sum \mathbb{C}[G]^2) \subseteq \sum \mathbb{C}[H]^2$. Let us fix precisely one element $k_t \in G$ in each left coset $t \in G/H$. Take an arbitrary element $a = \sum_{g \in G} \theta_g g$ of the group algebra $\mathbb{C}[G]$. Then there exist elements $a_i \in \mathbb{C}[H]$, $i \in G/H$, such that $a = \sum_{g \in G} \theta_g g = \sum_{i \in G/H} k_i a_i$. If $i, j \in G/H$ and $i \neq j$, then $k_i^{-1} k_j \notin H$ and hence $p(k_i^{-1} k_j) = 0$. Using this fact we obtain

$$\begin{aligned} p(a^* a) &= p\left(\left(\sum_{i \in G/H} k_i a_i\right)^* \left(\sum_{j \in G/H} k_j a_j\right)\right) = p\left(\sum_{i,j \in G/H} a_i^* k_i^{-1} k_j a_j\right) \\ &= \sum_{i,j \in G/H} p\left(a_i^* k_i^{-1} k_j a_j\right) = \sum_{i,j \in G/H} a_i^* \, p\left(k_i^{-1} k_j\right) a_j = \sum_{i \in G/H} a_i^* a_i, \end{aligned}$$

so $p(a^* a) \in \sum \mathbb{C}[H]^2$. That is, p is a strong conditional expectation.

From the preceding equality it follows also that p is faithful. Indeed, if $p(a^* a) = 0$, then $\sum_i a_i^* a_i = 0$ which implies that $a_i = 0$ for all $i \in G/H$ and hence $a = 0$. □

A large source of conditional expectations is obtained from groups of $*$-automorphisms. The idea is taken from the following standard construction of conditional expectations of C^*-algebras reproduced from [35], Example 1.5.

Example 6 Suppose that $\mathcal{A}$ is a C^*-algebra and G is a compact group such that there is a continuous action $g \mapsto \alpha_g$ of G as automorphism group of $\mathcal{A}$. Let dg denote the normalized Haar measure of G. Then the map

$$a \mapsto \int_G \alpha_g(a)dg, \ a \in \mathcal{A},$$

is a *strong conditional expectation* of $\mathcal{A}$ onto the C^*-subalgebra $\mathcal{B}$ of stable elements.

We now generalize this example to the case of general $*$-algebras.

Example 7 Suppose that G is a compact group which acts by $*$-automorphisms $\alpha_g, \ g \in G$, on a $*$-algebra $\mathcal{A}$. Assume in addition that the action is *locally finite-dimensional*, that is, for every $a \in \mathcal{A}$ there exists a finite-dimensional linear subspace $V \subset \mathcal{A}$ such that $a \in V$, $\alpha_g(V) \subseteq V$ for all $g \in G$, and the map $g \to \alpha_g(a)$ of G into V is continuous. Then the mapping p given by

$$p(a) = \int_G \alpha_g(a)dg, \ a \in \mathcal{A}, \tag{11}$$

is well-defined. One easily verifies that p is a $\mathcal{B}$-bimodule projection from $\mathcal{A}$ onto the $*$-subalgebra $\mathcal{B} := \{a \in \mathcal{A} : \alpha_g(a) = a \text{ for all } g \in G\}$ of stable elements.

Every G-invariant finite-dimensional subspace $V \subseteq \mathcal{A}$ is a unitarizable G-module. Since G is compact, $\mathcal{A}$ is a direct sum of submodules $\mathcal{A}_t, \ t \in \widehat{G}$, where $\mathcal{A}_t$ denotes the direct sum of submodules in $\mathcal{A}$ isomorphic to $t \in \widehat{G}$. In the case when $\mathcal{A}$ is a C^*-algebra, the subspaces $\mathcal{A}_t, \ t \in \widehat{G}$, are called *spectral subspaces,* see e.g. [17] and [9]. The mapping p is nothing but the projection of the direct sum $\mathcal{A} = \oplus_{t \in \widehat{G}} \mathcal{A}_t$ onto the spectral subspace $\mathcal{A}_0$ corresponding to the trivial representation.

An analogue of the map p was considered in [5]. Suppose R is a real closed field, $R[V]$ is the coordinate ring of an affine variety V and G is a linear algebraic group over R acting on $R[V]$. If G is reductive, there is a canonical projection ρ from $R[V]$ onto the subring $R[V]^G$ of G-invariants called *Reynolds operator* (see [5] for details). In the case when $G(R)$ semi-algebraically compact, Corollary 3.6 in [5] states that $\rho(\sum R[V]^2) \subseteq \sum R[V]^2$.

Proposition 5 *The map p defined by Eq. 11 is a conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$.*

Proof It remains to show that $p(\sum \mathcal{A}^2) \subseteq \sum \mathcal{A}^2$. Let $a \in \mathcal{A}$. Then there is a finite-dimensional G-invariant subspace V of $\mathcal{A}$ containing a. Then V is a finite direct sum of submodules $V^{(t)}$, where $V^{(t)}$ is multiple of $t \in \widehat{G}$. Fix $t \in \widehat{G}$ and let $V^{(t)} = \oplus_i V_i^{(t)}$ be a decomposition of $V^{(t)}$ into a direct sum of irreducible G-modules. We can choose an orthonormal base $a_{ij}^{(t)}$ in each space $V_i^{(t)}$ such that the matrices corresponding to α_g are unitary and equal for all i, i.e. we have

$$\alpha_g\left(a_{ij}^{(t)}\right) = \sum_k u_{kj}^{(t)}(g) a_{ik}^{(t)}, \ g \in G, \ t \in \widehat{G}.$$

Let us fix elements $a^{(t)}_{i_1 j_1}$, $a^{(s)}_{i_2 j_2} \in V \subseteq \mathcal{A}$. Using the orthogonality relations of matrix elements $u^{(t)}_{kj_1}$ and $u^{(s)}_{mj_2}$ on the compact group G we compute

$$p\left(\left(a^{(t)}_{i_1 j_1}\right)^* a^{(s)}_{i_2 j_2}\right) = \int \left(\sum_k \overline{u^{(t)}_{kj_1}(g)}\left(a^{(t)}_{i_1 k}\right)^*\right) \cdot \left(\sum_m u^{(s)}_{mj_2}(g) a^{(s)}_{i_2 m}\right) dg =$$
$$= \sum_{k,m} \int \overline{u^{(t)}_{kj_1}(g)} u^{(s)}_{mj_2}(g) dg \cdot \left(a^{(t)}_{i_1 k}\right)^* a^{(s)}_{i_2 m} =$$
$$= \frac{\delta_{ts}\, \delta_{j_1 j_2}}{\dim t} \sum_k \left(a^{(t)}_{i_1 k}\right)^* a^{(t)}_{i_2 k}.$$

Since $a \in V$, we can write a as a finite sum $a = \sum_{i,j,t} \lambda^{(t)}_{ij} a^{(t)}_{ij}$, where $\lambda^{(t)}_{ij} \in \mathbb{C}$. Applying the preceding equality we obtain

$$p(a^* a) = p\left(\sum_{i,j,t} \overline{\lambda^{(t)}_{ij}} \left(a^{(t)}_{ij}\right)^* \cdot \sum_{k,l,s} \lambda^{(s)}_{kl} a^{(s)}_{kl}\right) = \sum_{j,t} p\left(\sum_i \overline{\lambda^{(t)}_{ij}} \left(a^{(t)}_{ij}\right)^* \cdot \sum_k \lambda^{(t)}_{kj} a^{(t)}_{kj}\right) =$$
$$= \sum_{j,t} \frac{1}{\dim t} \left(\sum_i \lambda^{(t)}_{ij} a^{(t)}_{ij}\right)^* \cdot \left(\sum_k \lambda^{(t)}_{kj} a^{(t)}_{kj}\right) \in \sum \mathcal{A}^2.$$

In general this conditional expectation p is not strong, i.e. $p(\sum \mathcal{A}^2)$ is not contained in $\sum \mathcal{B}^2$. □

4 Group Graded $*$-Algebras

The algebraic representation theory of group graded algebras has been extensively studied, see e.g. the books [27] and [26]. The monograph [12] deals with $*$-algebraic bundles which can be considered as generalizations of G-graded $*$-algebras to the case when G is a topological group. However, in [12] only bounded Hilbert space representations are treated. As we shall see below, there are plenty of important G-graded $*$-algebras (Weyl algebra, enveloping algebras etc.) for which most $*$-representations are unbounded.

Definition 6 Let G be a (discrete) group. A *G-graded $*$-algebra* is a $*$-algebra $\mathcal{A}$ which is a direct sum $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ of vector spaces $\mathcal{A}_g$, $g \in G$, such that

$$\mathcal{A}_g \cdot \mathcal{A}_h \subseteq \mathcal{A}_{g\cdot h} \text{ and } (\mathcal{A}_g)^* \subseteq \mathcal{A}_{g^{-1}} \text{ for } g, h \in G. \tag{12}$$

From the two conditions in Eq. 12 it follows that a G-grading of a $*$-algebra $\mathcal{A}$ is completely determined if we know the corresponding components for a set of generators of the algebra $\mathcal{A}$. In what follows we shall describe most of our G-gradings of $*$-algebras in this manner.

Example 8 In this example we use some basics from the theory of semi-simple Lie algebras. All facts we need can be found in the monograph [7], 7.0 and 7.4.1. Suppose that $\mathfrak{g}$ is a semi-simple complex Lie algebra. We denote by $\mathfrak{h}$ a Cartan subalgebra, by

Q the root lattice and by $H_1, \ldots, H_l, X_{-\alpha_1}, \ldots, X_{-\alpha_n}, X_{\alpha_1}, \ldots, X_{\alpha_n}$ a Cartan-Weyl basis of the Lie algebra $\mathfrak{g}$. If we consider the complex universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ as a $\mathfrak{g}$-module and so as an $\mathfrak{h}$-module by the adjoint representation, we obtain a direct sum decomposition $\mathcal{U}(\mathfrak{g}) = \sum_{\lambda \in Q} \mathcal{U}(\mathfrak{g})_\lambda$. This means that $\mathcal{U}(\mathfrak{g})$ is a G-graded algebra, where G is the abelian group Q. If $\mathcal{U}(\mathfrak{g})$ is equipped with an involution such that $(X_{\alpha_j})^* = \varepsilon_j X_{-\alpha_j}$ and $(H_k)^* = H_k$ for all j, k, where $\varepsilon_j \in \{1, -1\}$, then we have $(\mathcal{U}(\mathfrak{g})_\lambda)^* = \mathcal{U}(\mathfrak{g})_{-\lambda}$ and hence $\mathcal{U}(\mathfrak{g})$ is a Q-graded $*$-algebra. The algebra $\mathcal{U}(\mathfrak{g})_0$ is just the commutant of the Cartan algebra $\mathfrak{h}$ in $\mathcal{U}(\mathfrak{g})$. Its structure is described in [7], 7.4.2.

Example 9 Let $\mathcal{F} = \mathbb{C}\langle z_1, \ldots, z_d, w_1, \ldots, w_d \rangle$ be the free polynomial algebra with generators $z_1, \ldots, z_d, w_1, \ldots, w_d$ and involution determined by $(z_j)^* = w_j$, $j = 1, \ldots, d$. Then $\mathcal{F}$ is a $\mathbb{Z}$-graded $*$-algebra with $\mathbb{Z}$-grading given by $z_j \in \mathcal{F}_1$.

To derive further examples we shall use the following lemma. We omit its simple proof.

Lemma 9 *If $\mathcal{F} = \bigoplus_{g \in G} \mathcal{F}_g$ is a G-graded $*$-algebra and $\mathcal{J}$ is a two-sided $*$-ideal of $\mathcal{F}$ generated by subsets of $\mathcal{F}_g$, $g \in G$, then the quotient $*$-algebra $\mathcal{F}/\mathcal{J}$ is also G-graded.*

The proofs of the existence of gradings for all examples occuring in this paper follow the same pattern: We first define the corresponding grading on the free $*$-algebra (Example 9). If the polynomials of the defining relations belong to single components of this grading, Lemma 9 applies and gives the grading of the $*$-algebra. We illustrate this by a number of examples in the last section.

Throughout the rest of this section G is a discrete group with unit element e, H denotes a subgroup of G and $\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ is a unital G-graded $*$-algebra. The subspace $\mathcal{A}_e$ is a $*$-subalgebra of $\mathcal{A}$ which will be denoted by $\mathcal{B}$. Clearly, $\mathbf{1}_{\mathcal{A}} \in \mathcal{B}$, so that $\mathbf{1}_{\mathcal{A}} = \mathbf{1}_{\mathcal{B}}$.

For a subset $X \subseteq G$ we denote by $\mathcal{A}_X$ the linear subspace $\bigoplus_{g \in X} \mathcal{A}_g$ of $\mathcal{A}$. From Eq. 12 we conclude that $\mathcal{A}_H$ is a $*$-subalgebra of $\mathcal{A}$ for the subgroup H of G.

Proposition 6 *Let p_H be the canonical projection of $\mathcal{A}$ onto $\mathcal{A}_H$, that is, $p_H(a) = \sum_{g \in H} a_g$ for $a = \sum_{g \in G} a_g$, where $a_g \in \mathcal{A}_g$. Then p_H is a conditional expectation of $\mathcal{A}$ onto $\mathcal{A}_H$.*

Proof Condition (i) of Definition 4 follows at once from Eq. 12. Our proof is complete once we have shown that $p_H(\sum \mathcal{A}^2) \subseteq \sum \mathcal{A}^2$.

We choose one element $k_i \in G$, $i \in G/H$, in each left coset of H in G. Let $a = \sum_{i \in G/H} b_i$, where $b_i \in \mathcal{A}_{k_i H}$. If $i, j \in G/H$, then $b_j^* b_i \in \mathcal{A}_{H k_j^{-1} k_i H}$, hence we have $p_H(b_i^* b_i) = b_i^* b_i$ and $p_H(b_j^* b_i) = 0$ if $i \neq j$. Using the latter facts we obtain

$$p_H(a^* a) = p_H \left(\sum_{i \in G/H} \sum_{j \in G/H} b_j^* b_i \right) = \sum_{i \in G/H} b_i^* b_i \in \sum \mathcal{A}^2. \tag{13}$$

□

The map p_H from Proposition 6 is called the *canonical conditional expectation* of the G-graded $*$-algebra $\mathcal{A}$ onto the $*$-subalgebra $\mathcal{A}_H$.

Equation 13 shows that p_H is faithful when $\sum_{k=1}^{n} a_k^* a_k = 0$ for arbitrary $a_1, \ldots, a_n \in \mathcal{A}$ implies that $a_1 = \cdots = a_n = 0$. In particular, p_H is faithful when $\mathcal{A}$ is an O^*-algebra.

Another immediate consequence of Eq. 13 is stated as

Corollary 1 *An element $a \in \mathcal{A}$ belongs to the cone $\sum \mathcal{A}^2 \cap \mathcal{A}_H$ if and only if it can be presented as a finite sum of squares $\sum b_i^* b_i$, where each b_i belongs to some $\mathcal{A}_{gH}$, $gH \in G/H$.*

Example 10 Let $\mathcal{A} = \langle a, a^* | aa^* - a^*a = 1 \rangle$ be the Weyl algebra (see Example 1). Then $\mathcal{A}$ is a $\mathbb{Z}$-graded $*$-algebra with $\mathbb{Z}$-grading defined by $a \in \mathcal{A}_1$, $a^* \in \mathcal{A}_{-1}$ and we have $\mathcal{B} = \mathbb{C}[N]$, where $N = a^*a$. We now use Corollary 1 to describe the cone $\sum \mathcal{A}^2 \cap \mathcal{B}$.

Suppose $k \in \mathbb{N}$. Let $a_k \in \mathcal{A}_k$. Then a_k is of the form $a_k = a^k p_k$, where $p_k \in \mathbb{C}[N]$, and

$$a_k^* a_k = p_k^* a^{k*} a^k p_k = N(N-1)\ldots(N-k+1) p_k^* p_k.$$

For $a_{-k} \in \mathcal{A}_{-k}$ we have $a_{-k} = a^{*k} p_{-k}$, where $p_{-k} \in \mathbb{C}[N]$, and

$$a_{-k}^* a_{-k} = p_{-k}^* a^k a^{*k} p_{-k} = (N+1)(N+2)\ldots(N+k) p_{-k}^* p_{-k}.$$

One easily verifies that $a_{-k}^* a_{-k}$ belongs to $\sum \mathcal{B}^2 + N \sum \mathcal{B}^2$. Hence from Corollary 1 we obtain

$$\sum \mathcal{A}^2 \cap \mathcal{B} = \sum \mathcal{B}^2 + N \sum \mathcal{B}^2 + N(N-1) \sum \mathcal{B}^2 + \ldots \tag{14}$$

This result was derived in [14] by other methods. Among others it shows that $\sum \mathcal{A}^2 \cap \mathcal{B} \neq \sum \mathcal{B}^2$ and that the canonical conditional expectation $p : \mathcal{A} \to \mathcal{B}$ is not strong.

Example 11 Let G be a discrete group and H a normal subgroup of G. Then the group algebra $\mathbb{C}[G]$ becomes a G/H-graded $*$-algebra in canonical manner. The canonical conditional expectation coincides with the one from the Example 5, so by Proposition 4 it is strong. In particular, we have $\sum \mathbb{C}[G]^2 \cap \mathbb{C}[H] = \sum \mathbb{C}[H]^2$.

Example 12 Let A be a unital $*$-algebra. Let G be a (discrete) group which acts as $*$-automorphism group $g \mapsto \alpha_g$ on A. Recall that the crossed product $*$-algebra $\mathcal{A} = A \times_\alpha G$ is defined as follows. As a linear space $\mathcal{A}$ is the tensor product $A \otimes \mathbb{C}[G]$ or equivalently the vector space of A-valued functions on G with finite support. Product and involution on $\mathcal{A}$ are determined by $(a \otimes g)(b \otimes h) = a\alpha_g(b) \otimes gh$ and $(a \otimes g)^* = \alpha_{g^{-1}}(a^*) \otimes g^{-1}$, respectively. If we identify b with $b \otimes e$ and g with $1 \otimes g$, then the $*$-algebra $A \times_\alpha G$ can be considered as the universal $*$-algebra generated by the two $*$-subalgebras A and $\mathbb{C}[G]$ with cross commutation relations $gb = \alpha_g(b)g$ for $b \in A$ and $g \in G$. Set $\mathcal{A}_g := A \otimes g$ for $g \in G$. Then $\mathcal{A}$ becomes a G-graded $*$-algebra with canonical conditional expectation p onto $\mathcal{B} = \mathcal{A}_e$ given by $p(a \otimes g) = \delta_{g,e} a \otimes e$.

Proposition 7 *The canonical conditional expectation $p : \mathcal{A} \times_\alpha G \to \mathcal{B}$ is strong.*

Proof Let $x = \sum_{g \in G} a_g \otimes g,\ a_g \in A$, be an element of the $\mathcal{A} \times_\alpha G$. Then

$$p(xx^*) = p\left(\sum_{g \in G}\sum_{h \in G}(a_g \otimes g)(a_h \otimes h)^*\right) = p\left(\sum_{g \in G}\sum_{h \in G} a_g \alpha_{gh^{-1}}\left(a_h^*\right) \otimes gh^{-1}\right) =$$

$$= \sum_{g \in G} a_g a_g^* \otimes e = \sum_{g \in G}(a_g \otimes e)(a_g \otimes e)^* \in \sum \mathcal{B}^2.$$

□

Example 13 Let G be a compact abelian group. Then the dual group $\widehat{G}$ is a discrete abelian group. We now establish a duality between actions of G and $\widehat{G}$-gradings on a $*$-algebra $\mathcal{A}$ (cf. Example 7).

Suppose that an action $\alpha : G \to Aut(\mathcal{A})$ is given. Assume, in addition, that the action is locally finite-dimensional (see Example 7). For $\psi \in \widehat{G},\ \psi : G \to \mathbb{T}$ put

$$\mathcal{A}_\psi = \left\{a \in \mathcal{A} |\ \alpha_g(a) = \psi(g)a,\ \text{for all } g \in G\right\}. \tag{15}$$

If $\mathcal{A}$ is a $\widehat{G}$-graded $*$-algebra, we define an action of $\widehat{\widehat{G}} = G$ on $\mathcal{A}$ as follows. For $a = \sum_{\psi \in \widehat{G}} a_\psi,\ a_\psi \in \mathcal{A}_\psi$ and $g \in G$, define a $*$-automorphism α_g by putting

$$\alpha_g(a) := \sum_{\psi \in G} \psi(g) a_\psi. \tag{16}$$

Proposition 8 *Equations 15 and 16 give a one-to-one correspondence between locally finite-dimensional actions of G on $\mathcal{A}$ and $\widehat{G}$-gradings of $\mathcal{A}$.*

Proof Let $\alpha : G \to Aut(\mathcal{A})$ be locally finite-dimensional action and let $\mathcal{A}_\psi$ be defined by Eq. 15. We consider $\mathcal{A}$ as G-module and $\mathcal{A}_\psi$ as unitary G-submodule. Take a finite-dimensional α-invariant linear subspace V of $\mathcal{A}$. Since G is compact, V is unitarizable and hence spanned by its subspaces $\mathcal{A}_\psi$. Since the action of G is locally finite-dimensional, $\mathcal{A}$ is spanned by such subspaces V and so by $\mathcal{A}_\psi,\ \psi \in \widehat{G}$. It is easily checked that $\mathcal{A} = \oplus_{\psi \in \widehat{G}} \mathcal{A}_\psi$ is a $\widehat{G}$-grading of $\mathcal{A}$.

Conversely, suppose $\mathcal{A}$ is a $\widehat{G}$-graded $*$-algebra. It is clear that Eq. 16 defines an action of G on $\mathcal{A}$. Each element $a \in \mathcal{A}$ is of the form $a = \sum_{i=1}^k a_{\psi_i}$, where $a_{\psi_i} \in \mathcal{A}_{\psi_i}$ and the elements $\psi_i \in \widehat{G}$ are pairwise distinct. The elements a_{ψ_i} span a finite-dimensional subspace of $\mathcal{A}$ which is obviously invariant under the action (Eq. 16). Hence the action (Eq. 16) is locally finite-dimensional. □

Remark For the study of modules over a G-graded ring $\mathcal{A} = \oplus_{g \in G}\mathcal{A}_g$, it is usually assumed that for all $g, h \in G$ the linear span of $\mathcal{A}_g\mathcal{A}_h$ is equal to $\mathcal{A}_{gh}$, see [26, 27]. Likewise in [12] it is supposed that this linear span is dense in $\mathcal{A}_{gh}$. We have not made such an assumption, because it is not satisfied in most of our standard examples. For instance, if $\mathcal{A}$ is the Weyl algebra (Example 10), then we have $\mathcal{B} = \mathbb{C}[N]$, $\mathcal{A}_1 = a\mathcal{B}$ and $\mathcal{A}_{-1} = a^*\mathcal{B} = \mathcal{B}a^*$. Therefore, the linear span of $\mathcal{A}_{-1} \cdot \mathcal{A}_1$ is equal to $N \cdot \mathbb{C}[N]$ which is different from $\mathcal{B}$.

5 Systems of Imprimitivity

Let $\mathcal{A} = \oplus_{g\in G} \mathcal{A}_g$ be a G-graded $*$-algebra. We retain the notation of the previous section. Recall that for a subgroup $H \subseteq G$, the left G-space of left H-cosets is denoted by G/H.

Definition 7 Let π be a $*$-representation of the $*$-algebra $\mathcal{A}$ and let E be a mapping from the set G/H to the set of projections of the underlying Hilbert space $\mathcal{H}_\pi$ such that

(i) $E(t_1)E(t_2) = 0$ for all $t_1, t_2 \in G/H,\ t_1 \neq t_2$, and $\sum_{t\in G/H} E(t) = I$,
(ii) $E(gt)\pi(a_g) \subseteq \pi(a_g)E(t)$ for all $g \in G,\ t \in G/H,\ a_g \in \mathcal{A}_g$.

We call the pair (π, E) a *system of imprimitivity* for the algebra $\mathcal{A}$ over G/H.

Let (π, E) be a system of imprimitivity. Let $t \in G/H$ and set $\mathcal{D}_t(\pi) := \operatorname{Ran} E(t) \cap \mathcal{D}(\pi)$. The conditions in Definition 7 imply that

$$E(t)\mathcal{D}(\pi) \subseteq \mathcal{D}(\pi),\ \ \pi(\mathcal{A}_g)\mathcal{D}_t(\pi) \subseteq \mathcal{D}_{gt}(\pi) \text{ for } g \in G,\ \text{ and } \mathcal{D}(\pi) \subseteq \widetilde{\oplus}_{t\in G/H}\mathcal{D}_t(\pi),$$

where $\widetilde{\oplus}$ denotes the direct Hilbert sum.

A system of imprimitivity (π, E) is called *non-degenerate* if for all $t \in G/H$ the subspace $\pi(\mathcal{A}_t)\mathcal{D}_H(\pi)$ is dense in $\mathcal{D}_t(\pi)$ with respect to the graph topology of π. Otherwise, we say that (π, E) is *degenerate*.

Lemma 10 *Let H be a subgroup of G and let (π, E) be a system of imprimitivity for the algebra $\mathcal{A}$ over G/H. Then the pair $(\overline{\pi}, E)$ is again a system of imprimitivity for $\mathcal{A}$ over G/H. Moreover, if (π, E) is non-degenerate, then $(\overline{\pi}, E)$ is also non-degenerate.*

Proof From condition (ii) we obtain $\|\pi(a_g)E(t)\varphi\| \leq \|\pi(a_g)\varphi\|$ for $a_g \in \mathcal{A}_g$ and $\varphi \in \mathcal{D}(\pi)$. This shows that $E(t)$ is a continuous mapping of $\mathcal{D}(\pi)$ with respect to the graph topology of π. Hence condition (2) extends by continuity to the closure $\overline{\pi}$ of π. Obviously, $(\overline{\pi}, E)$ is non-degenerate if (π, E) is. □

Systems of imprimitivity arise from induced representations in the following way (see e.g. [12], p. 1248, for the case of finite groups). Let ρ be a non-zero inducible representation of the algebra $\mathcal{A}_H$ on a dense domain $\mathcal{D}(\rho)$ of the Hilbert space $\mathcal{H}_\rho$ and let $\pi = \operatorname{Ind}_{\mathcal{A}_H\uparrow\mathcal{A}}\rho$.

Since $\mathcal{A} = \bigoplus_{t\in G/H} \mathcal{A}_t$, we get

$$\mathcal{A} \otimes_{\mathcal{A}_H} \mathcal{D}(\rho) = \bigoplus_{t\in G/H} \mathcal{A}_t \otimes_{\mathcal{A}_H} \mathcal{D}(\rho).$$

Recall that the representation space $\mathcal{H}_\pi$ of π is the completion of the quotient space of the tensor product $\mathcal{A} \otimes_{\mathcal{A}_H} \mathcal{D}(\rho)$ by the kernel $\mathcal{K}_\rho$ of the sesquilinear form $\langle\cdot,\cdot\rangle_0$ defined by Eq. 4. Let $\mathcal{H}_{t,0}$ denote the subspace of vectors $\xi_t \in \mathcal{A}_t \otimes_{\mathcal{A}_H} \mathcal{D}(\rho),\ t \in G/H$, such that $\langle\xi_t, \xi_t\rangle_0 = 0$. Take $\eta = \sum_{t\in G/H} \eta_t \in \mathcal{H}_0$, where $\eta_t \in \mathcal{A}_t \otimes_{\mathcal{A}_H} \mathcal{D}(\rho)$. Since $\langle\eta_t, \eta_s\rangle_0 = 0$ for $t \neq s$ we get

$$0 = \langle\eta, \eta\rangle_0 = \sum_{s,t\in G/H} \langle\eta_s, \eta_t\rangle_0 = \sum_{t\in G/H} \langle\eta_t, \eta_t\rangle_0,$$

that is, every η_t belongs to $\mathcal{H}_{t,0}$. This implies that $\mathcal{H}_0 = \bigoplus_{t\in G/H} \mathcal{H}_{t,0}$ and hence

$$(\mathcal{A} \otimes_{\mathcal{A}_H} \mathcal{D}(\rho))/\mathcal{H}_0 = \bigoplus_{t\in G/H} (\mathcal{A}_t \otimes_{\mathcal{A}_H} \mathcal{D}(\rho))/\mathcal{H}_{t,0}.$$

Note that for different left cosets $t \in G/H$ the subspaces $(\mathcal{A}_t \otimes_{\mathcal{A}_H} \mathcal{D}(\rho))/\mathcal{H}_{t,0}$ are pairwise orthogonal. For $t \in G/H$, we denote by $E(t)$ the orthogonal projection from $\mathcal{H}_\pi$ onto the completion of the subspace $(\mathcal{A}_t \otimes_{\mathcal{A}_H} \mathcal{H}_\rho)/\mathcal{H}_{t,0}$.

Proposition 9 *The pair (π, E) constructed above is a non-degenerate system of imprimitivity for the algebra $\mathcal{A}$ over G/H.*

Proof Because of Lemma 10 it suffices to check the conditions in Definition 7 for the restriction of π to its core $(\mathcal{A} \otimes_{\mathcal{A}_H} \mathcal{D}(\rho))/\mathcal{H}_0$. One easily verifies condition (i). We now show that condition (ii) is satisfied. Since the vectors $[a_t \otimes v],\ a_t \in \mathcal{A}_t,\ t \in G/H,\ v \in \mathcal{D}(\rho)$, span a core for π, it is enough to check (ii) for vectors of this form. Let us fix elements $g \in G,\ a_g \in \mathcal{A}_g,\ s, t \in G/H$ and $v \in \mathcal{D}(\rho)$. Then we have

$$\pi(a_g)E(s)[a_t \otimes v] = \begin{cases} [a_g a_t \otimes v], & \text{if } s = t; \\ 0, & \text{otherwise.} \end{cases}$$

Since the same result is obtained for $E(gs)\pi(a_g)[a_t \otimes v] = E(gs)[a_g a_t \otimes v]$, (ii) holds.

The equality $\pi(a_t)[\mathbf{1}_{\mathcal{A}} \otimes v] = [a_t \otimes v]$ implies that the span of $\pi(\mathcal{A}_t)\mathcal{D}_H(\pi)$ is equal to $\mathcal{D}_t(\pi)$, so (π, E) is non-degenerate. □

We call the pair (π, E) from Proposition 9 the *system of imprimitivity induced by ρ*.

Theorem 1 (First Imprimitivity Theorem) *Let $\mathcal{A} = \oplus_{g\in G} \mathcal{A}_g$ be a G-graded $*$-algebra and H a subgroup of G. Suppose that π is a closed $*$-representation of $\mathcal{A}$ and (π, E) is a non-degenerate system of imprimitivity for $\mathcal{A}$ over G/H. Then there exists a unique, up to unitary equivalence, closed $*$-representation ρ of $\mathcal{A}_H$ such that*

(i) *ρ is inducible,*
(ii) *(π, E) is unitarily equivalent to the system of imprimitivity induced by ρ.*

Proof By condition (ii) in Definition 7, the projection $E(H)$ commutes with the operators $\pi(a_H),\ a_H \in \mathcal{A}_H$. Hence the restriction of the representation $\mathrm{Res}_{\mathcal{A}_H}\pi$ to the subspace $\mathrm{Ran}\,E(H)$ is a well-defined $*$-representation of the $*$-algebra $\mathcal{A}_H$ denoted by ρ. The domain $\mathcal{D}(\rho)$ is equal to $\mathrm{Ran}\,E(H) \cap \mathcal{D}(\pi)$ and the representation space $\mathcal{H}_\rho$ is $\mathrm{Ran}\,E(H)$.

First we prove that ρ is inducible. We have to show that the form $\langle\cdot,\cdot\rangle_0$ is nonnegative. Take a vector $\xi = \sum_r a_r \otimes v_r \in \mathcal{A} \otimes_{\mathcal{A}_H} \mathcal{D}(\rho)$, where $v_r \in \mathcal{D}(\rho),\ a_r \in \mathcal{A}$.

Each a_r can be presented as a finite sum $a_r = \sum_{t\in G/H} a_{r,t}$, where $a_{r,t} \in \mathcal{A}_t,\ t \in G/H$. Then we have

$$\begin{aligned}\langle \xi, \xi\rangle_0 &= \left\langle \sum_r a_r \otimes v_r, \sum_s a_s \otimes v_s \right\rangle_0 = \sum_{r,s} \left\langle \rho\left(p_H\left(a_s^* a_r\right)\right) v_r, v_s\right\rangle = \\ &= \sum_{r,s} \left\langle \rho\left(\sum_{t\in G/H} a_{s,t}^* a_{r,t}\right) v_r, v_s\right\rangle = \sum_{t\in G/H}\sum_{r,s} \left\langle \rho\left(a_{s,t}^* a_{r,t}\right) v_r, v_s\right\rangle = \\ &= \sum_{t\in G/H}\sum_{r,s} \langle \pi(a_{s,t}) v_r, \pi(a_{r,t}) v_s\rangle = \sum_{t\in G/H} \left\langle \sum_r \pi(a_{r,t}) v_r, \sum_s \pi(a_{s,t}) v_s \right\rangle \geq 0.\end{aligned} \tag{17}$$

This shows that ρ is inducible.

Let (π_1, E_1) denote the system of imprimitivity on the space $\mathcal{H}_{\pi_1}$ induced by ρ. We have to prove that (π_1, E_1) is unitarily equivalent to (π, E). Define a linear mapping $F_0 : \mathcal{A} \otimes \mathcal{D}(\rho) \to \mathcal{D}(\pi)$ by putting $F_0(a \otimes v) = \pi(a)v$, where $v \in \mathcal{D}(\rho) \subseteq \mathcal{D}(\pi),\ a \in \mathcal{A}$. It is clear that F_0 maps $\mathcal{A} \otimes_{\mathcal{A}_H} \mathcal{D}(\rho)$ into $\mathcal{D}(\pi)$. Recall that $\mathcal{K}_\rho$ denotes the kernel of the sesquilinear form $\langle \cdot, \cdot\rangle_0$. Reasoning in the same manner as in Eq. 17 it follows that for any $\xi \in \mathcal{A} \otimes_{\mathcal{A}_H} \mathcal{D}(\rho)$ we have $\langle \xi, \xi\rangle_0 = \langle F_0(\xi), F_0(\xi)\rangle$. Therefore, the quotient mapping from $\mathcal{A} \otimes_{\mathcal{A}_H} \mathcal{D}(\rho)/\mathcal{K}_\rho$ to $\mathcal{H}_\pi$ is a well-defined isometric linear mapping. We extend this mapping by continuity to an isometry $F : \mathcal{H}_{\pi_1} \to \mathcal{H}_\pi$.

We claim that F intertwines the systems (π, E) and (π_1, E_1). Take $k \in G,\ a_k \in \mathcal{A}_k,\ a_t \in \mathcal{A}_t,\ t \in G/H,\ v \in \mathcal{D}(\rho)$. Then we obtain

$$\begin{gathered} F(\pi_1(a_k)([a_t \otimes v])) = F(a_k a_t \otimes v) = \pi(a_k a_t)v = \\ \pi(a_k)\pi(a_t)v = \pi(a_k)F([a_t \otimes v]) \end{gathered}$$

which means that F intertwines π and π_1.

For $v \in \mathcal{D}(\rho),\ a_t \in \mathcal{A}_t,\ t \in G/H$ condition (ii) in Definition 7 implies that $\pi(a_t)v \in \mathcal{D}_t(\pi)$. The subspace $\mathcal{D}_t(\pi_1)$, is spanned by the vectors $[a_t \otimes v],\ a_t \in \mathcal{A}_t,\ v \in \mathcal{D}(\rho)$, and we have $F([a_t \otimes v]) = \pi(a_t)v \in \mathcal{D}_t(\pi)$. Thus, $F(\mathcal{D}_t(\pi_1)) \subseteq \mathcal{D}_t(\pi)$ and F intertwines E and E_1.

Since (π, E) is non-degenerate, the vectors $F([a_t \otimes v]) = \pi(a_t)v,\ a_t \in \mathcal{A}_t,\ v \in \mathcal{D}(\rho)$, span a dense linear subspace $\mathcal{D}_t(\pi_1)$ of $\mathcal{D}_t(\pi)$ in the graph topology of π. In particular, we have $F(\operatorname{Ran} E_1(t)) = \operatorname{Ran} E(t)$, so that F is a unitary operator. Since the graph topology on $F(\mathcal{D}_t(\pi_1))$ is the same as that of π and π_1 is closed by definition, we have $F(\mathcal{D}_t(\pi_1)) = \mathcal{D}_t(\pi)$ for each $t \in G/H$, which implies that $F(\mathcal{D}(\pi_1)) = \mathcal{D}(\pi)$. That is, π and π_1 are unitarily equivalent.

Let ρ_1 be an inducible closed $*$-representation of $\mathcal{A}_H$ on the Hilbert space $\mathcal{H}_{\rho_1}$ and let (π_2, E_2) be the system of imprimitivity for $\mathcal{A}$ over G/H induced by ρ_1. It follows from the previous considerations that $\rho_2 := \operatorname{Res}_{\mathcal{A}_H}\pi_2 \upharpoonright \operatorname{Ran} E_2(H)$ is well-defined $*$-representation of $\mathcal{A}_H$. One immediately verifies that the canonical isomorphism $v \leftrightarrow [\mathbf{1}_\mathcal{A} \otimes v]$ of $\mathcal{H}_{\rho_1}$ and $\operatorname{Ran} E_2(H)$ defines a unitary equivalence of ρ_1 and ρ_2. □

Summarizing, we have shown that there is a one-to-one correspondence between unitary equivalence classes of inducible representations of $\mathcal{A}_H$ and unitary equivalence classes of non-degenerate closed systems of imprimitivity for $\mathcal{A}$ over G/H. In particular, the inducing representation ρ is determined uniquely up to unitary equivalence by the system of imprimitivity.

The following example shows that the non-degeneracy assumption of the system of imprimitivity is crucial in Theorem 1.

Example 14 Let $\mathcal{A}_q$ be the $*$-algebra $\mathbb{C}\langle a, a^* | aa^* - qa^*a = 1\rangle$, where $q > -1$. Put $\lambda_0 = 0$ and $\lambda_k = \sqrt{1 + q + q^2 + \cdots + q^{k-1}}$, $k \in \mathbb{N}$. Let $\mathcal{H}$ be a Hilbert space with orthonormal base $\{e_k,\ k \in \mathbb{N}_0\}$. There is a $*$-representations π of $\mathcal{A}_q$ on $\mathcal{D}(\pi) = \mathrm{Lin}\,\{\mathrm{e_k};\, \mathrm{k} \in \mathbb{N}_0\}$ such that

$$\pi(a)e_k = \lambda_k e_{k-1}, \pi(a^*)e_k = \lambda_{k+1}e_{k+1}, \text{ for } k \in \mathbb{N}_0,$$

where $e_{-1} := 0$. The representation π is bounded if and only if $-1 \le q \le 0$. Note that in the case $q = 1$ the algebra $\mathcal{A}_q$ is just the Weyl algebra and $\overline{\pi}$ is the Fock–Bargmann representation.

Let $E(n)$, $n \in \mathbb{N}$, be the orthogonal projection onto $\mathbb{C}\cdot e_{n-1}$ and put $E(n) := 0$ for $n \le 0$. Then the pair (π, E) is a system of imprimitivity for $\mathcal{A}$ over $G = \mathbb{Z}$. Since $E(0) = 0$, it follows immediately from the construction of the induced system of imprimitivity that (π, E) is not induced by a $*$-representation of $\mathcal{B}$.

We now define another construction of systems of imprimitivity. It will also include the system of imprimitivity in the latter example. Fix a system of imprimitivity (π, E) for $\mathcal{A}$ over G/H and an element $f \in G$. Define a mapping E^f from the set G/fHf^{-1} into the set of projections on the space $\mathcal{H}_\pi$ by $E^f(k(fHf^{-1})) := E(kfH)$, $k \in G$.

Proposition 10 *The pair (π, E^f) constructed above is a well-defined system of imprimitivity for $\mathcal{A}$ over G/fHf^{-1}.*

Proof Take $k_1(fHf^{-1}), k_2(fHf^{-1}) \in G/fHf^{-1}$, where $k_1, k_2 \in G$. The cosets $k_1(fHf^{-1})$ and $k_2(fHf^{-1})$ are equal if and only if $k_2^{-1}k_1 \in fHf^{-1}$ which is equivalent to $k_1 fH = k_2 fH$. This implies that E^f is well-defined. It is straightforward to verify that (π, E^f) satisfies the two conditions in Definition 7. □

Definition 8 If (π, E), $f \in G$, (π, E^f) are as above, we say that the system (π, E^f) is *conjugated* to the system (π, E) by the element $f \in G$.

Our second Imprimitivity Theorem describes systems of imprimitivity which are not necessarily non-degenerate. We prove it now for bounded representations (cf. also the Imprimitivity Theorem in [12], p. 1192). In Section 8 we formulate its analogue for well-behaved systems of imprimitivity (Theorem 4).

The following definition and the subsequent lemma are used in the proof of Theorem 2 below.

Definition 9 Let (π, E) be a system of imprimitivity for $\mathcal{A}$ over G/H and let $fH \in G/H$. We say that (π, E) is *generated by the projection* $E(fH)$ if for every $gH \in G/H$ the linear subspace $\pi(\mathcal{A}_{gHf^{-1}})(\mathcal{D}_{fH}(\pi))$ is dense in $\mathcal{D}_{gH}(\pi)$ with respect to the graph topology of π.

Lemma 11 *A system of imprimitivity* (π, E) *is generated by the projection* $E(fH)$, $fH \in G/H$, $f \in G$, *if and only if the conjugated system of imprimitivity* (π, E^f) *over* G/fHf^{-1} *is non-degenerate.*

The simple proof of Lemma 11 will be omitted. The next theorem says that for bounded representations each system of imprimitivity over G/H can be obtained as a direct sum of conjugated systems by elements of G.

Theorem 2 (Second Imprimitivity Theorem) *Let* $\mathcal{A} = \oplus_{g\in G}\, \mathcal{A}_g$ *a* G*-graded* $*$*-algebra,* H *a subgroup of* G *and* (π, E) *a system of imprimitivity for* $\mathcal{A}$ *over* G/H*. Suppose the* $*$*-representation* π *acts by bounded operators on* $\mathcal{D}(\pi) = \mathcal{H}_\pi$*. We fix one element* $k_t \in G$, $t \in G/H$, *in each left coset from* G/H*. Then for every* $t \in G/H$ *there exists a bounded* $*$*-representation* ρ_t *of* $\mathcal{A}_{k_tHk_t^{-1}}$ *on a Hilbert space* $\mathcal{H}_t$ *such that:*

(i) ρ_t *is inducible,*
(ii) (π, E) *is the direct sum of systems of imprimitivity* (π_t, E_t), $t \in G/H$, *where* (π_t, E_t) *is conjugated by the element* k_t *to the system of imprimitivity induced by* ρ_t, $t \in G/H$.

Proof Let (π_1, E_1) be an subsystem of imprimitivity of (π, E) over G/H, that is, $\pi_1 \subseteq \pi$ is a subrepresentation of π on a Hilbert subspace $\mathcal{H}_1 \subseteq \mathcal{H}_\pi$ and for all $gH \in G/H$ we have $\operatorname{Ran} E_1(gH) \subseteq \operatorname{Ran} E(gH)$. Since π is a bounded $*$-representation, there is a $*$-representation π_2 on $\mathcal{H}_2 := \mathcal{H}_\pi \ominus \mathcal{H}_1$ such that $\pi = \pi_1 \oplus \pi_2$. Put $E_2(gH) := E(gH) \ominus \operatorname{Ran} E_1(gH)$ for $gH \in G/H$. Then (π_2, E_2) is again a system of imprimitivity for $\mathcal{A}$ over G/H. Indeed, condition (i) in Definition 7 is obvious and condition (ii) follows immediately by subtracting the equation $\pi_1(a_g)E_1(fH) = E_1(gfH)\pi_1(a_g)$ from $\pi(a_g)E(fH) = E(gfH)\pi(a_g)$, where $g \in G, a_g \in \mathcal{A}_g$, $fH \in G/H$. That is, we have shown that every subsystem of imprimitivity has a complement.

Now we fix $fH \in G/H$. Let $E_1(gH)$ denote the orthogonal projection onto the closure of $\operatorname{Ran}\pi(\mathcal{A}_{gHf^{-1}})E(fH)$ and set $\mathcal{H}_1 := \oplus_{t\in G/H} \operatorname{Ran} E_1(t)$. It is easily checked that the family of projections $E_1(t)$, $t \in G/H$, satisfies condition (i) of Definition 7. Let $g \in G$, $a_g \in \mathcal{A}_g$ and $kH \in G/H$. Then we have

$$\pi(a_g)\operatorname{Ran} E_1(kH) = \pi(a_g)\overline{\operatorname{Ran}\pi(\mathcal{A}_{kHf^{-1}})E(fH)} \subseteq \overline{\operatorname{Ran}\pi(\mathcal{A}_{gkHf^{-1}})E(fH)} = E_1(gkH),$$

which shows that the subspace $\mathcal{H}_1$ is invariant under all operators $\pi(a)$, $a \in \mathcal{A}$. If we denote by π_1 the restriction of π to $\mathcal{H}_1$, then condition (ii) in Definition 7 holds for the pair (π_1, E_1). Therefore, (π_1, E_1) is an subsystem of imprimitivity for $\mathcal{A}$ over G/H. The system (π_1, E_1) is generated by $E_1(fH) = E(fH)$.

Combining the considerations of the preceding paragraphs with Zorn's lemma we conclude that there exist systems of imprimitivity (π_t, E_t), $t \in G/H$, for $\mathcal{A}$ over G/H such that every (π_t, E_t) is generated by the projection $E_t(k_tH)$, $t \in G/H$, and (π, E) is equal to the orthogonal direct sum of (π_t, E_t), $t \in G/H$.

Lemma 11 together with Theorem 1 imply that each conjugated system $(\pi_t, E_t^{k_t})$, $t \in G/H$, is induced by some representation ρ_t of the $*$-algebra $\mathcal{A}_{k_t H k_t^{-1}}$. By the construction of ρ_t (see the proof of the Theorem 1), ρ_t it is a bounded $*$-representation. □

Remark We do not know a generalization of Theorem 2 for *general unbounded* representations. The main difficulty lies in the fact that for a closed subrepresentation π_1 of a closed $*$-representation π in general there is no representation π_2 such that $\pi = \pi_1 \oplus \pi_2$.

6 A Partial Group Action Defined by the Grading

Throughout this section we assume that $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ is a G-graded unital $*$-algebra and that the *$*$-subalgebra $\mathcal{B} := \mathcal{A}_e$ is commutative*. The canonical conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$ is denoted by p.

Let $\widehat{\mathcal{B}}$ be the set of all characters of $\mathcal{B}$, that is, $\widehat{\mathcal{B}}$ is the set of nontrivial $*$-homomorphisms $\chi : \mathcal{B} \to \mathbb{C}$. The set of characters from $\widehat{\mathcal{B}}$ which are nonnegative on the cone $\sum \mathcal{A}^2 \cap \mathcal{B}$ is denoted by $\widehat{\mathcal{B}}^+$.

In addition we assume in this section that all characters $\chi \in \widehat{\mathcal{B}}^+$ satisfy the following condition:

$$\chi(c^*d)\chi(d^*c) = \chi(c^*c)\chi(d^*d) \text{ for all } \chi \in \widehat{\mathcal{B}}^+,\ g \in G, \text{ and } c, d \in \mathcal{A}_g. \tag{18}$$

Note that for $c, d \in \mathcal{A}_g$ we have $c^*d,\ d^*c,\ c^*c,\ d^*d \in \mathcal{A}_{g^{-1}} \cdot \mathcal{A}_g \subseteq \mathcal{A}_e = \mathcal{B}$. Hence all expressions in the Eq. 18 are well-defined. A condition similar to (18) was used in [4].

Proposition 11 *Let $\mathcal{A}$ denote the crossed product algebra $A \times_\alpha G$ from Example 12. Assume that A is commutative, so that $\mathcal{B} = A \otimes e$ is commutative. Then condition (18) is satisfied.*

Proposition 11 follows at once from the more general

Proposition 12 *Assume that for every $g \in G$ there exists an element $a_g \in \mathcal{A}_g$ such that $\mathcal{A}_g = a_g\mathcal{B}$ or $\mathcal{A}_g = \mathcal{B}a_g$. Then condition (18) is satisfied.*

Proof Fix a $g \in G$. Assume that there exists an element $a_g \in \mathcal{A}_g$ such that $\mathcal{A}_g = a_g\mathcal{B}$. Take $\chi \in \widehat{\mathcal{B}}^+$ and $c, d \in \mathcal{A}_g$. Then there exist $c_1, d_1 \in \mathcal{B}$ such that $c = a_g c_1$ and $d = a_g d_1$. We now compute

$$\chi(c^*d)\chi(d^*c) = \chi(c_1^*)\chi\left(a_g^*a_g\right)\chi(d_1)\chi(d_1^*)\chi\left(a_g^*a_g\right)\chi(c_1) = \chi(c^*c)\chi(d^*d).$$

In the same way one proves Eq. 18 in the case when $\mathcal{A}_g = \mathcal{B}a_g,\ a_g \in \mathcal{A}_g$. □

The main content of this section is the following partial action of G on the set $\widehat{\mathcal{B}}^+$.

Definition 10 Let $\chi \in \widehat{\mathcal{B}}^+$ and $g \in G$. We say that χ^g is defined if there exists an element $a_g \in \mathcal{A}_g$ such that $\chi(a_g^* a_g) \neq 0$. In this case we set

$$\chi^g(b) := \frac{\chi\left(a_g^* b a_g\right)}{\chi\left(a_g^* a_g\right)} \text{ for } b \in \mathcal{B}. \tag{19}$$

For $g \in G$ we denote by $\mathcal{D}_g$ the set of all characters $\chi \in \widehat{\mathcal{B}}^+$ such that χ^g is defined.

Remarks

1. One could also define χ^g as it was done in [12]. As noted in [12], the space $\mathcal{A}_g,\ g \in G$, has a natural structure of a $\mathcal{B}$-rigged $\mathcal{B}{-}\mathcal{B}$-bimodule, where $\mathcal{B}$ acts by the multiplication and the $\mathcal{B}$-valued product is

$$[\cdot,\cdot] : \mathcal{A}_g \times \mathcal{A}_g \to \mathcal{B},\ [c,d] := d^* c,\ c,d \in \mathcal{A}_g.$$

   Then χ^g is defined as the representation of $\mathcal{B}$ induced from χ via $\mathcal{A}_g$. Condition (18) ensures that χ^g is again a character.
2. Crossed-products defined by partial group actions on C^*-algebras appeared in [10]. Our G-graded $*$-algebra $\mathcal{A}$ can be considered as another generalization of crossed-product algebras. We shall not elaborate the details here.

Proposition 13 *The map $\chi \mapsto \chi^g$ is a well-defined partial action of G on the set $\widehat{\mathcal{B}}^+$, that is:*

(i) *$\chi^g(b)$ in Eq. 19 does not depend on the choice of a_g and we have $\chi^g \in \widehat{\mathcal{B}}^+$,*
(ii) *if χ^g and $(\chi^g)^h$ are defined, then χ^{hg} is defined and equal to $(\chi^g)^h$,*
(iii) *if χ^g is defined, then $(\chi^g)^{g^{-1}}$ is defined and equal to χ,*
(iv) *χ^e is defined and equal to χ.*

Proof

(i) Let $\chi \in \widehat{\mathcal{B}}^+,\ g \in G$, and $c, d \in \mathcal{A}_g$ such that $\chi(d^*d) \neq 0$ and $\chi(c^*c) \neq 0$. Since $\mathcal{B}$ is commutative, we have $bcd^* = cd^*b$ for $b \in \mathcal{B}$. Therefore we obtain

$$\chi(c^*bc)\chi(d^*d) = \chi(c^*bcd^*d) = \chi(c^*cd^*bd) = \chi(c^*c)\chi(d^*bd),$$

so that

$$\frac{\chi(c^*bc)}{\chi(c^*c)} = \frac{\chi(d^*bd)}{\chi(d^*d)}.$$

We show that χ^g is again a character belonging to $\widehat{\mathcal{B}}^+$. Let $b_1, b_2 \in \mathcal{B}$. Since $\mathcal{B}$ is commutative, we have $a_g a_g^* b_1 = b_1 a_g a_g^*$. Hence we get

$$\chi^g(b_1 b_2) = \frac{\chi\left(a_g^* b_1 b_2 a_g\right)}{\chi\left(a_g^* a_g\right)} = \frac{\chi\left(a_g^* a_g a_g^* b_1 b_2 a_g\right)}{\chi\left(a_g^* a_g\right)\chi\left(a_g^* a_g\right)} = \frac{\chi\left(a_g^* b_1 a_g a_g^* b_2 a_g\right)}{\chi\left(a_g^* a_g\right)\chi\left(a_g^* a_g\right)}$$
$$= \chi^g(b_1)\chi^g(b_2).$$

Next we prove the positivity of χ^g. For take $b \in \sum \mathcal{A}^2$. Since $\chi(\sum \mathcal{A}^2) \geq 0$ and $a_g^* b a_g \in \sum \mathcal{A}^2$ we have $\chi^g(b) > 0$.

(ii) Let $\chi \in \widehat{\mathcal{B}}^+$ and $g, h \in G$ such that $(\chi^g)^h$ is defined. Then there exists $a_g \in \mathcal{A}_g$ such that $\chi(a_g^* a_g) \neq 0$. Since $(\chi^g)^h$ is defined, there exists $a_h \in \mathcal{A}_h$ such that

$$\chi^g(a_h^* a_h) = \frac{\chi(a_g^* a_h^* a_h a_g)}{\chi(a_g^* a_g)} \neq 0,$$

that is, $\chi((a_h a_g)^* a_h a_g) \neq 0$. Since $a_h a_g \in \mathcal{A}_{hg}$, χ^{hg} is well-defined. It is straightforward to check that $(\chi^g)^h = \chi^{hg}$.

(iii) Assume that χ^g is defined. Then there exists $a_g \in \mathcal{A}_g$ such that $\chi(a_g^* a_g) \neq 0$. We have $a_g^* \in \mathcal{A}_{g^{-1}}$ and

$$\chi^g(a_g a_g^*) = \frac{\chi(a_g^* a_g a_g^* a_g)}{\chi(a_g^* a_g)} = \chi(a_g^* a_g) \neq 0.$$

Hence $(\chi^g)^{g^{-1}}$ is defined. One easily verifies that $(\chi^g)^{g^{-1}} = \chi$.

(iv) is trivial. □

Remark It follows from Proposition 13 that for each $g \in G$ the mapping $\chi \mapsto \chi^g$ defines a bijection $\alpha_g : \mathcal{D}_g \to \mathcal{D}_{g^{-1}}$ such that:

(i) $\mathcal{D}_e = \widehat{\mathcal{B}}^+$ and α_e is the identity mapping of $\widehat{\mathcal{B}}^+$,
(ii) $\alpha_g(\mathcal{D}_g \cap \mathcal{D}_h) = \mathcal{D}_{g^{-1}} \cap \mathcal{D}_{hg^{-1}}$,
(iii) $\alpha_g(\alpha_h(x)) = \alpha_{gh}(x)$, for $x \in \mathcal{D}_g \cap \mathcal{D}_{gh}$.

In what follows, we shall use both notations $\alpha_g(\chi)$ and χ^g for the partial action of $g \in G$ on $\chi \in \widehat{\mathcal{B}}^+$ and we freely use the properties (i)–(iii).

It should be emphasized that up to now condition (18) has not been used for the partial action. For the next proposition assumption (18) is needed.

Proposition 14 *Let $a_g, c_g \in \mathcal{A}_g$, $g \in G$, and $\chi \in \widehat{\mathcal{B}}^+$ be such that $\chi(a_g^* c_g) \neq 0$. Then we have $\chi \in \mathcal{D}_g$ and*

$$\chi^g(b) = \frac{\chi(a_g^* b c_g)}{\chi(a_g^* c_g)} \text{ for all } b \in \mathcal{B}. \tag{20}$$

Proof Since $\chi(a_g^* c_g) \neq 0$, we have $\chi(c_g^* a_g) = \overline{\chi(a_g^* c_g)} \neq 0$, so that Eq. 18 implies $\chi(a_g^* a_g) \neq 0$, i.e. $\chi \in \mathcal{D}_g$. Now Eq. 20 follows from the equality

$$\chi(a_g^* b a_g)\chi(a_g^* c_g) = \chi(a_g^* a_g a_g^* b c_g) = \chi(a_g^* a_g)\chi(a_g^* b c_g).$$

□

Examples developed below show that in general χ^g is not always defined, so that in general $\chi \mapsto \chi^g$ is not a group action.

We introduce some more notation which will be kept till the end of the paper. For a fixed $\chi \in \widehat{\mathcal{B}}^+$ let

$$G_\chi = \left\{ g \in G | \chi^g \text{ is defined} \right\}.$$

We denote by $\mathrm{Orb}_\chi \subseteq \widehat{\mathcal{B}}^+$ the *orbit* of the χ, that is,

$$\mathrm{Orb}_\chi = \left\{ \chi^g | \chi^g \text{ is defined} \right\}.$$

Further, let $\mathrm{St}\chi \subseteq G_\chi$ denote the *stabilizer* of the element χ, that is,

$$\mathrm{St}\chi = \left\{ g \in G | \chi^g \text{ is defined and equal to } \chi \right\}.$$

A number of elementary properties of the partial action of G are collected in the following

Proposition 15 *Let $\chi \in \widehat{\mathcal{B}}^+$. Then we have:*

(i) $\mathrm{St}\chi$ *is a subgroup of* G,
(ii) *The union of sets* G_ψ, $\psi \in \mathrm{Orb}\chi$ *equipped with the multiplication derived from* G *is a groupoid with identity element,*
(iii) *if* $\psi \in \widehat{\mathcal{B}}^+$, *then* $\psi \in \mathrm{Orb}\chi$ *if and only if* $\mathrm{Orb}_\psi = \mathrm{Orb}_\chi$,
(iv) *if* $\psi \in \mathrm{Orb}_\chi$, *then* $\mathrm{St}\chi$ *and* $\mathrm{St}\psi$ *are conjugate subgroups of* G.

Now we illustrate these concepts by a few examples.

Example 15 Let A be a commutative $*$-algebra and $\mathcal{A} = A \times_\alpha G$ be the crossed-product algebra from Example 12. It was shown therein that $\sum \mathcal{A}^2 \cap \mathcal{B} = \sum \mathcal{B}^2$. This implies that $\widehat{\mathcal{B}}^+ = \widehat{\mathcal{B}} = \widehat{A}$ and the partial action defined by Eq. 19 coincides with the usual group action of G on $\widehat{A}$ induced by the action of G on A.

Example 16 Let $\mathcal{A}$ be the Weyl algebra. We retain the notation from Examples 1 and 10. It follows from Eq. 14 that a character $\chi \in \widehat{\mathcal{B}}$ is non-negative on the cone $\sum \mathcal{A}^2 \cap \mathcal{B}$ if and only if $\chi(N) \in \mathbb{N}_0$. For $k \in \mathbb{N}_0$, let χ_k denote the character of $\widehat{\mathcal{B}}^+$ defined by $\chi_k(N) = k$.

Suppose that $n \in \mathbb{N}_0$. Clearly, any element of the $\mathcal{A}_n$ has the form $a^n p(N)$, where $p \in \mathbb{C}[N]$, and $\chi_k((a^n p(N))^* a^n p(N)) \neq 0$ implies $\chi_k(a^{*n} a^n) \neq 0$. So we obtain that

$$(\alpha_n(\chi_k))(N) = \frac{\chi_k(a^{*n} N a^n)}{\chi_k(a^{*n} a^n)} = \frac{\chi_k(N(N-1)\dots(N-n+1)(N-n))}{\chi_k(N(N-1)\dots(N-n+1))}$$

is defined if and only if $k \geq n$ and $(\alpha_n(\chi_k))(N) = \chi_{k-n}(N)$.

Analogously we conclude that

$$(\alpha_{-n}(\chi_k))(N) = \frac{\chi_k(a^n N a^{*n})}{\chi_k(a^n a^{*n})} = \frac{\chi_k((N+1)(N+2)\dots(N+n)^2)}{\chi_k((N+1)(N+2)\dots(N+n))}$$

is defined for all $n \in \mathbb{N}$ and $(\alpha_{-n}(\chi_k))(N) = \chi_{k+n}(N)$, i.e. $\alpha_{-n}(\chi_k) = \chi_{k+n}$.

The partial action is transitive, so $\widehat{\mathcal{B}}^+$ consists of a single orbit. The stabilizer $\mathrm{St}\chi_k$ of each character χ_k is trivial, the set G_{χ_k} is equal to $\{n \in \mathbb{Z} | n \leq k\}$.

The next proposition gives explicit formulas for representations induced from characters. Recall that a character $\chi \in \widehat{\mathcal{B}}^+$ is a one-dimensional $*$-representation of $\mathcal{B}$ on the space $\mathbb{C}$ and the representation space $\mathcal{H}_\pi$ of $\pi = \mathrm{Ind}\chi$ is spanned by the vectors $[a \otimes 1]$, $a \in \mathcal{A}$ (see Section 2).

Proposition 16 *Let $\chi \in \widehat{\mathcal{B}}^{+}$ and $\pi = \mathrm{Ind}\chi$. Fix elements $a_g \in \mathcal{A}_g$, $g \in G_\chi$, such that $\chi(a_g^* a_g) \neq 0$, $g \in G_\chi$. Then we have:*

(i) *The vectors*

$$e_g = \frac{[a_g \otimes 1]}{\sqrt{\chi(a_g^* a_g)}}, \; g \in G_\chi,$$

form an orthonormal base of the representation space $\mathcal{H}_\pi$ of $\mathrm{Ind}\chi$.

(ii) *For $b_h \in \mathcal{A}_h$ and $h \in G$ we have*

$$\pi(b_h)e_g = \frac{\chi(a_{hg}^* b_h a_g)}{\sqrt{\chi(a_{hg}^* a_{hg})\chi(a_g^* a_g)}} e_{hg}, \; \text{if } hg \in G_\chi$$

and $\pi(b_h)e_g = 0$ otherwise. In particular, if $b \in \mathcal{B}$, then we have

$$\pi(b)e_g = \frac{\chi(a_g^* b\, a_g)}{\chi(a_g^* a_g)} e_g = \chi^g(b)e_g.$$

Proof First suppose that $b_g \in \mathcal{A}_g$ and $g \notin G_\chi$. Then $\left\|[b_g \otimes 1]\right\|^2 = \chi(b_g^* b_g) = 0$, so $\mathcal{H}_\pi$ is spanned by the vectors $[b_g \otimes 1]$, where $b_g \in \mathcal{A}_g$ and $g \in G_\chi$.

For $b_g \in \mathcal{A}_g$ and $g \in G$ the equality (18) applied to a_g and b_g is equivalent to the equation

$$|\langle [a_g \otimes 1], [b_g \otimes 1]\rangle|^2 = \left\|[a_g \otimes 1]\right\|^2 \left\|[b_g \otimes 1]\right\|^2,$$

that is, we have equality in the Cauchy–Schwarz inequality. This implies that $[a_g \otimes 1] = \lambda[b_g \otimes 1]$ for some complex number λ. Hence it follows that the elements $[a_g \otimes 1]$, $g \in G_\chi$, span the space $\mathcal{H}_\pi$. Since $\langle [a_g \otimes 1], [a_h \otimes 1]\rangle = \chi(p(a_h^* a_g)) = \chi(0) = 0$ for $g \neq h$, the elements $[a_g \otimes 1]$ are pairwise orthogonal. The square of the norm of $[a_g \otimes 1]$ is equal to $\langle [a_g \otimes 1], [a_g \otimes 1]\rangle = \chi(a_g^* a_g)$. Thus we have shown that the elements e_g, $g \in G_\chi$, form an orthonormal base of $\mathcal{H}_\pi$.

Now let $b_h \in \mathcal{A}_h$, $h \in H$. If $hg \in G_\chi$ we have

$$\pi(b_h)e_g = \frac{[b_h a_g \otimes 1]}{\sqrt{\chi(a_g^* a_g)}} = \frac{\lambda[a_{hg} \otimes 1]}{\sqrt{\chi(a_g^* a_g)}} = \lambda \frac{\sqrt{\chi(a_{hg}^* a_{hg})}}{\sqrt{\chi(a_g^* a_g)}} e_{hg},$$

where λ is equal to

$$\frac{\langle [b_h a_g \otimes 1], [a_{hg} \otimes 1]\rangle}{\langle [a_{hg} \otimes 1][a_{hg} \otimes 1]\rangle} = \frac{\chi(a_{hg}^* b\, a_g)}{\chi(a_{hg}^* a_{hg})}.$$

This yields the second statement of the theorem. □

In Section 8 we will derive a simple criterion of the irreducibility of the induced representation by showing that $\mathrm{Ind}\chi$, $\chi \in \widehat{\mathcal{B}}^{+}$, is irreducible if and only if the stabilizer group $\mathrm{St}\chi$ is trivial.

7 Well-Behaved Representations

There is an essential difference between unbounded and bounded representation theory of $*$-algebras in Hilbert space. The problem of classifying *all* or even *all self-adjoint* unbounded $*$-representations is not well-posed for arbitrary $*$-algebras. Let us explain this for the $*$-algebra $\mathbb{C}[x_1, x_2]$ of polynomials in two variables. In [41] it was proved that for any properly infinite von Neumann algebra $\mathcal{N}$ on a separable Hilbert space there exists a self-adjoint $*$-representation π of $\mathbb{C}[x_1, x_2]$ such that the operators $\overline{\pi(x_1)}$ and $\overline{\pi(x_2)}$ are self-adjoint and their spectral projections generate $\mathcal{N}$. This result has been used in [38] to show the representation theory of $\mathbb{C}[x_1, x_2]$ is wild. Such a pathological behavior can be overcome if we restrict to integrable representations. For the $*$-algebra $\mathbb{C}[x_1, x_2]$ a self-adjoint representation π is integrable if and only the operators $\overline{\pi(x_1)}$ and $\overline{\pi(x_2)}$ are self-adjoint and their spectral projections commute. However, for arbitrary $*$-algebras no method is known to single out such a class of well-behaved representations. To define and classify well-behaved representations of general $*$-algebras is a fundamental problem in unbounded representation theory. One possible proposal was given in [40]. In this section we develop a concept of well-behaved representations for G-graded $*$-algebras $\mathcal{A}$ with commutative $*$-subalgebras $\mathcal{A}_e$. We begin with some simple technical facts.

Lemma 12 *Let π be a $*$-representation of a G-graded $*$-algebra $\mathcal{A}$ and $\mathcal{B} = \mathcal{A}_e$. Then the graph topologies of π and of* $\mathrm{Res}_{\mathcal{B}}\pi$ *coincide. In particular, π is closed if and only if* $\mathrm{Res}_{\mathcal{B}}\pi$ *is closed.*

Proof Since $\mathcal{B}$ is a $*$-subalgebra of $\mathcal{A}$, the graph topology of $\mathrm{Res}_{\mathcal{B}}\pi$ is obviously weaker than that of π. For $a_g \in \mathcal{A}_g$ and $\varphi \in \mathcal{D}(\pi)$, we have

$$\left\|\pi(a_g)\varphi\right\| = \langle \pi(a_g^* a_g)\varphi, \varphi\rangle^{1/2} \leq \left\|\pi(a_g^* a_g)\varphi\right\| + \|\varphi\| .$$

Since $a_g^* a_g \in \mathcal{B}$, the graph topology of π is weaker than the graph topology of $\mathrm{Res}_{\mathcal{B}}\pi$. Hence both topologies coincide. Since closedness of a $*$-representation is equivalent to the completeness in the graph topology (see [39], 8.1), it follows that π is closed if and only if $\mathrm{Res}_{\mathcal{B}}\pi$ is closed. □

Throughout the rest of this section we assume that $\mathcal{A} = \oplus_{g \in G} \mathcal{A}_g$ is a G-graded $*$-algebra such that $\mathcal{A}_e = \mathcal{B}$ is commutative and condition (18) is satisfied.

We begin with some preliminaries. An element $b \in \mathcal{B}$ can be viewed as a function f_b on the set $\widehat{\mathcal{B}}^+$, that is, $f_b(\chi) = \chi(b)$ for $b \in \mathcal{B}$ and $\chi \in \widehat{\mathcal{B}}^+$. Let τ denote the weakest topology on the set $\widehat{\mathcal{B}}^+$ for which all functions f_b, $b \in \mathcal{B}$, are continuous. This topology is generated by the sets $f_b^{-1}((c, d))$, $-\infty \leq c \leq d \leq \infty$. Clearly, the topology τ on $\widehat{\mathcal{B}}^+$ is Hausdorff. We assume in addition that the topology τ on $\widehat{\mathcal{B}}^+$ is *locally compact.*

The topology τ on $\widehat{\mathcal{B}}^+$ defines a Borel structure which is generated by all open sets. Since the domain $\mathcal{D}_g$ of the mapping α_g is the union of open sets $f_{a_g^* a_g}^{-1}((0, +\infty))$, $a_g \in \mathcal{A}_g$, the set $\mathcal{D}_g$ is open and hence Borel.

Lemma 13 *Let τ_g, $g \in G$, be the weakest topology on $\mathcal{D}_g$ for which all functions $f_{a_g^* a_g}$, $a_g \in \mathcal{A}_g$, are continuous. Then τ_g is induced from the topology τ on $\widehat{\mathcal{B}}^+$.*

Proof Let $\chi \in \mathcal{D}_g$. Since the topology τ on $\widehat{\mathcal{B}}^+$ is locally compact, there is a compact neighborhood Ω of χ. Since $\mathcal{D}_g$ is open, $\Omega_1 = \Omega \cap \mathcal{D}_g$ is again a neighborhood of χ. The elements of $\mathcal{B}$ separate the points of $\widehat{\mathcal{B}}^+$. The set $\{b^2 | b = b^*,\ b \in \mathcal{B}\}$ generates $\mathcal{B}$, so it also separates the points of $\widehat{\mathcal{B}}^+$. It follows that the set $\left\{a_g^* a_g,\ a_g \in \mathcal{A}_g\right\}$ separates the elements of $\mathcal{D}_g$. Since Ω is compact, $\overline{\Omega}_1$ is also compact. Since the functions $f_{a_g^* a_g}$ are continuous on Ω_1 and vanish on the set $\overline{\Omega}_1 \backslash \Omega_1$, they belong to the C^*-algebra $C_0(\Omega_1)$ of continuous functions vanishing at infinity. By the Stone–Weierstraß theorem, the functions $f_{a_g^* a_g}$, where $a_g \in \mathcal{A}_g$, generate a $*$-algebra which is dense in $C_0(\Omega_1)$. Hence the induced topology of τ_g on Ω_1 coincides with the induced topology of τ. Since $\chi \in \mathcal{D}_g$ is arbitrary, τ_g is induced from the topology τ on $\widehat{\mathcal{B}}^+$. □

For $\Delta \subseteq \widehat{\mathcal{B}}^+$ and $g \in G$, we define Δ^g by

$$\Delta^g = \left\{\chi^g | \chi \in \mathcal{D}_g \cap \Delta\right\}.$$

By definition, $\emptyset^g$ is $\emptyset$. In particular, if $\Delta \cap \mathcal{D}_g = \emptyset$, then $\Delta^g = \emptyset$. We also write $\alpha_g(\Delta)$ for Δ^g.

Lemma 14

(i) *For any $g \in G$, the mapping α_g is a homeomorphism of $\mathcal{D}_g$ onto $\mathcal{D}_{g^{-1}}$.*
(ii) *If $\Delta \subseteq \mathcal{D}_g$ is open (resp. Borel), then Δ^g is open (resp. Borel).*

Proof

(i) By Proposition 13, α_g is a bijection. The equality $f_{a_g^* a_g}(\chi) = f_{a_g a_g^*}(\chi^g)$, $a_g \in \mathcal{A}_g$, implies that for every open subset X of $\mathbb{R}$ the set $(f^{-1}_{a_g^* a_g}(X))^g = f^{-1}_{a_g a_g^*}(X)$ is open. Therefore, by Lemma 13, $\alpha_{g^{-1}}$ is continuous. Replacing g by g^{-1} we conclude that α_g is continuous. Since α_g and $\alpha_{g^{-1}}$ are inverse to each other, α_g is a homeomorphism.
(ii) As noted above, $\mathcal{D}_g$ is open. Therefore, if Δ is open (resp. Borel), then $\Delta \cap \mathcal{D}_g$ is open (resp. Borel). Since α_g is a homeomorphism, $\Delta^g = (\Delta \cap \mathcal{D}_g)^g$ is also open (resp. Borel). □

After these preliminaries we are ready to give the main definition of this section.

Definition 11 A $*$-representation π of $\mathcal{A}$ is *well-behaved* if the following two conditions are satisfied:

(i) The restriction $\mathrm{Res}_{\mathcal{B}}\pi$ of π to $\mathcal{B}$ is integrable and there exists a spectral measure E_π on the locally compact space $\widehat{\mathcal{B}}^+[\tau]$ such that

$$\overline{\pi(b)} = \int_{\widehat{\mathcal{B}}^+} f_b(\chi) dE_\pi(\chi) \text{ for } b \in \mathcal{B}. \tag{21}$$

(ii) For all $a_g \in \mathcal{A}_g,\ g \in G$, and all Borel subsets $\Delta \subseteq \widehat{\mathcal{B}}^+$, we have

$$E_\pi(\Delta^g)\pi(a_g) \subseteq \pi(a_g)E_\pi(\Delta).$$

If (i) is fulfilled, we shall say that the spectral measure E_π is associated with π.

We give some equivalent forms of the conditions in Definition 11. From Theorem 7 in the Appendix it follows that condition (i) is already fulfilled if $\mathrm{Res}_{\mathcal{B}}\pi$ is integrable and $\mathcal{B}$ is countably generated. The next proposition contains a number of reformulations of condition (ii).

Proposition 17 *Let π be a $*$-representation of $\mathcal{A}$ satisfying condition* (i) *of Definition* 11. *Let $\mathcal{F}_\pi$ denote the set of Borel functions f on $\widehat{\mathcal{B}}^+$ such that the operator $\int f dE_\pi$ maps the domain $\mathcal{D}(\pi)$ into itself. For $a_g \in \mathcal{A}_g,\ g \in G$, let U_gC_g be the polar decomposition of $\overline{\pi(a_g)}$. Then the following statements are equivalent:*

(i) *Condition* (ii) *of Definition* 11 *is fulfilled.*
(ii) *For all $a_g \in \mathcal{A}_g,\ g \in G$, and all Borel sets $\Delta \subseteq \widehat{\mathcal{B}}^+$ we have $U_gE_\pi(\Delta) = E_\pi(\Delta^g)U_g$.*
(iii) *For any E-measurable function f on $\widehat{\mathcal{B}}^+$ and $a_g \in \mathcal{A}_g,\ g \in G$, we have*

$$U_g \int f(\chi)dE_\pi(\chi) \subseteq \int_{\mathcal{D}_{g^{-1}}} f(\alpha_{g^{-1}}(\chi))dE_\pi(\chi)U_g.$$

(iv) *For any $f \in \mathcal{F}_\pi$, $a_g \in \mathcal{A}_g,\ g \in G$, and $\varphi \in \mathcal{D}(\pi)$, we have*

$$\pi(a_g) \int f(\chi)dE_\pi(\chi)\varphi = \int_{\mathcal{D}_{g^{-1}}} f(\alpha_{g^{-1}}(\chi))dE_\pi(\chi)\pi(a_g)\varphi.$$

Proof

(i) $\Rightarrow$ (ii) Fix $\Delta \subseteq \widehat{\mathcal{B}}^+$. Since $\mathrm{Res}_{\mathcal{B}}\pi$ is integrable, $\overline{\pi(a_g^*a_g)}$ is self-adjoint. But $\pi(a_g)^*\overline{\pi(a_g)}$ is a self-adjoint extension of $\overline{\pi(a_g^*a_g)}$, so that $C_g^2 = \pi(a_g)^*\overline{\pi(a_g)} = \overline{\pi(a_g^*a_g)}$. Since $\overline{\pi(a_g^*a_g)}$ commutes with the projections $E_\pi(\cdot)$, C_g^2 and hence C_g commutes with $E_\pi(\cdot)$. Thus we get $U_gE(\Delta^g)C_g \subseteq U_gC_gE(\Delta) = \overline{\pi(a_g)}E(\Delta)$. From Definition 11, (i) it follows that the kernel of $C_g^2 = \overline{\pi(a_g^*a_g)}$ is equal to $\mathrm{Ran}\,E_\pi(f_{a_g^*a_g}^{-1}(0))$. By the properties of the polar decomposition, this kernel equals to $\ker U_g = \ker C_g$. If $v \in \ker C_g$, then $E(\Delta^g)U_gv = 0$ and, since $P_0 := E_\pi(f_{a_g^*a_g}^{-1}(0))$ commutes with $E_\pi(\cdot)$, we get $U_gE(\Delta)v = U_gE(\Delta)P_0v = U_gP_0E(\Delta)v = 0$. Thus the bounded operators $U_gE(\Delta)$ and $E(\Delta^g)U_g$ coincide on the dense set $\mathrm{Ran}C_g + \ker C_g$, so they coincide everywhere.

(ii) $\Rightarrow$ (iii) From (ii) we get (iii) for characteristic functions, then for simple functions and by a limit procedure for arbitrary measurable functions $f \in \mathcal{F}_\pi$.

(iii) $\Rightarrow$ (iv) This follows from the relation $\pi(a_g)\varphi = U_gC_g\varphi$ combined with the fact that C_g and the first integral commute on vectors $\varphi \in \mathcal{D}(\pi)$.

(iv) $\Rightarrow$ (i) Since π is integrable, π is closed and so is $\mathrm{Res}_{\mathcal{B}}\pi$ by Lemma 12. Therefore, $\mathcal{D}(\pi) = \cap_{b\in\mathcal{B}}\mathcal{D}(\overline{\pi(b)})$. By Eq. 21 the latter implies that $E(\Delta)$ leaves the domain $\mathcal{D}(\pi)$ invariant. Hence the characteristic function of Δ belongs to $\mathcal{F}_\pi$ and (i) follows from (iv) applied to this characteristic function. □

Many notions on unbounded operators are derived from appropriate reformulations of the corresponding notions on bounded operators. The next proposition says that bounded $*$-representations satisfy the two conditions in Definition 11. This observation was in fact the starting point for our definition of well-behaved representations.

Proposition 18 *If π is a bounded $*$-representation of the $*$-algebra $\mathcal{A}$ such that $\mathcal{D}(\pi) = \mathcal{H}_\pi$, then π is well-behaved.*

Proof Since the representation π is bounded, the closure of $\pi(\mathcal{B})$ in the operator norm is a commutative C^*-algebra. Hence condition (i) follows from Theorem 12.22 in [36].

Fix $g \in G,\ a_g, b_g \in \mathcal{A}_g$. From assumption (18) we obtain that $f_{a_g^* a_g b_g^* b_g}(\chi) = f_{a_g^* b_g b_g^* a_g}(\chi)$ on $\widehat{\mathcal{B}}^+$. Therefore, by condition (i) we have $\pi(a_g^* a_g b_g^* b_g) = \pi(a_g^* b_g b_g^* a_g)$ which can be rewritten in the form

$$\pi(a_g^*)\pi(a_g b_g^* b_g) = \pi(a_g^*)\pi(b_g b_g^* a_g). \tag{22}$$

Since $\pi(b_g b_g^*)$ commutes with $\pi(a_g^* a_g)$, it also commutes with the projection onto the range of $\pi(a_g)$. This implies that $\pi(b_g b_g^*)(\mathrm{Ran}(\pi(a_g)))$ is contained in $\overline{\mathrm{Ran}(\pi(a_g))}$, so the range of the operator $\pi(b_g b_g^* a_g)$ is contained in $\overline{\mathrm{Ran}(\pi(a_g))}$. The range of the operator $\pi(a_g b_g^* b_g)$ is evidently contained in $\mathrm{Ran}\pi(a_g)$. From the relation $\overline{\mathrm{Ran}(\pi(a_g))} = \ker(\pi(a_g^*))^\perp$ it follows that $\pi(a_g^*)$ restricted to $\overline{\mathrm{Ran}(\pi(a_g))}$ is injective. Therefore, from Eq. 22 we get $\pi(a_g b_g^* b_g) = \pi(b_g b_g^* a_g)$ and so

$$\pi(a_g)\pi(b_g^* b_g) = \pi(b_g b_g^*)\pi(a_g)$$

for all $b_g \in \mathcal{A}_g$. Now we use a standard approximation procedure. The preceding relation yields

$$\pi(a_g) p_n(\pi(b_g^* b_g)) = p_n(\pi(b_g b_g^*))\pi(a_g)$$

for all polynomials $p_n \in \mathbb{C}[t]$ which implies that

$$\pi(a_g) E_{\pi(b_g^* b_g)}(X) = E_{\pi(b_g b_g^*)}(X)\pi(a_g),$$

where $E_{\pi(\cdot)}$ denotes the spectral measure of the self-adjoint operator $\pi(\cdot)$ and X is a Borel subset of $\mathbb{R}$. The spectral measure E_π on the space $\widehat{\mathcal{B}}^+$ associated with π is releated to the spectral measure of the operator $\pi(b_h^* b_h),\ b_h \in \mathcal{A}_h,\ h \in G$, by the equation

$$E_{\pi(b_h^* b_h)}(X) = E_\pi(f^{-1}_{b_h^* b_h}(X)),$$

where $f_{b_h^* b_h}$ is the function on $\widehat{\mathcal{B}}^+$ defined by the element $b_h^* b_h \in \mathcal{B}$. From the equality

$$\alpha_h(f^{-1}_{b_h^* b_h}(X)) = f^{-1}_{b_h b_h^*}(X)$$

we obtain

$$\pi(a_g)E_\pi(\Delta) = E_\pi(\Delta^g)\pi(a_g), \tag{23}$$

where $g \in G$, $a_g \in \mathcal{A}_g$, $\Delta = f^{-1}_{c_g^* c_g}(X)$, and X is a Borel subset $\mathbb{R}$. Since Eq. 23 is valid for such sets Δ, it holds for all sets from the σ-algebra generated by the sets Δ as well. From Lemma 13 we conclude that Eq. 23 holds for all Borel sets $\Delta \subseteq \mathcal{D}_g$.

In particular, Eq. 23 is true for $\Delta = \mathcal{D}_g$, so also for $\Delta = \widehat{\mathcal{B}}^+ \backslash \mathcal{D}_g$. Therefore we have $\pi(a_g)E_\pi(\widehat{\mathcal{B}}^+ \backslash \mathcal{D}_g) = 0$ which implies that $\pi(a_g)E_\pi(\Delta_0) = 0$ for all Borel subsets $\Delta_0 \subseteq \widehat{\mathcal{B}}^+ \backslash \mathcal{D}_g$. Since $E_\pi(\alpha_g(\Delta_0)) = E_\pi(\emptyset) = 0$, Eq. 23 is valid for all Borel sets Δ_0 of $\widehat{\mathcal{B}}^+ \backslash \mathcal{D}_g$. Hence condition (ii) of Definition 11 is satisfied. □

In the rest of this section we derive some basic properties of well-behaved representations.

Proposition 19 *Let π be a well-behaved representation of $\mathcal{A}$. Then any self-adjoint subrepresentation $\pi_0 \subseteq \pi$ is well-behaved.*

Proof Since π is well-behaved, it is self-adjoint. By Corollary 8.3.13 in [39], there exists a representation π_1 of $\mathcal{A}$ such that $\pi = \pi_0 \oplus \pi_1$. Since $\mathrm{Res}_{\mathcal{B}}\pi$ is integrable, $\mathrm{Res}_{\mathcal{B}}\pi_0$ is integrable by Proposition 9.1.17 (i) in [39]. Let $P \in \pi(\mathcal{A})'$ denote the projection on the representation space $\mathcal{H}_{\pi_0}$ of π_0. Then $PE_\pi(\cdot) \restriction \mathcal{H}_{\pi_0}$ is a spectral measure $E_{\pi_0}(\cdot)$ associated with π_0. Let $a_g \in \mathcal{A}_g,\ g \in G$, and let Δ be a Borel subset in $\widehat{\mathcal{B}}^+$ such that Δ^g is defined. Suppose that $\varphi \in \mathcal{D}(\pi_0)$. Using Definition 11, (ii) for π we obtain

$$E_{\pi_0}(\Delta^g)\pi_0(a_g)\varphi = PE_\pi(\Delta^g)\pi(a_g)\varphi = P\pi(a_g)E_\pi(\Delta)\varphi = \pi_0(a_g)E_{\pi_0}(\Delta)\varphi,$$

that is, $E_{\pi_0}(\Delta^g)\pi_0(a_g) \subseteq \pi_0(a_g)E_{\pi_0}(\Delta)$, so condition (ii) of Definition 11 holds for π_0. Hence π_0 is well-behaved. □

Lemma 15 *As above, H denotes a subgroup of G. Let ρ be a well-behaved inducible representation of $\mathcal{A}_H$, E_ρ a spectral measure on $\widehat{\mathcal{B}}^+$ associated with ρ and π the induced representation $\mathrm{Ind}_{\mathcal{A}_H \uparrow \mathcal{A}}\rho$. Suppose that $b \in \mathcal{B}$ and $g \in G$. Then the domain of the operator $\int_{\mathcal{D}_g} f_b(\alpha_g(\chi))dE_\rho(\chi)$ contains $\mathcal{D}(\rho)$ and for arbitrary $a_g \in \mathcal{A}_g$ and $v \in \mathcal{D}(\rho)$ we have*

$$\pi(b)[a_g \otimes v] = [b a_g \otimes v] = \left[a_g \otimes \left(\int_{\mathcal{D}_g} f_b(\alpha_g(\chi))dE_\rho(\chi) \right) v \right]. \tag{24}$$

Proof Let $[c_g \otimes w] \in \mathcal{H}_\pi$, where $c_g \in \mathcal{A}_g,\ w \in \mathcal{D}(\rho)$. Then we have

$$\begin{aligned}\langle \pi(b)[a_g \otimes v], [c_g \otimes w] \rangle &= \langle [b a_g \otimes v], [c_g \otimes w] \rangle = \langle \rho(c_g^* b a_g)v, w \rangle \\ &= \int_{\widehat{\mathcal{B}}^+} f_{c_g^* b a_g}(\chi) d\langle E_\rho(\chi)v, w \rangle.\end{aligned}$$

From Proposition 14 we obtain the equalities $f_{c_g^* b a_g}(\chi) = f_b(\alpha_g(\chi)) f_{c_g^* a_g}(\chi)$ for $\chi \in \mathcal{D}_g$ and $f_{c_g^* b a_g}(\chi) = 0$ for $\chi \in \widehat{\mathcal{B}}^+ \backslash \mathcal{D}_g$, so the preceding is equal to

$$\int_{\mathcal{D}_g} f_b(\alpha_g(\chi)) f_{c_g^* a_g}(\chi) d\langle E_\rho(\chi)v, w\rangle = \left\langle \left(\int_{\mathcal{D}_g} f_b(\alpha_g(\chi)) f_{c_g^* a_g}(\chi) dE_\rho(\chi) \right) v, w \right\rangle.$$

Since v belongs to the domains of $\int_{\mathcal{D}_g} f_b(\alpha_g(\chi)) f_{c_g^* a_g}(\chi) dE_\rho(\chi)$ and $\int_{\mathcal{D}_g} f_{c_g^* a_g}(\chi) dE_\rho(\chi)$, the multiplicativity property of the spectral integral (see e.g. [36], 13.24) implies that v belongs to the domain of $\int_{\mathcal{D}_g} f_b(\alpha_g(\chi)) dE_\rho(\chi)$ and we can proceed

$$\begin{aligned} \langle \pi(b)[a_g \otimes v], [c_g \otimes w]\rangle &= \left\langle \left(\int_{\mathcal{D}_g} f_{c_g^* a_g}(\chi) dE_\rho(\chi) \right) \left(\int_{\mathcal{D}_g} f_b(\alpha_g(\chi)) dE_\rho(\chi) \right) v, w \right\rangle \\ &= \left\langle \rho(c_g^* a_g) \left(\int_{\mathcal{D}_g} f_b(\alpha_g(\chi)) dE_\rho(\chi) \right) v, w \right\rangle \\ &= \left\langle \left[a_g \otimes \left(\int_{\mathcal{D}_g} f_b(\alpha_g(\chi)) dE_\rho(\chi) \right) v \right], [c_g \otimes w] \right\rangle. \end{aligned}$$

Since the linear span of vectors $[c_g \otimes w]$, where $c_g \in \mathcal{A}_g$ and $w \in \mathcal{D}(\rho)$, is dense in the closed subspace to which $[b a_g \otimes v]$ and $[a_g \otimes \left(\int f_b(\alpha_g(\chi)) dE_\rho(\chi)\right) v]$ belong, the assertion follows. □

Proposition 20 *Assume that $\mathcal{B}$ is countably generated. If ρ is a well-behaved inducible cyclic representation of the $*$-algebra $\mathcal{A}_H$, then the induced representation $\pi = \mathrm{Ind}_{\mathcal{A}_H \uparrow \mathcal{A}}(\rho)$ is a well-behaved representation of the $*$-algebra $\mathcal{A}$.*

Proof Let E_ρ be a spectral measure on $\widehat{\mathcal{B}}^+$ associated with ρ. It follows from the Theorem 7, (ii) that E_ρ is supported on $\widehat{\mathcal{B}}^+$. We first show that $\mathrm{Res}_{\mathcal{B}}\pi$ is defined by a spectral measure, i.e. condition (i) in Definition 11 holds for some spectral measure E_π on $\widehat{\mathcal{B}}^+$.

Let $a_g \in \mathcal{A}_g,\ g \in G,\ w \in \mathcal{D}(\rho)$, and let Δ be a Borel subset of $\widehat{\mathcal{B}}^+$. We define a linear operator $E_\pi(\Delta)$ on the tensor product $\mathcal{A} \otimes \mathcal{D}(\rho)$ by putting $E_\pi(\Delta)(a_g \otimes w) := a_g \otimes E_\rho(\Delta^{g^{-1}})w$. Note that the vector $E_\rho(\Delta^{g^{-1}})w$ belongs to $\mathcal{D}(\rho)$. Let $h \in H$ and $a_h \in \mathcal{A}_h$. Using Proposition 17 (i) we get that

$$\begin{aligned} E_\pi(\Delta)(a_g a_h \otimes w - a_g \otimes \rho(a_h)w) &= a_g a_h \otimes E_\rho\left(\Delta^{h^{-1}g^{-1}}\right) w - a_g \otimes E_\rho\left(\Delta^{g^{-1}}\right) \rho(a_h) w = \\ &= a_g a_h \otimes E_\rho\left(\Delta^{h^{-1}g^{-1}}\right) w - a_g \otimes \rho(a_h) E_\rho\left(\Delta^{h^{-1}g^{-1}}\right) w, \end{aligned}$$

belongs to the kernel of the quotient mapping $\mathcal{A} \otimes \mathcal{D}(\rho) \to \mathcal{A} \otimes_{\mathcal{A}_H} \mathcal{D}(\rho)$, so $E_\pi(\Delta)$ defines a linear operator on $\mathcal{A} \otimes_{\mathcal{A}_H} \mathcal{D}(\rho)$ which we denote again by $E_\pi(\Delta)$.

Let $v \in \mathcal{D}(\rho)$ be a cyclic vector for ρ. Take $a \otimes v \in \mathcal{A} \otimes_{\mathcal{A}_H} \mathcal{D}(\rho)$. We write a as a finite sum $\sum_{i,k} a_{ik},\ a_{ik} \in \mathcal{A}_{g_{ik}}$, where $g_{ik} \in G$ are pairwise distinct and $g_{ik}^{-1} g_{jm} \in H$ if and only if $k = m$. Then we have $\langle a_{ik} \otimes v, a_{jm} \otimes v\rangle_0 = 0$ for $k \neq m$ and remembering that ρ is well-behaved we get

$$\begin{aligned}
\langle E_\pi(\Delta)(a \otimes v), E_\pi(\Delta)(a \otimes v)\rangle_0 &= \left\langle \sum_{i,k} a_{ik} \otimes E_\rho\left(\Delta^{g_{ik}^{-1}}\right) v, \sum_{i,k} a_{ik} \otimes E_\rho\left(\Delta^{g_{ik}^{-1}}\right) v \right\rangle_0 \\
&= \sum_k \left\langle \sum_i a_{ik} \otimes E_\rho\left(\Delta^{g_{ik}^{-1}}\right) v, \sum_i a_{ik} \otimes E_\rho\left(\Delta^{g_{ik}^{-1}}\right) v \right\rangle_0 \\
&= \sum_k \sum_{i,j} \left\langle \rho\left(a_{kj}^* a_{ki}\right) E_\rho\left(\Delta^{g_{ik}^{-1}}\right) v, E_\rho\left(\Delta^{g_{jk}^{-1}}\right) v \right\rangle \\
&= \sum_k \sum_{i,j} \left\langle E_\rho\left(\Delta^{g_{jk}^{-1}}\right) \rho\left(a_{kj}^* a_{ki}\right) v, E_\rho\left(\Delta^{g_{jk}^{-1}}\right) v \right\rangle \\
&= \sum_k \sum_{i,j} \left\langle \rho\left(a_{kj}^* a_{ki}\right) v, E_\rho\left(\Delta^{g_{jk}^{-1}}\right) v \right\rangle \\
&= \langle a \otimes v, E_\pi(\Delta)(a \otimes v)\rangle_0 \qquad (25)
\end{aligned}$$

Assume that $a \otimes v \in \mathcal{K}_\rho$, that is, $\langle a \otimes v, a \otimes v\rangle_0 = 0$. The preceding calculation implies that $E_\pi(\Delta)(a \otimes v) \in \mathcal{K}_\rho$, so $E_\pi(\Delta)$ is a well-defined linear operator on the linear span of vectors $[a \otimes v] \in \mathcal{D}(\pi)$ defined by

$$E_\pi(\Delta)[a_g \otimes v] := [a_g \otimes E_\rho(\Delta^{g^{-1}})v]. \qquad (26)$$

Since v is cyclic, the set of vectors $[a \otimes v]$ is dense in $\mathcal{H}_\pi$ by Lemma 5. It follows from Eq. 25 that $E_\pi(\Delta)$ is bounded and can be extended by continuity to $\mathcal{H}_\pi$. From now on we consider $E_\pi(\Delta)$ on the subspace $\mathcal{H}_\pi$.

It can be easily seen that $E_\pi(\Delta)^2 = E_\pi(\Delta)$. We prove that $E_\pi(\Delta)$ is self-adjoint. For this it suffices to show that

$$\langle E_\pi(\Delta)[a_{g_1} \otimes v], [a_{g_2} \otimes v]\rangle = \langle [a_{g_1} \otimes v], E_\pi(\Delta)[a_{g_2} \otimes v]\rangle \qquad (27)$$

for $a_{g_1} \in \mathcal{A}_{g_1}$, $a_{g_2} \in \mathcal{A}_{g_2}$, $g_1, g_2 \in G$. First we consider the case when $g_1 H \neq g_2 H$. Then we get

$$\begin{aligned}
\langle E_\pi(\Delta)[a_{g_1} \otimes v], [a_{g_2} \otimes v]\rangle &= \langle [a_{g_1} \otimes E_\rho(\Delta^{g_1^{-1}})v], [a_{g_2} \otimes v]\rangle = \\
&= \langle \rho(p_H(a_{g_2}^* a_{g_1})) E_\rho(\Delta^{g_1^{-1}})v, v\rangle = 0,
\end{aligned}$$

since $p_H(a_{g_2}^* a_{g_1}) = 0$. Analogously, $\langle [a_{g_1} \otimes v], E_\pi(\Delta)[a_{g_2} \otimes v]\rangle = 0$, so that Eq. 27 holds in this case. Now suppose that $g_1 H = g_2 H$. Then we have

$$\langle E_\pi(\Delta)[a_{g_1} \otimes v], [a_{g_2} \otimes v]\rangle = \langle [a_{g_1} \otimes E_\rho(\Delta^{g_1^{-1}})v], [a_{g_2} \otimes v]\rangle = \langle \rho(a_{g_2}^* a_{g_1}) E_\rho(\Delta^{g_1^{-1}})v, v\rangle.$$

Since ρ is well-behaved and $a_{g_2}^* a_{g_1} \in \mathcal{A}_{g_2^{-1} g_1}$, the preceding equals to

$$= \langle E_\rho(\Delta^{g_2^{-1}})\rho(a_{g_2}^* a_{g_1})v, v\rangle = \langle \rho(a_{g_2}^* a_{g_1})v, E_\rho(\Delta^{g_2^{-1}})v\rangle = \langle [a_{g_1} \otimes v], E_\pi(\Delta)[a_{g_2} \otimes v]\rangle.$$

Thus, $E_\pi(\Delta)$ is self-adjoint.

Take $a_g \in \mathcal{A}_g$, a Borel set $\Delta \subseteq \widehat{\mathcal{B}}^+$ and $a_k \in \mathcal{A}_k$. Then we get

$$\begin{aligned}\pi(a_g)E_\pi(\Delta)[a_k \otimes v] &= \pi(a_g)\left[a_k \otimes E_\rho\left(\Delta^{k^{-1}}\right)v\right] = \left[a_g a_k \otimes E_\rho\left(\Delta^{k^{-1}}\right)v\right] \\ &= \left[a_g a_k \otimes E_\rho\left((\Delta^g)^{(gk)^{-1}}\right)v\right] = E_\pi(\Delta^g)\left[a_g a_k \otimes v\right] \\ &= E_\pi\left(\Delta^g\right)\pi(a_g)[a_k \otimes v]. \end{aligned} \tag{28}$$

Next we prove that $E_\pi(\Delta)\mathcal{D}(\pi) \subseteq \mathcal{D}(\pi)$. Take $d_g \in \mathcal{A}_g,\ g \in G$. Using Eq. 28 we obtain

$$\begin{aligned}\|E_\pi(\Delta)[a \otimes v]\|_{d_g}^2 &= \left\|\pi(d_g)E_\pi(\Delta)[a \otimes v]\right\|^2 \\ &= \langle \pi(d_g)E_\pi(\Delta)[a \otimes v], \pi(d_g)E_\pi(\Delta)[a \otimes v]\rangle = \\ &= \langle E_\pi(\Delta^g)\pi(d_g)[a \otimes v],\ E_\pi(\Delta^g)\pi(d_g)[a \otimes v]\rangle \\ &= \langle \pi(d_g)[a \otimes v],\ E_\pi(\Delta^g)\pi(d_g)[a \otimes v]\rangle = \\ &= \langle \pi(d_g), \pi(d_g)E_\pi(\Delta)[a \otimes v]\rangle \le \|[a \otimes v]\|_{d_g} \cdot \|E_\pi(\Delta)[a \otimes v]\|_{d_g}, \end{aligned}$$

and hence $\|E_\pi(\Delta)[a \otimes v]\|_{d_g} \le \|[a \otimes v]\|_{d_g}$. By Lemma 5, the set of vectors $[a \otimes v]$ is a core for π . Therefore, the preceding shows that $E_\pi(\Delta)$ is continuous in the graph topology of π. This in turn implies that $E_\pi(\Delta)\mathcal{D}(\pi) \subseteq \mathcal{D}(\pi)$.

Now we prove that $E_\pi(\cdot)$ defines a spectral measure on $\widehat{\mathcal{B}}^+$. For $a_g \in \mathcal{A}_g$ we have

$$\begin{aligned}\left\langle E_\pi\left(\widehat{\mathcal{B}}^+\right)\left[a_g \otimes v\right], \left[a_g \otimes v\right]\right\rangle &= \left\langle\left[a_g \otimes E_\rho(\mathcal{D}_g)v\right], \left[a_g \otimes v\right]\right\rangle = \\ &= \left\langle \rho\left(a_g^* a_g\right)E_\rho\left(\mathcal{D}_g\right)v, v\right\rangle = \left\langle \rho\left(a_g^* a_g\right)v, v\right\rangle \\ &= \left\langle\left[a_g \otimes v\right], \left[a_g \otimes v\right]\right\rangle \end{aligned}$$

which shows that $E_\pi(\widehat{\mathcal{B}}^+) = I$. The countable additivity $E_\pi(\cdot)$ follows at once from the countable additivity of $E_\rho(\cdot)$.

Next we show that $\mathrm{Res}_{\mathcal{B}}\pi$ is an integrable representation associated with spectral measure E_π. It suffices to prove that

$$\langle b[a_{g_1} \otimes v], [a_{g_2} \otimes v]\rangle = \int f_b(\chi)d\langle E_\pi(\chi)[a_{g_1} \otimes v], [a_{g_2} \otimes v]\rangle. \tag{29}$$

for all $[a_{g_1} \otimes v], [a_{g_2} \otimes v] \in \mathcal{H}_\pi$. In the case $g_1 H \neq g_2 H$ one easily checks that the both sides of Eq. 29 are equal to zero. In the case $g_1 H = g_2 H$ we use Eq. 24 and compute

$$\begin{aligned}\langle \pi(b)[a_{g_1} \otimes v], [a_{g_2} \otimes v]\rangle &= \left\langle\left[a_{g_1} \otimes \int_{\mathcal{D}_{g_1}} f_b\left(\alpha_{g_1}(\chi)\right)dE_\rho(\chi) \otimes v\right], \left[a_{g_2} \otimes v\right]\right\rangle \\ &= \left\langle \rho\left(a_{g_2}^* a_{g_1}\right)\int_{\mathcal{D}_{g_1}} f_b\left(\alpha_{g_1}(\chi)\right)dE_\rho(\chi)v, v\right\rangle. \end{aligned}$$

Applying Proposition 17 (iv) we continue

$$
\begin{aligned}
&= \left\langle \int_{\mathcal{D}_{g_2}} f_b(\alpha_{g_2}(\chi)) dE_\rho(\chi)\rho(a_{g_2}^* a_{g_1})v, v \right\rangle \\
&= \int_{\mathcal{D}_{g_2}} f_b(\alpha_{g_2}(\chi)) d\langle E_\rho(\chi)\rho(a_{g_2}^* a_{g_1})v, v\rangle \\
&= \int_{\mathcal{D}_{g_2}} f_b(\alpha_{g_2}(\chi)) d\langle \rho(a_{g_2}^* a_{g_1}) E_\rho(\alpha_{g_1^{-1}g_2}(\chi))v, v\rangle \\
&= \int_{\mathcal{D}_{g_2^{-1}}} f_b(\chi) d\langle \rho(a_{g_2}^* a_{g_1}) E_\rho(\alpha_{g_1^{-1}}(\chi))v, v\rangle \\
&= \int_{\mathcal{D}_{g_2^{-1}}} f_b(\chi) d\langle [a_{g_1} \otimes E_\rho(\alpha_{g_1^{-1}}(\chi))v], [a_{g_2} \otimes v]\rangle \\
&= \int_{\mathcal{D}_{g_2^{-1}}} f_b(\chi) d\langle E_\pi(\chi)[a_{g_1} \otimes v], [a_{g_2} \otimes v]\rangle \\
&= \int_{\mathcal{D}_{g_2^{-1}}} f_b(\chi) d\langle [a_{g_1} \otimes v], [a_{g_2} \otimes E_\rho(\alpha_{g_2^{-1}}(\chi))v]\rangle \\
&= \int_{\widehat{\mathcal{B}}^+} f_b(\chi) d\langle E_\pi(\chi)[a_{g_1} \otimes v], [a_{g_2} \otimes v]\rangle.
\end{aligned}
$$

It follows from Eq. 28 that the equality in the Definition 11, (ii) holds on the span of vectors $[a \otimes v] \in \mathcal{D}(\pi)$ which is a core of π by Lemma 5. Since $\pi(a_g)$ and $E_\pi(\Delta)$ are continuous in the graph topology of π, condition (ii) in Definition 11 holds for π. This completes the proof. □

In what follows, we want to induce from arbitrary well-behaved representations of subalgebras $\mathcal{A}_H$. For this reason we shall need the decomposition of well-behaved representations into direct sums of cyclic well-behaved representations. This aim will be achieved by Proposition 22 below. First we develop some more preliminaries.

Lemma 16 *Suppose that π is a well-behaved representation of $\mathcal{A}$. Let $a_g \in \mathcal{A}_g$ and let UC be the polar decomposition of $\overline{\pi(a_g)}$. Then U belongs to $\pi(\mathcal{A})''$.*

Proof Let $T \in \pi(\mathcal{A})'$. As noted already in the proof of Proposition 17, we have $C^2 = \overline{\pi(a_g^* a_g)}$. Since T commutes with $\pi(a_g^* a_g)$, it commutes with C^2 and therefore with C.

Take $\varphi \in \mathcal{D}(C)$. Then we obtain $TU(C\varphi) = T\overline{\pi(a_g)}\varphi = \overline{\pi(a_g)}T\varphi = UCT\varphi = UT(C\varphi)$. Now let $\psi \in \ker C = \ker U = \ker \overline{\pi(a_g)}$. Then we have $\overline{\pi(a_g)}T\psi = T\overline{\pi(a_g)}\psi = 0$, i.e. $T \ker U \subseteq \ker U$, so that $UT\psi = 0 = TU\psi$. Therefore, T and U commute on the linear dense subspace $\ker C + \operatorname{Ran} C$. Since T and U are bounded, they commute on $\mathcal{H}_\pi$. This shows that $U \in \pi(\mathcal{A})''$. □

Lemma 17 *If π is a well-behaved representation of $\mathcal{A}$, then we have:*

(i) $\overline{\pi(a_g^*)} = \pi(a_g)^*$ *for* $a_g \in \mathcal{A}_g$.
(ii) $\overline{\pi(a_g a_k)} = \overline{\overline{\pi(a_g)} \cdot \overline{\pi(a_k)}}$ *for* $a_g \in \mathcal{A}_g$ *and* $a_k \in \mathcal{A}_k$.

Proof

(i) It is clear that $\pi(a_g a_g^*) \subseteq \pi(a_g^*)^* \overline{\pi(a_g^*)}$. Since π is well-behaved, $\mathrm{Res}_{\mathcal{B}}\pi$ is integrable, so $\pi(a_g a_g^*)$ is essentially self-adjoint ([39], 9.1.2). Hence it follows that $\overline{\pi(a_g a_g^*)} = \pi(a_g^*)^* \overline{\pi(a_g^*)} = |\overline{\pi(a_g^*)}|^2$. By the same reasoning we obtain $\overline{\pi(a_g a_g^*)} = \overline{\pi(a_g)} \pi(a_g)^* = |\pi(a_g)^*|^2$. Combining these relations with the fact that $\mathcal{D}(T) = \mathcal{D}(|T|)$ for a closed operator T we get

$$\mathcal{D}(\overline{\pi(a_g^*)}) = \mathcal{D}(|\overline{\pi(a_g^*)}|) = \mathcal{D}((\overline{\pi(a_g a_g^*)})^{1/2}) = \mathcal{D}(|\pi(a_g)^*|) = \mathcal{D}(\pi(a_g)^*).$$

Since $\overline{\pi(a_g^*)} \subseteq \pi(a_g)^*$, the preceding implies that $\overline{\pi(a_g^*)} = \pi(a_g)^*$.

(ii) Clearly, $\pi(a_k^* a_g^* a_g a_k) \subseteq \left(\overline{\pi(a_g)} \cdot \overline{\pi(a_k)}\right)^* \overline{\overline{\pi(a_g)} \cdot \overline{\pi(a_k)}}$. Since $a_k^* a_g^* a_g a_k \in \mathcal{B}$, the operator $\overline{\pi(a_k^* a_g^* a_g a_k)}$ is self-adjoint, so we have the equality $\overline{\pi(a_k^* a_g^* a_g a_k)} = \left(\overline{\pi(a_g)} \cdot \overline{\pi(a_k)}\right)^* \overline{\overline{\pi(a_g)} \cdot \overline{\pi(a_k)}}$ which yields $\mathcal{D}((\overline{\pi(a_k^* a_g^* a_g a_k)})^{1/2}) = \mathcal{D}(\overline{\overline{\pi(a_g)} \cdot \overline{\pi(a_k)}})$. As shown in the proof of (i) we also have that $\mathcal{D}(\overline{\pi(a_g a_k)}) = \mathcal{D}((\overline{\pi(a_k^* a_g^* a_g a_k)})^{1/2}$. Combining these two equalities with the obvious inclusion $\overline{\pi(a_g a_k)} \subseteq \overline{\overline{\pi(a_g)} \cdot \overline{\pi(a_k)}}$, the assertion follows. □

Lemma 18 *Let π be a well-behaved $*$-representation of $\mathcal{A}$. We denote by $\mathcal{U}_\pi$ the set of all partial isometries in the polar decompositions of elements $\overline{\pi(a_g)}$, where $a_g \in \mathcal{A}_g$, $g \in G$. Then*

$$\mathfrak{A}_0 = \left\{ \sum_{i=1}^{n} \lambda_i U_i E_\pi(\Delta_i) : \lambda_i \in \mathbb{C},\ U_i \in \mathcal{U}_\pi,\ \Delta_i \subseteq \widehat{\mathcal{B}}^+,\ \Delta_i \text{ is a Borel set} \right\}$$

is a dense $*$-subalgebra of $\pi(\mathcal{A})''$ in the strong operator topology.

Proof Since $\mathcal{U}_\pi \subseteq \pi(\mathcal{A})''$ by Lemma 16 and the spectral projections $E_\pi(\cdot)$ belong to $\pi(\mathcal{B})'' \subseteq \pi(\mathcal{A})''$, we conclude that $\mathfrak{A}_0 \subseteq \pi(\mathcal{A})''$.

We prove that $\mathfrak{A}_0$ is a $*$-algebra. Take $a_g \in \mathcal{A}_g$ and let $U_g|\overline{\pi(a_g)}|$ be the polar decomposition of the closed operator $\overline{\pi(a_g)}$. By Lemma 17, (i) we have $\overline{\pi(a_g^*)} = \pi(a_g)^*$. It is well-known (see e.g. [21], p. 421), that $U_g^*|\overline{\pi(a_g^*)}|$ is the polar decomposition of the adjoint operator $\overline{\pi(a_g^*)} = \pi(a_g)^*$ of $\overline{\pi(a_g)}$. Therefore, $U_g^* \in \mathfrak{A}_0$ which proves that $\mathfrak{A}_0$ is $*$-invariant.

Take another element $a_k \in \mathcal{A}_k$, $k \in G$ and let $U_k C_k$ be the polar decomposition of $\overline{\pi(a_k)}$. Then using Lemma 17 and Proposition 17 (iii) we get

$$\overline{\pi(a_g a_k)} \supseteq U_g C_g U_k C_k \supseteq U_g U_k \int_{\mathcal{D}_k} f_{a_g^* a_g}(\alpha_k(\chi)) dE_\pi(\chi) \cdot C_k. \tag{30}$$

From the properties of the polar decomposition and the equality $\overline{\pi(a_g^*a_g)} = \int f_{a_g^*a_g} dE_\pi$ we conclude that $U_g^*U_g = E_\pi(f_{a_g^*a_g}^{-1}(0,+\infty))$. Similarly, $U_k^*U_k = E_\pi(f_{a_k^*a_k}^{-1}(0,+\infty))$. Using Proposition 17 (ii) it follows that

$$(U_gU_k)^*U_gU_k = U_k^*E_\pi(f_{a_g^*a_g}^{-1}(0,+\infty))U_k = U_k^*U_kE_\pi(\alpha_{k^{-1}}(\mathcal{D}_{k^{-1}} \cap f_{a_g^*a_g}^{-1}(0,+\infty))) =$$

$$= E_\pi(f_{a_k^*a_k}^{-1}(0,+\infty))E_\pi(\alpha_{k^{-1}}(\mathcal{D}_{k^{-1}} \cap f_{a_g^*a_g}^{-1}(0,+\infty))) \tag{31}$$

is a projection. Hence U_gU_k is a partial isometry. We denote by S_{gk} the closure of the operator $\int_{\mathcal{D}_k} f_{a_g^*a_g}(\alpha_k(\chi))dE_\pi(\chi) \cdot C_k$. From Eq. 31 and the properties of the partial action we conclude that the kernels of U_gU_k and S_{gk} are equal. Since S_{gk} is positive and its domain $\mathcal{D}(S_{gk})$ contains $\mathcal{D}(\pi)$, it follows from Eq. 30 that the polar decomposition of $\overline{\pi(a_ga_k)}$ is $U_gU_kS_{gk}$. Hence U_gU_k belongs to $\mathcal{U}_\pi$. By Proposition 17 (ii), $\mathfrak{A}_0$ is closed under multiplication. That is, $\mathfrak{A}_0$ is a unital ∗-algebra.

Since any $T \in \mathfrak{A}_0'$ commutes with $\mathcal{U}_\pi$ and with the spectral projections $E_\pi(\cdot)$, we have $T \in \pi(\mathcal{A})'$. That is, $\mathfrak{A}_0' \subseteq \pi(\mathcal{A})'$ and so $\mathfrak{A}_0'' \supseteq \pi(\mathcal{A})''$ which implies that $\mathfrak{A}_0'' = \pi(\mathcal{A})''$. Hence $\mathfrak{A}_0$ is dense in $\pi(\mathcal{A})''$ in the strong operator topology. □

Proposition 21 *Suppose that π is a well-behaved representation of algebra $\mathcal{A}$ such that the graph topology of π is metrizable. Then π is cyclic if and only if the von Neumann algebra $\pi(\mathcal{A})''$ is cyclic.*

Proof Suppose that $\varphi_0 \in \mathcal{H}_\pi$ is a cyclic vector for π. Let $\psi \in \mathcal{D}(\pi)$ and $\varepsilon > 0$. Then there exists an element $a \in \mathcal{A}$ such that $\|\pi(a)\varphi_0 - \psi\| < \varepsilon$. Clearly, a is a finite sum $a_1 + a_2 + \cdots + a_k$, where each a_i belong to some vector space $\mathcal{A}_g$, $g \in G$. Let $\overline{\pi(a_i)} = U_iC_i$ be the polar decomposition of $\overline{\pi(a_i)}$. Since the operators U_i (by Lemma 18) and the spectral projections $E_{C_i}(\cdot)$ of C_i belong to $\pi(\mathcal{A})''$, the operators

$$A_{i,r} := U_i \int_{-r}^{r} \lambda dE_{C_i}(\lambda), \ r \in \mathbb{N},$$

are in the von Neumann algebra $\pi(\mathcal{A})''$. We choose $r \in \mathbb{N}$ such that $\|(A_{i,r} - \pi(a_i))\varphi_0\| < \varepsilon/k$, $i = 1, \ldots, k$, and put $A_r := A_{1,r} + \cdots + A_{k,r}$. Then we have

$$\|A_r\varphi_0 - \psi\| \le \|(A_r - \pi(a))\varphi_0\| + \|\pi(a)\varphi_0 - \psi\|$$
$$\le \sum_{i=1}^{k} \|(A_{i,r} - \pi(a_i))\varphi\| + \|\pi(a)\varphi_0 - \psi\| < 2\varepsilon.$$

Since $A_r \in \pi(\mathcal{A})''$, this shows that φ_0 is cyclic for $\pi(\mathcal{A})''$.

Conversely, suppose that φ_0 is a cyclic vector for the von Neumann algebra $\pi(\mathcal{A})''$. Let P_0 be the orthogonal projection onto the closure of $\pi(\mathcal{B})''\varphi_0$. Obviously, $P_0 \in \pi(\mathcal{B})'$. Since $\mathrm{Res}_{\mathcal{B}}\pi$ is self-adjoint by Definition 11, $P_0\mathcal{H}_\pi$ reduces $\mathrm{Res}_{\mathcal{B}}\pi$ to a self-adjoint subrepresentation ρ ([39], 8.3.11) which is also integrable ([39], 9.1.17). The graph topology of π is metrizable by assumption, so are the graph topologies of $\mathrm{Res}_{\mathcal{B}}\pi$ and ρ by Lemma 12, (i). Therefore, a theorem of R.T. Powers ([30], see [39], 9.2.1) applies and states that ρ is cyclic, that is, there exists a vector $\psi_0 \in \mathcal{D}(\rho)$ such that $\rho(\mathcal{B})\psi_0$ is dense in $\mathcal{D}(\rho)$ in the graph topology. In particular $\overline{\rho(\mathcal{B})\psi_0} =$

$P_0\mathcal{H}_\pi = \overline{\pi(\mathcal{B})''\varphi_0}$. Hence ψ_0 is also cyclic for the commutative von Neumann algebra $\rho(\mathcal{B})'' = P_0\pi(\mathcal{B})''P_0$. Our aim is to show that ψ_0 is cyclic for π, that is, $\pi(\mathcal{A})\psi_0$ is dense in $\mathcal{D}(\pi)$ in the graph topology of π.

We first show that the subspace $\mathcal{H}_0 := \pi(\mathcal{A})\psi_0$ is dense in $\mathcal{H}_\pi$. Let $\mathfrak{A}_0$ be as in Lemma 18. Since $\mathfrak{A}_0$ is dense in $\pi(\mathcal{A})''$ in the strong operator topology, the vector φ_0 is also cyclic for $\mathfrak{A}_0$. Let $U_g \in \mathcal{U}_\pi$ and $a_g \in \mathcal{A}_g$, $g \in G$, be such that the polar decomposition of $\overline{\pi(a_g)}$ is U_gC_g. It suffices to show, that for any Borel $\Delta_0 \subseteq \widehat{\mathcal{B}}^+$ and $\varepsilon > 0$ there exists $b_1 \in \mathcal{B}$ such that

$$\left\| U_g E_\pi(\Delta_0)\varphi_0 - \pi(a_g b_1)\psi_0 \right\| < \varepsilon. \tag{32}$$

Let b_0 be such that $\|\rho(b_0)\psi_0 - E_\pi(\Delta_0)\varphi_0\| < \varepsilon/3$. Denote by E_{C_g} the spectral measure on $\mathbb{R}_+$ associated with C_g. Since $U_gE_{C_g}([0,+\infty)) = U_gE_{C_g}((0,+\infty))$, we can choose n such that

$$\left\| U_g(E_{C_g}([0,1/n]) + E_{C_g}([n,+\infty)))\rho(b_0)\psi_0 \right\| < \varepsilon/3. \tag{33}$$

Further, let f be the function on $\mathbb{R}$ defined by $f(x) = 1/x$ if $x \in (1/n, n)$ and $f(x) = 0$ otherwise. Then the bounded operator $f(C_g)$ is quasi-inverse to C_g, that is, we have

$$Id_{\mathcal{H}_\pi} = C_g f(C_g) + E_{C_g}([0,1/n]) + E_{C_g}([n,+\infty)).$$

Since ψ_0 is strongly cyclic and $\overline{\pi(a_g^*a_g)} = C_g^2$, there exists $b_1 \in \mathcal{B}$ such that

$$\left\| (1 + C_g^2)(f(C_g)\rho(b_0) - \rho(b_1))\psi_0 \right\| < \varepsilon/3. \tag{34}$$

Using Eqs. 33 and 34 we derive

$$\begin{aligned}
\left\| U_g E_\pi(\Delta_0)\varphi_0 - \pi(a_g b_1)\psi_0 \right\| &\le \left\| U_g(E_\pi(\Delta_0)\varphi_0 - \rho(b_0)\psi_0) \right\| \\
&\quad + \left\| U_g(\rho(b_0) - C_g\rho(b_1))\psi_0 \right\| \\
&\le \left\| U_g \right\| \varepsilon/3 + \left\| U_g \left(E_{C_g}([0,1/n]) + E_{C_g}([n,+\infty)) \right) \rho(b_0)\psi_0 \right\| \\
&\quad + \left\| U_g(C_g f(C_g)\rho(b_0) - C_g\rho(b_1))\psi_0 \right\| \\
&\le \varepsilon/3 + \varepsilon/3 + \left\| U_g C_g \left(1 + C_g^2\right)^{-1} \right\| \\
&\quad \cdot \left\| \left(1 + C_g^2\right)(f(C_g)\rho(b_0) - \rho(b_1))\psi_0 \right\| < \varepsilon.
\end{aligned}$$

Thus we have shown that $\mathcal{H}_0$ is dense in $\mathcal{H}_\pi$.

Let $\mathcal{D}_0$ denote the closure of $\pi(\mathcal{A})\psi_0$ in the graph topology of π. We show that the representation $\pi_0 := \pi \restriction \mathcal{D}_0$ of $\mathcal{A}$ is self-adjoint. Since ρ is a restriction of $\mathrm{Res}_{\mathcal{B}}\pi$, it is inducible. Let $\mathcal{H}_1$ denote the representation space of $\mathrm{Ind}\rho$. Define a linear operator $T : \mathcal{A} \otimes \mathcal{D}(\rho) \to \mathcal{D}_0 \subseteq \mathcal{D}(\pi)$ by $T(a \otimes \psi_0) := \pi(a)\psi_0$. One easily checks that T gives rise to a unitary operator $\widetilde{T}$ of $\mathcal{H}_1$ onto $\mathcal{H}_0$ such that $\widetilde{T}[a \otimes \psi_0] = \pi(a)\psi_0$ and that $\widetilde{T}$ defines a unitary equivalence of representations $\mathrm{Ind}\rho$ and π_0. Since ρ is cyclic and well-behaved, $\mathrm{Ind}\rho$ is well-behaved by Proposition 20 and hence self-adjoint by Lemma 12. Therefore, π_0 is self-adjoint. Since $\mathcal{D}(\pi_0) = \mathcal{D}_0$ is dense in $\mathcal{H}_\pi$ as shown in the preceding paragraph, the $*$-representation π of $\mathcal{A}$ is an extension of the self-adjoint representation π_0 acting on the same Hilbert space $\mathcal{H}_0$. By Corollary 8.3.12 in [39] this implies that $\mathcal{D}_0 = \mathcal{D}(\pi)$, that is, ψ_0 is a cyclic vector for π. □

Proposition 22 *Let π be a well-behaved representation of $\mathcal{A}$ on the Hilbert space $\mathcal{H}_\pi$ such that the graph topology of π is metrizable. Then π can be decomposed into a direct orthogonal sum of cyclic well-behaved representations.*

Proof The identity representation of the von Neumann algebra $\pi(\mathcal{A})''$ can be decomposed into a direct sum of cyclic representations, i.e. there exists a decomposition $\mathcal{H}_\pi = \oplus_{i\in I}\mathcal{H}_i$ such that the orthogonal projections P_i onto $\mathcal{H}_i$ belong to $\pi(\mathcal{A})'$ and each von Neumann algebra $P_i\pi(\mathcal{A})''$ is cyclic on $\mathcal{H}_i$. By Proposition 8.3.11 in [39] each representation $\pi_i := \pi \upharpoonright P_i\mathcal{D}(\pi)$ is self-adjoint. It is straightforward to check that $\pi = \oplus_{i\in I}\pi_i$. Since π is well-behaved, it follows from Proposition 19 that $\pi_i, i \in I$, is well-behaved. By Proposition 21, each representation π_i is cyclic. □

Proposition 22 combined with Lemmas 2 and 12 implies the following

Proposition 23 *Let H be a subgroup of G and let ρ be a well-behaved representation of $\mathcal{A}_H$ with metrizable graph topology. Then ρ is inducible to a $*$-representation of $\mathcal{A}$ if and only ρ is $\mathcal{C}$-positive, where $\mathcal{C} := \sum \mathcal{A}^2 \cap \mathcal{A}_H$.*

8 Well-Behaved Systems of Imprimitivity

In this section we shall prove an analogue of the Imprimitivity Theorem for well-behaved representations. A crucial step for this is to show that representations induced from well-behaved ones are again well-behaved. In the view of Proposition 20 we assume for this section that $\mathcal{B}$ is countably generated. We retain the notation from the previous section. Throughout H denotes a subgroup of the group G.

Definition 12 A system of imprimitivity (π, E) for $\mathcal{A}$ over G/H is called *well-behaved* if

(i) π is a well-behaved representation of $\mathcal{A}$,
(ii) the projections E and E_π commute, that is, $E(t)E_\pi(\Delta) = E_\pi(\Delta)E(t)$ for all $t \in G/H$ and all Borel subsets Δ of $\widehat{\mathcal{B}}^+$.

From Propositions 20 and 22 we obtain the following result.

Proposition 24 *If ρ is a well-behaved inducible representation of the $*$-algebra $\mathcal{A}_H$ with metrizable graph topology, then the induced representation $\pi = \mathrm{Ind}_{\mathcal{A}_H\uparrow\mathcal{A}}(\rho)$ is a well-behaved representation of the $*$-algebra $\mathcal{A}$.*

The next proposition is an analogue of Proposition 9.

Proposition 25 *If ρ is a well-behaved inducible $*$-representation of $\mathcal{A}_H$, then the system of imprimitivity induced by ρ is non-degenerate and well-behaved.*

Proof Let (π, E) be the system of imprimitivity induced by ρ and let $E_\pi(\cdot)$ be a spectral measure associated with π. It follows from Proposition 9 that (π, E) is non-degenerate. By Proposition 24 the representation π is well-behaved. From the construction of $E(\cdot)$ (see Section 4) and relation (26) it follows easily that $E(\cdot)$ and $E_\pi(\cdot)$ commute. □

Theorem 3 (Imprimitivity Theorem for well-behaved representations) *Let H be a subgroup of G and let (π, E) be a non-degenerate well-behaved system of imprimitivity for $\mathcal{A}$ over G/H. Then there exists a unique, up to unitary equivalence, inducible well-behaved representation ρ of $\mathcal{A}_H$ such that (π, E) is unitarily equivalent to the system of imprimitivity induced by ρ.*

Proof Define ρ as in the proof of the Theorem 1. By Theorem 1 we only need to prove that ρ is well-behaved. Recall that the representation space $\mathcal{H}_\rho$ is defined as $\operatorname{Ran} E(H)$ and the domain $\mathcal{D}(\rho)$ of ρ is $\mathcal{D}(\pi) \cap \operatorname{Ran} E(H)$. For a Borel set $\Delta \subseteq \widehat{\mathcal{B}}^+$ put $E_\rho(\Delta) := E_\pi(\Delta)E(H)$. Since $E_\pi(\cdot)$ commutes with $E(\cdot)$, E_ρ is a well-defined spectral measure on $\widehat{\mathcal{B}}^+$ whose values are projections in the Hilbert space $\operatorname{Ran} E(H) = \mathcal{H}_\rho$. One easily checks that $\operatorname{Res}_{\mathcal{B}}\rho$ is integrable and defined by $E_\rho(\cdot)$.

Let $a_h \in \mathcal{A}_h$, $h \in H$, $v \in \mathcal{D}(\rho)$, and let $\Delta \subseteq \widehat{\mathcal{B}}^+$ be a Borel set. Since $\pi(a_h)v = E(H)\pi(a_h)v$, we compute

$$\rho(a_h)E_\rho(\Delta)v = \pi(a_h)E_\pi(\Delta)v = E_\pi(\Delta^h)\pi(a_h)v = E_\rho(\Delta^h)\rho(a_h)v.$$

Hence ρ is well-behaved. □

For the sake of completeness we formulate an analogue of Theorem 2 for well-behaved representations. Using the fact that well-behaved subrepresentations have complements, the proof is similar to that of Theorem 2.

Theorem 4 *Let H be a subgroup of G and let (π, E) be a well-behaved system of imprimitivity for $\mathcal{A}$ over G/H. Fix one element $k_t \in G$, $t \in G/H$, in each left coset from G/H. Then for every $t \in G/H$ there exists a well-behaved $*$-representation ρ_t of $\mathcal{A}_{k_t H k_t^{-1}}$ on a Hilbert space $\mathcal{H}_t$ such that:*

(i) *ρ_t is inducible,*
(ii) *(π, E) is the direct sum of systems of imprimitivity (π_t, E_t), $t \in G/H$, where (π_t, E_t) is conjugated by the element k_t to the system of imprimitivity induced by ρ_t, $t \in G/H$.*

Definition 13 Let π be a well-behaved representation of $\mathcal{A}$. We say that π *is associated with an orbit* $\operatorname{Orb}\chi$, where $\chi \in \widehat{\mathcal{B}}^+$, if the spectral measure E_π associated with π is supported on the set $\operatorname{Orb}\chi$.

The next theorem is a central result of the Mackey analysis (cf. [12], p. 1251 and p. 1284).

Theorem 5 *Assume that the group G is countable. Let $\chi \in \widehat{\mathcal{B}}^+$ be a character and let $H = \operatorname{St}\chi$ be its stabilizer group. Then the map*

$$\rho \mapsto \operatorname{Ind}_{\mathcal{A}_H \uparrow \mathcal{A}}(\rho) = \pi \tag{35}$$

is a bijection from the set of unitary equivalence classes of inducible representations ρ *of* $\mathcal{A}_H$ *for which*

$$\mathrm{Res}_{\mathcal{B}}\rho \text{ corresponds to a multiple of the character } \chi \tag{36}$$

onto the set of unitary classes of well-behaved representations π *of* $\mathcal{A}$ *associated with* Orbχ. *A* $*$*-representation* ρ *satisfying Eq.* 36 *is bounded and inducible. Moreover, the von Neumann algebras* $\rho(\mathcal{A}_H)'$ *and* $\pi(\mathcal{A})'$ *are isomorphic. In particular,* π *is irreducible if and only if* ρ *is irreducible.*

Proof Let π be a well-behaved representation of $\mathcal{A}$ associated with Orbχ, $\chi \in \widehat{\mathcal{B}}^+$. Since G is countable, the orbit Orbχ is also countable. Therefore the spectral measure E_π is discrete. From the definition of E_π it follows that $E_\pi(\{\psi\})$, $\psi \in \mathrm{Orb}\chi$, is the eigenspace of each operator $\pi(b)$, $b \in \mathcal{B}$, corresponding to the eigenvalue $\psi(b)$. Hence for all $\psi \in \mathrm{Orb}\chi$ the range $\mathrm{Ran}\,E_\pi(\{\psi\})$ is contained in the domain of $\mathrm{Res}_{\mathcal{B}}\pi$ which is equal to $\mathcal{D}(\pi)$.

Since H is the stabilizer of χ, the projections $E_\pi(\{\chi\}^{g_1})$ and $E_\pi(\{\chi\}^{g_2})$ are equal if $g_1H = g_2H$ and for all $v \in \mathcal{D}(\pi)$ we have

$$\pi(a_g)E_\pi(\{\chi\}^k)v = E_\pi(\{\chi\}^{gk})\pi(a_g)v.$$

(Note that if $\chi \in \mathcal{D}_g$, then $E_\pi(\{\chi\}^g)$ is equal to $E_\pi(\{\alpha_g(\chi)\})$, otherwise it is zero projection.) Therefore, we can define a system of imprimitivity E for $\mathcal{A}$ over G/H by putting $E(gH) := E_\pi(\{\chi\}^g)$.

We show that (π, E) is non-degenerate. Let $g \in G$ be such that $\chi \in \mathcal{D}_g$ and let $e_{\chi^g} \in \mathrm{Ran}\,E(gH)$ be a non-zero vector. Since $\chi^g \in \mathcal{D}_{g^{-1}}$, there exists $a_{g^{-1}} \in \mathcal{A}_{g^{-1}}$ such that $\chi^g(a^*_{g^{-1}}a_{g^{-1}}) > 0$. Since e_{χ^g} belongs to $\mathrm{Ran}\,E(gH)$ and $a_{g^{-1}} \in \mathcal{A}_{g^{-1}}$, the vector $\pi(a_{g^{-1}})e_{\chi^g}$ belongs to $\mathrm{Ran}\,E(H)$. Set $e_\chi = (\chi^g(a^*_{g^{-1}}a_{g^{-1}}))^{-1}\pi(a_{g^{-1}})e_{\chi^g}$. Then, since $a^*_{g^{-1}} \in \mathcal{A}_g$ and $e_{\chi^g} \in \mathrm{Ran}\,E_\pi(\{\chi^g\})$, we obtain

$$\pi(a^*_{g^{-1}})e_\chi = (\chi^g(a^*_{g^{-1}}a_{g^{-1}}))^{-1}\pi(a^*_{g^{-1}}a_{g^{-1}})e_{\chi^g} = e_{\chi^g}.$$

Thus, we have shown that the set $\{\pi(a_g)e_\chi | a_g \in \mathcal{A}_g,\ e_\chi \in \mathrm{Ran}\,E(H)\}$ is equal to $\mathrm{Ran}\,E(gH)$, that is, (π, E) is non-degenerate. Since $E(H)$ is equal to $E_\pi(\{\chi\})$, condition (36) is satisfied.

Conversely, let ρ be a $*$-representation of $\mathcal{A}_H$ satisfying condition (36). Since $\rho(a^*_h a_h)$, $a_h \in \mathcal{A}_h$, $h \in H$, is a multiple of the identity, $\rho(a_h)$ is bounded. Therefore each $\rho(a)$, $a \in \mathcal{A}$, is bounded, in particular $\mathcal{D}(\rho) = \mathcal{H}_\rho$. We will show later (see Proposition 28) that every representation ρ satisfying Eq. 36 is positive on the cone $\sum \mathcal{A}^2$. Since ρ is bounded, it is a direct sum of cyclic representations and hence inducible by Lemma 2. Proposition 20 together with Lemma 6 imply that $\pi = \mathrm{Ind}_{\mathcal{A}_H \uparrow \mathcal{A}}\rho$ is well-behaved. Let E_π be the spectral measure associated with π. The equality (26) implies that E_π is supported on Orbχ which means that π is associated with Orbχ.

It was shown in the proof of the Theorem 1 that the map

$$\pi \mapsto \mathrm{Res}_{\mathcal{A}_H}\pi \restriction \mathrm{Ran}\,E(H)$$

is the inverse of the map (Eq. 35). Thus, we have proved that the mapping (Eq. 35) is indeed a bijection.

Now we prove that $\rho(\mathcal{A}_H)' = \pi(\mathcal{A})'$. Let $T \in \rho(\mathcal{A}_H)'$. Define the linear operator $\widetilde{T}$ on $\mathcal{A} \otimes \mathcal{H}_\rho$ by putting

$$\widetilde{T}(a \otimes v) = a \otimes Tv, \ a \in \mathcal{A}, v \in \mathcal{H}_\rho. \tag{37}$$

Let $c_H \in \mathcal{A}_H$. Then for arbitrary $a \in \mathcal{A}$ and $v \in \mathcal{H}_\rho$ we have

$$\widetilde{T}(ac_H \otimes v - a \otimes c_H v) = ac_H \otimes Tv - a \otimes Tc_H v = ac_H \otimes Tv - a \otimes c_H Tv,$$

so $\widetilde{T}$ defines a linear operator on $\mathcal{A} \otimes_{\mathcal{A}_H} \mathcal{H}_\rho$ which is also denoted by $\widetilde{T}$.

Let $a \in \mathcal{A}, \ v \in \mathcal{H}_\rho$. We denote by $\|\cdot\|_0$ the seminorm $\langle \cdot, \cdot \rangle_0^{1/2}$. Since ρ is inducible, $S := \rho(p_H(a^*a))$ is a positive operator on $\mathcal{H}_\rho$ commuting with T. Hence T commutes with $S^{1/2}$ and we get

$$\begin{aligned}
\left\| \widetilde{T}(a \otimes v) \right\|_0^2 &= \left\langle \widetilde{T}(a \otimes v), \widetilde{T}(a \otimes v) \right\rangle_0 = \left\langle \rho(p_H(a^*a))Tv, Tv \right\rangle = \left\langle S^{1/2}Tv, S^{1/2}Tv \right\rangle \\
&= \left\langle TS^{1/2}v, TS^{1/2}v \right\rangle \le \|T\|^2 \left\langle S^{1/2}v, S^{1/2}v \right\rangle \\
&= \|T\|^2 \left\langle \rho(p_H(a^*a))v, v \right\rangle = \|T\|^2 \|a \otimes v\|_0^2 .
\end{aligned}$$

Let ρ be a direct sum of cyclic representations ρ_i with cyclic vectors $v_i, \ i \in I$. Take $\xi = \sum a_k \otimes v_k \in \mathcal{A} \otimes_{\mathcal{A}_H} \mathcal{H}_\rho$, where $a_k \in \mathcal{A}$ and v_k are distinct, hence pairwise orthogonal, cyclic vectors. Then the vectors $a_k \otimes v_k$ are pairwise orthogonal with respect to $\langle \cdot, \cdot \rangle_0$. Using the preceding inequality and the latter fact we derive

$$\begin{aligned}
\left\| \widetilde{T}\xi \right\|^2 &= \left\| \widetilde{T}\left(\sum_k a_k \otimes v_k \right) \right\|_0^2 \le \left(\sum_k \|T\| \, \|a_k \otimes v_k\|_0 \right)^2 = \|T\|^2 \sum_k \langle a_k \otimes v_k, a_k \otimes v_k \rangle_0 \\
&= \|T\|^2 \left\langle \sum_k a_k \otimes v_k, \sum_k a_k \otimes v_k \right\rangle_0 = \|T\|^2 \left\| \sum_k a_k \otimes v_k \right\|_0^2 = \|T\|^2 \|\xi\|^2 .
\end{aligned}$$

This shows that $\widetilde{T}$ gives rise to a bounded operator on $\mathcal{H}_\pi$, which we denote again by $\widetilde{T}$. It is straightforward to check that $\widetilde{T}$ commutes with all operators $\pi(a), \ a \in \mathcal{A}$, and that the map $\beta : T \mapsto \widetilde{T}$ is a $*$-homomorphism from $\rho(\mathcal{A}_H)'$ into $\pi(\mathcal{A})'$.

If $\widetilde{T} = 0$, then in particular $\langle Tv, Tv \rangle = \left\| \widetilde{T}(1 \otimes v) \right\|^2 = 0$ for all $v \in \mathcal{D}(\rho)$ which implies that $T = 0$. That is, β is injective.

We prove that β is surjective. Let S be an operator from $\pi(\mathcal{A})'$. Then $S \in \pi(\mathcal{B})'$. Since the restrictions of $\mathrm{Res}_{\mathcal{B}}\pi$ to $\mathrm{Ran}\, E(gH) = \mathrm{Ran}\, E_\pi(\{\chi\}^g)$ are disjoint representations for distinct cosets $gH \in G/H$, S commutes with all operators $E(gH)$. In particular, $S_1 := S \restriction \mathrm{Ran}\, E(H)$ is a bounded operator on the Hilbert space $\mathrm{Ran}\, E(H)$ which commutes with all operators $\pi(a) \restriction \mathrm{Ran}\, E(H)$, where $a \in \mathcal{A}_H$. By the canonical isomorphism of $\mathcal{H}_\rho$ and $\mathrm{Ran}\, E(H)$, S_1 is a bounded operator on $\mathcal{H}_\rho$. By construction we have $S_1 \in \rho(\mathcal{A}_H)'$. One easily verifies that $\beta(S_1)$ is equal to S. This shows that β is surjective. Summarizing the preceding, we have proved that the mapping β is an isomorphism of von Neumann algebras $\rho(\mathcal{A}_H)'$ and $\pi(\mathcal{A})'$. □

Remark Suppose that ρ is an inducible well-behaved representation of $\mathcal{A}_H$. If condition (36) does not hold, then the mapping $\beta : T \mapsto \widetilde{T}$ of $\rho(\mathcal{A}_H)'$ into $\pi(\mathcal{A})'$ is not surjective in general.

We now derive an important corollary from the previous theorem.

Proposition 26 *Let $\chi \in \widehat{\mathcal{B}}^+$. Then the induced representation $\pi = \mathrm{Ind}\chi$ is irreducible if and only if its stabilizer group* $\mathrm{St}\chi$ *is trivial.*

Proof If the stabilizer $\mathrm{St}\chi$ is trivial, then π is irreducible by Theorem 5.

Assume that the stabilizer group is not trivial. Then there exists $h \in H{=}\mathrm{St}\chi$ such that $h \neq e$. We choose an element $a_h \in \mathcal{A}_h$ such that $\chi(a_h^* a_h) = 1$. Using similar arguments as in the proof of the Theorem 5, one shows that there is a linear operator T_h on the $\mathcal{H}_\pi$ defined by

$$T_h([a_g \otimes 1]) = [a_g a_h \otimes 1],\ a_g \in \mathcal{A}_g,\ g \in G.$$

For vectors $[a_1 \otimes 1], [a_2 \otimes 1] \in \mathcal{H}_\pi$, where $a_i \in \mathcal{A}_{g_i},\ g_i \in G, i = 1, 2$, we have

$$\langle T_h[a_1 \otimes 1], T_h[a_2 \otimes 1]\rangle = \langle [a_1 a_h \otimes 1], [a_2 a_h \otimes 1]\rangle = \chi(p(a_h^* a_2^* a_1 a_h)).$$

If $g_1 \neq g_2$., the latter is equal to $0 = \langle [a_1 \otimes 1], [a_2 \otimes 1]\rangle$. If $g_1 = g_2$, then $a_2^* a_1 \in \mathcal{B}$ and hence

$$\chi(p(a_h^* a_2^* a_1 a_h)) = \chi(a_h^* a_2^* a_1 a_h) = \chi(a_2^* a_1) = \langle [a_1 \otimes 1], [a_2 \otimes 1]\rangle.$$

This shows that T_h is unitary. Since T_h acts as a weighted shift (see Proposition 16), it is not a scalar multiple of the identity. One easily verifies that T_h commutes with all representation operators. Since the commutant of π contains a non-trivial unitary, π is not irreducible. □

We now classify all representations of $\mathcal{A}_H$ satisfying condition (36). The result is the same as in the case when $\mathcal{A}$ is the group algebra $\mathbb{C}[G]$ and $\mathcal{B}$ is the group algebra $\mathbb{C}[N]$ of a commutative normal subgroup (see [22] and [12], pp. 1252–1258). That is, we establish a correspondence between $*$-representations ρ of $\mathcal{A}_H$ satisfying Eq. 36 and unitary projective representations of H.

Let $\chi \in \widehat{\mathcal{B}}^+$ and let H be the stabilizer group of χ. Take a representation ρ satisfying Eq. 36. Since χ^h is defined for all $h \in H$, we can find elements a_h in each $\mathcal{A}_h,\ h \in H$, such that $\chi(a_h a_h^*) = \chi^h(a_h a_h^*) = \chi(a_h^* a_h) \neq 0$. From Eq. 36 it follows that for $h \in H$ the operator

$$\zeta(h) := \chi(a_h^* a_h)^{-1/2} \rho(a_h) \tag{38}$$

is unitary and for any $b_h \in \mathcal{A}_h$ the operator $\rho(b_h^* a_h)$ is a scalar multiple of the identity, so $\rho(a_h)$ differs from $\rho(b_h)$ by a scalar. Thus, the operators $\zeta(h)$ define a unitary projective representation of H. Hence (see [22]) there exists a 2-cocycle $\tau : H \times H \to \mathbb{T}$ such that

$$\zeta(hk) = \tau(h,k)\zeta(h)\zeta(k),\ h, k \in H. \tag{39}$$

For $k \in H$ we have the equality $\rho(a_k)^{-1} = \chi(a_k^* a_k)^{-1} \rho(a_k^*)$, in particular, $\chi(a_k^* a_k) = \chi(a_k a_k^*)$. Using this we calculate

$$\begin{aligned}\zeta(hk) &= \chi(a_{hk}^* a_{hk})^{-1/2} \rho(a_{hk}) = \chi(a_{hk}^* a_{hk})^{-1/2} \rho(a_h a_k) \rho(a_h a_k)^{-1} \rho(a_{hk}) = \\ &= \chi(a_{hk}^* a_{hk})^{-1/2} \chi(a_h^* a_h)^{1/2} \zeta(h) \chi(a_k^* a_k)^{1/2} \zeta(k) \chi(a_h^* a_h)^{-1} \rho(a_h^*) \chi(a_k^* a_k)^{-1} \rho(a_k^*) \rho(a_{hk}) = \\ &= \chi(a_{hk}^* a_{hk})^{-1/2} \chi(a_h^* a_h)^{-1/2} \chi(a_k^* a_k)^{-1/2} \chi(a_h^* a_k^* a_{hk}) \zeta(h) \zeta(k).\end{aligned}$$

Thus we have

$$\tau(h,k) = \chi(a_{hk}^*a_{hk})^{-1/2}\chi(a_h^*a_h)^{-1/2}\chi(a_k^*a_k)^{-1/2}\chi(a_h^*a_k^*a_{hk}),\ h,k\in H. \tag{40}$$

The mapping ζ satisfying Eq. 39 will be called τ-representation. Let t be the element of the cohomology group $Z^2(H,\mathbb{T})$ of H with values in $\mathbb{T}$ defined by the cocycle τ. Analogously to the group case we call t the *Mackey obstruction of* χ.

Conversely, having a cocycle τ of the form (40) and a τ-representation ζ of H it is straightforward to verify that Eq. 38 defines a $*$-representation ρ of $\mathcal{A}_H$ satisfying Eq. 36.

The proof of the following proposition is similar to the group case (see [12], pp. 1252–1258).

Proposition 27 *The Mackey obstruction t of χ is trivial if and only if χ can be extended to a character $\widetilde{\chi}$ of the algebra $\mathcal{A}_H$. Equation 38 defines a one-to-one correspondence between unitary equivalence classes of τ-representations ζ of H and unitary equivalence classes of $*$-representations ρ of $\mathcal{A}_H$ satisfying Eq. 36. Moreover, ρ is irreducible if and only if ζ is irreducible.*

We now show that condition (36) implies $\sum\mathcal{A}^2$-positivity.

Proposition 28 *Let $\chi\in\widehat{\mathcal{B}}^+$ and let H be its stabilizer. If ρ is a $*$-representation of $\mathcal{A}_H$ satisfying condition (36), then ρ is nonnegative on the cone $\sum\mathcal{A}^2\cap\mathcal{A}_H$.*

Proof It suffices to show that for any $a\in\mathcal{A},\ \rho(p_H(a^*a))$ is a positive operator. It is enough to consider the case when a belongs to $\mathcal{A}_{gH}$ for some $gH\in G/H$, i.e. $a=\sum_{h\in H}a_{gh},\ a_{gh}\in\mathcal{A}_{gh}$. Using that H is the stabilizer group of χ, we get

$$\chi(a_{gh}^*a_{gk}a_{gk}^*a_{gh}) = \chi^{gh}(a_{gk}a_{gk}^*)\chi(a_{gh}^*a_{gh}) = \chi^{gk}(a_{gk}a_{gk}^*)\chi(a_{gh}^*a_{gh}) = \chi(a_{gk}^*a_{gk})\chi(a_{gh}^*a_{gh}).$$

Using Eq. 38 and the latter equality we calculate

$$\begin{aligned}\rho(p_H(a^*a)) = \rho(a^*a) &= \sum_{k,h\in H}\rho(a_{gk}^*a_{gh}) = \sum_{k,h\in H}\chi(a_{gh}^*a_{gk}a_{gk}^*a_{gh})^{1/2}\zeta(k^{-1}h) = \\ &= \sum_{k,h\in H}\chi(a_{gk}a_{gk}^*)^{1/2}\chi(a_{gh}^*a_{gh})^{1/2}\zeta(k)^*\zeta(h)\\ &= \left(\sum_{h\in H}\chi(a_{gh}^*a_{gh})^{1/2}\zeta(h)\right)^*\sum_{h\in H}\chi(a_{gh}^*a_{gh})^{1/2}\zeta(h),\end{aligned}$$

which implies that $\rho(p_H(a^*a))$ is positive. □

Next we want to associate well-behaved irreducible representations with orbits. Under some technical assumption this aim will be achieved by Proposition 29 below. For this some preparations are necessary.

Definition 14 A Borel subset Δ of $\widehat{\mathcal{B}}^+$ is called *invariant* under the partial action of G if $\Delta^g\subseteq\Delta$ for every $g\in G$. A spectral measure E on $\widehat{\mathcal{B}}^+$ is called *ergodic* under the partial action of G on $\widehat{\mathcal{B}}^+$ if for every invariant Borel subset Δ of $\widehat{\mathcal{B}}^+$ either $E(\Delta)$ or $E(\widehat{\mathcal{B}}^+\backslash\Delta)$ is zero.

Lemma 19 *Let π be a well-behaved irreducible representation of the $*$-algebra $\mathcal{A}$ and let E_π be an associated spectral measure. Then E_π is ergodic.*

Proof Let Δ be a Borel subset of $\widehat{\mathcal{B}}^+$ which is invariant under the partial action of G. From Proposition 17 (i), it follows that $E_\pi(\Delta)$ is a projection commuting with $\pi(\mathcal{A}_g)$ for all $g \in G$ and hence with $\pi(\mathcal{A})$. Since π is irreducible, $E_\pi(\Delta)$ is trivial, i.e. $E_\pi(\Delta) = 0$ or $E_\pi(\Delta) = I$. □

The following concepts are taken from the paper [8].

We shall say that a measurable space $(Y, \mathfrak{B})$ is *countably separated* if there exists a countable subfamily $\mathfrak{B}_0$ of $\mathfrak{B}$ such that for any two points in Y there exists a member of $\mathfrak{B}_0$ containing one point but not the other. A measurable subset $\Gamma \subseteq Y$ is said to be *countably separated* if $(\Gamma, \mathfrak{B}_\Gamma)$ is countably separated, where $\mathfrak{B}_\Gamma$ is the induced Borel structure.

A subset $\Gamma \subseteq \widehat{\mathcal{B}}^+$ is called a *section of the partial action* of G on $\widehat{\mathcal{B}}^+$ if it contains precisely one point from each orbit. Recall that a (spectral) measure is called an *atom* if it attains only two values. An atom is called *trivial* if it is supported at a single point.

The proof of the following simple lemma is borrowed from the proof of Theorem 2.6 in [8].

Lemma 20 *Let E be a spectral measure on a countably separated measurable space $(X, \mathfrak{B})$. If E is an atom, then it is trivial.*

Proof Let $\{B_k; k \in \mathbb{N}\}$ be a countable family of Borel subsets of X which separates the points of X and is closed under taking complements. Let B_{k_n}, $n \in \mathbb{N}$, be those sets with $E(B_{k_n}) = I$ and put $B = \cap_{n\in\mathbb{N}} B_{k_n}$. Then we have $E(B_{k_1} \cap \cdots \cap B_{k_n}) = E(B_{k_1}) \dots E(B_{k_n}) = I$ which implies that $E(B) = I$ and $B \neq \emptyset$.

Assume to the contrary that there exist distinct points p and q in B. Then there exists $j \in \mathbb{N}$ such that $p \in B_j$ and $q \notin B_j$. Due to the latter relation, we have $B_j \notin \{B_{i_n}\}$ and $X \backslash B_j \notin \{B_{i_n}\}$ which implies that $E(B_j)$ and $E(X \backslash B_j)$ are zero. Hence $E(X) = 0$ which is a contradiction. □

Proposition 29 *Let G be a countable group. Suppose that the partial action of G on $\widehat{\mathcal{B}}^+$ possesses a measurable countably separated section Γ. Then every ergodic spectral measure E on $\widehat{\mathcal{B}}^+$ is supported on a single orbit. In particular, each irreducible well-behaved representation of $\mathcal{A}$ is associated with an orbit.*

Proof We first show that the spectral measure E restricted to Γ is either zero or an atom. Suppose that E restricted to Γ is non-zero. Assume to the contrary that E restricted to Γ is not an atom. Then Γ is a disjoint union of two Borel sets Γ_1 and Γ_2 such that $E(\Gamma_1) \neq 0$ and $E(\Gamma_2) \neq 0$. By Proposition 14, the sets $\Omega_i = \cup_{g\in G} \Gamma_i^g$, $i = 1, 2$, are Borel. The properties of the partial action imply that the sets Ω_i are invariant and both projections $E(\Omega_i)$ are non-zero which is a contradiction. Thus, E restricted to Γ is an atom.

Since Γ is countably separated, Proposition 14 implies that all Γ^g, $g \in G$, are countably separated. Since $\widehat{\mathcal{B}}^+$ is the union of sets Γ^g, it follows from Lemma 20 that there exist points $\chi_k \in \Gamma^k$, $k \in I \subseteq G$, such that $E(\chi_k) \neq 0$ for all $k \in I$ and E is supported on the (at most countable) set $\{\chi_k\}_{k\in I}$. Since the set $\mathrm{Orb}\chi_k$ is invariant and $E(\mathrm{Orb}\chi_k) \neq 0$ for all k, the ergodicity of E implies that all χ_k belong to a single orbit. □

9 Example: Enveloping Algebras of Some Complex Lie Algebras

In this section we illustrate the concepts of the previous sections on three examples: enveloping algebras $\mathcal{U}(su(2))$, $\mathcal{U}(su(1,1))$ and $\mathcal{U}(Vir)$, where Vir denotes the Virasoro algebra [6, 13]. Is is easily checked that in these cases condition (18) is satisfied and the space $\widehat{\mathcal{B}}^+$ is locally compact, so the theory developed in the preceding sections applies.

First let $\mathfrak{g}$ be one of the real Lie algebras $su(2)$ or $su(1,1)$ and let $\mathfrak{g}_{\mathbb{C}}$ be its complexification. Then $\mathfrak{g}_{\mathbb{C}} = sl(2,\mathbb{C})$ has a vector space basis $\{E, F, H\}$ with commutation relations

$$[H, E] = 2E,\ [H, F] = -2F,\ [E, F] = H. \tag{41}$$

From Eq. 41 it follows that in the complex universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ we have

$$Eq(H) = q(H-2)E,\ \ Fq(H) = q(H+2)F \tag{42}$$

$$HE^n = E^n(H+2n),\ \ FE^n = E^{n-1}(EF - n(H+n-1)),\ n \in \mathbb{N}, \tag{43}$$

$$HF^n = F^n(H-2n),\ \ EF^n = F^{n-1}(FE + n(H-n+1)),\ n \in \mathbb{N}. \tag{44}$$

for each polynomial $q \in \mathbb{C}[x]$ and that the Casimir element

$$C := 2(EF + FE) + H^2 = 4FE + H(H+2) = 4EF + H(H-2)$$

belongs to the center of $\mathcal{U}(\mathfrak{g})$.

The complex unital algebra $\mathcal{U}(\mathfrak{g})$ becomes a $*$-algebra with involution determined by $x^* = -x$ for $x \in \mathfrak{g}$. In terms of the generators $\{E, F, H\}$ of the algebra $\mathcal{U}(\mathfrak{g})$ this means that

$$E^* = F,\ \ H^* = H \text{ for } \mathfrak{g} = su(2), \tag{45}$$

$$E^* = -F,\ \ H^* = H \text{ for } \mathfrak{g} = su(1,1). \tag{46}$$

Using the commutation relation (41) it follows by induction that

$$\mathcal{U}(\mathfrak{g})_0 := \mathrm{Lin}\{E^l F^l H^k;\ k,l \in \mathbb{N}_0\} = \mathrm{Lin}\{(EF)^l H^k;\ k,l \in \mathbb{N}_0\} = \mathrm{Lin}\{C^l H^k;\ k,l \in \mathbb{N}_0\}.$$

In particular, $\mathcal{B} := \mathcal{U}(\mathfrak{g})_0$ is a commutative unital $*$-subalgebra of $\mathcal{A} = \mathcal{U}(\mathfrak{g})$. For $n \in \mathbb{N}_0$, let

$$\mathcal{A}_n = E^n\mathcal{B} = \mathrm{Lin}\left\{E^{n+l}F^l H^k; k,l \in \mathbb{N}_0\right\}, \mathcal{A}_{-n} = F^n\mathcal{B} = \mathrm{Lin}\left\{E^l F^{n+l} H^k; k,l \in \mathbb{N}_0\right\}.$$

By the Poincare–Birkhoff–Witt theorem, $\left\{E^i F^j H^l;\ i,j,l \in \mathbb{N}_0\right\}$ is a vector space basis of $\mathcal{U}(\mathfrak{g})$. From this fact and the definitions (45) and (46) of the involution we derive that

$$\mathcal{A} = \bigoplus_{n\in\mathbb{Z}} \mathcal{A}_n \tag{47}$$

is a $\mathbb{Z}$-graded $*$-algebra. Let $p : \mathcal{A} \to \mathcal{B}$ be the canonical conditional expectation (see Proposition 6). In both cases $\mathfrak{g} = su(2)$ and $\mathfrak{g} = su(1,1)$ the conditional expectation p is not strong, because we have $E^* E \in \sum \mathcal{A}^2 \cap \mathcal{B}$, but $E^* E \notin \sum \mathcal{B}^2$.

Remarks

1. The $\mathbb{Z}$-graded $*$-algebra (47) is the special case $\mathfrak{g} = sl(2, \mathbb{C})$ of Example 8. In this case, $Q = \mathbb{Z}$ and $\mathcal{B} = \mathcal{U}(\mathfrak{g})_0$ is just the commutant of the element H in the algebra $\mathcal{U}(\mathfrak{g})$. Note that $sl(2, \mathbb{C})$ is the only simple Lie algebra $\mathfrak{g}$ for which $\mathcal{B} = \mathcal{U}(\mathfrak{g})_0$ is commutative.
2. For the real Lie algebra $\mathfrak{g} = sl(2, \mathbb{R})$ the involution of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ is given by $E^* = E,\ F^* = F,\ H^* = -H$. In this case the decomposition (47) remains valid and shows that $\mathcal{U}(\mathfrak{g})$ is a $\mathbb{Z}$-graded algebra. But since $(\mathcal{U}(\mathfrak{g})_n)^* = \mathcal{U}(\mathfrak{g})_n$ for $n \in \mathbb{Z}$, $\mathcal{U}(\mathfrak{g}) = \oplus_n \mathcal{U}(\mathfrak{g})_n$ is not a $\mathbb{Z}$-graded $*$-algebra.

We derive three simple lemmas which will be needed below.

Lemma 21 *Let $\mathfrak{g}$ be one of the real Lie algebras $su(2)$ or $su(1,1)$. A character $\chi \in \widehat{\mathcal{B}}$ belongs to $\widehat{\mathcal{B}}^+$ if and only $\chi(F^{*k} F^k) \geq 0$ and $\chi(E^{*k} E^k) \geq 0$ for all $k \in \mathbb{N}$.*

Proof Recall that $\chi \in \widehat{\mathcal{B}}^+$ if and only if $\chi(b) \geq 0$ for all $b \in \mathcal{A}^2 \cap \mathcal{B}$. Hence the necessity of the condition is obvious. We prove that it is also sufficient. By Corollary 1, it suffices to show $\chi(a_n^* a_n) \geq 0$ for all homogeneous elements $a_n \in \mathcal{A}_n,\ n \in \mathbb{Z}$.

Let $n \in \mathbb{N}_0$ and take $a_n \in \mathcal{A}_n$. By the definition of $\mathcal{A}_n$ we have $a_n = E^n b$ for some $b \in \mathcal{B}$. Since $\chi(E^{*n} E^n) \geq 0$ by assumption, $\chi(a_n^* a_n) = \chi(b^* E^{*n} E^n b) = \chi(E^{*n} E^n)\chi(b^* b) \geq 0$. Similarly, for $n < 0$ the inequality $\chi(F^{*n} F^n) \geq 0$ implies that $\chi(a_n^* a_n) \geq 0$ for all $a_n \in \mathcal{A}_n$. □

Lemma 22 *For $n \in \mathbb{N}$ we have*

$$\begin{aligned} E^n F^n &= EF(EF + H{-}2)(EF + H{-}2 + H{-}4) \\ &\quad \cdots(EF + H{-}2 + \cdots + H{-}2(n-1)), \end{aligned} \tag{48}$$

$$\begin{aligned} F^n E^n &= (EF - H - (H{+}2) - \ldots - (H{+}2(n-1))) \\ &\quad \cdots(EF - H - (H{+}2))(EF - H) \\ &= FE(FE - (H{+}2))\cdots(FE - (H{+}2) - \cdots - (H{+}2(n{-}1))) \end{aligned} \tag{49}$$

Proof We prove the first equality (48) by induction on n. The two equalities concerning $F^n E^n$ are verified in a similar manner. Using the commutation relation (41) we compute

$$\begin{aligned} E^{n+1} F^{n+1} &= E^n (FE + H) F^n = E^n FEF^n + (H - 2n) E^n F^n = \\ &= E^{n-1}(FE + H) EF^n + (H - 2n) E^n F^n = \\ &= E^{n-1} FE^2 F^n + (H - 2(n-1)) E^n F^n + (H - 2n) E^n F^n = \ldots \\ \ldots &= (EF + H - 2 + \cdots + (H - 2n)) E^n F^n. \end{aligned}$$

Inserting the induction hypothesis (48) for n and remembering that all elements E^kF^k and H^l mutually commute, we obtain Eq. 48 for $n+1$. □

Lemma 23 $\mathcal{B} \equiv \mathcal{U}(\mathfrak{g})_0 = \mathbb{C}[EF, H] = \mathbb{C}[C, H]$.

Proof Since the elements EF and H of $\mathcal{U}(\mathfrak{g})$ commute, there is an algebra homomorphism $\sigma : \mathbb{C}[x_1, x_2] \to \mathcal{U}(\mathfrak{g})$ given by $\sigma(x_1) = EF$ and $\sigma(x_2) = H$. From the Poincaré–Birkhoff–Witt theorem we derive easily that σ is injective which gives $\mathcal{U}(\mathfrak{g})_0 = \mathbb{C}[EF, H]$. Clearly, we have also $\mathbb{C}[EF, H] = \mathbb{C}[C, H]$. □

Lemma 23 implies that the map $\widehat{\mathcal{B}} \ni \chi \mapsto (\chi(C), \chi(H)) \in \mathbb{R}^2$ is bijective. Denote by $\chi_{st} \in \widehat{\mathcal{B}},\ s, t \in \mathbb{R}$ a character such that

$$\chi_{st}(C) = s,\ \ \chi_{st}(H) = t. \tag{50}$$

Propositions 31 and 33 below describe the set of parameters $s, t \in \mathbb{R}$ for which $\chi_{st} \in \widehat{\mathcal{B}}^+$ in the cases $\mathfrak{g} = su(2)$ and $\mathfrak{g} = su(1, 1)$, respectively.

Proposition 30 *Let $\mathfrak{g}$ be one of the real Lie algebras $su(2)$ or $su(1, 1)$. If a character χ_{st} belongs to $\widehat{\mathcal{B}}^+$ and if χ^n_{st} is defined for $n \in \mathbb{Z}$, then we have*

$$\chi^n_{st} = \chi_{s,t+2n}. \tag{51}$$

Proof For $n = 0$ the proof is trivial. Assume that $n > 0$. In the case $n < 0$ the proof is similar. Since χ^n_{st} is defined, $\chi_{st}(E^{*n}E^n) > 0$. We compute

$$\chi^n_{st}(H) = \frac{\chi_{st}(F^nHE^n)}{\chi_{st}(F^nE^n)} = \frac{\chi_{st}(F^nE^n(H+2n))}{\chi_{st}(F^nE^n)} = \chi_{st}(H+2n) = t+2n = \chi_{s,t+2n}(H).$$

Since C belongs to the center of $\mathcal{A}$, we have $\chi^n_{st}(C) = \chi_{st}(C)$. By the definition of χ_{st} we obtain Eq. 51. □

9.1 The Case $\mathfrak{g} = su(2)$

In this subsection we let $\mathcal{A} = \mathcal{U}(su(2))$ and $\mathcal{B} = \mathcal{A}_0 = \mathbb{C}[EF, H] = \mathbb{C}[C, H]$. The next proposition describes the set $\widehat{\mathcal{B}}^+$.

Proposition 31 *A character χ_{st} defined by Eq. 50 belongs to $\widehat{\mathcal{B}}^+$ if and only if $t \in \mathbb{Z}$ and $s = (t+2n)(t+2n+2)$ for some $n \in \mathbb{N}_0$ such that $n+t \geq 0$.*

Proof Since $E^{*n} = F^n$, Lemmas 21 and 22 imply that χ belongs to $\widehat{\mathcal{B}}^+$ if and only if the following inequalities are fulfilled for arbitrary $k \in \mathbb{N}$:

$$\chi(E^kF^k) \equiv \chi(EF)\chi(EF+H-2)\dots\chi(EF+H-2+\dots+H-2k) \geq 0, \tag{52}$$

$$\chi(F^kE^k) \equiv \chi(EF-H)\chi(EF-H-(H+2)) \dots\chi(EF-H-\dots-(H+2k)) \geq 0. \tag{53}$$

We claim that for every $\chi \in \widehat{\mathcal{B}}^+$ there exist $m, n \in \mathbb{N}_0$ such that

$$\chi(EF+m(H-(m+1))) = 0,\ \ \chi(EF-(n+1)(H+n)) = 0. \tag{54}$$

Assume to the contrary that $\chi(EF + k(H - (k+1))) \neq 0$ for all $k \in \mathbb{N}_0$. It follows from Eq. 52 that χ is positive on all factors in Eq. 52, that is,

$$\begin{aligned}\chi(EF + H - 2 + \cdots + H - 2k) &= \chi(EF + k(H - (k+1))) \\ &= \chi(EF) + k(\chi(H) - (k+1))) > 0\end{aligned}$$

for all $k \in \mathbb{N}_0$ which is a contradiction. Hence $\chi(EF + m(H - (m+1))) = 0$ for some $m \in \mathbb{N}$. In the same way one proves the second equality in Eq. 54.

The solution of the system of Eq. 54 is

$$\chi(EF) = m(n+1), \ \ \chi(H) = m - n. \tag{55}$$

It is easy to verify that for all $m, n \in \mathbb{N}_0$ the characters χ defined by Eq. 55 satisfy both inequalities (52) and (53).

Putting $t = m - n$ in Eq. 55 we get

$$\begin{aligned}\chi(C) &= 4\chi(EF) + \chi(H^2 - 2H) = 4m(n+1) + (m-n)^2 - 2m + 2n = \\ &= (m+n)(m+n+2) = (t+2n)(t+2n+2),\end{aligned}$$

i.e. $\chi = \chi_{st}$ where $t = m - n \in \mathbb{Z}$ and $s = (t+2n)(t+2n+2)$. Clearly, we have $m, n \in \mathbb{N}_0$ if and only if $t \in \mathbb{Z},\ n + t \geq 0$. □

We denote by ψ_n, $n \in \mathbb{N}_0$, the character $\chi_{n(n+2),-n} \in \widehat{\mathcal{B}}^+$ and by Γ the subset $\{\psi_n,\ n \in \mathbb{N}_0\}$ of $\widehat{\mathcal{B}}^+$. By Propositions 30 and 31, each orbit under the partial action of $\mathbb{Z}$ on $\widehat{\mathcal{B}}^+$ contains precisely one of the characters from Γ, i.e. Γ is a section of the partial action of $\mathbb{Z}$ on $\widehat{\mathcal{B}}^+$.

Proposition 32 *The representations* Indχ, $\chi \in \Gamma$, *are pairwise non-equivalent and irreducible. Each irreducible well-behaved representation of $\mathcal{A}$ is unitarily equivalent to* Indχ *for some $\chi \in \Gamma$. A ∗-representation π of $\mathcal{A} = \mathcal{U}(su(2))$ is well-behaved (in the sense of Definition 11) if and only if π is integrable (that is, $\pi = dU$ for some unitary representation U of the Lie group $SU(2)$.)*

Proof Clearly, the bijection $\chi_{st} \mapsto (s, t)$ of the space $\widehat{\mathcal{B}}$ onto $\mathbb{R}^2$ (by Lemma 23) is a homeomorphism. Hence Proposition 31 implies that $\widehat{\mathcal{B}}^+$ is a discrete space. It follows from the formulas for the partial action of $\mathbb{Z}$ that Γ is a Borel section. By Proposition 29 all irreducible well-behaved representations are associated with orbits. Therefore, by Theorem 5 we have that Indχ, $\chi \in \Gamma$, are up to unitary equivalence all irreducible well-behaved representations. It follows from Proposition 31 that Orbψ_n, $n \in \mathbb{N}_0$ consists of $n+1$ elements, and Proposition 16 implies that Indψ_n, $n \in \mathbb{N}_0$ has dimension $n+1$. The latter implies in particular that each representation Indχ, $\chi \in \Gamma$ is integrable.

Let π be a well-behaved representation of $\mathcal{A}$ and let E_π be the associated spectral measure on $\widehat{\mathcal{B}}^+$. Denote by ρ the restriction of Res$_{\mathcal{B}}\pi$ to Ran$(E_\pi(\Gamma))$. It is easily checked that π is unitarily equivalent to Indρ. Since $\widehat{\mathcal{B}}^+$ is discrete, ρ is equivalent to a direct sum of characters $\chi \in \Gamma$ (taken with multiplicities), so that π is equivalent to a direct sum of representations Indχ, $\chi \in \Gamma$. Because Indχ is integrable as shown in the preceding paragraph, π is integrable.

Conversely, if π is an integrable representation, π is a direct sum of integrable irreducible representations π_i. Since each representation π_i is finite dimensional and hence well-behaved by Proposition 18, π is well-behaved. □

It is well-known that for each $n \in \mathbb{N}_0$ the spin $\frac{n}{2}$ representation is the unique (up to unitary equivalence) irreducible $(n+1)$-dimensional $*$-representation of $\mathcal{A}=\mathcal{U}(su(2))$. Since the $*$-representation $\mathrm{Ind}\psi_n$ of $\mathcal{A}$ is irreducible and of dimension $n+1$, $\mathrm{Ind}\psi_n$ is equivalent to the spin $\frac{n}{2}$ representation. We want to establish this equivalence by explicit formulas.

Recall from Proposition 16, (i) that the vectors

$$\left\{ \frac{[E^k \otimes 1]}{\|[E^k \otimes 1]\|},\ k = 0, 1 \dots n \right\}$$

form an orthonormal base of the representation space of $\mathrm{Ind}\psi_n$. By definition of ψ_n we have $\psi_n(H) = -n$ and $\psi_n(EF) = \frac{1}{4}\psi_n(C - H^2 + 2H) = 0$. Using Lemma 22 we compute

$$\begin{aligned} \|[E^k \otimes 1]\|^2 &= \psi_n(F^k E^k) = \psi_n((EF - H)(EF - 2(H+1)) \\ &\quad \dots (EF - k(H + k - 1))) = \\ &= n(2(n-1)) \dots (k(n-k+1)) = \frac{k! \cdot n!}{(n-k)!},\ k = 0, 1 \dots, n. \end{aligned}$$

Putting $l = \frac{n}{2}$, $\pi_l := \mathrm{Ind}\psi_n$ and

$$e_m := \frac{[E^{l+m} \otimes 1]}{\|[E^{l+m} \otimes 1]\|} = \sqrt{\frac{(l-m)!}{(2l)!(l+m)!}}\,[E^{l+m} \otimes 1],\ m = -l, l+1, \dots, l,$$

we calculate

$$\begin{aligned} \pi_l(E)e_m &= \frac{[E^{l+m+1} \otimes 1]}{\|[E^{l+m} \otimes 1]\|} = \frac{\|[E^{l+m+1} \otimes 1]\|}{\|[E^{l+m} \otimes 1]\|} e_{m+1} \\ &= \sqrt{\frac{(2l)!(l+m+1)!}{(l-m-1)!}} \sqrt{\frac{(l-m)!}{(2l)!(l+m)!}} e_{m+1} \\ &= \sqrt{(l-m)(l+m+1)} e_{m+1},\ m = -l, l+1, \dots, l. \end{aligned}$$

In the same manner we derive

$$\pi_l(F)e_m = \sqrt{(l-m+1)(l+m)} e_{m-1},\ \pi_l(H)e_m = 2me_m,\ m = -l, l+1, \dots, l.$$

These are the formulas for the actions of E, F, H in the spin l representation of $\mathcal{U}(su(2))$.

We now show that the representations π_l can be also induced from the $*$-subalgebra $\mathcal{C} = \mathbb{C}[H]$. Let $p_3 = p_2 \circ p_1$, where p_1 is the canonical conditional expectation $p_1 : \mathcal{A} \to \mathcal{B}$ and $p_2 : \mathcal{B} \to \mathcal{C}$ is conditional expectation defined by $p_2((EF)^k) = 0$, $p_2(H^k) = H^k$, $k \in \mathbb{N}$. Using Lemma 22 we obtain

$$
\begin{aligned}
p_3\left(\sum \mathcal{A}^2\right) &= \sum \mathcal{C}^2 - H \sum \mathcal{C}^2 + H(H + (H+2)) \sum \mathcal{C}^2 \\
&\quad - H(H + (H+2))(H + (H+2) + (H+4)) \sum \mathcal{C}^2 + \dots \\
&= \sum \mathcal{C}^2 - H \sum \mathcal{C}^2 + H(H+1) \sum \mathcal{C}^2 - H(H+1)(H+2) \sum \mathcal{C}^2 + \dots \\
&\quad + (-1)^k H(H+1)(H+2)\dots(H+k-1) \sum \mathcal{C}^2 + \dots .
\end{aligned}
$$

Obviously, p_3 is a $(\sum \mathcal{A}^2, p_3(\sum \mathcal{A}^2))$-conditional expectation. It is easy to check that $\sum \mathcal{A}^2 \cap \mathbb{C}[H] = \sum \mathcal{C}^2$. Since $p_3(\sum \mathcal{A}^2)$ is strictly larger than $\sum \mathcal{C}^2$, p_3 is not a conditional expectation according to Definition 4. In particular we have seen that the composition of two conditional expectations is not a conditional expectation in general.

It is clear from the preceding formulas that the set of characters on $\mathbb{C}[H]$ which are non-negative on the cone $p_3(\sum \mathcal{A}^2)$ and hence inducible via p_3 is the set $\{\chi_k,\ k \in \mathbb{N}_0\}$. Note that $\chi_k(H) = -k$. It is not difficult to compute that the corresponding induced representation $\mathrm{Ind}\chi_{2l}$, $l \in \frac{1}{2}\mathbb{N}_0$, is unitarily equivalent to π_l.

9.2 The Case $\mathfrak{g} = su(1, 1)$

In this subsection let $\mathcal{A} = \mathcal{U}(su(1, 1))$ and $\mathcal{B} = \mathcal{A}_0 = \mathbb{C}[EF, H] = \mathbb{C}[C, H]$.

We denote by $\chi_{st} \in \widehat{\mathcal{B}}$ the characters determined by Eq. 50. It is convenient to introduce the following subsets of $\widehat{\mathcal{B}}$:

$$
\begin{aligned}
X_{00} &= \{\chi_{00}\}, \\
X_{1k} &= \{\chi_{st} | 2k \le t < 2k+2,\ -\infty < s < (t-2k)(t-2(k+1))\},\ k \in \mathbb{Z}, \\
X_{2k} &= \{\chi_{st} | 2k < t < 2k+2,\ s = (t-2k)(t-2(k+1))\},\ k \in \mathbb{Z}, \\
X_{3k} &= \{\chi_{st} | t \ge 2k+2,\ s = (t-2k)(t-2(k+1))\},\ k \in \mathbb{N}_0, \\
X_{4k} &= \{\chi_{st} | t \le 2k,\ s = (t-2k)(t-2(k+1))\},\ k \in \mathbb{Z}\backslash\mathbb{N}_0.
\end{aligned}
$$

The following two propositions describe the set $\widehat{\mathcal{B}}^+$ and the partial action of $\mathbb{Z}$ on it.

Proposition 33 *The set $\widehat{\mathcal{B}}^+$ is equal to the disjoint union*

$$
X_{00} \cup \bigcup_{k\in\mathbb{Z}} X_{1k} \cup \bigcup_{k\in\mathbb{Z}} X_{2k} \cup \bigcup_{k\in\mathbb{N}_0} X_{3k} \cup \bigcup_{k\in\mathbb{Z}\backslash\mathbb{N}_0} X_{4k}.
$$

Proof The equality $E^{*n} = (-1)^n F^n$ and Lemmas 21 and 22 imply that a character $\chi \in \widehat{\mathcal{B}}$ belongs to $\widehat{\mathcal{B}}^+$ if and only if the following inequalities hold:

$$(-1)^k \chi(EF(EF + H - 2)\dots(EF + H - 2 + H - 4 + \dots + H - 2(k-1))) \\ \geq 0,\ k \in \mathbb{N}, \tag{56}$$

$$(-1)^k \chi((EF - H)(EF - H - (H+2)) \cdot \dots \\ \dots(EF - H - (H+2) - \dots - (H + 2(k-1)))) \geq 0,\ k \in \mathbb{N}. \tag{57}$$

Straightforward calculations show that the solutions of the latter system of inequalities are precisely the characters belonging to one of the above sets X_{ij}. One easily verifies that the sets X_{ij} are pairwise disjoint for different (i, j). □

Proposition 34

(i) *χ^n_{00} is defined only for $n = 0$.*
(ii) *For $\chi_{st} \in X_{1k} \cup X_{2k},\ k \in \mathbb{Z}$, the χ^n_{st} is defined for all $n \in \mathbb{Z}$.*
(iii) *For $\chi_{st} \in X_{3k},\ k \in \mathbb{N}_0$, the χ^n_{st} is defined for $n \geq -k$.*
(iv) *For $\chi_{st} \in X_{4k},\ k \in \mathbb{Z}$, the χ^n_{st} is defined for $n \leq k - 1$.*

Proof Follows directly from Propositions 30 and 33. □

Set

$$\Gamma := X_{00} \cup X_{10} \cup X_{20} \cup X_{30} \cup X_{4,-1} \subseteq \widehat{\mathcal{B}}^+.$$

It follows from the previous propositions that each orbit under the partial action of $\mathbb{Z}$ on $\widehat{\mathcal{B}}^+$ intersects Γ exactly in one point, i.e. Γ is a section of the partial action. As in the case of $su(2)$, the topology on $\widehat{\mathcal{B}}^+$ is induced from the standard topology on $\mathbb{R}^2$. Hence Γ is a countably separated Borel section of the partial action of $\mathbb{Z}$ on $\widehat{\mathcal{B}}^+$.

Explicit formulas for the representations $\mathrm{Ind}\chi,\ \chi \in \Gamma$, can be derived in a similar manner as in case of $su(2)$. We omit the details. In the standard terminology of representation theory of Lie algebras we have:

- the representation $\mathrm{Ind}\chi,\ \chi \in X_{00}$, is the trivial representation,
- the representations $\mathrm{Ind}\chi,\ \chi \in X_{10}$, form the principal unitary series,
- the representations $\mathrm{Ind}\chi,\ \chi \in X_{20}$, form the supplementary unitary series,
- the representations $\mathrm{Ind}\chi,\ \chi \in X_{30} \cup X_{40}$, form the discrete unitary series.

Using this description we obtain the following

Proposition 35 *The representations* $\mathrm{Ind}\chi,\ \chi \in \Gamma$, *are pairwise non-equivalent and irreducible. Each irreducible well-behaved representation of $\mathcal{A}$ is unitarily equivalent to* $\mathrm{Ind}\chi$ *for precisely one $\chi \in \Gamma$. A $*$-representation of $\mathcal{A} = \mathcal{U}(su(1,1))$ is well-behaved (in the sense of Definition 11) if and only it is of the form dU for some unitary representation U of the universal covering group of the Lie group $SU(1,1)$.*

We close this subsection with the following

Remark For a character $\chi \in \widehat{\mathcal{B}}^+$ the following three statements are equivalent:

(i) χ belongs to one of the series X_{1k} or X_{2k}, $k \in \mathbb{Z}$, corresponding to the principal or supplementary unitary series,
(ii) χ^k is defined for all $k \in \mathbb{Z}$,
(iii) $\chi(C) < 0$, where C is the Casimir element defined above.

9.3 Enveloping Algebra of the Virasoro Algebra

Recall that the Virasoro algebra is the complex Lie algebra *Vir* with generators L_n, $n \in \mathbb{Z}$, and C and defining relations

$$[L_n, L_m] = (m-n)L_{n+m} + \delta_{n,-m}(n^3 - n)/12 \cdot C \text{ and } [L_n, C] = 0 \text{ for } n, m \in \mathbb{Z}. \quad (58)$$

In this subsection we show that the unitary representations with finite-dimensional weight spaces of the Virasoro algebra can be identified with the well-behaved representations with respect to a canonical grading of a quotient algebra of its enveloping algebra. For results on unitary representations of *Vir* we refer to [6] and references therein.

Let $\mathcal{W}$ denote the enveloping algebra of *Vir*, that is, $\mathcal{W}$ is the unital $*$-algebra with generators L_n, $n \in \mathbb{Z}$, and C and the same defining relations (58). It is a $*$-algebra with involution determined by $L_n^* = L_{-n}$ for $n \in \mathbb{Z}$ and $C^* = C$. Lemma 9 implies that $\mathcal{W}$ is $\mathbb{Z}$-graded such that $L_n \in \mathcal{W}_n$ and $C \in \mathcal{W}_0$.

The main result in [6] states that there are precisely two families of irreducible unitary representations of $\mathcal{W}$ with finite-dimensional weight spaces. The first series consists of highest (resp. lowest) weight representations, i.e. representations generated by a vector v such that:

(i) $L_0 v = av$ for some $a \in \mathbb{C}$, (ii) $L_n v = 0$ for all $n > 0$ (resp. $n < 0$), (iii) $Cv = zv$ for some $z \in \mathbb{C}$.

These representations are uniquely defined by the pair $(a, z) \in \mathbb{C}^2$. The possible values of (a, z) for the representation to be unitary (that is, a $*$-representation in our terminology) are the following ones (see [13]):

$$a \geq 0, \ z \geq 1, \ \text{or } z_n = 1 - \frac{6}{n(n+1)}, \ a_n^{(p,q)} = \frac{(np+q)^2 - 1}{4n(n+1)}, \quad (59)$$

where the integers n, p, q satisfy $n \geq 2$ and $0 \leq p < q < n$.

The other series of unitary representations are defined on spaces of λ-densities (see [6]). They can be described as follows. Let $\{w_k\}_{k \in \mathbb{Z}}$ be an orthonormal base of $l^2(\mathbb{Z})$. Then the action of $\mathcal{W}$ on $l^2(\mathbb{Z})$ is given by

$$L_k w_n = (n + a + k\lambda) w_{n+k}, \ C w_n = 0, \ k, n \in \mathbb{Z}, \ \lambda \in \frac{1}{2} + i\mathbb{R}, \ a \in \mathbb{R}. \quad (60)$$

Let $\mathcal{I}$ denote the two-sided $*$-ideal of $\mathcal{W}$ generated by elements

$$bd - db, \ b, d \in \mathcal{W}_0 \text{ and } a_k^* c_k c_k^* a_k - a_k^* a_k c_k^* c_k, \ a_k, c_k \in \mathcal{W}_k, \ k \in \mathbb{Z}.$$

Lemma 24 *$\mathcal{I}$ is contained in the intersection of all kernels of representations described above.*

Proof We prove the assertion for $*$-representations defined by Eq. 60. For highest and lowest weight representations the proof is similar.

We fix a $*$-representation π given by Eq. 60, $k \in \mathbb{Z}$ and $a_k, c_k \in \mathcal{W}_k$. It follows from Eq. 60 that $\pi(a_k)w_m = \mu_m w_{m+k},\ \pi(c_k)w_m = \nu_m w_{m+k},\ m \in \mathbb{Z}$, for some $\mu_m,\ \nu_m \in \mathbb{C}$. This implies that

$$\pi(a_k^* c_k c_k^* a_k)w_m = \overline{\lambda_m}\nu_m\overline{\nu_m}\lambda_m \cdot w_m = \pi(a_k^* a_k c_k^* c_k)w_m,$$

for all $m \in \mathbb{Z}$. Taking $b, d \in \mathcal{W}_0$ the same reasoning shows that $\pi(bd)w_m = \pi(db)w_m,\ m \in \mathbb{Z}$. Therefore $\mathcal{I}$ is contained in $\ker \pi$. □

In view of Lemma 24 we introduce the $*$-algebra $\mathcal{A} = \mathcal{W}/\mathcal{I}$. Let $\iota : \mathcal{W} \to \mathcal{A}$ be the quotient mapping and put $l_k := \iota(L_k)$ for $k \in \mathbb{Z}$ and $c = \iota(C)$. Since the generators of $\mathcal{I}$ are homogeneous, Lemma 9 implies that $\mathcal{A}$ is again a $\mathbb{Z}$-graded $*$-algebra such that $l_k \in \mathcal{A}_k,\ k \in \mathbb{Z}$, and $c \in \mathcal{A}_0$. As usual we denote by $\mathcal{B}$ the subalgebra $\mathcal{A}_0$.

Because of the PBW-theorem there are two "natural" bases of the vector space $\mathcal{W}$:

$$\mathbf{B}_1 = \left\{C^k L_{n_1} L_{n_2} \dots L_{n_r} | n_1 \le n_2 \le \dots \le n_r,\ k, r \in \mathbb{N}_0, n_i \in \mathbb{Z}\right\},$$
$$\mathbf{B}_2 = \left\{C^k L_{n_1} L_{n_2} \dots L_{n_r} | n_1 \ge n_2 \ge \dots \ge n_r,\ k, r \in \mathbb{N}_0, n_i \in \mathbb{Z}\right\}.$$

Fix i=1, 2. Since all elements in $\mathbf{B}_i$ are homogeneous, the elements $C^k L_{n_1} L_{n_2} \dots L_{n_r} \in \mathbf{B}_i,\ \sum_j n_j = 0$, form a vector space base of the algebra $\mathcal{W}_0$. To define a character of $\mathcal{W}_0$, it is therefore sufficient to define it on these elements $C^k L_{n_1} L_{n_2} \dots L_{n_r} \in \mathbf{B}_i$.

Let π be an irreducible unitary highest weight representation of Vir with weight vector v. It defines a $*$-representation of $\mathcal{W}$ denoted also by π. One easily checks that the subspace $\mathbb{C} \cdot v$ is invariant under all operators $\pi(b),\ b \in \mathcal{W}_0$. Therefore it defines a character χ on $\mathcal{W}_0$ given by $\chi(L_{n_1} \dots L_{n_k}) = 0,\ \chi(L_0) = a,\ \chi(C) = z$, where $n_1 \le \dots \le n_k,\ \sum_r n_r > 0$, and (a, z) is one of the pairs defined by Eq. 59. By Lemma 24, χ annihilates the ideal $\mathcal{I}$, so it gives a character on the quotient algebra $\mathcal{B} = \iota(\mathcal{W}_0)$ which we denote again by χ. It is defined by

$$\chi(l_{n_1} \dots l_{n_k}) = 0,\ \chi(l_0) = a,\ \chi(c) = z,\ \text{where } n_1 \le \dots \le n_k \ne 0,\ \sum_r n_r = 0, \tag{61}$$

where (a, z) is given by Eq. 59. The character χ obviously belongs to $\widehat{\mathcal{B}}^+$.

From the lowest weight representations we get another series of characters $\chi \in \widehat{\mathcal{B}}^+$ determined by

$$\chi(l_{n_1} \dots l_{n_k}) = 0,\ \chi(l_0) = a,\ \chi(c) = z,\ \text{where } n_1 \ge \dots \ge n_k \ne 0,\ \sum_r n_r = 0, \tag{62}$$

where (a, z) is as in Eq. 59.

Let π be a representation given by Eq. 60. Considering the restriction of π to the subspace $\mathbb{C} \cdot w_0$ we obtain a series of characters $\chi \in \widehat{\mathcal{B}}^+$ defined by

$$\chi(l_{n_1} \dots l_{n_k}) = \prod_{r=1}^{k} \left(a - \sum_{s=1}^{r} n_s + n_r \lambda\right),\ \chi(c) = 0, \tag{63}$$

where $a \in \mathbb{R},\ \lambda \in \frac{1}{2} + i\mathbb{R}$.

Let $\Gamma \subseteq \widehat{\mathcal{B}}^+$ denote the union of all characters defined by the Eqs. 61, 62 and 63.

Proposition 36 *Each orbit under the partial action of $\mathbb{Z}$ on $\widehat{\mathcal{B}}^+$ contains precisely one character from Γ. The stabilizer of each character in $\widehat{\mathcal{B}}^+$ is trivial. For every $\chi \in \widehat{\mathcal{B}}^+$, $\iota \circ \operatorname{Ind}\chi$ is a $*$-representation of $\mathcal{W}$ with finite-dimensional weight spaces. Every irreducible $*$-representation of $\mathcal{W}$ with finite-dimensional weight spaces is unitarily equivalent to $\iota \circ \operatorname{Ind}\chi$ for precisely one $\chi \in \Gamma$.*

Proof A straightforward computation shows that

$$[l_0, l_{n_1} l_{n_2} \dots l_{n_r}] = (n_1 + n_2 + \dots + n_r) l_{n_1} l_{n_2} \dots l_{n_r},\ n_i \in \mathbb{Z},\ r \geq 1.$$

Since every $a_n \in \mathcal{A}_n$ is a linear combination of the elements $l_{n_1} l_{n_2} \dots l_{n_r},\ n_1 + n_2 + \dots + n_r = n$, it follows that

$$[l_0, a_n] = n a_n,\ \text{for all } a_n \in \mathcal{A}_n,\ n \in \mathbb{Z}. \tag{64}$$

Let $\chi \in \widehat{\mathcal{B}}^+$ and $n \in \mathbb{Z}$. Assume that χ^n is defined. Then there exists an $a_n \in \mathcal{A}_n$ such that $\chi(a_n^* a_n) > 0$. Using Eq. 64 we get

$$\chi^n(l_0) = \frac{\chi(a_n^* l_0 a_n)}{\chi(a_n^* a_n)} = \frac{\chi(a_n^* a_n l_0 + n a_n^* a_n)}{\chi(a_n^* a_n)} = \chi(l_0) + n. \tag{65}$$

Let $\pi := \operatorname{Ind}\chi$. Since χ satisfies condition (18), we can choose an orthonormal base of vectors e_k of the representation space $\mathcal{H}_\pi$ such that $\pi(l_0) e_k = \lambda_k e_k$, where $\lambda_k = \chi^k(l_0) = \chi(l_0) + k$. This implies that $\pi(l_0)$ acts as a semisimple operator and that all eigenspaces of $\pi(l_0)$ are finite dimensional. It is also clear that the stabilizer of χ is trivial, so the representation π is irreducible by Proposition 26. Therefore, by Theorem 0.5 in [6] the representation $\iota \circ \pi$ is unitarily equivalent either to a highest or lowest weight representation or to a representation defined by Eq. 60.

On the other hand, one easily verifies that $\operatorname{Ind}\chi$ gives rise via ι either to a highest or lowest weight representation or to a representation defined by Eq. 60. This implies that $\widehat{\mathcal{B}}^+$ is equal to the union of all orbits $\operatorname{Orb}\chi$, where $\chi \in \Gamma$. □

10 Example: Representations of Dynamical Systems

Let $f \in \mathbb{R}[x]$ be a fixed polynomial. In this section we consider the $*$-algebra

$$\mathcal{A} = \mathbb{C}\langle a, a^* | aa^* = f(a^* a)\rangle.$$

Representations of the relation $aa^* = f(a^*a)$ for a measurable real-valued function f have been studied in detail in [28] by other means. From the very beginning this important example gave us intuition for developing our theory.

By Lemma 9 the $*$-algebra $\mathcal{A}$ is $\mathbb{Z}$-graded with grading determined by $a \in \mathcal{A}_1$ and $a^* \in \mathcal{A}_{-1}$. From the definition of $\mathcal{A}$ it follows that every element of $\mathcal{A}$ is a linear combination of elements

$$a^m,\ m \geq 0;\ a^{*k},\ k > 0;\ a^{*k_1} a^{m_1} \dots a^{*k_r} a^{m_r},\ r \geq 1,\ k_1 > 0,\ m_r > 0.$$

This implies that $\mathcal{A}_n$ is the linear span of elements

$$a^{*k_1} a^{m_1} \dots a^{*k_r} a^{m_r},\ r \geq 1,\ k_1 \geq 0,\ m_r \geq 0,\ \sum m_j - \sum k_i = n.$$

From the defining relation $aa^* = f(a^*a)$ we easily derive that

$$ap(a^*a) = p(f(a^*a))a\ ,\ \ p(a^*a)a^* = a^*p(f(a^*a)) \text{ for } p \in \mathbb{C}[t]. \tag{66}$$

Lemma 25 *The $*$-algebra $\mathcal{B}$ is commutative and spanned by the Hermitian elements*

$$a^{*k_1}a^{m_1}\dots a^{*k_r}a^{m_r},\ r \geq 1,\ k_1 > 0,\ m_r > 0,\ \sum k_i = \sum m_j. \tag{67}$$

Proof For $k \in \mathbb{N}$, let $\mathcal{B}_k$ be the subalgebra of $\mathcal{B}$ generated by words w in a^* and a satisfying Eq. 67 and of length $|w|$ less or equal to $2k$.

We first prove by induction on k that the algebra $\mathcal{B}_k$ is generated by words $w,\ |w| \leq 2k$, of the form a^*Q for some word Q. For $k = 1$ the assertion holds, since $\mathcal{B}_1$ is generated by the element a^*a. Suppose that the assertion is valid for $k > 1$. Let $w \in \mathcal{B},\ |w| \leq 2k+2, k > 1$. If $w = a^*Q$ for some word Q, then the induction proof is complete. Let $w = a^r a^* P,\ r > 0$, for some word P. Using Eq. 66 we get

$$w = a^r a^* P = a^{r-1} f(a^*a) P = a^{r-2} f(f(a^*a))aP = \dots = f^r(a^*a)a^{r-1}P.$$

The word $a^{r-1}P$ belongs to the algebra $\mathcal{B}_{k-1}$ and the element $f^r(a^*a)$ belongs to $\mathcal{B}_1$. It follows that $w \in \mathcal{B}_{k-1}$ and the induction hypothesis applies. This completes our first induction proof.

A second similar induction proof shows that $\mathcal{B}_k,\ k \geq 1$, is generated by words $w,\ |w| \leq 2k$, of the form a^*Qa for some word Q.

We now prove by induction on k that $\mathcal{B}$ is commutative. The algebra $\mathcal{B}_1$ is generated by the single element a^*a, so it is commutative. Suppose that $\mathcal{B}_k,\ k \geq 1$, is commutative. Let w_1 and w_2 be words of length between $2k$ and $2k+2$. Then, it is enough to consider the case when the words w_i have the form $a^*P_ia,\ i = 1, 2$, for some words P_i. Remembering that $aa^* \in \mathcal{B}_1 \subseteq \mathcal{B}_k$ and using the induction hypothesis we compute

$$w_1w_2 = a^*P_1aa^*P_2a = a^*aa^*P_1P_2a = a^*aa^*P_2P_1a = a^*P_2aa^*P_1a = w_2w_1.$$

Thus, $\mathcal{B}_{k+1}$ is commutative. □

Remark The algebra $\mathcal{B}$ is in general rather "large" when the polynomial f is not linear. We shall see this from the description of the set $\widehat{\mathcal{B}}^+ \subseteq \widehat{\mathcal{B}}$ given below.

The following Proposition allows us to use the theory developed in the Section 6.

Proposition 37 *The $\mathbb{Z}$-grading of the algebra $\mathcal{A}$ introduced above satisfies condition* (18).

Proof Using a simple induction argument one can prove the equalities

$$\mathcal{A}_n = \mathcal{B}a^n,\ \mathcal{A}_{-n} = a^{*n}\mathcal{B},\ n \in \mathbb{N}. \tag{68}$$

Then Proposition 12 completes the proof. □

We now describe the set $\widehat{\mathcal{B}}^+$, the partial action of $\mathbb{Z}$ on it and the representations associated with orbits of this partial action.

Let $\chi \in \widehat{\mathcal{B}}^{+}$ be fixed and let π be the induced representation Indχ. Let h_k denote the vector $[a^k \otimes 1] \in \mathcal{H}_\pi$ for all $k \in \mathbb{Z}$. We always put $a^{-k} := a^{*k}$ for $k \in \mathbb{N}$ and $a^0 := \mathbf{1}_{\mathcal{A}}$.

If $h_k = 0$ for some $k > 0$, then for any $c_k \in \mathcal{A}_k$ we have $[c_k \otimes 1] = 0$. Indeed, by Eq. 68 there exists $b \in \mathcal{B}$ such that $c_k = b a^k$ which implies $[c_k \otimes 1] = [b a^k \otimes 1] = \pi(b)[a^k \otimes 1] = 0$. Moreover, for all $m > 0$ we have $h_{k+m} = \pi(a^m) h_k = 0$.

Analogously, if $h_{-k} = 0$ for some $k > 0$, then for any $c_{-k} \in \mathcal{A}_{-k}$ we have $[c_{-k} \otimes 1] = 0$. Indeed, by Eq. 68 there exists $b \in \mathcal{B}$ such that $c_{-k} = a^{*k} b$. It implies $[c_{-k} \otimes 1] = [a^{*k} b \otimes 1] = [a^{*k} \otimes \chi(b)] = \chi(b)[a^{*k} \otimes 1] = 0$. For all $m > 0$ we have $h_{-k-m} = \pi(a^{*m}) h_{-k} = 0$.

Summarizing the above considerations we conclude that there exist $K, M \in \mathbb{N} \cup \{\pm\infty\}$, $K < 0 < M$ such that $h_k \neq 0$ if and only if $K < k < M$. All h_k are pairwise orthogonal and Proposition 16 implies that the vectors h_k span $\mathcal{H}_\pi$. Using Proposition 16 we also conclude that $\pi(a) h_k = \mu_k h_{k+1}$ for some $\mu_k \in \mathbb{C}$. We choose numbers $\nu_k \in \mathbb{C} \setminus \{0\}$, $k \in \mathbb{Z}$, $\nu_0 = 1$, such that the vectors $e_k := \nu_k h_k$, $k \in \mathbb{Z}$ are of the norm 1 if $h_k \neq 0$ and

$$\pi(a) e_k = \lambda_k e_{k+1}, \; \pi(a^*) e_k = \lambda_{k-1} e_{k-1} \text{ for some } \lambda_k \geq 0, \; k \in \mathbb{Z}. \tag{69}$$

Thus the vectors e_k, $K < k < M$, form an orthonormal base of $\mathcal{H}_\pi$. Furthermore, $\lambda_k > 0$ for $K < k < M - 1$ and relation (69) together with the defining relation $aa^* = f(a^* a)$ imply $\lambda_{k-1}^2 = f(\lambda_k^2)$ for all $K < k < M$. In the case when K resp. M is finite we have also $f(\lambda_{K+1}^2) = \lambda_K^2 = 0$, resp. $\lambda_{M-1} = 0$, $f(0) = \lambda_{M-2}^2$.

For the fixed character $\chi \in \widehat{\mathcal{B}}^{+}$ we consider the possible cases depending on K and M.

1. Let $K < 0$ and $M > 0$ be finite, so that $\lambda_{k-1}^2 = f(\lambda_k^2)$ for $K < k < M$, $f(\lambda_{K+1}^2) = 0$, $f(0) = \lambda_{M-2}^2$. Since $\chi(c_k^* c_k) = \|[c_k \otimes 1]\|^2 = 0$ for all $c_k \in \mathcal{A}_k$, $k \leq K$, $k \geq M$, the character χ^k is defined only for $K < k < M$. It implies that the stabilizer of χ is trivial. Thus π is an irreducible finite-dimensional representation. Using Eq. 69 we get
$$\pi(a) e_k = \lambda_k e_{k+1}, \text{ for } K < k < M - 1, \; \pi(a) e_{M-1} = 0,$$
$$\pi(a^*) e_k = \lambda_{k-1} e_{k-1} \text{ for } K + 1 < k < M, \; \pi(a^*) e_{K+1} = 0.$$
2. Let only $M > 0$ be finite, so that $\lambda_{k-1}^2 = f(\lambda_k^2)$ for all $k < M$ and $f(0) = \lambda_{M-2}^2$. As in the previous case we have that the stabilizer of χ is trivial. Thus π is an irreducible infinite-dimensional representation. By Eq. 69 we have
$$\pi(a) e_k = \lambda_k e_{k+1}, \text{ for } k < M - 1, \; \pi(a) e_{M-1} = 0,$$
$$\pi(a^*) e_k = \lambda_{k-1} e_{k-1} \text{ for } k < M.$$
According to the terminology of [28], π is the *Fock representation.*
3. Let only $K < 0$ be finite, so that $\lambda_{k-1}^2 = f(\lambda_k^2)$ for $K < k$, $f(\lambda_{K+1}^2) = 0$. As in the case **1.** the stabilizer of χ is trivial. Thus π is an irreducible infinite-dimensional representation. From Eq. 69 we obtain
$$\pi(a) e_k = \lambda_k e_{k+1}, \text{ for } K < k,$$
$$\pi(a^*) e_k = \lambda_{k-1} e_{k-1} \text{ for } K + 1 < k, \; \pi(a^*) e_{K+1} = 0.$$
In the terminology of [28], π is called *anti-Fock representation.*

4. Let both K and M be infinite, so that $\lambda_{k-1}^2 = f(\lambda_k^2)$ for $k \in \mathbb{Z}$. Recall that a sequence $\{\lambda_k\}_{k\in\mathbb{Z}}$ is called periodic if there exists $m \in \mathbb{N}$, such that $\lambda_k = \lambda_{k+m}$ for all $k \in \mathbb{Z}$. The smallest such m is called period of the sequence $\{\lambda_k\}_{k\in\mathbb{Z}}$. We consider two subcases.

   4.1. Let $\{\lambda_k^2\}_{k\in\mathbb{Z}}$ be not periodic. Then, in particular all numbers $\lambda_k,\ k \in \mathbb{Z}$, are pairwise different. From Eq. 69 we have $\pi(a^*a)e_k = \lambda_k^2 e_k$ and Proposition 16 (ii) implies that $\chi^k(a^*a) = \lambda_k^2$. Since $\{\lambda_k^2\}_{k\in\mathbb{Z}}$ is not periodic, all characters $\chi^k,\ k \in \mathbb{Z}$, are different. Thus, the stabilizer of χ is trivial and the representation π defined by Eq. 69 is irreducible.

   4.2. Let $\{\lambda_k^2\}_{k\in\mathbb{Z}}$ be periodic with a period $m \in \mathbb{N}$. Repeating the arguments from the previous case it follows that the stabilizer H of χ is equal to $m\mathbb{Z} \subset \mathbb{Z}$. Let $\mathcal{H}_{\pi,m}$ be the Hilbert subspace spanned by the vectors $e_{rm},\ r \in \mathbb{Z}$. Let $p \in \mathbb{N}$ and $c_{pm} \in \mathcal{A}_{pm}$. Then Eq. 68 implies that $c_{pm} = b_1 a^{pm}$ for some $b_1 \in \mathcal{B}$. Using Eq. 69 and Proposition 16 (ii) we get

$$\pi(c_{pm})e_{rm} = \chi^{rm}(b_1)(\lambda_0\lambda_1 \dots \lambda_{m-1})^p e_{(r+p)m} = \chi(b_1)(\lambda_0\lambda_1 \dots \lambda_{m-1})^p e_{(r+p)m}.$$

   Thus $\pi(c_{pm})$ acts as a scalar multiple of the bilateral shift on $\mathcal{H}_{\pi,m}$. This implies that

$$\widetilde{\chi}(b_1 a^{pm}) := \chi(b_1)(\lambda_0\lambda_1 \dots \lambda_{m-1})^p,\ \ p \in \mathbb{N}, \tag{70}$$

   defines a character on the algebra $\mathcal{A}_H$. The restriction of $\widetilde{\chi}$ to $\mathcal{B}$ coincides with χ. Therefore, by Proposition 27 the Mackey obstruction of χ is trivial. We denote by $\zeta_z,\ z \in \mathbb{T}$, the character of the group $H = m\mathbb{Z}$ defined by $\zeta_z(m) = z$. Then, using Eqs. 38 and 70, we see that all representations $\rho_z,\ z \in \mathbb{T}$, of $\mathcal{A}_H$ satisfy condition (36). These representations are one-dimensional, that is, they are characters. For $c_{pm} = b a^{pm},\ p \in \mathbb{N},\ b \in \mathcal{B}$, we have

$$\begin{aligned}\rho_z(c_{pm}) &= \chi(c_{pm}^* c_{pm})^{1/2}\zeta_z(pm) = \widetilde{\chi}(c_{pm}^*)^{1/2}\widetilde{\chi}(c_{pm})^{1/2} z^p \\ &= \chi(b^*b)^{1/2}(\lambda_0\lambda_1 \dots \lambda_{m-1} z)^p,\end{aligned}$$

   where $z \in \mathbb{T}$.

We now compute the representations induced from $\rho_z,\ z \in \mathbb{T}$. Let π_z denotes the induced representation $\mathrm{Ind}_{\mathcal{A}_H \uparrow \mathcal{A}} \rho_z$ on the space $\mathcal{H}_z$. One easily verifies that the vectors

$$f_k = \chi(a^{*k}a^k)^{-1/2}[a^k \otimes 1],\ \ k = 0, \dots, m-1,$$

form an orthogonal base of the space $\mathcal{H}_z$. We calculate the action of $\pi(a)$ on the base vectors f_k. Using Proposition 16 (ii) and formulas (69) we find that $\chi(a^{*k}a^k) = \lambda_0^2\lambda_1^2 \dots \lambda_{k-1}^2,\ k \in \mathbb{N}$. Take $r = 0, \dots, m-2$. Then we have

$$\pi_z(a) f_r = \frac{\chi(a^{(r+1)*}a^{r+1})^{1/2}}{\chi(a^{r*}a^r)^{1/2}} f_{r+1} = \lambda_r f_{r+1}.$$

For f_{m-1} we get

$$\begin{aligned}\pi_z(a) f_{m-1} &= \chi(a^{*(m-1)}a^{m-1})^{-1/2}[a^m \otimes 1] = \chi(a^{*(m-1)}a^{m-1})^{-1/2}[\mathbf{1}_{\mathcal{A}} \otimes \rho_z(a^m)] = \\ &= \chi(a^{*(m-1)}a^{m-1})^{-1/2}\widetilde{\chi}(a^m)[\mathbf{1}_{\mathcal{A}} \otimes 1] = z\lambda_{m-1} f_0.\end{aligned}$$

Now suppose we are given a sequence $\lambda_k > 0,\ K < k < M-1$, where $-\infty \le K < 0 < M \le \infty$. Suppose also that $f(\lambda_{K+1}^2) = 0$ resp. $f(0) = \lambda_{M-2}^2$ in the case when K resp. M is finite. We call such a sequence *nonnegative orbit of the dynamical system* $(f, [0, +\infty))$. Then Eq. 69 defines a $*$-representation π of $\mathcal{A}$ and the restriction of $\mathrm{Res}_{\mathcal{B}}\pi$ to $\mathbb{C}\cdot e_0$ gives a character $\chi \in \widehat{\mathcal{B}}^+$. Let us describe this characters χ in the case **4.** explicitly. Take an element $a^{*k_1}a^{m_1}\dots a^{*k_r}a^{m_r} \in \mathcal{B},\ r \ge 1,\ k_1 > 0,\ m_r > 0,\ \sum k_i = \sum m_j$. Using formulas (69) we obtain

$$\chi(a^{*k_1}a^{m_1}\dots a^{*k_r}a^{m_r}) = \prod_{i=0}^{m_r-1}\lambda_i \prod_{i=1}^{k_r}\lambda_{m_r-i}\cdots\prod_{i=1}^{k_1}\lambda_{m_r-k_r+m_{r-1}-\dots+m_1-i}.$$

We summarize the above discussion in the following

Proposition 38 *The Eq. 69 give a one-to-one correspondence between nonnegative orbits of the dynamical system $(f, [0, +\infty))$ and orbits of the partial action of $\mathbb{Z}$ on $\widehat{\mathcal{B}}^+$. A representation π defined by Eq. 69 is reducible if and only if the sequence λ_k is periodic and $\lambda_k > 0$ for all $k \in \mathbb{Z}$.*

Finally, we consider the problem of associating irreducible well-behaved representations of $\mathcal{A}$ with orbits in $\widehat{\mathcal{B}}^+$ (cf. also [28]).

Proposition 39 *Assume that the function f is one-to-one and there exists a measurable set $\Gamma \subseteq [0, +\infty)$ containing precisely one point from each nonnegative orbit of the dynamical system $(f, [0, +\infty))$. Then every irreducible well-behaved representation of $\mathcal{A}$ is associated with an orbit in $\widehat{\mathcal{B}}^+$.*

Sketch of Proof Let π be an irreducible well-behaved representation of $\mathcal{A}$. Then $\pi(a^*a)$ is essentially self-adjoint. Using Proposition 33 in [28] we conclude that the spectral measure of $\overline{\pi(a^*a)}$ is ergodic with respect to f. Applying Proposition 34 in [28] it follows that the spectral measure of $\overline{\pi(a^*a)}$ is concentrated on a single orbit of the dynamical system $(f, [0, +\infty))$. □

For the case, when f is not bijective, we refer to Theorem 15 in [28].

11 Further Examples

In this section we mention and briefly discuss some other classes of examples, where the theory developed in the previous sections can be applied.

Example 17 (Compact Quantum Group Algebras) The simplest example is the quantum group $SU_q(2)$, $q \in \mathbb{R}$. The corresponding $*$-algebra $\mathcal{A}$ has two generators a and c and defining relations

$$ac = qca,\ ca^* = qa^*c,\ c^*c = cc^*,\ aa^* + q^2cc^* = 1,\ a^*a + c^*c = 1. \tag{71}$$

Then $\mathcal{A}$ is $\mathbb{Z}$-graded such that $a \in \mathcal{A}_1, a^* \in \mathcal{A}_{-1},\ c \in \mathcal{A}_0$.

Set $N := a^*a$. Then the subalgebra $\mathcal{B} = \mathcal{A}_0$ is equal to $\mathbb{C}[c, c^*, N]$. It follows from Eq. 71 that $\mathcal{B}$ is commutative and $\mathcal{A}_k = a^k\mathcal{B},\ k \in \mathbb{Z}$. Proposition 12 implies that condition (18) is satisfied and our theory applies. From the defining relations (71) it follows at once that every $*$-representation is bounded and hence well-behaved by Proposition 18.

Suppose that $q \in (-1, 1),\ q \neq 0$. In what follows many arguments are similar to the case of the Weyl algebra (see Examples 1, 10 and 16). The last two equations in Eq. 71 imply $aa^* - q^2a^*a = 1 - q^2$. By induction on $k \in \mathbb{Z}$ one proves the following formulas:

$$aa^{*k} = a^{*(k-1)}(q^{2k}(a^*a) - q^{2k} + 1),\ a^*a^k = \frac{1}{q^{2k}}a^{k-1}(aa^* + q^{2k} - 1), \tag{72}$$

$$a^na^{*n} = \prod_{k=1}^{n}(1 - q^{2k} + q^{2k}N),\ a^{*n}a^n = \frac{1}{q^{2n}}\prod_{k=0}^{n-1}(N + q^{2k} - 1). \tag{73}$$

From Corollary 1 and formula (73) we obtain

$$\sum \mathcal{A}^2 \cap \mathcal{B} = \sum \mathcal{B}^2 + N\sum \mathcal{B}^2 + \dots + N(N + q^2 - 1)\dots(N + q^{2k} - 1)\sum \mathcal{B}^2 + \dots \tag{74}$$

Equations 71 and 74 imply that the only characters $\chi \in \widehat{\mathcal{B}}$ which are positive on $\sum \mathcal{A}^2 \cap \mathcal{B}$ are:

- $\chi_{k,u}, k \in \mathbb{N}_0,\ u \in \mathbb{C},\ |u| = 1$, defined by $\chi_{k,u}(N) = 1 - q^{2k},\ \chi_{k,u}(c) = q^ku$, and
- χ_∞ defined by $\chi_\infty(N) = 1,\ \chi_\infty(c) = 0$.

From Eq. 73 we derive the partial action of $\mathbb{Z}$ on $\widehat{\mathcal{B}}^+$. For $\chi_{k,u}$, $\alpha_n(\chi_{k,u})$ is defined and then equal to $\chi_{k-n,u}$ if and only if $n \leq k$. For χ_∞ we have $\alpha_n(\chi_\infty) = \chi_\infty$ for all $n \in \mathbb{Z}$. The set $\{\chi_{0,u},\ |u| = 1\} \cup \{\chi_\infty\}$ is a section of the action, i.e. it contains exactly one point from each orbit. By Proposition 29 every irreducible representation is associated to some orbit.

The stabilizers of $\chi_{0,u},\ |u| = 1$, are trivial. Hence, by Theorem 5, $\pi_u := \mathrm{Ind}\chi_{0,u}$ is the only irreducible representation, up to unitary equivalence, associated with $\mathrm{Orb}\chi_{0,u}$. From Proposition 16 we obtain explicit formulas for the actions on some orthobase $(f_k, k \in \mathbb{N}_0)$, where $f_{-1} := 0$:

$$\pi_u(N)f_k = (1 - q^{2k})f_{k-1},\ \pi_u(a^*)f_k = (1 - q^{2k+2})^{1/2}f_{k+1},\ \pi_u(c)f_k = q^kuf_k,\ k \in \mathbb{N}_0.$$

The stabilizer of χ_∞ is $\mathbb{Z}$ and $\mathcal{A}_\mathbb{Z} = \mathcal{A}$. Let ρ be as in Theorem 5, that is, ρ is an irreducible representation of $\mathcal{A}$ such that $\mathrm{Res}_\mathcal{B}\rho$ is a multiple of χ_∞. Then $\rho(c) = \rho(c^*) = 0$ and $\rho(a^*a) = \rho(aa^*) = \mathbf{1}$. Hence ρ is one-dimensional and equal to ρ_u, where $\rho_u(c) = 0,\ \rho_u(a) = u,\ u \in \mathbb{C},\ |u| = 1$. Since $\mathrm{Ind}\rho \simeq \rho$, every irreducible representation associated with $\{\chi_\infty\}$ equals to some $\rho_u,\ |u| = 1$.

Example 18 (Quantum Disk Algebra) Suppose that $0 \leq \mu \leq 1,\ 0 \leq q \leq 1$, and $(\mu, q) \neq (0, 1)$. The two-parameter unit quantum disk $*$-algebra $\mathcal{A}$ has generators a and a^* and the defining relation

$$qaa^* - a^*a = q - 1 + \mu(1 - aa^*)(1 - a^*a).$$

Then $\mathcal{A}$ is $\mathbb{Z}$-graded such that $a \in \mathcal{A}_1$ and $a^* \in \mathcal{A}_{-1}$. As in the case of the dynamical systems in the previous section one shows that $\mathcal{B} = \mathcal{A}_0$ is commutative and condition

(18) is satisfied. There is a one-to-one correspondence between orbits in $\widehat{\mathcal{B}}^{+}$ and orbits of the dynamical system $(f, [0, +\infty))$ where

$$f(\lambda) = \frac{(q+\mu)\lambda + 1 - q - \mu}{\mu\lambda + 1 - \mu}.$$

For a more detailed analysis of this $*$-algebra see [23] and [28], p. 101.

Example 19 (Podles' Quantum Spheres) Let $q \in (0, \infty)$. For $r \in [0, \infty)$, $\mathcal{O}(\mathrm{S}^2_{qr})$ is the unital $*$-algebra with generators $A = A^*$, B, B^* and defining relations (see [29] or [24], 4.5)

$$AB = q^{-2}BA,\ AB^* = q^2B^*A,\ B^*B = A - A^2 + r,\ BB^* = q^2A - q^4A^2 + r.$$

For $r = \infty$, the defining relations of $\mathcal{O}(\mathrm{S}^2_{q,\infty})$ are

$$AB = q^{-2}BA,\ AB^* = q^2B^*A,\ B^*B = -A^2 + 1,\ BB^* = -q^4A^2 + 1.$$

In both cases $\mathcal{A} = \mathcal{O}(\mathrm{S}^2_{qr})$ is $\mathbb{Z}$-graded such that $B \in \mathcal{A}_1$, $B^* \in \mathcal{A}_{-1}$ and $A \in \mathcal{A}_0$. One can check that $\mathcal{B} = \mathcal{A}_0$ is commutative and condition (18) is fulfilled. It follows immediately from the defining relations that all $*$-representations of $\mathcal{A}$ are bounded.

Example 20 (Twisted CCR) Let $\mu \in (0, 1)$ be fixed. The twisted canonical commutation relations (briefly, TCCR) $*$-algebra $\mathcal{A} = \mathcal{A}_\mu$ is generated by elements a_i, a_i^*, $i = 1, \ldots, d$, with defining relations (see [34])

$$a_ia_i^* = 1 + \mu^2a_i^*a_i - (1-\mu^2)\sum_{k>i} a_k^*a_k,\ i = \overline{1, d},$$

$$a_ja_i = \mu a_ia_j,\ i < j,\ a_i^*a_j^* = \mu a_j^*a_i^*,\ i < j,\ a_ja_i^* = \mu a_ia_j^*,\ i \neq j.$$

For $\mu = 1$ we get the d-dimensional Weyl algebra. For all $\mu \in (0, 1]$, $\mathcal{A}$ is $\mathbb{Z}^d$-graded such that a_k, $a_k^* \in \mathcal{A}_{g_k}$, where $g_1, \ldots, g_d$ are generators of $\mathbb{Z}^d$, the subalgebra $\mathcal{B} = \mathcal{A}_0$ is commutative and condition (18) is satisfied.

Example 21 (Deformations of CAR Algebra) Let $q \in (0, 1)$ be fixed. The twisted canonical anti-commutation relations (briefly, TCAR) $*$-algebra $\mathcal{A} = \mathcal{A}_q$ is generated by elements a_i, a_i^*, $i = 1, \ldots, d$, with defining relations (see [33])

$$a_i^*a_i = 1 - a_ia_i^* - (1-q^2)\sum_{j<i} a_ja_j^*,\ i = 1, \ldots, d,$$

$$a_i^*a_j = -qa_ja_i^*,\ a_ja_i = -qa_ia_j,\ i < j, a_i^2 = 0,\ i = 1, \ldots, d.$$

For $q = 1$ we get the "usual" CAR algebra. For all $q \in (0, 1]$, $\mathcal{A}$ is $(\mathbb{Z}/2\mathbb{Z})^d$-graded such that a_k, $a_k^* \in \mathcal{A}_{g_k}$, where $g_1, \ldots, g_d$ are generators of $(\mathbb{Z}/2\mathbb{Z})^d$, the subalgebra $\mathcal{B} = \mathcal{A}_0$ is commutative and condition (18) is satisfied.

The Wick analogue of TCAR (denoted as WTCAR) was studied in [19, 31, 32]. The WTCAR $*$-algebra $\mathcal{A}$ is obtained from TCAR by omitting the relations between a_i and a_j. Hence $\mathcal{A}$ is $\mathbb{Z}^d$-graded such that $a_k \in \mathcal{A}_{g_k}$ where $g_1, \ldots, g_d$ are generators of $\mathbb{Z}^d$. In this case the $*$-subalgebra $\mathcal{B} = \mathcal{A}_0$ is not commutative. However, it was shown in [19, 31] that in any irreducible representation of WTCAR the relations

$$a_ja_i = -qa_ia_j,\ i < j,\ a_i^2 = 0,\ i = 1, \ldots, d-1,$$

hold automatically. Then our theory applies to the quotient of WTCAR $*$-algebra by the latter relations.

Example 22 (Quantum Algebras $\mathcal{U}_q(su(2))$ and $\mathcal{U}_q(su(1,1))$) For $q \in \mathbb{R}$, $q^2 \neq 1$, the q-deformed enveloping algebra $\mathcal{U}_q(sl(2))$ is the complex unital (associative) algebra with generators E, F, K, K^{-1} and defining relations

$$KK^{-1} = K^{-1}K = 1,\ KEK^{-1} = q^2E,\ KFK^{-1} = q^{-2}F,\ [E, F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

The involutions defining the $*$-algebras $\mathcal{U}_q(su(2))$ and $\mathcal{U}_q(su(1,1))$ are given by the formulas

$$E^* = F,\ F^* = E,\ K^* = K,\ K^{-1*} = K^{-1},$$
$$E^* = -F,\ F^* = -E,\ K^* = K,\ K^{-1*} = K^{-1},$$

respectively. Let $\mathcal{A}$ be one of the $*$-algebras $\mathcal{U}_q(su(2))$ or $\mathcal{U}_q(su(1,1))$. Then $\mathcal{A}$ is $\mathbb{Z}$-graded with grading determined by $E \in \mathcal{A}_1$, $F \in \mathcal{A}_{-1}$, and $K, K^{-1} \in \mathcal{A}_0$, the $*$-subalgebra $\mathcal{B} = \mathcal{A}_0$ is commutative, and condition (18) is valid. The Mackey analysis for $\mathcal{A}$ is similar to that of $\mathcal{U}(su(2))$ and $\mathcal{U}(su(1,1))$.

The algebra $\mathcal{U}_q(sl(2))$ was introduced in [25], see e.g. [24], 3.1. Representations of $\mathcal{U}_q(su(2))$ and $\mathcal{U}_q(su(1,1))$ have been investigated in [43] and [3], respectively.

Example 23 (CAR Algebras) Let $\mathcal{A}$ be the direct limit of matrix $*$-algebras $M_{2^k}(\mathbb{C})$, $k \in \mathbb{N}$, where the embedding $M_{2^k}(\mathbb{C}) \hookrightarrow M_{2^{k+1}}(\mathbb{C})$ is given by the canonical injection $M_{2^k}(\mathbb{C}) \otimes I_2 \hookrightarrow M_{2^{k+1}}(\mathbb{C})$. Here $I_2 \in M_2(\mathbb{C})$ is the identity matrix. The representation theory of $\mathcal{A}$ was studied in [15], see also [37] and [20].

Each matrix algebra $M_n(\mathbb{C})$ has a natural $\mathbb{Z}$-grading such that each matrix unit e_{ij} belongs to the $(i-j)$-component. Since the embeddings $M_{2^k}(\mathbb{C}) \hookrightarrow M_{2^{k+1}}(\mathbb{C})$ respect this grading, $\mathcal{A}$ is also $\mathbb{Z}$-graded. One checks that condition (18) is valid for $M_{2^k}(\mathbb{C})$ which implies that the $\mathbb{Z}$-grading on $\mathcal{A}$ also satisfies Eq. 18. The $*$-subalgebra $\mathcal{B} = \mathcal{A}_0$ is the direct limit of commutative algebras $\mathbb{C}^{2^k}$. It can be considered as a (dense) $*$-subalgebra of the $*$-algebra of all continuous functions on the Cantor set. The conditional expectation defined by the $\mathbb{Z}$-grading is strong, so $\widehat{\mathcal{B}}^+$ coincides with $\widehat{\mathcal{B}}$ which is equal to the Cantor set. All representations of $\mathcal{A}$ are bounded. The partial action of $\mathbb{Z}$ on $\widehat{\mathcal{B}}^+$ has trivial stabilizers. All irreducible representations associated with orbits in $\widehat{\mathcal{B}}^+$ are direct limits of representations. In this case the assumptions of Proposition 29 are not satisfied and there exist irreducible representations of $\mathcal{A}$ arising from ergodic measures under the partial action of $\mathbb{Z}$ on $\widehat{\mathcal{B}}^+$ which are not supported on single orbits.

Appendix

The main result of this Appendix (Theorem 7) is related to condition (i) of Definition 11, but it is also of interest in itself. Its proof is based on the spectral theorem for countable families of commuting self-adjoint operators, see [37], Theorem 1. We equip $\mathbb{R}^\infty = \mathbb{R} \times \mathbb{R} \times \dots$ with the product topology and denote by $\mathbf{B}(\mathbb{R}^\infty)$ the Borel structure on $\mathbb{R}^\infty$ induced by this topology.

Theorem 6 *For each family $A_k,\ k \in \mathbb{N}$, of strongly commuting self-adjoint operators there exists a unique resolution of the identity E on the Borel space $(\mathbb{R}^\infty, \mathbf{B}(\mathbb{R}^\infty))$ such that*

$$A_k = \int \lambda_k dE(\lambda_1, \lambda_2, \dots) \text{ for all } k \in \mathbb{N}.$$

In the notation of Theorem 6, the *joint spectrum* of the family $A_k,\ k \in \mathbb{N}$, is the intersection of all closed subsets X of $\mathbb{R}^\infty$ such that $E(X) = E(\mathbb{R}^\infty)$.

Let $\mathcal{B}$ be a commutative unital $*$-algebra. As in Section 7, we equip the set $\widehat{\mathcal{B}}$ of all characters of $\mathcal{B}$ with the weakest topology for which all functions $f_b,\ b \in \mathcal{B}$, are continuous, where f_b is defined by $f_b(\chi) = \chi(b)$ for $\chi \in \widehat{\mathcal{B}}$. Clearly, if $\mathcal{B}$ is generated by elements $b_n, n \in \mathbb{N}$, then this topology coincides with the weakest topology for which all functions $f_{b_n},\ n \in \mathbb{N}$, are continuous.

Theorem 7 *Suppose that $\mathcal{B}$ is a countably generated commutative unital $*$-algebra. We equip $\widehat{\mathcal{B}}$ with the Borel structure induced by the weak topology. Let $\mathcal{C}$ be a quadratic module of $\mathcal{B}$ and let $\widehat{\mathcal{B}}^+$ denote the set of all characters $\chi \in \widehat{\mathcal{B}}$ which are nonnegative on $\mathcal{C}$. If π is an integrable representation of $\mathcal{B}$, then:*

(i) *There exists a unique spectral measure E_π on $\widehat{\mathcal{B}}$ such that*

$$\overline{\pi(b)} = \int f_b(\lambda)\, dE_\pi(\lambda) \text{ for all } b \in \mathcal{B}.$$

(ii) *Assume in addition that $\langle \pi(c)\varphi, \varphi\rangle \geq 0$ for all $c \in \mathcal{C}$ and $\varphi \in \mathcal{D}(\pi)$. Then the spectral measure E_π is supported on $\widehat{\mathcal{B}}^+$ which is a closed subset of $\widehat{\mathcal{B}}$.*

Proof

(i) First we fix a sequence of self-adjoint generators $b_k, k \in \mathbb{N}$, of the $*$-algebra $\mathcal{B}$ and consider $\widehat{\mathcal{B}}$ as a subset of $\mathbb{R}^\infty$ by identifying

$$\widehat{\mathcal{B}} \ni \chi \longleftrightarrow (\chi(b_1), \chi(b_2), \chi(b_3), \dots) \in \mathbb{R}^\infty.$$

We prove that $\widehat{\mathcal{B}}$ is closed in $\mathbb{R}^\infty$, hence Borel. Let $\chi_n = (\chi_n(b_1), \chi_n(b_2), \dots) \in \widehat{\mathcal{B}},\ n \in \mathbb{N}$ be a sequence of characters converging to $\chi \in \mathbb{R}^\infty$ in the product topology. We claim that there is a character χ on $\mathcal{B}$ such that $\chi(b_k) := \lim_{n\to\infty} \chi_n(b_k)$. Indeed, let $m \in \mathbb{N}$ and $p \in \mathbb{C}[t_1, \dots, t_m]$ be a polynomial such that $p(b_1, \dots, b_m) = 0$. Since

$$p(\chi_n(b_1), \dots, \chi_n(b_m)) = \chi_n(p(b_1, \dots, b_m)) = 0,$$

we conclude that

$$p(\chi(b_1), \dots, \chi(b_m)) = p(\lim_{n\to\infty} \chi_n(b_1), \dots, \lim_{n\to\infty} \chi_n(b_m)) = 0$$

for all $n \in \mathbb{N}$. Therefore $\chi \in \mathbb{R}^\infty$ defines a character on $\mathcal{B}$, i.e. $\chi \in \widehat{\mathcal{B}}$.
A sequence $\chi_n \in \widehat{\mathcal{B}}$ converges to $\chi \in \widehat{\mathcal{B}}$ if and only if $\chi_n(b_k) = f_{b_k}(\chi_n)$ converges to $\chi(b_k) = f_{b_k}(\chi)$ for every fixed k as $n \to \infty$. Since the elements $b_k, k \in \mathbb{N}$, generate $\mathcal{B}$, it follows that the topology on $\widehat{\mathcal{B}}$ induced from $\mathbb{R}^\infty$ coincides with the weak topology. In particular, the Borel structure on $\widehat{\mathcal{B}}$ coincides with the one induced from $\mathbb{R}^\infty$.

Since π is integrable, the operators $\overline{\pi(b_k)}$, $k \in \mathbb{N}$, are self-adjoint and pairwise strongly commuting ([39], Corollary 9.1.14). Therefore, by Theorem 6 there exists a spectral measure E_π on the set $\mathbb{R}^\infty$ such that

$$\overline{\pi(b_k)} = \int \lambda_k dE_\pi(\lambda_1, \lambda_2, \dots).$$

for all $k \in \mathbb{N}$. For every polynomial $p \in \mathbb{R}[t_1, \dots, t_m]$ the operator $p(\pi(b_1), \dots, \pi(b_m))$ is essentially self-adjoint and from basic properties of spectral integrals we obtain

$$\overline{p(\pi(b_1), \dots, \pi(b_m))} = \int p(\lambda_1, \dots, \lambda_m) dE_\pi(\lambda_1, \lambda_2, \dots). \tag{75}$$

Next we show that the spectral measure E_π is supported on $\widehat{\mathcal{B}} \subseteq \mathbb{R}^\infty$, or equivalently, that the joint spectrum $\sigma(\overline{\pi(b_1)}, \overline{\pi(b_2)}, \dots)$ of the family $\overline{\pi(b_k)}$, $k \in \mathbb{N}$, is contained in $\widehat{\mathcal{B}}$. Let $x = (x_1, x_2, \dots) \in \mathbb{R}^\infty$ be a point in $\sigma(\overline{\pi(b_1)}, \overline{\pi(b_2)}, \dots)$. Again, let $m \in \mathbb{N}$ and $p_0 \in \mathbb{R}[t_1, \dots, t_m]$ be such that $p_0(b_1, \dots, b_m) = 0$. Then we obtain

$$\overline{\pi(p_0(b_1, \dots, b_m))} = 0.$$

Assume to the contrary that $p_0(x_1, x_2, \dots, x_m) \neq 0$. Then for every open neighborhood $O(x)$ we have $E_\pi(O(x)) \neq 0$. Using Eq. 75 we get

$$\begin{aligned} 0 = \overline{\pi(p_0(b_1, \dots, b_m))} &= \overline{p_0(\pi(b_1), \dots, \pi(b_m))} \\ &= \int p_0(\lambda_1, \dots, \lambda_m) dE_\pi(\lambda_1, \lambda_2, \dots) \neq 0, \end{aligned}$$

which is a contradiction. That is, we have $p_0(x_1, x_2, \dots, x_m) = 0$. Thus we have shown that $\chi(b_k) := x_k$ defines a character and E_π is supported on $\widehat{\mathcal{B}}$. The uniqueness of the spectral measure E_π follows at once from the corresponding assertion in Theorem 6.

(ii) Since $\widehat{\mathcal{B}}$ is a closed subset of the separable space $\mathbb{R}^\infty$, $\widehat{\mathcal{B}}$ is also separable. Similar arguments as used in the proof of (i), show that $\widehat{\mathcal{B}}^+$ is closed in $\widehat{\mathcal{B}}$.
Assume to the contrary that $E_\pi(\widehat{\mathcal{B}} \backslash \widehat{\mathcal{B}}^+) \neq 0$. Since $\widehat{\mathcal{B}}$ is separable and $\widehat{\mathcal{B}}^+$ is a closed subset of $\widehat{\mathcal{B}}$, there exists a countable dense subset $\{\chi_i\}_{i \in \mathbb{N}}$ of $\widehat{\mathcal{B}} \backslash \widehat{\mathcal{B}}^+$. For every χ_i there exists an element c_i of $\mathcal{C}$ such that $\chi_i(c_i) < 0$. Since $\{\chi_i\}_{i \in \mathbb{N}}$ is dense in $\widehat{\mathcal{B}} \backslash \widehat{\mathcal{B}}^+$, the open sets $f_{c_i}^{-1}((-\infty, 0))$ cover $\widehat{\mathcal{B}} \backslash \widehat{\mathcal{B}}^+$. From the latter it follows that there exists a $k \in \mathbb{N}$ such that $E_\pi(f_{c_k}^{-1}((-\infty, 0))) \neq 0$. Hence there exists a vector $\varphi \in \operatorname{Ran} E_\pi(f_{c_k}^{-1}((-\infty, 0))) \cap \mathcal{D}(\pi)$ such that $\langle \pi(c_k)\varphi, \varphi \rangle < 0$ which contradicts our assumption. □

Definition 15 If $\mathcal{B}$, π and E_π are as in the previous theorem, we shall say that the integrable representation π and the spectral measure E_π are *associated* with each other.

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