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One Dimensional Tilting Modules are of Finite Type

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Abstract We prove that every tilting module of projective dimension at most one is of finite type, namely that its associated tilting class is the Ext-orthogonal of a family of finitely presented modules of projective dimension at most one.

Keywords Tilting modules · Finite type · Ext-orthogonal

Mathematics Subject Classifications (2000) Primary 16D90 · 16E30 · Secondary 16G99

1 Introduction

The concept of tilting modules has its origin in the Representation Theory of finite dimensional algebras (see [10, 16]). In this context, tilting modules were assumed to be finitely generated and of projective dimension at most one. Later the theory was extended to arbitrary rings and to infinitely generated modules (see [11, 12]). We show that, even in the case of not necessarily finitely generated tilting modules, strong finiteness conditions are involved: a tilting class is determined by finitely presented data, namely there exists a family $\mathcal{R}$ of finitely presented modules, of projective dimension at most one, such that the tilting class is the Ext-orthogonal to $\mathcal{R}$. This implies that tilting classes are in bijective correspondence with resolving

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classes consisting of finitely presented modules of projective dimension at most one. This is a first step towards the classification of tilting classes.

Our starting point are the results in [9] that we collect in Theorem 2.1. In [9] it is shown that tilting modules of projective dimension at most one are of finite type if and only if the corresponding tilting classes are definable, namely closed under arbitrary direct products, direct limits, and pure submodules (cf. [13, Section 2.3]). In terms of model theory, this means that modules in a definable class are characterized by formulas of the first order in the language of model theory.

Since tilting classes are already closed under direct products and direct limits, *tilting modules are of finite type if and only if tilting classes are closed under pure submodules*. Also in [9], it is established that tilting classes are of countable type. More precisely, for any tilting class $\mathcal{B}$ there exists a family $\mathcal{S}$ of countably presented modules, of projective dimension at most one, such that $\mathcal{B}$ is the Ext-orthogonal to $\mathcal{S}$. Every module in $\mathcal{S}$ is a direct limit of a direct system of the form

$$C_1 \overset{f_1}{\to} C_2 \overset{f_2}{\to} C_3 \to \ldots \to C_n \overset{f_n}{\to} C_{n+1} \to \ldots$$

where C_n are *finitely presented* modules, and such a direct system fits in a (pure) exact sequence

$$0 \to \oplus_{n\in\mathbb{N}} C_n \overset{\phi}{\to} \oplus_{n\in\mathbb{N}} C_n \to \varinjlim C_n = A \to 0 \tag{1}$$

where, for every $n \in \mathbb{N}$, $\phi\varepsilon_n = \varepsilon_n - \varepsilon_{n+1} f_n$ and $\varepsilon_n \colon C_n \to \oplus_{n\in\mathbb{N}} C_n$ denotes the canonical map. Since $M \in \mathcal{B}$ if and only if $\mathrm{Ext}^1_R(A, M) = 0$ for any $A \in \mathcal{S}$, it follows from Eq. 1 that $\mathrm{Hom}_R(\phi, M)$ is onto for any module M in the tilting class $\mathcal{B}$. This means that any map $\oplus_{n\in\mathbb{N}} C_n \to M$ factors through ϕ. As every map $\oplus_{n\in\mathbb{N}} C_n \to M$ can be factored through a *diagonal* map $\oplus_{n\in\mathbb{N}} C_n \to M^{(\mathbb{N})}$ (cf. Lemma 4.2) and tilting classes are closed under direct sums, to solve our problem we need to study when all *diagonal* maps $\oplus_{n\in\mathbb{N}} C_n \to M^{(\mathbb{N})}$ factor through ϕ.

The characterization of when the exact sequence (1) splits, goes back to the fundamental paper by Bass [6] and it is based on stationary conditions of suitable descending chains. In Theorem 3.7 we extend this characterization by proving that, for a given module M and a given sequence $(C_n)_{n\in\mathbb{N}}$ of *small* modules, all *diagonal* maps $\oplus_{n\in\mathbb{N}} C_n \to M^{(\mathbb{N})}$ factor through ϕ if and only if the inverse system $(\mathrm{Hom}_R(C_n, M), \mathrm{Hom}_R(f_n, M))_{n\in\mathbb{N}}$ satisfies the Mittag–Leffler condition. It is interesting to note that our result holds more generally for a sequence $(C_n)_{n\in\mathbb{N}}$ of *compact* objects in an additive category with countable coproducts.

To conclude that tilting classes are closed under pure submodules, hence of finite type, the crucial point is to observe that, when $(C_n)_{n\in\mathbb{N}}$ is a sequence of finitely presented modules, the Mittag–Leffler condition is inherited by pure submodules.

The paper is organized as follows: in Section 2 we state our main result Theorem 2.5, which is proved assuming the results in Sections 3 and 4 and that has as an immediate consequence in Theorem 2.6 that tilting modules are of finite type. In Section 3 we characterize when diagonal maps factor through ϕ, and in Section 4 we show that the characterization is inherited by pure submodules.

In Section 5 we present further applications of our results to tilting modules of projective dimension greater than one. We prove that the weak and the projective dimension of a tilting module must coincide.

Our analysis has consequences also in other directions. In Theorem 5.1 we describe a setting in which the Mittag–Leffler condition is equivalent to the exactness of countable inverse limits and also to the fact that countable direct limits of Ext-orthogonal objects are Ext-orthogonal.

2 Tilting Modules and Tilting Classes

In what follows R is always an associative ring with unit. We first recall some definitions and results.

Let $\mathcal{C}$ be a class of right R-modules. Define $\mathcal{C}^{\perp}=\{M\in\text{Mod-}R \mid \text{Ext}^1_R(C, M)=0$ for any $C \in \mathcal{C}\}$ and ${}^{\perp}\mathcal{C} = \{M \in \text{Mod-}R \mid \text{Ext}^1_R(M, C) = 0$ for any $C \in \mathcal{C}\}$.

A pair of classes of modules $(\mathcal{A}, \mathcal{B})$ is a *cotorsion pair* provided that $\mathcal{A} = {}^{\perp}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp}$.

Following [12], a right R-module T is said to be 1-tilting if and only if $T^{\perp} = \text{Gen}\, T$, where Gen T is the class of modules generated by T. A class of modules $\mathcal{B}$ is 1*-tilting* provided there is a 1-tilting module T such that $\mathcal{B} = T^{\perp}$. In this case, $({}^{\perp}\mathcal{B}, \mathcal{B})$ is a cotorsion pair, called the *cotorsion pair cogenerated by* T. All modules in ${}^{\perp}\mathcal{B}$ have projective dimension at most 1, and 1-tilting classes $\mathcal{B}$ are characterized by the properties: $\mathcal{B}$ is closed under direct sums and $\mathcal{B} = M^{\perp}$ for a module M with projective dimension at most 1 (cf. [4] and [14]).

As introduced in [3], a 1-tilting module T is said to be of *finite type* (*countable type*) provided there is a set $\mathcal{S}$ of *finitely presented* (*countably presented*) right R-modules of projective dimension at most 1 such that $T^{\perp} = \mathcal{S}^{\perp}$. A 1-tilting class $T^{\perp}$ is of *finite type* (*countable type*) in case T is.

A class of modules is called *definable* if it is closed under arbitrary direct products, direct limits, and pure submodules (cf. [13, Section 2.3]). Any 1-tilting class of finite type is definable; because for any finitely presented module G of projective dimension at most 1, $G^{\perp}$ is closed under pure submodules and the functor $\text{Ext}^1_R(G, -)$ commutes with direct products and direct limits. In general, a 1-tilting class is definable if and only if it is closed under pure submodules, since it is always closed under direct limits and direct products.

We collect in the following theorem a reformulation of some results proved in [9].

Theorem 2.1 *Let T be a* 1*-tilting module and $(\mathcal{A}, \mathcal{B})$ the cotorsion pair cogenerated by T. Then T is of countable type, and T is of finite type if and only if the class $\mathcal{B}$ is definable.*

Proof See [9, Proposition 1.1, Theorem 1.2 and Theorem 2.5]. □

Remark 2.2 If A is a countably presented right R-module, then it is a countable direct limit of finitely presented right R-modules. Moreover, it can be assumed that A is a direct limit of a direct system of the form

$$C_1 \xrightarrow{f_1} C_2 \xrightarrow{f_2} C_3 \to \dots \to C_n \xrightarrow{f_n} C_{n+1} \to \dots$$

where, for every $n \in \mathbb{N}$, C_n is finitely presented.

We are interested in studying direct systems of the form appearing in Remark 2.2. To this aim we fix the following notation which will be used in the rest of the paper, sometimes without previous acknowledgment.

Notation 2.3 Given a countable direct system

$$C_1 \xrightarrow{f_1} C_2 \xrightarrow{f_2} C_3 \to \dots \to C_n \xrightarrow{f_n} C_{n+1} \to \dots$$

of right R-modules, we consider the pure exact sequence:

$$0 \to \oplus_{n\in\mathbb{N}} C_n \xrightarrow{\phi} \oplus_{n\in\mathbb{N}} C_n \to \varinjlim C_n \to 0$$

where, for every $n \in \mathbb{N}$, $\phi\varepsilon_n = \varepsilon_n - \varepsilon_{n+1} f_n$ and $\varepsilon_n \colon C_n \to \oplus_{n\in\mathbb{N}} C_n$ denotes the canonical map.

The next two sections will be devoted to study factorization properties of the map ϕ. Our final goal will be achieved in Theorem 4.3.

We recall the definition of a (countable) Mittag–Leffler inverse system, see [15, 13.1.1] or [21, Definition 3.5.6].

Definition 2.4 A countable inverse system of abelian groups

$$\dots \to H_3 \xrightarrow{h_2} H_2 \xrightarrow{h_1} H_1$$

satisfies the Mittag–Leffler condition if, for every $m \in \mathbb{N}$, the chain of subgroups of H_m

$$H_m \supseteq h_m(H_{m+1}) \supseteq \dots \supseteq h_m h_{m+1} \cdots h_{m+n-1}(H_{m+n}) \supseteq \dots$$

is stationary. Equivalently, for each $m \in \mathbb{N}$, there exists $l(m) > m$ such that

$$h_m \cdots h_k(H_{k+1}) = h_m \cdots h_{l(m)-1}(H_{l(m)})$$

for any $k \geq l(m)$.

The Mittag–Leffler condition for an inverse system of abelian groups was introduced by Grothendieck in [15, Section 13]. For countable inverse systems of abelian groups the Mittag–Leffler condition implies the exactness of the inverse limit (see [15, Proposition 13.2.2] for the precise statement) and, more generally, that the first derived functor of the inverse limit is zero (cf. [21, Proposition 3.5.7]).

Now we are ready to state and prove our main theorem, assuming Theorem 4.3.

Theorem 2.5 *Let R be a ring. Let $\mathcal{C}$ be a class of right R-modules satisfying that $M^{(\mathbb{N})} \in \mathcal{C}$ whenever $M \in \mathcal{C}$. If A is a countably presented right R-module such that $\operatorname{Ext}^1_R(A, M) = 0$ for any $M \in \mathcal{C}$, then $\operatorname{Ext}^1_R(A, N) = 0$ for any right module N isomorphic to a pure submodule of a module in $\mathcal{C}$.*

Proof By Remark 2.2 and Notation 2.3, there is an exact sequence

$$0 \to \oplus_{n\in\mathbb{N}} C_n \xrightarrow{\phi} \oplus_{n\in\mathbb{N}} C_n \to \varinjlim C_n \cong A \to 0 \qquad (2)$$

where C_n are finitely presented right modules.

Let $M \in \mathcal{C}$, and let N be a pure submodule of M. Set $M_n = M$ for any $n \in \mathbb{N}$. Since $\oplus_{n\in\mathbb{N}} M_n \in \mathcal{C}$, $\mathrm{Ext}^1_R(A, \oplus_{n\in\mathbb{N}} M_n) = 0$ by hypothesis. Thus, for every homomorphism $\gamma\colon \oplus_{n\in\mathbb{N}} C_n \to \oplus_{n\in\mathbb{N}} M_n$ there exists $\psi\colon \oplus_{n\in\mathbb{N}} C_n \to \oplus_{n\in\mathbb{N}} M_n$ such that $\psi\phi = \gamma$. By Theorem 4.3, the inverse system of abelian groups $(\mathrm{Hom}_R(C_n, N), \mathrm{Hom}_R(f_n, N))_{n\in\mathbb{N}}$ is Mittag–Leffler.

As the modules C_n are finitely presented, when we apply the functor $\mathrm{Hom}_R(C_n, -)$ to the pure exact sequence

$$0 \to N \to M \to M/N \to 0$$

we obtain an inverse system of pure exact sequences of the form

$$0 \to \mathrm{Hom}_R(C_n, N) \to \mathrm{Hom}_R(C_n, M) \to \mathrm{Hom}_R(C_n, M/N) \to 0.$$

As $(\mathrm{Hom}_R(C_n, N), \mathrm{Hom}_R(f_n, N))_{n\in\mathbb{N}}$ is Mittag–Leffler we can apply [15, Proposition 13.2.2] to conclude that there is an exact sequence

$$0 \to \varprojlim \mathrm{Hom}_R(C_n, N) \to \varprojlim \mathrm{Hom}_R(C_n, M) \to \varprojlim \mathrm{Hom}_R(C_n, M/N) \to 0,$$

which in turn gives the exact sequence

$$0 \to \mathrm{Hom}_R(A, N) \to \mathrm{Hom}_R(A, M) \to \mathrm{Hom}_R(A, M/N) \to 0.$$

Therefore, we also have the exact sequence

$$0 \to \mathrm{Ext}^1_R(A, N) \to \mathrm{Ext}^1_R(A, M) = 0$$

from which we deduce that $\mathrm{Ext}^1_R(A, N) = 0$ as desired. □

Theorem 2.6 *Let R be a ring. Then any* 1*-tilting class is definable and any* 1*-tilting module is of finite type.*

Proof Let $(\mathcal{A}, \mathcal{B})$ be a 1-tilting cotorsion pair. Since there exists a (1-tilting) module such that $\mathcal{B} = \mathrm{Gen}(T) = T^{\perp}$, $\mathcal{B}$ is closed under products and under direct limits. Combining Theorem 2.1 with Theorem 2.5 it follows that $\mathcal{B}$ is also closed under pure submodules, hence $\mathcal{B}$ is definable.

The fact that 1-tilting modules are of finite type is now a consequence of Theorem 2.1. □

It is known that a combination of Ziegler's result [23, Corollary 6.9] and the Keisler–Shelah Theorem (cf. [17] and [20]) implies that a definable class is determined by the indecomposable pure injective modules it contains. Hence we have:

Corollary 2.7 *Let $\mathcal{B}$ and $\mathcal{B}'$ be two* 1*-tilting classes over a ring R. Then $\mathcal{B} = \mathcal{B}'$ if and only if they contain the same indecomposable pure injective modules.*

Corollary 2.8 *Every* 1*-tilting flat module is projective.*

Proof Let T be a 1-tilting flat module, and let $(\mathcal{A}, \mathcal{B})$ be the cotorsion pair cogenerated by T. Then, by the well known $\mathrm{Ext} - \mathrm{Tor}$ formulas, $\mathcal{B}$ contains all pure injective modules and, as $\mathcal{B}$ is definable by Theorem 2.6, it is closed under pure submodules. As any module is a pure submodule of its pure-injective envelope, $T^{\perp} = \mathcal{B} = \mathrm{Mod}-R$ and, hence, T is projective. □

3 Factoring Diagonal Maps through ϕ

We recall that a right R-module C_R is said to be *small* if for any family of right R-modules $\{M_i\}_{i\in I}$ and for any morphism $f\colon C \to \oplus_{i\in I} M_i$ there exists a finite subset $F \subseteq I$ such that $f(C) \subseteq \varepsilon_F(\oplus_{i\in F} M_i)$ where $\varepsilon_F\colon \oplus_{i\in F} M_i \to \oplus_{i\in I} M_i$ denotes the canonical map. Equivalently, $\mathrm{Hom}_R(C, \oplus_{i\in I} M_i) \cong \oplus_{i\in I}\mathrm{Hom}_R(C, M_i)$.

Let $(M_n)_{n\in\mathbb{N}}$ and $(C_n)_{n\in\mathbb{N}}$ be two sequences of right R-modules, and let

$$\alpha\colon \oplus_{n\in\mathbb{N}} C_n \to \oplus_{n\in\mathbb{N}} M_n$$

be an R-homomorphism. For any $i, j \in \mathbb{N}$, define $\alpha_{ij}\colon C_j \to M_i$ to be the map

$$C_j \stackrel{\varepsilon_j}{\to} \oplus_{n\in\mathbb{N}} C_n \stackrel{\alpha}{\to} \oplus_{n\in\mathbb{N}} M_n \stackrel{\pi_i}{\to} M_i,$$

where ε_j and π_i denote the canonical inclusion and the canonical projection, respectively. We associate the matrix (α_{ij}) to the map α, and we will identify the map α with its associated matrix (α_{ij}). We say that α is a diagonal map if $\alpha_{ij} = 0$ whenever $i \neq j$ and we will say that α is upper (lower) triangular if $\alpha_{ij} = 0$ whenever $i > j$ $(i < j)$.

In the case that each C_n is small for every $n \in \mathbb{N}$, the matrix (α_{ij}) is column finite. This is the fundamental reason why we consider the factorization of the morphism ϕ in the setting of small modules.

Following Notation 2.3, the map ϕ in the exact sequence:

$$0 \to \oplus_{n\in\mathbb{N}} C_n \stackrel{\phi}{\to} \oplus_{n\in\mathbb{N}} C_n \to \varinjlim C_n \to 0$$

is identified with the matrix

$$\phi = \begin{pmatrix} 1 & 0 \dots & & 0 \dots \\ -f_1 & 1 & & \\ 0 & -f_2 & \ddots & \\ \vdots & & \ddots & 1 \\ \vdots & & & -f_n & \ddots \\ \vdots & & & & \ddots \end{pmatrix}.$$

Given a homomorphism $\gamma\colon \oplus_{n\in\mathbb{N}} C_n \to \oplus_{n\in\mathbb{N}} M_n$ of right R-modules, we are interested in finding conditions under which there exists $\psi\colon \oplus_{n\in\mathbb{N}} C_n \to \oplus_{n\in\mathbb{N}} M_n$ such that $\psi\phi = \gamma$. We will focus on the case in which the C_n's are small modules.

It is convenient to recall when ϕ is invertible.

Lemma 3.1 *Assuming the modules C_n in the Notation 2.3 are small, the following statements are equivalent:*

(1) *ϕ is invertible.*
(2) $\varinjlim C_n = 0$.

(3) *For every* $m \in \mathbb{N}$ *there exists an integer* $l(m) > m$ *such that* $f_{k-1} \cdots f_m = 0$ *for any* $k \geq l(m)$.
(4) *The matrix* $\sum_{n\geq 0}(1-\phi)^n$ *is column finite.*

Moreover, if these equivalent statements hold then $\phi^{-1} = \sum_{n\geq 0}(1-\phi)^n$.

Remark 3.2 The matrix $L = \sum_{n\geq 0}(1-\phi)^n$ is lower triangular and its entries are

$$L_{ij} = \begin{cases} f_{i-1}\cdots f_j & \text{if } i > j \\ 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Thus, in general, L is not associated to an element of $\mathrm{End}_R(\oplus_{n\in\mathbb{N}} C_n)$.

The next lemma gives a necessary condition to factor a diagonal map γ through ϕ. It is inspired by the characterization of the case when $C_n = M_n$ and γ the identity, that is when ϕ has a left inverse. The argument in the proof, that we repeat for completeness' sake, was used first by Bass [6] for the case $C_n = M_n = R$, later it was completed and extended to more general situations in work by Zimmermann [24], Whitehead [22], Azumaya [5], Angeleri-Hügel and Saorin [2].

Lemma 3.3 *Assume that the modules* C_n *in the Notation* 2.3 *are small. Let* $\{M_n\}_{n\in\mathbb{N}}$ *be right* R*-modules and let* $\gamma\colon \oplus_{n\in\mathbb{N}} C_n \to \oplus_{n\in\mathbb{N}} M_n$ *be a diagonal map. Assume that there is* $\psi\colon \oplus_{n\in\mathbb{N}} C_n \to \oplus_{n\in\mathbb{N}} M_n$ *such that* $\psi\phi = \gamma$. *Then,*

(*) *There exist a sequence of maps* $(g_n\colon C_{n+1} \to M_n)_{n\in\mathbb{N}}$ *and a sequence of natural numbers* $(l(m))_{m\in\mathbb{N}}$, *with* $l(m) > m$ *for every* $m \in \mathbb{N}$, *satisfying the following property:*

$$\gamma_{kk} f_{k-1} f_{k-2} \cdots f_m = g_k f_k f_{k-1} f_{k-2} \cdots f_m,$$

for all $k \geq l(m)$.

Proof Fix $m \geq 1$. Let $(\psi)_k$ denote the k^{th}-row of the matrix ψ and $[\phi]_k$ denote the k^{th}-column of the matrix ϕ. Since γ is diagonal and $\psi\phi = \gamma$, we have $(\psi)_k[\phi]_j = 0$ whenever $k \neq j$. Thus, if $k > m$, $\psi_{kj} = \psi_{k\,j+1} f_j$ for every $m \leq j \leq k-1$. Hence,

(a) $\psi_{km} = \psi_{kk} f_{k-1} f_{k-2} \cdots f_m$ and $(\psi)_k[\phi]_k = \gamma_{kk}$ yields
(b) $\psi_{kk} = \psi_{k\,k+1} f_k + \gamma_{kk}$. Since ψ is column finite, there exists an index $l(m) > m$ such that $\psi_{km} = 0$ for every $k \geq l(m)$. Thus, multiplying (b) by $f_{k-1} f_{k-2} \cdots f_m$, we obtain
(c) $-\psi_{k\,k+1} f_k f_{k-1} f_{k-2} \cdots f_m = \gamma_{kk} f_{k-1} f_{k-2} \cdots f_m$, for every $k \geq l(m)$.

So the sequence of maps $(g_n = -\psi_{n\,n+1}\colon C_{n+1} \to M_n)_{n\in\mathbb{N}}$ and the sequence $(l(m))_{m\in\mathbb{N}}$ satisfy condition $(*)$. □

Remark 3.4

(1) As seen in the proof of Lemma 3.3, the only requirement to define the sequence $(l(m))_{m\in\mathbb{N}}$ is that $\psi_{km} = 0$ for any $k \geq l(m)$ and $m \geq 1$. Therefore, $(l(m))_{m\in\mathbb{N}}$ can

be chosen to be increasing and, in this case, condition $(*)$ is empty on the maps $g_1, \ldots, g_{l(1)-1}$, so they can be taken to be zero.

(2) The same computations as in Lemma 3.3 give analogous conditions to factor an upper triangular map γ through ϕ. Moreover, if $\gamma\colon \oplus_{n\in\mathbb{N}} C_n \to \oplus_{n\in\mathbb{N}} M_n$ is arbitrary then, grouping together the M_n's, it can always be assumed that γ is upper triangular.

Lemma 3.3 can be understood in terms of matrices.

Remark 3.5 Let $(g_n\colon C_{n+1} \to M_n)_{n\in\mathbb{N}}$ be a sequence of maps. Consider the matrix

$$G' = \begin{pmatrix} 0 & g_1 & 0 & \dots & & & \\ 0 & 0 & g_2 & & & & \\ \vdots & & \ddots & \ddots & & & \\ & & & 0 & g_n & & \\ & & & & 0 & \ddots & \\ \vdots & & & & & \ddots & \end{pmatrix}.$$

Condition $(*)$ of Lemma 3.3 is equivalent to say that the matrix

$$G = (\gamma - G')\left(\sum_{n\geq 0}(1-\phi)^n\right)$$

is column finite (cf. Lemma 3.1). In fact, $G_{km} = 0$ for $k \geq l(m)$ because the entries of $G\colon \oplus_{n\in\mathbb{N}} C_n \to \oplus_{n\in\mathbb{N}} M_n$ are

$$G_{km} = \begin{cases} -g_k & \text{if } m = k+1 \\ \gamma_{kk} - g_k f_k & \text{if } k = m \\ (\gamma_{kk} - g_k f_k) f_{k-1} \dots f_m & \text{if } m < k \\ 0 & \text{otherwise} \end{cases}$$

In the next lemma we collect and extend to our setting the different ideas available in the literature to factor a map γ through ϕ. In particular, conditions (1) and (2) will be helpful in the proof of Theorem 3.7.

Lemma 3.6 *Let $(g_n\colon C_{n+1} \to M_n)_{n\in\mathbb{N}}$ be a sequence of maps satisfying the conclusion of Lemma* 3.3*. With the notation of Remark* 3.5 *we have:*

(1) $G\phi = \gamma - G'$.
(2) *If $\beta\colon \oplus_{n\in\mathbb{N}} C_n \to \oplus_{n\in\mathbb{N}} M_n$ is such that $\beta\phi = G'$ then*

$$(G + \beta)\,\phi = \gamma.$$

(3) *If γ has a left inverse then there exists $\psi\colon \oplus_{n\in\mathbb{N}} C_n \to \oplus_{n\in\mathbb{N}} M_n$ such that $\psi\phi = \gamma$. In the particular case that $\gamma = 1$, $\psi = \left(\sum_{n\geq 0}(G')^n\right) G$.*

Proof

(1) Follows from direct computation using the formula for the entries of G given in Remark 3.5.

(2) Follows from (1).
(3) Let $\alpha\colon \oplus_{n\in\mathbb{N}} M_n \to \oplus_{n\in\mathbb{N}} C_n$ be a left inverse of γ. As γ is a diagonal map, α can also be assumed to be diagonal. Hence it makes sense to consider $\sum_{n\geq 0}(\alpha G')^n$ which is the inverse of $1-\alpha G'$. By (1), $\alpha\, G\phi = 1-\alpha\, G'$. Then

$$1=\left(\sum_{n\geq 0}(\alpha G')^n\right)(1-\alpha\, G')=\left(\sum_{n\geq 0}(\alpha G')^n\right)\alpha\, G\phi.$$

Hence we can take

$$\psi=\gamma\left(\sum_{n\geq 0}(\alpha G')^n\right)\alpha\, G.$$ □

In the next theorem we give a necessary and sufficient condition to factor certain type of diagonal maps through ϕ. To prove the implication (2) $\Rightarrow$ (1) we adapt Azumaya's argument in [5, Theorem 26] to our setting.

Theorem 3.7 *Assume that the modules C_n in the Notation* 2.3 *are small. Let M be a right R-module. For each $n\in\mathbb{N}$, we set $M_n\cong M$. Then the following statements are equivalent:*

(1) *For every diagonal homomorphism $\gamma\colon \oplus_{n\in\mathbb{N}} C_n \to \oplus_{n\in\mathbb{N}} M_n$ there exists a homomorphism $\psi\colon \oplus_{n\in\mathbb{N}} C_n \to \oplus_{n\in\mathbb{N}} M_n$ such that $\psi\phi=\gamma$.*
(2) *For every $m\in\mathbb{N}$, the chain of subgroups of* $\operatorname{Hom}_R(C_m, M)$:

$$\operatorname{Hom}_R(C_{m+1}, M)\, f_m \supseteq \operatorname{Hom}_R(C_{m+2}, M)\, f_{m+1} f_m \supseteq \dots$$
$$\dots \supseteq \operatorname{Hom}_R(C_{m+n}, M)\, f_{m+n-1} f_{m+n-2}\cdots f_m \supseteq \dots$$

is stationary.
(3) *The inverse system of abelian groups*

$$\dots \longrightarrow \operatorname{Hom}_R(C_3, M) \overset{\operatorname{Hom}_R(f_2, M)}{\longrightarrow} \operatorname{Hom}_R(C_2, M) \overset{\operatorname{Hom}_R(f_1, M)}{\longrightarrow} \operatorname{Hom}_R(C_1, M)$$

satisfies the Mittag–Leffler condition.

Proof To prove that (2) and (3) are equivalent, observe that for every $m\in\mathbb{N}$ and for any $k>m$

$$\operatorname{Hom}_R(f_m, M)\cdots\operatorname{Hom}_R(f_{k-1}, M)\operatorname{Hom}_R(C_k, M)=\operatorname{Hom}_R(C_k, M)\, f_{k-1}\cdots f_m.$$

To prove (1) $\Rightarrow$ (2), assume by way of contradiction that there exists an integer m for which the chain is not stationary. Then, there exists an infinite set $N\subseteq\mathbb{N}$ such that, for any $n\in N$, there is a map $\alpha_n\in\operatorname{Hom}_R(C_n, M)$ such that $\alpha_n f_{n-1} f_{n-2}\cdots f_m\notin \operatorname{Hom}_R(C_{n+1}, M)\, f_n f_{n-1}\cdots f_m$. Consider the diagonal homomorphism $\alpha\colon \oplus_{n\in\mathbb{N}} C_n \to \oplus_{n\in\mathbb{N}} M_n$ defined by $\alpha_{nn}=\alpha_n$ for $n\in N$ and $\alpha_{mn}=0$ otherwise. By hypothesis, α factors through ϕ, hence, by Lemma 3.3, there exists an integer $l(m)>m$ such that for all $k\geq l(m)$, $\alpha_{kk} f_{k-1} f_{k-2}\cdots f_m\in\operatorname{Hom}_R(C_{k+1}, M)\, f_k f_{k-1} f_{k-2}\cdots f_m$ contradicting the choice of the infinite family $(\alpha_n)_{n\in N}$.

(2) ⇒ (1). For every $m \in \mathbb{N}$, let $l(m)$ be the minimal integer such that $l(m) > m$ and

$$\begin{aligned}\operatorname{Hom}_R(C_{l(m)}, M)\, f_{l(m)-1}\, f_{l(m)-2} \cdots f_m &= \operatorname{Hom}_R(C_{l(m)+1}, M)\, f_{l(m)}\, f_{l(m)-1} \cdots f_m = \\ &= \operatorname{Hom}_R(C_{l(m)+2}, M)\, f_{l(m)+1}\, f_{l(m)} \cdots f_m = \dots\end{aligned}$$

Denote this subgroup of $\operatorname{Hom}_R(C_m, M)$ by Λ_m. As

$$\begin{aligned}\Lambda_{m+1} &= \operatorname{Hom}_R(C_{l(m+1)}, M)\, f_{l(m+1)-1}\, f_{l(m+1)-2} \cdots f_{m+1} = \\ &= \operatorname{Hom}_R(C_{l(m+1)+1}, M)\, f_{l(m+1)}\, f_{l(m+1)-1} \cdots f_{m+1} = \dots,\end{aligned}$$

then,

$$\begin{aligned}\Lambda_{m+1} f_m &= \operatorname{Hom}_R(C_{l(m+1)}, M)\, f_{l(m+1)-1}\, f_{l(m+1)-2} \cdots f_{m+1} f_m = \\ &= \operatorname{Hom}_R(C_{l(m+1)+1}, M)\, f_{l(m+1)}\, f_{l(m+1)-1} \cdots f_{m+1} f_m = \dots\end{aligned}$$

implies $l(m+1) \geq l(m)$ and $\Lambda_{m+1} f_m = \Lambda_m$. Thus we have $l(1) \leq l(2) \leq \dots$ and

$$\Lambda_m = \Lambda_{m+1} f_m = \Lambda_{m+2} f_{m+1} f_m = \cdots = \Lambda_{n+1} f_n f_{n-1} \cdots f_m,$$

for every $n \geq m$.

Fix a diagonal map $\gamma\colon \oplus_{n\in\mathbb{N}} C_n \to \oplus_{n\in\mathbb{N}} M_n$. First we construct a sequence of maps $(g_k\colon C_{k+1} \to M_k)_{k\in\mathbb{N}}$ satisfying condition (∗) in Lemma 3.3 with respect to the sequence $(l(k))_{k\in\mathbb{N}}$.

We set $g_1, \dots, g_{l(1)-1}$ to be zero (cf. Remark 3.4(1)). Choose $k \geq l(1)$. Let $h \in \mathbb{N}$ be maximal such that $l(h) \leq k$ (h exists because $l(m) > m$). Since $k \geq l(h)$, $\Lambda_h = \operatorname{Hom}_R(C_k, M)\, f_{k-1}\, f_{k-2} \cdots f_h$ and moreover $\Lambda_h = \Lambda_{k+1} f_k f_{k-1} \cdots f_h$. Since $\gamma_{kk} f_{k-1} f_{k-2} \cdots f_h \in \Lambda_h$, there exists a map $g_k \in \Lambda_{k+1}$, such that

$$g_k f_k f_{k-1} \cdots f_h = \gamma_{kk} f_{k-1} f_{k-2} \cdots f_h.$$

This finishes the construction of the sequence $(g_k)_{k\in\mathbb{N}}$ with $g_k \in \Lambda_{k+1}$ for any $k \in \mathbb{N}$. It is immediate to check that it satisfies the desired condition.

Using γ and the sequence $(g_k)_{k\in\mathbb{N}}$, we construct the matrix G given by Remark 3.5. In view of Lemma 3.6 (2), if we construct $\beta\colon \oplus_{n\in\mathbb{N}} C_n \to \oplus_{n\in\mathbb{N}} M_n$ such that

$$\beta\phi = \begin{pmatrix} 0 & g_1 & 0 & \dots & & \\ 0 & 0 & g_2 & & & \\ \vdots & & \ddots & \ddots & & \\ & & & 0 & g_n & \\ & & & & 0 & \ddots \\ \vdots & & & & & \ddots \end{pmatrix},$$

then $\psi = G + \beta$ will satisfy $\psi\phi = \gamma$. Equivalently, we have to define β_{kn} such that:

$$\begin{cases} \beta_{kn} - \beta_{k\,n+1} f_n = 0 & n \neq k+1 \\ \beta_{k\,k+1} - \beta_{k\,k+2} f_{k+1} = g_k & \end{cases}$$

To this aim, fix $k \in \mathbb{N}$. Since, for every $n \geq k+2$,

$$g_k \in \Lambda_{k+1} = \Lambda_{k+2} f_{k+1} \text{ and } \Lambda_n = \Lambda_{n+1} f_n$$

we can construct inductively the sequence $\beta_{kn} \in \Lambda_n$ letting

(1) $\beta_{kn} = 0$, for $n \leq k+1$;
(2) $\beta_{k\,k+2}\, f_{k+1} = g_k$ and
(3) $\beta_{k\,n+1}\, f_n = \beta_{k\,n}$, for $n \geq k+2$.

Then the homomorphism $\beta\colon \oplus_{n\in\mathbb{N}} C_n \to \oplus_{n\in\mathbb{N}} M_n$ defined by

$$\beta_{kn} = \begin{cases} \beta_{kn} & \text{if } n \geq k+2 \\ 0 & \text{otherwise} \end{cases}$$

satisfies the desired property. □

Remark 3.8 Recall that an object C in an additive category $\mathcal{A}$ with arbitrary coproducts is called *compact* if $\operatorname{Hom}_R(C, \oplus_{i\in I} M_i) \cong \oplus_{i\in I} \operatorname{Hom}_R(C, M_i)$ for every family of objects $\{M_i \mid i \in I\}$ in $\mathcal{A}$. For example, if $\mathcal{A} = \operatorname{Mod}-R$ the compact objects are exactly the small modules.

In the proof of Theorem 3.7 we use only the hypothesis that $(C_n)_{n\in\mathbb{N}}$ is a sequence of compact objects in an additive category. Thus the result holds in this more general context.

We reformulate the descending chain condition of Theorem 3.7(2). We follow the spirit of [22, Theorems 1.9 and 2.1] and of [18, Lemma 3.1], where this type of statement is used to give a description of countably generated projective modules.

Proposition 3.9 *Assume that the modules C_n are as in the Notation* 2.3*. Let M be a right R-module. Then the following statements are equivalent:*

(1) *For every $m \in \mathbb{N}$, the chain of subgroups of* $\operatorname{Hom}_R(C_m, M)$*:*

$$\operatorname{Hom}_R(C_{m+1}, M)\, f_m \supseteq \operatorname{Hom}_R(C_{m+2}, M)\, f_{m+1} f_m \supseteq \dots$$
$$\dots \supseteq \operatorname{Hom}_R(C_{m+n}, M)\, f_{m+n-1} f_{m+n-2} \cdots f_m \supseteq \dots$$

is stationary.

(2) *There is a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers such that, putting $f'_{n_k} = f_{n_{k+1}-1} \cdots f_{n_k}$,*

$$\operatorname{Hom}_R(C_{n_{k+1}}, M)\, f'_{n_k} f'_{n_{k-1}} = \operatorname{Hom}_R(C_{n_k}, M)\, f'_{n_{k-1}}$$

for every $k > 1$.

Proof To prove (1) ⇒ (2), construct the sequence $(n_k)_{k\in\mathbb{N}}$ inductively as follows. Let $n_1 = 1$; if $k > 1$ and assuming n_i has been defined for every $i \leq k-1$, define $n_k = l(n_{k-1})$, where for every $m \in \mathbb{N}$, $l(m)$ is the minimum integer such that $l(m) > m$ and

$$\operatorname{Hom}_R(C_{l(m)}, M)\, f_{l(m)-1} f_{l(m)-2} \cdots f_m = \operatorname{Hom}_R(C_k, M)\, f_{k-1} f_{k-2} \cdots f_m$$

for every $k \geq l(m)$.

Since $n_{k+1} = l(n_k) > n_k = l(n_{k-1}) > n_{k-1}$,

$$\operatorname{Hom}_R(C_{n_{k+1}}, M) f_{n_{k+1}-1} \cdots f_{n_k} f_{n_k-1} \cdots f_{n_{k-1}} =$$
$$= \operatorname{Hom}_R(C_{n_{k+1}-1}, M) f_{n_{k+1}-2} \cdots f_{n_k} f_{n_k-1} \cdots f_{n_{k-1}} = \dots$$
$$\dots = \operatorname{Hom}_R(C_{n_k}, M) f_{n_k-1} \cdots f_{n_{k-1}}.$$

Hence, by the definition of f'_{n_k},

$$\operatorname{Hom}_R(C_{n_{k+1}}, M) f'_{n_k} f'_{n_{k-1}} = \operatorname{Hom}_R(C_{n_k}, M) f'_{n_{k-1}}$$

for $k > 1$, as wanted.

To show that (2) $\Rightarrow$ (1) observe that condition (2) and the definition of f'_{n_k} imply that

$$(1) \qquad \operatorname{Hom}_R(C_n, M) f_{n-1} \cdots f_{n_k} f'_{n_{k-1}} f_{n_{k-1}-1} \cdots f_m =$$
$$= \operatorname{Hom}_R(C_{n_k}, M) f'_{n_{k-1}} f_{n_{k-1}-1} \cdots f_m$$

for any $n, m \in \mathbb{N}$ such that $n_{k+1} \geq n > n_k > n_{k-1} \geq m$.

Let $m \in \mathbb{N}$, and choose $\ell \in \mathbb{N}$ such that $n_{\ell-1} \geq m$. We claim that

$$\operatorname{Hom}_R(C_{n_\ell+j}, M) f_{n_\ell+j-1} \cdots f_m = \operatorname{Hom}_R(C_{n_\ell}, M) f_{n_\ell-1} f_{n_\ell-2} \cdots f_m,$$

for any $j \in \mathbb{N}$, so that the sequence of subgroups of $\operatorname{Hom}_R(C_m, M)$:

$$\operatorname{Hom}_R(C_{m+1}, M) f_m \supseteq \operatorname{Hom}_R(C_{m+2}, M) f_{m+1} f_m \supseteq \dots$$
$$\dots \supseteq \operatorname{Hom}_R(C_{m+n}, M) f_{m+n-1} f_{m+n-2} \cdots f_m \supseteq \dots$$

is stationary from the position n_ℓ.

Fix $j \in \mathbb{N}$. As the sequence $(n_k)_{k\in\mathbb{N}}$ is strictly increasing, we can choose $n_k \geq n_\ell$ such that $n_{k+1} \geq n_\ell + j > n_k$. Then applying (1) we obtain that

$$\operatorname{Hom}_R(C_{n_\ell+j}, M) f_{n_\ell+j-1} \cdots f_m = \operatorname{Hom}_R(C_{n_k}, M) f_{n_k-1} f_{n_k-2} \cdots f_m.$$

As $n_\ell + j > n_k \geq n_\ell$ the claim follows by induction on j. □

In Proposition 3.9, as the sequence $(n_k)_{k\in\mathbb{N}}$ is cofinal in $\mathbb{N}$, the direct limit $\varinjlim C_{n_k}$ of the direct system

$$C_{n_1} \xrightarrow{f'_{n_1}} C_{n_2} \xrightarrow{f'_{n_2}} C_{n_3} \to \dots \to C_{n_k} \xrightarrow{f'_{n_k}} C_{n_{k+1}} \to \dots$$

is isomorphic to $A = \varinjlim C_n$. So that we obtain the exact sequence

$$0 \to \oplus_{k\in\mathbb{N}} C_{n_k} \xrightarrow{\phi'} \oplus_{k\in\mathbb{N}} C_{n_k} \to A = \varinjlim C_n \to 0$$

where

$$\phi' = \begin{pmatrix} 1 & 0 & \dots & & 0 \dots \\ -f'_{n_1} & 1 & & & \\ 0 & -f'_{n_2} & \ddots & & \\ \vdots & & \ddots & 1 & \\ \vdots & & & -f'_{n_k} & \ddots \\ \vdots & & & & \ddots \end{pmatrix}.$$

Let M be a right R-module, and, for each $n \in \mathbb{N}$, set $M_n = M$. In view of Theorem 3.7 and Proposition 3.9, any diagonal map $\gamma\colon \oplus_{n\in\mathbb{N}} C_n \to \oplus_{n\in\mathbb{N}} M_n$ factors through ϕ if and only if any diagonal map $\gamma'\colon \oplus_{k\in\mathbb{N}} C_{n_k} \to \oplus_{k\in\mathbb{N}} M_{n_k}$ factors through ϕ'. Let ψ' be the morphism constructed in the proof of Theorem 3.7 satisfying that $\psi'\phi' = \gamma'$, for $\gamma'\colon \oplus_{k\in\mathbb{N}} C_{n_k} \to \oplus_{k\in\mathbb{N}} M_{n_k}$. As for the direct system $\{C_{n_k}\}$, $l(n_k) = n_{k+1}$, it follows that ψ' is an upper triangular matrix.

4 Inheritance by Pure Submodules

We start this section by proving that the factoring condition given in Theorem 3.7 is inherited by pure submodules.

Lemma 4.1 *Let C_1 and C_2 be finitely generated right R-modules such that C_2 is finitely presented. Let Y, Z be right R-modules such that Y is a pure submodule of Z, and let $\varepsilon\colon Y \to Z$ denote the inclusion. Let $f\colon C_1 \to C_2$ and $h\colon C_1 \to Y$ be homomorphisms of right R-modules. If $\hat{g}\colon C_2 \to Z$ is such that $\varepsilon h = \hat{g} f$ then there exists $g\colon C_2 \to Y$ such that $h = gf$.*

Proof We choose a set of generators of C_1, $\{c_1^1, \dots, c_n^1\}$ say. We fix a (finite) presentation of C_2,

$$R^t \overset{\alpha_1}{\to} R^m \overset{\alpha_2}{\to} C_2 \to 0,$$

where α_1 is given by left multiplication by a matrix $A \in M_{m\times t}(R)$. Denote by $\{e_1, \dots, e_m\}$ the canonical basis of R^m. For $i = 1, \dots, m$, let $c_i^2 = \alpha_2(e_i)$.

For each $j \in \{1, \dots, n\}$, $f(c_j^1) = \sum_{i=1}^m c_i^2 r_{ij}$, for suitable $r_{ij} \in R$, and $h(c_j^1) = y_j \in Y$. Set $B = (r_{ij}) \in M_{m\times n}(R)$.

For $i \in \{1, \dots, m\}$, $\hat{g}(c_i^2) = z_i \in Z$.

As $\hat{g} f = \varepsilon h$, $(z_1, \dots, z_m)B = (y_1, \dots, y_n) \in Y^n$. As $\hat{g}$ defines a morphism $\hat{g}\colon C_2 \to Z$, $(z_1, \dots, z_m)A = (0, \dots, 0)$. So that

$$(z_1, \dots, z_m)\left(\begin{array}{c|c} A & B \end{array}\right) = \left(\begin{array}{c|c} 0, \dots, 0 & y_1, \dots, y_n \end{array}\right).$$

Since Y is pure in Z, there exists $(x_1, \dots, x_m) \in Y^m$ such that

$$(x_1, \dots, x_m)\left(\begin{array}{c|c} A & B \end{array}\right) = \left(\begin{array}{c|c} 0, \dots, 0 & y_1, \dots, y_n \end{array}\right).$$

The relation $(x_1, \dots, x_m)A = (0, \dots, 0)$ implies that the right R-module homomorphism $g_1\colon R^m \to Y$ defined by $e_i \mapsto x_i$, for $i \in \{1, \dots, m\}$, satisfies $g_1\alpha_1 = 0$. Hence we obtain a right R-module homomorphism $g\colon C_2 \to Y$ such that $c_i^2 \mapsto x_i$, for $i \in \{1, \dots, m\}$.

The relation $(x_1, \dots, x_m)B = (y_1, \dots, y_n)$ implies that $(gf)(c_j^1) = h(c_j^1)$, $j \in \{1, \dots, n\}$. Hence $gf = h$, as we wanted to show. □

The next lemma gives the final step to prove Theorem 4.3.

Lemma 4.2 *Let $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ be two sequences of right R-modules. We fix $\rho \in \mathrm{Hom}_R(\oplus_{n\in\mathbb{N}} A_n, \oplus_{n\in\mathbb{N}} B_n)$. Let Y be a right R-module, we set $Y_n = Y$ for any $n \in \mathbb{N}$. If for any diagonal map $\gamma\colon \oplus_{n\in\mathbb{N}} A_n \to \oplus_{n\in\mathbb{N}} Y_n$ there exists $\sigma\colon \oplus_{n\in\mathbb{N}} B_n \to \oplus_{n\in\mathbb{N}} Y_n$ such that $\sigma\rho = \gamma$, then for any $\Gamma\colon \oplus_{n\in\mathbb{N}} A_n \to Y$ there exists $\sigma'\colon \oplus_{n\in\mathbb{N}} B_n \to Y$ such that $\sigma'\rho = \Gamma$.*

Proof Let $\Gamma\colon \oplus_{n\in\mathbb{N}} A_n \to Y$. We define $\Gamma_i = \Gamma\varepsilon_i$, where $\varepsilon_i\colon A_i \to \oplus_{n\in\mathbb{N}} A_n$ is the canonical inclusion. We define $\gamma\colon \oplus_{n\in\mathbb{N}} A_n \to \oplus_{n\in\mathbb{N}} Y_n$ to be the diagonal map such that $\gamma_{nn} = \Gamma_n$ for every $n \in \mathbb{N}$. If we denote by $\sum\colon \oplus_{n\in\mathbb{N}} Y_n \to Y$ the summation map, then $\Gamma = \sum\gamma$.

By hypothesis, there exists, $\sigma\colon \oplus_{n\in\mathbb{N}} B_n \to \oplus_{n\in\mathbb{N}} Y_n$ such that $\sigma\rho = \gamma$. To conclude take $\sigma'\colon \oplus_{n\in\mathbb{N}} B_n \to Y$ to be $\sigma' = \sum\sigma$. □

Theorem 4.3 *Let Y be a pure submodule of a right R-module Z, and let Y_n, Z_n be copies of Y, Z, respectively, for every $n \in \mathbb{N}$. In the Notation 2.3 assume that the modules C_n are finitely presented and that for every diagonal map $\gamma\colon \oplus_{n\in\mathbb{N}} C_n \to \oplus_{n\in\mathbb{N}} Z_n$ there is $\psi\colon \oplus_{n\in\mathbb{N}} C_n \to \oplus_{n\in\mathbb{N}} Z_n$ such that $\psi\phi = \gamma$. Then the following hold:*

(1) *All diagonal maps $\gamma'\colon \oplus_{n\in\mathbb{N}} C_n \to \oplus_{n\in\mathbb{N}} Y_n$ factor through ϕ;*
(2) *The inverse system of abelian groups*

$$\dots \longrightarrow \mathrm{Hom}_R(C_3, Y) \overset{\mathrm{Hom}_R(f_2,Y)}{\longrightarrow} \mathrm{Hom}_R(C_2, Y) \overset{\mathrm{Hom}_R(f_1,Y)}{\longrightarrow} \mathrm{Hom}_R(C_1, Y)$$

satisfies the Mittag–Leffler condition;
(3) $\mathrm{Hom}_R(\phi, Y)$ *is onto.*

Proof By Theorem 3.7, for every $m \in \mathbb{N}$, the chain of subgroups of $\mathrm{Hom}_R(C_m, Z)$

$$\mathrm{Hom}_R(C_{m+1}, Z)\, f_m \supseteq \mathrm{Hom}_R(C_{m+2}, Z)\, f_{m+1} f_m \supseteq \dots$$

$$\dots \supseteq \mathrm{Hom}_R(C_{m+n}, Z)\, f_{m+n-1} f_{m+n-2} \cdots f_m \supseteq \dots$$

is stationary. Let $\varepsilon\colon Y \to Z$ be the inclusion. By Lemma 4.1,

$$\mathrm{Hom}_R(C_{m+n}, Z)\, f_{m+n-1} f_{m+n-2} \cdots f_m \bigcap \varepsilon\,\mathrm{Hom}_R(C_m, Y) =$$

$$= \mathrm{Hom}_R(C_{m+n}, Y)\, f_{m+n-1} f_{m+n-2} \cdots f_m$$

so we get the stationary condition for the corresponding chain of subgroups of $\mathrm{Hom}_R(C_m, Y)$. Hence (1) and (2) follow from Theorem 3.7.

(3) follows from (1) and Lemma 4.2. □

5 Further Applications

Our discussion about factoring the map ϕ gives the following result.

Theorem 5.1 *In the Notation 2.3, assume that the modules C_n are small. Let $\mathcal{C}$ be a class of modules satisfying that $M^{(\mathbb{N})} \in \mathcal{C}$ whenever $M \in \mathcal{C}$. Then the following statements are equivalent:*

(1) *For any $M \in \mathcal{C}$, the map $\operatorname{Hom}_R(\phi, M)$ is onto.*
(2) *For any $M \in \mathcal{C}$, the inverse system of abelian groups*

$$\ldots \longrightarrow \operatorname{Hom}_R(C_3, M) \overset{\operatorname{Hom}_R(f_2,M)}{\longrightarrow} \operatorname{Hom}_R(C_2, M) \overset{\operatorname{Hom}_R(f_1,M)}{\longrightarrow} \operatorname{Hom}_R(C_1, M)$$

is Mittag–Leffler.
(3) *For any $M \in \mathcal{C}$, $\varprojlim^1 \operatorname{Hom}_R(C_n, M) = 0$.*

Assume moreover that $\operatorname{Ext}^1_R(C_n, M) = 0$ for any $n \in \mathbb{N}$ and any $M \in \mathcal{C}$, then the above conditions are equivalent to:

(4) *For any $M \in \mathcal{C}$, $\operatorname{Ext}^1_R(\varinjlim C_n, M) = 0$.*

Proof Since $M^{(\mathbb{N})} \in \mathcal{C}$ for every $M \in \mathcal{C}$, the implication (1) $\Rightarrow$ (2) follows from Theorem 3.7. As the Mittag–Leffler property implies that $\varprojlim^1 \operatorname{Hom}_R(C_n, M) = 0$ (cf. [21, Proposition 3.5.7]), (2) $\Rightarrow$ (3).

By the definition of the first derived functor of a countable inverse system of abelian groups (cf. [21, Definition 3.5.1]), applying the functor $\operatorname{Hom}_R(-, M)$ to the exact sequence

$$0 \to \oplus_{n\in\mathbb{N}} C_n \overset{\phi}{\to} \oplus_{n\in\mathbb{N}} C_n \to \varinjlim C_n = A \to 0$$

we obtain the exact sequence

$$0 \to \varprojlim \operatorname{Hom}_R(C_n, M) = \operatorname{Hom}_R(A, M) \to \prod_{n\in\mathbb{N}} \operatorname{Hom}_R(C_n, M) \overset{\operatorname{Hom}_R(\phi,M)}{\to}$$
$$\to \prod_{n\in\mathbb{N}} \operatorname{Hom}_R(C_n, M) \to \varprojlim{}^1 \operatorname{Hom}_R(C_n, M) \to 0.$$

Therefore (1) and (3) are equivalent statements.

We prove now that if $\operatorname{Ext}^1_R(C_n, M) = 0$ for any $n \in \mathbb{N}$, then (1) and (4) are equivalent. This is immediate if we take into account that applying the functor $\operatorname{Hom}_R(-, M)$ to the exact sequence

$$0 \to \oplus_{n\in\mathbb{N}} C_n \overset{\phi}{\to} \oplus_{n\in\mathbb{N}} C_n \to \varinjlim C_n = A \to 0$$

we obtain the exact sequence

$$\prod_{n\in\mathbb{N}} \operatorname{Hom}_R(C_n, M) \overset{\operatorname{Hom}_R(\phi,M)}{\to} \prod_{n\in\mathbb{N}} \operatorname{Hom}_R(C_n, M)$$
$$\to \operatorname{Ext}^1_R(\varinjlim C_n, M) \to \prod_{n\in\mathbb{N}} \operatorname{Ext}^1_R(C_n, M) = 0. \qquad \square$$

Following [19], a right R-module A is said to be Mittag–Leffler provided that the canonical map $A \bigotimes_R N^I \to (A \bigotimes_R N)^I$, $a \otimes (n_i) \mapsto (a \otimes n_i)$ is injective for any left R-module N. Equivalently, if A is the direct limit of a direct system $(C_i, f_{ij})_{i \in I}$ of finitely presented modules then, for any right R-module M, the inverse system $(\mathrm{Hom}_R(C_i, M), \mathrm{Hom}_R(f_{ij}, M))_{ij}$ satisfies the Mittag–Leffler condition.

Raynaud and Gruson proved that countably generated Mittag–Leffler modules are pure-projective (cf. [19, Corollaire 2.2.2 p. 74]). Theorem 5.1 gives also a generalization of this result, because if $\oplus_{n\in\mathbb{N}} C_n \in \mathcal{C}$, then $\varinjlim C_n$ is a direct summand of $\oplus_{n\in\mathbb{N}} C_n$.

Now we go back to illustrate some consequences of Theorem 2.6.

Recall that a *torsion pair* is a pair $(\mathcal{T}, \mathcal{F})$ of classes of modules which are mutually orthogonal with respect to the Hom_R functor, i.e., such that

$$\mathcal{T} = \{T \in \mathrm{Mod}-R \mid \mathrm{Hom}_R(T, F) = 0 \text{ for all } F \in \mathcal{F}\}$$

$$\mathcal{F} = \{F \in \mathrm{Mod}-R \mid \mathrm{Hom}_R(T, F) = 0 \text{ for all } T \in \mathcal{T}\}.$$

T is called a torsion class and its objects are the torsion modules; $\mathcal{F}$ is called a torsion-free class and its objects are the torsion-free modules. $\mathcal{T}$ is a torsion class if and only if it is closed under epimorphic images, direct sums and extensions; while $\mathcal{F}$ is a torsion-free class if and only if it is closed under submodules direct products and extensions. A torsion class is a tilting torsion class if it coincides with the class $T^{\perp}$, for a 1-tilting module T.

The following result is dual to the result (proved in [7]) stating that cotilting torsion free classes are closed under direct limits, hence definable.

Corollary 5.2 *Tilting torsion classes are closed under pure submodules, hence definable.*

Proof The result follows from the previous remarks and Theorem 2.6. □

We recall that a class $\mathcal{R}$ of finitely generated modules is said to be *resolving* provided that

(R1) $\mathcal{R}$ contains all finitely generated projective modules,
(R2) $\mathcal{R}$ is closed under direct summands and extensions, and
(R3) $X \in \mathcal{R}$ whenever there is an exact sequence $0 \to X \to Y \to Z \to 0$ with $Y, Z \in \mathcal{R}$.

As the modules in a resolving class $\mathcal{R}$ are finitely generated, (R3) implies that the modules in $\mathcal{R}$ have a projective resolution consisting of finitely generated projective modules.

In [3, Theorem 2.2] it was proved that there is a bijective correspondence between 1-tilting classes of finite type and resolving subclasses of modules of projective dimension at most 1. If $\mathcal{B}$ is a 1-tilting class of finite type, its corresponding resolving subcategory is

$$\alpha(\mathcal{B}) = \{C \in {}^{\perp}\mathcal{B} \mid C \text{ is finitely presented }\}.$$

If $\mathcal{R}$ is a resolving subclass of modules of projective dimension at most 1 then its corresponding tilting class is

$$\beta(\mathcal{R}) = \mathcal{R}^{\perp}.$$

In this setting, an immediate consequence of Theorem 2.6 is the following.

Corollary 5.3 *Let R be a ring. There exists a bijective correspondence between* 1*-tilting classes and resolving subclasses of modules of projective dimension at most one.*

We recall the notion of n-tilting modules, as introduced in [1].

Let $n \in \mathbb{N}$. A module T is *n-tilting* provided

(T1) The projective dimension of T is at most n,
(T2) $\mathrm{Ext}^i_R(T, T^{(I)}) = 0$ for each $i \geq 1$ and all sets I, and
(T3) There exist $r \geq 0$ and a long exact sequence

$$0 \to R \to T_0 \to \cdots \to T_r \to 0$$

such that $T_i \in \mathrm{Add}(T)$ for each $0 \leq i \leq r$.

Here, $\mathrm{Add}(T)$ denotes the class of all direct summands of arbitrary direct sums of copies of T.

The above definition for $n = 1$ is equivalent to $\mathrm{Gen}\, T = T^{\perp}$ (see [12]). For a characterization of n-tilting modules in terms of the class of modules generated by T and some classes of Ext-orthogonal modules, see [8].

Proposition 5.4 *Let R be a ring, and let T be an n-tilting right module over R. Then the projective dimension and the weak dimension of T coincide.*

Proof Assume that the weak dimension of T is m. We only need to prove that the projective dimension of T is not bigger than m.

Let $n > m$, and consider a projective resolution of T

$$0 \to P_n \to P_{n-1} \to P_{n-2} \to \ldots \to P_1 \to P_0 \to T \to 0$$

with projective modules P_i. Let $H_{n-1} = \mathrm{Ker}(P_{n-2} \to P_{n-3})$ be the $(n-1)^{\text{st}}$-syzygy module of T and let $H_m = \mathrm{Ker}(P_{m-1} \to P_{m-2})$. Then H_m is a flat module and the exact sequence

$$0 \to H_{n-1} \to P_{n-2} \to \ldots \to P_m \to H_m \to 0$$

yields that H_{n-1} is a flat module. Clearly, the projective dimension of H_{n-1} is 1 and, by shift dimension, $H_{n-1}^{\perp} = \{X \in \mathrm{Mod}-R \mid \mathrm{Ext}^n_R(T, X) = 0\}$. By [8, Lemma 3.4], $H_{n-1}^{\perp}$ is closed under direct sums and, by [14, Theorem 10] $H_{n-1}^{\perp}$ is a special pre-enveloping class, namely every R-module M fits in a short exact sequence $0 \to M \to B \to A \to 0$, where $B \in H_{n-1}^{\perp}$ and $A \in {}^{\perp}(H_{n-1}^{\perp})$. Hence by the characterization of 1-tilting classes (see [4, Theorem 2.1]), $H_{n-1}^{\perp}$ is a 1-tilting class, that is $H_{n-1}^{\perp} = T_1^{\perp}$, for a 1-tilting module T_1. Since H_{n-1} is flat we can argue as in the proof of Corollary 2.8 to conclude that T_1 and, hence, H_{n-1} are projective. That is, the class $H_{n-1}^{\perp} = T_1^{\perp}$ contains all the pure injective modules and since it is closed by pure submodules, by Theorem 2.6, $H_{n-1}^{\perp} = T_1^{\perp} = \mathrm{Mod}-R$. Hence, both modules must be projective.

This shows that the projective dimension of T is bounded by its weak dimension. Thus they must coincide. □

Corollary 5.5 *Over a von Neumann regular ring, n-tilting modules are projective.*

Proof As von Neumann regular rings are the rings with weak dimension 0, the statement follows from Proposition 5.4. □

Corollary 5.6 *Let R be a ring of weak dimension* 1 *then every n-tilting module has projective dimension at most* 1*, hence it is* 1*-tilting. In particular, n-tilting modules over R are of finite type.*

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