Algebr Represent Theor (2012) 15:675–696
DOI 10.1007/s10468-010-9258-y

Combinatorial Hopf Algebras and Towers of Algebras—Dimension, Quantization and Functorality

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Received: 17 October 2010 / Accepted: 25 October 2010 / Published online: 24 November 2010

Abstract Bergeron and Li have introduced a set of axioms which guarantee that the Grothendieck groups of a tower of algebras $\bigoplus_{n\geq 0} A_n$ can be a pair of graded dual Hopf algebras. Hivert and Nzeutchap, and independently Lam and Shimozono constructed dual graded graphs from primitive elements in Hopf algebras. In this paper we apply the composition of these constructions to towers of algebras. We show that if a tower $\bigoplus_{n\geq 0} A_n$ gives rise to a pair of graded dual Hopf algebras, then $\dim(A_n) = r^n n!$ where $r = \dim(A_1)$. In the case of $r = 1$ we give a conjectural classification. We then investigate a quantum version of the main theorem. We conclude with some open problems and a categorification of these constructions.

Presented by Peter Littelmann.

N. Bergeron is supported in part by CRC and NSERC. T. Lam is partially supported by NSF grants DMS-0600677 and DMS-0652641. H. Li is supported in part by CRC, NSERC and NSF grant DMS-0652641.

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Keywords Graded algebra · Hopf algebra · Grothendieck group · Dual graded graphs

Mathematics Subject Classifications (2010) Hopf algebras 16W30; Grothendieck groups 18F30

1 Introduction

This paper[1] is concerned with the interplay between towers of associative algebras, pairs of graded dual combinatorial Hopf algebras, and dual graded graphs. Our point of departure is the study of the composition of two constructions: (1) the construction of dual Hopf algebras from towers of algebras satisfying some axioms, due to Bergeron and Li [6]; and (2) the construction of dual graded graphs from primitive elements in dual Hopf algebras, discovered independently by Hivert and Nzeutchap [15], and Lam and Shimozono:

$$\text{tower of algebras} \longrightarrow \text{combinatorial Hopf algebra} \longrightarrow \text{dual graded graph} \qquad (1.1)$$

1.1 From Towers of Algebras to Combinatorial Hopf Algebras

A *tower* $A = \bigoplus_{n\geq 0} A_n$ is a direct sum of associative algebras A_n over $\mathbb{C}$, equipped with an external multiplication $\rho_{m,n} : A_m \otimes A_n \to A_{m+n}$ satisfying a number of axioms (see Section 2). A *combinatorial Hopf algebra* is a graded, connected Hopf algebra with a distinguished basis such that all product and coproduct structure constants are non-negative. The Grothendieck groups $G(A)$ and $K(A)$ of a certain tower A give rise to a pair of graded dual combinatorial Hopf algebras: the product and coproduct structures come from induction and restriction of modules, while the distinguished bases come from the classes of simple modules, and of indecomposable projective modules. Our notion of combinatorial Hopf algebra is related to, but different from the one in [1] (see Section 8).

1.2 From Combinatorial Hopf Algebras to Dual Graded Graphs

The notion of a pair (Γ, Γ') of *dual graded graphs* (see Section 3) was introduced by Fomin [11] (see also [27]) to encode the enumerative properties of the Robinson-Schensted correspondence and its generalizations. A pair of graded graphs is equipped with linear operators D and U satisfying the relation $DU - UD = r\mathrm{Id}$, where r is a non-negative integer, and Id is the identity. The second arrow of Eq. 1.1 is obtained by using (some of the) structure constants of a combinatorial Hopf algebra as edge multiplicities for a graph. This construction depends on the choices of primitive elements, but for combinatorial Hopf algebras arising from towers of algebras there is a natural choice (see Eq. 4.1).

[1] A summary of an earlier shorter version of this paper appeared in [5].

1.3 The Dimension Theorem

Using the natural choice (see Eq. 4.1) we relate the dimensions of simple modules and indecomposable projective modules of A_n to the numbers of some paths in the dual graded graphs via Eq. 1.1. We obtain the following result, that towers of algebras giving rise to combinatorial Hopf algebras are much more rigid than they first appear.

Theorem 1.1 *Let $A = \bigoplus_{n\geq 0} A_n$ be a tower of algebras. If the Grothendieck groups of A form a pair of graded dual Hopf algebras, then* $\dim(A_n) = r^n n!$ *where* $r = \dim(A_1)$.

The notion of "forming a pair of graded dual Hopf algebras" is made precise in Section 2. The number $r^n n!$ counts certain paths in a pair of dual graded graphs. It is also the number of permutation matrices with entries in a finite group of size r.

1.4 Symmetric Groups, Symmetric Functions and Young's Graph

The fundamental example of all three classes of objects arises from the representation theory of symmetric groups and the theory of symmetric functions. The symmetric group algebras give rise to a tower $\bigoplus_{n\geq 0} \mathbb{C}\mathfrak{S}_n$, the Grothendieck groups of which form a graded self-dual combinatorial Hopf algebra (see [8] and [29]), which can be identified with the ring *Sym* of symmetric functions. The Hopf structure of *Sym* was studied by Geissinger [12]. The corresponding dual graded graph is the (self-dual) *Young's graph* of partitions, which is the motivating example [11, 27] of dual graded graphs. Indeed Young's graph can be obtained from Young's branching rule for the symmetric group, or equivalently the Pieri rule for symmetric functions.

1.5 Towards a Classification?

In recent years it has been shown that other graded dual Hopf algebras can be obtained from towers of algebras. In [25] Malvenuto and Reutenauer establish the duality between the Hopf algebra *NSym* of noncommutative symmetric functions and the Hopf algebra *QSym* of quasi-symmetric functions. Krob and Thibon [18] then showed that this duality can be interpreted as the duality of the Grothendieck groups associated with $\bigoplus_{n\geq 0} H_n(0)$ the tower of Hecke algebras at zero. For more examples, see [4, 13, 26].

It is very tempting, as suggested by J. Y. Thibon, to classify all combinatorial Hopf algebras which arise as Grothendieck groups associated with towers of algebras. The list of axioms given by Bergeron and Li in [6] guarantees that the Grothendieck groups of a tower of algebras form a pair of graded dual Hopf algebras. This list of axioms is far from a classification.

The rigidity proved in Theorem 1.1 suggests however that there may be a classification theorem for towers of algebras which give rise to combinatorial Hopf algebras. For the case $r = 1$, we give a conjectural classification in Section 6, which includes symmetric group algebras, 0-Hecke algebras, nilCoxeter algebras (studied by Khovanov [17]) and the infinite families of Hecke algebras at a root of unity (see [21]).

In general, Theorem 1.1 suggests that to perform the inverse constructions of the arrows in Eq. 1.1 one should study algebras related to symmetric groups (or wreath products of symmetric groups). There are many combinatorial Hopf algebras for which one may attempt to perform the inverse construction, but there are even more dual graded graphs. The general construction of [20] produces dual graded graphs from Bruhat orders of Weyl groups of Kac-Moody algebras and it is unclear whether there are Hopf algebras, or towers of algebras giving rise to these graphs.

1.6 Quantization and Categorification

Our work should also be put into the context of the general notion of categorification: Theorem 1.1 provides a condition on when a Hopf algebra can be categorified by the Grothendieck groups of a tower of algebras. The idea of categorifying Hopf algebras, in particular quantum groups, has been around for some time. For example, Crane and Frenkel [9] introduced a notion of a *Hopf-category* in the context of four-dimensional topological quantum field theory. We remark that quantum groups are not graded, and so do not fit into the class of Hopf algebras that we consider.

In the last two sections, we turn our attention to some generalizations. In Section 7, we give a quantum version of Theorem 1.1: replacing towers of algebras with filtered towers of algebras, Hopf algebras with q-twisted Hopf algebras [23], and dual graded graphs with quantized dual graded graphs [19].

In Section 8, we relate our work to the combinatorial Hopf algebras of Aguiar, Bergeron and Sottile [1]. We also discuss different notions of towers of algebras, and describe how to categorify the constructions in Eq. 1.1, in particular making the arrows into functors.

2 From Towers of Algebras to Combinatorial Hopf Algebras

We recall here the work of Bergeron and Li [6] on towers of algebras. For B an arbitrary algebra we denote by ${}_B\mathrm{mod}$, the category of all finitely generated left B-modules, and by $\mathcal{P}(B)$, the category of all finitely generated projective left B-modules. For some category $\mathcal{C}$ of left B-modules (${}_B\mathrm{mod}$ or $\mathcal{P}(B)$) let $\mathbf{F}$ be the free abelian group generated by the symbols (M), one for each isomorphism class of modules M in $\mathcal{C}$. Let $\mathbf{F_0}$ be the subgroup of $\mathbf{F}$ generated by all expressions $(M)-(L)-(N)$ one for each exact sequence

$$0 \to L \to M \to N \to 0$$

in $\mathcal{C}$. The *Grothendieck group* $\mathcal{K}_0(\mathcal{C})$ of the category $\mathcal{C}$ is defined by the quotient $\mathbf{F}/\mathbf{F_0}$, an abelian additive group. For $M \in \mathcal{C}$, we denote by $[M]$ its image in $\mathcal{K}_0(\mathcal{C})$. We then set

$$G_0(B) = \mathcal{K}_0({}_B\mathrm{mod}) \quad \text{and} \quad K_0(B) = \mathcal{K}_0(\mathcal{P}(B)).$$

For B a finite-dimensional algebra over a field K, let $\{V_1, \cdots, V_s\}$ be a complete list of nonisomorphic simple B-modules. The projective covers $\{P_1, \cdots, P_s\}$ of the simple modules V_i's is a complete list of nonisomorphic indecomposable projective B-modules. Then $G_0(B) = \bigoplus_{i=1}^{s} \mathbb{Z}[V_i]$ and $K_0(B) = \bigoplus_{i=1}^{s} \mathbb{Z}[P_i]$.

Let $\varphi\colon B \to A$ be an injection of algebras preserving unities, and let M be a (left) A-module and N a (left) B-module. The *induction* of N from B to A is $\mathrm{Ind}_B^A N = A \otimes_\varphi N$, the (left) A-module $A \otimes N$ modulo the relations $a \otimes bn \equiv a\varphi(b) \otimes n$, and the *restriction* of M from A to B is $\mathrm{Res}_B^A M = \mathrm{Hom}_A(A, M)$, the (left) B-module with the B-action defined by $b\, f(a) = f(a\varphi(b))$.

Let $A = \bigoplus_{n\geq 0} A_n$ be a graded algebra over $\mathbb{C}$ with multiplication $\rho\colon A \otimes A \to A$. Bergeron and Li studied five axioms for A (we refer to [6] for full details):

(1) For each $n \geq 0$, A_n is a finite-dimensional algebra by itself with (internal) multiplication $\mu_n\colon A_n \otimes A_n \to A_n$ and unit 1_n. $A_0 \cong \mathbb{C}$.
(2) The (external) multiplication $\rho_{m,n} : A_m \otimes A_n \to A_{m+n}$ is an injective homomorphism of algebras, for all m and n (sending $1_m \otimes 1_n$ to 1_{m+n}).
(3) A_{m+n} is a two-sided projective $A_m \otimes A_n$-module with the action defined by $a \cdot (b \otimes c) = a\rho_{m,n}(b \otimes c)$ and $(b \otimes c) \cdot a = \rho_{m,n}(b \otimes c)a$, for all $m, n \geq 0$, $a \in A_{m+n}$, $b \in A_m$, $c \in A_n$ and $m, n \geq 0$.
(4) A relation between the decomposition of A_{n+m} as a left $A_m \otimes A_n$-module and as a right $A_m \otimes A_n$-module holds.
(5) An analogue of Mackey's formula relating induction and restriction of modules holds.

We say here that $A = \bigoplus_{n\geq 0} A_n$ is a *tower of algebras* if it satisfies conditions (1)–(3).

Condition (1) guarantees that the Grothendieck groups $G(A) = \bigoplus_{n\geq 0} G_0(A_n)$ and $K(A) = \bigoplus_{n\geq 0} K_0(A_n)$ are graded connected. Conditions (2) and (3) ensure that induction and restriction are well defined on $G(A)$ and $K(A)$, defining a product and coproduct, as follows. For $[M] \in G_0(A_m)$ (or $K_0(A_m)$) and $[N] \in G_0(A_n)$ (or $K_0(A_n)$) we let

$$[M][N] = \left[\mathrm{Ind}_{A_m\otimes A_n}^{A_{m+n}} M \otimes N\right] \qquad \text{and} \qquad \Delta([N]) = \sum_{k+l=n} \left[\mathrm{Res}_{A_k\otimes A_l}^{A_n} N\right].$$

The pairing between $K(A)$ and $G(A)$ is given by $\langle\ ,\ \rangle : K(A) \times G(A) \to \mathbb{Z}$ where

$$\langle [P], [M]\rangle = \begin{cases} \dim_K\big(\mathrm{Hom}_{A_n}(P, M)\big) & \text{if } [P] \in K_0(A_n) \text{ and } [M] \in G_0(A_n),\\ 0 & \text{otherwise.}\end{cases}$$

Thus with (only) conditions (1), (2), and (3), $G(A)$ and $K(A)$ are dual free $\mathbb{Z}$-modules both endowed with a product and coproduct.

Theorem 2.1 (Bergeron and Li [6]) *If a graded algebra $A = \bigoplus_{n\geq 0} A_n$ over $\mathbb{C}$ satisfies conditions* (1)–(5)*, then $G(A)$ and $K(A)$ are graded dual Hopf algebras.*

In particular Theorem 1.1 applies to graded algebras which satisfy conditions (1)–(5). Note that the dual Hopf algebras $G(A)$ and $K(A)$ come with distinguished bases consisting of the isomorphism classes of simple and indecomposable projective modules.

3 From Combinatorial Hopf Algebras to Dual Graded Graphs

This section recounts work of Fomin [11], Hivert and Nzeutchap [15], and Lam and Shimozono. A *graded graph* $\Gamma = (V, E, h, m)$ consists of a set of vertices V, a set of (directed) edges $E \subset V \times V$, a height function $h : V \to \{0, 1, \ldots\}$ and an edge multiplicity function $m : V \times V \to \{0, 1, \ldots\}$. If $(v, u) \in E$ is an edge, then we must have $h(u) = h(v) + 1$. The multiplicity function determines the edge set: $(v, u) \in E$ if and only if $m(v, u) \neq 0$. We assume always that there is a single vertex v_0 of height 0.

Let $\mathbb{Z}V = \bigoplus_{v \in V} \mathbb{Z} \cdot v$ be the free $\mathbb{Z}$-module generated by the vertex set. Given a graded graph $\Gamma = (V, E, h, m)$ we define up and down operators $U, D \colon \mathbb{Z}V \to \mathbb{Z}V$ by

$$U_\Gamma(v) = \sum_{u \in V} m(v, u)\, u \quad D_\Gamma(v) = \sum_{u \in V} m(u, v)\, u$$

and extending by linearity over $\mathbb{Z}$. We will assume that Γ is locally-finite, so that these operators are well defined. A pair (Γ, Γ') of graded graphs with the same vertex set V and height function h is called *dual* with *differential coefficient* r if

$$D_{\Gamma'} U_\Gamma - U_\Gamma D_{\Gamma'} = r\, \mathrm{Id}.$$

We shall need the following result of Fomin. For a graded graph Γ, let f_Γ^v denote the number of paths from v_0 to v, where for two vertices $w, u \in V$, we think that there are $m(w, u)$ edges connecting w to u.

Theorem 3.1 (Fomin [11]) *Let (Γ, Γ') be a pair of dual graded graphs with differential coefficient r. Then*

$$r^n n! = \sum_{v \colon h(v)=n} f_\Gamma^v f_{\Gamma'}^v.$$

Let $H_\bullet = \bigoplus_{n \geq 0} H_n$ and $H^\bullet = \bigoplus_{n \geq 0} H^n$ be a pair of graded dual Hopf algebras over $\mathbb{Z}$ with respect to the pairing $\langle\, .\, ,\, .\, \rangle : H_\bullet \times H^\bullet \to \mathbb{Z}$. We assume that we are given dual sets of homogeneous free $\mathbb{Z}$-module generators $\{p_\lambda \in H_\bullet\}_{\lambda \in \Lambda}$ and $\{s_\lambda \in H^\bullet\}_{\lambda \in \Lambda}$, such that all structure constants are non-negative integers. We also assume that $\dim(H_i) = \dim(H^i) < \infty$ for each $i \geq 0$ and $\dim(H_0) = \dim(H^0) = 1$, so that H_0 and H^0 are spanned by distinguished elements the unit 1. Let us suppose we are given non-zero homogeneous elements $\alpha \in H_1$ and $\beta \in H^1$ of degree 1 such that αp_μ (resp. βs_μ) is a linear combination of $\{p_\lambda\}$ (resp. $\{s_\lambda\}$) with non-negative integer coefficients for any $\mu \in \Lambda$.

We now define a graded graph $\Gamma(\beta) = (V, E, h, m)$ where $V = \{s_\lambda\}_{\lambda \in \Lambda}$ and $h \colon V \to \mathbb{Z}$ is defined by $h(s_\lambda) = \deg(s_\lambda)$. The map $m \colon V \times V \to \mathbb{Z}$ is defined by

$$m(s_\lambda, s_\mu) = \langle p_\mu, \beta s_\lambda \rangle = \langle \Delta(p_\mu), \beta \otimes s_\lambda \rangle$$

and E is determined by m. The grading of $\Gamma(\beta)$ follows from the assumption that β has degree 1. Similarly, we define a graded graph $\Gamma'(\alpha) = (V', E', h', m')$ where $V' = V$, $h' = h$, and

$$m'(s_\lambda, s_\mu) = \langle \alpha\, p_\lambda, s_\mu \rangle = \langle \alpha \otimes p_\lambda, \Delta(s_\mu) \rangle.$$

The following theorem is due independently to Hivert and Nzeutchap [15] and Lam and Shimozono (unpublished).

Theorem 3.2 *The graded graphs $\Gamma = \Gamma(\beta)$ and $\Gamma' = \Gamma'(\alpha)$ form a pair of dual graded graphs with differential coefficient $\langle \alpha, \beta \rangle$.*

Proof We identify $\mathbb{Z}V$ with $H^\bullet$ and note that $U_\Gamma(x) = \beta\, x$ where $x \in H^\bullet$ and we use the product in $H^\bullet$. Also,

$$D_{\Gamma'}(x) = \sum_{\mu \in \Lambda} \langle \alpha \otimes p_\mu, \Delta x \rangle\, s_\mu = \sum \langle \alpha, x^{(1)} \rangle\, x^{(2)}.$$

where $\Delta x = \sum x^{(1)} \otimes x^{(2)}$. Now observe that by our hypotheses on the degree of α and β they are primitive elements: $\Delta\alpha = 1 \otimes \alpha + \alpha \otimes 1$ and $\Delta\beta = 1 \otimes \beta + \beta \otimes 1$. We first calculate

$$\langle \alpha, \beta\, x \rangle = \langle \Delta\alpha, \beta \otimes x \rangle = \langle 1, \beta \rangle \langle \alpha, x \rangle + \langle \alpha, \beta \rangle \langle 1, x \rangle = \langle \alpha, \beta \rangle \langle 1, x \rangle$$

and then compute

$$\begin{aligned} D_{\Gamma'} U_\Gamma(x) &= D_{\Gamma'}(\beta x) \\ &= \sum \left(\langle \alpha, \beta\, x^{(1)} \rangle\, x^{(2)} + \langle \alpha, x^{(1)} \rangle\, \beta\, x^{(2)} \right) \\ &= \langle \alpha, \beta \rangle x + U_\Gamma D_{\Gamma'}(x) \end{aligned}$$

where to obtain $\langle \alpha, \beta \rangle x$ in the last line we use $\Delta x = 1 \otimes x +$ terms of other degrees. □

4 Proof of Theorem 1.1

We are given a graded algebra $A = \bigoplus_{n \geq 0} A_n$ over $\mathbb{C}$ with multiplication ρ satisfying conditions (1)–(3). Moreover we assume that the two Grothendieck groups $G(A)$ and $K(A)$ form a pair of graded dual Hopf algebras as in Section 2. Under these assumptions we show that

$$\dim(A_n) = r^n n!$$

where $r = \dim(A_1)$.

Let $H^\bullet = G(A)$ and $H_\bullet = K(A)$. Let $\{s_1^{(1)} = [S_1^{(1)}], \ldots, s_t^{(1)} = [S_t^{(1)}]\}$ and $\{p_1^{(1)} = [P_1^{(1)}], \ldots, p_t^{(1)} = [P_t^{(1)}]\}$ denote the isomorphism classes of simple and indecomposable projective A_1-modules, so that $H^1 = \bigoplus_{i=1}^t \mathbb{Z}s_i^{(1)}$ and $H_1 = \bigoplus_{i=1}^t \mathbb{Z}p_i^{(1)}$. Define $a_i = \dim(S_i^{(1)})$ and $b_i = \dim(P_i^{(1)})$ for $1 \leq i \leq t$. We set for the remainder of this paper

$$\alpha = \sum_{i=1}^t a_i p_i^{(1)} \in H_1 \qquad \text{and} \qquad \beta = \sum_{i=1}^t b_i s_i^{(1)} \in H^1. \tag{4.1}$$

Since $A_0 \cong \mathbb{C}$, we let $s_1^{(0)}$ (respectively, $p_1^{(0)}$) be the unique simple (respectively, indecomposable projective) module representative in H^0 (respectively, H_0). Similarly, let $\{s_i^{(n)} = [S_i^{(n)}]\}$ be all isomorphism classes of simple A_n-modules and $\{p_i^{(n)} = [P_i^{(n)}]\}$ be all isomorphism classes of indecomposable projective A_n-modules. The sets $\bigcup_{n \geq 0}\{s_i^{(n)}\}$ and $\bigcup_{n \geq 0}\{p_i^{(n)}\}$ form dual free $\mathbb{Z}$-module bases of $H^\bullet$ and $H_\bullet$.

Now define $\Gamma = \Gamma(\beta)$ and $\Gamma' = \Gamma'(\alpha)$ as in Section 3.

Lemma 4.1 *The numbers of paths from $s_1^{(0)}$ to $s_j^{(n)}$ in Γ and Γ' are*

$$f_\Gamma^{s_j^{(n)}} = \dim P_j^{(n)} \quad \text{and} \quad f_{\Gamma'}^{s_j^{(n)}} = \dim S_j^{(n)}.$$

Proof From the definition of β in Eq. 4.1,

$$m(s_i^{(n-1)}, s_j^{(n)}) = \sum_{l=1}^{t} b_l c_l,$$

where c_l is the number of copies of the indecomposable projective module $P_l^{(1)} \otimes P_i^{(n-1)}$ as a summand in $\mathrm{Res}^{A_n}_{A_1 \otimes A_{n-1}} P_j^{(n)}$. Note that $s_1^{(0)}$ is the unit of $H^\bullet$ and $m(s_1^{(0)}, s_i^{(1)}) = b_i = \dim P_i^{(1)}$ for all $1 \le i \le t$. The dimension of an indecomposable projective module $P_j^{(n)}$ is given by

$$\dim P_j^{(n)} = \sum_{i,l} c_l \dim \left(P_l^{(1)} \otimes P_i^{(n-1)}\right) = \sum_i m(s_i^{(n-1)}, s_j^{(n)}) \dim P_i^{(n-1)}.$$

By induction on n, we deduce that $\dim P_j^{(n)}$ is the number of paths from $s_1^{(0)}$ to $s_j^{(n)}$ in Γ. The claim for Γ' is similar. □

For any finite dimensional algebra B let $\{S_\lambda\}_\lambda$ be a complete set of simple B-modules. For each λ let P_λ be the projective cover of S_λ. It is well known (see [10]) that we can find minimal idempotents $\{e_i\}$ such that $B = \bigoplus Be_i$ where each Be_i is isomorphic to a P_λ. Moreover, the quotient of B by its radical shows that the multiplicity of P_λ in B is equal to $\dim S_\lambda$. This implies the following lemma.

Lemma 4.2 *Let B be a finite dimensional algebra and $\{S_\lambda\}_\lambda$ be a complete set of simple B-modules.*

$$\dim B = \sum_\lambda (\dim P_\lambda)(\dim S_\lambda),$$

where P_λ is the projective cover of S_λ.

By Lemma 4.2, $r = \sum_{i=1}^{t} a_i b_i = \langle \alpha, \beta \rangle$. By Theorem 3.2 we apply Theorem 3.1 to (Γ, Γ'). Using Lemmas 4.2 and 4.1, Theorem 3.1 says

$$\dim(A_n) = \sum_i (\dim P_i^{(n)})(\dim S_i^{(n)}) = \sum_i f_\Gamma^{s_i^{(n)}} f_{\Gamma'}^{s_i^{(n)}} = r^n n!.$$

Remark 4.3 If the tower consists of semisimple algebras A_i, then $\Gamma = \Gamma'$. So we obtain a *self-dual graded graph* Γ. In this case the graph would be a weighted version of a *differential poset* in the sense of Stanley [27]. If furthermore the branching of irreducible modules from A_n to $A_1 \otimes A_{n-1}$ is multiplicity free, then we get a true differential poset.

Remark 4.4 The Hopf algebras $H^\bullet$ and $H_\bullet$ are not in general either commutative or co-commutative. Thus in the definitions of Section 3 we could have obtained a different pair of dual graded graphs by setting $m(s_\lambda, s_\mu) = \langle p_\mu, s_\lambda\, \beta\rangle$ or $m'(s_\lambda, s_\mu) = \langle p_\lambda\, \alpha, s_\mu\rangle$.

5 Examples

In this section we explain four examples of the constructions in Sections 2–4, all with $r = \dim(A_1) = 1$.

Tower of algebras A	$K(A)$	$G(A)$	Γ	Γ'
$\mathfrak{S}_n$-group algebras	*Sym*	*Sym*	Young's graph	Young's graph
NilCoxeter algebras	$\mathbb{Z}[x]$	$\mathbb{Z}[x, x^2/2, x^3/3!, \ldots]$	Weighted chain	Chain
0-Hecke algebras	*NSym*	*QSym*	BinWord graph	Lifted binary tree
Hecke algebras at $\sqrt[r]{1}$	$(\mathcal{J}^{(r)})^\perp$	$Sym/\mathcal{J}^{(r)}$	???	???

5.1 Symmetric Group Algebras

Let $A = \bigoplus_{n\geq 0} \mathbb{C}\mathfrak{S}_n$ be the tower of symmetric group algebras. Since $\mathbb{C}\mathfrak{S}_n$ is semisimple, $K(A) = G(A)$. Indeed both $K(A)$ and $G(A)$ can be identified with the (self-dual) Hopf algebra *Sym* of symmetric functions, and the classes of the simple modules and the indecomposable projective modules are identified with the Schur functions s_λ. The corresponding self-dual graded graph is Young's lattice of partitions. We refer the reader to [29] for further details of this well-known example.

5.2 NilCoxeter Algebras

The *nilCoxeter algebra* N_n is the unital algebra over $\mathbb{C}$ generated by $T_1, T_2, \ldots, T_{n-1}$ with relations

$$\begin{aligned} T_i^2 &= 0 \\ T_iT_j &= T_jT_i \qquad\qquad \text{for } |i-j| > 1 \\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1}. \end{aligned}$$

It has a basis $\{T_w \mid w \in \mathfrak{S}_n\}$ labeled by permutations of $\{1, 2, \ldots, n\}$, where $T_w = T_{i_1}T_{i_2}\cdots T_{i_\ell}$ if $w = s_{i_1}s_{i_2}\cdots s_{i_\ell}$ is a reduced factorization of w. An explicit realization of this algebra is obtained by *divided difference operators*. The external multiplication $N_i \otimes N_j \to N_{i+j}$ is defined in the same way as for symmetric group algebras. The representation theory of the tower $N = \bigoplus_{n\geq 0} N_n$ was worked out by Khovanov [17].

The unique simple module (up to isomorphism) S_n of N_n has dimension 1, with projective cover $P_n = N_n$. Then $K(N) = \mathbb{Z}[x]$ with $[P_n] = x^n$ and $G(N) = \mathbb{Z}[x, x^2/2, x^3/3!, \ldots]$ with $[S_n] = x^n/n!$. Here the algebra $\mathbb{Z}[x, x^2/2, x^3/3!, \ldots]$ is the free divided powers algebra on one generator over $\mathbb{Z}$. The coproduct is given by $\Delta(x) = 1 \otimes x + x \otimes 1$ for both $K(N)$ and $G(N)$. The graph Γ' is a chain, with vertices $\{0, 1, 2, \ldots\}$ and multiplicities $m(i, i+1) = 1$ for $i = 0, 1, 2, \ldots$. The graph Γ is a weighted chain, with vertices $\{0, 1, 2, \ldots\}$ and multiplicities $m(i, i+1) = i + 1$. The "up" operators in these graphs correspond to product by x in $\mathbb{Z}[x]$ and $\mathbb{Z}[x, x^2/2, x^3/3!, \ldots]$. This pair of dual graded graphs occurred as Example 2.2.1 in [11].

5.3 0-Hecke Algebras

The *0-Hecke algebra* $H_n(0)$ is the unital algebra over $\mathbb{C}$ generated by $T_1, T_2, \ldots, T_{n-1}$ with relations

$$\begin{aligned} T_i^2 &= -T_i \\ T_iT_j &= T_jT_i \qquad \text{for } |i-j|>1 \\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1}. \end{aligned}$$

The 0-Hecke algebra has a basis $\{T_w \mid w \in \mathfrak{S}_n\}$ and an external multiplication $H_i(0) \otimes H_j(0) \to H_{i+j}(0)$ defined in a manner similar to the nilCoxeter algebras. An explicit realization of $H_n(0)$ is obtained by the *isobaric divided difference operators*. The representation theory of the tower $H(0) = \bigoplus_{n\geq 0} H_n(0)$ was worked out by Krob and Thibon [18].

A composition $I = (i_1, i_2, \ldots, i_r)$ of n is a finite sequence of positive integers summing to n. The algebra $H_n(0)$ is not semi-simple and it has 2^{n-1} non-isomorphic simple modules S_I all of dimension 1, as I ranges over the compositions of n. The projective cover P_I of S_I has dimension $\dim(P_I) = \{w \in \mathfrak{S}_n \mid \mathrm{Des}(w) = \mathrm{Des}(I) := \{i_1, i_1+i_2, \ldots, i_1+\cdots+i_{r-1}\}$. It is known that $G(H(0)) = QSym$, the Hopf algebra of quasi-symmetric functions, and $K(H(0)) = NSym$, the Hopf algebra of noncommutative symmetric functions. The class of $[S_I]$ in $QSym$ is given by the fundamental quasi-symmetric function $F_I \in QSym$. The class of $[P_I]$ in $NSym$ is given by the ribbon Schur function $R_I \in NSym$.

The graph Γ' is an infinite binary tree with vertices of height n identified with compositions of n. There are edges (with multiplicity 1) joining the composition $(i_1, i_2, \ldots, i_r)$ with the compositions $(1, i_1, i_2, \ldots, i_r)$ and with $(i_1+1, i_2, \ldots, i_r)$. The graph Γ has edges (multiplicity 1) joining $(i_1, i_2, \ldots, i_r)$ with

$$\{(i_1, \ldots, i_{j-1}, i_j+1, i_{j+1}, \ldots, i_r), (i_1, \ldots, i_{j-1}, k+1, i_j-k, i_{j+1}, \ldots, i_r)\}$$

for each $j = 1, 2, \ldots, r$ and $k = 0, \ldots, i_j - 1$. We reproduce these graphs in Fig. 1 of Section 7.5 (the edge labels should be ignored for now). This pair of dual graded graphs occurred as Example 2.3.6 in [11].

5.4 Hecke Algebras at Roots of Unity

Let $v \in \mathbb{C}$. The Hecke algebra $H_n(v)$ is generated by $T_1, T_2, \ldots, T_{n-1}$ with relations

$$\begin{aligned} T_i^2 &= (v-1)T_i + v \\ T_iT_j &= T_jT_i \qquad \text{for } |i-j|>1 \\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1}. \end{aligned}$$

The Hecke algebra has a basis $\{T_w \mid w \in \mathfrak{S}_n\}$ and an external multiplication $H_i(v) \otimes H_j(v) \to H_{i+j}(v)$ defined in a manner similar to the nilCoxeter algebras. If $v = 0$, then we recover the 0-Hecke algebra $H_n(0)$. If $v = 1$, then we recover the symmetric groups algebras. If v is neither 0 nor a root of unity, then the tower

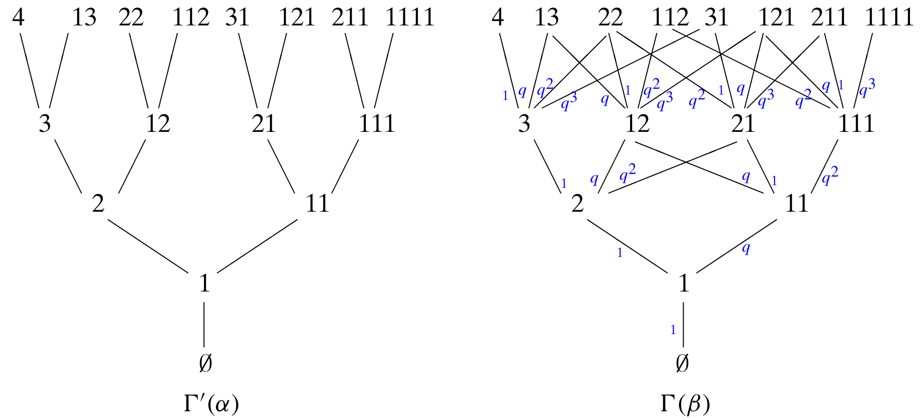

Fig. 1 Quantizations of the dual graded graphs for the 0-Hecke algebra

$H(v) = \bigoplus_{n\geq 0} H_n(v)$ has representation theory identical to that of $\bigoplus_{n\geq 0} \mathbb{C}\mathfrak{S}_n$. In this case we say that v is generic.

We now let $v = \zeta$ be a primitive r-th root of unity, and let $H(\zeta) = \bigoplus_{n\geq 0} H_n(\zeta)$ denote the corresponding tower of algebras. The representation theory of this infinite family of towers of algebras is not completely understood. We refer the reader to [21] for the following discussion. Let $\mathcal{J}^{(r)} \subset Sym$ be the ideal in Sym generated by the power symmetric functions $p_r, p_{2r}, p_{3r}, \ldots$. Then the graded dual Hopf algebras $G(H(\zeta)) = Sym/\mathcal{J}^{(r)}$ and $K(H(\zeta)) = (\mathcal{J}^{(r)})^{\perp}$, where $(\mathcal{J}^{(r)})^{\perp} \subset Sym$ is the set of elements annihilated by $\mathcal{J}^{(r)}$ under the usual pairing of Sym with itself. Ariki [3], proving a conjecture from [21], showed that the symmetric functions representing the classes of the simple modules, or the projective indecomposable modules, can be expressed in terms of Schur functions via the (lower and upper) global bases at $q = 1$ of the Fock space representation of $U_q(\hat{sl}_r)$.

The graded graphs Γ and Γ' are not known explicitly to our knowledge, though they have been the subject of much recent work; see for example [7, 21]. In particular, these branching graphs are closely related to the crystal graphs of quantum affine algebras of type A. It follows from Theorem 3.2 that

Corollary 5.1 *The branching graph Γ for the simple modules, and the branching graph Γ' for the projective indecomposable modules of $H(\zeta)$ form a pair of dual graded graphs with differential coefficient $r = 1$.*

Remark 5.2 The case $r \geq 2$ is abundant. In particular, for $r = 2$ one can consider towers of super-algebras and super modules. This is how Sergeev [26] constructed the combinatorial Hopf algebra of Q-Schur functions from the tower of Sergeev algebras. This is also how Bergeron et al. [4] constructed the combinatorial Hopf algebra of Θ-peak quasisymmetric functions from the tower of Hecke–Clifford algebras. Theorem 1.1 also holds for towers of super-algebras; the proof is a direct adaptation of the one presented here.

6 Two Parameter Hecke Algebras and Conjectural Classification

Let $a, b \in \mathbb{C}$. Let $H_n(a, b)$ denote the *two-parameter Hecke algebra* with generators $T_1, T_2, \ldots, T_{n-1}$ and relations

$$\begin{aligned} T_i^2 &= aT_i + b \\ T_iT_j &= T_jT_i \qquad\qquad \text{for } |i-j|>1 \\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1}. \end{aligned}$$

Proposition 6.1 *The $\mathbb{C}$-algebra $H_n(a, b)$ is isomorphic to one of the following four families of algebras:*

(H1) *a Hecke algebra $H_n(v)$ at a generic* (*see Section* 5.4) *value of v, or*
(H2) *a Hecke algebra $H_n(\zeta)$ at a root of unity ζ, or*
(H3) *the 0-Hecke algebra $H_n(0)$ (when $a \neq 0$ but $b = 0$), or*
(H4) *the nilCoxeter algebra N_n (when $a = b = 0$).*

Proof If $(a, b) = (0, 0)$, then $H_n(a, b) = N_n$ the nilCoxeter algebra. Otherwise, we can find a non-zero $z \in \mathbb{C}$ satisfying

$$az = bz^2 - 1. \tag{6.1}$$

The elements $T_i' = zT_i$ then satisfy

$$(T_i')^2 = (q-1)T_i' + q$$

where $q = bz^2$. Note that the braid relation for the T_i implies the braid relation for the T_i'. Thus $H_n(a, b)$ is isomorphic to $H_n(bz^2)$.

If $b = 0$, then $H_n(a, 0)$ is isomorphic to the 0-Hecke algebra $H_n(0)$ and we are in Case (H3). Otherwise we are in Case (H1) or (H2). Note that if z and z' are the two roots of Eq. 6.1, then bz^2 is a r-th root of unity if and only if $b(z')^2$ is. This follows from the fact that $zz' = -1/b$. □

Note that the isomorphism of Proposition 6.1 is compatible with the external multiplication of the obvious construction of the tower $H(a, b) = \bigoplus_{n \geq 0} H_n(a, b)$. It thus follows that for any $a, b \in \mathbb{C}$, the tower $H(a, b)$ gives rise to one of the graded dual Hopf algebras, and dual graded graphs, discussed in Section 5.

Based on this and Theorem 1.1, we conjecture

Conjecture 6.2

(1) *(Weak version) Suppose A is a tower of algebras with* $\dim(A_1) = 1$, *giving rise to graded dual Hopf algebras $K(A)$ and $G(A)$. Then the pair $(K(A), G(A))$ is isomorphic, together with their distinguished bases* (*classes of simples and indecomposable projectives*), *to one of the examples in Section* 5.
(2) (*Strong version*) *Suppose A is a tower of algebras with* $\dim(A_1) = 1$, *giving rise to graded dual Hopf algebras. Then A is isomorphic to one of the towers $H(a, b)$.*

Zelevinsky [29] shows that a graded connected self-dual Hopf algebra H, with a self-dual basis $\{b_\lambda\}$ such that all product and coproduct structure constants are positive with respect to this basis, must be a tensor product of the Hopf algebra of symmetric functions, together with the tensor product of the Schur function basis. Thus Conjecture 6.2(1) holds when A is a tower of semisimple algebras.

7 Quantum Version

In this section, we describe a "quantum" version of our theorem. We replace Eq. 1.1 with

$$\text{tower of filtered algebras} \to q\text{-twisted Hopf algebra} \to q\text{-dual graded graph}$$

We shall not make the first arrow completely axiomatic here.

7.1 From Filtered Towers of Algebras to q-Twisted Hopf Algebras

We first recall the notion of a q-twisted Hopf algebra [14, 23]. Let H be a graded connected algebra over $\mathbb{Z}[q]$, equipped with an associative graded coproduct $\Delta : H \to H \otimes_{\mathbb{Z}[q]} H$. The formula for the *q-twisted product of tensors* is

$$(a \otimes b) \cdot_q (a' \otimes b') = q^{\deg(b)\cdot\deg(a')}(aa' \otimes bb').$$

We say that H is a *q-twisted Hopf algebra* if $\Delta(a) \cdot_q \Delta(b) = \Delta(ab)$ for every $a, b \in H$. The other structure maps (unit, counit, antipode) will not concern us here.

The notion of q-twisted Hopf algebras is a particular instance of *twisted* Hopf algebras (or Hopf monoids) for braided monoidal categories [16]. Indeed, a q-twisted Hopf algebra is a Hopf monoid in the category of graded vector spaces with braiding induced by $\tau_q : V \otimes W \to W \otimes V$ where $\tau_q(v \otimes w) = q^{nm} w \otimes v$ for v (resp. w) a homogeneous element of degree n (resp. m).

Now let $A = \bigoplus_{n \geq 0} A_n$ be a tower of algebras as in Section 2. We suppose that each A_n is equipped with a filtration $A_n = A_n^{(0)} \supset A_n^{(1)} \supset \cdots$ such that each $A_n^{(k)}$ is a left ideal in A_n. We call this a *filtered tower of algebras*. Let M be a left A_n-module and $M' \subset M$ be a subset. If $M = A_n \cdot M'$, then the sequence $M^{(0)} = A_n^{(0)} \cdot M' \supset M^{(1)} = A_n^{(1)} \cdot M' \supset M^{(2)} = A_n^{(2)} \cdot M' \supset \cdots$ is a filtration of M by left submodules of A_n. The *graded character* $[M]_q \in G_0(A) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$ is defined by

$$[M]_q = \sum_{i \geq 0} q^i \, [M^{(i)}/M^{(i+1)}].$$

Obviously $[M]_q$ depends on M' even though it is suppressed in the notation.

We now define a multiplication $*$ in $G(A)_q = G(A) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$. For $[M] \in G_0(A_m)$ and $[N] \in G_0(A_n)$ we let

$$[M] * [N] = \left[\mathrm{Ind}_{A_m \otimes A_n}^{A_{m+n}} M \otimes N\right]_q$$

with respect to the subset $M \otimes N \subset \mathrm{Ind}_{A_m \otimes A_n}^{A_{m+n}} M \otimes N$. We also equip $G(A)_q$ with the usual coproduct of $G(A)$, extended by linearity to $G(A)_q$. Assume:

(Q1) The multiplication $*$ is a well-defined associative product on $G(A)_q$.
(Q2) $G(A)_q$ is a q-twisted Hopf algebra.

(Q3) Graded characters and inductions are sufficiently compatible such that

$$\beta^{*n} = [A_n]_q = \sum_{i\geq 0} q^i \, [A_n^{(i)}/A_n^{(i+1)}]$$

where β is as in Section 4.

Remark 7.1 For example, (Q3) would follow from the more general compatibility equation

$$[M_1] * [M_2] * \cdots * [M_r] = \left[\mathrm{Ind}_{A_{m_1}\otimes\cdots\otimes A_{m_r}}^{A_{m_1+m_2+\cdots+m_r}} M_1 \otimes \cdots \otimes M_r\right]_q$$

and some assumption about the structure map $A_1 \otimes \cdots \otimes A_1 \to A_n$. We believe that this compatibility relation is the most natural one, but do not need such generality here.

For our purposes, the precise construction of $G(A)_q$ is not crucial, as long as (Q1)–(Q3) are satisfied. We say that a multiplication on $G(A) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$ quantizes $G(A)$ if it reduces to the usual product of $G(A)$ at $q = 1$. Let $[r] = 1 + q + \cdots + q^{r-1}$, and $[r]! = [r][r-1]\cdots[1]$ be the usual q-analogues.

Theorem 7.2 *Let A be a filtered tower of algebras, and suppose a quantization $G(A)_q$ of $G(A)$ exists, satisfying* (Q1)–(Q3) *above. Then*

$$\dim_q(A_n) = \sum_{i\geq 0} q^i \dim(A_n^{(i)}/A_n^{(i+1)}) = r^n[n]!$$

where $r = \dim(A_1)$.

Theorem 7.2 will be proved in Section 7.3 below.

7.2 From q-Twisted Hopf Algebras to Quantized Dual Graded Graphs

Quantized dual graded graphs are defined and studied in [19]. We now allow our graded graphs $\Gamma = (V, E, h, m)$ to have multiplicities taking values in $\mathbb{N}[q]$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$ (for some purposes $\mathbb{N}[q^{1/2}, q^{-1/2}]$ could also be considered). Making definitions analogous to those in Section 3, we say that (Γ, Γ') is a pair of *quantized dual graded graphs* with differential coefficient r, if the linear operators U_Γ and $D_{\Gamma'}$ satisfy

$$D_{\Gamma'} U_\Gamma - q\, U_\Gamma D_{\Gamma'} = r\mathrm{Id}.$$

For now we allow r to lie in $\mathbb{Z}[q]$, though in the end we have no need for such generality.

We define $f_\Gamma^v \in \mathbb{N}[q]$ as before. The following is the quantized analogue of Theorem 3.1.

Theorem 7.3 (Lam [19]) *Let* (Γ, Γ') *be a pair of quantized dual graded graphs with differential coefficient r. Then*

$$r^n[n]! = \sum_{v\,:\,h(v)=n} f_\Gamma^v f_{\Gamma'}^v.$$

We now generalize Theorem 3.2 to the quantized setting. We first make the general observation that the graded dual of a graded q-twisted Hopf algebra over $\mathbb{Z}[q]$ is again a graded q-twisted Hopf algebra.

Let $H_\bullet = \bigoplus_{n\geq 0} H_n$ and $H^\bullet = \bigoplus_{n\geq 0} H^n$ be graded dual q-twisted Hopf algebras over $\mathbb{Z}[q]$ with respect to the pairing $\langle\, .\, , .\, \rangle : H_\bullet \times H^\bullet \to \mathbb{Z}[q]$. We assume that we are given dual sets of homogeneous free $\mathbb{Z}[q]$-module generators $\{p_\lambda \in H_\bullet\}_{\lambda\in\Lambda}$ and $\{s_\lambda \in H^\bullet\}_{\lambda\in\Lambda}$, such that all structure constants lie in $\mathbb{N}[q]$. We also assume that $\dim(H_i) = \dim(H^i) < \infty$ for each $i \geq 0$ and $\dim(H_0) = \dim(H^0) = 1$, so that H_0 and H^0 are spanned by distinguished elements the unit 1. Let us suppose we are given non-zero homogeneous elements $\alpha \in H_1$ and $\beta \in H^1$ of degree 1 such that αp_μ (resp. βs_μ) is a linear combination of $\{p_\lambda\}$ (resp. $\{s_\lambda\}$) with $\mathbb{N}[q]$-coefficients for any $\mu \in \Lambda$.

We now define graded graphs $\Gamma(\beta)$ and $\Gamma'(\alpha)$ exactly as in Section 3. The following theorem generalizes Theorem 3.2.

Theorem 7.4 *The graded graphs $\Gamma = \Gamma(\beta)$ and $\Gamma' = \Gamma'(\alpha)$ form a pair of quantized dual graded graphs with differential coefficient $\langle\alpha, \beta\rangle$.*

Proof The proof is identical to that of Theorem 3.2 until the final calculation, which proceeds

$$\begin{aligned}
D_{\Gamma'} U_\Gamma(x) &= D_{\Gamma'}(\beta x) \\
&= \sum_{\mu\in\Lambda} \langle \alpha \otimes p_\mu, \Delta(\beta) ._q \Delta(x)\rangle s_\mu \\
&= \sum \langle \alpha \otimes p_\mu, \beta\, x^{(1)} \otimes x^{(2)} + q^{\deg(x^{(1)})} x^{(1)} \otimes \beta\, x^{(2)}\rangle s_\mu \\
&= \sum \left(\langle \alpha, \beta\, x^{(1)}\rangle\, x^{(2)} + q^{\deg(x^{(1)})}\, \langle \alpha, x^{(1)}\rangle\, \beta\, x^{(2)} \right) \\
&= \langle\alpha, \beta\rangle x + \sum_{\deg(x^{(1)})=1} q\, \beta\, \langle \alpha, x^{(1)}\rangle\, x^{(2)} \\
&= \langle\alpha, \beta\rangle x + q U_\Gamma D_{\Gamma'}(x).
\end{aligned}$$

□

7.3 Proof of Theorem 7.2

The proof is analogous to that of Theorem 1.1. By Theorems 7.3 and 7.4, and assumptions (Q1) and (Q2), it suffices to show that $\dim_q(A_n) = \sum_i f_\Gamma^{s_i^{(n)}} f_{\Gamma'}^{s_i^{(n)}}$. By assumption (Q3),

$$[A_n]_q = \beta^{*n} = \sum f_\Gamma^{s_i^{(n)}} [S_i^{(n)}]$$

in $G(A)_q$. But coproduct in $G(A)_q$ is the same as in $G(A)$, so by Lemma 4.1, we have

$$\dim_q(A_n) = \sum_i f_\Gamma^{s_i^{(n)}} \dim(S_i^{(n)}) = \sum_i f_\Gamma^{s_i^{(n)}} f_{\Gamma'}^{s_i^{(n)}},$$

as required.

7.4 Filtered nilCoxeter Algebras

Recall the nilCoxeter algebra N_n from Section 5.2. The algebras N_n are filtered by $N_n^{(k)} = \bigoplus_{\ell(w)\geq k} \mathbb{C}T_w$, where $\ell(w)$ denotes the length of the permutation w. In fact, N_n is a graded algebra, and the graded representation theory was also considered by Khovanov [17]. (The formulae for our q-twisted Hopf algebra below differs somewhat from that considered in [17].)

The construction of Section 7.1 produces a q-twisted Hopf algebra $G(N)_q$ with multiplication as follow. Let S_n and S_m be the (unique) simple modules of N_n and N_m respectively. Then

$$\left[\mathrm{Ind}_{N_m\otimes N_n}^{N_{m+n}} S_m \otimes S_n\right]_q = \frac{[n+m]!}{[n]![m]!}[S_{n+m}].$$

We can thus identify $G(N)_q$ with $\mathbb{Z}[q][x/[1], x^2/[2]!, x^3/[3]!, \cdots]$, equipped with the usual multiplicative structure. The coproduct is then defined by

$$\Delta\left(\frac{x^c}{[c]!}\right) = \sum_{r=0}^{c} \frac{x^r}{[r]!} \otimes \frac{x^{c-r}}{[c-r]!}.$$

The q-twisted structure reduces to a well-known identity for q-binomial coefficients. Let $\binom{m}{n}_q = [m]!/([n]![m-n]!)$ be the usual q-binomial coefficients, where by convention $\binom{m}{n}_q = 0$ if $m < n$. Then the following identity is standard (and follows easily from the interpretation of $\binom{m}{n}_q$ as the rank-generating function of the product of two chains):

$$\binom{m}{n}_q = \sum_{i=0}^{n} q^{(n-i)(r-i)} \binom{r}{i}_q \binom{m-r}{n-i}_q \tag{7.1}$$

for any $1 \leq r \leq n$. We then calculate

$$\Delta\left(\frac{x^a}{[a]!}\right) \cdot_q \Delta\left(\frac{x^{c-a}}{[c-a]!}\right) = \left(\sum_{i=0}^{a} \frac{x^i}{[i]!} \otimes \frac{x^{a-i}}{[a-i]!}\right) \cdot_q \left(\sum_{j=0}^{c-a} \frac{x^j}{[j]!} \otimes \frac{x^{c-a-j}}{[c-a-j]!}\right)$$

$$= \sum_{i=0}^{a}\sum_{j=0}^{c-a} q^{(a-i)j} \binom{i+j}{i}_q \binom{c-i-j}{a-i}_q \frac{x^{i+j}}{[i+j]!} \otimes \frac{x^{c-i-j}}{[c-i-j]!}.$$

Now take the coefficient of $x^r/[r]! \otimes x^{c-r}/[c-r]!$ and use Eq. 7.1 with $m = c$ and $n = a$ to see that this is equal to $\binom{c}{a}_q \Delta(x^c/[c]!)$. Theorem 7.2 reduces to the well-known combinatorial identity $\sum_{w\in\mathfrak{S}_n} q^{\ell(w)} = [n]!$.

The graph Γ' is still a chain, as in Section 5.2. The graph Γ has edge multiplicities $m(i, i+1) = [i+1]$. The pair (Γ, Γ') is a pair of quantized dual graded graphs with differential coefficient $r = 1$.

7.5 Filtered 0-Hecke Algebras

Now consider the tower $H(0)$ of 0-Hecke algebras, which we equip with the filtrations $H_n(0)^{(k)} = \bigoplus_{\ell(w)\geq k} \mathbb{C}T_w$, where $\ell(w)$ denotes the length of the permutation

w. The representation theory of this filtered tower of algebras was considered by Thibon and Ung [28], to which we refer for further details.

The construction of Section 7.1 produces a q-twisted Hopf algebra $G(H(0))_q = QSym_q$ known as the *quantum quasi-symmetric functions*. $QSym_q$ is spanned over $\mathbb{Z}[q]$ by the fundamental quasi-symmetric functions F_I, equipped with the *quantum shuffle product*, which we now describe. Let $I = (i_1, i_2, \ldots, i_r)$ and $J = (j_1, j_2, \ldots, j_s)$ be compositions of m and n respectively. Let $w = w_1 w_2 \cdots w_m \in \mathfrak{S}_m$ and $v = v_1 v_2 \cdots v_n \in \mathfrak{S}_n$ be permutations with descent sets $\mathrm{Des}(w) = \mathrm{Des}(I) := \{i_1, i_1 + i_2, \ldots, i_1 + i_2 + \cdots + i_{r-1}\}$ and $\mathrm{Des}(v) = \mathrm{Des}(J)$. Denote the shuffles of w and v by $\mathrm{Shuf}(w, v) \subset \mathfrak{S}_{m+n}$, which we illustrate with an example: if $w = 132$ and $v = 21$, then

$$\mathrm{Shuf}(w, v) = \{13254, 13524, 15324, 51324, 13542, 15342, 51342, 15432, 51432, 54132\}.$$

For $u \in \mathfrak{S}_n$, write $C(u)$ for the composition C of n such that $\mathrm{Des}(C) = \mathrm{Des}(u)$. Then in $QSym_q$,

$$F_I * F_J = \sum_{u \in \mathrm{Shuf}(w,v)} q^{\theta(u)} F_{C(u)}$$

where

$$\theta(u) = \#\{(i, j) \in \{1, 2, \ldots, m\} \times \{m+1, \ldots, m+n\} \mid i \text{ occurs after } j \text{ in } u\}.$$

With this product, and the usual coproduct of quasi-symmetric functions, $QSym_q$ becomes a q-twisted Hopf algebra. At $q = 1$, the algebra $QSym_q$ reduces to $QSym$. We caution that while $QSym$ is commutative, the algebra $QSym_q$ is noncommutative, and in fact is isomorphic to $NSym$ as a ring ([28]). Theorem 7.2 reduces, again, to the well-known combinatorial identity $\sum_{w \in \mathfrak{S}_n} q^{\ell(w)} = [n]!$.

The graph Γ' is the infinite lifted binary tree as before. The edge multiplicities for Γ are powers of q, which is illustrated in Fig. 1. The edge joining $(i_1, i_2, \ldots, i_r)$ with $(i_1, \ldots, i_{j-1}, i_j + 1, i_{j+1}, \ldots, i_r)$ has multiplicity $q^{i_1 + i_2 + \cdots + i_{j-1}}$; the edge joining $(i_1, i_2, \ldots, i_r)$ with $(i_1, \ldots, i_{j-1}, k + 1, i_j - k, i_{j+1}, \ldots, i_r)$ has multiplicity $q^{i_1 + \cdots + i_{j-1} + k + 1}$. This gives a pair of quantized dual graded graphs with differential coefficient $r = 1$, which is not difficult to verify directly.

8 Further Directions

There are many new avenues that could be explored from the point of view developed in this paper.

8.1 Generalized Bialgebras, and Hopf Monoids

In this paper, we have three worlds connected by some constructions:

$$\text{tower of algebras} \longrightarrow \text{combinatorial Hopf algebra} \longrightarrow \text{dual graded graph}. \tag{8.1}$$

It is natural to ask if this is possible for other triples of similar objects. In particular, in [6], Bergeron and Li described how a general tower of algebras gives rise to generalized bialgebras in the sense of Loday [24]. Fomin's notion of dual graded graphs is naturally related to Hopf algebras. It is thus natural to ask what generalization of dual graded graph is related to other generalized bialgebras. In particular, what relation replaces $DU - UD = r\mathrm{Id}$?

The theory developed in Section 7 suggests another direction. The construction there replaces Hopf algebras with Hopf monoids which lie in a different braided monoidal category. It would be interesting to study if it is possible to construct a triple for other types of Hopf monoids.

8.2 Category of Combinatorial Hopf Algebras

In [1], Aguiar, Bergeron, and Sottile considered the category of combinatorial Hopf algebras, consisting of pairs (H, ζ) where H is a graded connected Hopf algebra and $\zeta : H \to \mathbb{C}$ is a character (multiplicative homomorphism to the ground field $\mathbb{C}$).

The first arrow of Eq. 8.1 allows us to construct very natural pairs (H, ζ). More precisely, let $A = \bigoplus_{n\geq 0} A_n$ be a tower of algebras satisfying Theorem 1.1 and suppose we are given a family $\{P_n^0\}$ of one-dimensional projective modules, satisfying the following compatibility relation:

$$\mathrm{Res}_{A_n\otimes A_m}^{A_{n+m}} P_{m+n}^0 = P_n^0 \otimes P_m^0. \tag{8.2}$$

We can thus define the linear map

$$\begin{aligned} \zeta^0 : \qquad & G_0(A) \to \mathbb{Z} \\ [M] \in\ & G_0(A_n) \mapsto \langle [P_n^0], [M] \rangle. \end{aligned}$$

Using Eq. 8.2, it is clear that ζ^0 is a character (taking values on $\mathbb{Z}$). We thus have that $(G_0(A), \zeta^0)$ is a combinatorial Hopf algebra in the sense of [1].

In our key examples, $(G_0(A), \zeta^0)$ satisfies some universal properties. For instance, for the tower of 0-Hecke algebras, a natural family of one-dimensional projective modules exists and in *NSym* are encoded by the ribbon Schur functions $R_{(n)}$. The resulting character ζ^0 is precisely what is needed to get the universal property of *QSym* as in [1]. For the tower of $\mathfrak{S}_n$-group algebras, the one-dimensional projective modules are encoded by the Schur functions $s_{(n)}$ in *Sym*. As shown in [1], (Sym, ζ^0) is universal among cocomutative Hopf algebras.

In both cases, this family of one-dimensional projective modules are "trivial" – both symmetric group algebras and 0-Hecke algebras have a distinguished basis (see Section 5) which acts trivially on these modules. It is tempting to say that as soon as a tower has "trivial" projective modules, then it is a universal object in some category. This seems to be the case for most of the examples we know, and P. Choquette[2] has some results along this line for our quantum example in Section 7. Yet we do not have such a result for the tower of Hecke algebras at root of unity. It would be very interesting to find a category for which $Sym/\mathcal{J}^{(r)}$ and its character is a universal object.

Remark 8.1 The towers given in Remark 5.2 also have a natural compatible family of one-dimensional projective modules. The Hopf algebra of Q-Schur functions (with

[2]Personal communication with P. Choquette, 2009.

the associated character) is the universal object in the category of cocommutative odd combinatorial Hopf algebras. The Hopf algebra of peak quasisymetric functions is universal in the category of odd combinatorial Hopf algebras.

Remark 8.2 The tower of nilCoxeter algebras does not have one-dimensional projective modules as described above. Yet, the simple modules S_n^0 do satisfy Eq. 8.2, giving rise to a character ζ^0 on $K(A) = \mathbb{Z}[x]$, given by $\zeta^0(x^n) = 1$. The universal property satisfied by $\mathbb{Z}[x]$ is somehow trivial: for any combinatorial Hopf algebra (H, ζ), with ζ taking values in $\mathbb{Z}$, there is a unique algebra morphism $\hat{\zeta}\colon H \to \mathbb{Z}[x]$ defined by $\hat{\zeta}(h) = \zeta(h)x^n$ for $h \in H_n$, satisfying $\zeta^0 \circ \hat{\zeta} = \zeta$.

8.3 Bi-Tower of Algebras, and Categorification

One can consider Fomin's work on dual graded graphs [11] as generalizing Stanley's notion of differential posets by considering different posets for the up and for the down operators. In this context it seems natural to allow two distinct tower structures on a family of algebras in order to define induction and restriction with a compatibility relation. More precisely, let us say that a bi-tower of algebras (A, ρ, ρ') is a tower $A = \bigoplus_{n\geq 0} A_n$ such that $\rho\colon A \otimes A \to A$ is a tower of algebras satisfying conditions (1)–(3) of Section 2, and A with $\rho'\colon A \otimes A \to A$ also satisfies (1)–(3). Now, we use ρ to define the product of $G(A)$ and the coproduct of $K(A)$ but we use ρ' to define the coproduct of $G(A)$ and the product of $K(A)$.

We may now ask whether $(K(A), G(A))$ form a pair of dual graded Hopf algebras. It is straightforward to check that Theorem 1.1 also holds for bi-towers of algebras which give rise to a pair of graded dual Hopf algebras.

Theorem 8.3 *Let (A, ρ, ρ') be a bi-tower of algebras such that its associated Grothendieck groups form a pair of graded dual Hopf algebras. Then* $\dim(A_n) = r^n n!$ *where* $r = \dim(A_1)$.

Example 8.4 We give one interesting example of a bi-tower of algebras. Let $A_n = \mathbb{C}^{n!}$ be the commutative semisimple algebra of dimension $n!$. This implies that the Grothendieck groups $G(A) = K(A) = A$. The canonical basis of A_n is given by $\{e_\sigma : \sigma \in \mathfrak{S}_n\}$. To define ρ and ρ', let $\mathrm{Shuf}(n, m) = \{\zeta \in \mathfrak{S}_{n+m} : \zeta(1) < \cdots < \zeta(n), \zeta(n+1) < \cdots < \zeta(n+m)\}$. We also consider the canonical imbedding $\mathfrak{S}_n \times \mathfrak{S}_m \hookrightarrow \mathfrak{S}_{n+m}$ and denote by $\sigma \times \pi \in \mathfrak{S}_{n+m}$ the element corresponding to $(\sigma, \pi) \in \mathfrak{S}_n \times \mathfrak{S}_m$. We define $\rho\colon A \otimes A \to A$ by

$$\rho(e_\sigma \otimes e_\pi) = \sum_{\zeta \in \mathrm{Shuf}(n,m)} e_{(\sigma\times\pi)\zeta^{-1}}.$$

We define $\rho'\colon A \otimes A \to A$ by

$$\rho'(e_\sigma \otimes e_\pi) = \sum_{\zeta \in \mathrm{Shuf}(n,m)} e_{\zeta(\sigma\times\pi)}.$$

With these two maps, the reader can easily verify that $G(A)$ and its dual $K(A)$ will correspond to the two descriptions of the Malvenuto-Reutenauer Hopf algebras as in [2]. The pair of dual graphs corresponding to it is the fundamental pair given in Section 2.6 of [11].

We now want to see the diagram in Eq. 8.1 as functors between categories.

$$\mathcal{T} \longrightarrow \mathcal{H} \longrightarrow \mathcal{G} \tag{8.3}$$

It is very natural to allow bi-towers of algebras in our constructions. The objects in the first category are bi-towers of algebras (A, ρ, ρ') which give rise to graded dual Hopf algebras. A morphism $F\colon (A, \rho, \rho') \to (B, \overline{\rho}, \overline{\rho}')$ is given by a family of algebra homomorphisms $F_n\colon A_n \to B_n$ such that $\overline{\rho} \circ (F \otimes F) = F \circ \rho$ and $\overline{\rho}' \circ (F \otimes F) = F \circ \rho'$. Moreover we require that for every primitive idempotent g of A_{n+m}, we can find idempotents e's and f's such that

$$gA_{n+m} \cong \bigoplus eA_n \otimes fA_m \qquad \text{and} \qquad F(g)B_{n+m} \cong \bigoplus F(e)B_n \otimes F(f)B_m.$$

As proven by Li in [22], this induces graded dual Hopf algebra morphisms $F_*\colon K(A) \to K(B)$ and $F^*\colon G(B) \to G(A)$.

We now consider the category $\mathcal{H}$ with objects $(H_\bullet, \{p_\lambda\}, \alpha, \beta)$ where $H_\bullet = \bigoplus H_n$ is a graded connected Hopf algebra over $\mathbb{Z}$, the set $\{p_\lambda\}$ is a homogeneous basis of $H_\bullet$ such that all structure constants for product and coproduct are non-negative and $\alpha \in H_1$ is a non-negative $\mathbb{Z}$-linear combination of the basis $\{p_\lambda\}$. We denote by $H^\bullet$ the graded dual of $H_\bullet$ and by $\{s_\lambda\}$ the homogeneous basis dual to $\{p_\lambda\}$. The element $\beta \in H^1$ is a non-negative $\mathbb{Z}$-linear combination of the basis $\{s_\lambda\}$. A morphism $T\colon (H_\bullet, \{p_\lambda\}, \alpha, \beta) \to (\overline{H}_\bullet, \{\overline{p}_\lambda\}, \overline{\alpha}, \overline{\beta})$ in this category corresponds to a graded Hopf algebra morphism $T_\bullet\colon H_\bullet \to \overline{H}_\bullet$ such that $T_\bullet(p_\lambda)$ is a non-negative linear combination of the $\{\overline{p}_\lambda\}$ and $T_\bullet(\alpha) = \overline{\alpha}$. By duality this induces a graded Hopf algebra morphism $T^\bullet\colon \overline{H}^\bullet \to H^\bullet$ for which $T^\bullet(\overline{s}_\lambda)$ is a non-negative linear combination of the $\{s_\lambda\}$. We require that $T^\bullet(\overline{\beta}) = \beta$.

From the above construction and Section 2, if $F\colon (A, \rho, \rho') \to (\overline{A}, \overline{\rho}, \overline{\rho}')$ is a morphism of bi-towers of algebras, then $F_*\colon K(A) \to K(\overline{A})$ is a graded Hopf algebra morphism such that $F_*(p_\lambda)$ decomposes into a non-negative linear combination of projective $\overline{A}$-modules, which is a non-negative linear combination of the $\{\overline{p}_\lambda\}$. Using the $\alpha \in K(A_1)$ and $\beta \in G(A_1)$ as defined in Section 4, we obtain that $F_*(\alpha) = \overline{\alpha}$ and $F^*(\overline{\beta}) = \beta$. Hence the construction $(A, \rho, \rho') \mapsto (K(A), \{p_\lambda\}, \alpha, \beta)$ is functorial.

The third category $\mathcal{G}$ consists of dual graded graphs (Γ, Γ'). A morphism $\varphi\colon (\overline{\Gamma}, \overline{\Gamma}') \to (\Gamma, \Gamma')$ is a $\mathbb{Z}$-linear map $\varphi\colon \mathbb{Z}\overline{V} \to \mathbb{Z}V$ on the $\mathbb{Z}$-module of vertices such that $h \circ \varphi = \overline{h}$ where h and $\overline{h}$ are extended linearly, $U_\Gamma \circ \varphi = \varphi \circ U_{\overline{\Gamma}}$ and $D_{\Gamma'} \circ \varphi = \varphi \circ D_{\overline{\Gamma}'}$. We also require that $\varphi(\overline{v})$ is a non-negative linear combination of V for all $\overline{v} \in \overline{V}$.

Given a morphism $T\colon (H_\bullet, \{p_\lambda\}, \alpha, \beta) \to (\overline{H}_\bullet, \{\overline{p}_\lambda\}, \overline{\alpha}, \overline{\beta})$ in the category $\mathcal{H}$, we obtain a morphism of the category $\mathcal{G}$ as follows. First we remark that if $T_\bullet(p_\lambda) = \sum_\mu c_{\lambda,\mu}\overline{p}_\mu$, then $T^\bullet\colon \overline{H}^\bullet \to H^\bullet$ is a graded $\mathbb{Z}$-linear map such that $T^\bullet(\overline{s}_\mu) = \sum_\lambda c_{\lambda,\mu}s_\lambda$ is a non-negative linear combination. From Section 3,

$$U_{\Gamma(\beta)} \circ T^\bullet(\overline{x}) = \beta T^\bullet(\overline{x}) = T^\bullet(\overline{\beta}\overline{x}) = T^\bullet \circ U_{\Gamma(\overline{\beta})}(\overline{x}).$$

Also

$$\begin{aligned} D_{\Gamma(\alpha)} \circ T^\bullet(\overline{x}) &= \sum_\lambda \langle \alpha p_\lambda, T^\bullet(\overline{x})\rangle s_\lambda = \sum_\lambda \langle T_\bullet(\alpha p_\lambda), \overline{x}\rangle s_\lambda \\ &= \sum_\lambda \sum_\mu c_{\lambda,\mu} \langle \overline{\alpha}\, \overline{p}_\mu, \overline{x}\rangle s_\lambda = T^\bullet\Big(\sum_\mu \langle \overline{\alpha}\, \overline{p}_\mu, \overline{x}\rangle \overline{s}_\mu\Big) \\ &= T^\bullet \circ D_{\Gamma(\overline{\alpha})}(\overline{x}). \end{aligned}$$

Hence the construction $(H_\bullet, \{p_\lambda\}, \alpha, \beta) \mapsto (\Gamma(\beta), \Gamma(\alpha))$ is a (contravariant) functor from $\mathcal{H}$ to $\mathcal{G}$. We have thus shown the following theorem.

Theorem 8.5 *The two constructions $\mathcal{T} \to \mathcal{H}$ and $\mathcal{H} \to \mathcal{G}$ are functorial.*

Remark 8.6 For $r = 1$, it should be possible to map the minimal idempotents of a bi-tower (A, ρ, ρ') (giving rise to graded dual Hopf algebras) into the bi-tower of Example 8.4 in such a way that we get a morphism. This would be a good way to see the fundamental role played by the dual graded graphs given in Section 2.6 of [11]. It also would explain the importance of the Malvenuto-Reutenauer Hopf algebra. This is conceptually plausible but in practice likely to be very hard. For example, such a morphism is not known for the tower of 0-Hecke algebras.

Acknowledgements We thank Susumu Ariki for comments. Lam would like to thank Mark Shimozono for the collaboration which led to the line of thinking in this paper. Li thanks LaCIM and NSF/FRG for providing support to her as a postdoctoral fellow at UQAM and Drexel University.

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