Double Poisson Structures on Finite Dimensional Semi-Simple Algebras

Geert Van de Weyer

Abstract We give a description of the bimodule of double derivations of a finite dimensional semi-simple algebra S and its double Schouten bracket in terms of a quiver. This description is used to determine which degree two monomials induce double Poisson brackets on S. In case $S = \mathbb{C}^{\oplus n}$, a criterion for any degree two element to give a double Poisson bracket is deduced. For $S = \mathbb{C}^{\oplus n}$ and $S' = \mathbb{C}^{\oplus m}$ the induced Poisson bracket on the variety of isomorphism classes of semi-simple representations $\mathsf{iss}_n(S * T)$ of the free product $S * T$ is given.

1 Introduction

Throughout this note, we will work over an algebraically closed field of characteristic 0 which we denote by $\mathbb{C}$. Unadorned tensor products will be over the base field $\mathbb{C}$.

Double Poisson algebras were introduced by M. Van den Bergh in [4] as a generalization of classical Poisson geometry to the setting of noncommutative geometry. The

G. Van de Weyer
Department of Mathematics and Computer Science,
University of Antwerp, 2020 Antwerp, Belgium
e-mail: geert.vandeweyer@ua.ac.be

key fact to this generalization is that an algebra A equipped with a double Poisson bracket has a canonical Poisson structure on all its finite dimensional representation spaces $\mathsf{rep}_n(A)$. More specifically, a double Poisson algebra A is an associative unital algebra equipped with a linear map

$$\{\!\{-,-\}\!\} : A \otimes A \to A \otimes A$$

that is a derivation in its second argument for the outer A-bimodule structure on $A \otimes A$, where the outer action of A on $A \otimes A$ is defined as $a.a' \otimes a''b := (aa') \otimes (a''b)$. Furthermore, we must have that $\{\!\{a,b\}\!\} = -\{\!\{b,a\}\!\}^o$ and that the double Jacobi identity holds for all $a,b,c \in A$:

$$\{\!\{a, \{\!\{b,c\}\!\}'\}\!\} \otimes \{\!\{b,c\}\!\}'' + \{\!\{c,a\}\!\}'' \otimes \{\!\{b, \{\!\{c,a\}\!\}'\}\!\}$$

$$+ \{\!\{c, \{\!\{a,b\}\!\}'\}\!\}'' \otimes \{\!\{a,b\}\!\}'' \otimes \{\!\{c, \{\!\{a,b\}\!\}'\}\!\}' = 0,$$

where we used Sweedler notation, that is $\{\!\{x,y\}\!\} = \sum \{\!\{x,y\}\!\}' \otimes \{\!\{x,y\}\!\}''$ for all $x,y \in A$. Such a map is called a *double Poisson bracket*.

A double Poisson bracket yields, for each n, a classical Poisson bracket on the coordinate ring $\mathbb{C}[\mathsf{rep}_n(A)]$ of the variety of n-dimensional representations of A through $\{a_{ij}, b_{k\ell}\} := \{\!\{a,b\}\!\}'_{kj}\{\!\{a,b\}\!\}_{i\ell}$. This bracket restricts to a Poisson bracket on $\mathbb{C}[\mathsf{rep}_n(A)]^{\mathsf{GL}_n}$, the coordinate ring of the quotient variety $\mathsf{iss}_n(A)$ under the action of the natural symmetry group GL_n of $\mathsf{rep}_n(A)$.

In this paper, we study double Poisson brackets on a direct sum $S = M_{d_1}(\mathbb{C}) \oplus \cdots \oplus M_{d_k}(\mathbb{C})$ of matrix algebras over $\mathbb{C}$. Because such algebras are smooth, we know from [4] that all double Poisson brackets are determined by double Poisson tensors. That is, elements of degree 2 in $\mathbb{D}S = T_S \mathbb{D}\mathrm{er}(S)$ where $\mathbb{D}\mathrm{er}(S) = \mathrm{Der}(S, S \otimes S)$ is the module of double derivations. That is, the module of derivations from S to the S-bimodule $S \otimes S$, where the S-action on $S \otimes S$ is the outer action. $\mathbb{D}\mathrm{er}(S)$ is an S-bimodule through the inner action: $(s.\vartheta.t)(u) = \vartheta(u)'t \otimes s\vartheta(u)''$.

A first important result is the explicit description of $\mathbb{D}\mathrm{er}(S)$ and $\mathbb{D}\mathrm{er}_T(S)$. Here, $\mathbb{D}\mathrm{er}_T(S)$ is the bimodule of T-linear double derivations with $T \subset S$ a subalgebra. That is, double derivations that are identically zero on T. We have that

$$\mathbb{D}\mathrm{er}(S) \cong \bigoplus_{i=1}^k M_{d_i}(\mathbb{C})^{\oplus d_i^2 - 1} \oplus \bigoplus_{i \neq j} M_{d_i \times d_j}(\mathbb{C})^{\oplus d_i d_j}$$

as S-bimodules where S acts on the right hand side expression by matrix multiplication. If $T = M_{e_1}(\mathbb{C}) \oplus \ldots M_{e_l}(\mathbb{C})$ a finite dimensional semi-simple subalgebra of S with Bratelli diagram with respect to S given by $(a_{ij})_{(i,j)=(1,1)}^{(k,l)}$, then

$$\mathbb{D}\mathrm{er}_T(S) \cong \bigoplus_{i=1}^k M_{d_i}(\mathbb{C})^{\oplus r_i} \oplus \bigoplus_{i \neq j} M_{d_i \times d_j}(\mathbb{C})^{\oplus r_{ij}}$$

as S-bimodules, with $r_i = \sum_{u=1}^l a_{iu}^2 - 1$ and $r_{ij} = \sum_{u=1}^l a_{iu}a_{ju}$. These descriptions are formulated in Theorems 1 and 2.

Using these two theorems, we are able to formulate in Theorem 3 an explicit description of the graded Lie algebra $\mathbb{D}S/[\mathbb{D}S,\mathbb{D}S][1]$, where the bracket on $\mathbb{D}S/[\mathbb{D}S,\mathbb{D}S][1]$ is the bracket associated to the double Schouten-Nijenhuis bracket on $\mathbb{D}S$. This description is formulated in terms of a duo-tone quiver, defined in

Section 3. Briefly, this is a quiver where each arrow $i \leftarrow j$ is marked by an index in I_{ji} where I_{ji} is a finite set for all pairs of vertices (i, j). Assign to S a quiver $\overline{Q}_S$ on k vertices with one arrow between each two vertices $i \neq j$ and one loop in all vertices i for which $d_i > 1$. Choose as index sets $C_{ji} = \{1, \ldots, d_j\} \times \{1, \ldots, d_i\}$ if $i \neq j$ and $C_{ii} = \{1, \ldots, d_j\} \times \{1, \ldots, d_i\}\backslash\{(1, 1)\}$, then $\mathbb{D}S/[\mathbb{D}S, \mathbb{D}S][1]$ is isomorphic as a graded Lie algebra to $\mathbb{C}\overline{Q}_S/[\mathbb{C}\overline{Q}_S, \mathbb{C}\overline{Q}_S]_{super}$, where the bracket on two words $\omega_1 = v_1 \ldots v_n$ and $\omega_2 = u_1 \ldots u_m$ in $\mathbb{C}\overline{Q}_S/[\mathbb{C}\overline{Q}_S, \mathbb{C}\overline{Q}_S]_{super}$ is given by

$$\sum_{a \in (\overline{Q}_S)_0} (-1)^{(i+j)(n-1)}$$

$$= -(-1)^{(i+j+1)(n-1)}$$
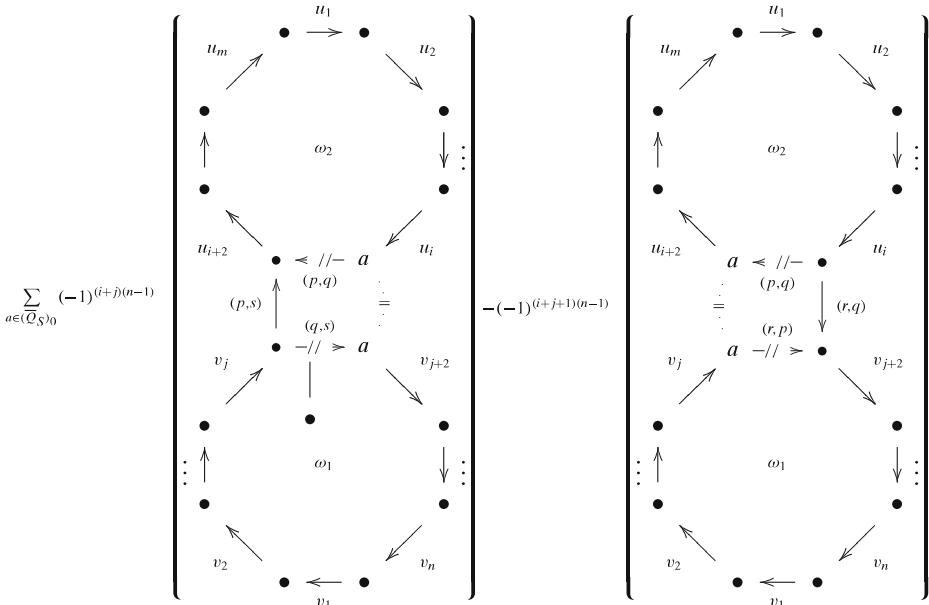
This result can then be used to determine all monomials of degree 2 in $\mathbb{D}S$ that yield nontrivial double Poisson structures on S. In Section 4 we use this result to compute the first double Poisson-Lichnerowicz cohomology groups for S.

Although the representation varieties of finite dimensional semi-simple algebras are rather simple and the quotient varieties consist of a finite number of points, double Poisson structures on these algebras yield interesting noncommutative geometry as they can be extended to double Poisson structures on the free product of such algebras. For such a free product $S * T$, the quotient variety $\mathrm{iss}_n(S * T)$ is no longer trivial and double Poisson structures can yield nontrivial Poisson structures on this variety. In the final section of this paper, we give an explicit description of the Poisson brackets on the quotient variety $\mathrm{iss}_n(\mathbb{C}^{\oplus p} * \mathbb{C}^{\oplus q})$.

2 Double Derivations and Quivers

Consider a finite dimensional semi-simple algebra $S = M_{d_1}(\mathbb{C}) \oplus \cdots \oplus M_{d_k}(\mathbb{C})$. As seen in the introduction, the set of double derivations $\mathbb{D}\mathrm{er}(S) := \mathrm{Der}(S, S \otimes S)$ for the outer action of S on $S \otimes S$ can be equipped with an S-bimodule structure using the inner action of S on $S \otimes S$, defined as $a \circ a' \otimes a'' \circ b := (a'b) \otimes (aa'')$. The S-bimodule of double derivations can be described as

Theorem 1

$$\mathbb{D}\mathrm{er}(S) \cong \bigoplus_{i=1}^{k} M_{d_i}(\mathbb{C})^{\oplus d_i^2-1} \oplus \bigoplus_{i\neq j} M_{d_i \times d_j}(\mathbb{C})^{\oplus d_i d_j}$$

as S-modules where the actions on $\mathrm{Der}(S, S \otimes S)$ *are the inner actions and the actions on the right hand side are just matrix multiplication:*

$$(s_1, \ldots, s_k).(m_1, \ldots, m_k, m_{12}, \ldots, m_{ij}, \ldots, m_{(k-1)k}, m_{k(k-1)}).(t_1, \ldots, t_k)$$

$$= (s_1 m_1 t_1, \ldots, s_k m_k t_k, s_1 m_{12} t_2, \ldots, s_i m_{ij} t_j, \ldots, s_{k-1} m_{(k-1)k} t_k, s_k m_{k(k-1)} t_{k-1}).$$

Proof First of all note that because S is semi-simple all derivations must be inner, that is, any derivation $d : S \to S \otimes S$ is of the form

$$d(y) = d_x(y) = x' \otimes (x''y) - (yx') \otimes x''$$

for some $x = x' \otimes x'' \in S \otimes S$. So we consider the morphism

$$D : S \otimes S \twoheadrightarrow \mathbb{D}\mathrm{er}(S) : x \mapsto d_x$$

of $S \otimes S$ modules for the inner action and will compute its kernel.

First of all note that $S \otimes S$ for the inner action is isomorphic to $\bigoplus_{i=1}^{k} M_{d_i}(\mathbb{C})^{\oplus d_i^2} \oplus \bigoplus_{i\neq j} M_{d_i \otimes d_j}(\mathbb{C})^{\oplus d_i d_j}$. Indeed, denote by e_{pq}^r the standard basis elements for $M_{d_r}(\mathbb{C})$, then for $s \in M_{d_j}(\mathbb{C})$ and $t \in M_{d_i}(\mathbb{C})$ we have

$$s. \left(e_{pq}^i \otimes e_{rs}^j \right).t = \left(e_{pq}^i t \right) \otimes \left(s e_{rs}^j \right)$$

$$= \sum_{a=1}^{d_j} \sum_{b=1}^{d_i} e_{pb}^i t_{qb} \otimes e_{as}^j s_{ar}.$$

That is, on the subspace spanned by

$$f_{ab}^{ps} := e_{pb}^i \otimes e_{as}^j, \ 1 \leq a \leq d_j, \ 1 \leq b \leq d_i$$

we have

$$s. f_{rq}^{ps}.t = \sum_{a=1}^{d_j} \sum_{b=1}^{d_i} s_{ar} f_{ab}^{ps} t_{qb}.$$

But this is nothing else than taking the $n \times m$ matrix with 1 on row r and column q and zeroes elsewhere, and multiplying it on the left by the matrix s and on the right by the matrix t, and we have $d_i d_j$ such subspaces for each i and j.

In order to compute the kernel of the map D, decompose $x \in S \otimes S$ as in the previous paragraph, that is, $x = (x^p, x^{qr})_{1\leq p\leq k, 1\leq q\neq r\leq k}$ with $x^p \in M_{d_p}(\mathbb{C}) \otimes M_{d_p}(\mathbb{C})$ and $x^{qr} \in M_{d_q}(\mathbb{C}) \otimes M_{d_r}(\mathbb{C})$. For this decomposition we have

$$d_x(y_1, \ldots, y_k) = (x^p y_p - y_p x^p, x^{qr} y_r - y_q x^{qr})_{1\leq p\leq k, 1\leq q\neq r\leq k}.$$

Now if $d_x = 0$, fix r and take $y_r = 1$ and $y_s = 0$ for $s \neq r$ which yields $x^{qr} = 0$ for all q. Taking $y_r = e^r_{ii}$ yields $x^r e^r_{ii} - e^r_{ii} x^r = 0$, whence

$$x^r = \sum_{a,b,p=1}^{d_r} x^r_{abp} f^{pp}_{ab}.$$

Finally, taking $y_r = e^r_{ij}$ yields $x^r e^r_{ij} - e^r_{ij} x^r = 0$, which gives $x_{abp} = x_{abq} = x_{ab}$ for all $1 \leq p, q, a, b \leq d_r$. But then

$$x^r = \sum_{a,b,p=1}^{d_r} x^r_{ab} f^{pp}_{ab}.$$

So for the decomposition of $S \otimes S$ as a direct sum of modules of matrices, we get $Ker(D) = M_{d_1}(\mathbb{C}) \oplus \cdots \oplus M_{d_k}(\mathbb{C})$ from which the theorem follows. □

This description can be encoded using the language of quivers.

Definition 1 Let S be a finite dimensional semi-simple algebra $M_{d_1}(\mathbb{C}) \oplus \cdots \oplus M_{d_k}(\mathbb{C})$, then the *double derivation quiver* Q_S for S is the quiver with k vertices, $d_i^2 - 1$ loops in vertex i and $d_i d_j$ arrows from vertex i to vertex j for all $i \neq j$.

As we already indicated in the introduction, the tensor algebra $\mathbb{D}S = T_S \text{Der}(S, S \otimes S)$ plays an important role in the study of double Poisson brackets. This algebra can be seen as a 'path algebra' of the double derivation quiver Q_S of S. In order to clarify what we mean, note first of all that $\mathbb{D}\text{er}(S)$ is generated as an S-module by the elements

$$x^i_{pq} = e^i_{p1} \otimes e^i_{1q} \in M_{d_i}(\mathbb{C})$$

with $(p, q) \neq (1, 1)$ and

$$y^{ij}_{pq} = e^i_{p1} \otimes e^j_{1q} \in M_{d_i}(\mathbb{C})$$

with $i \neq j$. Identifying x^i_{pq} with the loops in vertex i of Q_S and y^{ij}_{pq} with the arrows from vertex j to vertex i, the multiplication of two elements of degree 1 in D_S, say $u = sat$ and $v = s'a't'$ with a and a' arrows in Q_S is then easily seen to be zero unless $h(a) = t(a')$.

We can easily see that in case $S = \mathbb{C}^{\oplus n}$ the algebra $\mathbb{D}S$ is indeed equal to the path algebra of the double derivation quiver.

Proposition 1 *Let* $S = \mathbb{C}^{\oplus n}$, *then* $D_S = \mathbb{C}Q_S$.

Recall that for a subalgebra $T \subset S$, a double derivation ϑ is called T-linear if and only if $\vartheta(T) = 0$. Next, recall that a finite dimensional semi-simple subalgebra $T = M_{e_1}(\mathbb{C}) \oplus \ldots M_{e_\ell}(\mathbb{C})$ of S is completely determined by its Bratelli diagram $(a_{ij})^{(k,\ell)}_{(i,j)=(1,1)}$, listing for each component $M_{e_j}(\mathbb{C})$ its multiplicity in $M_{d_i}(\mathbb{C})$. For T-linear derivations we can state the following result.

Theorem 2 *Let $T = M_{e_1}(\mathbb{C}) \oplus \ldots M_{e_l}(\mathbb{C})$ be a finite dimensional semi-simple subalgebra of S with Bratelli diagram with respect to S given by $(a_{ij})_{(i,j)=(1,1)}^{(k,l)}$, then*

$$\mathrm{Der}_T(S, S \otimes S) \cong \bigoplus_{i=1}^k M_{d_i}(\mathbb{C})^{\oplus r_i} \oplus \bigoplus_{i \neq j} M_{d_i \times d_j}(\mathbb{C})^{\oplus r_{ij}}$$

as S-modules, with $r_i = \sum_{u=1}^l a_{iu}^2 - 1$ and $r_{ij} = \sum_{u=1}^l a_{iu} a_{ju}$.

Proof We use the identification from Theorem 1. Write $f_{pq}^{rs}(i)$ for the basis elements in $M_{d_i}(\mathbb{C})^{d_i^2-1}$ ($1 \leq r, s \leq d_i$, $1 \leq p \leq d_i$, $2 \leq q \leq d_i$) and $f_{pq}^{rs}(i,j)$ for the basis elements in $M_{d_i \times d_j}(\mathbb{C})^{d_i d_j}$ ($i \neq j$, $1 \leq p, s \leq d_j$, $1 \leq q, r \leq d_i$). Let $x \in \mathrm{Der}(S, S \otimes S)$. Let y_{vw}^u be the (v, w)-th basis element of the u-th component of T, then y_{vw}^u is embedded in the i-th component of S as

$$\sum_{h=0}^{a_{iu}-1} e_{(n_{iu}+he_u+v)(n_{iu}+he_u+w)},$$

where $n_{iu} = a_{i1}e_1 + \ldots a_{i(u-1)}e_{u-1}$ for $i > 1$ and $n_{i1} = 0$. Then the term of $x(y_{vw}^u)$ in $S_i \otimes S_j$ for $i \neq j$ equals

$$\sum_{r=1}^{n_{iu}} \sum_{(p,q)=(1,1)}^{(d_i,d_j)} \sum_{h=0}^{a_{ju}-1} x_{pq}^{r(n_{ju}+v+he_u)}(i,j) f_{pq}^{r(n_{ju}+w+he_u)}(i,j) \tag{1}$$

$$+ \sum_{r=n_{iu}+1}^{n_{iu}+a_{iu}e_u} \sum_{(p,q)=(1,1)}^{(d_i,d_j)} \sum_{h=0}^{a_{ju}-1} x_{pq}^{r(n_{ju}+v+he_u)}(i,j) f_{pq}^{r(n_{ju}+w+he_u)}(i,j) \tag{2}$$

$$- \sum_{s=1}^{d_j} \sum_{(p,q)=(1,1)}^{(d_i,d_j)} \sum_{h=0}^{a_{iu}-1} x_{pq}^{(n_{iu}+v+he_u)s}(i,j) f_{pq}^{(n_{iu}+w+he_u)s}(i,j) \tag{3}$$

$$+ \sum_{r=n_{i(u+1)}+1}^{d_i} \sum_{(p,q)=(1,1)}^{(d_i,d_j)} \sum_{h=0}^{a_{ju}-1} x_{pq}^{r(n_{ju}+v+he_u)}(i,j) f_{pq}^{r(n_{ju}+w+he_u)}(i,j). \tag{4}$$

Now $x \in \mathrm{Der}_T(S, S \otimes S)$ if and only if $x(y) = 0$ for all $y \in T$. Letting u, v and w run over all possible values, lines (1) and (4) in the expression above then yield that for $r \in [n_{iu} + 1, n_{i(u+1)}]$ and $s \notin [n_{ju} + 1, n_{j(u+1)}]$ the term of x in the (r, s)-th component of $M_{d_i \times d_j}(\mathbb{C})^{\oplus d_i d_j}$ is zero. Choosing $v = w$ and running over all possible values, lines (2) and (3) imply that for $r \in [n_{iu} + 1, n_{i(u+1)}]$ and $s \in [n_{ju} + 1, n_{j(u+1)}]$ with $r = n_{iu} + xe_u + y$ and $s = n_{ju} + ae_u + b$, $y, b < e_u$ the term in the (r, s)-th component of $M_{d_i \times d_j}(\mathbb{C})^{\oplus d_i d_j}$ is zero unless $y = b$. Choosing $v \neq w$ and letting these indices run over all possible values again then yields that for $y = b$ we have that the term in the (r, s)-th component of $M_{d_i \times d_j}(\mathbb{C})^{\oplus d_i d_j}$ is equal to the term in the (r', s)-th component for $r' = n_{iu} = xe_u + y'$. This means that we only have $a_{iu} a_{ju}$ nonzero components of x for each u, yielding the multiplicities r_{ij} of the theorem.

The multiplicities r_i are obtained through a completely analogous reasoning, taking into account that because of Theorem 1 we start with $d_i^2 - 1$ copies of $M_{d_i}(\mathbb{C})$ instead of $d_i d_i$ copies. □

3 Double Poisson Tensors and Cycles

Because a finite dimensional semi-simple algebra is formally smooth, all double Poisson brackets are determined by linear combinations of degree two elements $\delta\Delta$ in $\mathbb{D}S$ that are nilpotent (modulo commutators) with respect to the Schouten bracket on $\mathbb{D}S$ [4, Proposition 4.1.2]. The double bracket corresponding to such a degree 2 element being defined as

$$\{a, b\} := \Delta(b)'\delta(a)'' \otimes \delta(a)'\Delta(b)'' - \delta(b)'\Delta(a)'' \otimes \Delta(a)'\delta(b)''.$$

Recall from [4] that the double Schouten bracket $\{\{-, -\}\}_s$ on $\mathbb{D}S$ is a *double Gerstenhaber bracket* defined as

$$\{\{s, t\}\}_s = 0$$

$$\{\{\delta, s\}\}_s = \delta(s)$$

$$\{\{\delta, \Delta\}\}_s = \{\{\delta, \Delta\}\}_l + \{\{\delta, \Delta\}\}_r$$

where

$$\{\{\delta, \Delta\}\}_l = \tau_{(23)}\{\{\delta, \Delta\}\}_{\bar{l}} \text{ and } \{\{\delta, \Delta\}\}_r = \tau_{(12)}\{\{\delta, \Delta\}\}_{\bar{r}}$$

with

$$\{\{\delta, \Delta\}\}_{\bar{l}} := (\delta \otimes 1)\Delta - (1 \otimes \Delta)\delta \text{ and } \{\{\delta, \Delta\}\}_{\bar{r}} := (1 \otimes \delta)\Delta - (\Delta \otimes 1)\delta$$

viewed as elements of $\mathbb{D}\text{er}(S) \otimes S$ respectively $S \otimes \mathbb{D}\text{er}(S)$. The Schouten bracket on $\mathbb{D}S$ becomes

Proposition 2 *For the generators of $\mathbb{D}S$ defined in the previous section, we have*

$$\{\{x_{pq}^i, x_{rs}^j\}\}_S = \begin{cases} 0 & i \neq j \\ e_{p1}^i \otimes e_{1q}x_{rs}^i - x_{rs}^i e_{p1}^i \otimes e_{1q}^i \\ + e_{1s}^i \otimes x_{pq}^i e_{r1}^i - e_{1s}^i x_{pq}^i \otimes e_{r1}^i & i = j \\ + \delta_{rq}x_{ps}^i \otimes e_{11}^i - \delta_{ps}e_{11}^i \otimes x_{rq}^i \end{cases}$$

$$\{\{x_{pq}^i, y_{rs}^{uv}\}\}_S = \begin{cases} 0 & i \neq u, v \\ e_{1s}^v \otimes x_{pq}^i e_{r1}^i + \delta_{qr}y_{ps}^{iv} \otimes e_{11}^i - y_{rs}^{iv}e_{p1}^i \otimes e_{1q}^i & u = i \\ e_{p1}^i \otimes e_{1q}^i y_{rs}^{ui} - \delta_{ps}e_{11}^i \otimes y_{rq}^{ui} - e_{1s}^i x_{pq}^i \otimes e_{r1}^u & v = i \end{cases}$$

$$\{\{y_{pq}^{rs}, y_{ab}^{cd}\}\}_S = \begin{cases} e_{p1}^r \otimes e_{1q}^s y_{ab}^{cd} + e_{1b}^d \otimes y_{pq}^{rs}e_{a1}^c - e_{1b}^d y_{pq}^{rs} \otimes e_{a1}^c - y_{ab}^{cd}e_{p1}^r \otimes e_{1q}^s & r \neq d, c \neq s \\ -\delta_{bp}e_{11}^d \otimes y_{aq}^{cs} & r = d, c \neq s \\ -\delta_{bp}e_{11}^r \otimes x_{aq}^s + \delta_{qa}x_{pb}^r \otimes e_{11}^s & r = d, c = s \end{cases}$$

where we use the shorthand notation $x_{11}^i = -\sum_{r=2}^{d_i} x_{rr}^i$.

Proof We show the first equality holds. The other computations are analogous. By definition of the double Schouten bracket it is obvious that $\{\!\{x^i_{pq}, x^j_{rs}\}\!\} = 0$ if $i \neq j$. Now assume $i = j$ and denote $x^i_{pq} = x_{pq}$ and $x^i_{rs} = x_{rs}$, then for $z \in M_{d_i}(\mathbb{C})$ we get

$$(1 \otimes x_{pq})x_{rs}(z) - (x_{rs} \otimes 1)x_{pq}(z) = e_{r1} \otimes e_{p1} \otimes e_{1q}e_{1s}z - e_{r1} \otimes e_{1s}ze_{p1} \otimes e_{1q}$$
$$- ze_{r1} \otimes e_{p1} \otimes e_{1q}e_{1s} + ze_{r1} \otimes e_{1s}e_{p1} \otimes e_{1q}$$
$$- e_{r1} \otimes e_{1s}e_{p1} \otimes e_{1q}z + e_{p1}e_{r1} \otimes e_{1s} \otimes e_{1q}z$$
$$+ e_{r1} \otimes e_{1s}ze_{p1} \otimes e_{1q} - ze_{p1}e_{r1} \otimes e_{1s} \otimes e_{1q}$$

Applying $\tau_{(12)}$ to this expression yields

$$e_{p1} \otimes e_{1q}x_{rs}(z) + e_{1s} \otimes x_{pq}(z)e_{r1} - \delta_{ps}e_{11} \otimes x_{rq}(z).$$

Changing the sign, interchanging the indices and applying to the previous computation $\tau_{(23)}$ then yields

$$-e_{1s}x_{pq}(z) \otimes_{r1} -\delta_{rq}x_{ps}(z) \otimes e_{11} + x_{rs}(z)e_{p1} \otimes e_{1q}.$$

Adding this expression to the one found in the previous paragraph then yields the expression in the statement of the proposition.   □

For the remainder of the paper, we will use the shorthand notation introduced in the preceding proposition for the index $(1, 1)$ at a loop.

For $S = \mathbb{C}^{\oplus n}$ this structure descends to the following double Gerstenhaber bracket on the quiver Q_S.

Corollary 1 *The double Gerstenhaber bracket on $\mathbb{C}Q_S$ for $S = \mathbb{C}^{\oplus n}$ is determined by*

$$\{\!\{i \leftarrow j, k \leftarrow i\}\!\} = -i \otimes (k \leftarrow j) \ (j \neq k)$$
$$\{\!\{k \leftarrow i, i \leftarrow j\}\!\} = (k \leftarrow j) \otimes i \ (j \neq k)$$
$$\{\!\{i \leftarrow j, k \leftarrow \ell\}\!\} = 0 \ (otherwise)$$

More generally, we know $\mathbb{D}S/[\mathbb{D}S, \mathbb{D}S][1]$ is a graded Lie algebra with graded Lie bracket $\{-, -\}_s$. This algebra can be described in terms of a *duo-tone quiver*. That is, a quiver where each arrow $j \xrightarrow{a} i$ has an bi-index in the *colour index set* $C_{ji} = \{1, \ldots, n_j\} \times \{1, \ldots, n_i\}$. For S a finite dimensional semi-simple algebra as before, let $\overline{Q}_S$ be the duo-tone quiver with k vertices, one arrow between each pair of different vertices, one loop in each vertex not corresponding to a component $\mathbb{C}$ and colour index sets $C_{ji} = \{1, \ldots, d_j\} \times \{1, \ldots, d_i\}$ if $i \neq j$ and $C_{ii} = \{1, \ldots, d_i\} \times \{1, \ldots, d_i\}\backslash\{(1, 1)\}$. The first index will be called the *primary colour* of the arrow and the second index will be called the *secondary colour* of the arrow.

Theorem 3 *For a semi-simple algebra S, we have that $\mathbb{D}S/[\mathbb{D}S, \mathbb{D}S]$ with bracket $\{-, -\}_s$ is isomorphic as a graded Lie algebra to $\mathbb{C}\overline{Q}_S/[\mathbb{C}\overline{Q}_S, \mathbb{C}\overline{Q}_S]_{super}$ with bracket defined as follows*

$$\sum_{a \in (\overline{Q}_S)_0} (-1)^{(i+j)(n-1)} \left[\cdots \right] = -(-1)^{(i+j+1)(n-1)} \left[\cdots \right]$$
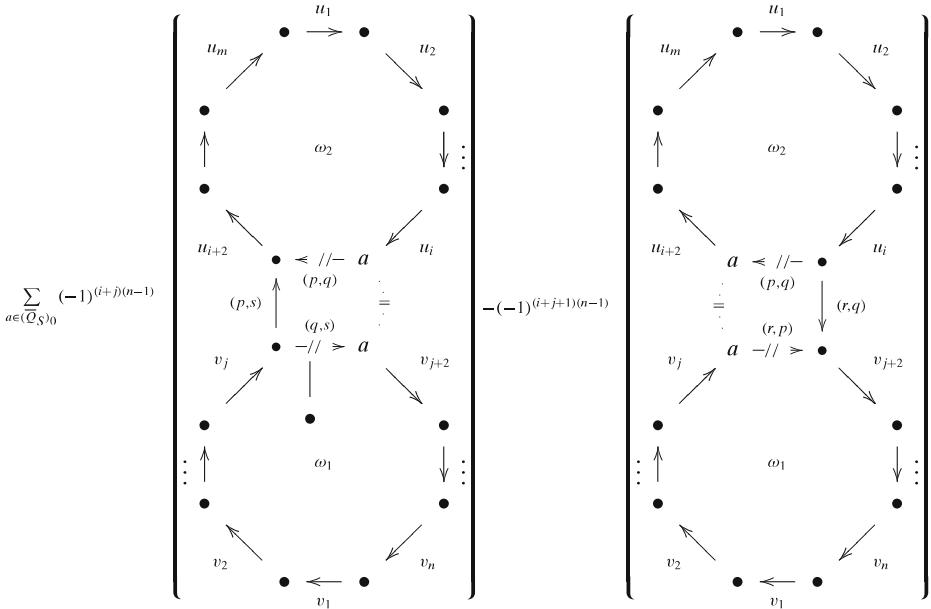
That is, we fix representatives for ω_1 and ω_2 in $\mathbb{C}\overline{Q}_S$ and for every vertex $a \in (\overline{Q}_S)_0$ we look for an occurrence of a in ω_1 and in ω_2. If $a = h(u_i)$ in ω_1 and $a = t(v_{j+2})$ in ω_2 and the secondary colour of u_{i+1} equals the primary colour of v_{j+1}, we remove u_{i+1} from ω_1 and v_{j+1} from ω_2, glue ω_1 to ω_2 in a and connect the loose ends $t(v_{j+1})$ and $h(u_{i+1})$ with an arrow with primary colour equal to that of u_{i+1} and secondary colour equal to that of v_{j+1}. The necklace thus obtained is multiplied by a factor $(-1)^{(i+j)(n-1)}$. An analogous procedure is followed each time a is the tail of an arrow in ω_1 and the head of an arrow in ω_2, with the roles of primary and secondary colours interchanged and factor $-(-1)^{(i+j+1)(n-1)}$.

Proof First of all, note that $\mathbb{D}S/[\mathbb{D}S, \mathbb{D}S]$ is generated as a vector space by all necklace words for Q_S because for generators $u_i \in \{x_{pq}^j, y_{rs}^{k\ell}\}$ and elements $c_i \in S$ we have for any word

$$c_1 u_1 c_2 u_2 \ldots c_p u_p c_{p+1} = c_1 e_{11}^{h(u_1)} u_1 e_{11}^{t(u_1)} c_2 \ldots c_1 e_{11}^{h(u_p)} u_p e_{11}^{t(u_p)} c_{p+1}$$

$$= (c_1)_{11}^{h(u_1)} \ldots (c_p)_{11}^{h(u_p)} (c_{p+1})_{11}^{h(u_1)} u_1 \ldots u_p \bmod [\mathbb{D}S, \mathbb{D}S].$$

Now let $u_1 u_2 \ldots u_m$ and $v_1 v_2 \ldots v_n$ be two necklaces in $\mathbb{D}S$, then

$$\{\{v_1 v_2 \ldots v_n, u_1 u_2 \ldots u_m\}\}$$

$$= \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} -(-1)^{i(n-1)} u_1 \ldots u_i \sigma_{(12)}(v_1 \ldots v_j \{\{u_{i+1}, v_{j+1}\}\} v_{j+2} \ldots v_n) u_{i+2} \ldots u_m.$$

We have four different settings to consider. First of all assume $u_{i+1} = x^a_{pq}$ and $v_{j+1} = x^b_{rs}$. Proposition 2 then yields $\{\{u_{i+1}, v_{j+1}\}\} = 0$ if $a \neq b$ and

$$u_1 \ldots u_i \sigma_{(12)}(v_1 \ldots v_j \{\{u_{i+1}, v_{j+1}\}\} v_{j+2} \ldots v_n) u_{i+2} \ldots u_m$$

$$= (-1)^{j(n-j)} u_1 \ldots u_i e_{1q} x^a_{rs} v_{j+2} \ldots v_n \otimes v_1 \ldots v_j e^a_{p1} u_{i+2} \ldots u_m$$

$$- (-1)^{(j+1)(n-j-1)} u_1 \ldots u_i e^a_{1q} v_{j+2} \ldots v_n \otimes v_1 \ldots v_j x^a_{rs} e^a_{p1} u_{i+2} \ldots u_m$$

$$+ (-1)^{j(n-j)} u_1 \ldots u_i x^a_{pq} e^a_{r1} v_{j+2} \ldots v_n \otimes v_1 \ldots v_j e^i_{1s} u_{i+2} \ldots u_m$$

$$- (-1)^{(j+1)(n-j-1)} u_1 \ldots u_i e^a_{r1} v_{j+2} \ldots v_n \otimes v_1 \ldots v_j e^a_{1s} x^a_{pq} u_{i+2} \ldots u_m$$

$$+ (-1)^{(j+1)(n-j-1)} \delta_{rq} u_1 \ldots u_i e^a_{11} v_{j+2} \ldots v_n \otimes v_1 \ldots v_j x^a_{ps} u_{i+2} \ldots u_m$$

$$- (-1)^{j(n-j)} \delta_{ps} u_1 \ldots u_i x^a_{rq} v_{j+2} \ldots v_n \otimes v_1 \ldots v_j e^a_{11} u_{i+2} \ldots u_m.$$

Now by definition $(p, q) \neq (1, 1)$ and $(r, s) \neq (1, 1)$, so modulo commutators this expression is mapped by the multiplication to

$$\delta_{rq}(-1)^{(j+1)(n-j-1)} u_1 \ldots u_i v_{j+2} \ldots v_n v_1 \ldots v_j x^a_{ps} u_{i+2} \ldots u_m$$

$$- \delta_{ps}(-1)^{j(n-j)} u_1 \ldots u_i x^a_{rq} v_{j+2} \ldots v_n v_1 \ldots v_j u_{i+2} \ldots u_m.$$

Next, assume $u_{i+1} = x^a_{pq}$ and $v_{i+1} = y^{bc}_{rs}$. This yields $\{\{u_{i+1}, v_{j+1}\}\} = 0$ if $a \neq b, c$ and a similar computation as the previous yields

$$u_1 \ldots u_i \sigma_{(12)}(v_1 \ldots v_j \{\{u_{i+1}, v_{j+1}\}\} v_{j+2} \ldots v_n) u_{i+2} \ldots u_m \mapsto$$

$$\delta_{qr}(-1)^{(j+1)(n-j-1)} u_1 \ldots u_i v_{j+2} \ldots v_n v_1 \ldots v_j y^{bc}_{ps} u_{i+2} \ldots u_m$$

if $a = b$ and

$$u_1 \ldots u_i \sigma_{(12)}(v_1 \ldots v_j \{\{u_{i+1}, v_{j+1}\}\} v_{j+2} \ldots v_n) u_{i+2} \ldots u_m \mapsto$$

$$- \delta_{ps}(-1)^{j(n-j)} u_1 \ldots u_i v_{j+2} \ldots v_n v_1 \ldots v_j y^{bc}_{rq} u_{i+2} \ldots u_m$$

if $a = c$. For $u_{i+1} = y^{bc}_{rs}$ and $v_{i+1} = x^a_{pq}$ we get

$$u_1 \ldots u_i \sigma_{(12)}(v_1 \ldots v_j \{\{u_{i+1}, v_{j+1}\}\} v_{j+2} \ldots v_n) u_{i+2} \ldots u_m \mapsto$$

$$- \delta_{qr}(-1)^{j(n-j)} u_1 \ldots u_i v_{j+2} \ldots v_n v_1 \ldots v_j y^{bc}_{ps} u_{i+2} \ldots u_m$$

if $a = b$ and

$$u_1 \ldots u_i \sigma_{(12)}(v_1 \ldots v_j \{\{u_{i+1}, v_{j+1}\}\} v_{j+2} \ldots v_n) u_{i+2} \ldots u_m \mapsto$$

$$\delta_{ps}(-1)^{(j+1)(n-j-1)} u_1 \ldots u_i v_{j+2} \ldots v_n v_1 \ldots v_j y^{bc}_{rq} u_{i+2} \ldots u_m$$

if $a = c$. Finally, for $u_{i+1} = y^{ab}_{pq}$ and $v_{i+1} = y^{cd}_{rs}$ we get $\{u_{i+1}, v_{i+1}\}_s = 0$ unless $b = c$ or $a = d$ in which case we get either

$$u_1 \ldots u_i \sigma_{(12)}(v_1 \ldots v_j \{\{u_{i+1}, v_{j+1}\}\} v_{j+2} \ldots v_n) u_{i+2} \ldots u_m \mapsto$$

$$- \delta_{qr}(-1)^{j(n-j)} u_1 \ldots u_i y^{cb}_{ps} v_{j+2} \ldots v_n v_1 \ldots v_j u_{i+2} \ldots u_m,$$

if $a = d$ and $b \neq c$,

$$u_1 \ldots u_i \sigma_{(12)}(v_1 \ldots v_j \{\{u_{i+1}, v_{j+1}\}\} v_{j+2} \ldots v_n) u_{i+2} \ldots u_m \mapsto$$

$$\delta_{ps}(-1)^{(j+1)(n-j-1)} u_1 \ldots u_i v_{j+2} \ldots v_n v_1 \ldots v_j y_{rq}^{ad} u_{i+2} \ldots u_m,$$

if $b = c$ and $a \neq d$ or

$$u_1 \ldots u_i \sigma_{(12)}(v_1 \ldots v_j \{\{u_{i+1}, v_{j+1}\}\} v_{j+2} \ldots v_n) u_{i+2} \ldots u_m \mapsto$$

$$- \delta_{sp}(-1)^{j(n-j)} u_1 \ldots u_i x_{rq}^b v_{j+2} \ldots v_n v_1 \ldots v_j u_{i+2} \ldots u_m$$

$$+ \delta_{qr}(-1)^{(j+1)(n-j-1)} u_1 \ldots u_i v_{j+2} \ldots v_n v_1 \ldots v_i x_{ps}^a u_{i+2} \ldots u_m.$$

if $b = c$ and $a = d$. □

Now, in order to determine which length two elements in $\mathbb{D}S$ yield nontrivial double Poisson brackets on S, note that

Lemma 1 *For* $P = x_{pq}^i y_{uv}^{rs}$ *or* $P = y_{uv}^{rs} x_{pq}^i$ *with* i, p, q, r, s, u, v *arbitrary and for* $P = y_{pq}^{rs} y_{tu}^{vw}$ *with* p, q, t, u *arbitrary and* $s \neq v$ *or* $w \neq r$ *we have that*

$$\{\{-, -\}\}_P = 0.$$

Proof This is due to the fact that in the definition of $\{\{-, -\}\}_P$ the tensor products are obtained through componentwise multiplication. □

This lemma, in combination with the proposition preceding it, proves that

Proposition 3 *Let* $\{\{-, -\}\}$ *be a nonzero double bracket on S, then* $\{\{-, -\}\}$ *is completely determined by a linear combination*

$$\sum \alpha_{abcd}^{pq} y_{ab}^{pq} y_{cd}^{qp} + \beta_{efgh}^i x_{ef}^i x_{gh}^i$$

with all α_{abcd}^{pq} *and* β_{efgh}^i *in* $\mathbb{C}$.

The next two lemmas determine which cycles of length two yield double Poisson brackets.

Lemma 2 *For* $P = y_{ab}^{pq} y_{cd}^{qp}$ *we have that*

(1) *if* $S_p \neq \mathbb{C}$ *and* $S_q \neq \mathbb{C}$ *then*

$$\{P, P\} = 0 \bmod [\mathbb{D}S, \mathbb{D}S] \Leftrightarrow (a - d)(b - c) \neq 0$$

(2) *if* $S_p \neq \mathbb{C}$ *and* $S_q = \mathbb{C}$ *then*

$$\{P, P\} = 0 \bmod [\mathbb{D}S, \mathbb{D}S] \Leftrightarrow a - d \neq 0$$

(3) *if* $S_p = \mathbb{C}$ *and* $S_q \neq \mathbb{C}$ *then*

$$\{P, P\} = 0 \bmod [\mathbb{D}S, \mathbb{D}S] \Leftrightarrow b - c \neq 0$$

(4) *if* $S_p = \mathbb{C}$ *and* $S_q = \mathbb{C}$ *then*

$$\{P, P\} = 0 \bmod [\mathbb{D}S, \mathbb{D}S]$$

and in each of these cases $\{\{-, -\}\}_P$ *is nontrivial.*

Proof Using the description of the bracket on $\mathbb{C}\overline{Q}_S/[\mathbb{C}\overline{Q}_S, \mathbb{C}\overline{Q}_S]$ from Proposition 3, we compute

$$\left\{ p \underset{(a,b)}{\overset{(c,d)}{\rightleftarrows}} q \ , \ p \underset{(a,b)}{\overset{(c,d)}{\rightleftarrows}} q \right\} = 2\delta_{ad} \ \begin{matrix} p \xrightarrow{(a,b)} q \\ {}^{(c,b)}\diagdown \quad \diagup {}_{(c,d)} \\ p \end{matrix} \ + 2\delta_{bc} \ \begin{matrix} q \xrightarrow{(c,d)} p \\ {}^{(a,d)}\diagdown \quad \diagup {}_{(a,b)} \\ q \end{matrix} \ .$$

Which is nonzero if $a = d$ and the loop at p exists or if $b = c$ and the loop at p exists, yielding the four situations described in the lemma. The fact that the bracket obtained from this double Poisson tensor is non-zero is easy to verify.   □

Lemma 3 *For $P = x^u_{pq} x^u_{rs}$ we have that*

$$\{P, P\} = 0 \ mod \ [\mathbb{D}S, \mathbb{D}S]$$

and $\{\{-, -\}\}_P$ is nontrivial if and only if either $(p-q)(p-s)(r-s)(r-q) \neq 0$, or $p = q = r$, or $r = s = p$.

Proof Again we use the description of the bracket from Proposition 3. We have

$$\left\{ u \underset{(r,s)}{\overset{(p,q)}{\rightleftarrows}} u \right\} = 2\delta_{ps} \ \begin{matrix} u \xrightarrow{(p,q)} u \\ {}^{(r,q)}\diagdown \ \diagup {}_{(r,s)} \\ u \end{matrix} \ - 2\delta_{rs} \ \begin{matrix} u \xrightarrow{(p,q)} u \\ {}^{(r,r)}\diagdown \ \diagup {}_{(p,q)} \\ u \end{matrix}$$

$$+2\delta_{rq} \ \begin{matrix} u \xrightarrow{(p,q)} u \\ {}^{(r,s)}\diagdown \ \diagup {}_{(p,s)} \\ u \end{matrix} \ - 2\delta_{pq} \ \begin{matrix} u \xrightarrow{(p,p)} u \\ {}^{(r,s)}\diagdown \ \diagup {}_{(r,s)} \\ u \end{matrix}$$

Which is exactly nonzero when the conditions formulated in the lemma are satisfied. Again, nontriviality is straightforward.   □

For $S = \mathbb{C}^{\oplus n}$, we can easily show the following

Proposition 4 *Let*

$$P = \sum_{i<j} \alpha_{ij} y^{ij} y^{ji} \in \mathbb{D}S,$$

then P determines a double Poisson bracket on S if and only if the following relation between the α holds:

$$\forall i < j < k : \alpha_{ij}\alpha_{ik} + \alpha_{ik}\alpha_{jk} - \alpha_{ij}\alpha_{jk} = 0.$$

Proof A straightforward computation shows that for i, j, k, ℓ pairwise different we have

$$\{y^{ij}y^{ji}, y^{ki}y^{ik}\} = -y^{ki}y^{jk}y^{ij} - y^{ji}y^{kj}y^{ik} \bmod [\mathbb{D}S, \mathbb{D}S]$$

$$\{y^{ij}y^{ji}, y^{jk}y^{kj}\} = -y^{jk}y^{ij}y^{ki} - y^{ik}y^{ji}y^{kj} \bmod [\mathbb{D}S, \mathbb{D}S]$$

$$\{y^{ij}y^{ji}, y^{ik}y^{ki}\} = y^{ik}y^{ji}y^{kj} + y^{jk}y^{ij}y^{ki} \bmod [\mathbb{D}S, \mathbb{D}S]$$

$$\{y^{ji}y^{ij}, y^{ki}y^{ik}\} = y^{ik}y^{ji}y^{kj} + y^{jk}y^{ij}y^{ki} \bmod [\mathbb{D}S, \mathbb{D}S]$$

$$\{y^{ij}y^{ji}, y^{k\ell}y^{\ell k}\} = 0 \bmod [\mathbb{D}S, \mathbb{D}S]$$

This yields

$$
\begin{aligned}
\{P, P\} &= \sum_{i<j}\sum_{k<\ell} \alpha_{ij}\alpha_{k\ell} \{y^{ij}y^{ji}, y^{k\ell}y^{\ell k}\} \\
&= \sum_{i,k<j} \alpha_{ij}\alpha_{kj} \left(y^{jk}y^{ij}y^{ki} + y^{ik}y^{ji}y^{kj} \right) \\
&\quad - \sum_{i<j<\ell} \alpha_{ij}\alpha_{j\ell} \left(y^{j\ell}y^{ij}y^{\ell i} + y^{ji}y^{\ell j}y^{i\ell} \right) \\
&\quad + \sum_{i<j,\ell} \alpha_{ij}\alpha_{i\ell} \left(y^{i\ell}y^{ji}y^{\ell j} + y^{j\ell}y^{ij}y^{\ell i} \right) \\
&\quad - \sum_{k<i<j} \alpha_{ij}\alpha_{ki} \left(y^{ki}y^{jk}y^{ij} + y^{ji}y^{kj}y^{ik} \right) \bmod [\mathbb{D}S, \mathbb{D}S] \\
&= \sum_{i<j<k} 2(\alpha_{ij}\alpha_{ik} + \alpha_{ik}\alpha_{jk} - \alpha_{ij}\alpha_{jk}) \left(y^{ji}y^{kj}y^{ik} + y^{ki}y^{jk}y^{ij} \right) \bmod [\mathbb{D}S, \mathbb{D}S].
\end{aligned}
$$

So we have a double Poisson bracket if and only if for all $i < j < k$ we have

$$\alpha_{ij}\alpha_{ik} + \alpha_{ik}\alpha_{jk} - \alpha_{ij}\alpha_{jk} = 0. \qquad \square$$

The cycles of length two can be seen as being analogous to the classical Poisson tensors $\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$. In classical Poisson geometry, these Poisson tensors play an important role in the Splitting Theorem [5], where any Poisson structure can be split into a symplectic part and a totally degenerate part, the totally degenerate part being of the form $\varphi_{ij}(x)\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$. If $\varphi_{ij}(x) = c_{ij}$, the totally degenerate part gives rise to an affine Poisson structure for any choice of c_{ij} [2]. In the case of double Poisson structures, the previous proposition shows that this no longer holds.

Another important concept introduced in [4] is that of a moment map for a double Poisson bracket. In the case of a finite dimensional semi-simple algebra $S = \mathbb{C}^{\oplus n}$ we have

Proposition 5 *Let $S = \mathbb{C}^{\oplus n}$ with orthogonal idempotents e_i and let $P = \sum_{i<j} c_{ij}d_{ij}d_{ji}$ determine a double Poisson bracket with d_{ij} the derivation corresponding to $e_i \otimes e_j$,*

then there exists a moment map μ for P if and only if all c_{ij} are nonzero. In this case, the moment map μ is unique up to a constant term and can be written as

$$\mu = -\sum_{i=2}^{n} \frac{1}{c_{1i}} e_i.$$

Proof First of all note that for the double Schouten bracket we have

$$\{\{\delta\Delta, \mu\}\} = \{\{\mu, \delta\Delta\}\}^{op}$$
$$= (-\delta\{\{\mu, \Delta\}\} + \{\{\mu, \delta\}\}\Delta)^{op}$$
$$= -\delta * \Delta(\mu) + \delta(\mu) * \Delta.$$

So for $\mu = \sum_i \mu_i e_i$ we get

$$\{\{P, \mu\}\} = \sum_{i<j} c_{ij}\{\{d_{ij}d_{ji}, \mu\}\}$$
$$= \sum_{i<j} c_{ij}(-d_{ij} * d_{ji}(\mu) + d_{ij}(\mu)d_{ji})$$
$$= \sum_{i<j} c_{ij}(-(\mu_i - \mu_j)e_j \otimes d_{ij} + (\mu_j - \mu_i)d_{ji} \otimes e_i)$$
$$= \sum_{i<j} c_{ij}(\mu_j - \mu_i)(e_j \otimes d_{ij} + d_{ji} \otimes e_i).$$

Under the multiplication this maps to

$$\sum_{i<j} c_{ij}(\mu_j - \mu_i)(d_{ij} + d_{ji}),$$

which on e_i is equal to

$$\sum_{i<j} c_{ij}(\mu_j - \mu_i)(-e_i \otimes e_j + e_j \otimes e_i) + \sum_{i>j} c_{ji}(\mu_j - \mu_i)(e_i \otimes e_j - e_j \otimes e_i).$$

Now

$$E(e_i) = 1 \otimes e_i - e_i \otimes 1 = \sum_{j\neq i} e_j \otimes e_i - e_i \otimes e_j,$$

so $\{P, \mu\} = -E$ implies $c_{ij}(\mu_j - \mu_i) = -1$ if $j > i$ and $c_{ij}(\mu_j - \mu_i) = 1$ if $j < i$. This means all c_{ij} must be nonzero and for $i < j$ we have

$$\mu_i - \mu_j = \frac{1}{c_{ij}},$$

which means the c_{ij} have to satisfy the additional relation for $i < j < k$:

$$c_{ij}c_{jk} = c_{ij}c_{jk} - c_{ik}c_{jk}.$$

But as P determines a double Poisson bracket, by Proposition 4 we know this condition is automatically satisfied. The uniqueness follows from the fact that $e_1 + \cdots + e_n = 1$ and the fact that one μ_i determines all other coefficients. □

4 Double Poisson-Lichnerowicz Cohomology

In [3], Lichnerowicz observed that $d_\pi = \{\pi, -\}$ with π a Poisson tensor for a Poisson manifold M is a square 0 derivation of degree $+1$, which yields a complex

$$0 \xrightarrow{d_\pi} \mathcal{O}(M) \xrightarrow{d_\pi} D\mathcal{O}(M) \xrightarrow{d_\pi} \wedge^2 D\mathcal{O}(M) \xrightarrow{d_\pi} \ldots,$$

the homology of which is called the *Poisson-Lichnerowicz cohomology*. In this section, we show there is an analogous cohomology on $\mathbb{D}S$ that descends to the classical Poisson-Lichnerowicz cohomology on the quotient spaces of the representation spaces of the algebra.

Let A be an associative algebra with unit. From [4, §7] we know that the Poisson bracket on $\mathsf{rep}_n(A)$ and $\mathsf{iss}_n(A)$ induced by a double Poisson tensor P corresponds to the Poisson tensor $tr(P)$. We furthermore know that the map $tr : \mathbb{D}S/[\mathbb{D}S, \mathbb{D}S] \to \bigwedge \mathrm{Der}(\mathcal{O}(\mathsf{rep}_n(A))$ is a morphism of graded Lie algebras, so we have a morphism of complexes

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathbb{D}S/[\mathbb{D}S, \mathbb{D}S])^0 & \xrightarrow{d_P} & (\mathbb{D}S/[\mathbb{D}S, \mathbb{D}S])^1 & \xrightarrow{d_P} & (\mathbb{D}S/[\mathbb{D}S, \mathbb{D}S])^2 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle tr} & & \downarrow{\scriptstyle tr} & & \downarrow{\scriptstyle tr} & & \\
0 & \longrightarrow & \mathcal{O}(\mathsf{rep}_n(A)) & \xrightarrow{d_{tr(P)}} & \mathrm{Der}(\mathcal{O}(\mathsf{rep}_n(A))) & \xrightarrow{d_{tr(P)}} & \wedge^2\mathrm{Der}(\mathcal{O}(\mathsf{rep}_n(A))) & \longrightarrow & \cdots
\end{array}
$$

which restricts to a morphism of complexes

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathbb{D}S/[\mathbb{D}S, \mathbb{D}S])^0 & \xrightarrow{d_P} & (\mathbb{D}S/[\mathbb{D}S, \mathbb{D}S])^1 & \xrightarrow{d_P} & (\mathbb{D}S/[\mathbb{D}S, \mathbb{D}S])^2 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle tr} & & \downarrow{\scriptstyle tr} & & \downarrow{\scriptstyle tr} & & \\
0 & \longrightarrow & \mathcal{O}(\mathsf{iss}_n(A)) & \xrightarrow{d_{tr(P)}} & \mathrm{Der}(\mathcal{O}(\mathsf{iss}_n(A))) & \xrightarrow{d_{tr(P)}} & \wedge^2\mathrm{Der}(\mathcal{O}(\mathsf{iss}_n(A))) & \longrightarrow & \cdots
\end{array}
$$

So there is a map from the homology $H^\bullet_P(A)$ of the upper chain complex, which we call the *double Poisson-Lichnerowicz cohomology* to the classical Poisson-Lichnerowicz cohomology on $\mathsf{rep}_n(A)$ and $\mathsf{iss}_n(A)$.

Using the description of $\mathbb{D}S/[\mathbb{D}S, \mathbb{D}S]$ from Proposition 3 we can compute the double Poisson-Lichnerowicz cohomology for a semi-simple algebra S when the double Poisson bracket is given by a single necklace in a straightforward way. We illustrate this for the zero-th and first cohomology groups. It is easy to see that $H^0_P(S) = 0$ from the description of the double Schouten bracket. For $H^1_P(S)$ we get

Proposition 6 *For a double Poisson bracket $P = i \underset{(r,s)}{\overset{(p,q)}{\rightleftarrows}} j$ with $q \neq r$ and $p \neq s$*

we have that $H^1_P(S)$ is generated by

- *all loops of all bi-colours in vertices $k \neq i, j$ in $\overline{Q}_S$,*
- *the loop in i with all possible bi-colourings where the primary colour is different from q and the secondary colour is different from r,*

- *the loop in j with all possible bi-colourings where the primary colour is different from s and the secondary colour is different from p,*

- *the sum* (r,r) $\circlearrowright i + j \circlearrowleft$ (s,s) *and*

- *the sum* (p,p) $\circlearrowright i + j \circlearrowleft$ (q,q) ,

and hence has dimension $\sum_{k \neq i,j} (n_k^2 - 1) + (n_i - 1)^2 + (n_j - 1)^2.$

For a double Poisson bracket $P = i \overset{(p,q)}{\underset{(r,s)}{\rightleftarrows}} i$ *we have that* $H^1_P(S)$ *is generated by all loops of all bi-colours in vertices* $j \neq i$ *and the loop in i with all possible bi-colours where the primary colour differs from s and q and the secondary colour differs from p and r if* $(p - q)(p - s)(r - s)(r - q) \neq 0.$

Proof The first claim follows immediately from the fact that

$$\left\{ i \overset{(p,q)}{\underset{(r,s)}{\rightleftarrows}} j \ , \ i \overset{(u,v)}{\longrightarrow} i \right\} = \delta_{vr} \ i \overset{(p,q)}{\underset{(u,s)}{\rightleftarrows}} j \ - \delta_{uq} \ i \overset{(p,v)}{\underset{(r,s)}{\rightleftarrows}} j \ ,$$

$$\left\{ i \overset{(p,q)}{\underset{(r,s)}{\rightleftarrows}} j \ , \ j \overset{(u',v')}{\longrightarrow} j \right\} = -\delta_{su'} \ i \overset{(p,q)}{\underset{(r,v')}{\rightleftarrows}} j \ + \delta_{v'p} \ i \overset{(u',q)}{\underset{(r,s)}{\rightleftarrows}} j \ ,$$

and

$$\left\{ i \overset{(p,q)}{\underset{(r,s)}{\rightleftarrows}} j \ , \ k \overset{(u,v)}{\longrightarrow} k \right\} = 0$$

if $k \neq i, j$, in combination with the fact that $\{P, S\} = 0$. The second claim follows immediately from

$$\left\{ i \overset{(p,q)}{\underset{(r,s)}{\rightleftarrows}} i \ , \ i \overset{(u,v)}{\longrightarrow} i \right\} = - \delta_{pv} \ i \overset{(u,q)}{\underset{(r,s)}{\rightleftarrows}} i \ + \delta_{us} \ i \overset{(p,q)}{\underset{(r,v)}{\rightleftarrows}} i$$

$$+\delta_{vr} \ i \overset{(p,q)}{\underset{(u,s)}{\rightleftarrows}} i \ - \delta_{uq} \ i \overset{(p,v)}{\underset{(r,s)}{\rightleftarrows}} i$$

$\square$

It becomes clear from the proof that the double Poisson-Lichnerowicz cohomology for a generic finite dimensional semi-simple algebra will never become zero. For the simplest non-trivial semi-simple algebra $S = \mathbb{C} \oplus \mathbb{C}$ one easily sees that

$$\dim H_P^i(S) = \begin{cases} 1 & i \neq 0 \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

5 Double Derivations on Amalgamated Products

In [1], the notion of a tree of semi-simple algebras was introduced and linked to the representation theory of $\mathsf{SL}_2(\mathbb{Z})$ via the amalgamated product $\mathbb{Z}_6 *_{\mathbb{Z}_2} \mathbb{Z}_4$. The study of the finite dimensional representations of this amalgamated product can be seen as equivalent to the study of the finite dimensional representations of the amalgamated product of semi-simple algebras $\mathbb{C}^{\oplus 6} *_{\mathbb{C}^{\oplus 2}} \mathbb{C}^{\oplus 4}$. More generally, the representation theory of any torus knot group can be reduced via the representation theory of $\mathbb{Z}_p * \mathbb{Z}_q$ to the representation theory of $\mathbb{C}^{\oplus p} * \mathbb{C}^{\oplus q}$. Adriaenssens and Le Bruyn show in [1] that the representation spaces of these amalgamated products have an étale cover characterized by a symmetric quiver and hence has a double Poisson structure as well.

From [4, Prop 2.4.1] we know that each pair of double Poisson brackets on the factors of an amalgamated product induces a unique double Poisson bracket on the product, so combining the nonzero double Poisson brackets we determined in the previous section of the paper, we obtain double Poisson structures on the representation spaces of the amalgamated products themselves rather than on an étale cover. In this section, we will formulate an explicit description of the Poisson brackets they induce on the quotient space $\text{iss}_n(S * T)$

We begin by fixing notation for the rest of this section. We let $S = \mathbb{C}^{\oplus n}$ with orthogonal idempotents $e_1, \ldots, e_n$ and $T = \mathbb{C}^{\oplus m}$ with orthogonal idempotents $f_1, \ldots, f_m$. By Q_S and Q_T we denote as before the double derivation quivers of S and T. The arrows in Q_S will be denoted by a_{ij} and the arrows in Q_T will be b_{kl} with $h(a_{ij}) = i$, $t(a_{ij}) = j$, $h(b_{kl}) = k$ and $t(b_{kl}) = l$. Finally, we let $P = \sum_{i,j} c_{ij} a_{ij} a_{ji}$ be a double Poisson tensor in $\mathbb{C}Q_S$ and $P' = \sum_{k,l} d_{kl} b_{kl} b_{lk}$ a double Poissson tensor in $\mathbb{C}Q_T$. We have the following expression for the double Poisson bracket induced by P and P' on $S * T$.

Lemma 4 *Let $x = e_{i_1} * f_{i_1'} * \cdots * e_{i_p} * f_{i_p'}$ and $y = e_{j_1} * f_{j_1'} * \cdots * e_{j_q} * f_{j_q'}$. Define for all $1 \leq i, j \leq n$ and $1 \leq k, \ell \leq m$ the elements $\bar{c}_{ij} = c_{ij} - c_{ji}$ and $\bar{d}_{k\ell} = d_{k\ell} - d_{\ell k}$ in $\mathbb{C}$ and the elements $\bar{e}_i = -\sum_{s \neq i} \bar{c}_{is} e_s$ and $\bar{f}_k = -\sum_{s \neq i} \bar{d}_{ks} e_s$ then*

$$\{\{x, y\}\} = \sum_{\ell=1}^{p} \sum_{\substack{k=1 \\ j_k' \neq i_\ell'}}^{q} \bar{d}_{i_\ell' j_k'} \left(e_{j_1} * \cdots * e_{j_k} * \left(f_{i_\ell'} * \cdots * f_{i_p'} \otimes e_{i_1} * \cdots * e_{i_\ell} * f_{j_k'} \right. \right.$$

$$\left. \left. - f_{j_k'} * e_{i_{\ell+1}} * \cdots * f_{i_p'} \otimes e_{i_1} * \cdots * f_{i_\ell'} \right) * e_{j_{k+1}} * \cdots * f_{j_q'} \right)$$

$$+ \sum_{\substack{k=1 \\ j_k \neq i_\ell}}^{q} \bar{c}_{i_\ell j_k} \Big(e_{j_1} * \cdots * f_{j'_{k-1}} * \big(e_{i_\ell} * \cdots * f_{i'_p} \otimes e_{i_1} * \cdots * f_{i'_{\ell-1}} * e_{j_k}$$

$$- e_{j_k} * f_{i'_\ell} * \cdots * f_{i'_p} \otimes e_{i_1} * \cdots * e_{i_\ell} \big) * f_{j'_k} * \cdots * f_{j'_q} \Big)$$

$$+ \sum_{\substack{k=1 \\ j'_k = i'_\ell}}^{q} e_{j_1} * \cdots * e_{j_k} * \Big(f_{i'_\ell} * \cdots * f_{i'_p} \otimes e_{i_1} * \cdots * e_{i_\ell} * \bar{f}_{j'_k}$$

$$- \bar{f}_{j'_k} * e_{i_{\ell+1}} * \cdots * f_{i'_p} \otimes e_{i_1} * \cdots * f_{i'_\ell} \big) * e_{j_{k+1}} * \cdots * f_{j'_q}$$

$$+ \sum_{\substack{k=1 \\ j_k = i_\ell}}^{q} e_{j_1} * \cdots * f_{j'_{k-1}} * \big(e_{i_\ell} * \cdots * f_{i'_p} \otimes e_{i_1} * \cdots * f_{i'_{\ell-1}} * \bar{e}_{j_k}$$

$$- \bar{e}_{j_k} * f_{i'_\ell} * \cdots * f_{i'_p} \otimes e_{i_1} * \cdots * e_{i_\ell} \big) * f_{j'_k} * \cdots * f_{j'_q}.$$

Proof We will prove the statement by induction on the length $2p$ of x. First of all note that

$$\{\!\{ f_b, e_{j_1} * f_{j_1} * \cdots * e_{j_q} * f_{j_q} \}\!\} = \sum_{k=1}^{q} e_{j_1} * \cdots * e_{j_k} * \{\!\{ f_b, f_{j_k} \}\!\} * e_{j_{k+1}} * \cdots * f_{j_q}$$

and

$$\{\!\{ e_a, e_{j_1} * f_{j_1} * \cdots * e_{j_q} * f_{j_q} \}\!\} = \sum_{k=1}^{q} e_{j_1} * \cdots * f_{j_{k-1}} * \{\!\{ e_a, e_{j_k} \}\!\} * f_{j_k} * \cdots * f_{j_q}$$

because $\{\!\{ f_b, - \}\!\}$ is an S-linear double derivation and $\{\!\{ e_a, - \}\!\}$ is a T-linear double derivation. This means for $p = 1$ that

$$\{\!\{ e_a * f_b, y \}\!\} = e_a \circ \{\!\{ f_b, y \}\!\} + \{\!\{ e_a, y \}\!\} \circ f_b$$

$$= e_a \circ \sum_{k=1}^{q} e_{j_1} * \cdots * e_{j_k} * \{\!\{ f_b, f_{j_k} \}\!\} * e_{j_{k+1}} * \cdots * f_{j_q}$$

$$+ \sum_{k=1}^{q} e_{j_1} * \cdots * f_{j_{k-1}} * \{\!\{ e_a, e_{j_k} \}\!\} * f_{j_k} * \cdots * f_{j_q}) \circ f_b.$$

Now by definition of the double brackets associated to P and P' we get

$$\{\!\{ f_b, f_{j_k} \}\!\} = (d_{b j_k} - d_{j_k b}) (f_b \otimes f_{j_k} - f_{j_k} \otimes f_b) \, b \neq j_k$$

$$\{\!\{ f_b, f_b \}\!\} = - \sum_{s \neq b} (d_{bs} - d_{sb}) (f_b \otimes f_s - f_s \otimes f_b)$$

$$= f_b \otimes \bar{f}_b - \bar{f}_b \otimes f_b$$

and

$$\{\{e_a, e_{j_k}\}\} = (c_{a j_k} - c_{j_k a})(e_a \otimes e_{j_k} - e_{j_k} \otimes e_a) \qquad a \neq j_k$$

$$\{\{e_a, e_a\}\} = -\sum_{s \neq a}(c_{as} - c_{sa})(e_a \otimes e_s - e_s \otimes e_a)$$

$$= e_a \otimes \bar{e}_a - \bar{e}_a \otimes e_a.$$

So

$$\{\{e_a * f_b, y\}\} = \sum_{\substack{k=1 \\ j_k \neq b}}^{q} \bar{d}_{b j_k} e_{j_1} * \cdots * e_{j_k} * \left(f_b \otimes e_a * f_{j_k} - f_{j_k} \otimes e_a * f_b \right) * e_{j_{k+1}} * \cdots * f_{j_q}$$

$$+ \sum_{\substack{k=1 \\ j_k \neq a}}^{q} \bar{c}_{a j_k} e_{j_1} * \cdots * f_{j_{k-1}} * \left(e_a * f_b \otimes e_{j_k} - e_{j_k} * f_b \otimes e_a \right) * f_{j_k} * \cdots * f_{j_q}$$

$$+ \sum_{\substack{k=1 \\ j_k = b}}^{q} e_{j_1} * \cdots * e_{j_k} * \left(f_b \otimes e_a * \overline{f}_b - \overline{f}_b \otimes e_a * f_b \right) * e_{j_{k+1}} * \cdots * f_{j_q}$$

$$+ \sum_{\substack{k=1 \\ j_k = a}}^{q} e_{j_1} * \cdots * f_{j_{k-1}} * (e_a * f_b \otimes \bar{e}_a - \bar{e}_a * f_b \otimes e_a) * f_{j_k} * \cdots * f_{j_q}$$

which means the claim holds for $p = 1$. Now assume the lemma is correct up to $p - 1$, then

$$\{\{x, y\}\} = \left(e_{i_1} * \cdots * f_{i'_{p-1}} \right) \circ \{\{e_{i_p} * f_{i'_p}, y\}\} + \{\{e_{i_1} * \cdots * f_{i'_{p-1}}, y\}\} \circ \left(e_{i_p} * f_{i'_p} \right).$$

The first term in this expression is equal to

$$\sum_{\substack{k=1 \\ j_k \neq i'_\ell}}^{q} \bar{d}_{i_p j_k} \left(e_{j_1} * \cdots * e_{j_k} * \left(f_{i'_p} * \cdots * f_{i'_p} \otimes e_{i_1} * \cdots * e_{i_p} * f_{j_k} \right. \right.$$

$$\left. \left. - f_{j_k} * e_{i_{p+1}} * \cdots * f_{i'_p} \otimes e_{i_1} * \cdots * f_{i'_p} \right) * e_{j_{k+1}} * \cdots * f_{j_q} \right)$$

$$+ \sum_{\substack{k=1 \\ j_k \neq i_p}}^{q} \bar{c}_{i_p j_k} \left(e_{j_1} * \cdots * f_{j_{k-1}} * \left(e_{i_p} * \cdots * f_{i'_p} \otimes e_{i_1} * \cdots * f_{i'_{p-1}} * e_{j_k} \right. \right.$$

$$\left. \left. - e_{j_k} * f_{i'_p} * \cdots * f_{i'_p} \otimes e_{i_1} * \cdots * e_{i_p} \right) * f_{j_k} * \cdots * f_{j_q} \right)$$

$$+ \sum_{\substack{k=1 \\ j_k = i'_p}}^{q} e_{j_1} * \cdots * e_{j_k} * \left(f_{i'_p} * \cdots * f_{i'_p} \otimes e_{i_1} * \cdots * e_{i_p} * \overline{f}_{j_k} \right.$$

$$\left. - \overline{f}_{j_k} * e_{i_{p+1}} * \cdots * f_{i'_p} \otimes e_{i_1} * \cdots * f_{i'_p} \right) * e_{j_{k+1}} * \cdots * f_{j_q}$$

$$
+ \sum_{\substack{k=1 \\ j_k = i_p}}^{q} e_{j_1} * \cdots * f_{j_{k-1}} * \left(e_{i_p} * \cdots * f_{i'_p} \otimes e_{i_1} * \cdots * f_{i'_{p-1}} * \overline{e}_{j_k} \right.
$$

$$
\left. - \overline{e}_{j_k} * f_{i'_p} * \cdots * f_{i'_p} \otimes e_{i_1} * \cdots * e_{i_p} \right) * f_{j'_k} * \cdots * f_{j'_q}.
$$

and the second term becomes

$$
\{\!\{x, y\}\!\} = \sum_{\ell=1}^{p-1} \sum_{\substack{k=1 \\ j'_k \neq i'_\ell}}^{q} \overline{d}_{i_\ell j_k} \left(e_{j_1} * \cdots * e_{j_k} * \left(f_{i'_\ell} * \cdots * f_{i'_p} \otimes e_{i_1} * \cdots * e_{i_\ell} * f_{j'_k} \right. \right.
$$

$$
\left. \left. - f_{j'_k} * e_{i_{\ell+1}} * \cdots * f_{i'_p} \otimes e_{i_1} * \cdots * f_{i'_\ell} \right) * e_{j_{k+1}} * \cdots * f_{j'_q} \right)
$$

$$
+ \sum_{\substack{k=1 \\ j_k \neq i_\ell}}^{q} \overline{c}_{i_\ell j_k} \left(e_{j_1} * \cdots * f_{j'_{k-1}} * \left(e_{i_\ell} * \cdots * f_{i'_p} \otimes e_{i_1} * \cdots * f_{i'_{\ell-1}} * e_{j_k} \right. \right.
$$

$$
\left. \left. - e_{j_k} * f_{i'_\ell} * \cdots * f_{i'_p} \otimes e_{i_1} * \cdots * e_{i_\ell} \right) * f_{j'_k} * \cdots * f_{j'_q} \right)
$$

$$
+ \sum_{\substack{k=1 \\ j'_k = i'_\ell}}^{q} e_{j_1} * \cdots * e_{j_k} * \left(f_{i'_\ell} * \cdots * f_{i'_p} \otimes e_{i_1} * \cdots * e_{i_\ell} * \overline{f}_{j'_k} \right.
$$

$$
\left. - \overline{f}_{j'_k} * e_{i_{\ell+1}} * \cdots * f_{i'_p} \otimes e_{i_1} * \cdots * f_{i'_\ell} \right) * e_{j_{k+1}} * \cdots * f_{j'_q}
$$

$$
+ \sum_{\substack{k=1 \\ j_k = i_\ell}}^{q} e_{j_1} * \cdots * f_{j'_{k-1}} * \left(e_{i_\ell} * \cdots * f_{i'_p} \otimes e_{i_1} * \cdots * f_{i'_{\ell-1}} * \overline{e}_{j_k} \right.
$$

$$
\left. - \overline{e}_{j_k} * f_{i'_\ell} * \cdots * f_{i'_p} \otimes e_{i_1} * \cdots * e_{i_\ell} \right) * f_{j'_k} * \cdots * f_{j'_q}.
$$

Adding these two terms together then yields the expression from the lemma. □

Proposition 7 *With notations as before, the Poisson bracket induced by P and P' on* iss$(S * T)$ *is equal to*

$$
\{tr(x), tr(y)\} = \sum_{\ell=1}^{p} tr \left(\sigma^{2\ell-1}(x) * \left(\sum_{\substack{k=1 \\ j'_k \neq i'_\ell}}^{q} \overline{d}_{i'_\ell j'_k} \sigma^{2k-1}(y) - \sum_{\substack{k=1 \\ j_k \neq i_\ell}}^{q} \overline{c}_{i_\ell j_k} \sigma^{2k-1}(y) \right) \right)
$$

$$
- tr \left(\sigma^{2\ell}(x) * \left(\sum_{\substack{k=1 \\ j'_k \neq i'_\ell}}^{q} \overline{d}_{i'_\ell j'_k} \sigma^{2k}(y) - \sum_{\substack{k=1 \\ j_k \neq i_\ell}}^{q} \overline{c}_{i_{\ell+1} j_k} \sigma^{2k}(y) \right) \right)
$$

$$+ \sum_{\substack{k=1 \\ j_k' = i_\ell'}}^{q} tr\left(\sigma^{2\ell-1}(x) * \sigma^{2k-1}(y_k)\right) - tr\left(\sigma^{2\ell}(x) * \sigma^{2k}(y_k)\right)$$

$$+ \sum_{\substack{k=1 \\ j_k = i_\ell}}^{q} tr\left(\sigma^{2(\ell-1)}(x) * \sigma^{2(k-1)}(_k y)\right) - tr\left(\sigma^{2\ell-1}(x) * \sigma^{2k-1}(_k y)\right).$$

with $\sigma(a * x) = x * a$ the cyclic permutation and where $y_k = e_{j_1} * f_{j_1'} * \cdots * f_{j_{k-1}'} * e_{j_k} * \overline{f}_{j_k} * e_{j_{k+1}} * \cdots * e_{j_q} * f_{j_q'}$ and $_k y = e_{j_1} * f_{j_1'} * \cdots * f_{j_{k-1}'} * \overline{e}_{j_k} * f_{j_k} * \cdots * e_{j_q} * f_{j_q'}$.

Proof By definition of the bracket induced on $\mathrm{iss}_n(S * T)$ we have

$$\{tr(x), tr(y)\} = \{\!\{x, y\}\!\}_{ij} \{\!\{x, y\}\!\}_{ji}.$$

Using the expression found in the previous lemma this becomes

$$\sum_{\ell=1}^{p} \sum_{\substack{k=1 \\ j_k' \neq i_\ell'}}^{q} \overline{d}_{i_\ell j_k} \left(\left(e_{j_1} * \cdots * e_{j_k} * f_{i_\ell'} * \cdots * f_{i_p'}\right)_{rs} \left(e_{i_1} * \cdots * e_{i_\ell} * f_{j_k'} * e_{j_{k+1}} * \cdots * f_{j_q'}\right)_{sr} \right.$$

$$\left. - \left(e_{j_1} * \cdots * e_{j_k} * f_{j_k'} * e_{i_{\ell+1}} * \cdots * f_{i_p'}\right)_{rs} \left(e_{i_1} * \cdots * f_{i_\ell'} * e_{j_{k+1}} * \cdots * f_{j_q'}\right)_{sr} \right)$$

$$+ \sum_{\substack{k=1 \\ j_k \neq i_\ell}}^{q} \overline{c}_{i_\ell j_k} \left(\left(e_{j_1} * \cdots * f_{j_{k-1}'} * e_{i_\ell} * \cdots * f_{i_p'}\right)_{rs} \left(e_{i_1} * \cdots * f_{i_{\ell-1}'} * e_{j_k} * f_{j_k'} * \cdots * f_{j_q'}\right)_{sr} \right.$$

$$\left. - \left(e_{j_1} * \cdots * f_{j_{k-1}'} * e_{j_k} * f_{i_\ell'} * \cdots * f_{i_p'}\right)_{rs} \left(e_{i_1} * \cdots * e_{i_\ell} * f_{j_k'} * \cdots * f_{j_q'}\right)_{sr} \right)$$

$$+ \sum_{\substack{k=1 \\ j_k' = i_\ell'}}^{q} \left(e_{j_1} * \cdots * e_{j_k} * f_{i_\ell'} * \cdots * f_{i_p'}\right)_{rs} \left(e_{i_1} * \cdots * e_{i_\ell} * \overline{f}_{j_k'} * e_{j_{k+1}} * \cdots * f_{j_q'}\right)_{sr}$$

$$- \left(e_{j_1} * \cdots * e_{j_k} * \overline{f}_{j_k'} * e_{i_{\ell+1}} * \cdots * f_{i_p'}\right)_{rs} \left(e_{i_1} * \cdots * f_{i_\ell'} * e_{j_{k+1}} * \cdots * f_{j_q'}\right)_{sr}$$

$$+ \sum_{\substack{k=1 \\ j_k = i_\ell}}^{q} \left(e_{j_1} * \cdots * f_{j_{k-1}'} * e_{i_\ell} * \cdots * f_{i_p'}\right)_{rs} \left(e_{i_1} * \cdots * f_{i_{\ell-1}'} * \overline{e}_{j_k} * f_{j_k'} * \cdots * f_{j_q'}\right)_{sr}$$

$$- \left(e_{j_1} * \cdots * f_{j_{k-1}'} * \overline{e}_{j_k} * f_{i_\ell'} * \cdots * f_{i_p'}\right)_{rs} \left(e_{i_1} * \cdots * e_{i_\ell} * f_{j_k'} * \cdots * f_{j_q'}\right)_{sr}.$$

This is equal to

$$\sum_{\ell=1}^{p} \sum_{\substack{k=1 \\ j_k' \neq i_\ell'}}^{q} \overline{d}_{i_\ell j_k} \left(tr\left(e_{j_1} * \cdots * e_{j_k} * f_{i_\ell'} * \cdots * f_{i_p'} * e_{i_1} * \cdots * e_{i_\ell} * f_{j_k'} * e_{j_{k+1}} * \cdots * f_{j_q'} \right) \right.$$

$$\left. - tr\left(e_{j_1} * \cdots * e_{j_k} * f_{j_k'} * e_{i_{\ell+1}} * \cdots * f_{i_p'} * e_{i_1} * \cdots * f_{i_\ell'} * e_{j_{k+1}} * \cdots * f_{j_q'} \right) \right)$$

$$+ \sum_{\substack{k=1 \\ j_k \neq i_\ell}}^{q} \overline{c}_{i_\ell j_k} \left(tr\left(e_{j_1} * \cdots * f_{j_{k-1}'} * e_{i_\ell} * \cdots * f_{i_p'} * e_{i_1} * \cdots * f_{i_{\ell-1}'} * e_{j_k} * f_{j_k'} * \cdots * f_{j_q'} \right) \right.$$

$$\left. - tr\left(e_{j_1} * \ldots * f_{j_{k-1}'} * e_{j_k} * f_{i_\ell'} * \cdots * f_{i_p'} * e_{i_1} * \cdots * e_{i_\ell} * f_{j_k'} * \cdots * f_{j_q'} \right) \right)$$

$$+ \sum_{\substack{k=1 \\ j_k' = i_\ell'}}^{q} tr\left(e_{j_1} * \cdots * e_{j_k} * f_{i_\ell'} * \cdots * f_{i_p'} * e_{i_1} * \cdots * e_{i_\ell} * \overline{f}_{j_k'} * e_{j_{k+1}} * \cdots * f_{j_q'} \right)$$

$$- tr\left(e_{j_1} * \cdots * e_{j_k} * \overline{f}_{j_k'} * e_{i_{\ell+1}} * \cdots * f_{i_p'} * e_{i_1} * \cdots * f_{i_\ell'} * e_{j_{k+1}} * \cdots * f_{j_q'} \right)$$

$$+ \sum_{\substack{k=1 \\ j_k = i_\ell}}^{q} tr\left(e_{j_1} * \cdots * f_{j_{k-1}'} * e_{i_\ell} * \cdots * f_{i_p'} * e_{i_1} * \cdots * f_{i_{\ell-1}'} * \overline{e}_{j_k} * f_{j_k'} * \cdots * f_{j_q'} \right)$$

$$- tr\left(e_{j_1} * \cdots * f_{j_{k-1}'} * \overline{e}_{j_k} * f_{i_\ell'} * \cdots * f_{i_p'} * e_{i_1} * \cdots * e_{i_\ell} * f_{j_k'} * \cdots * f_{j_q'} \right).$$

Using the fact that traces are invariant under cyclic permutation, we may rewrite this as

$$\sum_{\ell=1}^{p} \sum_{\substack{k=1 \\ j_k' \neq i_\ell'}}^{q} \overline{d}_{i_\ell j_k} \left(tr\left(\underbrace{f_{i_\ell'} * \cdots * f_{i_p'} * e_{i_1} * \cdots * e_{i_\ell}}_{\sigma^{2\ell-1}(x)} * \underbrace{f_{j_k'} * e_{j_{k+1}} * \cdots * f_{j_q'} * e_{j_1} * \cdots * e_{j_k}}_{\sigma^{2k-1}(y)} \right) \right.$$

$$\left. - tr\left(\underbrace{e_{i_{\ell+1}} * \cdots * f_{i_p'} * e_{i_1} * \cdots * f_{i_\ell'}}_{\sigma^{2\ell}(x)} * \underbrace{e_{j_{k+1}} * \cdots * f_{j_q'} * e_{j_1} * \cdots * e_{j_k} * f_{j_k'}}_{\sigma^{2k}(y)} \right) \right)$$

$$+ \sum_{\substack{k=1 \\ j_k \neq i_\ell}}^{q} \overline{c}_{i_\ell j_k} \left(tr\left(\underbrace{e_{i_\ell} * \cdots * f_{i_p'} * e_{i_1} * \cdots * f_{i_{\ell-1}'}}_{\sigma^{2(\ell-1)}(x)} * \underbrace{e_{j_k} * f_{j_k'} * \cdots * f_{j_q'} * e_{j_1} * \cdots * f_{j_{k-1}'}}_{\sigma^{2(k-1)}(y)} \right) \right.$$

$$\left. - tr\left(\underbrace{* f_{i_\ell'} * \cdots * f_{i_p'} * e_{i_1} * \cdots * e_{i_\ell}}_{\sigma^{2\ell-1}(x)} * \underbrace{f_{j_k'} * \cdots * f_{j_q'} * e_{j_1} * \ldots * f_{j_{k-1}'} * e_{j_k}}_{\sigma^{2k-1}(y)} \right) \right)$$

$$+ \sum_{\substack{k=1 \\ j'_k = i'_\ell}}^{q} tr\left(\underbrace{f_{i'_\ell} * \cdots * f_{i'_p} * e_{i_1} * \cdots * e_{i_\ell}}_{\sigma^{2\ell-1}(x)} * \underbrace{\overline{f}_{j'_k} * e_{j_{k+1}} * \cdots * f_{j'_q} * e_{j_1} * \cdots * e_{j_k}}_{\sigma^{2k-1}(y_k)} \right)$$

$$- tr\left(\underbrace{e_{i_{\ell+1}} * \cdots * f_{i'_p} * e_{i_1} * \cdots * f_{i'_\ell}}_{\sigma^{2\ell}(x)} * \underbrace{e_{j_{k+1}} * \cdots * f_{j'_q} * e_{j_1} * \cdots * e_{j_k} * \overline{f}_{j'_k}}_{\sigma^{2k}(y_k)} \right)$$

$$+ \sum_{\substack{k=1 \\ j_k = i_\ell}}^{q} tr\left(\underbrace{e_{i_\ell} * \cdots * f_{i'_p} * e_{i_1} * \cdots * f_{i'_{\ell-1}}}_{\sigma^{2(\ell-1)}(x)} * \underbrace{\overline{e}_{j_k} * f_{j'_k} * \cdots * f_{j'_q} * e_{j_1} * \cdots * f_{j'_{k-1}}}_{\sigma^{2(k-1)}(_k y)} \right)$$

$$- tr\left(\underbrace{f_{i'_\ell} * \cdots * f_{i'_p} * e_{i_1} * \cdots * e_{i_\ell}}_{\sigma^{2\ell-1}(x)} * \underbrace{f_{j'_k} * \cdots * f_{j'_q} * e_{j_1} * \cdots * f_{j'_{k-1}} * \overline{e}_{j_k}}_{\sigma^{2k-1}(_k y)} \right).$$

Regrouping this expression then gives the expression from the proposition. □

Although this expression yields nontrivial Poisson brackets in general, note that we have the following corollary.

Corollary 2 *For $S = \mathbb{C}^{\oplus 2}$ and $T = \mathbb{C}^{\oplus d}$ with d arbitrary, any Poisson bracket induced by double Poisson tensors P and P' becomes the zero-bracket.*

Proof First of all note that for $S = \mathbb{C}^{\oplus 2}$ the ring of invariants $\mathbb{C}[\text{rep}_n(S * T)]^{\text{GL}_n}$ is generated by expressions of the form $tr(e_1 * f_{i_1} * e_1 * f_{i_2} \cdots * e_1 * f_{i_p})$. But then

$$\sigma^{2\ell-1}(x) * \sigma^{2k-1}(y) = \sigma^{2\ell}(x)\sigma^{2k}(y)$$
$$\sigma^{2\ell-1}(x) * \sigma^{2k-1}(y_k) = \sigma^{2\ell}(x) * \sigma^{2k}(y_k)$$
$$\sigma^{2(\ell-1)}(x) * \sigma^{2(k-1)}(_k y) = \sigma^{2\ell-1}(x) * \sigma^{2k-1}(_k y)$$

up to cyclic permutation, so the bracket becomes zero. □

From Theorem 2 we know the module of $\mathbb{C}^{\oplus 2}$-relative double derivations on $\mathbb{C}^{\oplus 4}$ can be depicted as

and the module of $\mathbb{C}^{\oplus 2}$-relative double derivations on $\mathbb{C}^{\oplus 6}$ can be depicted as
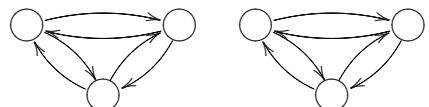
The last corollary then also means that we only get the trivial bracket on this amalgamated product, so in case of $\text{SL}_2(\mathbb{Z})$ we do not obtain a Poisson structure on the quotient spaces from the double Poisson structures on the factors, whereas Adriaenssens and Le Bruyn obtain nontrivial Poisson structures on an étale cover. For higher dimensional factors, however, the above formula yields nontrivial Poisson brackets on the quotient variety.

References

1. Adriaenssens, J., Le Bruyn, L.: Trees of semi-simple algebras. math.RA/0507503 (2005)
2. Bhaskara, K.H.: Affine Poisson structures. Proc. Indian Acad. Sci. Math. Sci. **100**(3), 189–202 (1990)
3. Lichnerowicz, A.: Les variétés de Poisson et leurs algèbres de Lie associées. J. Differential Geom. **12**(2), 253–300 (1977)
4. Van den Bergh, M.: Double poisson algebras. math.AG/0410528 (2004)
5. Weinstein, A.: The local structure of Poisson manifolds. J. Differential Geom. **18**(3), 523–557 (1983)