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Euler Characteristics of Quiver Grassmannians and Ringel-Hall Algebras of String Algebras

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Abstract We compute the Euler characteristics of quiver Grassmannians and quiver flag varieties of tree and band modules and prove their positivity. This generalizes some results by G. Cerulli Irelli (2010). As an application we consider the Ringel-Hall algebra $\mathcal{C}(A)$ of some string algebras A and compute in combinatorial terms the products of arbitrary functions in $\mathcal{C}(A)$. These results are transferred to covering theory.

Keywords Covering theory · Euler characteristic · Quiver Grassmannian · Quiver flag variety · Quiver representation · Ringel-Hall algebra · String algebra · Band module · String module · Tree module

Mathematics Subject Classification (2010) 06B15

1 Introduction

1.1 Motivation

Fomin and Zelevinsky (see [11–13]) have introduced cluster algebras. For their studies the Euler characteristics of a class of projective varieties, called quiver Grassmannians, are important (see [5, 10]). For instance, Caldero and Keller have shown in [6] and [7] that the Euler characteristic plays a central role for the categorification of cluster algebras.

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1.2 Basic Concepts

We use and improve a technique of Cerulli Irelli [8] to compute Euler characteristics of such projective varieties. In general it is hard to compute the Euler characteristic of a quiver Grassmannian, but in the case of tree and band modules we show that it is only a combinatorial task.

Let Q be a locally finite quiver, M a finite-dimensional representation of Q over $\mathbb{C}$ and $\mathbf{d}$ a dimension vector of Q (see Section 2.1). Then the *quiver Grassmannian* $\mathrm{Gr}_{\mathbf{d}}(M)$ is the complex projective variety of subrepresentations of M with dimension vector $\mathbf{d}$ (see Definition 2.1). Our aim is to compute its Euler characteristic $\chi_{\mathbf{d}}(M)$. This computation can be simplified by certain algebraic actions of the one-dimensional torus $\mathbb{C}^*$, since the Euler characteristic of a variety equals the Euler characteristic of the subset of fixed points under certain $\mathbb{C}^*$-actions.

To find suitable $\mathbb{C}^*$-actions we introduce gradings of representations of Q in Section 4. A map $\partial\colon E \to \mathbb{Z}$ with a basis E of a representation M is called a *grading of the representation* M. A grading ∂ of a representation M induces an action of $\mathbb{C}^*$ on the vector space M. If this grading ∂ induces also an action on some locally closed subset X of $\mathrm{Gr}_{\mathbf{d}}(M)$, it is called *stable* on X (see Definition 4.7). The linear combinations of the basis vectors with the same values under a grading ∂ are called ∂-*homogeneous*. For a locally closed subset X of $\mathrm{Gr}_{\mathbf{d}}(M)$ let

$$X^{\partial} := \left\{U \in X \middle| U \text{ has a } \partial\text{-homogeneous vector space basis.}\right\}. \tag{1}$$

Theorem 1.1 *Let Q be a locally finite quiver, M a finite-dimensional representation of Q, $X \subseteq \mathrm{Gr}_{\mathbf{d}}(M)$ a locally closed subset and ∂ a stable grading on X. Then X^{∂} is a locally closed subset of $\mathrm{Gr}_{\mathbf{d}}(M)$ and the Euler characteristic of X equals the Euler characteristic of X^{∂}. If the subset X is non-empty and closed in $\mathrm{Gr}_{\mathbf{d}}(M)$, then X^{∂} is also non-empty and closed in $\mathrm{Gr}_{\mathbf{d}}(M)$.*

This theorem can be used for more than one grading at the same time or in an iterated way.

1.3 Tree and Band Modules

Some special morphisms of quivers $F\colon S \to Q$ are called *windings of quivers* (for further details see Section 2.3). Each winding induces a functor $F_*\colon \mathrm{rep}(S) \to \mathrm{rep}(Q)$ of the categories of finite-dimensional representations and a map $\mathbf{F}\colon \mathbb{N}^{S_0} \to \mathbb{N}^{Q_0}$ of the dimension vectors of S and Q. If S is a finite quiver and a tree and every vector space of a representation of S is one-dimensional and every map non-zero, this representation is denoted by $\mathbb{1}_S$ and the image of this representation under the functor F_* is called a *tree module* (see Definition 2.4). Let $n \in \mathbb{Z}_{>0}$, S be a quiver of type $\tilde{A}_{l-1}$ and $\mathcal{I}_S^n$ the set of indecomposable representations $V = (V_i, V_a)_{i\in S_0, a\in S_1}$ of S with V_a is an isomorphism for any $a \in S_1$ and $\dim_{\mathbb{C}}(V_i) = n$ for some $i \in S_0$. The representation $F_*(V)$ of Q is called a *band module* if $V \in \mathcal{I}_S^n$ and $F_*(V)$ is indecomposable (see Definition 2.7).

In Theorem 1.2 we compute the Euler characteristics of quiver Grassmannians of all tree and band modules and prove their positivity (see Corollaries 3.1 and 3.2).

1.4 Quiver Flag Varieties

The projective variety $\mathcal{F}_{\mathbf{d}^{(1)},\ldots,\mathbf{d}^{(r)}}(M)$ of flags of subrepresentations of M with dimension vectors $\mathbf{d}^{(1)},\ldots,\mathbf{d}^{(r)}$ is called *quiver flag variety* (see Definition 3.8). Theorems 1.1 and 1.2 are also generalized to analogous statements for such quiver flag varieties (see Corollary 3.10).

1.5 Coverings of Quivers

Let $\hat{Q}$ be a locally finite quiver and G a group. An *action of the group G on $\hat{Q}$* is a pair of maps $G \times \hat{Q}_0 \to \hat{Q}_0$, $(g,i) \mapsto gi$ and $G \times \hat{Q}_1 \to \hat{Q}_1$, $(g,a) \mapsto ga$ such that $gs(a) = s(ga)$ and $gt(a) = t(ga)$ for all $g \in G$ and $a \in \hat{Q}_1$. We say, the group G acts *freely* on the quiver $\hat{Q}$ if for all $i \in \hat{Q}_0$ and all $a \in \hat{Q}_1$ the stabilizers are trivial. Let $Q = \hat{Q}/G$ be the orbit quiver of an action and $\pi : \hat{Q} \to Q$ the canonical projection map. If G acts freely on the quiver $\hat{Q}$, then π is a winding.

In Theorem 1.2(c) we show that the Euler characteristic of a quiver Grassmannian of some finite-dimensional Q-representation $\pi_*(V)$ is determined by the Euler characteristics of the quiver Grassmannians of the $\hat{Q}$-representation V. Since this theorem holds for free and free abelian groups, we write “free (abelian) group”.

1.6 Main Result

In this section we state the main result of this preprint.

Theorem 1.2

(a) *Let Q and S be finite quivers, $F\colon S \to Q$ a tree or a band, $\mathbf{d}$ a dimension vector of Q and V any finite-dimensional S-representation. Then*

$$\chi_{\mathbf{d}}(F_*(V)) = \sum\nolimits_{\mathbf{t}\in\mathbf{F}^{-1}(\mathbf{d})} \chi_{\mathbf{t}}(V). \tag{2}$$

(b) *Let S be a quiver of type $\tilde{A}_{l-1}$, $\mathbf{t} = (t_i)_{i\in S_0}$ a dimension vector of S and $V \in \mathcal{I}_S^n$, i.e. V is a band module of S and* $\dim_{\mathbb{C}}(V_i) = n$ *for some $i \in S_0$. Then*

$$\chi_{\mathbf{t}}(V) = \left(\prod_{\substack{i\in S_0\\ \text{source}}} \frac{(n-t_i)!}{t_i!}\right)\left(\prod_{\substack{i\in S_0\\ \text{sink}}} \frac{t_i!}{(n-t_i)!}\right)\left(\prod_{a\in S_1} \frac{1}{(t_{t(a)}-t_{s(a)})!}\right) \tag{3}$$

with $0! = 1$, $s! = 0$ *and* $\frac{1}{s!} = 0$ *for all negative* $s \in \mathbb{Z}$.

(c) *Let $\hat{Q}$ be a locally finite quiver and G a free (abelian) group, which acts freely on $\hat{Q}$. Let $\mathbf{d}$ be a dimension vector of $Q = \hat{Q}/G$ and V a finite-dimensional $\hat{Q}$-representation. Then*

$$\chi_{\mathbf{d}}(\pi_*(V)) = \sum\nolimits_{\mathbf{t}\in\pi^{-1}(\mathbf{d})} \chi_{\mathbf{t}}(V). \tag{4}$$

It is easy to see that *relations I* of a quiver Q do not affect these results. Let M be a representation of a quiver Q with relations I. So M is also a representation of the quiver Q without relations. Any subrepresentation of the representation M of Q is also a subrepresentation of the representation M of Q with the relations I. Thus the variety $\mathrm{Gr}_{\mathbf{d}}(M)$ for a finite-dimensional representation M of a quiver Q with relations I equals the variety $\mathrm{Gr}_{\mathbf{d}}(M)$ for the representation M of the quiver Q without relations.

1.7 Morphisms of Ringel-Hall Algebras

Let Q be a locally finite quiver, I an admissible ideal and $A = \mathbb{C}Q/I$ the corresponding $\mathbb{C}$-algebra (for further details see Section 2.1). We associate to the algebra A the *Ringel-Hall algebra* $\mathcal{H}(A)$, its subalgebra $\mathcal{C}(A)$ and its completions $\hat{\mathcal{H}}(A)$ and $\hat{\mathcal{C}}(A)$ (for further details see Section 2.4). Trees, bands and coverings induce morphisms of Ringel-Hall algebras.

Theorem 1.3

(a) *Let $F\colon S \to Q$ be a tree or a band and $A = \mathbb{C}Q/I$ and $B = \mathbb{C}S/J$ finite-dimensional algebras. If F induces a functor $F_*\colon \mathrm{mod}(B) \to \mathrm{mod}(A)$, then the map*

$$\mathcal{C}(F)\colon \mathcal{C}(A) \to \mathcal{C}(B),\ f \mapsto f \circ F_* \tag{5}$$

is a Hopf algebra homomorphism. If F is injective, this map $\mathcal{C}(F)$ is surjective. If any A-module can be lifted to a B-module, i.e. F_* is dense, the map $\mathcal{C}(F)$ is injective.

(b) *Let $\hat{Q}$ be a locally finite quiver and G a free (abelian) group, which acts freely on $\hat{Q}$. Let $Q = \hat{Q}/G$, $A = \mathbb{C}Q/I$ and $B = \mathbb{C}\hat{Q}/J$ be algebras and $\pi : \hat{Q} \to Q$ the canonical projection. If π induces a functor $\pi_*\colon \mathrm{mod}(B) \to \mathrm{mod}(A)$, then the map*

$$\mathcal{C}(\pi)\colon \mathcal{C}(A) \to \hat{\mathcal{C}}(B),\ f \mapsto f \circ \pi_* \tag{6}$$

is a Hopf algebra homomorphism. If any A-module can be lifted to a B-module, this map is injective.

In both cases these maps are functorial, since $\mathcal{C}(\mathrm{id}_A) = \mathrm{id}_{\mathcal{C}(A)}$ and $\mathcal{C}(F \circ G) = \mathcal{C}(G) \circ \mathcal{C}(F)$ for some maps F and G occurring in the theorem. Moreover these maps $\mathcal{C}(F)$ and $\mathcal{C}(\pi)$ can be extended to the bigger Ringel-Hall algebras:

$$\mathcal{H}(F)\colon \mathcal{H}(A) \to \mathcal{H}(B),\ f \mapsto f \circ F_*, \quad \mathcal{H}(\pi)\colon \mathcal{H}(A) \to \hat{\mathcal{H}}(B),\ f \mapsto f \circ \pi_*.$$

But these maps are in general no algebra morphisms. Nevertheless we use this notation.

Let $X \subseteq \mathrm{rep}_{\mathbf{d}}(A)$ and $Y \subseteq \mathrm{rep}_{\mathbf{c}}(A)$ be constructible and $\mathrm{GL}(\mathbb{C})$-stable subsets. To consider the multiplication of the Ringel-Hall algebra $\mathcal{H}(A)$ we have to compute the Euler characteristic of the following constructible subset of $\mathrm{Gr}_{\mathbf{d}}(M)$

$$\left\{N \in \mathrm{Gr}_{\mathbf{d}}(M) \middle| N \in X, M/N \in Y\right\}.$$

The gradings used in the proof of Theorem 1.2 are also stable on the Grassmannians appearing in the product $\mathcal{C}(F)(f * g)$ for some tree or band F and in the product

$\mathcal{C}(\pi)(f * g)$ for some covering π. This simplifies the calculations of the Euler characteristics of these Grassmannians.

1.8 Ringel-Hall-Algebras, Tree and Band Modules

Let $\mathbf{F} = \left(F^{(1)}, \ldots, F^{(r)}\right)$ with $F^{(i)}\colon S^{(i)} \to Q$ be a tuple of trees, $\mathbf{B} = \left(B^{(1)}, \ldots, B^{(s)}\right)$ with $B^{(i)}\colon T^{(i)} \to Q$ a tuple of bands and $\mathbf{n} = (n_1, \ldots, n_s)$ a tuple of positive integers (see Definition 2.4 and 2.7). Let

$$\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}(M) = \begin{cases} 1 \text{ if } \exists V_i \in \mathcal{I}^{n_i}_{T^{(i)}},\ M \cong \bigoplus_{i=1}^r F^{(i)}_*(\mathbb{1}_{S^{(i)}}) \oplus \bigoplus_{i=1}^s B^{(i)}_*(V_i), \\ 0 \text{ otherwise.} \end{cases}$$

This defines constructible functions in $\mathcal{H}(A)$, which are not necessarily in $\mathcal{C}(A)$.

Let F be a winding and $\mathbf{F} = \left(F^{(1)}, \ldots, F^{(r)}\right)$ with $F^{(i)}\colon S^{(i)} \to Q$ be a tuple of windings. We define a set of tuples by the following: Let $\mathcal{G}_F(\mathbf{F})$ be a set of representatives of the equivalence classes of the set

$$\left\{\widetilde{\mathbf{F}} = \left(\widetilde{F}^{(1)}, \ldots, \widetilde{F}^{(r)}\right) \middle| \widetilde{F}^{(i)}\colon S^{(i)} \to Q \text{ winding}, F\widetilde{F}^{(i)} = F^{(i)}\ \forall i\right\}$$

with the equivalence relation $\sim$ defined by $\widetilde{\mathbf{F}} \sim \widetilde{\mathbf{F}}'$ iff $\mathbb{1}_{\widetilde{\mathbf{F}}} = \mathbb{1}_{\widetilde{\mathbf{F}}'}$ in $\mathcal{H}(A)$. Thus for all i the diagram in Fig. 1 commutes. If $r = 0$, the set $\mathcal{G}_F(\mathbf{F})$ consists by convention of one trivial element.

For the next statement it is not important if we compute the products in $\mathcal{H}(\mathbb{C}Q)$ or in $\mathcal{H}(\mathbb{C}Q/I)$ for some admissible ideal I. It is only essential that F resp. π induces a well-defined functor $\text{mod}(B) \to \text{mod}(A)$.

Proposition 1.4 *Let* $\mathbf{F}$ *be a tuple of trees,* $\mathbf{B}$ *a tuple of bands and* $\mathbf{n}$ *a tuple of positive integers.*

(a) *Let Q be a finite quiver and $F\colon S \to Q$ a tree or a band. Then*

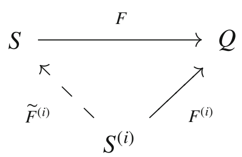
$$\mathcal{H}(F)\left(\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}\right) = \sum\nolimits_{\widetilde{\mathbf{F}} \in \mathcal{G}_F(\mathbf{F}), \widetilde{\mathbf{B}} \in \mathcal{G}_F(\mathbf{B})} \mathbb{1}_{\widetilde{\mathbf{F}},\widetilde{\mathbf{B}},\mathbf{n}}. \tag{7}$$

(b) *Let $\hat{Q}$ be a locally finite quiver, G a free (abelian) group, which acts freely on $\hat{Q}$, $Q = \hat{Q}/G$ and $\pi : \hat{Q} \to Q$ the canonical projection. Then*

$$\mathcal{H}(\pi)\left(\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}\right) = \sum\nolimits_{\widetilde{\mathbf{F}} \in \mathcal{G}_\pi(\mathbf{F}), \widetilde{\mathbf{B}} \in \mathcal{G}_\pi(\mathbf{B})} \mathbb{1}_{\widetilde{\mathbf{F}},\widetilde{\mathbf{B}},\mathbf{n}}. \tag{8}$$

Combining this proposition with the previous Theorem 1.3 we get useful corollaries to compute the products of these functions $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}$ in $\mathcal{H}(\mathbb{C}Q)$ (see Corollaries 3.17 and 3.25). Actually for a string algebra $A = \mathbb{C}Q/I$ (see Section 2.5) the computation

Fig. 1 Factorization property



of arbitrary products of functions in $\mathcal{C}(A)$ is reduced to a purely combinatorial task (see Corollary 3.22).

1.9 Outline of Paper

The paper is organized as follows: First we define our main objects in Section 2 and then we give some examples, generate some corollaries and explain our results in Section 3. After that we introduce the gradings as a useful tool in Section 4 and then we prove Theorem 1.1 in Section 5.1. Both are used to prove the remaining results of Sections 1 and 3 in the remaining sections.

2 Main Definitions

2.1 Quivers and Path Algebras

Let $Q = (Q_0, Q_1, s, t)$ be a *locally finite quiver* (or quiver for short), i.e. an oriented graph with vertex set Q_0, arrow set Q_1 and maps $s, t\colon Q_1 \to Q_0$ indicating the start and terminal point of each arrow such that in any vertex only finitely many arrows start and end. A *finite-dimensional representation* $M = (M_i, M_a)_{i\in Q_0, a\in Q_1}$ of Q (or *Q-representation* for short) is a set of finite-dimensional $\mathbb{C}$-vector spaces $\{M_i | i \in Q_0\}$ and a set of $\mathbb{C}$-linear maps $\{M_a\colon M_{s(a)} \to M_{t(a)} | a \in Q_1\}$ such that only finitely many of the vector spaces are non-zero. A *morphism* $f = (f_i)_{i\in Q_0}\colon M \to N$ of Q-representations is a set of $\mathbb{C}$-linear maps $\{f_i\colon M_i \to N_i | i \in Q_0\}$ such that $f_{t(a)} M_a = N_a f_{s(a)}$ for all $a \in Q_1$. Let $\mathrm{rep}(Q)$ denote the *category of finite-dimensional Q-representations.*

A *subrepresentation* $N = (N_i)_{i\in Q_0}$ of $M = (M_i, M_a)_{i\in Q_0, a\in Q_1}$ is a set of subspaces $\{N_i \subseteq M_i | i \in Q_0\}$ such that $M_a(N_{s(a)}) \subseteq N_{t(a)}$ for all $a \in Q_1$. So every subrepresentation $N = (N_i)_{i\in Q_0}$ of a Q-representation $M = (M_i, M_a)_{i\in Q_0, a\in Q_1}$ is again a Q-representation by $(N_i, M_a|_{N_{s(a)}})_{i\in Q_0, a\in Q_1}$. In this case we write $N \subseteq M$. The *dimension* of a Q-representation M is $\dim(M) := \sum_{i\in Q_0} \dim_{\mathbb{C}}(M_i)$ and its *dimension vector* is the tuple $\mathbf{dim}(M) := (\dim_{\mathbb{C}}(M_i))_{i\in Q_0} \in \mathbb{N}^{Q_0}$. So we assume for each dimension vector $\mathbf{d} = (d_i)_{i\in Q_0} \in \mathbb{N}^{Q_0}$ of Q that $|\mathbf{d}| := \sum_{i\in Q_0} d_i < \infty$.

Let Q be a quiver. An *oriented path* $\rho = a_1 \dots a_n$ of Q is the concatenation of some arrows $a_1, \dots, a_n \in Q_1$ such that $t(a_{i+1}) = s(a_i)$ for all $1 \le i < n$. Additionally we introduce a path e_i of length zero for each vertex $i \in Q_0$. The *path algebra* $\mathbb{C}Q$ of a quiver Q is the $\mathbb{C}$-vector space with the set of oriented paths as a basis. The product of basis vectors is given by the concatenation of paths if possible or by zero otherwise.

Let Q be a locally finite quiver and $\mathbb{C}Q^+$ the ideal in the path algebra $\mathbb{C}Q$, which is generated by all arrows. An ideal I of the path algebra $\mathbb{C}Q$ is called *admissible* if a $k \in \mathbb{Z}_{>0}$ exists such that $(\mathbb{C}Q^+)^k \subseteq I \subseteq (\mathbb{C}Q^+)^2$. If the quiver Q is finite and I admissible, $A = \mathbb{C}Q/I$ is a finite-dimensional $\mathbb{C}$-algebra such that the isomorphism classes of simple representations are in bijection with the vertices of the quiver Q.

For a quiver Q it is well known that the *category of finite-dimensional* $\mathbb{C}Q$-*modules* $\mathrm{mod}(\mathbb{C}Q)$ is equivalent to the category $\mathrm{rep}(Q)$. So we think of Q-representations as $\mathbb{C}Q$-modules and vice versa. Let I be an admissible ideal, $A = \mathbb{C}Q/I$ and $\mathrm{mod}(A)$ the *category of finite-dimensional A-modules*. Again we think of A-modules as Q-representations and some Q-representations as A-modules.

An expanded introduction to (finite-dimensional) algebras over an arbitrary field can be found in [1].

2.2 Quiver Grassmannians

Definition 2.1 Let Q be a quiver, M a Q-representation and $\mathbf{d}$ a dimension vector. Then the closed subvariety

$$\mathrm{Gr}_{\mathbf{d}}(M) := \left\{ U \subseteq M \middle| \mathbf{dim}(U) = \mathbf{d} \right\}$$

of a product of classical Grassmannians is called the *quiver Grassmannian*.

Hence this is a projective complex variety, which is by [21] in general neither smooth nor irreducible. We denote the Euler characteristic of a quasi-projective variety X by $\chi(X)$ and the Euler characteristic of $\mathrm{Gr}_{\mathbf{d}}(M)$ by $\chi_{\mathbf{d}}(M)$ for short.

Example 2.2 Let $Q = 1 \xrightarrow{a} 2$, $M_1 = M_2 = \mathbb{C}^2$ and $M_a \colon M_1 \to M_2$ a linear map with $\mathrm{rk}(M_a) = 1$. Then $M = (M_1, M_2, M_a)$ is a Q-representation such that $\mathrm{Gr}_{(1,1)}(M)$ can be described as $(\{*\} \times \mathbb{P}^1) \cup (\mathbb{P}^1 \times \{*\}) \subseteq \mathbb{P}^1 \times \mathbb{P}^1$. This projective variety is neither smooth nor irreducible and $\chi_{(1,1)}(M) = 3$.

A proof of the following proposition is given in Section 5.2.

Proposition 2.3 (Riedtmann [19]) *Let Q be a quiver,* $\mathbf{d}$ *a dimension vector and M and N Q-representations. Then*

$$\chi_{\mathbf{d}}(M \oplus N) = \sum\nolimits_{0 \leq \mathbf{c} \leq \mathbf{d}} \chi_{\mathbf{c}}(M) \chi_{\mathbf{d}-\mathbf{c}}(N) \tag{9}$$

with $(c_i)_{i \in Q_0} = \mathbf{c} \leq \mathbf{d} = (d_i)_{i \in Q_0}$ if and only if $c_i \leq d_i$ for all $i \in Q_0$.

Thus it is enough to consider the Euler characteristic of Grassmannians associated to indecomposable representations.

2.3 Tree and Band Modules

Let $Q = (Q_0, Q_1, s, t)$ and $S = (S_0, S_1, s', t')$ be two quivers. A *winding of quivers* $F \colon S \to Q$ (or *winding* for short) is a pair of maps $F_0 \colon S_0 \to Q_0$ and $F_1 \colon S_1 \to Q_1$ such that the following hold:

(a) F is a morphism of quivers, i.e. $sF_1 = F_0 s'$ and $tF_1 = F_0 t'$.
(b) If $a, b \in S_1$ with $a \neq b$ and $s'(a) = s'(b)$, then $F_1(a) \neq F_1(b)$.
(c) If $a, b \in S_1$ with $a \neq b$ and $t'(a) = t'(b)$, then $F_1(a) \neq F_1(b)$.

This generalizes Krause's definition of a winding [17]. Let V be an S-representation. For $i \in Q_0$ and $a \in Q_1$ set

$$(F_*(V))_i = \bigoplus\nolimits_{j \in F_0^{-1}(i)} V_j \qquad \text{and} \qquad (F_*(V))_a = \bigoplus\nolimits_{b \in F_1^{-1}(a)} V_b.$$

This induces a functor $F_* \colon \mathrm{rep}(S) \to \mathrm{rep}(Q)$ and a map of dimension vectors $\mathbf{F} \colon \mathbb{N}^{S_0} \to \mathbb{N}^{Q_0}$. The concatenation of windings behaves very well: Let $F \colon S \to Q$ and

$G\colon T \to S$ be windings then $FG\colon T \to Q$ is again a winding and the functors $(FG)_*$ and F_*G_* are equivalent.

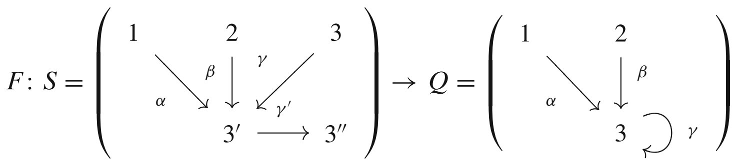
Let Q be a finite quiver. Then the Q-representation $(M_i, M_a)_{i\in Q_0, a\in Q_1}$ with $M_i = \mathbb{C}$ for all $i \in Q_0$ and $M_a = \mathrm{id}_{\mathbb{C}}$ for all $a \in Q_1$ is denoted $\mathbb{1}_Q$. For $n \in \mathbb{Z}_{>0}$ let $\mathcal{I}_Q^n$ be the set of all indecomposable Q-representations $(M_i, M_a)_{i\in Q_0, a\in Q_1}$ with $\dim_{\mathbb{C}}(M_i) = n$ for all $i \in Q_0$ and M_a is an isomorphism for all $a \in Q_1$.

A simply connected and finite quiver S is called a *tree*, i.e. for any two vertices in S there exists a unique not necessarily oriented path from one vertex to the other.

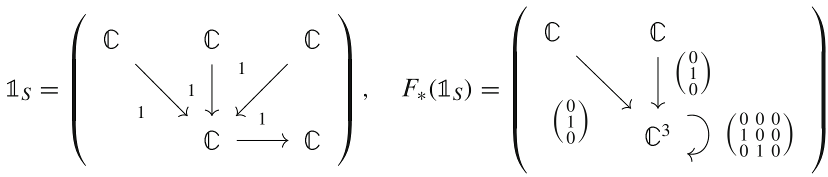
Definition 2.4 Let Q and S be quivers and $F\colon S \to Q$ a winding. If S is a tree, then the representation $F_*(\mathbb{1}_S)$ is called a *tree module*. We call such a winding F a *tree*, too.

By [14] all tree modules are indecomposable.

Example 2.5 Let Q, S and F be described by the following picture.



Let $\mathbb{1}_S$ and $F_*(\mathbb{1}_S)$ be described by the following pictures.



Then $F\colon S \to Q$ is a tree and $F_*(\mathbb{1}_S)$ a tree module.

A quiver S is called *of type* A_l for some $l \in \mathbb{Z}_{>0}$ if $S_0 = \{1, \ldots, l\}$ and $S_1 = \{s_1, \ldots, s_{l-1}\}$ such that for all $i \in S_0$ with $i \neq l$ there exists a $\varepsilon_i \in \{-1, 1\}$ with $s(s_i^{\varepsilon_i}) = i + 1$ and $t(s_i^{\varepsilon_i}) = i$. (We use here the convention $s(a^{-1}) = t(a)$ and $t(a^{-1}) = s(a)$ for all $a \in S_1$.) Figure 2 visualizes a quiver S of type A_l.

Definition 2.6 Let Q and S be quivers, S of type A_l, $F\colon S \to Q$ a winding and $F_*(\mathbb{1}_S)$ a tree module. Then F is called a *string* and $F_*(\mathbb{1}_S)$ a *string module*.

A quiver S is called *of type* $\tilde{A}_{l-1}$ for some $l \in \mathbb{Z}_{>0}$ if $S_0 = \{1, \ldots, l\}$ and $S_1 = \{s_1, \ldots, s_l\}$ such that for all $i \in S_0$ a $\varepsilon_i \in \{-1, 1\}$ exists with $s(s_i^{\varepsilon_i}) = i + 1$ and $t(s_i^{\varepsilon_i}) = i$. (We set $l + i := i$ in S_0.) We draw a picture of a quiver of type $\tilde{A}_{l-1}$ in Fig. 4.

Fig. 2 A quiver of type A_l

$$1 \xleftarrow{s_1^{\varepsilon_1}} 2 \xleftarrow{s_2^{\varepsilon_2}} 3 \longleftarrow \cdots \longleftarrow l-1 \xleftarrow{s_{l-1}^{\varepsilon_{l-1}}} l$$

Definition 2.7 Let Q and S be quivers, $B\colon S \to Q$ a winding and $V \in \mathcal{I}_S^n$. If S is of type $\tilde{A}_{l-1}$ and $B_*(V)$ is indecomposable, then $B_*(V)$ is called a *band module*. B is called a *band* if S is of type $\tilde{A}_{l-1}$ and $B_*(\mathbb{1}_S)$ is indecomposable.

Let S be a quiver of type $\tilde{A}_{l-1}$ and $B\colon S \to Q$ a winding. The module $B_*(\mathbb{1}_S)$ is not necessarily indecomposable. This well known feature is explained in the following example.

Example 2.8 Let Q and S be quivers, S of type $\tilde{A}_{l-1}$, $B\colon S \to Q$ a winding and $V \in \mathcal{I}_S^n$. We set $s_{l+i} := s_i$ in S_1 and $\varepsilon_{l+i} := \varepsilon_i$ for all $i \in S_0$.

(a) If there is no integer r with $1 \le r < l$, $B_1(s_i) = B_1(s_{i+r})$ and $\varepsilon_i = \varepsilon_{i+r}$ for all $1 \le i \le l$ and the Jordan normal form of the map $V_{s_1}^{\varepsilon_1} \dots V_{s_l}^{\varepsilon_l}$ is an indecomposable Jordan matrix, then $B_*(V)$ is indecomposable.
(b) If there is an integer r with $r > 0$ as above, then $B_*(V) \cong \bigoplus_{i=1}^{s} M^{(i)}$ with $s = \frac{l}{\gcd(r,l)}$ and Q-representations $M^{(i)}$ of dimension $n \gcd(r, l)$.

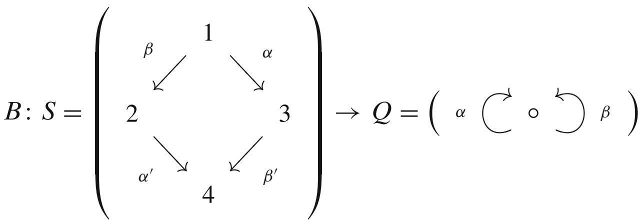
Remark 2.9 Let $r \in \mathbb{Z}_{>0}$. Using the Jordan normal form, the indecomposable modules of the polynomial ring $\mathbb{C}[T, T^{-1}]$ of dimension r are canonically parametrized by $\mathbb{C}^*$. Let $\varphi_r\colon \mathbb{C}^* \to \mathrm{mod}(\mathbb{C}[T, T^{-1}])$ describe this parametrization and $\varphi\colon \mathbb{C}^* \times \mathbb{Z}_{>0} \to \mathrm{mod}(\mathbb{C}[T, T^{-1}])$ with $\varphi(\lambda, r) = \varphi_r(\lambda)$.

Let $B\colon S \to Q$ be a band and $\mathrm{mod}(\mathbb{C}[T, T^{-1}])$ the category of finite-dimensional $\mathbb{C}[T, T^{-1}]$-modules. There exists a full and faithful functor $F\colon \mathrm{mod}(\mathbb{C}[T, T^{-1}]) \to \mathrm{rep}(S)$ such that $F(V) \in \mathcal{I}_S^{\dim\,(V)}$ for any indecomposable $V \in \mathrm{mod}(\mathbb{C}[T, T^{-1}])$.

The map $\mathbb{C}^* \times \mathbb{Z}_{>0} \xrightarrow{\varphi} \mathrm{mod}(\mathbb{C}[T, T^{-1}]) \xrightarrow{F} \mathrm{rep}(S) \xrightarrow{B_*} \mathrm{rep}(Q)$ is a parametrization of all band modules of the form $B_*(V)$. The image of $(\lambda, r) \in \mathbb{C}^* \times \mathbb{Z}_{>0}$ under this map is denoted $B_*(\lambda, r)$. Additionally, we define $B_*(\lambda, 0) = 0$ for all $\lambda \in \mathbb{C}^*$. We remark that neither the functor F nor our parametrization of band modules of the form $B_*(V)$ is unique.

Let $\lambda \in \mathbb{C}^*$ and $r, s \in \mathbb{N}$ with $r \ge s$. Then a surjective morphism $B_*(\lambda, r) \twoheadrightarrow B_*(\lambda, s)$ and an injective morphism $B_*(\lambda, s) \hookrightarrow B_*(\lambda, r)$ exist. Let $\varphi\colon B_*(\lambda, r) \to B_*(\lambda, s)$ be such a morphism. Then the kernel and the image of φ are independent of φ. So for all $r, s \in \mathbb{N}$ with $r \ge s$ there exists a unique sub- and a unique factormodule of $B_*(\lambda, r)$ isomorphic to $B_*(\lambda, s)$.

Example 2.10 Let $Q = (\{\circ\}, \{\alpha, \beta\}, s, t)$, $\lambda \in \mathbb{C}^*$ and B the band described by the following picture.



In this case we can assume that the band module $B_*(\lambda, 3)$ is visualized by Fig. 3.

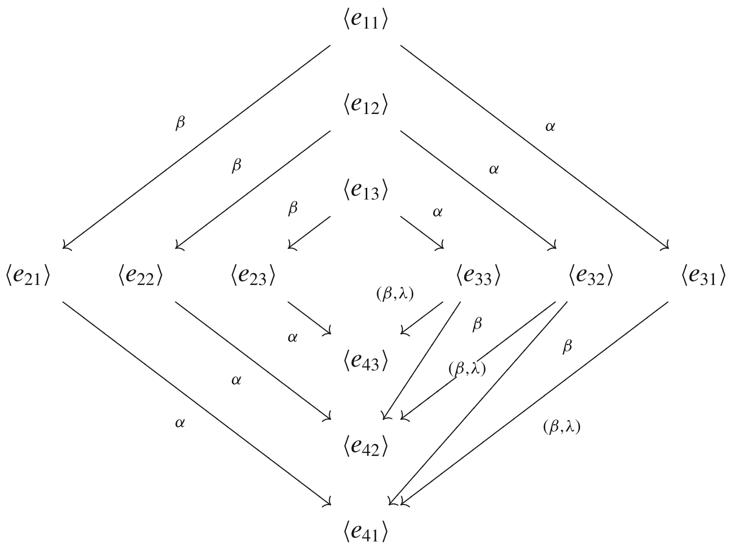


Fig. 3 A band module $B_*(\lambda, 3)$

Crawley-Boevey [9] and Krause [17] constructed a basis of the homomorphism spaces of tree and band modules. This description yields the following lemma.

Lemma 2.11 *Let Q, S, T be connected quivers, $F\colon S \to Q$, $G\colon T \to Q$ tree or band, $V \in \mathcal{I}_S^n$ and $W \in \mathcal{I}_T^m$. If $F_*(V) \cong G_*(W)$ then a unique bijective winding $H\colon S \to T$ exists such that $F = GH$ and $H_*(V) \cong W$.*

Proof Since $F_*(V)$ is indecomposable the endomorphism ring $\operatorname{End}_Q(F_*(V))$ is local. Thus by [9] and [17] such a winding H exists. Since F and G are trees or bands the modules $H_*(V)$ and W are isomorphic. The winding H is unique since there is no non-trivial automorphism $H'\colon S \to S$ with $F = FH'$ for a connected tree or band $F\colon S \to Q$. □

Remark 2.12 Let Q be a quiver of type $\tilde{A}_{l-1}$. The category $\operatorname{rep}(Q)$ is well known and described in [22]. The indecomposable representations are divided into three classes of representations: The classes of preprojective, regular and preinjective representations. Let M be a band module and N a string module of Q. Then the following hold:

- The band module M is regular.
- $\operatorname{Hom}_Q(N, M) \neq 0$ and $\operatorname{Hom}_Q(M, N) = 0$ if and only if N is preprojective.
- $\operatorname{Hom}_Q(N, M) = 0$ and $\operatorname{Hom}_Q(M, N) = 0$ if and only if N is regular.
- $\operatorname{Hom}_Q(N, M) = 0$ and $\operatorname{Hom}_Q(M, N) \neq 0$ if and only if N is preinjective.
- If N is preprojective and $\dim(N) \leq \dim(M)$, then an injective map $N \hookrightarrow M$ and an indecomposable preinjective representation with dimension vector $\mathbf{dim}(M) - \mathbf{dim}(N)$ exist.
- If N is non-regular, then N is determined up to isomorphism by its dimension vector.

- If N is preprojective, then all short exact sequences $0 \to M \to L \to N \to 0$ with some Q-representation L split.
- If N is preinjective, then all short exact sequences $0 \to N \to L \to M \to 0$ with some Q-representation L split.

Let M and N be indecomposable preprojective Q-representations with $\dim(M) \geq \dim(N)$. Then $\mathrm{Hom}_Q(M, N) = 0$ if $M \ncong N$ and all short exact sequences $0 \to M \to L \to N \to 0$ with some Q-representation L split.

2.4 Ringel-Hall Algebras

The Ringel-Hall algebras of finite-dimensional hereditary algebras over finite fields are well known objects (see [20] for an introduction). We now consider the Ringel-Hall algebra $\mathcal{H}(A)$ of constructible functions over the module varieties $\mathrm{rep}_{\mathbf{d}}(A)$ of $\mathbb{C}$-algebra $A = \mathbb{C}Q/I$ with a locally finite quiver Q and an admissible ideal I (see definition below). This is an idea due to Schofield (unpublished manuscript), which also appears in works of Lusztig [18] and Riedtmann [19]. An introduction to the construction of Kapranov and Vasserot [16] and Joyce [15], which we are using here, can be found in [4]. For completeness we review the definition.

Let $A = \mathbb{C}Q/I$ be a path algebra of a locally finite quiver Q and an admissible ideal I. For a dimension vector $\mathbf{d} \in \mathbb{N}^{Q_0}$ let

$$\mathrm{rep}_{\mathbf{d}}(A) := \left\{ (M_\alpha)_\alpha \in \prod\nolimits_{\alpha \in Q_1} \mathrm{Hom}_{\mathbb{C}}\left(\mathbb{C}^{d_{s(\alpha)}}, \mathbb{C}^{d_{t(\alpha)}}\right) \middle| \left(\mathbb{C}^{d_i}, M_\alpha\right)_{i,\alpha} \in \mathrm{mod}(A) \right\}$$

be the *module variety* of the A-modules with dimension vector $\mathbf{d}$. The algebraic group $\mathrm{GL}_{\mathbf{d}}(\mathbb{C}) = \prod_{i \in Q_0} \mathrm{GL}_{d_i}(\mathbb{C})$ acts by conjugation on the variety $\mathrm{rep}_{\mathbf{d}}(A)$ such that the $\mathrm{GL}_{\mathbf{d}}(\mathbb{C})$-orbits are in bijection to the isomorphism classes of A-modules with dimension vector $\mathbf{d}$.

A function $f\colon X \to \mathbb{C}$ on a variety X is called *constructible* if the image is finite and every fibre is constructible. A constructible function $f\colon \mathrm{rep}_{\mathbf{d}}(A) \to \mathbb{C}$ is called $\mathrm{GL}_{\mathbf{d}}(\mathbb{C})$-*stable* (or $\mathrm{GL}(\mathbb{C})$-*stable* for short) if the fibres are $\mathrm{GL}_{\mathbf{d}}(\mathbb{C})$-stable sets.

Let $\mathcal{H}_{\mathbf{d}}(A)$ be the vector space of constructible and $\mathrm{GL}_{\mathbf{d}}(\mathbb{C})$-stable functions on $\mathrm{rep}_{\mathbf{d}}(A)$. Let $\mathcal{H}(A) = \bigoplus_{\mathbf{d} \in \mathbb{N}^{Q_0}} \mathcal{H}_{\mathbf{d}}(A)$ and $*\colon \mathcal{H}(A) \otimes \mathcal{H}(A) \to \mathcal{H}(A)$ with

$$(\mathbb{1}_X * \mathbb{1}_Y)(M) = \chi\left(\left\{0 \subseteq N \subseteq M \middle| N \in X, M/N \in Y\right\}\right)$$

for all $M \in \mathrm{rep}_{\mathbf{c}+\mathbf{d}}(A)$ and all constructible and $\mathrm{GL}(\mathbb{C})$-stable subsets $X \subseteq \mathrm{rep}_{\mathbf{d}}(A)$ and $Y \subseteq \mathrm{rep}_{\mathbf{c}}(A)$. For a dimension vector $\mathbf{d}$ let $\mathbb{1}_{\mathbf{d}}$ be the characteristic function of all representations with dimension vector $\mathbf{d}$ and $\mathbb{1}_{S^{\mathbf{d}}}$ the characteristic function of the semisimple representations with dimension vector $\mathbf{d}$. For an A-module M let $\mathbb{1}_M$ be the characteristic function of the orbit of the module M.

Proposition 2.13 *The vector space $\mathcal{H}(A)$ with the product $*$ is an associative, $\mathbb{N}^{Q_0}$-graded algebra with unit $\mathbb{1}_0$.*

Let $\mathcal{C}(A)$ be the subalgebra of $\mathcal{H}(A)$ generated by the set $\left\{\mathbb{1}_{\mathbf{d}} \middle| \mathbf{d} \in \mathbb{N}^{Q_0}\right\}$. The algebra $\mathcal{C}(A)$ is a cocommutative Hopf algebra with the coproduct $\Delta\colon \mathcal{C}(A) \to \mathcal{C}(A) \otimes \mathcal{C}(A)$ defined by $\Delta(f)(M, N) = f(M \oplus N)$ for all $f \in \mathcal{C}(A)$. This is known by Joyce [15] and also stated in [4]. Moreover for a dimension vector $\mathbf{d}$ and the antipode S holds $S(\mathbb{1}_{\mathbf{d}}) = (-1)^{|\mathbf{d}|}\mathbb{1}_{S^{\mathbf{d}}}$ in $\mathcal{C}(A)$.

Let $\hat{\mathcal{H}}(A) = \prod_{\mathbf{d}\in\mathbb{N}^{Q_0}} \mathcal{H}_{\mathbf{d}}(A)$ be the completion of the Ringel-Hall algebra $\mathcal{H}(A)$ and $\hat{\mathcal{C}}(A)$ the one of $\mathcal{C}(A)$.

2.5 String Algebras

Let Q be a finite quiver and I an admissible ideal. Then $A = \mathbb{C}Q/I$ is called a *string algebra* if the following hold:

(a) At most two arrows start in each vertex of Q.
(b) At most two arrows end in each vertex of Q.
(c) If $\alpha, \beta, \gamma \in Q_1$ with $\alpha \neq \beta$, then $\alpha\gamma \in I$ or $\beta\gamma \in I$.
(d) If $\alpha, \beta, \gamma \in Q_1$ with $\beta \neq \gamma$, then $\alpha\beta \in I$ or $\alpha\gamma \in I$.
(e) The ideal I is generated by oriented paths of Q.

Example 2.14 Let Q be as in Example 2.10. Then $A = \mathbb{C}Q/(\alpha^2, \beta^2, \alpha\beta\alpha)$ is a string algebra and the set $\{e_\circ, \alpha, \beta, \alpha\beta, \beta\alpha, \beta\alpha\beta\}$ of paths is a basis of the vector space A.

Let A be a string algebra. Then it is well known that every indecomposable A-module is a string or a band module.

3 Corollaries and Examples

In this section we give some examples, generate some corollaries and explain our results in more detail.

3.1 Tree and Band Modules

All the corollaries and examples of this section are strictly related to Theorem 1.2(a) and (b).

Corollary 3.1 *Let $F\colon S \to Q$ be a tree or a band and $\mathbf{d}$ a dimension vector of Q. Then we have to count successor closed subquivers of S with dimension vectors in $\mathbf{F}^{-1}(\mathbf{d})$ to compute $\chi_{\mathbf{d}}(F_*(\mathbb{1}_S))$.*

This corollary follows immediately from Theorem 1.2(a).

Corollary 3.2 *Let Q be a quiver, M a tree or band module and $\mathbf{d}$ a dimension vector of Q such that the variety $\mathrm{Gr}_{\mathbf{d}}(M)$ is non-empty. Then $\chi_{\mathbf{d}}(M) > 0$.*

Proof The inequality $\chi_{\mathbf{d}}(M) \geq 0$ is clear by Theorem 1.2. We prove the statement of Theorem 1.2(a) by applying Theorem 1.1 several times. So also the stronger inequality $\chi_{\mathbf{d}}(M) > 0$ follows. □

If the quiver S is an oriented cycle, each band module $B_*(V)$ has a unique filtration with $n = \dim_{\mathbb{C}}(V_i)$ pairwise isomorphic simple factors of dimension $|S_0|$. In this case Theorem 1.2(b) holds (see Example 3.3).

Therefore we assume without loss of generality that S is not an oriented cycle. Let $\{i_1, \ldots, i_r\}$ be the sources of S and $\{i'_1 \ldots, i'_r\}$ the sinks. We assume that $r > 0$ and $1 \le i_1 < i'_1 < i_2 < i'_2 \ldots < i_r < i'_r \le l$. Then the quiver S is visualized in Fig. 4.

Example 3.3 Let S, V and $\mathbf{t}$ be as in Theorem 1.2(b). Let $t_1 = t_2 = \ldots = t_l \le n$. Then $\chi_{\mathbf{t}}(V) = 1$.

The next example shows one result of [8, Proposition 3] as a special case of Theorem 1.2(b).

Example 3.4 Let S, V and $\mathbf{t}$ be as in Theorem 1.2(b). Let $r = 1$, $i_1 = 1$ and $i'_1 = l$. Then

$$\chi_{\mathbf{t}}(V) = \binom{t_l}{t_1}\binom{n-t_1}{n-t_l}\frac{(t_l - t_1)!}{\prod_{i=1}^{l-1}(t_{i+1}-t_i)!}.$$

Example 3.5 Let Q, B be as in Example 2.10 and $V \in \mathcal{I}_S^2$. Using Theorem 1.2, it is easy to calculate the Euler characteristics $\chi_d(B_*(V))$. For instance,

$$\begin{aligned}\chi_4(B_*(V)) &= \chi_{(0,0,2,2)}(V) + \chi_{(0,2,0,2)}(V) + \chi_{(0,1,1,2)}(V) + \chi_{(1,1,1,1)}(V) \\ &= 1 + 1 + 4 + 1 = 7\end{aligned}$$

since $\mathbf{F}^{-1}(4) = \{\mathbf{t} = (t_1, t_2, t_3, t_4) \in \mathbb{N}^{S_0} | t_1 + t_2 + t_3 + t_4 = 4\}$ and $\mathrm{Gr}_{\mathbf{t}}(V) = \varnothing$ if $s(a) > t(a)$ for some $a \in S_1$ or $t_i > 2$ for some $i \in S_0$.

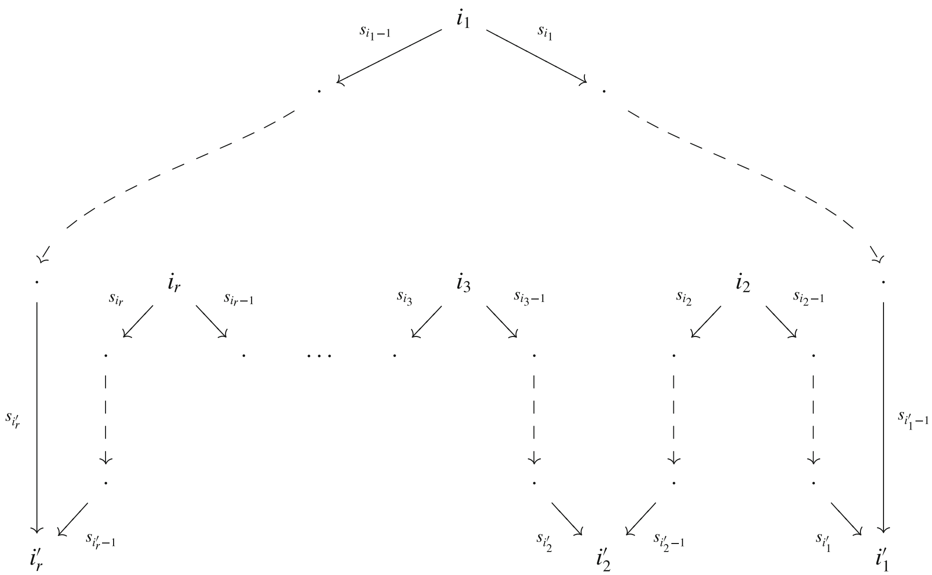
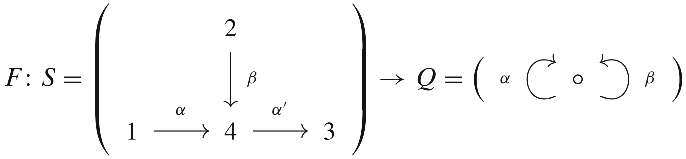


Fig. 4 A quiver of type $\tilde{A}_{l-1}$

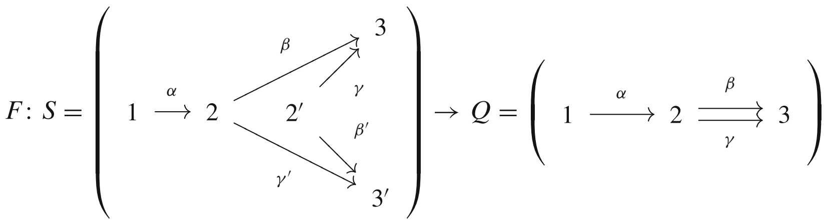
Example 3.6 If F is a tree or a band, Theorem 1.2(a) holds for any S-representation V. Let F be the tree described by the following picture.



Let V be an indecomposable S-representation with dimension vector $(1, 1, 1, 2)$. Then the dimension vector of a subrepresentation U of V with $\dim(U) = 3$ is in $\{(1, 0, 1, 1), (0, 1, 1, 1), (0, 0, 1, 2)\}$. Thus

$$\chi_3\left(F_*(V)\right) = \chi_{(1,0,1,1)}(V) + \chi_{(0,1,1,1)}(V) + \chi_{(0,0,1,2)}(V) = 3.$$

Example 3.7 If S is not a tree and not a band, Eq. 2 does not hold in general. To see this we consider the winding F described by the following picture.



Then $F_*(\mathbb{1}_S)$ is indecomposable and

$$\chi_{(0,1,1)}\left(F_*(\mathbb{1}_S)\right) = 2 \neq 0 = \sum\nolimits_{\mathbf{t}\in\mathbf{F}^{-1}((0,1,1))} \chi_{\mathbf{t}}(\mathbb{1}_S).$$

It is easy to see that there exists no quiver S and no winding F such that a formula similar to Eq. 2 holds. So it is not possible to describe these Euler characteristics purely combinatorially using our techniques.

3.2 Quiver Flag Varieties

Definition 3.8 Let Q be a quiver, M a Q-representation and $\mathbf{d}^{(1)}, \ldots, \mathbf{d}^{(r)}$ dimension vectors. Then the closed subvariety

$$\mathcal{F}_{\mathbf{d}^{(1)},\ldots,\mathbf{d}^{(r)}}(M) := \left\{0 \subseteq U^{(1)} \subseteq \ldots \subseteq U^{(r)} \subseteq M \middle| U^{(i)} \in \mathrm{Gr}_{\mathbf{d}^{(i)}}(M)\ \forall i\right\}$$

of the classical partial flag variety is called the *quiver flag variety*.

We denote the Euler characteristic of $\mathcal{F}_{\mathbf{d}^{(1)},\ldots,\mathbf{d}^{(r)}}(M)$ by $\chi_{\mathbf{d}^{(1)},\ldots,\mathbf{d}^{(r)}}(M)$. The following corollaries of Theorem 1.2(a) follow immediately from the analogous statements for the quiver Grassmannians.

Corollary 3.9 (Riedtmann) *Let Q be a quiver, $\mathbf{d}^{(1)}, \ldots, \mathbf{d}^{(r)}$ dimension vectors and M and N Q-representations. Then*

$$\chi_{\mathbf{d}^{(1)},\ldots,\mathbf{d}^{(r)}}(M \oplus N) = \sum\nolimits_{0\leq\mathbf{c}^{(i)}\leq\mathbf{d}^{(i)}} \chi_{\mathbf{c}^{(1)},\ldots,\mathbf{c}^{(r)}}(M)\,\chi_{\mathbf{d}^{(1)}-\mathbf{c}^{(1)},\ldots,\mathbf{d}^{(r)}-\mathbf{c}^{(r)}}(N).$$

Corollary 3.10 *Let Q and S be quivers, $\mathbf{d}^{(1)}, \ldots, \mathbf{d}^{(r)}$ dimension vectors of Q and V an S-representation. If $F\colon S \to Q$ is a tree or a band, then*

$$\chi_{\mathbf{d}^{(1)},\ldots,\mathbf{d}^{(r)}}(F_*(V)) = \sum\nolimits_{\mathbf{t}^{(i)} \in \mathbf{F}^{-1}(\mathbf{d}^{(i)})} \chi_{\mathbf{t}^{(1)},\ldots,\mathbf{t}^{(r)}}(V).$$

In particular we have to count flags of successor closed subquivers of S with dimension vectors in $\mathbf{F}^{-1}(\mathbf{d}^{(i)})$ to compute $\chi_{\mathbf{d}^{(1)},\ldots,\mathbf{d}^{(r)}}(F_*(\mathbb{1}_S))$.

Example 3.11 Let $Q = (1 \rightrightarrows 2)$, $n \in \mathbb{N}$ with $n \geq 3$ and M an indecomposable module with dimension $2n$. Then

$$\chi_{(1,2),(2,3)}(M) = 8(n-2).$$

A detailed proof of this equation is given in Section 7. For this calculation it is enough to count flags of successor closed subquivers of the quiver

associated to the dimension vectors (1, 2) and (2, 3).

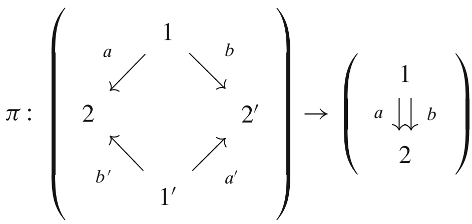
Corollary 3.12 *Let Q be a quiver, M a tree module and $\mathbf{d}^{(1)}, \ldots, \mathbf{d}^{(r)}$ dimension vectors of Q such that $\mathrm{Gr}_{\mathbf{d}^{(1)},\ldots,\mathbf{d}^{(r)}}(M)$ is non-empty. Then $\chi_{\mathbf{d}^{(1)},\ldots,\mathbf{d}^{(r)}}(M) > 0$.*

3.3 Coverings of Quivers

We give two examples of coverings. In one case Theorem 1.2(c) holds and in the other it fails. This shows that for this it is necessary that G is free and acts freely on $\hat{Q}$.

Example 3.13 Let $\hat{Q} = (\mathbb{Z}, \mathbb{Z})$ and $G = \mathbb{Z}$ with $s(n) = n$, $t(n) = n + 1$ and $gk = g + k$ for all $k \in \hat{Q}_0 \dot{\cup} \hat{Q}_1$ and $g \in G$. Let $\hat{I}$ be a ideal of $\mathbb{C}\hat{Q}$ generated by the passes of $\hat{Q}$ of length m and $I = \hat{I}/G$. Then $Q = \hat{Q}/G$ is the one loop quiver and $\mathbb{C}Q/I$ is isomorphic to $\mathbb{C}[T]/T^m$. Let $l \leq m$. For any indecomposable $\mathbb{C}Q/I$-module M of length l there is an indecomposable $\mathbb{C}\hat{Q}/\hat{I}$-module N with $\pi_*(N) \cong M$. Then for $0 \leq k \leq l$ we have $\chi_k(M) = \chi(\{U \subseteq N \mid \dim U = k\}) = 1$.

Example 3.14 Let $\pi : \hat{Q} \to Q$ be the winding described by the following picture:



Then $\mathbb{1}_{\hat{Q}}$ is indecomposable and has only one two-dimensional subrepresentation. But $\pi_*(\mathbb{1}_{\hat{Q}})$ is decomposable and has three two-dimensional subrepresentations. Thus

$$\chi_{(1,1)}\left(\pi_*\left(\mathbb{1}_{\hat{Q}}\right)\right) = 2 \neq 0 = \sum\nolimits_{\mathbf{t}\in\boldsymbol{\pi}^{-1}((1,1))} \chi_{\mathbf{t}}\left(\mathbb{1}_{\hat{Q}}\right).$$

3.4 Ringel-Hall Algebras

We are studying the products of functions of the form $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}$ in $\mathcal{H}(A)$. Using the following lemma and following example, it is enough to consider the images of indecomposable A-modules.

Lemma 3.15 *Let $A = \mathbb{C}Q/I$ be an algebra, $f, g \in \mathcal{C}(A)$ and M and N be A-modules. Then*

$$(f * g)(M \oplus N) = \sum\nolimits_{i,j}\left(f_i^{(1)} * g_j^{(1)}\right)(M)\left(f_i^{(2)} * g_j^{(2)}\right)(N),$$

where $\Delta(f) = \sum_i f_i^{(1)} \otimes f_i^{(2)}$ and $\Delta(g) = \sum_j g_j^{(1)} \otimes g_j^{(2)}$.

Proof By definition $\Delta(f)(M, N) = f(M \oplus N)$ for any $f \in \mathcal{C}(A)$. Since $\mathcal{C}(A)$ is a bialgebra the comultiplication Δ is an algebra homomorphism. □

Example 3.16 Let $\mathbf{F}$ be a tuple of trees, $\mathbf{B}$ a tuple of bands and $\mathbf{n}$ a tuple of positive integers. Then

$$\Delta\left(\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}\right) = \sum_{\mathbf{F}^{(1)}\dot{\cup}\mathbf{F}^{(2)}=\mathbf{F},\mathbf{B}^{(1)}\dot{\cup}\mathbf{B}^{(2)}=\mathbf{B},\mathbf{n}^{(1)}\dot{\cup}\mathbf{n}^{(2)}=\mathbf{n}} \mathbb{1}_{\mathbf{F}^{(1)},\mathbf{B}^{(1)},\mathbf{n}^{(1)}} \otimes \mathbb{1}_{\mathbf{F}^{(2)},\mathbf{B}^{(2)},\mathbf{n}^{(2)}}.$$

In this example we have been a little bit lazy: $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}$ is not necessarily in $\mathcal{C}(A)$, but we can extend the comultiplication in a natural way to all functions of the form $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}$.

Using Theorem 1.3 and Proposition 1.4 we get the following corollary. The proof of this corollary is given in Section 9.

Corollary 3.17 *Let $\mathbf{F}$ and $\mathbf{F}'$ be tuples of trees, $\mathbf{B}$ and $\mathbf{B}'$ tuples of bands and $\mathbf{n}$ and $\mathbf{n}'$ tuples of positive integers.*

(a) *Let Q be a finite quiver, $F\colon S \to Q$ a tree or a band and $V \in \operatorname{rep}(S)$. Then*

$$\left(\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}} * \mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}\right)(F_*(V)) = \sum \left(\mathbb{1}_{\widetilde{\mathbf{F}},\widetilde{\mathbf{B}},\mathbf{n}} * \mathbb{1}_{\widetilde{\mathbf{F}}',\widetilde{\mathbf{B}}',\mathbf{n}'}\right)(V), \tag{10}$$

where the sum is over $(\widetilde{\mathbf{F}}, \widetilde{\mathbf{F}}', \widetilde{\mathbf{B}}, \widetilde{\mathbf{B}}') \in \mathcal{G}_F(\mathbf{F}) \times \mathcal{G}_F(\mathbf{F}') \times \mathcal{G}_F(\mathbf{B}) \times \mathcal{G}_F(\mathbf{B}')$.

(b) *Let $\hat{Q}$ be a locally finite quiver, G a free (abelian) group, which acts freely on $\hat{Q}$, $Q = \hat{Q}/G$, $\pi : \hat{Q} \to Q$ the canonical projection and $V \in \operatorname{rep}(S)$. Then*

$$\left(\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}} * \mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}\right)(\pi_*(V)) = \sum \left(\mathbb{1}_{\widetilde{\mathbf{F}},\widetilde{\mathbf{B}},\mathbf{n}} * \mathbb{1}_{\widetilde{\mathbf{F}}',\widetilde{\mathbf{B}}',\mathbf{n}'}\right)(V), \tag{11}$$

where the sum is over $(\widetilde{\mathbf{F}}, \widetilde{\mathbf{F}}', \widetilde{\mathbf{B}}, \widetilde{\mathbf{B}}') \in \mathcal{G}_\pi(\mathbf{F}) \times \mathcal{G}_\pi(\mathbf{F}') \times \mathcal{G}_\pi(\mathbf{B}) \times \mathcal{G}_\pi(\mathbf{B}')$.

The functions $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}$, $\mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}$ and the corresponding products are in $\mathcal{H}(\mathbb{C}Q)$. The other functions $\mathbb{1}_{\widetilde{\mathbf{F}},\widetilde{\mathbf{B}},\mathbf{n}}$, $\mathbb{1}_{\widetilde{\mathbf{F}}',\widetilde{\mathbf{B}}',\mathbf{n}'}$ are in $\mathcal{H}(\mathbb{C}S)$ or $\hat{\mathcal{H}}(\mathbb{C}\hat{Q})$. So this corollary shows: To calculate $\left(\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}} * \mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}\right)(F_*(V))$ it is enough to consider some combinatorics and S-representations, where S is a tree or a quiver of type $\tilde{A}_{l-1}$.

Proposition 3.18 *Let A be a finite-dimensional algebra, $\mathbf{F}$ and $\mathbf{F}'$ tuples of trees, $\mathbf{B}$ and $\mathbf{B}'$ tuples of bands and $\mathbf{n}$ and $\mathbf{n}'$ tuples of positive integers.*

(a) *Let $F_*(\mathbb{1}_S)$ be a tree module of A such that*

$$\left(\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}} * \mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}\right)(F_*(\mathbb{1}_S)) \neq 0.$$

Then $l(\mathbf{B}) = l(\mathbf{B}') = 0$.

(b) *Let $B_*(V)$ be a band module of A such that*

$$\left(\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}} * \mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}\right)(B_*(V)) \neq 0.$$

Then $\mathbf{B}, \mathbf{B}' \in \{0, (B)\}$, $\mathbf{F}$ and $\mathbf{F}'$ are tuples of strings and $l(\mathbf{F}) = l(\mathbf{F}')$, where $l(\mathbf{F})$ denotes the length of the tuple $\mathbf{F}$.

Proof Corollary 3.17 shows: To compute $\left(\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}} * \mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}\right)(F_*(V))$ with a tree or band $F\colon S \to Q$ we have only to consider the products $\left(\mathbb{1}_{\tilde{\mathbf{F}},\tilde{\mathbf{B}},\mathbf{n}} * \mathbb{1}_{\tilde{\mathbf{F}}',\tilde{\mathbf{B}}',\mathbf{n}'}\right)(V)$, where S is a tree or a quiver of type $\tilde{A}_{l-1}$, and some combinatorics.

Thus for part (a) we assume without loss of generality that Q is a tree and F is the identity winding. So we have to compute $\left(\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}} * \mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}\right)(\mathbb{1}_Q)$ in $\mathcal{H}(\mathbb{C}Q)$. All sub- and factormodules of the tree module $\mathbb{1}_Q$ are again tree modules. If $\left(\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}} * \mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}\right)(\mathbb{1}_Q) \neq 0$, then $l(\mathbf{B}) = l(\mathbf{B}') = 0$.

For part (b) we assume without loss of generality that $A = \mathbb{C}Q$, where Q is a quiver of type $\tilde{A}_{l-1}$. All Q-modules are string or band modules $B'_*(V')$ such that $B'\colon Q \to Q$ is the identity winding. If $\left(\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}} * \mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}\right)(V) \neq 0$, then $l(\mathbf{B}), l(\mathbf{B}') \leq 1$ by Remark 2.9.

The equality $l(\mathbf{F}) = l(\mathbf{F}')$ is shown by induction. Let V be a band module and U a submodule, which is isomorphic to a string module. It is enough to show that for the representation $V/U = (W_i, W_a)_{i\in Q_0, a\in Q_1}$ the equality

$$\dim(V/U) - 1 = \sum\nolimits_{a\in Q_1} \mathrm{rk}(W_a)$$

holds, where $\mathrm{rk}(W_a)$ is the rank of the linear map W_a. This is clear since V is a band and U a string module with $\mathbf{dim}(U) \notin \mathbb{Z}(1, \ldots, 1)$. □

Remark 3.19 The calculation of the image of a tree module under a product $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}} * \mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}$ is now a purely combinatorial task: Using this proposition it is enough to consider $\left(\mathbb{1}_{(F^{(1)},\ldots,F^{(r)})} * \mathbb{1}_{(F'^{(1)},\ldots,F'^{(s)})}\right)(F_*(\mathbb{1}_S))$. By Corollary 3.17 it is even enough to count successor closed subquivers T of S with $F_*(\mathbb{1}_T) \cong \bigoplus_{i=1}^r F^{(i)}(\mathbb{1}_{S^{(i)}})$ and $F_*(\mathbb{1}_S/\mathbb{1}_T) \cong \bigoplus_{i=1}^s F'^{(i)}(\mathbb{1}_{S'^{(i)}})$.

Example 3.20 Let F be the string described by the following picture.

$$F\colon S=\left(\begin{array}{ccccccccc} & & 1 & & & & 1' & & \\ & \overset{\alpha}{\swarrow} & & \overset{\beta}{\searrow} & & \overset{\beta'}{\swarrow} & & \overset{\alpha'}{\searrow} & \\ 2 & & & & 3 \overset{\gamma}{\longrightarrow} 3' & & & & 2' \end{array}\right)$$

$$\to Q=\left(\begin{array}{ccccc} & & 1 & & \\ & \overset{\alpha}{\swarrow} & & \overset{\beta}{\searrow} & \\ 2 & & & & 3 \circlearrowright \gamma \end{array}\right)$$

Let $\mathbf{F}=(2\to Q,(3\overset{\gamma}{\to}3')\to Q)$ and $\mathbf{F}'=(1\to Q,(1\overset{\alpha}{\to}2)\to Q)$. We compute $(\mathbb{1}_{\mathbf{F}}*\mathbb{1}_{\mathbf{F}'})\,(F_*(\mathbb{1}_S))$ with Corollary 3.17. Then

$$\mathcal{G}_F(\mathbf{F})=\left\{\left(2\to S,\left(3\overset{\gamma}{\to}3'\right)\to S\right),\left(2'\to S,\left(3\overset{\gamma}{\to}3'\right)\to S\right)\right\},$$

$$\mathcal{G}_F(\mathbf{F}')=\left\{\left(1\to S,\left(1'\overset{\alpha'}{\to}2'\right)\to S\right),\left(1'\to S,\left(1\overset{\alpha}{\to}2\right)\to S\right)\right\}$$

and thus $(\mathbb{1}_{\mathbf{F}}*\mathbb{1}_{\mathbf{F}'})\,(F_*(\mathbb{1}_S))=2$ by counting these subquivers.

Proposition 3.21 *Let Q be a quiver of type $\tilde{A}_{l-1}$, $\mathbf{F}$ and $\mathbf{F}'$ be tuples of strings, $B\colon Q\to Q$ the identity winding, $m\in\mathbb{N}$ and $\lambda\in\mathbb{C}^*$.*

(a) *Let $n,n'\in\mathbb{N}$ with $n+n'\leq m$. Then*

$$\left(\mathbb{1}_{\mathbf{F},B,n}*\mathbb{1}_{\mathbf{F}',B,n'}\right)(B_*(\lambda,m))=(\mathbb{1}_{\mathbf{F}}*\mathbb{1}_{\mathbf{F}'})\,(B_*(\lambda,m-n-n')). \tag{12}$$

(b) *Let $n\in\mathbb{N}$, F a string and $\mathbf{F}(n)=(F,\ldots,F)$ with $l\,(\mathbf{F}(n))=n$ such that $F_*(\mathbb{1}_S)$ and $F_*^{(i)}(\mathbb{1}_{S^{(i)}})$ are preprojective, $\dim(F_*(\mathbb{1}_S))\geq\dim(F_*^{(i)}(\mathbb{1}_{S^{(i)}}))$ and $F_*(\mathbb{1}_S)\ncong F_*^{(i)}(\mathbb{1}_{S^{(i)}})$ for all i. Then $(\mathbb{1}_{\mathbf{F}(n)\dot{\cup}\mathbf{F}}*\mathbb{1}_{\mathbf{F}'})(B_*(\lambda,m))=$*

$$\sum\nolimits_{k_1,\ldots,k_n\in\mathbb{N}}(\mathbb{1}_{\mathbf{F}}*\mathbb{1}_{\mathbf{F}'})\left(B_*\left(\lambda,m-\sum\nolimits_{i=1}^{n}k_i\right)\oplus\bigoplus\nolimits_{i=1}^{n}I_{k_i}\right) \tag{13}$$

with I_{k_i} an indecomposable representation with $\mathbf{dim}(I_{k_i})=(k_i,\ldots,k_i)-\mathbf{dim}(F_*(\mathbb{1}_S))$ for all i.

A proof of this is given in Section 9. If $\mathbf{dim}(B_*(\lambda,k_i))>\mathbf{dim}(F_*(\mathbb{1}_S))$, the module I_k exists, is preinjective and determined up to isomorphism uniquely by Remark 2.12.

Let Q be a quiver of type $\tilde{A}_{l-1}$, $\mathbf{F}''$ and $\mathbf{F}'$ be tuples of strings and $V\in\mathcal{I}_Q^n$ such that $(\mathbb{1}_{\mathbf{F}''}*\mathbb{1}_{\mathbf{F}'})(V)\neq 0$. Without loss of generality we assume that $\dim(F_*''^{(1)}(\mathbb{1}_S))\geq\dim(F_*''^{(i)}(\mathbb{1}_S))$ for all i. Then $F_*''^{(i)}(\mathbb{1}_S)$ is preprojective for all i and we apply Theorem 3.21(b) with $F=F''^{(1)}$ and $\mathbf{F}=\{F''^{(i)}\,|\,F''^{(i)}(\mathbb{1}_S)\ncong F(\mathbb{1}_S)\}$. Thus we get the following corollary.

Corollary 3.22 *Let $A=\mathbb{C}Q/I$ be an algebra, M a direct sum of tree and band modules of Q such that M is an A-module. Let $\mathbf{F}$ and $\mathbf{F}'$ be tuples of trees, $\mathbf{B}$ and $\mathbf{B}'$ tuples of bands and $\mathbf{n}$ and $\mathbf{n}'$ tuples of positive integers. Then $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}*\mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}(M)$ can be computed combinatorially.*

3.5 String Algebras

In this section we consider the Ringel-Hall algebras of string algebras. The proofs of these corollaries and of Eq. 14 are stated in Section 10.

Corollary 3.23 *Let A be a string algebra. Let $\mathbf{F}$ be a tuple of strings, $\mathbf{B}$ a tuple of bands and $\mathbf{n}$ a tuple of positive integers. Then*

$$\mathbb{1}_{\mathbf{F}} * \mathbb{1}_{\mathbf{B},\mathbf{n}} = \mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}} = \mathbb{1}_{\mathbf{B},\mathbf{n}} * \mathbb{1}_{\mathbf{F}}.$$

Example 3.24 Let $Q = (1 \rightrightarrows 2)$, $\mathbf{F}$ and $\mathbf{F}'$ tuples of strings, $B\colon Q \to Q$ the identity winding and $m \in \mathbb{N}$ such that $(\mathbb{1}_{\mathbf{F}} * \mathbb{1}_{\mathbf{F}'})\,(B_*(\lambda, m)) \neq 0$. Then

$$(\mathbb{1}_{\mathbf{F}} * \mathbb{1}_{\mathbf{F}'})\,(B_*(\lambda, m)) = \frac{l(\mathbf{F})!}{\prod_{\{F^{(i)}|i\}/\cong} |[F^{(i)}]|!} \frac{l(\mathbf{F}')!}{\prod_{\{F'^{(i)}|i\}/\cong} |[F'^{(i)}]|!}, \tag{14}$$

where $\{F^{(i)}|i\}/\cong$ is the set of isomorphism classes and $|[F^{(i)}]|$ is the number of elements in the isomorphism class of $F^{(i)}$. For instance,

$$\left(\mathbb{1}_{S_2^{m-r} \oplus P(S_1)^r} * \mathbb{1}_{S_1^{m-s} \oplus I(S_2)^s}\right)(B_*(\lambda, m+r+s)) = \binom{m}{r}\binom{m}{s}$$

with $m, r, s \in \mathbb{N}$, $S_i \in \operatorname{mod}(A)$ is the simple representation associated to the vertex $i \in Q_0$, $P(S_i) \in \operatorname{mod}(A)$ is the projective cover of S_i and $I(S_i) \in \operatorname{mod}(A)$ is the injective hull of S_i.

In general it is much harder to give an explicit formula for $(\mathbb{1}_{\mathbf{F}} * \mathbb{1}_{\mathbf{F}'})\,(B_*(\lambda, m))$.

Corollary 3.25 *Let A be a string algebra. Then every function in $\mathcal{C}(A)$ is a linear combination of functions of the form $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}$ with some tuple $\mathbf{F}$ of strings, some tuple $\mathbf{B}$ of bands and some tuple $\mathbf{n}$ of positive integers.*

4 Gradings

4.1 Definitions

Let Q be a quiver and $M = (M_i, M_a)_{i \in Q_0, a \in Q_1}$ a Q-representation. Let $I = \{1, 2, \ldots, \dim(M)\}$ and $E = \{e_j | j \in I\}$ be a basis of $\bigoplus_{i \in Q_0} M_i$ such that $E \subseteq \bigcup_{i \in Q_0} M_i$.

Definition 4.1 A map $\partial\colon E \to \mathbb{Z}$ is called a *grading* of M.

So every grading depends on the choice of a basis E. It is useful to change the basis during calculations. A vector $m = \sum_{j \in I} m_j e_j \in M$ with $m_j \in \mathbb{C}$ is called ∂-*homogeneous* of degree $n \in \mathbb{Z}$ if $\partial(e_j) = n$ for all $j \in I$ with $m_j \neq 0$. If $m \in M$ is ∂-homogeneous of degree $n \in \mathbb{Z}$, we set $\partial(m) = n$.

The following grading has been studied by Riedtmann [19]: Let $M = \bigoplus_{k=1}^r N_k$, where N_k is a subrepresentation of M for all k and $E \subseteq \bigcup_{k=1}^r N_k$. Then the grading $\partial\colon E \to \mathbb{Z}$ with $\partial(e_j) = k$ if $e_j \in N_k$ is called *Riedtmann grading* (or *R-grading* for short).

Definition 4.2 Let ∂ and $\partial_1, \ldots, \partial_r$ be gradings of M and $\Delta(\mathbf{y}, \mathbf{z}, a) \in \mathbb{Z}$ for all $\mathbf{y}, \mathbf{z} \in \mathbb{Z}^r$ and $a \in Q_1$ such that

$$\Delta\big((\partial_m(e_j))_{1\leq m\leq r}, (\partial_m(e_i))_{1\leq m\leq r}, a\big) = \partial(e_i) - \partial(e_j) \tag{15}$$

for all $i, j \in I$ and $a \in Q_1$ with $e_i \in M_{t(a)}$, $e_j \in M_{s(a)}$ and $m_i \neq 0$ for $M_a(e_j) = \sum_{k\in I} m_k e_k$. Then ∂ is called a *nice* $\partial_1, \ldots, \partial_r$-grading.

Example 4.3 Let $Q = \left(1 \overset{a}{\underset{b}{\rightrightarrows}} 2 \right)$, $M_1 = M_2 = \mathbb{C}^2$, $M_a = \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$ and $M_b = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. Then $M = (M_1, M_2, M_a, M_b)$ is a Q-representation. Let $\{e_i = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), f_i = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\}$ be the canonical basis of M_i for each $i \in Q_0$ and $E = \{e_1, f_1, e_2, f_2\}$. Let $\partial, \partial_1 \colon E \to \mathbb{Z}$ be gradings with $\partial(e_1) = \partial(f_1) = 0$, $\partial(e_2) = 3$, $\partial(f_2) = 5$, $\partial_1(e_i) = 0$ and $\partial_1(f_i) = 1$ for each $i \in Q_0$. Then ∂ is a nice ∂_1-grading, since $\Delta(0, 0, b) = \partial(e_2) - \partial(e_1) = 3$ and $\Delta(1, 1, b) = \partial(f_2) - \partial(f_1) = 5$. But ∂ is a not a nice $\varnothing$-grading, since for $\Delta(\varnothing, \varnothing, b)$ we have: $\partial(e_2) - \partial(e_1) = 3 \neq 5 = \partial(f_2) - \partial(f_1)$.

Example 4.4 In this example we state two extreme cases of gradings.

- Let ∂ and ∂' be gradings such that $\partial' \colon E \to \mathbb{Z}$ is an injective map. Then ∂ is a nice ∂'-grading.
- Let ∂ be a grading such that $\partial(e_i) = \partial(e_j)$ for all $i, j \in I$. Then ∂ is a nice grading.

The definition of $\partial_1, \ldots, \partial_r$-nice gradings generalizes the gradings introduced by Cerulli Irelli [8]. He only considers the nice $\varnothing$-gradings, i.e. $r = 0$. (We say *nice grading* for short.) Now we successively apply these gradings.

By the following remark, we describe a way to visualize a nice $\partial_1, \ldots, \partial_r$-grading ∂ of some representations of the form $F_*(\mathbb{1}_S)$.

Remark 4.5 Let Q and S be quivers and $F \colon S \to Q$ a winding. Let $M = F_*(\mathbb{1}_S)$ and $\{f_i \in (\mathbb{1}_S)_i | i \in S_0\}$ be a basis of $\bigoplus_{i\in S_0} (\mathbb{1}_S)_i$. Then $E := \{F_*(f_i) | i \in S_0\}$ is a basis of $\bigoplus_{i\in Q_0} M_i$.

- Now we illustrate each grading $\partial \colon E \to \mathbb{Z}$ of M by a labelling of the quiver S. For this we extend ∂ to $E \cup S_0$ by $\partial(i) = \partial(F_*(f_i))$ for each $i \in S_0$.
- For each nice $\partial_1, \ldots, \partial_r$-grading ∂ we further extend ∂ in a meaningful way to $E \cup S_0 \cup S_1$ by

$$\partial(a) = \Delta\Big(\big(\partial_m(s(a))\big)_{1\leq m\leq r}, \big(\partial_m(t(a))\big)_{1\leq m\leq r}, F_1(a)\Big)$$

  for all $a \in S_1$. Then by Eq. 15

$$\partial(a) = \partial(t(a)) - \partial(s(a)) \tag{16}$$

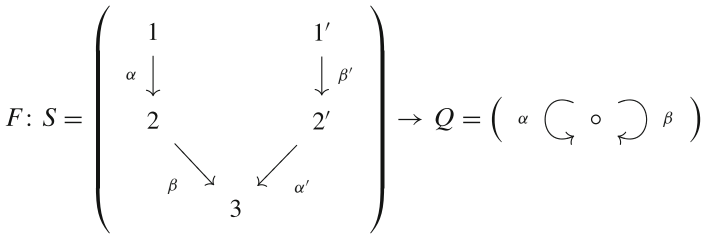
  for all $a \in S_1$.
- Let $\partial \colon S_0 \cup S_1 \to \mathbb{Z}$ be a map with the following conditions:

  (S1) The Eq. 16 holds for all $a \in S_1$.
  (S2) $\partial(a) = \partial(b)$ for all $a, b \in S_1$ with $F_1(a) = F_1(b)$, $\partial_m(s(a)) = \partial_m(s(b))$ and $\partial_m(t(a)) = \partial_m(t(b))$ for all m.

  Then the map ∂ induces a nice $\partial_1, \ldots, \partial_r$-grading $\partial \colon E \to \mathbb{Z}$ on M.

- Let S be a tree and $\partial\colon S_1 \to \mathbb{Z}$ be a map such that the condition (S2) holds. Then the map ∂ induces a nice $\partial_1, \ldots, \partial_r$-grading $\partial\colon E \to \mathbb{Z}$ on M.
- Let $\partial\colon S_1 \to \mathbb{Z}$. If S is connected, such an induced grading ∂ is unique up to shift.

Example 4.6 Let $F_*(\mathbb{1}_S)$ be the tree module described by the following picture.



Then $F_*(\mathbb{1}_S)$ has a basis $E = \{F_*(f_1), F_*(f_{1'}), F_*(f_2), F_*(f_{2'}), F_*(f_3)\}$ as above.

Let $\partial_1\colon S_1 \to \mathbb{Z}, \gamma \mapsto 1$ for all $\gamma \in S_1$ and $\partial_1(F_*(f_1)) = 0$. This induces by the previous remark a unique nice grading ∂_1 of $F_*(\mathbb{1}_S)$. In particular $\partial_1(1) = \partial_1(1') = 0$, $\partial_1(2) = \partial_1(2') = 1$ and $\partial_1(3) = 2$. Let $\partial_2\colon S_1 \to \mathbb{Z}, \beta \mapsto 1, \gamma \mapsto 0$ for all $\beta \neq \gamma \in S_1$ and $\partial_2(F_*(f_1)) = 0$. This induces a unique nice ∂_1-grading ∂_2 of $F_*(\mathbb{1}_S)$. So in particular $\partial_1(1) = \partial_1(2) = 0$, $\partial_1(1') = \partial_1(2') = \partial_1(3) = 1$.

4.2 Stable Gradings

Let Q be a quiver, M a Q-representation and ∂ a grading. The algebraic group $\mathbb{C}^*$ acts by

$$\varphi_\partial\colon \mathbb{C}^* \to \operatorname{End}_{\mathbb{C}}(M),\ \varphi_\partial(\lambda)(e_j) := \lambda^{\partial(e_j)} e_j \tag{17}$$

on the vector space M. This defines in some cases a $\mathbb{C}^*$-action on the quiver Grassmannian $\operatorname{Gr}_{\mathbf{d}}(M)$.

Definition 4.7 Let X be a locally closed subset of $\operatorname{Gr}_{\mathbf{d}}(M)$ and ∂ a grading of M. If for all $U \in X$ and $\lambda \in \mathbb{C}^*$ the vector space $\varphi_\partial(\lambda)U$ is in X, then the grading ∂ is called *stable on* X.

Let X be a locally closed subset of $\operatorname{Gr}_{\mathbf{d}}(M)$ and $\partial_1, \ldots, \partial_r$ gradings. Let

$$X^{\partial_1,\ldots,\partial_r} := \big\{U \in X \big| U \text{ has a basis, which is } \partial_i\text{-homogeneous for each } i\big\}. \tag{18}$$

This equation is a generalization of Eq. 1. By definition, each stable grading on X is also a stable grading on $X^{\partial_1,\ldots,\partial_r}$.

Lemma 4.8 *Let Q be a quiver, M a Q-representation and $\mathbf{d}$ a dimension vector. Let $U \in \operatorname{Gr}_{\mathbf{d}}(M)$ and $\partial_1, \ldots, \partial_r$ gradings. Then $U \in \operatorname{Gr}_{\mathbf{d}}(M)^{\partial_1,\ldots,\partial_r}$ if and only if $\varphi_{\partial_i}(\lambda)U = U$ as vector spaces for all i and $\lambda \in \mathbb{C}^*$.*

Proof If $U \in \mathrm{Gr}_{\mathbf{d}}(M)$ has a basis, which is ∂_i-homogeneous for each i, we get $\varphi_{\partial_i}(\lambda)U = U$ for each i and $\lambda \in \mathbb{C}^*$.

Let $U \in \mathrm{Gr}_{\mathbf{d}}(M)$ such that $\varphi_{\partial_i}(\lambda)U = U$ for all i and $\lambda \in \mathbb{C}^*$. Our aim is to find a basis for U, which is ∂_i-homogeneous for each i. Let $s \in \mathbb{N}$ with $1 \leq s \leq r$ and $\{m_1, \ldots, m_t\}$ be a basis of U, which is ∂_i-homogeneous for each i with $1 \leq i < s$. For each j with $1 \leq j \leq t$ let $m_j = \sum_{i \in I} \lambda_{ij} e_i$ with $\lambda_{ij} \in \mathbb{C}$. For each $z \in \mathbb{Z}$ and $j \in \mathbb{N}$ with $1 \leq j \leq t$ define $m_{j,z} := \sum_{i \in I, \partial(e_i) = z} \lambda_{ij} e_i \in M$. Then $m_{j,z}$ is ∂_i-homogeneous for each i with $1 \leq i \leq s$, $\varphi_{\partial_s}(\lambda)(m_{j,z}) = \lambda^z m_{j,z}$ for all $\lambda \in \mathbb{C}^*$ and $m_j = \sum_{z \in \mathbb{Z}} m_{j,z}$. Then $\varphi_{\partial_s}(\lambda)(m_j) = \sum_{z \in \mathbb{Z}} \lambda^z m_{j,z} \in U$ for all $\lambda \in \mathbb{C}^*$ and so $m_{j,z} \in U$ for all $z \in \mathbb{Z}$ and all j. Since $\{m_{j,z} | 1 \leq j \leq t, z \in \mathbb{Z}\}$ generates U, a subset of this set is a basis of the vector space U, which is ∂_i-homogeneous for each i with $1 \leq i \leq s$. The statement follows by an induction argument. □

We will show that all R-gradings and all nice gradings are stable on $\mathrm{Gr}_{\mathbf{d}}(M)$.

Lemma 4.9 *Let Q be a quiver, M a Q-representation and $\mathbf{d}$ a dimension vector. Then each R-grading ∂ is stable on $\mathrm{Gr}_{\mathbf{d}}(M)$.*

For this lemma it is enough to show that M_a and $\varphi_\partial(\lambda)$ commute for all $a \in Q_1$ and $\lambda \in \mathbb{C}^*$.

Lemma 4.10 *Let $\partial_1, \ldots, \partial_r$ and ∂ be gradings of M. Then ∂ is stable on the variety $\mathrm{Gr}_{\mathbf{d}}(M)^{\partial_1,\ldots,\partial_r}$ for all $\mathbf{d} \in \mathbb{N}^{Q_0}$ if and only if for all $\lambda \in \mathbb{C}^*$, $a \in Q_1$ and $\partial_1, \ldots, \partial_r$-homogeneous elements $u \in M$ we have*

$$M_a\left(\varphi_\partial(\lambda)u\right) \in \varphi_\partial(\lambda)U_{\partial_1,\ldots,\partial_r}(u), \tag{19}$$

where $U_{\partial_1,\ldots,\partial_r}(u)$ is the minimal subrepresentation of M such that $u \in U_{\partial_1,\ldots,\partial_r}(u)$ and $U_{\partial_1,\ldots,\partial_r}(u) \in \mathrm{Gr}_{\mathbf{d}}(M)^{\partial_1,\ldots,\partial_r}$ for some $\mathbf{d} \in \mathbb{N}^{Q_0}$.

If $U \in \mathrm{Gr}_{\mathbf{d}}(M)^{\partial_1,\ldots,\partial_r}$ and $V \in \mathrm{Gr}_{\mathbf{c}}(M)^{\partial_1,\ldots,\partial_r}$, then Lemma 4.8 implies $U \cap V \in \mathrm{Gr}_{\mathbf{dim}(U \cap V)}(M)^{\partial_1,\ldots,\partial_r}$. So the submodule $U_{\partial_1,\ldots,\partial_r}(u)$ is well-defined and unique.

Proof If ∂ is stable on $\mathrm{Gr}_{\mathbf{d}}(M)^{\partial_1,\ldots,\partial_r}$ for all $\mathbf{d} \in \mathbb{N}^{Q_0}$, then $\varphi_\partial(\lambda)U_{\partial_1,\ldots,\partial_r}(u)$ is a subrepresentation of M for all $\lambda \in \mathbb{C}^*$ and $u \in M$.

Let $U \in \mathrm{Gr}_{\mathbf{d}}(M)^{\partial_1,\ldots,\partial_r}$. If Eq. 19 holds for all $\lambda \in \mathbb{C}^*$, $a \in Q_1$ and $\partial_1, \ldots, \partial_r$-homogeneous $u \in M$, then $M_a\left(\varphi_\partial(\lambda)U_{s(a)}\right) \subseteq \varphi_\partial(\lambda)U_{t(a)}$ for all $\lambda \in \mathbb{C}^*$ and $a \in Q_1$, since U is generated by $\partial_1, \ldots, \partial_r$-homogeneous elements. Thus $\varphi_\partial(\lambda)U \in \mathrm{Gr}_{\mathbf{d}}(M)$. □

Lemma 4.11 *Let Q be a quiver, M a Q-representation and $\mathbf{d}$ a dimension vector. Then every nice $\partial_1, \ldots, \partial_r$-grading ∂ is stable on $\mathrm{Gr}_{\mathbf{d}}(M)^{\partial_1,\ldots,\partial_r}$.*

Proof By Lemma 4.10, it is enough to consider $\lambda \in \mathbb{C}^*$, $a \in Q_1$ and a homogeneous $u \in M$. We write $u = \sum_{k \in I} u_k e_k$ with $u_k \in \mathbb{C}$, $M_a(e_k) = \sum_{j \in I} m_{jk} e_j$ with $m_{jk} \in \mathbb{C}$

for all $k \in I$ and $M_a(u) = \sum_{\mathbf{z}\in\mathbb{Z}^r} m_{\mathbf{z}}$ with $(\partial_m(m_{\mathbf{z}}))_m = \mathbf{z}$. So $m_{\mathbf{z}} = \sum_{k,j\in I,(\partial_m(e_j))_m=\mathbf{z}} u_k m_{jk} e_j$ and

$$\begin{aligned} M_a\left(\varphi_\partial(\lambda)u\right) &= \sum\nolimits_{k\in I} u_k M_a\left(\lambda^{\partial(e_k)} e_k\right) = \sum\nolimits_{k,j\in I} u_k \lambda^{\partial(e_k)} m_{jk} e_j \\ &= \sum\nolimits_{k,j\in I} \lambda^{\partial(e_k)-\partial(e_j)} u_k m_{jk} \varphi_\partial(\lambda) e_j \\ &= \varphi_\partial(\lambda)\left(\sum\nolimits_{k,j\in I} \lambda^{\Delta((\partial_m(u))_m,(\partial_m(e_j))_m,a)} u_k m_{jk} e_j\right) \\ &= \varphi_\partial(\lambda)\left(\sum\nolimits_{\mathbf{z}\in\mathbb{Z}^r} \lambda^{\Delta((\partial_m(u))_m,\mathbf{z},a)} m_{\mathbf{z}}\right) \in \varphi_\partial(\lambda) U_{\partial_1,\ldots,\partial_r}(u). \end{aligned}$$

□

4.3 Ringel-Hall Algebras

In the theory of Ringel-Hall algebras one has to compute the Euler characteristic of the following locally closed subsets of the projective variety $\mathrm{Gr}_{\mathbf{d}}(M)$. Let U and M be Q-representations and X a locally closed subset of $\mathrm{Gr}_{\mathbf{d}}(M)$. Let

$$X^U := \left\{V \in X \middle| V \cong U,\, M/V \cong M/U\right\}.$$

Lemma 4.12 *Let Q be a quiver, U and M be Q-representations. Then every R-grading ∂ is stable on* $\mathrm{Gr}_{\mathbf{d}}(M)^U$.

Proof The linear map $\varphi_\partial(\lambda)\colon M \to M$ is an automorphism of Q-representations for all $\lambda \in \mathbb{C}^*$. □

Lemma 4.13 *Let Q be a quiver, M a Q-representation and $F_*(\mathbb{1}_S) \subseteq M$ with $F\colon S \to Q$ a tree such that $M/F_*(\mathbb{1}_S)$ is a tree module, too. Let ∂ be a nice grading on* $\mathrm{Gr}_{\mathbf{d}}(M)$. *Then ∂ is also stable on* $\mathrm{Gr}_{\mathbf{d}}(M)^{F_*(\mathbb{1}_S)}$.

Proof Let $a \in Q_1$, $\lambda \in \mathbb{C}^*$ and $U \in \mathrm{Gr}_{\mathbf{d}}(M)^{F_*(\mathbb{1}_S)}$. Since ∂ is a nice grading we know $M_a\varphi_\partial(\lambda) = \lambda^{\partial(a)}\varphi_\partial(\lambda)M_a$ by the proof of Lemma 4.11. Let $i \in S_0$ and ρ_j the unique not necessarily oriented path in S from i to some $j \in S_0$. Then we associate an integer $\partial(\rho_j)$ to each path ρ_j such that $f_j \mapsto \lambda^{\partial(\rho_j)} f_j$ induces an isomorphism $U \to \varphi_\partial(\lambda)(U)$ of quiver representations. The same holds for the quotient. □

Lemma 4.14 *Let Q and S be quivers, $B\colon S \to Q$ a winding, M a Q-representation and ∂ a nice grading on* $\mathrm{Gr}_{\mathbf{d}}(M)$. *Let*

$$X = \left\{U \in \mathrm{Gr}_{\mathbf{d}}(M) \middle| \exists B_*(V) \text{ band module}, U \cong B_*(V)\right\},$$

a locally closed subset of $\mathrm{Gr}_{\mathbf{d}}(M)$. *Then ∂ is also stable on X.*

Proof We use the proof of Lemma 4.13. In this case the representations U and $\varphi_\partial(\lambda)(U)$ are in general non-isomorphic. But they are both band modules for the same quiver S and the same winding $B\colon S \to Q$. □

The next example shows that this lemma is not true if we restrict the action to one orbit of a band module.

Example 4.15 Let $F_*(\mathbb{1}_S)$ be the tree module described by the following picture.

$$F:\left(\begin{array}{ccc} 1 & & 1' \\ & {}_{\alpha}\searrow \quad \swarrow_{\beta} & \\ & 2 & \end{array}\right) \to \left(\begin{array}{c} 1 \\ \alpha \downdownarrows \beta \\ 2 \end{array}\right)$$

Let U be the subrepresentation of $F_*(\mathbb{1}_S)$ generated by $F_*(f_1 + f_{1'})$. Let $\lambda \in \mathbb{C}^*$ with $\lambda \neq 1$ and ∂ a nice grading of $F_*(\mathbb{1}_S)$ with $\partial(\alpha) = 1$ and $\partial(\beta) = 0$ (see Remark 4.5). Then $\varphi_\partial(\lambda)U$ is generated by $F_*(f_1 + \lambda f_{1'})$, and U and $\varphi_\partial(\lambda)U$ are non-isomorphic band modules.

5 Quiver Grassmannians

5.1 Proof of Theorem 1.1

The following proposition is well known.

Proposition 5.1 (Bialynicki-Birula [2]) *Let $\mathbb{C}^*$ act on a locally closed subset X of a projective variety Y. Then the subset of fixed points $X^{\mathbb{C}^*}$ under this action is a locally closed subset of Y and $\chi(X) = \chi\left(X^{\mathbb{C}^*}\right)$. If the subset X is non-empty and closed in Y, then $X^{\mathbb{C}^*}$ is also non-empty and closed in Y.*

Proof The subset of fixed points $X^{\mathbb{C}^*}$ is closed in X. By [3], this is non-empty if X is non-empty and closed in Y.

So we decompose X into the locally closed subset of fixed points $X^{\mathbb{C}^*}$ and its complement $U = X - X^{\mathbb{C}^*}$ in X. So $\chi(X) = \chi\left(X^{\mathbb{C}^*}\right) + \chi(U)$. Since U is the union of the non trivial orbits in X, the projection $U \to U/\mathbb{C}^*$ is an algebraic morphism. Since $\chi(\mathbb{C}^*) = 0$ the Euler characteristic of U is also zero. □

The action φ_∂ of the algebraic group $\mathbb{C}^*$ on the projective variety X is well-defined. Thus Proposition 5.1 yields the equality of the Euler characteristic of X and the Euler characteristic of the set of fixed points under this action. By Lemma 4.8, a subrepresentation U of M in X is a fixed point of φ_∂ if and only if U has a basis of ∂-homogeneous elements. This proves Theorem 1.1. □

Corollary 5.2 *Let Q be a quiver, M a Q-representation and $\partial_1, \ldots, \partial_r$ gradings of M such that for all $1 \leq i \leq r$ the grading ∂_i is a stable grading on $\mathrm{Gr}_{\mathbf{d}}(M)^{\partial_1,\ldots,\partial_{i-1}}$. Then $\chi_{\mathbf{d}}(M) = \chi\left(\mathrm{Gr}_{\mathbf{d}}(M)^{\partial_1,\ldots,\partial_r}\right)$.*

This corollary follows directly from Theorem 1.1, since different $\mathbb{C}^*$-actions commute.

5.2 Proof of Proposition 2.3

We choose any basis of M and any basis of N. So the union induces a basis of $M \oplus N$. Using a R-grading ∂, we have to compute the Euler characteristic of the set of fixed points. This variety $\mathrm{Gr}_{\mathbf{d}}(M \oplus N)^\partial$ can be decomposed into a union of locally closed sets $X_{\mathbf{c}}$, where the subrepresentation of M has the dimension vector $\mathbf{c}$ and the subrepresentation of N has the dimension vector $\mathbf{d}-\mathbf{c}$. Then $\chi_{\mathbf{d}}(M)=\sum_{\mathbf{c}}\chi(X_{\mathbf{c}})=\sum_{\mathbf{c}}\chi_{\mathbf{c}}(M)\chi_{\mathbf{d}-\mathbf{c}}(N)$. □

6 Tree and Band Modules

6.1 Proof of Theorem 1.2(a)

If F is a tree, Theorem 1.2(c) yields this theorem by the following property: If $F\colon S\to Q$ is a tree and $\pi:\hat{Q}\to Q$ a universal covering, then a factorization $F=\pi\iota$ exists such that $\iota\colon S\to\hat{Q}$ is injective. But we give in this section an independent proof.

For this it is enough to consider $V=\mathbb{1}_S$. By Remark 4.5 the set $E=\{F_*(f_i)|i\in S_0\}$ is a basis of $F_*(\mathbb{1}_S)$. We write $\partial(i)$ instead of $\partial(F_*(f_i))$ for all $i\in S_0$. To prove Theorem 1.2(a) we will use the following proposition inductively. This proposition holds in general and not only for trees and bands. But in the case of trees and bands there exist enough nice gradings such that Theorem 1.2(a) follows (see Lemmas 6.3 and 6.4).

Proposition 6.1 *Let Q and S be locally finite quivers, T a finite subquiver of S, $F\colon S\to Q$ a winding of quivers and $\mathbf{d}$ a dimension vector of Q. Let ∂ be a nice grading of $F_*(\mathbb{1}_T)$. Define a quiver Q' by*

$$\begin{aligned}Q'_0 &=\{(F_0(i),\partial(i))|i\in S_0\}\\ Q'_1 &=\{(\partial(s(a)),\partial(t(a)),F_1(a))|a\in S_1\}\\ s'(\partial(s(a)),\partial(t(a)),F_1(a)) &=(s(F_1(a)),\partial(s(a)))\\ t'(\partial(s(a)),\partial(t(a)),F_1(a)) &=(t(F_1(a)),\partial(t(a)))\ \textit{for all}\ a\in Q_1.\end{aligned}$$

Define windings $F'\colon S\to Q'$ by $i\mapsto(F_0(i),\partial(i))$, $a\mapsto(\partial(s(a)),\partial(t(a)),F_1(a))$ and $G\colon Q'\to Q$ by $(F_0(i),\partial(i))\mapsto F_0(i)$, $(\partial(s(a)),\partial(t(a)),F_1(a))\mapsto F_1(a)$. Then

$$\chi_{\mathbf{d}}(F_*(\mathbb{1}_T))=\sum\nolimits_{\mathbf{t}\in\mathbf{G}^{-1}(\mathbf{d})}\chi_{\mathbf{t}}({F'}_*(\mathbb{1}_T)). \tag{20}$$

Proof By definition of Q', F' and G holds $F=GF'$ and $\mathrm{Gr}_{\mathbf{d}}(F_*(\mathbb{1}_T))^{\partial}=$

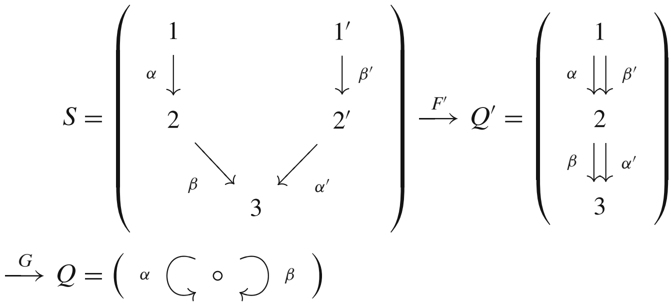
$$\begin{aligned}&\left\{U\subseteq F_*(\mathbb{1}_T)\middle|\mathbf{dim}(U)=\mathbf{d},\ U\text{ has a }\partial\text{-homogeneous vector space basis.}\right\}\\ &=\dot{\bigcup}_{\mathbf{t}\in\mathbf{G}^{-1}(\mathbf{d})}\mathrm{Gr}_{\mathbf{t}}(F'_*(\mathbb{1}_T)).\end{aligned}$$

Thus Theorem 1.1 implies

$$\chi_{\mathbf{d}}(F_*(\mathbb{1}_T))=\chi\left(\mathrm{Gr}_{\mathbf{d}}(F_*(\mathbb{1}_T))^{\partial}\right)=\sum\nolimits_{\mathbf{t}\in\mathbf{G}^{-1}(\mathbf{d})}\chi_{\mathbf{t}}(F'_*(\mathbb{1}_T)).$$

□

Example 6.2 We have a look at Example 4.6. Let Q' and F' be described by the following picture.



Using the nice grading ∂_1, it is enough to observe $F'_*(\mathbb{1}_S)$ and $\chi_{\mathbf{t}}\left(F'_*(\mathbb{1}_S)\right)$ to compute $\chi_{\mathbf{d}}(F_*(\mathbb{1}_S))$. So the nice ∂_1-grading ∂_2 induces a nice grading of $F'_*(\mathbb{1}_S)$.

Lemma 6.3 *Equation 2 holds for each tree module $F_*(\mathbb{1}_S)$.*

Proof By Proposition 6.1 it is enough to treat the cases when $F_0\colon S_0 \to Q_0$ is surjective and not injective. If $i, j \in S_0$ exist with $F_0(i) = F_0(j)$ and $i \neq j$, we construct a nice grading ∂ of $F_*(\mathbb{1}_S)$ such that $\partial(i) \neq \partial(j)$. So we do an induction over $|S_0| - |Q_0|$.

Let S' be a minimal connected subquiver of S such that there exist $i, j \in S'_0$ with $F_0(i) = F_0(j)$ and $i \neq j$. Then S' is of type A_l. Let $F'\colon S' \to Q$ be the winding induced by F. Since S is a tree, every nice grading of $F'_*(\mathbb{1}_{S'})$ can be extended to a nice grading of $F_*(\mathbb{1}_S)$.

So without loss of generality let S' be equal to S. So $S_0 = \{1, \ldots, l\}$ and $S_1 = \{s_1, s_2, \ldots, s_{l-1}\}$ as in Section 2.3 and $F_0(1) = F_0(l)$ and $1 < l$. So $\partial\colon S_0 \to \mathbb{Z}, i \mapsto \delta_{i1}$ defines a grading of $F_*(\mathbb{1}_S)$ with $\partial(1) = 1 \neq 0 = \partial(l)$. Since $F_0(2) \neq F_0(l-1)$, we have $F_1(s_1)^{-\varepsilon_1} \neq F_1(s_{l-1})^{\varepsilon_{l-1}}$ and so for all $1 < k < l$ the equation $F_1(s_1) \neq F_1(s_k)$ holds by the minimality of S. Therefore ∂ is a nice grading. □

Lemma 6.4 *Equation 2 holds for each band module $F_*(\mathbb{1}_S)$.*

Proof Let $i, j \in S_0$ with $F_0(i) = F_0(j)$, $i < j$ and $j - i$ minimal (i.e. $F_0(k) \neq F_0(m)$ for all $k, m \in S_0$ with $i \leq k < m \leq j$ and $(i, j) \neq (k, m)$). If no such tuple $(i, j) \in S_0^2$ exists, we are done. By Proposition 6.1 it is again enough to construct a nice grading ∂ of $F_*(\mathbb{1}_S)$ such that $\partial(i) \neq \partial(j)$.

For each $a \in Q_1$ let $\rho(a) := \sum_{i=1}^{l} \varepsilon_i \delta_{a, F_1(s_i)}$ and $\partial^{(a)}\colon Q_1 \to \mathbb{Z}, b \mapsto \delta_{ab}$ a map.

- If $a \in Q_1$ with $\rho(a) = 0$, then $\partial^{(a)}$ induces a nice grading $\partial^{(a)}$ of $F_*(\mathbb{1}_S)$ such that
$$\partial^{(a)}(i) - \partial^{(a)}(j) = \sum_{k=i}^{j-1} \varepsilon_k \delta_{a, F_1(s_k)}.$$
- If $a, b \in Q_1$, then $\partial^{(a,b)} := \rho(a)\partial^{(b)} - \rho(b)\partial^{(a)}$ induces similarly a nice grading $\partial^{(a,b)}$ of $F_*(\mathbb{1}_S)$.

If $\rho(F_1(s_i)) = 0$, then $\partial^{(F_1(s_i))}(i) - \partial^{(F_1(s_i))}(j) = \varepsilon_i$ since $j - i$ is minimal. Thus $F_1(s_i) \neq F_1(s_k)$ for all $k \in S_0$ with $i < k < j$.

If $\rho(F_1(s_i)) \neq 0$, we should have a look at $\partial^{(F_1(s_i), F_1(s_k))}$ for all $k \in S_0$. If $\partial^{(F_1(s_i), F_1(s_k))}(i) - \partial^{(F_1(s_i), F_1(s_k))}(j) \neq 0$ for some $k \in S_0$, we are done. So let us assume

$\partial^{(F_1(s_i),F_1(s_k))}(i) - \partial^{(F_1(s_i),F_1(s_k))}(j) = 0$ for all $k \in S_0$ and for all tuples $(i, j) \in S_0^2$ with $0 < j - i$ minimal. If $F_1(s_i) \neq F_1(s_k)$, then

$$\begin{aligned} 0 =&\partial^{(F_1(s_i),F_1(s_k))}(i) - \partial^{(F_1(s_i),F_1(s_k))}(j) \\ =&\rho(F_1(s_i))\left(\sum\nolimits_{m=i+1}^{j-1}\varepsilon_m\delta_{F_1(s_k),F_1(s_m)}\right) - \rho(F_1(s_k))\varepsilon_i \\ =&\rho(F_1(s_i))\varepsilon_{k'} - \rho(F_1(s_k))\varepsilon_i \end{aligned}$$

for some $k' \in S_0$ with $i < k' < j$ and $F_1(s_k) = F_1(s_{k'})$. So $\varepsilon_k\rho(F_1(s_k)) = \varepsilon_m\rho(F_1(s_m))$ for all $k, m \in S_0$. In other words, $\rho(F_1(s_k)) \neq 0$ for all $k \in S_1$ and $\varepsilon_k = \varepsilon_m$ for all $k, m \in S_0$ with $F_1(s_k) = F_1(s_m)$. So some $r \in \mathbb{Z}_{>0}$ exists such that $F_1(s_k) = F_1(s_{k+r})$ for all $k \in S_0$. By Example 2.8, the representation $F_*(\mathbb{1}_S)$ is decomposable if $r < l$. This is a contradiction. □

6.2 Proof of Theorem 1.2(b)

Let S be a quiver of type $\widetilde{A}_{l-1}$ and $\{i_1, \ldots, i_r\}$ be the sources and $\{i'_1, \ldots, i'_r\}$ be the sinks of S. It is visualized in Fig. 4. For all $i, j \in S_0$ with $i \leq j$ let S_{ij} be the full subquiver of S with $(S_{ij})_0 = \{i, i+1, \ldots, j\}$.

Lemma 6.5 *Let S be a quiver as above, $V = (V_i, V_{s_i})_{i \in S_0} \in \mathcal{I}_S^n$ and $\mathbf{t} = (t_1, \ldots, t_l)$ be a dimension vector of S. Then*

$$\chi_{\mathbf{t}}(V) = \sum\nolimits_{k\in\mathbb{Z}}\binom{t_{i_1}}{k}\binom{n-t_{i_1}}{k}X^{(1,1)}_{t_{i_1}-k,k,k,n-t_{i_1}-k}(\mathbf{t}') \tag{21}$$

with

$$\mathbf{t}' = \begin{cases}(0, t_{i_1+1} - t_{i_1}, \ldots, t_{i'_1-1} - t_{i_1}, t_{i'_1} - t_{i_1} - k, t_{i'_1+1} - t_{i_1}, \ldots, t_{i_1-1} - t_{i_1}) & \text{if } r = 1,\\ (0, t_{i_1+1} - t_{i_1}, \ldots, t_{i'_1} - t_{i_1}, t_{i'_1+1}, \ldots, t_{i'_r-1}, t_{i'_r} - t_{i_1}, \ldots, t_{i_1-1} - t_{i_1}) & \text{if } r > 1.\end{cases}$$

For all $s, t \in S_0$ and $\alpha, \beta, \gamma, \delta \in \mathbb{N}$ we define $X^{(s,t)}_{\alpha,\beta,\gamma,\delta}(\mathbf{t})$ to be

$$\chi_{\mathbf{t}}\left(M\left(S_{i'_s+1,i'_r-1}\right)^{\alpha} \oplus M\left(S_{i'_s+1,i_1-1}\right)^{\beta} \oplus M\left(S_{i_t+1,i'_r-1}\right)^{\gamma} \oplus M\left(S_{i_t+1,i_1-1}\right)^{\delta}\right),$$

where $M(S_{ij})$ is an indecomposable S_{ij}-representation with dimension $j - i + 1$ for all $i, j \in S_0$ with $i \leq j$.

(We use here the convention $\binom{r}{s} = 0$ for all $r, s \in \mathbb{Z}$ if $s < 0$ or $s > r$.) We visualize the representations $M(S_{i'_s+1,i'_r-1})$, $M(S_{i'_s+1,i_1-1})$, $M(S_{i_t+1,i'_r-1})$ and $M(S_{i_t+1,i_1-1})$ in Fig. 5.

Proof of Lemma 6.5 Using Remark 2.9 we get a basis $\{e_{ik} | i \in S_0, 1 \leq k \leq n\}$ of V such that the following hold.

(a) For all $1 \leq m \leq n$, the vector space $V^{(m)} := \langle e_{i,k} | i \in S_0, 1 \leq k \leq m\rangle$ is a subrepresentation of V and a band module.

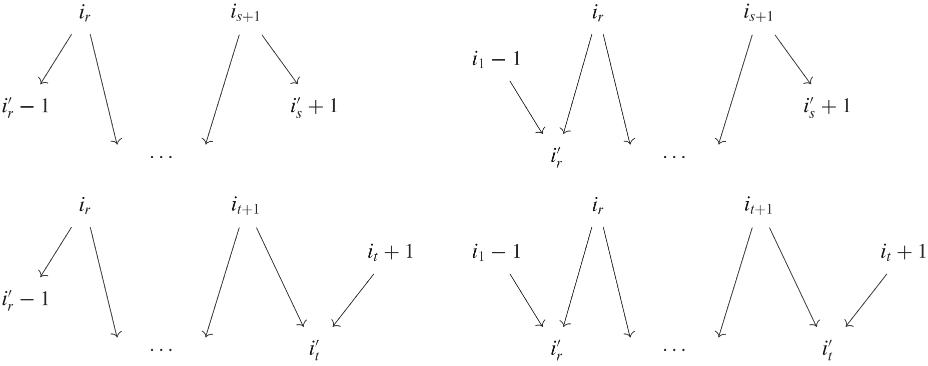


Fig. 5 Modules occurring in the definition of $X^{(s,t)}_{\alpha,\beta,\gamma,\delta}(\mathbf{t})$

(b) There exists a nilpotent endomorphism ψ of V such that $\psi(e_{i1}) = 0$ and $\psi(e_{ik}) = e_{i,k-1}$ for all $1 < k \leq n$ and all $i \in S_0$.

Let $U = (U_i, V_{s_i}|_{U_i})_{i \in S_0} \in \mathrm{Gr}_{\mathbf{t}}(V)$. Using the Gauss algorithm, a unique tuple

$$\mathbf{j}(U) := \left(1 \leq j_1 < j_2 < \ldots < j_{t_{i_1}} \leq n\right) \tag{22}$$

and unique $\lambda_{kj}(U) \in \mathbb{C}$ exist such that

$$\left\{ e_{i_1 j_m} + \sum\nolimits_{j=1,\, j \neq j_k \forall k}^{j_m - 1} \lambda_{mj}(U) e_{i_1 j} \middle| 1 \leq m \leq t_{i_1} \right\}$$

is a basis of the vector space U_{i_1}. The variety $\mathrm{Gr}_{\mathbf{t}}(V)$ is decomposed into a disjoint union of locally closed subsets

$$\mathrm{Gr}_{\mathbf{t}}(V)_{\mathbf{j}} := \left\{ U \subseteq V \middle| \mathbf{dim}(U) = \mathbf{t}, \mathbf{j}(U) = \mathbf{j} \right\},$$

where $\mathbf{j} \in \mathbb{N}^{t_{i_1}}$. For each such tuple $\mathbf{j}$ let

$$\mathrm{Gr}_{\mathbf{t}}(V)^0_{\mathbf{j}} := \left\{ U \subseteq V \middle| \mathbf{dim}(U) = \mathbf{t}, \mathbf{j}(U) = \mathbf{j}, \lambda_{1j}(U) = 0 \forall j \right\}.$$

These are locally closed subsets of $\mathrm{Gr}_{\mathbf{t}}(V)$. The projection $\pi \colon \mathrm{Gr}_{\mathbf{t}}(V)_{\mathbf{j}} \to \mathrm{Gr}_{\mathbf{t}}(V)^0_{\mathbf{j}}$ with

$$U \mapsto \prod\nolimits_{j=1}^{j_1 - 1} \left(1 + \lambda_{1j}(U) \psi^{j_1 - j}\right)^{-1} (U) \tag{23}$$

is an algebraic morphism with affine fibres, since the map $\mathrm{Gr}_{\mathbf{t}}(V)_{\mathbf{j}} \to \mathrm{Gr}_{\mathbf{t}}(V)_{\mathbf{j}}$ with $U \mapsto \left(1 + \lambda_{1j}(U) \psi^{j_1 - j}\right)(U)$ for each $1 \leq j < j_1$ can be described by polynomials and for each $U \in \mathrm{Gr}_{\mathbf{t}}(V)^0_{\mathbf{j}}$ holds:

$$\pi^{-1}(U) = \left\{ \prod\nolimits_{j=1}^{j_1 - 1} \left(1 + \mu_j \psi^{j_1 - j}\right)(U) \middle| \mu_1, \ldots, \mu_{j_1 - 1} \in \mathbb{C} \right\} \cong \mathbb{C}^{j_1 - 1}.$$

Thus $\chi(\mathrm{Gr}_{\mathbf{t}}(V)_{\mathbf{j}}) = \chi(\mathrm{Gr}_{\mathbf{t}}(V)^0_{\mathbf{j}})$.

For $U \in \mathrm{Gr}_{\mathbf{t}}(V)^0_{\mathbf{j}}$ let $U_{\mathbf{j}}$ be the subrepresentation of V generated by $e_{i_1 j_1}$. Let $V_{\mathbf{j}}$ be the subrepresentation of V with vector space basis

$$\left\{ e_{i_1 k} \middle| j_1 \leq k \leq n \right\} \cup \left\{ e_{ik} \middle| i_1 \neq i \in S_0, 1 \leq k \leq n \right\}.$$

Then $U_{\mathbf{j}} \subseteq U \subseteq V_{\mathbf{j}} \subseteq V$ and thus $\mathrm{Gr}_{\mathbf{t}}(V)^0_{\mathbf{j}} \cong \mathrm{Gr}_{\mathbf{t}-\mathbf{dim}(U_{\mathbf{j}})}\left(V_{\mathbf{j}}/U_{\mathbf{j}}\right)$. This implies

$$\chi_{\mathbf{t}}(V) = \sum\nolimits_{\mathbf{j}\in\mathbb{N}^{t_{i_1}}} \chi\left(\mathrm{Gr}_{\mathbf{t}}(V)^0_{\mathbf{j}}\right) = \sum\nolimits_{\mathbf{j}\in\mathbb{N}^{t_{i_1}}} \chi_{\mathbf{t}-\mathbf{dim}(U_{\mathbf{j}})}\left(V_{\mathbf{j}}/U_{\mathbf{j}}\right). \tag{24}$$

Using the representation theory of S, we get

$$\mathbf{dim}\left(U_{\mathbf{j}}\right)_i = \begin{cases} 0 & \text{if } i'_1 < i < i'_r, \\ 2 & \text{if } r = 1, i'_1 = i \text{ and } j_1 > 1, \\ 1 & \text{otherwise.} \end{cases}$$

So if $j_1 = 1$ we get

$$V_{\mathbf{j}}/U_{\mathbf{j}} \cong V^{(n-1)} \oplus M\left(S_{i'_1+1,i'_r-1}\right), \tag{25}$$

and if $j_1 > 1$ we get

$$V_{\mathbf{j}}/U_{\mathbf{j}} \cong V^{(n-j_1)} \oplus M\left(S_{i'_1+1,i_1-1}\right) \oplus M\left(S_{i_1+1,i'_r-1}\right) \oplus M\left(S_{i_1+1,i_1-1}\right)^{j_1-2}. \tag{26}$$

Let $n_{\mathbf{j}} := |\{1 \le i \le n | i \ne j_m \forall m, \exists m : i+1 = j_m\}|$. A simple calculation shows

$$\left|\{\mathbf{j} | n_{\mathbf{j}} = k\}\right| = \binom{t_{i_1}}{k}\binom{n-t_{i_1}}{k}.$$

We do an induction over t_{i_1}. Then Eq. 26 occurs $n_{\mathbf{j}}$-times, Eq. 25 occurs $(t_{i_1} - n_{\mathbf{j}})$-times and so Eq. 21 holds in general by an inductive version of Eq. 24. □

The rest of the proof of Theorem 1.2(b) is done in the next two combinatorial lemmas.

Lemma 6.6 *Let $a, b, c, d, e, f \in \mathbb{N}$. Then*

$$\binom{a}{b}\binom{b}{c} = \binom{a-c}{b-c}\binom{a}{c}$$

and

$$\sum\nolimits_{m\in\mathbb{Z}} \binom{a}{b-m}\binom{b-m}{c}\binom{d}{e+m}\binom{e+m}{f} = \binom{a}{c}\binom{d}{f}\binom{a+d-c-f}{a+d-b-e}.$$

Proof The first equation can be shown using the definition. The second equation is a consequence of the first one. □

Lemma 6.7 *Let S, V, $\mathbf{t}$, n as above and $1 \le m \le r$. Then*

$$\chi_{\mathbf{t}}(V) = \Lambda_m \Gamma_{i_1 i_m} \sum\nolimits_{k\in\mathbb{Z}} \binom{t_{i_1}}{t_{i_m}-k}\binom{n-t_{i_1}}{k} X^{(m,m)}_{t_{i_m}-k,k,t_{i_1}-t_{i_m}+k,n-t_{i_1}-k}(\mathbf{t}') \tag{27}$$

with

$$\Lambda_m := \prod_{k=1}^{m-1} \frac{(n-t_{i_{k+1}})!}{t_{i_k}!} \frac{t_{i'_k}!}{(n-t_{i'_k})!}, \qquad \Gamma_{ij} := \prod_{k=i}^{j-1} \frac{1}{(\varepsilon_k(t_k - t_{k+1}))!}$$

and

$$\mathbf{t}' = \begin{cases} (0, \ldots, 0, t_{i_r+1} - t_{i_r}, \ldots, t_{i'_r-1} - t_{i_r}, \\ \quad t_{i'_r} - t_{i_1} - k, t_{i'_r+1} - t_{i_1}, \ldots, t_{i_1-1} - t_{i_1}) & \text{if } m = r, \\ (0, \ldots, 0, t_{i_m+1} - t_{i_m}, \ldots, t_{i'_m} - t_{i_m}, t_{i'_m+1}, \ldots, t_{i'_r-1}, \\ \quad t_{i'_r} - t_{i_1}, \ldots, t_{i_1-1} - t_{i_1}) & \text{if } m < r. \end{cases}$$

Proof For $m = 1$ this is the statement of Lemma 6.5. We prove the lemma by induction. Let $1 < m \leq r$. Then

$$\begin{aligned} \chi_{\mathbf{t}}(V) =& \Lambda_{m-1}\Gamma_{i_1 i_{m-1}} \sum_k \binom{t_{i_1}}{t_{i_{m-1}}-k}\binom{n-t_{i_1}}{k} X^{(m-1,m-1)}_{t_{i_{m-1}}-k,k,t_{i_1}-t_{i_{m-1}}+k,n-t_{i_1}-k}(\mathbf{t}') \\ =& \Lambda_{m-1}\Gamma_{i_1 i_{m-1}} \sum_k \binom{t_{i_1}}{t_{i_{m-1}}-k}\binom{n-t_{i_1}}{k} \sum_p \binom{t_{i_1}-t_{i_{m-1}}+k}{t_{i'_{m-1}}-t_{i_{m-1}}-p}\binom{n-t_{i_1}-k}{p} \\ & \left(t_{i'_{m-1}} - t_{i_{m-1}}\right)!\Gamma_{i_{m-1}i'_{m-1}} X^{(m-1,m)}_{t_{i'_{m-1}}-k-p,k+p,t_{i_1}-t_{i'_{m-1}}+k+p,n-t_{i_1}-k-p}(\mathbf{t}'') \\ =& \Lambda_{m-1}\Gamma_{i_1 i'_{m-1}} \sum_p \left(\sum_k \binom{t_{i_1}}{t_{i_{m-1}}-k}\binom{n-t_{i_1}}{k}\binom{t_{i_1}-t_{i_{m-1}}+k}{t_{i_1}-t_{i'_{m-1}}+p}\binom{n-t_{i_1}-k}{n-t_{i_1}-p}\right) \\ & \left(t_{i'_{m-1}} - t_{i_{m-1}}\right)! X^{(m-1,m)}_{t_{i'_{m-1}}-p,p,t_{i_1}-t_{i'_{m-1}}+p,n-t_{i_1}-p}(\mathbf{t}'') \end{aligned}$$

with $\mathbf{t}'' = (0, \ldots, 0, t_{i'_{m-1}+1}, \ldots, t_{i'_r+1} - t_{i_1}, \ldots, t_{i_1-1} - t_{i_1})$. Lemma 6.6 yields

$$\begin{aligned} \chi_{\mathbf{t}}(V) =& \Lambda_{m-1}\Gamma_{i_1 i'_{m-1}} \sum_p \binom{n-t_{i_1}}{p}\binom{t_{i_1}}{t_{i'_{m-1}}-p}\binom{t_{i'_{m-1}}}{t_{i_{m-1}}} \\ & \left(t_{i'_{m-1}} - t_{i_{m-1}}\right)! X^{(m-1,m)}_{t_{i'_{m-1}}-p,p,t_{i_1}-t_{i'_{m-1}}+p,n-t_{i_1}-p}(\mathbf{t}'') \\ =& \Lambda_{m-1}\frac{t_{i'_{m-1}}!}{t_{i_{m-1}}!}\Gamma_{i_1 i'_{m-1}} \sum_p \binom{n-t_{i_1}}{p}\binom{t_{i_1}}{t_{i'_{m-1}}-p} \sum_k \binom{t_{i'_{m-1}}-p}{t_{i_m}-k}\binom{p}{k} \\ & \left(t_{i'_{m-1}} - t_{i_m}\right)!\Gamma_{i'_{m-1}i_m} X^{(m,m)}_{t_{i_m}-k,k,t_{i_1}-t_{i_m}+k,n-t_{i_1}-k}(\mathbf{t}') \\ =& \Lambda_{m-1}\frac{t_{i'_{m-1}}!}{t_{i_{m-1}}!}\Gamma_{i_1 i_m} \sum_k \left(\sum_p \binom{n-t_{i_1}}{p}\binom{t_{i_1}}{t_{i'_{m-1}}-p}\binom{t_{i'_{m-1}}-p}{t_{i_m}-k}\binom{p}{k}\right) \\ & \left(t_{i'_{m-1}} - t_{i_m}\right)! X^{(m,m)}_{t_{i_m}-k,k,t_{i_1}-t_{i_m}+k,n-t_{i_1}-k}(\mathbf{t}'). \end{aligned}$$

Using Lemma 6.6 again, we get

$$\chi_{\mathbf{t}}(V) = \Lambda_{m-1}\frac{t_{i'_{m-1}}!}{t_{i_{m-1}}!}\Gamma_{i_1 i_m}\sum_k \binom{t_{i_1}}{t_{i_m}-k}\binom{n-t_{i_1}}{k}\binom{n-t_{i_m}}{n-t_{i'_{m-1}}}$$

$$\left(t_{i'_{m-1}} - t_{i_m}\right)! X^{(m,m)}_{t_{i_m}-k,k,t_{i_1}-t_{i_m}+k,n-t_{i_1}-k}(\mathbf{t}')$$

$$= \Lambda_m \Gamma_{i_1 i_m}\sum_k \binom{t_{i_1}}{t_{i_m}-k}\binom{n-t_{i_1}}{k} X^{(m,m)}_{t_{i_m}-k,k,t_{i_1}-t_{i_m}+k,n-t_{i_1}-k}(\mathbf{t}').$$

□

Corollary 6.8 *Let S, V,* $\mathbf{t}$ *and n as above. Then Eq. 3 holds.*

Proof We have to show $\chi_{\mathbf{t}}(V) = \Lambda_{r+1}\Gamma_{1,l+1}$ with $t_{i_{r+1}} = t_{i_1}$. Lemma 6.7 implies

$$\chi_{\mathbf{t}}(V) = \Lambda_r \Gamma_{i_1 i_r}\sum_k \binom{t_{i_1}}{t_{i_r}-k}\binom{n-t_{i_1}}{k} X^{(r,r)}_{t_{i_r}-k,k,t_{i_1}-t_{i_r}+k,n-t_{i_1}-k}(\mathbf{t}')$$

with $\mathbf{t}' = (0, \dots, 0, t_{i_r+1} - t_{i_r}, \dots, t_{i'_r-1} - t_{i_r}, t_{i'_r} - t_{i_1} - k, t_{i'_r+1} - t_{i_1}, \dots, t_{i_1-1} - t_{i_1})$. So we have

$$\chi_{\mathbf{t}}(V) = \Lambda_r \Gamma_{i_1 i_r}\sum_k \binom{t_{i_1}}{t_{i_r}-k}\binom{n-t_{i_1}}{k}\binom{n-t_{i_1}-k}{t_{i'_r}-t_{i_1}-k} X^{(r,r)}_{t_{i_r}-k,t_{i'_r}-t_{i_1},t_{i'_r}-t_{i_r},n-t_{i'_r}}(\mathbf{t}''')$$

$$= \Lambda_r \Gamma_{i_1 i_r}\sum_k \binom{t_{i_1}}{t_{i_r}-k}\binom{n-t_{i_1}}{k}\binom{n-t_{i_1}-k}{t_{i'_r}-t_{i_1}-k}\left(t_{i'_r} - t_{i_1}\right)!\Gamma_{i'_r i_1}\left(t_{i'_r} - t_{i_r}\right)!\Gamma_{i_r i'_r}$$

with $\mathbf{t}''' = (0, \dots, 0, t_{i_r+1} - t_{i_r}, \dots, t_{i'_r-1} - t_{i_r}, 0, t_{i'_r+1} - t_{i_1}, \dots, t_{i_1-1} - t_{i_1})$. Using Lemma 6.6 again, we obtain

$$\chi_{\mathbf{t}}(V) = \Lambda_r \Gamma_{1l}\binom{n-t_{i_1}}{n-t_{i'_r}}\binom{t_{i'_r}}{t_{i_r}}\left(t_{i'_r} - t_{i_1}\right)!\left(t_{i'_r} - t_{i_r}\right)! = \Lambda_{r+1}\Gamma_{1,l+1}.$$

□

7 Quiver Flag Varieties

In this section we explain and justify Example 3.11. Let $B\colon Q \to Q$ be the identity winding. For each $\mu \in \mathbb{C}$ there is an automorphism of the algebra $\mathbb{C}Q$ such that

$B_*(\lambda, n)$ is mapped to $B_*(\lambda - \mu, n)$. This is not necessarily a band module. So we assume without loss of generality that M is a string module. Let

$$T^{(n)} = \left(\begin{array}{ccccccccc} 1^{(1)} & & & & 1^{(2)} & & & & 1^{(n)} & & \\ & \searrow & & \swarrow & & \searrow & & & & \swarrow & & \searrow & \\ & & 2^{(1)} & & & & 2^{(2)} & \cdots & 2^{(n-1)} & & & & 2^{(n)} \end{array} \right),$$

$$U^{(n)} = \left(\begin{array}{ccccccccc} 1^{(1)} & & & & 1^{(2)} & & & & 1^{(n)} \\ & \searrow & & \swarrow & & \searrow & & & & \swarrow \\ & & 2^{(1)} & & & & 2^{(2)} & \cdots & 2^{(n-1)} \end{array} \right).$$

For any subquiver V of $T^{(n)}$ let

$$\mathbf{dim}(V) = \left(\left|\left\{ i \middle| 1^{(i)} \in V_0 \right\}\right|, \left|\left\{ i \middle| 2^{(i)} \in V_0 \right\}\right| \right).$$

For any dimension vectors $\mathbf{c}$ and $\mathbf{d}$ of Q we set

$$X_{\mathbf{c}}(V) = \left\{ R \subseteq V \middle| R \text{ is a successor closed subquiver of } V, \mathbf{dim}(R) = \mathbf{c} \right\},$$
$$X_{\mathbf{c},\mathbf{d}}(V) = \left\{ R \subseteq S \subseteq V \middle| R \in X_{\mathbf{c}}(S), S \in X_{\mathbf{d}}(V) \right\}.$$

Using Corollary 3.10, it is enough to show the following equality.

$$\begin{aligned} \left| X_{(1,2),(2,3)}\left(T^{(n)}\right) \right| &= \sum_{i=1}^{n} \left| \left\{ (R \subseteq S) \in X_{(1,2),(2,3)}\left(T^{(n)}\right) \middle| 1^{(i)} \in R_0 \right\} \right| \\ &= \left| X_{(0,1),(1,2)}\left(T^{(n-1)}\right) \right| + \sum_{i=2}^{n} \left| X_{(1,1)}\left(U^{(i-1)} \dot\cup T^{(n-i)} \right) \right| \\ &= \binom{2}{1} \left| X_{(1,2)}\left(T^{(n-1)}\right) \right| + \left(\left| X_{(0,1)}\left(T^{(n-2)}\right) \right| + \left| X_{(1,1)}\left(T^{(n-2)}\right) \right| \right) \\ &\quad + \sum_{i=3}^{n-1} \left(\left| X_{(1,1)}\left(U^{(i-1)}\right) \right| + \left| X_{(1,1)}\left(T^{(n-i)}\right) \right| \right) + \left| X_{(1,1)}\left(U^{(n-1)}\right) \right| \\ &= 2\left(\binom{n-2}{1} + \binom{n-2}{1} \right) + \left(\binom{n-2}{1} + 1 \right) + (n-3)(2+1) + 2 \\ &= 8(n-2). \end{aligned}$$

8 Coverings of Quivers

8.1 Proof of Theorem 1.2(c) for a Free Abelian Group

It is enough to consider the representation $V = \mathbb{1}_T$ with some finite subquiver $T = (T_0, T_1)$ of the quiver $\hat{Q}$. So without loss of generality we assume that G is of finite rank, e.g. $G = \mathbb{Z}^k$. By induction and Proposition 6.1 it is enough to assume $G = \mathbb{Z}$. Let R_0 be a set of representatives of the $\mathbb{Z}$-orbits in $\hat{Q}_0$. Since $\mathbb{Z}$ acts freely on $\hat{Q}$

there is a unique $z_i \in \mathbb{Z}$ and $r_i \in R_0$ for any $i \in \hat{Q}_0$ with $i = z_i r_i$. The Q-representation $\pi_*(\mathbb{1}_T)$ has a basis $\{f_i | i \in T_0\}$. We define a grading of $\pi_*(\mathbb{1}_T)$ by $\partial(f_i) = z_i$. This grading is well-defined and so it is enough to show that ∂ is a nice grading. Let $a, b \in T_1$ such that they are lying in the same $\mathbb{Z}$-orbit, i.e., there exists $z \in \mathbb{Z}$ with $za = b$. Thus $zs(a) = s(za) = s(b)$ and $z_{s(a)} + z = z_{s(b)}$. So

$$\begin{aligned}\partial\left(f_{t(b)}\right) - \partial\left(f_{s(b)}\right) &= z_{t(b)} - z_{s(b)} = (z_{t(a)} + z) - (z_{s(a)} + z) \\ &= z_{t(a)} - z_{s(a)} = \partial\left(f_{t(a)}\right) - \partial\left(f_{s(a)}\right).\end{aligned}$$

□

8.2 Proof of Theorem 1.2(c) for a Free Group

This part of Theorem 1.2 can be proven again by using Theorem 1.2(c) in the case with $G = \mathbb{Z}$. In order to prevent a tangled mass of notation we only give an example instead of proving it.

Example 8.1 Let $Q = (\{x\}, \{\alpha, \beta\}, s, t)$ and $\pi : \hat{Q} \to Q$ the universal covering. The fundamental group of Q is a free group with two generators. For $m \in \mathbb{N}$ let $Q^{(m)}$ be the quiver with

$$Q_0^{(m)} = \mathbb{Z}^m,\ Q_1^{(m)} = \left\{\alpha_{\mathbf{c}}, \beta_{\mathbf{c}} \middle| \mathbf{c} \in Q_0^{(m)}\right\},$$

$$s(\alpha_{\mathbf{c}}) = \mathbf{c},\ t(\alpha_{\mathbf{c}}) = \begin{cases} \mathbf{c} + e_1 & \text{if } m > 0 \\ \mathbf{c} & \text{otherwise} \end{cases},$$

$$s(\beta_{\mathbf{c}}) = \mathbf{c},\ t(\beta_{\mathbf{c}}) = \begin{cases} \mathbf{c} + e_{\mathbf{c}_1+1} & \text{if } 1 \le \mathbf{c}_1 < m \\ \mathbf{c} & \text{otherwise} \end{cases}.$$

for any $\mathbf{c} \in Q_0^{(m)}$. Then $Q^{(0)} \cong Q$ and $Q^{(1)}$ is described by the picture in Fig. 6. Let $\pi^{(m)} \colon Q^{(m)} \to Q$ be the canonical projection. Thus Theorem 1.2 holds for $\pi^{(m)}$ by induction. Let V be any $\hat{Q}$-representation. To prove Theorem 1.2(c) for $\pi_*(V)$ it is enough to consider V as a $Q^{(m)}$ representation for some large m. Thus Theorem 1.2 holds also for π.

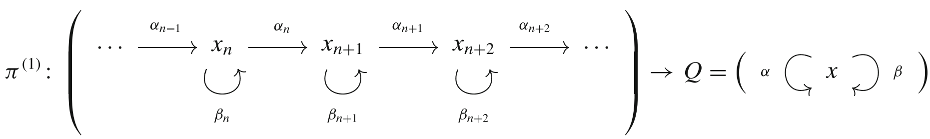


Fig. 6 The covering $\pi^{(1)} \colon Q^{(1)} \to Q$

9 Ringel-Hall Algebras

9.1 Proof of Theorem 1.3(a)

For $V \in \mathrm{mod}(B)$ and dimension vectors $\mathbf{d}^{(i)} \in \mathbb{N}^{Q_0}$ we have:

$$\begin{aligned}
&\mathcal{C}(F)\left(\prod_{i=1}^{n} \mathbb{1}_{\mathbf{d}^{(i)}}\right)(V)\\
&= \chi\left(\left\{0 = U^{(0)} \subseteq \ldots \subseteq U^{(n)} = F_*(V) \middle| \mathbf{dim}\left(U^{(i)}/U^{(i-1)}\right) = \mathbf{d}^{(i)} \forall i\right\}\right)\\
&= \chi\left(\left\{0 = U^{(0)} \subseteq \ldots \subseteq U^{(n)} = F_*(V) \middle| \mathbf{dim}\left(U^{(i)}/U^{(i-1)}\right) = \mathbf{d}^{(i)} \forall i\right\}^{\partial_1,\ldots,\partial_n}\right)\\
&= \chi\left(\left\{0 = U^{(0)} \subseteq \ldots \subseteq U^{(n)} = V \middle| \mathbf{dim}\left(F_*\left(U^{(i)}/U^{(i-1)}\right)\right) = \mathbf{d}^{(i)} \forall i\right\}\right)\\
&= \left(\prod_{i=1}^{n} \mathcal{C}(F)\left(\mathbb{1}_{\mathbf{d}^{(i)}}\right)\right)(V)
\end{aligned}$$

with some gradings $\partial_1, \ldots, \partial_n$ as in the proofs of Theorem 1.2, Lemmas 6.3 and 6.4. For $\mathbf{d} \in \mathbb{N}^{Q_0}$ holds

$$\mathcal{C}(F)(\mathbb{1}_{\mathbf{d}}) = \sum_{\mathbf{t}\in\mathbf{F}^{-1}(\mathbf{d})} \mathbb{1}_{\mathbf{t}}, \quad \mathcal{C}(F)(\mathbb{1}_{S^{\mathbf{d}}}) = \sum_{\mathbf{t}\in\mathbf{F}^{-1}(\mathbf{d})} \mathbb{1}_{S^{\mathbf{t}}}.$$

Thus $\mathcal{C}(F)$ is a well-defined algebra homomorphism. For any $f \in \mathcal{C}(A)$ we have

$$\begin{aligned}
(\mathcal{C}(F) \otimes \mathcal{C}(F))(\Delta(f))(V, W) &= (\Delta(f) \circ (F_*, F_*))(V, W) = \Delta(f)(F_*(V), F_*(W))\\
&= f(F_*(V) \oplus F_*(W)) = f(F_*(V \oplus W))\\
&= \mathcal{C}(F)(f)(V \oplus W) = \Delta(\mathcal{C}(F)(f))(V, W)
\end{aligned}$$

and for $\mathbf{d} \in \mathbb{N}^{Q_0}$

$$\begin{aligned}
S(\mathcal{C}(F)(\mathbb{1}_{\mathbf{d}})) &= S\left(\sum_{\mathbf{t}\in\mathbf{F}^{-1}(\mathbf{d})} \mathbb{1}_{\mathbf{t}}\right) = (-1)^{|\mathbf{d}|} \sum_{\mathbf{t}\in\mathbf{F}^{-1}(\mathbf{d})} \mathbb{1}_{S^{\mathbf{t}}}\\
&= (-1)^{|\mathbf{d}|} \mathcal{C}(F)(\mathbb{1}_{S^{\mathbf{d}}}) = \mathcal{C}(F)(S(\mathbb{1}_{\mathbf{d}})).
\end{aligned}$$

By this $\mathcal{C}(F)$ is actually a Hopf algebra homomorphism.

If $F\colon S \to Q$ is injective, $F_*\colon \mathrm{mod}(B) \to \mathrm{mod}(A)$ is injective and $\mathcal{C}(F)(\mathbb{1}_{\mathbf{F}(\mathbf{d})}) = \mathbb{1}_{\mathbf{d}}$ holds for each dimension vector $\mathbf{d} \in \mathbb{N}^{S_0}$. The functor F_* induces an embedding of varieties $\mathrm{mod}(B, \mathbf{d}) \to \mathrm{mod}(A, \mathbf{F}(\mathbf{d}))$. Thus $\mathcal{C}(F)$ is surjective.

Let $f \in \mathrm{Ker}\,\mathcal{C}(F)$ and $W \in \mathrm{mod}(A)$. If $F_*\colon \mathrm{mod}(B) \to \mathrm{mod}(A)$ is dense, a B-module V with $F_*(V) \cong W$ exists. By $f(W) = f(F_*(V)) = \mathcal{C}(F)(f)(V) = 0$ is $\mathrm{Ker}\,\mathcal{C}(F) = 0$ and $\mathcal{C}(F)$ is injective. □

9.2 Proof of Theorem 1.3(b)

This part can be proven in a very similar way as Theorem 1.3(a). □

9.3 Proof of Proposition 1.4

This proposition is a direct consequence of the following factorization property.

Proposition 9.1 (Factorization Property) *Let* $\mathbf{F}$ *be a tuple of trees,* $\mathbf{B}$ *a tuple of bands and* $\mathbf{n}$ *a tuple of positive integers.*

(a) *Let* $F\colon S \to Q$ *be a tree or a band and* $V \in \operatorname{rep}(S)$ *such that* $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}(F_*(V)) = 1$. *Then there exists a tuple* $(\widetilde{\mathbf{F}}, \widetilde{\mathbf{B}}) \in \mathcal{G}_F(\mathbf{F}) \times \mathcal{G}_F(\mathbf{B})$ *with* $\mathbb{1}_{\widetilde{\mathbf{F}},\widetilde{\mathbf{B}},\mathbf{n}}(V) = 1$.

(b) *Let* $\hat{Q}$ *be a locally finite quiver, G a free (abelian) group, which acts freely on* $\hat{Q}$, $Q = \hat{Q}/G$ *and* $\pi : \hat{Q} \to Q$ *the canonical projection. Let* $V \in \operatorname{rep}(\hat{Q})$ *such that* $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}(\pi_*(V)) = 1$. *Then there is a tuple* $(\widetilde{\mathbf{F}}, \widetilde{\mathbf{B}}) \in \mathcal{G}_\pi(\mathbf{F}) \times \mathcal{G}_\pi(\mathbf{B})$ *with* $\mathbb{1}_{\widetilde{\mathbf{F}},\widetilde{\mathbf{B}},\mathbf{n}}(V) = 1$.

Proof of Part (a) If F is a tree, a factorization $F = \pi\iota$ with the universal covering $\pi\colon \hat{Q} \twoheadrightarrow Q$ and an embedding $\iota\colon S \hookrightarrow \hat{Q}$ exists (see Fig. 7, left hand side). By the additivity of F_* and [14, Lemma 3.5] we assume without loss of generality that V, $\iota_*(V)$ and $F_*(V)$ are indecomposable. The module $F_*(V)$ can be lifted to a $\hat{Q}$-module. By $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}(F_*(V)) = 1$ the module $F_*(V)$ is a tree or a band module. If $F_*(V)$ is a band module, it cannot be lifted to a $\hat{Q}$-module since $\hat{Q}$ is a tree. This is a contradiction.

So $F_*(V) \cong F_*^{(1)}(\mathbb{1}_{S^{(1)}})$ is a tree module. Since $F^{(1)}$ is a tree, there exists another factorization $F^{(1)} = \pi\iota'$ with an embedding $\iota'\colon S^{(1)} \hookrightarrow \hat{Q}$ (see Fig. 7, left hand side). Using the proof of [14, Theorem 3.6(c)] we get $\iota_*(V)$ is (up to shift by some group element) isomorphic to $\iota'_*(\mathbb{1}_{S^{(1)}})$. So we can modify ι' such that $\iota_*(V) \cong \iota'_*(\mathbb{1}_{S^{(1)}})$ and some winding $\widetilde{F}^{(1)}\colon S^{(1)} \to S$ exists such that the diagram in Fig. 7 commutes. In particular V is a tree module.

If F is a band, then V is a direct sum of some string and band modules. Since F_* is additive we assume again without loss of generality that V is indecomposable. Thus V is a tree or a band module and there exists a winding $G\colon T \to S$ with $G_*(W) \cong V$ for some $W \in \mathcal{I}_T^{n'}$. Since $FG\colon T \to Q$ is a winding and $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}((FG)_*(W)) = 1$ we get $l(\mathbf{F}) + l(\mathbf{B}) = 1$.

If $l(\mathbf{F}) = 1$, then $(FG)_*(W) \cong F_*^{(1)}(\mathbb{1}_{S^{(1)}})$. By Lemma 2.11 there exists an isomorphism of quivers $H\colon S^{(1)} \to T$ such that $(FG)H = F^{(1)}$. By setting $\widetilde{F}^{(1)} = GH$ the statement follows (see Fig. 7, right hand side). For $l(\mathbf{B}) = 1$ the result follows analogously. □

Proof of Part (b) By the additivity of F_* and [14, Lemma 3.5] we assume without loss of generality that V and $\pi_*(V)$ are indecomposable. And by $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}(\pi_*(V)) = 1$ the module $\pi_*(V)$ is a tree or a band module.

If $\pi_*(V)$ is a tree module, we get $\pi_*(V) \cong F_*^{(1)}(\mathbb{1}_{S^{(1)}})$. Since G acts on $\hat{Q}$ and $Q = \hat{Q}/G$ the tree $F^{(1)}$ factors through π. Let $\widetilde{F}^{(1)}$ be the factorization, e.g. $\pi\widetilde{F}^{(1)} = F^{(1)}$ (see Fig. 8, left hand side). Then $\pi_*(\widetilde{F}_*^{(1)}(\mathbb{1}_{S^{(1)}})) = F_*^{(1)}(\mathbb{1}_{S^{(1)}}) \cong \pi_*(V)$ and by the proof of [14, Theorem 3.6(c)] we get $\widetilde{F}_*^{(1)}(\mathbb{1}_{S^{(1)}})$ is (up to shift by some group element)

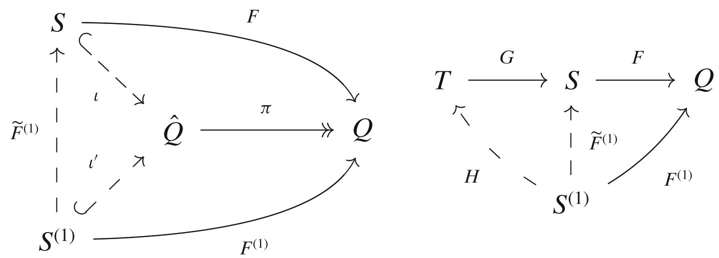


Fig. 7 Windings occurring in the proof of Proposition 9.1(a)

isomorphic to V. So we can modify again $\widetilde{F}^{(1)}$ such that $\widetilde{F}_*^{(1)}(\mathbb{1}_{S^{(1)}}) \cong V$ and still $\pi \widetilde{F}^{(1)} = F^{(1)}$.

If $\pi_*(V)$ is a band module, we get $\pi_*(V) \cong B_*^{(1)}(V_1)$ for some $V_1 \in \mathcal{I}_{T^{(1)}}^{n_1}$. Let $\rho\colon \hat{T}^{(1)} \to T^{(1)}$ be the universal covering of $T^{(1)}$ (see Fig. 8, right hand side). Since $\pi_*(V) \cong B_*^{(1)}(V_1)$ and G is a free (abelian) group, which acts freely on $\hat{Q}$, we get not only a lifting $\hat{B}^{(1)}$ of $B^{(1)}\rho$ but also a lifting $\widetilde{B}^{(1)}$ of $B^{(1)}$. Then the result follows as above. □

9.4 Proof of Corollary 3.17(a)

We consider the products of functions of the form $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}$ in $\mathcal{H}(A)$. By the proof of Theorem 1.3(a), Lemmas 4.13 and 7.14, and Proposition 9.1(a) we get

$$\begin{aligned}
&(\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}} * \mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'})(F_*(V)) \\
&\quad = \chi\left(\left\{U \subseteq F_*(V) \middle| \mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}(U) = 1, \mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}(F_*(V)/U) = 1\right\}^{\partial_1,\ldots,\partial_n}\right) \\
&\quad = \chi\left(\left\{U \subseteq V \middle| \mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}(F_*(U)) = 1, \mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}(F_*(V/U)) = 1\right\}\right) \\
&\quad = \left(\mathcal{H}(F)\left(\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}\right) * \mathcal{H}(F)\left(\mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}\right)\right)(V) \\
&\quad = \sum \left(\mathbb{1}_{\widetilde{\mathbf{F}},\widetilde{\mathbf{B}},\mathbf{n}} * \mathbb{1}_{\widetilde{\mathbf{F}}',\widetilde{\mathbf{B}}',\mathbf{n}'}\right)(V).
\end{aligned}$$

So we only have to use the representation theory of trees and quivers of type $\widetilde{A}_{l-1}$, to calculate the Euler characteristics of the occurring varieties. □

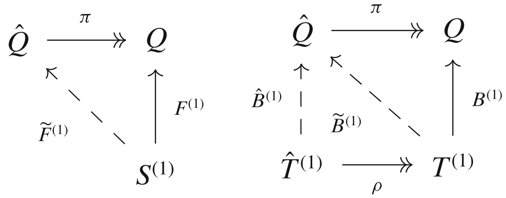
9.5 Proof of Corollary 3.17(b)

This corollary can be proven in a very similar way as Corollary 3.17(a). □

9.6 Proof of Proposition 3.21(a)

Let $M := B_*(\lambda, m)$, $\pi\colon M \to B_*(\lambda, m-n)$ a projection, $K := \bigoplus_i F_*^{(i)}(\mathbb{1}_{S^{(i)}})$ and $K' := \bigoplus_i F_*'^{(i)}(\mathbb{1}_{S'^{(i)}})$. By Remark 2.9, there exists a unique $U \subseteq B_*(\lambda, m)$ with

Fig. 8 Windings occurring in the proof of Proposition 9.1(b)



$U \cong B_*(\lambda, n)$, so we can assume $B_*(\lambda, n) \subseteq B_*(\lambda, m-n') \subseteq B_*(\lambda, m)$. Define the varieties

$$X := \left\{ U \subseteq M \middle| U \cong B_*(\lambda, n) \oplus K,\ M/U \cong B_*(\lambda, n') \oplus K' \right\}$$

$$\overline{X} := \left\{ V \subseteq \overline{M} \middle| V \cong K,\ \overline{M}/V \cong K' \right\}$$

with $\overline{U} := \big(U \cap B_*(\lambda, m-n')\big)/B_*(\lambda, n)$ for all $B_*(\lambda, n) \subseteq U \subseteq M$ and an algebraic morphism $\phi\colon X \to \overline{X}$ by $U \mapsto \overline{U}$. Using Remark 2.9 again, $B_*(\lambda, n) \subseteq U \subseteq B_*(\lambda, m-n')$ for all $U \in X$. So ϕ is well-defined and injective.

Let $V \in \overline{X}$. Since $V \cong K$ and $\overline{M}/V \cong K'$ we have $B_*(\lambda, m-n')/\pi^{-1}(V) \cong K'$ and $M/B_*(\lambda, m-n') \cong B_*(\lambda, n')$. There exist two short exact sequences

$$0 \to B_*(\lambda, n) \to \pi^{-1}(V) \to K \to 0$$

$$0 \to K' \to M/\pi^{-1}(V) \to B_*(\lambda, n') \to 0$$

Using Remark 2.12, we assume without loss of generality that the direct summands of K are preprojective Q-representations and the direct summands of K' are preinjective ones. So both sequences split and this means that $\pi^{-1}(V) \cong B_*(\lambda, n) \oplus K$ and $M/\pi^{-1}(V) \cong K' \oplus B_*(\lambda, n')$. Thus $\pi^{-1}(V) \in X$ and $\overline{\pi^{-1}(V)} = V$. This shows that the Euler characteristics of both varieties are equal. □

9.7 Proof of Proposition 3.21(b)

Proposition 3.21(b) follows inductively by the following lemma.

Lemma 9.2 *Let Q, m, $B\colon Q \to Q$, $F\colon S \to Q$, $\mathbf{F}(n)$, $\mathbf{F}$ and $\mathbf{F}'$ as in Proposition 3.21(b). Let $M = B_*(\lambda, m)$ and $n \in \mathbb{Z}_{>0}$. Then*

$$\left(\mathbb{1}_{\mathbf{F}(n)\dot\cup\mathbf{F}} * \mathbb{1}_{\mathbf{F}'}\right)(M) = \sum_{k\in\mathbb{N}} \left(\left(\mathbb{1}_{\mathbf{F}(n-1)} \otimes 1\right) * \Delta\left(\mathbb{1}_{\mathbf{F}}\right) * \Delta\left(\mathbb{1}_{\mathbf{F}'}\right)\right)\left(B_*(\lambda, m-k), I_k\right)$$

with I_k an indecomposable module and $\mathbf{dim}(I_k) = \mathbf{dim}(B_*(\lambda, k)) - \mathbf{dim}(F_*(\mathbb{1}_S))$.

Proof of Part (b) Let $M = (M_i, M_a)_{i\in Q_0, a\in Q_1}$, $\mathbf{c} := \mathbf{dim}(F_*(\mathbb{1}_S))$, $\mathbf{d}^{(i)} := \mathbf{dim}(F_*^{(i)}(\mathbb{1}_{S^{(i)}}))$ for all i and $\mathbf{d} = n\mathbf{c} + \sum_i \mathbf{d}^{(i)}$. By Remark 2.12, we know $\mathbb{1}_{\mathbf{F}(n)} * \mathbb{1}_{\mathbf{F}} = \mathbb{1}_{\mathbf{F}(n)\dot\cup\mathbf{F}}$. So we have to calculate the Euler characteristic of

$$X = \left\{(0 \subseteq U \subseteq W \subseteq M) \in \mathcal{F}_{n\mathbf{c},\mathbf{d}}(M) \middle| \mathbb{1}_{\mathbf{F}(n)}(U) = \mathbb{1}_{\mathbf{F}}(W/U) = \mathbb{1}_{\mathbf{F}'}(M/W) = 1\right\}.$$

We use now the arguments of the proof of Lemma 6.5 in Section 6.2. Let $\{e_{ik} | i \in Q_0, 1 \le k \le m\}$ be a basis of M such that the following hold.

(a) For all $1 \le p \le m$, the vector space $M^{(p)} := \langle e_{i,k} | i \in Q_0, 1 \le k \le p\rangle$ is a subrepresentation of M isomorphic to $B_*(\lambda, p)$.
(b) There exists a nilpotent endomorphism ψ of M such that $\psi(e_{i1}) = 0$ and $\psi(e_{ik}) = e_{i,k-1}$ for all $1 < k \le m$ and all $i \in Q_0$.

The quiver S is of type $A_{|\mathbf{c}|}$ such that $S_0 = \{1, \ldots, |\mathbf{c}|\}$ and $S_1 = \{s_1, \ldots, s_{|\mathbf{c}|-1}\}$.

Let $(0 \subseteq U \subseteq W \subseteq M) \in X$. Then $U \cong F_*(\mathbb{1}_S)^n$. Using the Gauss algorithm, there exists a unique tuple $\mathbf{j}(U) = (1 \leq j_1 < j_2 < \ldots < j_n \leq m)$ as in Eq. 22 and unique $\lambda_{kj}(U) \in \mathbb{C}$ such that the vector space U is spanned by

$$\left(M^{\varepsilon_1}_{F_1(s_1)} \ldots M^{\varepsilon_q}_{F_1(s_q)}\right)^{-1} \left(e_{F_0(1),j_p} + \sum\nolimits_{j=1,\, j \neq j_k \forall k}^{j_p - 1} \lambda_{pj}(U) e_{F_0(1),j}\right)$$

with $1 \leq p \leq n$ and $0 \leq q < |\mathbf{c}|$. This is well-defined since all linear maps M_i are isomorphisms. The variety X can be decomposed into a disjoint union of locally closed subsets

$$X_k := \left\{(U \subseteq W) \in X \middle| \left(\mathbf{j}(U)\right)_1 = k\right\}.$$

For each k let

$$X_k^0 := \left\{(U \subseteq W) \in X_k \middle| \lambda_{1j}(U) = 0 \,\forall j\right\},$$

a locally closed subset of X. Equation 23 defines again an algebraic morphism $\pi : X_k \to X_k^0$ with affine fibres.

For each k there exists a $U_k \subseteq M$ such that $U_k \cong F_*(\mathbb{1}_S)$, $U_k \subseteq U$ for all $(U \subseteq W) \in X_k^0$ and $M/U_k \cong M^{(m-k)} \oplus I_k$ with an indecomposable module I_k as in the lemma. Since $|\mathbf{c}| \geq |\mathbf{d}^{(i)}|$ for all i and $F_*(\mathbb{1}_S)$ is preprojective, all sequences of the form

$$0 \to F_*(\mathbb{1}_S) \to \pi^{-1}(W) \to F_*(\mathbb{1}_S)^{n-1} \oplus \bigoplus\nolimits_i F_*^{(i)}(\mathbb{1}_{S^{(i)}}) \to 0$$

with a projection $\pi : M \to M/F_*(\mathbb{1}_S)$ and a submodule $W \subseteq M/F_*(\mathbb{1}_S)$ split. Let

$$\overline{X_k^0} := \Big\{\left(U \subseteq W \subseteq M^{(m-k)}\right), \left(W' \subseteq I_k\right) \Big| \\ \mathbb{1}_{\mathbf{F}(n-1)}(U) = \mathbb{1}_{\mathbf{F}}(W/U \oplus W') = \mathbb{1}_{\mathbf{F}'}\left(M^{(m-k)}/W \oplus I_k/W'\right) = 1\Big\}.$$

Using an R-grading, we conclude, as in the proof of Proposition 3.21(a), that

$$\chi\left(X_k^0\right) = \chi\left(\overline{X_k^0}\right) = \left(\left(\mathbb{1}_{\mathbf{F}(n-1)} \otimes 1\right) * \Delta\left(\mathbb{1}_{\mathbf{F}}\right) * \Delta\left(\mathbb{1}_{\mathbf{F}'}\right)\right)\left(M^{(m-k)}, I_k\right)$$

and by $\chi(X) = \sum_{k \in \mathbb{N}} \chi\left(X_k^0\right)$ this proves the lemma. □

10 String Algebras

10.1 Proof of Corollary 3.23

If A is a string algebra, then every indecomposable A-module is a string or a band module. So Corollary 3.23 follows directly from Proposition 3.18, Lemma 3.15 and Example 3.16. □

10.2 Proof of Eq. 14

This can be proven by iterated use of Proposition 3.21(b). We give here an alternative proof. By Section 7, it is enough to show Eq. 14 for a string module $F_*(\mathbb{1}_S)$ with dimension vector (m, m). Using Theorem 1.3, this can be computed by counting all orderings of the strings in $\mathbf{F}$ and in $\mathbf{F}'$. □

10.3 Proof of Corollary 3.25

We use an induction over the dimension vectors of Q. Let $\mathbf{d}$ be a dimension vector. Then the set

$$H_{\mathbf{d}} := \left\{\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}} \middle| \exists M \in \mathrm{rep}(Q), \mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}(M) \neq 0, \mathbf{dim}(M) = \mathbf{d}\right\}$$

is finite and the function $\mathbb{1}_{\mathbf{d}}$ is the sum of all functions in $H_{\mathbf{d}}$.

It remains to show that each product $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}} * \mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'} \in \mathcal{H}_{\mathbf{d}}(A)$ is a linear combination of functions in $H_{\mathbf{d}}$. Using Lemma 3.15 and Example 3.16, we have to check that for all bands B and $m \in \mathbb{N}$ the product $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}} * \mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}(B_*(\lambda, m))$ is independent of $\lambda \in \mathbb{C}^*$. This is clear by Proposition 3.21(b) and an induction argument. □

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