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The Prime Spectrum and the Extended Prime Spectrum of Noncommutative Rings

Igor Klep · Marcus Tressl

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Abstract We investigate the prime spectrum of a noncommutative ring and its spectral closure, the extended prime spectrum. We construct a ring for which the prime spectrum is a spectral space different from the extended prime spectrum and we construct a von Neumann regular ring for which the prime spectrum is not a spectral space.

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1. Introduction

To each associative ring with identity R a topological space is associated: The so-called *prime spectrum* Spec R consisting of all prime ideals of R (an ideal $\mathfrak{p} \subsetneq R$ is *prime* if $aRb \subseteq \mathfrak{p}$ implies $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$); by definition, the subbasis for the topology on Spec R are the sets $U(a) := \{\mathfrak{p} \in \operatorname{Spec} R \,|\, a \notin \mathfrak{p}\}$ for $a \in R$.

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I. Klep (✉)
Department of Mathematics, University of Ljubljana, IMFM, Jadranska 19, SI-1111 Ljubljana, Slovenia
e-mail: igor.klep@fmf.uni-lj.si

M. Tressl
Universität Regensburg, NWF I-Mathematik, D-93040 Regensburg, Germany
e-mail: marcus.tressl@mathematik.uni-regensburg.de

The prime spectrum of a commutative ring is always a *spectral space*, i.e., a compact T_0-space, such that the open and compact subsets are a basis of the topology, closed under finite intersections and every closed and irreducible subset has a generic point. In Hochster [6] it is proved that every spectral space is the spectrum of a commutative ring.

Prime spectra of noncommutative rings have not had the attention given their commutative counterparts. They are not as well behaved for a variety of reasons. In general, Spec R will be T_0, compact and irreducible closed subsets will have a generic point. However, there need not be a basis of compact open subsets which is closed under finite intersections. Therefore several authors studied classes of noncommutative rings whose prime spectra are spectral spaces. For example, Kaplansky [7] introduced neo-commutative rings as those rings, for which the product of two finitely generated ideals is finitely generated, and proved that the prime spectrum of such a ring is spectral. Belluce [1] generalized Kaplansky's theorem by introducing quasi-commutative rings and proving that the prime spectrum of such a ring is spectral. The question of whether the quasi-commutative rings are precisely those rings for which the prime spectrum is spectral, was left open.

In 1997 Belluce [2] introduced the spectral closure of Spec R, the so-called extended prime spectrum XSpec R. He gave an example of a ring R with Spec $R \neq$ XSpec R and claimed that Spec $R \neq$ XSpec R implies that Spec R is not spectral. It turns out that this is not true (cf. Theorem 18 below). We introduce the extended prime spectrum from the viewpoint of spectral spaces, which gives a simplification of the matter compared with the construction of XSpec R from [2]. Also, many properties of XSpec R are more easily accessible from our construction.

The paper is organized as follows. In Section 2 we recall the basics about spectral spaces and we provide the topological tools needed for our purposes (cf. Proposition 1). Section 3 contains the definition and some basic properties of the extended prime spectrum of a noncommutative ring. In Section 4 we consider *spectral rings* (i.e., rings R for which Spec R is a spectral space) and *quasi-commutative rings* (i.e., rings R for which Spec R = XSpec R) and we prove that certain products of simple rings are quasi-commutative with Boolean Spec R. Section 5 contains our first main result. We show that for every prime ideal $\mathfrak{p}$ of a ring R which is not completely prime (an ideal $\mathfrak{a}$ is *completely prime* if $ab \in \mathfrak{a} \Rightarrow a \in \mathfrak{a}$ or $b \in \mathfrak{a}$) one can construct a ring S which is not quasi-commutative. This is used in Section 6 to give our second main result, an example of a spectral ring, which is not quasi-commutative (the ring even has a Boolean spectrum). In the final section we show that the ring proposed by Goodearl and studied by Belluce [1, 2] is not spectral thereby improving Belluce's result (he showed that this ring is not quasi-commutative).

2. Spectral Spaces

For a topological space X, let $\overset{\circ}{\mathcal{K}}(X) := \{U \subseteq X | U \text{ open and compact}\}$, $\overline{\mathcal{K}}(X) := \{X \backslash U | U \in \overset{\circ}{\mathcal{K}}(X)\}$ and let $\mathcal{K}(X)$ be the algebra of subsets of X, generated by $\overset{\circ}{\mathcal{K}}(X) \cup \overline{\mathcal{K}}(X)$. If $x, y \in X$ and $y \in \overline{\{x\}}$, then we say that x *specializes* to y.

A spectral space is a topological space X, which is compact and T_0, such that $\overset{\circ}{\mathcal{K}}(X)$ is a basis of X closed under finite intersections and such that every closed and irreducible subset $A \subseteq X$ has a generic point, i.e., $A = \overline{\{x\}}$ for some $x \in A$.

If X is a spectral space, then another topology is defined on X, which has $\overset{\circ}{\mathcal{K}}(X) \cup \overline{\mathcal{K}}(X)$ as a subbasis of open sets. This topology is called the *constructible topology* (in [6] it is called the *patch topology*) and X^{con} denotes X when viewed with this topology. The first fundamental theorem on spectral spaces says that X^{con} is a Boolean space (i.e., compact, Hausdorff and totally disconnected). A subset of X which is closed and open in X^{con} is called *constructible* and the constructible subsets of X are exactly those from $\mathcal{K}(X)$.

A map $f\colon X \to Y$ between spectral spaces X and Y is called *spectral* if $f^{-1}(V) \in \overset{\circ}{\mathcal{K}}(X)$ for all $V \in \overset{\circ}{\mathcal{K}}(Y)$. In other words, f is spectral iff f is continuous and continuous with respect to the constructible topologies. The category of spectral spaces has the spectral maps as morphisms. We now describe the Stone duality for spectral spaces. Let $L = (L, \wedge, \vee, 0, 1)$ be a distributive lattice. Here, we always assume that lattices contain a least element 0, a largest element 1 and that all lattice homomorphisms map 0 to 0 and 1 to 1. Let Prim L be the set of prime filters $\mathscr{F}$ of L (a proper filter $\mathscr{F}$ is *prime* if it contains a or b whenever $a \vee b \in \mathscr{F}$). We view Prim L as a topological space where a subbasis of open sets consists of all $D(a) := \{\mathscr{F} \in \text{Prim } L | a \notin \mathscr{F}\}$, where $a \in L$. It turns out that Prim L is a spectral space with $\overset{\circ}{\mathcal{K}}(\text{Prim } L) = \{D(a) | a \in L\}$. We write $V(a) := \text{Prim } L \backslash D(a)$, hence $\overline{\mathcal{K}}(\text{Prim } L) = \{V(a) | a \in L\}$ is a lattice of subsets of Prim L. The Stone representation says that the map $L \to \overline{\mathcal{K}}(\text{Prim } L)$, which sends a to $V(a)$ is a lattice isomorphism (respecting 0 and 1).

Finally, we describe the anti-equivalence between spectral spaces and distributive lattices. For a lattice homomorphism $\phi\colon L \to L'$ the map Prim ϕ: Prim $L' \to$ Prim L, Prim $\phi(\mathscr{F}) := \phi^{-1}(\mathscr{F})$ is a spectral map and Prim is a contravariant functor from the category of distributive lattices into the category of spectral spaces. The second fundamental theorem on spectral spaces says that Prim is an anti-equivalence; the inverse is given by $X \mapsto \overline{\mathcal{K}}(X)$ for a spectral space X. We refer to [5] for proofs of these facts.

In this paper, we are concerned with subsets of a spectral space that are dense in the constructible topology. We first collect some properties of this situation on the level of spectral spaces:

PROPOSITION 1. *Let X be a spectral space and $Y \subseteq X$ a subset which is dense in the constructible topology of X. Then:*

(1) *For every constructible subset D of X, D is the closure of $C := D \cap Y$ in X^{con} and is the unique constructible subset D' of X with the property $D' \cap Y = C$.*
(2) *For every $C \in \mathcal{K}(Y)$, the closure C_X of C in the constructible topology of X is constructible in X. C_X is the unique constructible subset C' of X with the property $C' \cap Y = C$.*
(3) *The map $\mathcal{K}(Y) \to \mathcal{K}(X)$ which sends C to C_X is an embedding of Boolean algebras.*
(4) *$C \in \overset{\circ}{\mathcal{K}}(Y)$ implies $C_X \in \overset{\circ}{\mathcal{K}}(X)$ and $C \in \overline{\mathcal{K}}(Y)$ implies $C_X \in \overline{\mathcal{K}}(X)$.*
(5) *Now suppose that Y, equipped with the topology induced from X, is a spectral space. If $x \in X$, then the closure $\overline{\{x\}}$ of $\{x\}$ in X meets Y and $\overline{\{x\}} \cap Y$ has a generic point in Y. The spectral map $\Phi\colon X \to Y$ induced by the lattice homomorphism $\overline{\mathcal{K}}(Y) \to \overline{\mathcal{K}}(X)$, $C \mapsto C_X$ sends a point $x \in X$ to the generic point of $\overline{\{x\}} \cap Y$. In particular, Φ is a retract of the inclusion $Y \hookrightarrow X$ (which is not spectral if $Y \neq X$).*

Proof. First observe that $y_1 \in Y$ specializes to $y_2 \in Y$ in the topology of Y iff y_1 specializes to y_2 in the topology of X.

Since Y is dense in X^{con}, assertion (1) certainly holds. Before proving the other statements we prove a few claims.

CLAIM 1. *For every $V \in \overset{\circ}{\mathcal{K}}(Y)$ there is some $U \in \overset{\circ}{\mathcal{K}}(X)$ with $U \cap Y = V$.*

Proof of the claim. This is so, since V is the union of sets of the form $U \cap Y$ with $U \in \overset{\circ}{\mathcal{K}}(X)$; as V is compact, V is a finite union of such sets, hence indeed we can find a single $U \in \overset{\circ}{\mathcal{K}}(X)$ with $U \cap Y = V$.

CLAIM 2. *For every $B \in \overline{\mathcal{K}}(Y)$ there is some $A \in \overline{\mathcal{K}}(X)$ with $A \cap Y = B$.*

Proof of the claim. By Claim 1 there is some $U \in \overset{\circ}{\mathcal{K}}(X)$ with $U \cap \mathrm{Y} = \mathrm{Y} \backslash \mathrm{B}$. Now take $A := X \backslash U$.

CLAIM 3. *If $C \in \mathcal{K}(Y)$, then there is a unique $C_X \in \mathcal{K}(X)$ with $C_X \cap Y = C$.*

Proof of the claim. Say $C = \bigcup_{i=1}^{k} \bigcap_{j=1}^{k} B_{ij} \cap V_{ij}$, with $B_{ij} \in \overline{\mathcal{K}}(Y)$, $V_{ij} \in \overset{\circ}{\mathcal{K}}(Y)$. By Claims 1 and 2 there are $A_{ij} \in \overline{\mathcal{K}}(X)$, $U_{ij} \in \overset{\circ}{\mathcal{K}}(X)$ with $A_{ij} \cap Y = B_{ij}$ and $U_{ij} \cap Y = V_{ij} (1 \leqslant i, j \leqslant k)$. Now take $C_X := \bigcup_{i=1}^{k} \bigcap_{j=1}^{k} A_{ij} \cap U_{ij}$.

Now we prove the proposition. By Claim 3 and assertion (1), we know that assertion (2) holds. Assertion (3) holds, since taking constructible closures preserves finite unions and complements: If $C \in \mathcal{K}(Y)$ then C_X and $(Y \backslash C)_X$ are constructible, hence by (1) they have empty intersection. Assertion (4) holds by Claims 1 and 2, and the uniqueness property of the sets C_X.

It remains to show assertion (5), where Y is assumed to be a spectral space. By assertions (3) and (4), the map $\overline{\mathcal{K}}(Y) \to \overline{\mathcal{K}}(X),\ C \mapsto C_X$ is a lattice homomorphism. Let Φ: $X \to Y$ be the corresponding spectral map. Let $U \in \overset{\circ}{\mathcal{K}}(X)$ with $\Phi(x) \in U$. Since Y is spectral, there is some $V \in \overset{\circ}{\mathcal{K}}(Y)$ with $\Phi(x) \in V \subseteq U$. Thus $x \in \Phi^{-1}(V) = V_X \subseteq U$. This shows that $x \rightsquigarrow \Phi(x)$.

On the other hand, if $y \in Y$ and $V \in \overset{\circ}{\mathcal{K}}(y)$ with $y \in V$, then $\Phi(y) \in V$; otherwise $y \in \Phi^{-1}(Y \backslash V) = X \backslash V_X$, a contradiction. This shows that $\Phi(y) \rightsquigarrow y$, thus $\Phi(y) = y$ for $y \in Y$.

For each $y \in Y$ with $x \rightsquigarrow y$ it follows that $\Phi(x) \rightsquigarrow \Phi(y) = y$. Hence $\Phi(x) \in Y$ is the generic point of $\{\overline{x}\} \cap \mathrm{Y}$. □

Now, we present an example of a spectral space X such that the set of maximal points (w.r.t. specialization) Y of X is a Boolean space in the topology induced from X and such that Y is a proper, dense subset of X^{con}.

EXAMPLE 2. Let X be an infinite set and take $x, y \in X$ with $x \neq y$. Define

$$\tau := \{O \subseteq X \mid y \in O \Rightarrow x \in O \text{ and } x \in O \Rightarrow O \text{ is cofinite}\}.$$

Certainly τ is the set of open sets of a topology on X. Moreover, $\overset{\circ}{\mathcal{K}}(X) = \{O \in \tau | O \text{ is finite or } x \in O\}$ is a lattice of subsets of X separating points of X and every element in τ is a union of elements of $\overset{\circ}{\mathcal{K}}(X)$. The closed subsets of X are those from $\{A \subseteq X | x \in A \Rightarrow y \in A$ and A infinite $\Rightarrow x \in A\}$, so every closed and irreducible subset A of X is either a singleton or x is a generic point of A. Thus every closed and irreducible subset of X has a generic point. All this shows that X is a spectral space.

Let Y be the set of maximal points of X, thus $Y = X \backslash \{x\}$. Write τ_Y for the topology of Y induced from X. Then Y is dense in X^{con}, $\tau_Y = \{U \subseteq Y \mid y \in U \Rightarrow U$ is cofinite$\}$ and $\overset{\circ}{\mathcal{K}}(Y) = \{U \in \tau_Y | U$ is finite or $y \in U\}$. It follows that Y is a Boolean space. But $X \backslash \{y\} \in \overset{\circ}{\mathcal{K}}(X)$ and $(X \backslash \{y\}) \cap Y \notin \overset{\circ}{\mathcal{K}}(Y)$.

3. The Extended Prime Spectrum

Let R be a ring and let $L(R)$ be the lattice of subsets generated by all the $Z(a) :=$ Spec $R \backslash U(a)$, $a \in R$. Then $L(R)$ is a distributive lattice with smallest element $Z(1) = \emptyset$ and largest element $Z(0) =$ Spec R.

The map Φ: Spec $R \to$ Prim $L(R)$ which sends $\mathfrak{p}$ to $\{S \in L(R) | \mathfrak{p} \in S\}$ is obviously well defined, injective and a homeomorphism onto its image. Moreover, by definition, the image of Φ in Prim $L(R)$ is dense in the constructible topology. In this sense, the spectral space Prim $L(R)$ is a spectral closure of Spec R.

In general Φ is not surjective, since for $\mathscr{F} \in$ Prim $L(R)$ the set $\mathfrak{J} := \{a \in R | Z(a) \in \mathscr{F}\}$ need not be a prime ideal of R. We add these sets to Spec R in order to get an algebraic description of points in Prim $L(R)$. Observe that the set $\mathfrak{J}$ above has the property

$$\begin{aligned} & a_1, \ldots, a_k \in \mathfrak{J},\ b_1, \ldots, b_\ell \notin \mathfrak{J} \Rightarrow \\ & \Rightarrow \exists \mathfrak{p} \in \text{Spec } R : a_1, \ldots, a_k \in \mathfrak{p} \wedge b_1, \ldots, b_\ell \notin \mathfrak{p} \end{aligned} \tag{$\star$}$$

for all $a_1, \ldots, a_k, b_1, \ldots, b_\ell \in R$.

DEFINITION. A proper ideal $\mathfrak{J}$ of R is called *locally prime* (compare [2, Proposition 33]) if property ($\star$) holds for all $a_1, \ldots, a_k, b_1, \ldots, b_\ell \in R$.

The set of all locally prime ideals of R is denoted by XSpec R (motivated by *extended spectrum*). Moreover, for $a \in R$, we define $XU(a) := \{\mathfrak{J} \in \text{XSpec } R | a \notin \mathfrak{J}$ and $XZ(a) := \{\mathfrak{J} \in \text{XSpec } R | a \in \mathfrak{J}\}$. Finally, $XL(R)$ denotes the lattice of subsets generated by all the $XZ(a)$, $a \in R$.

LEMMA 3. *The map* $X\Phi$: XSpec $R \to$ Prim $XL(R)$*, which sends a locally prime ideal* $\mathfrak{J}$ *to* $\{S | \mathfrak{J} \in S\}$*, is bijective.*

Proof. Clearly, $X\Phi$ is well defined and injective. In order to show that $X\Phi$ is surjective, take $\mathscr{F} \in$ Prim $XL(R)$ and define

$$\mathfrak{J} := \{x \in R | XZ(x) \in \mathscr{F}\}.$$

We prove that $\mathfrak{J}$ is a locally prime ideal. If $x, y \in \mathfrak{J}$, then $XZ(x - y) \supseteq XZ(x) \cap XZ(y)$. Since $\mathscr{F}$ is a filter, $x - y \in \mathfrak{J}$. Moreover if $a, b \in R$, then $XZ(axb) \supseteq XZ(x)$, hence $axb \in \mathfrak{J}$, too. Clearly, $\emptyset \notin \mathscr{F}$ implies $1 \notin \mathfrak{J}$. This shows that $\mathfrak{J}$ is a proper ideal of R.

Now assume $a_1, \ldots, a_n \in \mathfrak{J}$ and $b_1, \ldots, b_m \notin \mathfrak{J}$. Since $\mathscr{F}$ is a prime filter, $\bigcup_j XZ(b_j) \notin \mathscr{F}$. From $\bigcap_i XZ(a_i) \in \mathscr{F}$ and since $\mathscr{F}$ is a proper filter we get

$(\bigcap_i XZ(a_i))\backslash(\bigcup_j XZ(b_j)) \neq \emptyset$. Hence there is a locally prime ideal of R containing $a_1, \ldots, a_n$ and avoiding $b_1, \ldots, b_m$. By definition, there is a prime ideal $\mathfrak{p}$ satisfying $a_1, \ldots, a_n \in \mathfrak{p}$ and $b_1, \ldots, b_m \notin \mathfrak{p}$.

This shows $\mathfrak{J} \in \mathrm{XSpec}\, R$ and it follows easily that $\mathscr{F} = X\Phi(\mathfrak{J})$. □

DEFINITION. The topology on XSpec R is the one induced by the bijection $X\Phi$: XSpec $R \to$ Prim $XL(R)$ from Lemma 3.

Hence XSpec R is a spectral space and $\mathring{\mathcal{K}}(\mathrm{XSpec}\, R)$ is the set of finite unions of finite intersections of the sets $XU(a)$, $a \in R$.

THEOREM 4. *We have the following commutative diagram:*

$$\begin{array}{ccc}
\mathrm{XSpec}\, R & \xrightarrow[X\Phi]{\cong} & \mathrm{Prim}\, XL(R) \\
\uparrow & & \cong \uparrow \mathrm{Prim}\,\pi \\
\mathrm{Spec}\, R & \xrightarrow[\Phi]{} & \mathrm{Prim}\, L(R)
\end{array}$$

The prime spectrum Spec R *is dense in* XSpec R *with respect to the constructible topology. The map* $\pi : XL(R) \to L(R)$ *which sends a set* S *to* $S \cap \mathrm{Spec}\, R$ *is a lattice isomorphism.*

Proof. This follows directly from the definitions and Lemma 3. □

Applying Proposition 1 (5) to our situation yields:

COROLLARY 5. *If* Spec R *is spectral, then for every locally prime ideal* $\mathfrak{a}$ *of* R, *the ideal* $\sqrt{\mathfrak{a}}$ *is prime and the map* Φ: XSpec $R \to$ Spec R, $\mathfrak{a} \mapsto \sqrt{\mathfrak{a}}$ *is a spectral retract of the inclusion* Spec $R \hookrightarrow$ XSpec R.

Proof. By Theorem 4, $Y :=$ Spec R is dense in the constructible topology of $X :=$ XSpec R and the topology of Spec R is the one induced from XSpec R. Now apply Proposition 1 (5). □

REMARK 6. Another topological space can be associated to a ring A, namely the set $\mathcal{C}$Spec A of all completely prime ideals. $\mathcal{C}$Spec A with the topology induced from Spec A is a spectral space and $\mathring{\mathcal{K}}(\mathcal{C}\,\mathrm{Spec}\, A)$ is the set of all finite unions of finite intersections of the sets of the form $U(a) \cap \mathcal{C}\mathrm{Spec}\, A$ for $a \in A$.

4. Spectral and Quasi-commutative Rings

Recall from the introduction that a ring R is called spectral if Spec R is a spectral space; R is called quasi-commutative if Spec $R =$ XSpec R. Belluce proves

$$R \text{ is quasi-commutative} \Leftrightarrow R \text{ is spectral and } U(a) \text{ is compact } (a \in R),$$

which follows quickly from our Theorem 4, too.

PROPOSITION 7. *Let R be a ring such that* Spec R *is Hausdorff. Then for every* $\mathfrak{a} \in$ XSpec R, $\sqrt{\mathfrak{a}} \in$ Spec R. *If* XSpec R *is Hausdorff, then R is quasi-commutative.*

Proof. Let $\mathfrak{a} \in$ XSpec R. We have to show that there is a unique prime ideal of R containing $\mathfrak{a}$. Suppose there are $\mathfrak{p}, \mathfrak{q} \in$ Spec R containing $\mathfrak{a}$ with $\mathfrak{p} \neq \mathfrak{q}$. Since Spec R is Hausdorff, there are open and compact subsets U, V of XSpec R with $\mathfrak{p} \in U$, $\mathfrak{q} \in V$ and $U \cap V \cap$ Spec $R = \emptyset$. Since $\mathfrak{p}$ and $\mathfrak{q}$ are specializations of $\mathfrak{a}$ we get $\mathfrak{a} \in U \cap V$. Hence $U \cap V$ is a nonempty constructible subset of XSpec R, which does not contain a point of Spec R. Since Spec R is dense in the constructible topology of XSpec R, this is not possible.

Therefore, for every $\mathfrak{a} \in$ XSpec R, $\sqrt{\mathfrak{a}} \in$ Spec R. If XSpec R is Hausdorff, then Spec R is a closed subset of XSpec R, since Spec R is compact. Since Spec R is dense in XSpec R, both sets must be equal. □

Proposition 40 in [2] claims that R is spectral iff R is quasi-commutative. This is not true, even if Spec R is Boolean, as we show in Theorem 18 below. If R is *von Neumann regular* (i.e., for all $x \in R$ there is some $y \in R$ satisfying $xyx = x$; for more on von Neumann regular rings we refer the reader to [4]) then both notions coincide:

PROPOSITION 8. *Let R be a von Neumann regular ring. Then*

(1) *R is quasi-commutative if and only if R is spectral.*
(2) *If* Spec R *is Hausdorff, then R is quasi-commutative.*

Proof. First note that every ideal $\mathfrak{a}$ of R satisfies $\mathfrak{a} = \sqrt{\mathfrak{a}}$: if $x \in \sqrt{\mathfrak{a}}$ and $y \in R$ with $xyx = x$, then $xy \in \sqrt{\mathfrak{a}}$ and by Lam [8, Chapter 4, (10.6) and (10.7)], there is some $d \in \mathbb{N}$ such that $(xy)^d \in \mathfrak{a}$. Since $xyx = x$, xy is idempotent, thus $xy \in \mathfrak{a}$, so $x = xyx \in \mathfrak{a}$.

Now we prove (1). Assume that R is spectral and let $\mathfrak{a} \in$ XSpec R. By Corollary 5, $\sqrt{\mathfrak{a}}$ is prime, hence $\mathfrak{a} = \sqrt{\mathfrak{a}} \in$ Spec R.

(2) Since Spec R is Hausdorff, we know $\sqrt{\mathfrak{a}} \in$ Spec R for every $\mathfrak{a} \in$ XSpec R from Proposition 7. Since $\sqrt{\mathfrak{a}} = \mathfrak{a}$ for all ideals $\mathfrak{a}$ we get XSpec $R =$ Spec R. □

In Theorem 20 below, we construct a von Neumann regular ring R for which Spec R is T_1 but not a spectral space.

DEFINITION. Let R be a ring and let I be a nonempty set. If $Z \subseteq I$, then we write $\chi(Z)$ for the element of R^I, which has i-th coordinate 0 if $i \in Z$ and 1 if $i \in I \backslash Z$. For an ideal $\mathfrak{a} \subseteq R^I$ define $\Phi(\mathfrak{a}) := \{Z \subseteq I | \chi(Z) \in \mathfrak{a}\}$. For a filter $\mathscr{F}$ on I we define $\Psi(\mathscr{F}) := \bigcup_{J \in \mathscr{F}} \mathfrak{a}_J$, where $\mathfrak{a}_J$ denotes the ideal $R^{I \backslash J} \times \{0\}^J$ of R^I.

PROPOSITION 9. *Let R be a ring and I a nonempty set.*

(1) *If $\mathfrak{a} \subseteq R^I$ is an ideal, then $\Phi(\mathfrak{a})$ is a filter on I.*
(2) *If $\mathscr{F}$ is a filter on I, then $\Psi(\mathscr{F}) \subseteq R^I$ is an ideal.*
(3) *$\Phi(\Psi(\mathscr{F})) = \mathscr{F}$ and $\Psi(\Phi(\mathfrak{a})) \subseteq \mathfrak{a}$.*
(4) *Φ: Spec $R^I \to$ Prim 2^I is continuous and surjective.*

Proof. (1) Take any ideal $\mathfrak{a} \subseteq R^I$. Assume $A \in \Phi(\mathfrak{a})$ and $B \supseteq A$. Then $\chi(A) \in \mathfrak{a}$ and $\chi(B) = \chi(A) \cdot \chi(B) \in \mathfrak{a}$. Hence $B \in \Phi(\mathfrak{a})$. If $A, B \in \Phi(\mathfrak{a})$, then $\chi(A \cap B) = \chi(A \cup (I \backslash B)) + \chi(B) \in \mathfrak{a}$, thus $A \cap B \in \Phi(\mathfrak{a})$. This shows that $\Phi(\mathfrak{a})$ is a filter.

(2) Let $\mathscr{F}$ be a filter on I. Take $x, y \in \Psi(\mathscr{F})$, say $x \in \mathfrak{a}_{J_1}$, and $y \in \mathfrak{a}_{J_2}$ for $J_1, J_2 \in \Psi(\mathscr{F})$. As $x - y \in \mathfrak{a}_{J_1 \cap J_2}$ and $J_1 \cap J_2 \in \Psi(\mathscr{F})$, this shows $x - y \in \Psi(\mathscr{F})$. If $r \in K^I$ then clearly $rx, xr \in \mathfrak{a}_{J_1}$. Hence $\Psi(\mathscr{F})$ is an ideal.

(3) Let $\mathscr{F}$ be a filter on I. Then

$$(\Phi \circ \Psi)(\mathscr{F}) = \Phi\left(\bigcup_{J \in \mathscr{F}} \mathfrak{a}_J\right) = \left\{Z \subseteq I | (1)_{I \backslash Z} \times (0)_Z \in \bigcup_{J \in \mathscr{F}} \mathfrak{a}_J\right\} = \mathscr{F}.$$

For the second part of the statement, take some ideal $\mathfrak{a} \subseteq R^I$. Then

$$(\Psi \circ \Phi)(\mathfrak{a}) = \Psi\Big(\underbrace{\{Z \subseteq I | (1)_{I \backslash Z} \times (0)_Z \in \mathfrak{a}\}}_{=:\mathscr{F}}\Big) = \bigcup_{Z \in \mathscr{F}} \mathfrak{a}_Z.$$

If $x \in \mathfrak{a}_Z$ for some $Z \in \mathscr{F}$, then $x = x \cdot \chi(Z) \in \mathfrak{a}$. Hence $(\Psi \circ \Phi)(\mathfrak{a}) \subseteq \mathfrak{a}$.

(4) To prove that Φ maps prime ideals to prime filters, let $\mathfrak{p} \in$ Spec R^I. Take $A, B \subseteq I$ such that $A \cup B \in \Phi(\mathfrak{p})$. In other words, $\chi(A) \cdot \chi(B) \in \mathfrak{p}$. For every $x \in R^I$ we have $\chi(A) \cdot x \cdot \chi(B) = x \cdot \chi(A) \cdot \chi(B) \in \mathfrak{p}$. As $\mathfrak{p}$ is a prime ideal, this implies $\chi(A) \in \mathfrak{p}$ or $\chi(B) \in \mathfrak{p}$. Let us prove that Φ: Spec $R^I \to$ Prim 2^I is surjective. For this let $\mathscr{F} \in$ Prim 2^I be arbitrary. Then $\Psi(\mathscr{F})$ is a proper ideal of R and is thus contained in a prime ideal $\mathfrak{p}$. As $(\Phi \circ \Psi)(\mathscr{F}) = \mathscr{F}$ and Ψ is monotone with respect to inclusion, we get $\Phi(\mathfrak{p}) = \mathscr{F}$. The continuity is clear. □

DEFINITION. Let K be a ring and let $x \in K$. We say that x is of *principal generation height* $\leq n \in \mathbb{N}$ if the ideal generated by x in K is equal to

$$\underbrace{KxK + \cdots + KxK}_{n-\text{times}}.$$

We define *the principal generation height of* x as the least $n \in \mathbb{N} \cup \{\infty\}$ such that x is of principal generation height $\leq n$. We say that a ring K is of *finite principal generation height* if there is some $n \in \mathbb{N}$ such that every element of K is of principal generation height $\leq n$. The smallest such n (if it exists) is called the *principal generation height* of K. A similar notion for simple rings was studied by Cohn (see, e.g., [3]). In our notation his n-simple rings are simple rings of principal generation height $\leq n$.

EXAMPLE 10. Let D be a division ring. Then $M_n(D)$ is a simple ring of principal generation height n: If $e_{11} \in M_n(D)$ denotes the matrix which has exactly one entry different from 0, namely 1 in the upper left corner, then there are no elements $a_i, b_i \in M_n(D)$ with $1 = \sum_{i=1}^{n-1} a_i e_{11} b_i$.

EXAMPLE 11. To see an example of a simple ring of infinite principal generation height (i.e., not of finite p.g.h.), take

$$R := \left\{x \in M_\infty(D) | \exists y \in M_{2^k}(D) : x = \text{diag}(y, y, \ldots)\right\}.$$

So R is the direct limit of the system $M_{2^n}(D) \to M_{2^{n+1}}(D)$, $x \mapsto \text{diag}(x, x)$. Therefore R is a simple ring. Using the idea from the previous example it turns out that R is not of finite principal generation height.

THEOREM 12. *Let K be a simple ring and let I be a nonempty set. Let* Φ: Spec $K^I \to$ Prim 2^I *be as above.*

(1) *If I is finite, then Φ is a homeomorphism with inverse Ψ : Prim $2^I \to$ Spec K^I and K^I is quasi-commutative.*
(2) *If I is infinite, then the following are equivalent*:
    (a) *K is of finite principal generation height,*
    (b) *Φ is injective,*
    (c) *Φ is a homeomorphism.*

In this case $\Phi^{-1} = \Psi$ and K^I is quasi-commutative.

Proof. (1) Let us prove that $\Psi = \Phi^{-1}$. We start by showing that Ψ maps prime filters to prime ideals. Let $\mathscr{F} \in$ Prim 2^I. As I is finite, $\mathscr{F}$ is a principal filter, generated by, say, $i_0 \in I$. Hence $\Psi(\mathscr{F}) = \mathfrak{a}_{i_0}$. Clearly, $\mathfrak{a}_{i_0} \in$ Spec K^I.

It remains to be seen that all prime ideals of K^I are of this form. If an element $x \in K^I$ has only nonzero entries, then $(x) = K^I$. Take some $\mathfrak{p} \in$ Spec K^I. We claim that there exists $i_0 \in I$ with $\mathfrak{p} \subseteq \mathfrak{a}_{i_0}$. Otherwise we have $x^{(i)} \in \mathfrak{p}$ with $x_i^{(i)} = 0$ for $i \in I$. Hence $\chi(\{i\}) \in \mathfrak{p}$ and thus $1 \in \mathfrak{p}$ since I is finite. This is clearly a contradiction showing that $\mathfrak{p} \subseteq \mathfrak{a}_{i_0}$ for some $i_0 \in I$. Similarly as before, one can show that this implies $\mathfrak{p} = \mathfrak{a}_{i_0}$.

(2) If K is not of finite principal generation height, then there is some $x = (x_i)_{i \in I} \in K^I$ such that $x_i \neq 0$ for all i and such that the set

$$I_n := \left\{\text{i} \in I | x_i \text{ is of principal generation height } > n\right\}$$

is nonempty for every $n \in \mathbb{N}$. Since $I_1 \supseteq I_2 \supseteq ...$, there is some $\mathscr{F} \in$ Prim 2^I containing every I_n. Then the ideal $(\Psi(\mathscr{F}), x)$ is proper: To see this, take $n \in \mathbb{N}$ and suppose there are $a_1, ..., a_n, b_1, ..., b_n \in K^I$ and $c \in \Psi(\mathscr{F})$ with

$$1 = a_1 x b_1 + \ldots + a_n x b_n + c.$$

Take $J \in \mathscr{F}$ with $c \in \mathfrak{a}_J$, thus $c_j = 0$ for all $j \in J$. Since $J \in \mathscr{F}$, also $J \cap I_n \in \mathscr{F}$. For $j \in J \cap I_n$ we have $1 = a_{1j}x_jb_{1j} + ... + a_{nj}x_jb_{nj} + c_j = a_{1j}x_jb_{1j} + ... + a_{nj}x_jb_{nj}$. But x_j is of principal generation height $> n$ (by definition of I_n), a contradiction.

Hence the ideal $(\Psi(\mathscr{F}),\ x)$ is proper and there is a prime ideal $\mathfrak{p}_1$ of R containing x and $\Psi(\mathscr{F})$. We write $\mathfrak{a} := \Psi(\mathscr{F})$. Observe that K is prime (as it is simple). Hence for every nonzero $a \in K$ there is some $b \in K$ such that $aba \neq 0$. Write $M := \{y = (y_i)_i \in K^I \mid \forall i \in I : y_i \neq 0\}$. In the terminology of [8], M is an m-system containing x. As $\mathfrak{a} \cap M = \emptyset$, by [8, Proposition 10.5] there is some prime ideal $\mathfrak{p}_2$ of K^I containing $\mathfrak{a}$ and avoiding x. Since $\Phi(\mathfrak{a}) = \mathscr{F}$, Φ is inclusion preserving and $\mathfrak{p}_1, \mathfrak{p}_2 \supseteq \mathfrak{a},\ \Phi(\mathfrak{p}_1) = \Phi(\mathfrak{p}_2)$. As $\mathfrak{p}_1 \neq \mathfrak{p}_2$, this proves that Φ is not injective.

Conversely we assume that K is of finite principal generation height and we first show that $\Psi(\text{Prim } 2^I) \subseteq \text{Spec } K^I$ with $\Psi(\Phi(\mathfrak{p})) = \mathfrak{p}$. Let $\mathscr{F} \in \text{Prim } 2^I$ be arbitrary. We claim that $\Psi(\mathscr{F})$ is prime. First note that $(1)_I \notin \Psi(\mathscr{F})$ since $\emptyset \notin \mathscr{F}$. Assume $x, y \in K^I$ and $xry \in \Psi(\mathscr{F})$ for all $r \in K^I$. Let $J_1 := \{i \in I \mid x_i = 0)$ and $J_2 := \{j \in I \mid y_j = 0\}$. For $i \in I \backslash (J_1 \cup J_2)$, we have $x_i, y_i \neq 0$. Since K is simple, the zero ideal of K is prime. Hence there must be some $r_i \in K$ with $x_i r_i y_i \neq 0$. Since K is simple, the ideal generated by $x_i r_i y_i$ in K contains 1. Let $r \in K^I$ which has r_i as the i-th component for $i \in I \setminus (J_1 \cup J_2)$. From $xry \in \Psi(\mathscr{F})$ and since K has finite principal generation height, we get $\chi(J_1 \cup J_2) \in \Psi(\mathscr{F})$. In particular, $\chi(J_1 \cup J_2) \in \mathfrak{a}_J$ for some $J \in \mathscr{F}$. Hence $J \subseteq J_1 \cup J_2$ and thus $J_1 \cup J_2 \in \mathscr{F}$. This implies $J_1 \in \mathscr{F}$ or $J_2 \in \mathscr{F}$ and $x = x \cdot \chi(J_1) \in \Psi(\mathscr{F})$ or $y = y \cdot \chi(J_2) \in \Psi(\mathscr{F})$.

By the previous proposition, $\Psi(\Phi(\mathfrak{p})) \subseteq \mathfrak{p}$. For the converse inclusion, take $x \in \mathfrak{p}$ and define $J := \{i \in I \mid x_i = 0\}$. Since K is simple and has finite principal generation height we get $\chi(J) \in \mathfrak{p}$, thus $J \in \Phi(\mathfrak{p})$ and $x \in \mathfrak{a}_J$. This proves that $\Phi^{-1} = \Psi$.

Hence Φ is bijective. Since Φ is continuous, Spec K^I is compact and Prim 2^I is Hausdorff, Φ is a homeomorphism.

It remains to show that K^I is quasi-commutative if (a)–(c) holds. Take $x \in K^I$ and let $Z \subseteq I$ be the set of all $i \in I$ with $x_i = 0$. Since K has finite principal generation height we clearly have $U(x) = U(\chi(Z))$. Since $\Phi(U(x)) = D(Z)$ and Φ is a homeomorphism it follows that $U(x)$ is compact. Thus K^I is quasi-commutative. □

REMARK 13. If R is a ring with center $Z(R)$, then the canonical embedding $Z(R) \hookrightarrow R$ induces a continuous mapping Spec $R \to$ Spec $Z(R)$. If K and I are as above, then the mapping Φ: Spec $K^I \to$ Prim 2^I factorizes through Spec $K^I \to$ Spec $Z(K)^I$ and Spec $Z(K)^I \to$ Prim 2^I. Observe that the latter mapping is a homeomorphism since K is simple and thus $Z(K)$ is a field.

5. A Method to Produce Non Quasi-commutative Rings

In this section we consider a prime ideal $\mathfrak{p}$ of a ring R which is not completely prime and we construct a ring out of these data which is not quasi-commutative (cf. Remark 17 and Proposition 16 below). This will give a tool to produce a ring which is not spectral (cf. Theorem 20) and a tool to produce a spectral ring, which is not quasi-commutative (cf. Theorem 18).

LEMMA 14. *Let I be a nonempty set and let $\mathfrak{B}$ be a filter on I containing all cofinite subsets of I. Take a prime ideal $\mathfrak{p}$ of a ring k and let R be a subring of k^I such that for all $x, y \in k^I$ we have*

$$\{i \in I \mid x_i = y_i\} \in \mathfrak{B} \text{ and } x \in R \Rightarrow y \in R.$$

Fix an index $j \in I$. Then $\{x \in R \mid x_j \in \mathfrak{p}\}$ is a prime ideal of R.

Proof. Set $\mathfrak{q} := \{x \in R \mid x_j \in \mathfrak{p}\}$. Take $x, y \in R$ with $xRy \subseteq \mathfrak{q}$. We must show $x_j \in \mathfrak{p}$ or $y_j \in \mathfrak{p}$ and since $\mathfrak{p}$ is prime it is enough to show $x_j k y_j \subseteq \mathfrak{p}$. If $a \in k$, then by the assumption on the filter $\mathfrak{B}$ and the ring R the element $z \in k^I$ defined by $z_j = a$ and $z_i = 0$ $(i \neq j)$ is in R. Hence $xzy \in \mathfrak{q}$ says $x_j a y_j \in \mathfrak{p}$ as desired. □

DEFINITION. Let R be a ring and take $x, y \in R$. We call the set $\{U(xry) \mid r \in R\}$ the *base cover* defined by x, y on Spec R. Note that $\bigcup_{r\in R} U(xry) = U(x) \cap U(y)$. We call the base cover *finitary* if there is a finite set $E \subseteq R$ with $\bigcup_{r\in E} U(xry) = U(x) \cap U(y)$.

REMARK 15. R is quasi-commutative iff all base covers defined by elements $x, y \in R$ are finitary.

Proof. If R is quasi-commutative then the $U(x), x \in X$ are open and compact. Since Spec R is spectral, $U(x) \cap U(y)$ is also compact. In particular, the base cover of x, y is finitary.

If R is not quasi-commutative, then there is a locally prime ideal $\mathfrak{a}$ of R which is not prime. Hence there are $x, y \in R$ with $xRy \subseteq \mathfrak{a}$ such that $x, y \notin \mathfrak{a}$. The base cover of x, y is not finitary. To see this take $r_1, \ldots, r_n \in R$. Since $\mathfrak{a}$ is a locally prime ideal, $xr_iy \in \mathfrak{a}$ for each i and $x, y \notin a$ there is a prime ideal $\mathfrak{p}$ of R with $xr_iy \in \mathfrak{p}$ for each i and $x, y \notin \mathfrak{p}$. Thus $\mathfrak{p} \in U(x) \cap U(y) \setminus (U(xr_1y) \cup \ldots \cup U(xr_ny))$ as desired. □

PROPOSITION 16. *Let k be a ring and let $\mathfrak{p}$ be a prime ideal of k. Take $\xi, \eta \in k$ with $\xi, \eta \notin \mathfrak{p}$. Let k' be a* subring *of k such that $\xi k' \eta \subseteq \mathfrak{p}$. Let k_0 be the* subring *of k generated by k'* and *ξ, η.*

(1) *We have $\xi k_0 \eta \subseteq \mathfrak{p}$.*
(2) *Let I be an infinite set and let $\mathfrak{B}$ be a proper filter on I containing all* cofinite *subsets of I. Define $x, y \in k^I$* by $x_i := \xi$, $y_i := \eta$ $(i \in I)$ and set

$$R := \{r \in k^I | \{i \in I | r_i \in k_0\} \in \mathfrak{B}\}.$$

Then R is a subring *of k^I containing x, y and the base cover defined by x and y on* Spec *R is not finitary. In particular,* Spec $R \neq$ XSpec R.

Proof. (1) In order to see $\xi k_0 \eta \subseteq \mathfrak{p}$, it is enough to show $\xi a_1 \ldots a_n \eta \in \mathfrak{p}$ for all $n \in \mathbb{N}$, where each a_i is from k', or equal to ξ or equal to η; this holds true as one sees immediately by induction on n.

(2) Clearly, R is a subring of k^I and since $\xi, \eta \in k_0$, we have $x, y \in R$. Let $E \subseteq R$ be finite. Since $\mathfrak{B}$ is a proper filter, there is some $j \in I$ such that $r_j \in k_0$ for each $r \in E$.

Let $\mathfrak{q} := \{r \in R \mid r_j \in \mathfrak{p}\}$. Since R and $\mathfrak{B}$ fulfill the requirements of Lemma 14, we know that $\mathfrak{q}$ is a prime ideal of R. As $\xi, \eta \in \mathfrak{p}$ we have $\mathfrak{q} \in U(x) \cap U(y)$. If $r \in E$, then $(xry)_j = \xi r_j \eta \in \mathfrak{p}$ since $r_j \in k_0$. We get $xry \in \mathfrak{q}$, thus $\mathfrak{q} \notin \bigcup_{r\in E} U(xry)$. This shows that $\bigcup_{r\in E} U(xry) \subsetneq U(x) \cap U(y)$, hence the base cover defined by x and y on Spec R is not finitary. □

REMARK 17. Observe that the situation in Proposition 16 can be produced for every prime ideal $\mathfrak{p}$ of k, which is not completely prime: take $\xi, \eta \in k \setminus \mathfrak{p}$ with $\xi\eta \in \mathfrak{p}$ and let k_0 be the subring of k generated by the center Z of k and ξ, η. Then, as $\xi\eta \in \mathfrak{p}$, also $\xi Z \eta \subseteq \mathfrak{p}$, hence $\xi k_0 \eta \subseteq \mathfrak{p}$.

Another choice for k_0 is the subring of k generated by ξ and the subring k' of all elements of k that commute with η.

6. A Spectral Ring which is not Quasi-commutative

THEOREM 18. *There exists a ring R such that* Spec R *is a Boolean space (in particular, R is spectral) and such that* Spec $R \neq$ XSpec R.

Proof. Let K be a field, $k := M_2(K)$ and $\xi := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in k$. Denote by k_0 the subring of k generated by K and ξ. Let I be an infinite set and let $\mathfrak{B}$ be a proper filter on I containing all cofinite subsets of I. Set $R := \{r \in k^I \mid \{i \in I \mid r_i \in k_0\} \in \mathfrak{B}\}$.

We claim that Spec R is Boolean and Spec $R \neq$ XSpec R. By Proposition 16 applied to ξ, $\eta = \xi$ and $\mathfrak{p} = \{0\}$, the base cover of ξ^I and ξ^I defined on Spec R is not finitary (i.e., $U(\xi^I)$ is not compact). Hence it is enough to show that Spec R is Boolean.

CLAIM 1. $k_0 = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a, b \in K \right\}$ is commutative, k_0 is the centralizer of ξ and $k_0^\times = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a, b \in K, a \neq 0 \right\}$.

Proof of the claim: Clearly, k_0 is commutative. Since $\xi^2 = 0$, k_0 is of the desired form. For a, $b \in K$, $a \neq 0$ we have $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{a} & -\frac{b}{a^2} \\ 0 & \frac{1}{a} \end{bmatrix} \in k_0$, hence the units of k_0 have the form as claimed. A straightforward computation shows that k_0 is the centralizer of ξ.

CLAIM 2. Take $r, s \in R$ and let $e \in \{0, 1\}^I$ be defined by $e_i = 1 \Leftrightarrow r_i s_i r_i \neq 0$. Then $U(rsr) \subseteq U(e) \subseteq U(r)$.

Proof of the claim: As $rsr = rsre$ we have $U(rsr) \subseteq U(e)$. In order to prove $U(e) \subseteq U(r)$ we define elements $a, b, c, d \in k^I$ as follows. If $i \in I$ with $r_i s_i r_i = 0$ we take $a_i = b_i = c_i = d_i = 0$. If $i \in I$ with $r_i s_i r_i \neq 0$ such that $s_i \notin k_0$ or $r_i \notin k_0$, then we take $a_i, b_i, c_i, d_i \in k$ so that $a_i r_i b_i + c_i r_i d_i = 1$. If $i \in I$ with $r_i s_i r_i \neq 0$, such that $s_i, r_i \in k_0$, then $0 \neq r_i s_i r_i = s_i r_i^2$ and r_i must be a unit in k_0 (all squares of non-units in k_0 are zero!); thus we may take $a_i = r_i^{-1} \in k_0$, $b_i = 1$ and $c_i = d_i = 0$.

It follows that $e = arb + crd$. Since $\{i \in I \mid a_i, b_i, c_i, d_i \in k_0\}$ contains $\{i \in I \mid s_i \in k_0\} \cap \{i \in I \mid r_i \in k_0\}$ and $r, s \in R$, it follows that $a, b, c, d \in R$. Hence $e = arb + crd$ implies $U(e) \subseteq U(r)$.

CLAIM 3. The mapping ρ: Spec $R \to$ Spec K^I which sends $\mathfrak{q}$ to $\mathfrak{q} \cap K^I$ is a homeomorphism. In particular, Spec R is Boolean.

Proof of the claim: Since K^I is central, $\mathfrak{q} \cap K^I$ is indeed a prime ideal of K^I for every $\mathfrak{q} \in$ Spec R. First we show

$$\mathfrak{p} \in \text{Spec } K^I \Rightarrow \mathfrak{p}R \subsetneq R.$$

By Theorem 12, $\mathfrak{p}$ is of the form $\Psi(\mathscr{F}) = \bigcup_{J \in \mathscr{F}} \mathfrak{a}_J$ for some $\mathscr{F} \in$ Prim 2^I, where $\mathfrak{a}_J = \prod_{i \in I \setminus J} K \times \prod_{j \in J} \{0\}$. As $\mathfrak{p}R \lhd R$, it suffices to show $1 \notin \mathfrak{p}R$. Assume otherwise.

Then for some $p^{(k)} \in \mathfrak{p}$ and $s_k \in R$ we have $1 = \sum_{k=1}^{m} p^{(k)} s_k$. Hence $p^{(k)} \in \mathfrak{a}_{J_k}$ for some $J_k \in \mathscr{F}$. In particular, $p_j{}^{(k)} = 0$ for $j \in J_k$. So $p_j{}^{(k)} = 0$ for all $j \in J_1 \cap \cdots \cap J_m \neq \emptyset$. This shows $1 \neq \sum_{k=1}^{m} p^{(k)} s_k$.

As $\mathfrak{p}R$ is a proper ideal of R, there is a maximal (and hence prime) ideal $\mathfrak{m}$ of R above $\mathfrak{p}R$. On the other hand, $\mathfrak{m} \cap K^I$ is a prime ideal of Spec K^I that contains $\mathfrak{p}$ (K^I is central). As Spec K^I is Boolean by Theorem 12, we have $\mathfrak{m} \cap K^I = \mathfrak{p}$.

This shows that ρ is surjective. In order to show that ρ is injective let $\mathfrak{p}_1, \mathfrak{p}_2 \in$ Spec R with $\mathfrak{p}_1 \not\subseteq \mathfrak{p}_2$. Let $r \in \mathfrak{p}_1 \backslash \mathfrak{p}_2$. Then $\mathfrak{p}_2 \in U(r)$ and $\mathfrak{p}_1 \notin U(r)$. Let $s \in R$ satisfy $\mathfrak{p}_2 \in U(rsr) \subseteq U(r)$. By Claim 2, there is some $a \in K^I$ with $\mathfrak{p}_2 \in U(a) \subseteq U(r)$. Hence also $\mathfrak{p}_1 \notin U(a)$ and $a \in (\mathfrak{p}_1 \cap K^I) \backslash \mathfrak{p}_2$ as desired.

So we know that ρ is bijective. For each $a \in K^I$ we certainly have $\rho^{-1}(U(a)) = U(a)$, hence ρ is continuous. As Spec K^I is Hausdorff and Spec R is compact, ρ is indeed a homeomorphism. This concludes the proof of the theorem. □

REMARK 19. For the ring R constructed in the proof of Theorem 18, the open and compact subsets of Spec R are precisely the sets $U(a)$, where $a \in \{0, 1\}^I \subseteq R$. This follows easily from Theorem 12 and Claim 3 of the proof of Theorem 18.

7. A von Neumann Regular Ring which is not Spectral

At the end of [1] (see also [2] after Proposition 40) an example of a ring which is not quasi-commutative is given. Together with the claim that every spectral ring is quasi-commutative the author deduces the existence of non-spectral rings. As we have seen in Theorem 18, this implication fails in general. Nevertheless, not every ring is spectral:

THEOREM 20. *There is a von Neumann regular ring R such that* Spec R *is T_1 but not a spectral space.*

Proof. The following example was proposed by Goodearl and then studied by Belluce in [1, 2]. Let K be a field and R the ring of all sequences $(a_1, a_2, ...)$ of 2×2 matrices over K that are eventually diagonal. Let $\xi := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \eta := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in k := M_2(K)$. Since matrix rings over fields are von Neumann regular it follows easily that R is von Neumann regular. We show that R is not quasi-commutative; by Proposition 8, this implies that R is not spectral.

With $k_0 := \{a \in k \mid a$ is diagonal) we have $\xi, \eta \in k_0$ and $\xi k_0 \eta = \xi \eta k_0 = \{0\} \in$ Spec k. Let $x := \xi^{\mathbb{N}}$, $y := \eta^{\mathbb{N}}$. By Proposition 16 applied to the filter of cofinite subsets of $\mathbb{N}$, the base cover defined by x and y on Spec R is not finitary. Hence R is not quasi-commutative (cf. Remark 15).

It remains to show that Spec R is T_1, i.e., that all prime ideals of R are maximal. Observing that R is a simplified version of [4, Example 6.5] we may apply [4, Theorem 6.2] together with [4, Corollary 6.7] to see that all prime, primitive and maximal ideals of R coincide. □

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