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# Integral Theory for Hopf Algebroids

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**Abstract.** The theory of integrals is used to analyze the structure of Hopf algebroids. We prove that the total algebra of a Hopf algebroid is a separable extension of the base algebra if and only if it is a semi-simple extension and if and only if the Hopf algebroid possesses a normalized integral. It is a Frobenius extension if and only if the Hopf algebroid possesses a nondegenerate integral. We give also a sufficient and necessary condition in terms of integrals, under which it is a quasi-Frobenius extension, and illustrate by an example that this condition does not hold true in general. Our results are generalizations of classical results on Hopf algebras.

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**Key words:** Hopf algebroid, integral, Maschke theorem, (quasi-)Frobenius extension.

## 1. Introduction

The notion of *integrals* in Hopf algebras has been introduced by Sweedler [33]. The integrals in Hopf algebras over principal ideal domains were analyzed in [19, 32] where the following – by now classical – results have been proven:

- A free, finite-dimensional bialgebra over a principal ideal domain is a Hopf algebra if and only if it possesses a nondegenerate left integral. (Larson–Sweedler Theorem.)
- The antipode of a free, finite-dimensional Hopf algebra over a principal ideal domain is bijective.
- A Hopf algebra over a field is finite-dimensional if and only if it possesses a nonzero left integral.
- The left integrals in a finite-dimensional Hopf algebra over a field form a one dimensional subspace.
- A Hopf algebra over a field is semi-simple if and only if it possesses a normalized left integral. (Maschke's Theorem.)

There are numerous generalizations of these results in the literature. Historically the first is due to Pareigis [27] who proved the following statements on a finitely generated and projective Hopf algebra $(H, \Delta, \epsilon, S)$ over a commutative ring $k$:

– $H$ is a Frobenius extension of $k$ if and only if there exists a Frobenius functional $\psi$: $H \to k$ satisfying $(H \otimes \psi) \circ \Delta = 1_H \psi(\_)$.
– The antipode, $S$, is bijective.
– The left integrals form a projective rank 1 direct summand of the $k$-module $H$.
– $H$ is a quasi-Frobenius extension of $k$.
– A finitely generated and projective bialgebra over a commutative ring $k$, such that $\mathrm{pic}(k) = 0$, is a Hopf algebra if and only if it possesses a nondegenerate left integral.

The generalization of the Maschke theorem to Hopf algebras $H$ over commutative rings $k$ states that the existence of a normalized left integral in $H$ is equivalent to the separability of $H$ over $k$, which is further equivalent to its relative semi-simplicity in the sense of [15, 17] that any $H$-module is $(H, k)$-projective [12, 20]. This is equivalent to the true semi-simplicity of $H$ (i.e. the true projectivity of any $H$-module [28]) if and only if $k$ is a semi-simple ring [20].

As a nice review on these results we recommend Section 3.2 in [13].

Similar results are known also for the generalizations of Hopf algebras. Integrals for finite-dimensional quasi-Hopf algebras [14] over fields were studied in [16, 25, 26, 11] and for finite-dimensional weak Hopf algebras [4, 3] over fields in [3, 40].

The purpose of the present paper is to investigate which of the above results generalizes to *Hopf algebroids*.

Hopf algebroids with bijective antipode have been introduced in [1, 5]. It is important to emphasize that this notion of Hopf algebroid is not equivalent to the one introduced under the same name by Lu in [21]. Here we generalize the definition of [5, 1] by relaxing the requirement of the bijectivity of the antipode. A Hopf algebroid consists of a compatible pair of a left and a right bialgebroid structure [21, 34, 35, 38] on the common total algebra $A$. The antipode relates these two left- and right-handed structures. Left/right integrals *in* a Hopf algebroid are defined as the invariants of the left/right regular $A$-module in terms of the counit of the left/right bialgebroid. Integrals *on* a Hopf algebroid are the comodule maps from the total algebra to the base algebra (reproducing the integrals *in* the dual bialgebroids, provided the duals possess bialgebroid structures).

The total algebra of a bialgebroid can be looked at as an extension of the base algebra or its opposite via the source and target maps, respectively. This way there are four algebra extensions associated to a Hopf algebroid. The main results of the paper relate the properties of these extensions to the existence of integrals with special properties:

- A Maschke-type theorem, proving that the separability, and also the (in two cases left in two cases right) semi-simplicity of any of the four extensions is equivalent to the existence of a normalized integral *in* the Hopf algebroid (Theorem 3.1).

- Any of the four extensions is a Frobenius extension if and only if there exists a nondegenerate integral *in* the Hopf algebroid (Theorem 4.7).
- Any of the four extensions is (in two cases a left in two cases a right) quasi-Frobenius extension if and only if the total algebra is a finitely generated and projective module, and the (left or right) integrals *on* the Hopf algebroid form a flat module, over the base algebra (Theorem 5.2).

Our main tool in proving the latter two points is the Fundamental Theorem for Hopf modules over Hopf algebroids (Theorem 4.2).

The paper is organized as follows: We start Section 2 with reviewing some results on bialgebroids from [9, 18, 21, 30, 31, 34–36, 38], the knowledge of which is needed for the understanding of the paper. Then we give the definition of Hopf algebroids and discuss some of its immediate consequences. Integrals both *in* and *on* Hopf algebroids are introduced and some equivalent characterizations are given.

In Section 3 we prove two Maschke-type theorems. The first collects some equivalent properties (in particular the separability) of the inclusion of the base algebra in the total algebra of a Hopf algebroid. These equivalent properties are related to the existence of a normalized integral *in* the Hopf algebroid. The second collects some equivalent properties (in particular the coseparability) of the coring, underlying the Hopf algebroid. These equivalent properties are shown to be equivalent to the existence of a normalized integral *on* the Hopf algebroid.

In Section 4 we prove the Fundamental Theorem for Hopf modules over a Hopf algebroid. This theorem is somewhat stronger than the one that can be obtained by the application of [7, Theorem 5.6], to the present situation. The main result of the section is Theorem 4.7. In proving it we follow an analogous line of reasoning as in [19]. That is, assuming that one of the module structures of the total algebra over the base algebra is finitely generated and projective, we apply the Fundamental Theorem to the Hopf module, constructed on the dual of the Hopf algebroid (w.r.t. the base algebra). Similarly to the case of Hopf algebras, our result implies the existence of nonzero integrals *on* any finitely generated projective Hopf algebroid. Since the dual of a (finitely generated projective) Hopf algebroid is not known to be a Hopf algebroid in general, we have no dual result, that is, we do not know whether there exist nonzero integrals *in* any finitely generated projective Hopf algebroid. As a byproduct, also a sufficient and necessary condition on a finitely generated projective Hopf algebroid is obtained, under which the antipode is bijective. We do not know, however, whether this condition follows from the axioms.

In Section 5 we use the results of Section 4 to obtain conditions which are equivalent to the (either left or right) quasi-Frobenius property of any of the four extensions behind a Hopf algebroid. In order to show that these conditions do not hold true in general, we construct a counterexample.

Throughout the paper we work over a commutative ring $k$. That is, the total and base algebras of our Hopf algebroids are $k$-algebras. For an (always associative and unital) $k$-algebra $A \equiv (A, m_A, 1_A)$ we denote by ${}_A\mathcal{M}$, $\mathcal{M}_A$ and ${}_A\mathcal{M}_A$ the categories of left, right, and bimodules over $A$, respectively. For the $k$-module of morphisms in ${}_A\mathcal{M}$, $\mathcal{M}_A$ and ${}_A\mathcal{M}_A$ we write ${}_A\mathrm{Hom}(\ ,\ )$, $\mathrm{Hom}_A(\ ,\ )$ and ${}_A\mathrm{Hom}_A(\ ,\ )$, respectively.

## 2. Integrals for Hopf Algebroids

Hopf algebroids with bijective antipode have been introduced in [5], where several equivalent reformulations of the definition [5, Definition 4.1] have been given. The definition we give in this section generalizes the form in [5, Proposition 4.2(iii)] by allowing the antipode not to be bijective.

Integrals *in* Hopf algebroids have also been introduced in [5]. As we shall see, the definition [5, Definition 5.1] applies also in our more general setting. In this section we introduce integrals also *on* Hopf algebroids.

In order for the paper to be self-contained we recall some results on bialgebroids from [38, 21, 34, 35, 18]. For more on bialgebroids we refer to the papers [30, 9, 31, 36].

The notions of Takeuchi's $\times_R$-bialgebra [38], Lu's bialgebroid [21] and Xu's bialgebroid with anchor [41] have been shown to be equivalent in [9]. We are going to use the definition in the following form:

DEFINITION 2.1. A *left bialgebroid* $\mathcal{A}_L = (A, B, s, t, \gamma, \pi)$ consists of two algebras $A$ and $B$ over the commutative ring $k$, which are called the total and base algebras, respectively. $A$ is a $B \underset{k}{\otimes} B^{\mathrm{op}}$-ring (i.e. a monoid in ${}_{B\otimes B^{\mathrm{op}}}\mathcal{M}_{B\otimes B^{\mathrm{op}}}$) via the algebra homomorphisms $s\colon B \to A$ and $t\colon B^{\mathrm{op}} \to A$, called the source and target maps, respectively. In terms of $s$ and $t$ one equips $A$ with a $B$–$B$ bimodule structure ${}_BA_B$ as

$$b \cdot a \cdot b' := s(b)t(b')a \quad \text{for } a \in A,\ b, b' \in B.$$

The triple $({}_BA_B, \gamma, \pi)$ is a $B$-coring, that is a comonoid in ${}_B\mathcal{M}_B$. Introducing Sweedler's convention $\gamma(a) = a_{(1)} \underset{B}{\otimes} a_{(2)}$ for $a \in A$, the axioms

$$a_{(1)}t(b) \underset{B}{\otimes} a_{(2)} = a_{(1)} \underset{B}{\otimes} a_{(2)}s(b), \tag{2.1}$$

$$\gamma(1_A) = 1_A \underset{B}{\otimes} 1_A, \tag{2.2}$$

$$\gamma(aa') = \gamma(a)\gamma(a'), \tag{2.3}$$

$$\pi(1_A) = 1_B, \tag{2.4}$$

$$\pi(a\ s \circ \pi(a')) = \pi(aa'), \tag{2.5}$$

$$\pi(a\ t \circ \pi(a')) = \pi(aa') \tag{2.6}$$

are required for all $b \in B$ and $a, a' \in A$.

Notice that – although $A \underset{B}{\otimes} A$ is not an algebra – axiom (2.3) makes sense in view of (2.1).

The homomorphisms of left bialgebroids $\mathcal{A}_L = (A, B, s, t, \gamma, \pi) \to \mathcal{A}'_L = (A', B', s', t', \gamma', \pi')$ are pairs of $k$-algebra homomorphisms $(\Phi\colon A \to A',$

$\phi\colon B \to B'$) satisfying

$$s' \circ \phi = \Phi \circ s, \tag{2.7}$$

$$t' \circ \phi = \Phi \circ t, \tag{2.8}$$

$$\gamma' \circ \Phi = (\Phi \underset{B}{\otimes} \Phi) \circ \gamma, \tag{2.9}$$

$$\pi' \circ \Phi = \phi \circ \pi. \tag{2.10}$$

The bimodule ${}_B A_B$, appearing in Definition 2.1, is defined in terms of multiplication on the left. Hence – following the terminology of [18] – we use the name *left* bialgebroid for this structure. In terms of right multiplication one defines right bialgebroids analogously. For the details we refer to [18].

Once the map $\gamma\colon A \to A \underset{B}{\otimes} A$ is given we can define $\gamma^{\text{op}}\colon A \to A \underset{B^{\text{op}}}{\otimes} A$ via $a \mapsto a_{(2)} \otimes a_{(1)}$. It is straightforward to check that if $\mathcal{A}_L = (A, B, s, t, \gamma, \pi)$ is a left bialgebroid then $\mathcal{A}_{L\ \text{cop}} = (A, B^{\text{op}}, t, s, \gamma^{\text{op}}, \pi)$ is also a left bialgebroid and $\mathcal{A}_L^{\text{op}} = (A^{\text{op}}, B, t, s, \gamma, \pi)$ is a right bialgebroid.

In the case of a left bialgebroid $\mathcal{A}_L = (A, B, s, t, \gamma, \pi)$ the category ${}_A\mathcal{M}$ of left $A$-modules is a monoidal category. As a matter of fact, any left $A$-module is a $B$–$B$ bimodule via $s$ and $t$. The monoidal product in ${}_A\mathcal{M}$ is defined as the $B$-module tensor product with $A$-module structure

$$a \cdot (m \underset{B}{\otimes} m') := a_{(1)} \cdot m \underset{B}{\otimes} a_{(2)} \cdot m' \quad \text{for } a \in A,\ m \underset{B}{\otimes} m' \in M \underset{B}{\otimes} M'.$$

Just the same way as axiom (2.3), also this definition makes sense in the view of (2.1). The monoidal unit is $B$ with $A$-module structure

$$a \cdot b := \pi(as(b)) \quad \text{for } a \in A,\ b \in B.$$

Analogously, in the case of a right bialgebroid $\mathcal{A}_R$ the category $\mathcal{M}_A$ of right $A$-modules is a monoidal category.

The $B$-coring structure $({}_B A_B, \gamma, \pi)$, underlying the left bialgebroid $\mathcal{A}_L = (A, B, s, t, \gamma, \pi)$, gives rise to a $k$-algebra structure on any of the $B$-duals of ${}_B A_B$ [10, 17.8]. The multiplication on the $k$-module ${}_*\mathcal{A} := {}_B\text{Hom}(A, B)$, for example, is given by

$$({}_*\phi\, {}_*\psi)(a) = {}_*\psi(t \circ {}_*\phi(a_{(2)})a_{(1)}) \quad \text{for } {}_*\phi, {}_*\psi \in {}_*\mathcal{A},\ a \in A \tag{2.11}$$

and the unit is $\pi$. ${}_*\mathcal{A}$ is a left $A$-module and $A$ is a right ${}_*\mathcal{A}$-module via

$$a \rightharpoonup {}_*\phi := {}_*\phi(\_\, a) \quad \text{and} \quad a \leftharpoonup {}_*\phi := t \circ {}_*\phi(a_{(2)})\, a_{(1)} \tag{2.12}$$

for ${}_*\phi \in {}_*\mathcal{A}$, $a \in A$. As it is well known [39, 18], ${}_*\mathcal{A}$ is also a $B \underset{k}{\otimes} B^{\text{op}}$-ring via the inclusions

$$ {}_*s\colon B \to {}_*\mathcal{A}, \qquad b \mapsto \pi(\_)b,$$

$$ {}_*t\colon B^{\text{op}} \to {}_*\mathcal{A}, \qquad b \mapsto \pi(\_\, s(b)).$$

Both maps ${}_*s$ and ${}_*t$ are split injections of $B$-modules with common left inverse ${}_*\pi\colon {}_*\mathcal{A} \to B, {}_*\phi \mapsto {}_*\phi(1_A)$. What is more, if $A$ is finitely generated and projective as a left $B$-module, then ${}_*\mathcal{A}$ has also a right bialgebroid structure (with source and target maps ${}_*s$ and ${}_*t$, respectively, and counit ${}_*\pi$).

Notice that the algebra ${}_*\mathcal{A}$ reduces to the opposite of the usual dual algebra if $({}_BA_B, \gamma, \pi)$ is a coalgebra over a commutative ring $B$. In the case when $A$ is a finitely generated projective left $B$-module, also the coproduct specializes to the opposite of the usual one in the case when $\mathcal{A}$ is a bialgebra. This convention is responsible for duality to flip the notions of left and right bialgebroids.

Applying the above formulae to the left bialgebroid $(\mathcal{A}_L)_{\mathrm{cop}}$ we obtain a $B \underset{k}{\otimes} B^{\mathrm{op}}$-ring structure on $\mathcal{A}_* := \mathrm{Hom}_B(A, B)$. The inclusions $B \to \mathcal{A}_*$ and $B^{\mathrm{op}} \to \mathcal{A}_*$ will be denoted by $s_*$ and $t_*$, respectively. In particular, $\mathcal{A}_*$ is a left $A$-module and $A$ is a right $\mathcal{A}_*$-module via

$$a \rightharpoonup \phi_* := (\_\, a) \quad \text{and} \quad a \leftharpoonup \phi_* := s \circ \phi_*(a_{(1)})\, a_{(2)}. \tag{2.13}$$

If the module $A$ is finitely generated and projective as a right $B$-module then $\mathcal{A}_*$ is also a right bialgebroid.

In the case of a right bialgebroid $\mathcal{A}_R = (A, B, s, t, \gamma, \pi)$ the application of the opposite of the multiplication formula (2.11) to $(\mathcal{A}_R)^{\mathrm{op}}_{\mathrm{cop}}$ and to $(\mathcal{A}_R)^{\mathrm{op}}$ results $B \underset{k}{\otimes} B^{\mathrm{op}}$-ring structures on $\mathcal{A}^* := \mathrm{Hom}_B(A, B)$ and ${}^*\mathcal{A} := {}_B\mathrm{Hom}(A, B)$, respectively. We have the inclusions $s^*\colon B \to \mathcal{A}^*$, $t^*\colon B^{\mathrm{op}} \to \mathcal{A}^*$, ${}^*s\colon B \to {}^*\mathcal{A}$ and ${}^*t\colon B^{\mathrm{op}} \to {}^*\mathcal{A}$.

In particular, $\mathcal{A}^*$ and ${}^*\mathcal{A}$ are right $A$-modules and $A$ is a left $\mathcal{A}^*$-module and a left ${}^*\mathcal{A}$-module via the formulae

$$\phi^* \leftharpoonup a := \phi^*(a\, \_) \quad \text{and} \quad \phi^* \rightharpoonup a := a^{(2)}\, t \circ \phi^*(a^{(1)}), \tag{2.14}$$

$${}^*\phi \leftharpoondown a := {}^*\phi(a\, \_) \quad \text{and} \quad {}^*\phi \rightharpoondown a := a^{(1)}\, s \circ {}^*\phi(a^{(2)}) \tag{2.15}$$

for $\phi^* \in \mathcal{A}^*$, ${}^*\phi \in {}^*\mathcal{A}$ and $a \in A$. If $A$ is finitely generated and projective as a right, or as a left $B$-module then the corresponding dual is also a left bialgebroid.

Before defining the structure that is going to be the subject of the paper let us stop here and introduce some notations. Analogous notations were already used in [5].

When dealing with a $B \underset{k}{\otimes} B^{\mathrm{op}}$-ring $A$, we have to face the situation that $A$ carries different module structures over the base algebra $B$. In this situation the usual notation $A \underset{B}{\otimes} A$ would be ambiguous. Therefore we make the following notational convention. In terms of the maps $s\colon B \to A$ and $t\colon B^{\mathrm{op}} \to A$ we introduce four $B$-modules

$$\begin{aligned} {}_BA &: \quad b \cdot a := s(b)a, \\ A_B &: \quad a \cdot b := t(b)a, \\ A^B &: \quad a \cdot b = as(b), \\ {}^BA &: \quad b \cdot a = at(b). \end{aligned} \tag{2.16}$$

(Our notation can be memorized as left indices stand for left modules and right indices for right modules. Upper indices for modules defined in terms of right multiplication and lower indices for the ones defined in terms of left multiplication.)

In writing $B$-module tensor products we write out explicitly the module structures of the factors that are taking part in the tensor products, and do not put marks under the symbol $\otimes$. For example, we write $A_B \otimes {}_B A$. Normally we do not denote the module structures that are not taking part in the tensor product, this should be clear from the context. In writing elements of tensor product modules we do not distinguish between the various module tensor products. That is, we write both $a \otimes a' \in A_B \otimes {}_B A$ and $c \otimes c' \in A^B \otimes {}_B A$, for example.

A left $B$-module can be considered as a right $B^{\text{op}}$-module, and sometimes we want to take a module tensor product over $B^{\text{op}}$. In this case we use the name of the corresponding $B$-module and the fact that the tensor product is taken over $B^{\text{op}}$ should be clear from the order of the factors. For example, ${}_B A \otimes A_B$ is the $B^{\text{op}}$-module tensor product of the right $B^{\text{op}}$ module defined via multiplication by $s(b)$ on the left, and the left $B^{\text{op}}$-module defined via multiplication by $t(b)$ on the left.

In writing multiple tensor products we use different types of letters to denote which module structures take part in the same tensor product. For example, the $B$-module tensor product $A_B \otimes {}^B A$ can be given a right $B$ module structure via multiplication by $t(b)$ on the left in the second factor. The tensor product of this right $B$-module with ${}_B A$ is denoted by $A_B \otimes {}^B A_{\mathbf{B}} \otimes {}_{\mathbf{B}} A$.

We are ready to introduce the structure that is going to be the subject of the paper:

DEFINITION 2.2. A *Hopf algebroid* $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ consists of a left bialgebroid $\mathcal{A}_L = (A, L, s_L, t_L, \gamma_L, \pi_L)$, a right bialgebroid $\mathcal{A}_R = (A, R, s_R, t_R, \gamma_R, \pi_R)$ and a $k$-module map $S\colon\ A \to A$, called the antipode, such that the following axioms hold true:

(i) $s_L \circ \pi_L \circ t_R = t_R, \qquad t_L \circ \pi_L \circ s_R = s_R$ and
$$s_R \circ \pi_R \circ t_L = t_L, \qquad t_R \circ \pi_R \circ s_L = s_L, \tag{2.17}$$

(ii) $(\gamma_L \otimes {}^R A) \circ \gamma_R = (A_L \otimes \gamma_R) \circ \gamma_L$
as maps $A \to A_L \otimes {}_L A^R \otimes {}^R A$ and
$(\gamma_R \otimes {}_L A) \circ \gamma_L = (A^R \otimes \gamma_L) \circ \gamma_R$
$$\text{as maps } A \to A^R \otimes {}^R A_L \otimes {}_L A, \tag{2.18}$$

(iii) $S$ is both an $L$–$L$ bimodule map ${}^L A_L \to {}_L A^L$
$$\text{and an } R\text{–}R \text{ bimodule map } {}^R A_R \to {}_R A^R, \tag{2.19}$$

(iv) $m_A \circ (S \otimes {}_L A) \circ \gamma_L = s_R \circ \pi_R$ and
$$m_A \circ (A^R \otimes S) \circ \gamma_R = s_L \circ \pi_L. \tag{2.20}$$

If $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ is a Hopf algebroid then so is $\mathcal{A}^{\text{op}}_{\text{cop}} = ((\mathcal{A}_R)^{\text{op}}_{\text{cop}}, (\mathcal{A}_L)^{\text{op}}_{\text{cop}}, S)$ and if $S$ is bijective then also $\mathcal{A}_{\text{cop}} = ((\mathcal{A}_L)_{\text{cop}}, (\mathcal{A}_R)_{\text{cop}}, S^{-1})$ and $\mathcal{A}^{\text{op}} = ((\mathcal{A}_R)^{\text{op}}, (\mathcal{A}_L)^{\text{op}}, S^{-1})$.

The following modification of Sweedler's convention will turn out to be useful. For a Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ we use the notation $\gamma_L(a) = a_{(1)} \otimes a_{(2)}$

with lower indices, and $\gamma_R(a) = a^{(1)} \otimes a^{(2)}$ with upper indices for $a \in A$ in the case of the coproducts of $\mathcal{A}_L$ and of $\mathcal{A}_R$, respectively. The axioms (2.18) read in this notation as

$$\begin{aligned} a^{(1)}{}_{(1)} \otimes a^{(1)}{}_{(2)} \otimes a^{(2)} &= a_{(1)} \otimes a_{(2)}{}^{(1)} \otimes a_{(2)}{}^{(2)}, \\ a_{(1)}{}^{(1)} \otimes a_{(1)}{}^{(2)} \otimes a_{(2)} &= a^{(1)} \otimes a^{(2)}{}_{(1)} \otimes a^{(2)}{}_{(2)} \end{aligned}$$

for $a \in A$.

Examples of Hopf algebroids (with bijective antipode) are collected in [5].

PROPOSITION 2.3. (1) *The base algebras $L$ and $R$ of the left and right bialgebroids in a Hopf algebroid are anti-isomorphic.*

(2) *For a Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ the pair $(S, \pi_L \circ s_R)$ is a left bialgebroid homomorphism $(\mathcal{A}_R)^{\mathrm{op}}_{\mathrm{cop}} \to \mathcal{A}_L$ and $(S, \pi_R \circ s_L)$ is a left bialgebroid homomorphism $\mathcal{A}_L \to (\mathcal{A}_R)^{\mathrm{op}}_{\mathrm{cop}}$.*

*Proof.* (1) Both $\pi_R \circ s_L$ and $\pi_R \circ t_L$ are anti-isomorphisms $L \to R$ with inverses $\pi_L \circ t_R$ and $\pi_L \circ s_R$, respectively.

(2) We have seen that the map $\pi_L \circ s_R \colon R^{\mathrm{op}} \to L$ is an algebra homomorphism. It follows from (2.19), (2.20) and some bialgebroid identities that $S\colon A^{\mathrm{op}} \to A$ is an algebra homomorphism, as for $a, b \in A$ we have

$$\begin{aligned} S(1_A) &= 1_A\, S(1_A) = s_L \circ \pi_L(1_A) = 1_A \quad \text{and} \\ S(ab) &= S[t_L \circ \pi_L(a_{(2)})\, a_{(1)}\, b] \\ &= S[a_{(1)}\, t_L \circ \pi_L(b_{(2)})\, b_{(1)}]\, a_{(2)}{}^{(1)} S(a_{(2)}{}^{(2)}) \\ &= S[a^{(1)}{}_{(1)} b^{(1)}{}_{(1)}]\, a^{(1)}{}_{(2)} b^{(1)}{}_{(2)} S(b^{(2)}) S(a^{(2)}) \\ &= s_R \circ \pi_R(a^{(1)} b^{(1)})\, S(b^{(2)})\, S(a^{(2)}) \\ &= S[b^{(2)}\, t_R \circ \pi_R(t_R \circ \pi_R(a^{(1)})\, b^{(1)})] S(a^{(2)}) \\ &= S(b)\, s_R \circ \pi_R(a^{(1)})\, S(a^{(2)}) = S(b)S(a). \end{aligned}$$

The properties (2.7)–(2.8) follow from (2.19) and (2.17) as

$$\begin{aligned} s_L \circ \pi_L \circ s_R &= S \circ t_L \circ \pi_L \circ s_R = S \circ s_R, \\ t_L \circ \pi_L \circ s_R &= s_R = S \circ t_R. \end{aligned}$$

The properties (2.9)–(2.10) are checked on an element $a \in A$ as

$$\begin{aligned} \gamma_L \circ S(a) &= S(a_{(1)})_{(1)}\, s_L \circ \pi_L(a_{(2)}) \otimes S(a_{(1)})_{(2)} \\ &= S(a^{(1)}{}_{(1)})_{(1)} a^{(1)}{}_{(2)} S(a^{(2)}) \otimes S(a^{(1)}{}_{(1)})_{(2)} \\ &= S(a^{(1)}{}_{(1)})_{(1)}\, t_L \circ \pi_L(a^{(1)}{}_{(2)(2)})\, a^{(1)}{}_{(2)(1)}\, S(a^{(2)}) \otimes S(a^{(1)}{}_{(1)})_{(2)} \\ &= S(a^{(1)(1)}{}_{(1)})_{(1)} a^{(1)(1)}{}_{(2)(1)} S(a^{(2)}) \\ &\quad \otimes S(a^{(1)(1)}{}_{(1)})_{(2)} a^{(1)(1)}{}_{(2)(2)} S(a^{(1)(2)}) \\ &= S(a^{(2)}) \otimes s_R \circ \pi_R(a^{(1)(1)})\, S(a^{(1)(2)}) = (S \otimes S) \circ \gamma_R^{\mathrm{op}}(a) \quad \text{and} \\ \pi_L \circ S(a) &= \pi_L[S(a_{(1)})\, s_L \circ \pi_L(a_{(2)})] = \pi_L \circ s_R \circ \pi_R(a). \end{aligned}$$

The proof is completed by the observation that in passing from the Hopf algebroid $\mathcal{A}$ to $\mathcal{A}^{\mathrm{op}}_{\mathrm{cop}}$ the roles of $(S, \pi_L \circ s_R)$ and $(S, \pi_R \circ s_L)$ become interchanged. □

PROPOSITION 2.4. *The left bialgebroid $\mathcal{A}_L$ in a Hopf algebroid $\mathcal{A}$ is a $\times_L$-Hopf algebra in the sense of* [30]. *That is, the map*

$$\alpha\colon\ {}^{L}A \otimes A_L \to A_L \otimes {}_{L}A, \quad a \otimes b \to a_{(1)} \otimes a_{(2)} b$$

*is bijective.*

*Proof.* The inverse of $\alpha$ is given by

$$\alpha^{-1}\colon\ A_L \otimes {}_{L}A \to {}^{L}A \otimes A_L, \quad a \otimes b \mapsto a^{(1)} \otimes S(a^{(2)}) b. \qquad \square$$

The relation between the left and the right bialgebroids in a Hopf algebroid $\mathcal{A}$ implies relations between the dual algebras $\mathcal{A}^* \equiv \mathrm{Hom}_R(A^R, R)$ and $\mathcal{A}_* \equiv \mathrm{Hom}_L(A_L, L)$ and also between ${}^*\mathcal{A} \equiv {}_R\mathrm{Hom}({}^RA, R)$ and ${}_*\mathcal{A} \equiv {}_L\mathrm{Hom}({}_LA, L)$:

LEMMA 2.5. *For a Hopf algebroid $\mathcal{A}$ there exist algebra anti-isomorphisms $\sigma\colon {}_*\mathcal{A} \to {}^*\mathcal{A}$ and $\chi\colon \mathcal{A}^* \to \mathcal{A}_*$ satisfying*

$$a \leftharpoondown {}_*\phi = \sigma({}_*\phi) \rightharpoondown a \tag{2.21}$$

*and*

$$\phi^* \rightharpoonup a = a \leftharpoonup \chi(\phi^*) \tag{2.22}$$

*for all ${}_*\phi \in {}_*\mathcal{A}$, $\phi^* \in \mathcal{A}^*$ and $a \in A$.*

*Proof.* We leave it to the reader to check that the maps

$$\sigma\colon {}_*\mathcal{A} \to {}^*\mathcal{A}, \quad {}_*\phi \mapsto \pi_R(\_ \leftharpoondown {}_*\phi)$$

and

$$\chi\colon \mathcal{A}^* \to \mathcal{A}_*, \quad \phi^* \mapsto \pi_L(\phi^* \rightharpoonup \_)$$

are algebra anti-homomorphisms satisfying (2.21)–(2.22). The inverses are given by

$$\sigma^{-1}\colon {}^*\mathcal{A} \to {}_*\mathcal{A}, \quad {}^*\phi \mapsto \pi_L({}^*\phi \rightharpoondown \_)$$

and

$$\chi^{-1}\colon \mathcal{A}_* \to \mathcal{A}^*, \quad \phi_* \mapsto \pi_R(\_ \leftharpoonup \phi_*). \qquad \square$$

LEMMA 2.6. *The following properties of a Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ are equivalent*:

(1.a) *The module $A_L$ is finitely generated and projective.*
(1.b) *The module $A^R$ is finitely generated and projective.*

*The following are also equivalent*:

(2.a) *The module* $_LA$ *is finitely generated and projective.*
(2.b) *The module* $^RA$ *is finitely generated and projective.*

*If furthermore S is bijective then all the four properties* (1.a)*,* (1.b)*,* (2.a) *and* (2.b) *are equivalent.*

*Proof.* (1.a) $\Rightarrow$ (1.b) In terms of the dual bases, $\{b_i\} \subset A$ and $\{\beta_*^i\} \subset \mathcal{A}_*$ for the module $A_L$, the dual bases, $\{k_j\} \subset A$ and $\{\kappa_j^*\} \subset \mathcal{A}^*$ for the module $A^R$, can be constructed by the requirement that

$$\sum_j k_j \otimes \kappa_j^* = \sum_i b_i^{(1)} \otimes [\chi^{-1}(\beta_*^i) \leftharpoonup s_R \circ \pi_R \circ s_L \circ \pi_L(b_i^{(2)})]$$

as elements of $A^R \otimes {}_R\mathcal{A}^*$, where $\chi$ is the isomorphism (2.22). The expression on the right-hand side is well defined since – though the map

$$A_L \otimes {}^L\mathcal{A}_* \to \mathcal{A}^*, \quad a \otimes \phi_* \mapsto \chi^{-1}(\phi_*) \leftharpoonup s_R \circ \pi_R \circ s_L \circ \pi_L(a)$$

is not a left $R$-module map ${}^RA_L \otimes {}^L\mathcal{A}_* \to {}_R\mathcal{A}^*$ – its restriction to the $R$-submodule $\{\sum_k a_k \otimes \phi_*^k \in A_L \otimes {}^L\mathcal{A}_* \mid \sum_k a_k t_L(l) \otimes \phi_*^k = \sum_k a_k \otimes \phi_*^k s_*(l)\ \forall l \in L\}$ is so.

(2.a) $\Rightarrow$ (2.b) Similarly, in terms of the dual bases, $\{b_i\} \subset A$ and $\{{}_*\beta^i\} \subset {}_*\mathcal{A}$ for the module ${}_LA$, the dual bases, $\{k_j\} \subset A$ and $\{{}^*\kappa_j\} \subset {}^*\mathcal{A}$ for the module ${}^RA$, can be constructed by the requirement that

$$\sum_j {}^*\kappa_j \otimes k_j = \sum_i [\sigma({}_*\beta^i) \leftharpoonup t_R \circ \pi_R \circ t_L \circ \pi_L(b_i^{(1)})] \otimes b_i^{(2)}$$

as elements of ${}^*\mathcal{A}_R \otimes {}^RA$, where $\sigma$ is the isomorphism (2.21).

(1.b) $\Rightarrow$ (1.a) follows by applying (2.a) $\Rightarrow$ (2.b) to the Hopf algebroid $\mathcal{A}_{\text{cop}}^{\text{op}}$.

(2.b) $\Rightarrow$ (2.a) follows by applying (1.a) $\Rightarrow$ (1.b) to the Hopf algebroid $\mathcal{A}_{\text{cop}}^{\text{op}}$.

Now suppose that $S$ is bijective.

(1.a) $\Rightarrow$ (2.b) In terms of the dual bases, $\{b_i\} \subset A$ and $\{\beta_*^i\} \subset \mathcal{A}_*$ for the module $A_L$, the dual bases, $\{k_j\} \subset A$ and $\{{}^*\kappa_j\} \subset {}^*\mathcal{A}$ for the module ${}^RA$, can be constructed by the requirement that

$$\sum_j {}^*\kappa_j \otimes k_j = \sum_i \pi_R \circ t_L \circ \beta_*^i \circ S \otimes S^{-1}(b_i) \quad \text{as elements of } {}^*\mathcal{A}_R \otimes {}^RA.$$

(2.b) $\Rightarrow$ (1.a) follows by applying (1.a) $\Rightarrow$ (2.b) to the Hopf algebroid $\mathcal{A}_{\text{cop}}^{\text{op}}$. □

Now we turn to the study of the notion of integrals *in* Hopf algebroids. For a left bialgebroid $\mathcal{A}_L = (A, L, s_L, t_L, \gamma_L, \pi_L)$ and a left $A$-module $M$ the *invariants* of $M$ with respect to $\mathcal{A}_L$ are the elements of

$$\text{Inv}(M) := \{n \in M \mid a \cdot n = s_L \circ \pi_L(a) \cdot n\ \forall a \in A\}.$$

Clearly, the invariants of $M$ with respect to $(\mathcal{A}_L)_{\rm cop}$ coincide with its invariants with respect to $\mathcal{A}_L$. The invariants of a right $A$-module $M$ with respect to a right bialgebroid $\mathcal{A}_R$ are defined as the invariants of $M$ (viewed as a left $A^{\rm op}$-module) with respect to $(\mathcal{A}_R)^{\rm op}$.

DEFINITION 2.7. The *left integrals in a left bialgebroid* $\mathcal{A}_L$ are the invariants of the left regular $A$-module with respect to $\mathcal{A}_L$.

The *right integrals in a right bialgebroid* $\mathcal{A}_R$ are the invariants of the right regular $A$-module with respect to $\mathcal{A}_R$.

The *left/right integrals in a Hopf algebroid* $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ are the left/right integrals in $\mathcal{A}_L/\mathcal{A}_R$, that is the elements of

$$\mathcal{L}(\mathcal{A}) = \{\ell \in A \mid a\ell\ = s_L \circ \pi_L(a)\ \ell\ \forall a \in A\}$$

and

$$\mathcal{R}(\mathcal{A}) = \{\wp \in A \mid \wp a = \wp\ s_R \circ \pi_R(a)\ \forall a \in A\}.$$

For any Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ we have $\mathcal{L}(\mathcal{A}) = \mathcal{R}(\mathcal{A}^{\rm op}_{\rm cop})$ and if $S$ is bijective then also $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}_{\rm cop}) = \mathcal{R}(\mathcal{A}^{\rm op})$. Since for $\ell \in \mathcal{L}(\mathcal{A})$ and $a \in A$,

$$S(\ell)a = S[t_L \circ \pi_L(a_{(1)})\ \ell]a_{(2)} = S(a_{(1)}\ell)a_{(2)} = S(\ell)\ s_R \circ \pi_R(a),$$

we have $S(\mathcal{L}(\mathcal{A})) \subseteq \mathcal{R}(\mathcal{A})$ and, similarly, $S(\mathcal{R}(\mathcal{A})) \subseteq \mathcal{L}(\mathcal{A})$.

SCHOLIUM 2.8. *The following properties of an element* $\ell \in A$ *are equivalent*:

(1.a) $\ell \in \mathcal{L}(\mathcal{A})$,
(1.b) $S(a)\ell^{(1)} \otimes \ell^{(2)} = \ell^{(1)} \otimes a\ell^{(2)} \quad \forall a \in A$,
(1.c) $a\ell^{(1)} \otimes S(\ell^{(2)}) = \ell^{(1)} \otimes S(\ell^{(2)})a \quad \forall a \in A$.

*The following properties of the element* $\wp \in A$ *are also equivalent*:

(2.a) $\wp \in \mathcal{R}(\mathcal{A})$,
(2.b) $\wp_{(1)} \otimes \wp_{(2)}S(a) = \wp_{(1)}a \otimes \wp_{(2)} \quad \forall a \in A$,
(2.c) $S(\wp_{(1)}) \otimes \wp_{(2)}a = aS(\wp_{(1)}) \otimes \wp_{(2)} \quad \forall a \in A$.

By comodules over a left bialgebroid $\mathcal{A}_L = (A, L, s_L, t_L, \gamma_L, \pi_L)$ we mean comodules over the $L$-coring $({}_LA_L, \gamma_L, \pi_L)$, and by comodules over a right bialgebroid $\mathcal{A}_R = (A, R, s_R, t_R, \gamma_R, \pi_R)$ comodules over the $R$-coring $({}^RA^R, \gamma_R, \pi_R)$. The pair $({}_LA, \gamma_L)$ is a left comodule, and $(A_L, \gamma_L)$ is a right comodule over the left bialgebroid $\mathcal{A}_L$. Since the $L$-coring $({}_LA_L, \gamma_L, \pi_L)$ possesses a grouplike element $1_A$, also $(L, s_L)$ is a left comodule and $(L, t_L)$ is a right comodule over $\mathcal{A}_L$ (see [10, 28.2]). Similarly, $(A^R, \gamma_R)$ and $(R, s_R)$ are right comodules, and $({}^RA, \gamma_R)$ and $(R, t_R)$ are left comodules over $\mathcal{A}_R$.

DEFINITION 2.9. An *s-integral on a left bialgebroid* $\mathcal{A}_L = (A, L, s_L, t_L, \gamma_L, \pi_L)$ is a left $\mathcal{A}_L$-comodule map ${}_*\rho$: $({}_LA, \gamma_L) \to (L, s_L)$. That is, an element of

$$\mathcal{R}({}_*\mathcal{A}) := \{{}_*\rho \in {}_*\mathcal{A} \mid (A_L \otimes {}_*\rho) \circ \gamma_L = s_L \circ {}_*\rho\}.$$

A *t-integral on* $\mathcal{A}_L$ is a right $\mathcal{A}_L$-comodule map $(A_L, \gamma_L) \to (L, t_L)$. That is, an element of

$$\mathcal{R}(\mathcal{A}_*) := \{\rho_* \in \mathcal{A}_* \mid (\rho_* \otimes {}_L A) \circ \gamma_L = t_L \circ \rho_*\}.$$

An *s-integral on a right bialgebroid* $\mathcal{A}_R = (A, R, s_R, t_R, \gamma_R, \pi_R)$ is a right $\mathcal{A}_R$-comodule map $(A^R, \gamma_R) \to (R, s_R)$. That is, an element of

$$\mathcal{L}(\mathcal{A}^*) := \{\lambda^* \in \mathcal{A}^* \mid (\lambda^* \otimes {}^R A) \circ \gamma_R = s_R \circ \lambda^*\}.$$

A *t-integral on* $\mathcal{A}_R$ is a left $\mathcal{A}_R$-comodule map $({}^R A, \gamma_R) \to (R, t_R)$. That is, an element of

$$\mathcal{L}({}^*\mathcal{A}) := \{{}^*\lambda \in {}^*\mathcal{A} \mid (A^R \otimes {}^*\lambda) \circ \gamma_R = t_R \circ {}^*\lambda\}.$$

The *right/left s- and t-integrals on a Hopf algebroid* $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ are the $s$- and $t$-integrals on $\mathcal{A}_L/\mathcal{A}_R$.

The integrals on a *left/right* bialgebroid are checked to be invariants of the appropriate *right/left* regular module – justifying our usage of the terms *'right'* and *'left'* integrals for them (cf. the remark in Section 2 about using the opposite–co-opposite of the convention, usual in the case of bialgebras, when defining the dual bialgebroids ${}_*\mathcal{A}$ and $\mathcal{A}^*$). As a matter of fact, for example, if ${}_*\rho \in \mathcal{R}({}_*\mathcal{A})$ then

$$\begin{aligned}[{}_*\rho\ {}_*\phi](a) &= {}_*\phi(a \leftharpoonup {}_*\rho) = {}_*\phi(s_L \circ {}_*\rho(a)) = {}_*\rho(a)\ {}_*\phi(1_A) \\ &= [{}_*\rho\ \ {}_*s \circ {}_*\pi({}_*\phi)](a) \end{aligned} \tag{2.23}$$

for all ${}_*\phi \in {}_*\mathcal{A}$ and $a \in A$. If the module ${}_L A$ is finitely generated and projective (hence ${}_*\mathcal{A}$ is a right bialgebroid) then also the converse is true, so in this case the $s$-integrals on $\mathcal{A}_L$ are the same as the right integrals in ${}_*\mathcal{A}$. Similar statements hold true on the elements of $\mathcal{R}(\mathcal{A}_*)$, $\mathcal{L}(\mathcal{A}^*)$ and $\mathcal{L}({}^*\mathcal{A})$.

The reader should be warned that integrals on Hopf algebras $H$ over commutative rings $k$ are defined in the literature sometimes as comodule maps $H \to k$ – similarly to our Definition 2.9 –, sometimes by the analogue of the weaker invariant condition (2.23).

For any Hopf algebroid $\mathcal{A}$ we have $\mathcal{R}({}_*\mathcal{A}) = \mathcal{L}((\mathcal{A}^{\mathrm{op}}_{\mathrm{cop}})^*)$ and $\mathcal{R}(\mathcal{A}_*) = \mathcal{L}({}^*(\mathcal{A}^{\mathrm{op}}_{\mathrm{cop}}))$. If the antipode is bijective then also $\mathcal{R}({}_*\mathcal{A}) = \mathcal{R}((\mathcal{A}_{\mathrm{cop}})_*) = \mathcal{L}({}^*(\mathcal{A}^{\mathrm{op}}))$.

SCHOLIUM 2.10. *Let* $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ *be a Hopf algebroid. The following properties of an element* ${}_*\rho \in {}_*\mathcal{A}$ *are equivalent*:

(1.a) ${}_*\rho \in \mathcal{R}({}_*\mathcal{A})$,
(1.b) $\pi_R \circ s_L \circ {}_*\rho \in \mathcal{L}({}^*\mathcal{A})$,
(1.c) $s_L \circ {}_*\rho(aS(b_{(1)}))\, b_{(2)} = t_L \circ {}_*\rho(a_{(2)}S(b))\, a_{(1)} \quad \forall a, b \in A$.

*The following properties of an element $\rho_* \in \mathcal{A}_*$ are equivalent*:

(2.a) $\rho_* \in \mathcal{R}(\mathcal{A}_*)$,
(2.b) $\pi_R \circ t_L \circ \rho_* \in \mathcal{L}(\mathcal{A}^*)$,
(2.c) $t_L \circ \rho_*(ab^{(1)})\, S(b^{(2)}) = s_L \circ \rho_*(a_{(1)}b)\, a_{(2)} \quad \forall a, b \in A$.

*The following properties of an element $\lambda^* \in \mathcal{A}^*$ are equivalent*:

(3.a) $\lambda^* \in \mathcal{L}(\mathcal{A}^*)$,
(3.b) $\pi_L \circ s_R \circ \lambda^* \in \mathcal{R}(\mathcal{A}_*)$,
(3.c) $a^{(1)}\, s_R \circ \lambda^*(S(a^{(2)})b) = b^{(2)}\, t_R \circ \lambda^*(S(a)b^{(1)}) \quad \forall a, b \in A$.

*The following properties of an element ${}^*\lambda \in {}^*\mathcal{A}$ are equivalent*:

(4.a) ${}^*\lambda \in \mathcal{L}({}^*\mathcal{A})$,
(4.b) $\pi_L \circ t_R \circ {}^*\lambda \in \mathcal{R}({}_*\mathcal{A})$,
(4.c) $S(a_{(1)})\, t_R \circ {}^*\lambda(a_{(2)}b) = b^{(1)}\, s_R \circ {}^*\lambda(ab^{(2)}) \quad \forall a, b \in A$.

*In particular, for ${}_*\rho \in \mathcal{R}({}_*\mathcal{A})$ the element ${}_*\rho \circ S$ belongs to $\mathcal{R}(\mathcal{A}_*)$ and for $\lambda^* \in \mathcal{L}(\mathcal{A}^*)$ the element $\lambda^* \circ S$ belongs to $\mathcal{L}({}^*\mathcal{A})$.*

## 3. Maschke Type Theorems

The most classical version of Maschke's theorem [22] considers group algebras over fields. It states that the group algebra of a finite group $G$ over a field $F$ is semi-simple if and only if the characteristic of $F$ does not divide the order of $G$. This result has been generalized to finite-dimensional Hopf algebras $H$ over fields $F$ by Sweedler [32] proving that $H$ is a separable $F$-algebra if and only if it is semi-simple and if and only if there exists a normalized left integral in $H$. The proof goes as follows. It is a classical result that a separable algebra over a field is semi-simple. If $H$ is semi-simple then, in particular, the $H$-module on $F$, defined in terms of the counit, is projective. This means that the counit, as an $H$-module map $H \to F$, splits. Its right inverse maps the unit of $F$ into a normalized integral. Finally, in terms of a normalized integral one can construct an $H$-bilinear right inverse for the multiplication map $H \underset{F}{\otimes} H \to H$.

The only difficulty in the generalization of Maschke's theorem to Hopf algebras over commutative rings comes from the fact that in the case of an algebra $A$ over a commutative base ring $k$, separability does not imply the semi-simplicity of $A$ in the sense [28] that every (left or right) $A$-module was projective. It implies [15, 17], however, that every $A$-module is $(A, k)$-projective, i.e. that every epimorphism of $A$-modules which is $k$-split, is also $A$-split. In order to avoid confusion, we will say that the $k$-algebra $A$ is *semi-simple* [28] if it is an Artinian semi-simple ring, i.e. if any $A$-module is projective. By the terminology of [15] we call $A$ a (left or right) *semi-simple extension* of $k$ if any (left or right) $A$-module is $(A, k)$-projective.

Since the counit of a Hopf algebra $H$ over a commutative ring $k$ is a split epimorphism of $k$-modules, the Maschke theorem generalizes to this case in the following form [12, 20]. The extension $k \to H$ is separable if and only if it is (left and right) semi-simple and if and only if there exist normalized (left and right) integrals in $H$.

In this section we investigate the properties of the total algebra of a Hopf algebroid, as an extension of the base algebra, that are equivalent to the existence of normalized integrals *in* the Hopf algebroid. Dually, we investigate also the properties of the coring over the base algebra, underlying a Hopf algebroid, that are equivalent to the existence of normalized integrals *on* the Hopf algebroid (in any of the four possible senses).

A Maschke-type theorem on certain Hopf algebroids can be obtained also by the application of [37, Theorem 4.2]. Notice, however, that the Hopf algebroids occurring this way are only the Frobenius Hopf algebroids (discussed in Section 4 below), that is the Hopf algebroids possessing nondegenerate integrals (which are called Frobenius integrals in [37]).

The following Theorem 3.1 generalizes results from [12, Proposition 4.7] and [20, Theorem 3.3].

THEOREM 3.1 (Maschke Theorem for Hopf algebroids). *The following assertions on a Hopf algebroid* $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ *are equivalent*:

(1.a) *The extension* $s_R\colon R \to A$ *is separable. That is, the multiplication map* $A^R \otimes_R A \to A$ *splits as an* $A$–$A$ *bimodule map.*

(1.b) *The extension* $t_R\colon R^{\mathrm{op}} \to A$ *is separable. That is, the multiplication map* ${}^R A \otimes A_R \to A$ *splits as an* $A$–$A$ *bimodule map.*

(1.c) *The extension* $s_L\colon L \to A$ *is separable. That is, the multiplication map* $A^L \otimes_L A \to A$ *splits as an* $A$–$A$ *bimodule map.*

(1.d) *The extension* $t_L\colon L^{\mathrm{op}} \to A$ *is separable. That is, the multiplication map* ${}^L A \otimes A_L \to A$ *splits as an* $A$–$A$ *bimodule map.*

(2.a) *The extension* $s_R\colon R \to A$ *is right semi-simple. That is, any right* $A$*-module is* $(A, R)$*-projective.*

(2.b) *The extension* $t_R\colon R^{\mathrm{op}} \to A$ *is right semi-simple. That is, any right* $A$*-module is* $(A, R^{\mathrm{op}})$*-projective.*

(2.c) *The extension* $s_L\colon L \to A$ *is left semi-simple. That is, any left* $A$*-module is* $(A, L)$*-projective.*

(2.d) *The extension* $t_L\colon L^{\mathrm{op}} \to A$ *is left semi-simple. That is, any left* $A$*-module is* $(A, L^{\mathrm{op}})$*-projective.*

(3.a) *There exists a normalized left integral in* $\mathcal{A}$*. That is, an element* $\ell \in \mathcal{L}(\mathcal{A})$ *such that* $\pi_L(\ell) = 1_L$*.*

(3.b) *There exists a normalized right integral in* $\mathcal{A}$*. That is, an element* $\wp \in \mathcal{R}(\mathcal{A})$ *such that* $\pi_R(\wp) = 1_R$*.*

(4.a) *The epimorphism* $\pi_R\colon A \to R$ *splits as a right* $A$*-module map.*

(4.b) *The epimorphism* $\pi_L\colon A \to L$ *splits as a left* $A$*-module map.*

*Proof.* (1.a) $\Rightarrow$ (2.a), (1.b) $\Rightarrow$ (2.b), (1.c) $\Rightarrow$ (2.c) and (1.d) $\Rightarrow$ (2.d) It is proven in [17, Proposition 2.6] that a separable extension is both left and right semi-simple.

(2.a) $\Rightarrow$ (4.a) ((2.b) $\Rightarrow$ (4.a)) The epimorphism $\pi_R$ is split as a right (left) $R$-module map by $s_R$ (by $t_R$), hence it is split as a right $A$-module map.

(4.a) $\Rightarrow$ (3.b) Let $\nu$: $R \to A$ be the right inverse of $\pi_R$ in $\mathcal{M}_A$. Then $\wp := \nu(1_R)$ is a normalized right integral in $\mathcal{A}$.

(3.a) $\Leftrightarrow$ (3.b) By part (2) of Proposition 2.3 the antipode takes a normalized left/right integral to a normalized right/left integral.

(3.a) $\Rightarrow$ (1.a) and (3.b) $\Rightarrow$ (1.b) If $\ell$ is a normalized left integral in $\mathcal{A}$ then, by Scholium 2.8, the required right inverse of the multiplication map $A^R \otimes_R A \to A$ is given by the $A$–$A$ bimodule map $a \mapsto a\ell^{(1)} \otimes S(\ell^{(2)}) \equiv \ell^{(1)} \otimes S(\ell^{(2)})a$. Similarly, if $\wp$ is a normalized right integral in $\mathcal{A}$ then the right inverse of the multiplication map ${}^R A \otimes A_R \to A$ is given by $a \mapsto aS(\wp_{(1)}) \otimes \wp_{(2)} \equiv S(\wp_{(1)}) \otimes \wp_{(2)}a$.

The proof is completed by applying the above arguments to the Hopf algebroid $\mathcal{A}^{\mathrm{op}}_{\mathrm{cop}}$. □

Let us make a comment on the semi-simplicity of the algebra $A$ (cf. [17, Proposition 1.3]). If $R$ is a semi-simple algebra and the equivalent conditions of Theorem 3.1 hold true, then $A$ – being a semi-simple extension of a semi-simple algebra – is a semi-simple algebra. On the other hand, notice that condition (4.a) in Theorem 3.1 is equivalent to the projectivity of the right $A$-module $R$. Hence if $A$ is a semi-simple $k$-algebra then the equivalent conditions of the theorem hold true. It is not true, however, that the semi-simplicity of the total algebra implies the semi-simplicity of the base algebra (which was shown by Lomp to be the case in Hopf algebras [20]). A counterexample can be constructed as follows: If $B$ is a Frobenius algebra over a commutative ring $k$ then $A := \mathrm{End}_k(B)$ has a Hopf algebroid structure over the base $B$ [6]. If $B$ is a Frobenius algebra over a field – which can be non-semi-simple! – then $A$ is a Hopf algebroid with semi-simple total algebra.

The following Theorem 3.2 can be considered as a dual of Theorem 3.1 in the sense that it speaks about corings over the base algebras instead of algebra extensions. It is important to emphasize, however, that the two theorems are independent results. Even in the case of Hopf algebroids such that all module structures (2.16) are finitely generated and projective, the duals are not known to be Hopf algebroids.

Recall that the dual notion of that of a relative projective module is the relative injective comodule. Namely, a comodule $M$ for an $R$-coring $A$ is called $(A, R)$-*injective* [10, 18.18] if any monomorphism of $A$-comodules from $M$, which splits as an $R$-module map, splits also as an $A$-comodule map.

THEOREM 3.2 (Dual Maschke Theorem for Hopf algebroids). *The following assertions on a Hopf algebroid* $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ *are equivalent*:

(1.a) *The* $R$*-coring* $({}^{R}A^{R}, \gamma_R, \pi_R)$ *is coseparable. That is, the comultiplication* $\gamma_R\colon A \to A^R \otimes {}^{R}A$ *splits as an* $\mathcal{A}_R$*–*$\mathcal{A}_R$ *bicomodule map.*
(1.b) *The* $L$*-coring* $({}_{L}A_{L}, \gamma_L, \pi_L)$ *is coseparable. That is, the comultiplication* $\gamma_L\colon A \to A_L \otimes {}_{L}A$ *splits as an* $\mathcal{A}_L$*–*$\mathcal{A}_L$ *bicomodule map.*
(2.a) *Any right* $\mathcal{A}_R$*-comodule is* $(\mathcal{A}_R, R)$*-injective.*
(2.b) *Any left* $\mathcal{A}_R$*-comodule is* $(\mathcal{A}_R, R)$*-injective.*
(2.c) *Any left* $\mathcal{A}_L$*-comodule is* $(\mathcal{A}_L, L)$*-injective.*
(2.d) *Any right* $\mathcal{A}_L$*-comodule is* $(\mathcal{A}_L, L)$*-injective.*
(3.a) *There exists a normalized left* $s$*-integral on* $\mathcal{A}$*. That is, an element* $\lambda^* \in \mathcal{L}(\mathcal{A}^*)$ *such that* $\lambda^*(1_A) = 1_R$*.*
(3.b) *There exists a normalized left* $t$*-integral on* $\mathcal{A}$*. That is, an element* ${}^*\lambda \in \mathcal{L}({}^*\mathcal{A})$ *such that* ${}^*\lambda(1_A) = 1_R$*.*
(3.c) *There exists a normalized right* $s$*-integral on* $\mathcal{A}$*. That is, an element* ${}_*\rho \in \mathcal{R}({}_*\mathcal{A})$ *such that* ${}_*\rho(1_A) = 1_L$*.*
(3.d) *There exists a normalized right* $t$*-integral on* $\mathcal{A}$*. That is, an element* $\rho_* \in \mathcal{R}(\mathcal{A}_*)$ *such that* $\rho_*(1_A) = 1_L$*.*
(4.a) *The monomorphism* $s_R\colon R \to A$ *splits as a right* $\mathcal{A}_R$*-comodule map.*
(4.b) *The monomorphism* $t_R\colon R \to A$ *splits as a left* $\mathcal{A}_R$*-comodule map.*
(4.c) *The monomorphism* $s_L\colon L \to A$ *splits as a left* $\mathcal{A}_L$*-comodule map.*
(4.d) *The monomorphism* $t_L\colon L \to A$ *splits as a right* $\mathcal{A}_L$*-comodule map.*

*Proof.* (1.a) $\Rightarrow$ (2.a), (2.b) is proven in [10, 26.1].

(2.a) $\Rightarrow$ (4.a) ((2.b) $\Rightarrow$ (4.b)) The monomorphism $s_R$ $(t_R)$ is split as a right (left) $R$-module map by $\pi_R$ hence it is split as a right (left) $\mathcal{A}_R$-comodule map.

(4.a) $\Rightarrow$ (3.a) and (4.b) $\Rightarrow$ (3.b) The left inverse $\lambda^*$ of $s_R$ in the category of right $\mathcal{A}_R$-comodules is a normalized $s$-integral on $\mathcal{A}_R$ by its very definition. Similarly, the left inverse ${}^*\lambda$ of $t_R$ in the category of left $\mathcal{A}_R$-comodules is a normalized $t$-integral on $\mathcal{A}_R$.

(3.a) $\Rightarrow$ (3.b) If $\lambda^*$ is a normalized $s$-integral on $\mathcal{A}_R$ then $\lambda^* \circ S$ is a normalized $t$-integral on $\mathcal{A}_R$ by Scholium 2.10.

(3.b) $\Rightarrow$ (1.a) In terms of the normalized $t$-integral ${}^*\lambda$ on $\mathcal{A}_R$ the required right inverse of the coproduct $\gamma_R$ is constructed as the map

$$A^R \otimes {}^{R}A \to A, \quad a \otimes b \mapsto t_R \circ {}^*\lambda(aS(b_{(1)}))\, b_{(2)}.$$

It is checked to be an $\mathcal{A}_R$–$\mathcal{A}_R$ bicomodule map using that by Scholium 2.10, (4.b) and (1.c) we have $t_R \circ {}^*\lambda(aS(b_{(1)}))\, b_{(2)} = a^{(1)}\, s_R \circ \pi_R[t_R \circ {}^*\lambda(a^{(2)}S(b_{(1)}))\, b_{(2)}]$ for all $a, b$ in $A$.

(3.a) $\Leftrightarrow$ (3.d) follows from Scholium 2.10, (2.b).

The remaining equivalences are proven by applying the above arguments to the Hopf algebroid $\mathcal{A}^{\mathrm{op}}_{\mathrm{cop}}$. □

The proofs of Theorem 3.1 and 3.2 can be unified if one formulates them as equivalent statements on the forgetful functors from the category of $A$-modules, and from the category of $\mathcal{A}_L$ or $\mathcal{A}_R$-comodules, respectively, to the category of $L$- or $R$-modules – as it is done in the case of Hopf algebras over commutative rings in [12]. We believe (together with the referee), however, that the above formulation in terms of algebra extensions and corings, respectively, is more appealing.

## 4. Frobenius Hopf Algebroids and Nondegenerate Integrals

A left or right integral $\ell$ in a Hopf algebra $(H, \Delta, \epsilon, S)$ over a commutative ring $k$ is called nondegenerate [19] if the maps

$$\mathrm{Hom}_k(H,k) \to H, \quad \phi \mapsto (\phi \otimes H) \circ \Delta(\ell) \quad \text{and}$$
$$\mathrm{Hom}_k(H,k) \to H, \quad \phi \mapsto (H \otimes \phi) \circ \Delta(\ell)$$

are bijective.

The notion of nondegenerate integrals is made relevant by the Larson–Sweedler Theorem [19] stating that a free and finite-dimensional bialgebra over a principal ideal domain is a Hopf algebra if and only if there exists a nondegenerate left integral in $H$.

The Larson–Sweedler Theorem has been extended by Pareigis [27] to Hopf algebras over commutative rings with trivial Picard group. He proved also that a bialgebra over an arbitrary commutative ring $k$, which is a Frobenius $k$-algebra, is a Hopf algebra if and only if there exists a Frobenius functional $\psi\colon H \to k$ satisfying

$$(H \otimes \psi) \circ \Delta = 1_H \psi(\_).$$

As a matter of fact, based on the results of [27] the following variant of [13, 3.2 Theorem 31] can be proven:

THEOREM 4.1. *The following properties of a Hopf algebra $(H, \Delta, \epsilon, S)$ over a commutative ring $k$ are equivalent*:

(1) *$H$ is a Frobenius $k$-algebra.*
(2) *There exists a nondegenerate left integral in $H$.*
(3) *There exists a nondegenerate right integral in $H$.*
(4) *There exists a nondegenerate left integral on $H$. That is, a Frobenius functional $\psi\colon H \to k$ satisfying $(H \otimes \psi) \circ \Delta = 1_H \psi(\_)$.*
(5) *There exists a nondegenerate right integral on $H$. That is, a Frobenius functional $\psi\colon H \to k$ satisfying $(\psi \otimes H) \circ \Delta = 1_H \psi(\_)$.*

The main subject of the present section is the generalization of Theorem 4.1 to Hopf algebroids.

The most important tool in the proof of Theorem 4.1 is the Fundamental Theorem for Hopf modules [19]. A very general form of it has been proven by Brzeziński [7, Theorem 5.6], see also [10, 28.19] in the framework of corings. It can be applied in our setting as follows.

Hopf modules over bialgebroids are examples of Doi–Koppinen modules over algebras, studied in [8]. A left-left Hopf module over a left bialgebroid $\mathcal{A}_L = (A, L, s_L, t_L, \gamma_L, \pi_L)$ is a left comodule for the comonoid $(A, \gamma_L, \pi_L)$ in the category of left $A$-modules. That is, a pair $(M, \tau)$ where $M$ is a left $A$-module, hence a left $L$-module ${}_L M$ via $s_L$. The pair $({}_L M, \tau)$ is a left $\mathcal{A}_L$-comodule such that $\tau\colon M \to A_L \otimes {}_L M$ is a left $A$-module map to the module

$$a \cdot (b \otimes m) := a_{(1)} b \otimes a_{(2)} \cdot m \quad \text{for } a \in A,\ b \otimes m \in A_L \otimes {}_L M.$$

The right–right Hopf modules over a right bialgebroid $\mathcal{A}_R$ are the left–left Hopf modules over $(\mathcal{A}_R)^{\mathrm{op}}_{\mathrm{cop}}$.

It follows from [8, Proposition 4.1] that the left–left Hopf modules over $\mathcal{A}_L$ are the left comodules over the $A$-coring

$$\mathcal{W} := (A_L \otimes {}_L A, \gamma_L \otimes {}_L A, \pi_L \otimes {}_L A), \tag{4.1}$$

where the $A$–$A$ bimodule structure is given by

$$a \cdot (b \otimes c) \cdot d := a_{(1)} b \otimes a_{(2)} c d \quad \text{for } a, d \in A,\ b \otimes c \in A_L \otimes {}_L A.$$

The coring (4.1) was studied in [2]. It was shown to possess a group-like element $1_A \otimes 1_A \in A_L \otimes {}_L A$ and corresponding coinvariant subalgebra $t_L(L)$ in $A$. The coring (4.1) is Galois (w.r.t. the group-like element $1_A \otimes 1_A$) if and only if $\mathcal{A}_L$ is a $\times_L$-Hopf algebra in the sense of [30]. Since in a Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ the left bialgebroid $\mathcal{A}_L$ is a $\times_L$-Hopf algebra by Proposition 2.4, the $A$-coring (4.1) is Galois in this case. Denote the category of left–left Hopf modules over $\mathcal{A}_L$ (i.e. of left comodules over the coring (4.1)) by ${}^{\mathcal{W}}\mathcal{M}$. The application of [7, Theorem 5.6] results that if $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ is a Hopf algebroid, such that the module ${}^L A$ is faithfully flat, then the functor

$$\begin{aligned} &G\colon {}^{\mathcal{W}}\mathcal{M} \to \mathcal{M}_L, \\ &(M, \tau) \mapsto \mathrm{Coinv}(M)_L := \{m \in M \mid \tau(m) = 1_A \otimes m \in A_L \otimes {}_L M\} \end{aligned} \tag{4.2}$$

(where the right $L$-module structure on $\mathrm{Coinv}(M)$ is given via $t_L$) and the induction functor

$$F\colon \mathcal{M}_L \to {}^{\mathcal{W}}\mathcal{M}, \quad N_L \mapsto ({}^L A \otimes N_L, \gamma_L \otimes N_L) \tag{4.3}$$

(where the left $A$-module structure on ${}^L A \otimes N_L$ is given by left multiplication in the first factor) are inverse equivalences.

In the case of Hopf algebras $H$ over commutative rings $k$, these arguments lead to the Fundamental Theorem only for faithfully flat Hopf algebras. The proof of the Fundamental Theorem in [19], however, does not rely on any assumption on the $k$-module structure of $H$.

Since the Hopf algebroid structure is more restrictive than the $\times_L$-Hopf algebra structure, one hopes to prove the Fundamental Theorem for Hopf algebroids also under milder assumptions – using the whole strength of the Hopf algebroid structure.

THEOREM 4.2 (Fundamental Theorem for Hopf algebroids). *Let* $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ *be a Hopf algebroid and* $\mathcal{W}$ *the* $A$*-coring* (4.1). *The functors* $G\colon {}^{\mathcal{W}}\mathcal{M} \to \mathcal{M}_L$ *in* (4.2) *and* $F\colon \mathcal{M}_L \to {}^{\mathcal{W}}\mathcal{M}$ *in* (4.3) *are inverse equivalences.*

*Proof.* We construct the natural isomorphisms $\alpha\colon F\circ G \to {}^{\mathcal{W}}\mathcal{M}$ and $\beta\colon G\circ F \to \mathcal{M}_L$. The map

$$\alpha_M\colon {}^{L}A \otimes \mathrm{Coinv}(M)_L \to M, \quad a \otimes m \mapsto a \cdot m$$

is a left $\mathcal{W}$-comodule map and natural in $M$. The isomorphism property is proven by constructing the inverse

$$\alpha_M^{-1}\colon M \to {}^{L}A \otimes \mathrm{Coinv}(M)_L, \quad m \mapsto {m_{\langle -1\rangle}}^{(1)} \otimes S({m_{\langle -1\rangle}}^{(2)}) \cdot m_{\langle 0\rangle},$$

where we used the standard notation $\tau(m) = m_{\langle -1\rangle} \otimes m_{\langle 0\rangle}$. It requires some work to check that $\alpha_M^{-1}(m)$ belongs to ${}^{L}A \otimes \mathrm{Coinv}(M)_L$. Let us introduce the right $L$-submodule $X$ of $A_L \otimes {}_L A_{\mathbf{L}} \otimes {}_{\mathbf{L}} M$ as

$$X := \Bigg\{ \sum_i a_i \otimes b_i \otimes m_i \in A_L \otimes {}_L A_{\mathbf{L}} \otimes {}_{\mathbf{L}} M \;\Bigg|\; \sum_i a_i t_L(l) \otimes b_i \otimes m_i = \sum_i a_i \otimes b_i s_L(l) \otimes m_i \ \forall l \in L \Bigg\}$$

with $L$-module structure $[\sum_i a_i \otimes b_i \otimes m_i] \cdot l := \sum_i a_i\ t_R \circ \pi_R \circ t_L(l) \otimes b_i \otimes m_i$, and the map

$$\omega\colon A_L \otimes {}_L A_{\mathbf{L}} \otimes {}_{\mathbf{L}} M \to M,$$
$$\sum_i a_i \otimes b_i \otimes m_i \mapsto \sum_i S[s_L \circ \pi_L(a_i)\, b_i] \cdot m_i.$$

Making $M$ a right $L$-module via $t_L$, the restriction of $\omega$ becomes a right $L$-module map $X \to M_L$. The image of the map $\omega \circ (A_L \otimes \tau)\colon A_L \otimes {}_L M \to M$ lies in $\mathrm{Coinv}(M)$, since for any $a \otimes m \in A_L \otimes {}_L M$ we have

$$\begin{aligned}
&\tau \circ \omega \circ (A_L \otimes \tau)(a \otimes m) \\
&\quad = S({m_{\langle -1\rangle}}^{(2)}) m_{\langle 0\rangle\langle -1\rangle} \otimes S[s_L \circ \pi_L(a)\, {m_{\langle -1\rangle}}^{(1)}] \cdot m_{\langle 0\rangle\langle 0\rangle} \\
&\quad = s_R \circ \pi_R({m_{\langle -1\rangle}}^{(2)}) \otimes S[s_L \circ \pi_L(a)\, {m_{\langle -1\rangle}}^{(1)}] \cdot m_{\langle 0\rangle} \\
&\quad = 1_A \otimes \omega \circ (A_L \otimes \tau)(a \otimes m).
\end{aligned}$$

Since $\alpha_M^{-1} = [{}^{L}A \otimes \omega \circ (A_{\mathbf{L}} \otimes \tau)] \circ (\gamma_R \otimes_{\mathbf{L}} M) \circ \tau$, it follows that $\alpha_M^{-1}(m)$ belongs to ${}^{L}A \otimes \mathrm{Coinv}(M)_L$ for all $m \in M$, as stated.

The coinvariants of the left $\mathcal{W}$-comodule ${}^{L}A \otimes N_L$ are the elements of

$$\mathrm{Coinv}({}^{L}A \otimes N_L) = \Bigg\{ \sum_i a_i \otimes n_i \in {}^{L}A \otimes N_L \;\Bigg|\; \sum_i a_i \otimes n_i = \sum_i s_R \circ \pi_R(a_i) \otimes n_i \Bigg\},$$

hence the map

$$\beta_N\colon \operatorname{Coinv}({}^L A \otimes N_L) \to N,$$
$$\sum_i a_i \otimes n_i \mapsto \sum_i n_i \cdot \pi_L \circ S(a_i) \equiv \sum_i n_i \cdot \pi_L(a_i)$$

is a right $L$-module map and is natural in $N$. It is an isomorphism with inverse

$$\beta_N^{-1}\colon N \to \operatorname{Coinv}({}^L A \otimes N_L), \quad n \mapsto 1_A \otimes n. \qquad \square$$

An analogous result for right–right Hopf modules over $\mathcal{A}_R$ can be obtained by applying Theorem 4.2 to the Hopf algebroid $\mathcal{A}^{\mathrm{op}}_{\mathrm{cop}}$.

PROPOSITION 4.3. *Let $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ be a Hopf algebroid and $(M, \tau)$ a left–left Hopf module over $\mathcal{A}_L$. Then* Coinv$(M)$ *is a $k$-direct summand of $M$.*

*Proof.* The canonical inclusion Coinv$(M) \to M$ is split by the $k$-module map

$$E_M\colon M \to \operatorname{Coinv}(M), \quad m \mapsto S(m_{\langle -1\rangle}) \cdot m_{\langle 0\rangle}. \tag{4.4}$$

$\square$

As the next step towards our goal, let us assume that $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ is a Hopf algebroid such that the module $A^R$ – and hence by Lemma 2.6 also $A_L$ – is finitely generated and projective. Under this assumption we are going to equip $\mathcal{A}^*$ with the structures of a left–left Hopf module over $\mathcal{A}_L$ and a right–right Hopf module over $\mathcal{A}_R$.

Let $\{b_i\} \subset A$ and $\{\beta_*^i\} \subset \mathcal{A}_*$ be dual bases for the module $A_L$. A left $\mathcal{A}_L$-comodule structure on $\mathcal{A}^*$ can be introduced via the $L$-module structure

$${}_L\mathcal{A}^*\colon\ l \cdot \phi^* := \phi^* \leftharpoonup S \circ s_L(l) \quad \text{for } l \in L,\ \phi^* \in \mathcal{A}^*$$

and the left coaction

$$\tau_L\colon \mathcal{A}^* \to A_L \otimes {}_L\mathcal{A}^*, \quad \phi^* \mapsto \sum_i b_i \otimes \chi^{-1}(\beta_*^i)\phi^*. \tag{4.5}$$

Similarly, a right $\mathcal{A}_R$-comodule structure on $\mathcal{A}^*$ can be introduced by the right $R$-module structure

$$\mathcal{A}^*_R\colon\ \phi^* \cdot r := \phi^* \leftharpoonup s_R(r) \quad \text{for } r \in R,\ \phi^* \in \mathcal{A}^*$$

and the right coaction

$$\tau_R\colon \mathcal{A}^* \to \mathcal{A}^*_R \otimes {}^R A, \quad \phi^* \mapsto \sum_i \chi^{-1}(\beta_*^i)\phi^* \otimes S(b_i), \tag{4.6}$$

where $\chi\colon \mathcal{A}^* \to \mathcal{A}_*$ is the algebra anti-isomorphism (2.22).

PROPOSITION 4.4. *Let $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ be a Hopf algebroid such that the module $A^R$ is finitely generated and projective.*

(1) *Introduce the left $A$-module*

$$ {}_A\mathcal{A}^*\colon\ a\cdot\phi^* := \phi^* \leftharpoondown S(a) \quad \textit{for } a\in A,\ \phi^*\in\mathcal{A}^*. $$

*Then* $({}_A\mathcal{A}^*, \tau_L)$ *– where* $\tau_L$ *is the map* (4.5) *– is a left–left Hopf module over* $\mathcal{A}_L$.

(2) *Introduce the right $A$-module*

$$ \mathcal{A}^*_A\colon\ \phi^*\cdot a := \phi^* \leftharpoondown a \quad \textit{for } a\in A,\ \phi^*\in\mathcal{A}^*. $$

*Then* $(\mathcal{A}^*_A, \tau_R)$ *– where* $\tau_R$ *is the map* (4.6) *– is a right–right Hopf module over* $\mathcal{A}_R$.

*The coinvariants of both Hopf modules* $({}_A\mathcal{A}^*, \tau_L)$ *and* $(\mathcal{A}^*_A, \tau_R)$ *are the elements of* $\mathcal{L}(\mathcal{A}^*)$.

*Proof.* (1) We have to show that $\tau_L$ is a left $A$-module map. That is, for all $a\in A$ and $\phi^*\in\mathcal{A}^*$,

$$ \sum_i b_i\otimes\chi^{-1}(\beta^i_*)(\phi^*\leftharpoondown S(a)) = \sum_i a_{(1)}b_i\otimes(\chi^{-1}(\beta^i_*)\phi^*)\leftharpoondown S(a_{(2)}) \quad (4.7) $$

as elements of $A_L\otimes{}_L\mathcal{A}^*$. Since for any $\phi_*\in\mathcal{A}_*$ and $a\in A$,

$$ \sum_i \chi^{-1}(\beta^i_*)\leftharpoondown S[s_L\circ\phi_*(a_{(1)}b_i)\,a_{(2)}] = \chi^{-1}(\phi_*)\leftharpoondown s_L\circ\pi_L(a), $$

the following identity holds true in $A_L\otimes{}_L\mathcal{A}^*$ for all $a\in A$:

$$ \begin{aligned} &\sum_i a_{(1)}b_i\otimes\chi^{-1}(\beta^i_*)\leftharpoondown S(a_{(2)}) \\ &\quad=\sum_{i,j} t_L\circ\beta^j_*(a_{(1)}b_i)\,b_j\otimes\chi^{-1}(\beta^i_*)\leftharpoondown S(a_{(2)}) \\ &\quad=\sum_{i,j} b_j\otimes\chi^{-1}(\beta^i_*)\leftharpoondown S[s_L\circ\beta^j_*(a_{(1)}b_i)\,a_{(2)}] \\ &\quad=\sum_j b_j\otimes\chi^{-1}(\beta^j_*)\leftharpoondown s_L\circ\pi_L(a). \end{aligned} \quad (4.8) $$

Since for all $\phi^*, \psi^*\in\mathcal{A}^*$ and $a\in A$,

$$ (\phi^*\psi^*)\leftharpoondown a = (\phi^*\leftharpoondown a^{(2)})(\psi^*\leftharpoondown a^{(1)}), \quad (4.9) $$

the identity (4.8) is equivalent to (4.7).

(2) We have to show that $\tau_R$ is a right $A$-module map. That is, for all $a\in A$ and $\phi^*\in\mathcal{A}^*$,

$$ \sum_i \chi^{-1}(\beta^i_*)(\phi^*\leftharpoondown a)\otimes S(b_i) = \sum_i(\chi^{-1}(\beta^i_*)\phi^*)\leftharpoondown a^{(1)}\otimes S(b_i)a^{(2)} \quad (4.10) $$

as elements of $\mathcal{A}^*_R \otimes {}^R A$. Recall from the proof of Lemma 2.6 that the dual bases, $\{b_i\} \subset A$ and $\{\beta^i_*\} \subset \mathcal{A}_*$ for the module $A_L$, and the dual bases, $\{k_j\} \subset A$ and $\{\kappa^*_j\} \subset \mathcal{A}^*$ for $A^R$, are related to each other by

$$\sum_i b_i \otimes \beta^i_* = \sum_j k_{j(1)} \otimes \chi[s^* \circ \pi_R(k_{j(2)})\, \kappa^*_j] \quad \text{as elements of } A_L \otimes {}^L \mathcal{A}_*.$$

This implies that $\tau_R(\phi^*) = \sum_j s^* \circ \pi_R(k_{j(2)})\, \kappa^*_j\, \phi^* \otimes S(k_{j(1)})$. The following identity holds true in $\mathcal{A}^*_R \otimes {}^R A$ for all $a \in A$:

$$\begin{aligned}
&\sum_j [s^* \circ \pi_R(k_{j(2)})\, \kappa^*_j] \leftharpoonup a^{(1)} \otimes S(k_{j(1)})a^{(2)} \\
&\quad = \sum_j s^* \circ \pi_R({a^{(1)}}_{(2)} k_{j(2)})\, \kappa^*_j \otimes S({a^{(1)}}_{(1)} k_{j(1)})a^{(2)} \\
&\quad = \sum_j s^* \circ \pi_R[s_R \circ \pi_R({a_{(2)}}^{(1)})\, k_{j(2)}]\kappa^*_j \otimes S(a_{(1)} k_{j(1)}){a_{(2)}}^{(2)} \\
&\quad = \sum_j [s^* \circ \pi_R(k_{j(2)})\, \kappa^*_j] \leftharpoonup s_R \circ \pi_R({a_{(2)}}^{(1)}) \otimes S(a_{(1)} k_{j(1)}){a_{(2)}}^{(2)} \\
&\quad = \sum_j s^* \circ \pi_R(k_{j(2)})\, \kappa^*_j \otimes S(k_{j(1)})\, s_R \circ \pi_R(a) \\
&\quad = \sum_j [s^* \circ \pi_R(k_{j(2)})\, \kappa^*_j] \leftharpoonup t_R \circ \pi_R(a) \otimes S(k_{j(1)}).
\end{aligned} \tag{4.11}$$

Here we used the identity $(s^*(r)\phi^*) \leftharpoonup a = s^*(r)\, (\phi^* \leftharpoonup a)$ for $r \in R$, $\phi^* \in \mathcal{A}^*$ and $a \in A$, the property of the dual bases $\sum_j k_j \otimes \kappa^*_j \leftharpoonup a = \sum_j ak^j \otimes \kappa^*_j$ for all $a \in A$ as elements of $A^R \otimes_R \mathcal{A}^*$, the right analogue of the bialgebroid axiom (2.6) and the Hopf algebroid axioms (2.19) and (2.20). In view of (4.9) the identity (4.11) is equivalent to (4.10).

In the cases of the Hopf modules $({}_A\mathcal{A}^*, \tau_L)$ and $(\mathcal{A}^*_A, \tau_R)$ a projection onto the coinvariants is given by the map (4.4) and its right–right version, respectively, both yielding

$$E_{\mathcal{A}^*}\colon \mathcal{A}^* \to \mathrm{Coinv}(\mathcal{A}^*), \quad \phi^* \mapsto \sum_i \chi^{-1}(\beta^i_*)\phi^* \leftharpoonup S^2(b_i). \tag{4.12}$$

A left $s$-integral $\lambda^*$ on $\mathcal{A}$ is a coinvariant, since it is an invariant of the left regular $\mathcal{A}^*$-module and so for all $a \in A$,

$$\begin{aligned}
E_{\mathcal{A}^*}(\lambda^*)(a) &= \sum_i \chi^{-1}(\beta^i_*)(1_A)\lambda^*(S^2(b_i)a) \\
&= \lambda^*[S^2(t_L \circ \beta^i_*(1_A)\, b_i)a] = \lambda^*(a).
\end{aligned}$$

On the other hand, for all $a \in A$,

$$\sum_i S(b_i)(a \leftharpoonup \beta^i_*) = S[t_L \circ \beta^i_*(a_{(1)})\, b_i]a_{(2)} = s_R \circ \pi_R(a), \tag{4.13}$$

hence for all $\phi^* \in \mathcal{A}^*$,

$$\begin{aligned}
&E_{\mathcal{A}^*}(\phi^*) \rightharpoonup a \\
&= \sum_i a^{(2)}\, t_R \circ \pi_R\{[\phi^* \rightharpoonup S^2(b_i)a^{(1)}] \leftharpoonup \beta_*^i\} \\
&= \sum_i t_R \circ \pi_R \circ S^2(b_i^{(2)})\, a^{(2)}\, t_R \circ \pi_R\{[\phi^* \rightharpoonup S^2(b_i^{(1)})a^{(1)}] \leftharpoonup \beta_*^i\} \\
&= \sum_i S(b_{i(2)})\, (S^2(b_{i(1)})a)^{(2)}\, t_R \circ \pi_R\{[\phi^* \rightharpoonup (S^2(b_{i(1)})a)^{(1)}] \leftharpoonup \beta_*^i\} \\
&= \sum_{i,j} S(b_{i(2)})\{[\phi^* \rightharpoonup S^2(t_L \circ \beta_*^j(b_{i(1)})\, b_j)a] \leftharpoonup \beta_*^i\} \\
&= \sum_{i,j} S(b_i \leftharpoonup \beta_*^j)\{[\phi^* \rightharpoonup S^2(b_j)a] \leftharpoonup \beta_*^i\} \\
&= \sum_j s_R \circ \pi_R\{[\phi^* \rightharpoonup S^2(b_j)a] \leftharpoonup \beta_*^j\} = s_R \circ E_{\mathcal{A}^*}(\phi^*)(a).
\end{aligned}$$

That is, any coinvariant is an $s$-integral on $\mathcal{A}_R$. Here we used (4.12), the right analogue of (2.1), the identity $t_R \circ \pi_R \circ S^2 = S \circ s_R \circ \pi_R$, (2.20), the right analogue of (2.3), the identity $\gamma_R[(\phi^* \rightharpoonup a) \leftharpoonup \psi_*] = (\phi^* \rightharpoonup a^{(1)}) \leftharpoonup \psi_* \otimes a^{(2)}$, holding true for all $a \in A$, $\phi^* \in \mathcal{A}^*$ and $\psi_* \in \mathcal{A}_*$, the right $L$-linearity of the map $(\phi^* \rightharpoonup \_) \leftharpoonup \psi_*\colon A_L \to A_L$ and (4.13). □

The application of Theorem 4.2 to the Hopf modules of Proposition 4.4 results in isomorphisms

$$\alpha_L\colon {}^L A \otimes \mathcal{L}(\mathcal{A}^*)^L \to \mathcal{A}^*, \quad a \otimes \lambda^* \mapsto \lambda^* \leftharpoonup S(a) \quad \text{and} \tag{4.14}$$

$$\alpha_R\colon {}^R\mathcal{L}(\mathcal{A}^*) \otimes A_R \to \mathcal{A}^*, \quad \lambda^* \otimes a \mapsto \lambda^* \leftharpoonup a \tag{4.15}$$

of left–left Hopf modules over $\mathcal{A}_L$ and of right–right Hopf modules over $\mathcal{A}_R$, respectively. (The right $L$-module structure on $\mathcal{L}(\mathcal{A}^*)$ is given by $\lambda^* \cdot l := \lambda^* \leftharpoonup s_L(l)$ and the left $R$-module structure is given by $r \cdot \lambda^* := \lambda^* \leftharpoonup t_R(r)$ – see the explanation after (4.2).)

COROLLARY 4.5. *For a Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$, such that any of the modules $A^R$, ${}^R A$, ${}_L A$ and $A_L$ is finitely generated and projective, there exist nonzero elements in all of $\mathcal{L}(\mathcal{A}^*)$, $\mathcal{L}({}^*\mathcal{A})$, $\mathcal{R}({}_*\mathcal{A})$ and $\mathcal{R}(\mathcal{A}_*)$.*

*Proof.* Suppose that the module $A^R$ (equivalently, by Proposition 2.6 the module $A_L$) is finitely generated and projective. It follows from Proposition 4.4 and Theorem 4.2 that the map (4.14) is an isomorphism, hence there exist nonzero elements in $\mathcal{L}(\mathcal{A}^*)$.

For any element $\lambda^*$ of $\mathcal{L}(\mathcal{A}^*)$, $\lambda^* \circ S$ is a (possibly zero) element of $\mathcal{L}({}^*\mathcal{A})$ by Scholium 2.10. Now we claim that it is excluded by the bijectivity of the map (4.14) that $\lambda^* \circ S = 0$ for all $\lambda^* \in \mathcal{L}(\mathcal{A}^*)$. For if so, then by the surjectivity of the map (4.14) we have $\phi^*(1_A) = 0$ for all $\phi^* \in \mathcal{A}^*$. But this is impossible, since $\pi_R(1_A) = 1_R$, by definition.

It follows from Scholium 2.10, (3.b) and (4.b) that also $\mathcal{R}({}_*\mathcal{A})$ and $\mathcal{R}(\mathcal{A}_*)$ must contain nonzero elements.

The case when the module ${}_LA$ (equivalently, by Proposition 2.6 the module ${}^RA$) is finitely generated and projective can be treated by applying the same arguments to the Hopf algebroid $\mathcal{A}^{\mathrm{op}}_{\mathrm{cop}}$. □

Since none of the duals of a Hopf algebroid is known to be a Hopf algebroid, it does not follow from Theorem 4.2, however, that for a Hopf algebroid, in which the total algebra is finitely generated and projective as a module over the base algebra, also $\mathcal{L}(A)$ and $\mathcal{R}(A)$ contain nonzero elements. At the moment we do not know under what necessary conditions the existence of nonzero integrals in a Hopf algebroid follows.

It is well known [27, Proposition 4] that the antipode of a finitely generated and projective Hopf algebra over a commutative ring is bijective. We do not know whether a result of the same strength holds true on Hopf algebroids. Our present understanding on this question is formulated in

PROPOSITION 4.6. *The following statements on a Hopf algebroid* $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ *are equivalent*:

(1) *The antipode* $S$ *is bijective and any of the modules* ${}_LA$, $A_L$, $A^R$ *and* ${}^RA$ *is finitely generated and projective.*
(2) *There exists an invariant* $\sum_k x_k \otimes \lambda_k^*$ *of the left* $A$*-module* ${}^RA \otimes \mathcal{L}(\mathcal{A}^*)^R$ *– defined via left multiplication in the first factor – with respect to* $\mathcal{A}_L$, *satisfying* $\sum_k \lambda_k^*(x_k) = 1_R$. (*The right* $R$*-module structure of* $\mathcal{L}(\mathcal{A}^*)$ *is defined by the restriction of the one of* $(\mathcal{A}^*)^R$, *i.e. as* $\lambda^* \cdot r := \lambda^*(\_\, t_R(r))$.)

*Proof.* For any invariant $\sum_k x_k \otimes \lambda_k^*$ of the left $A$-module ${}^RA \otimes \mathcal{L}(\mathcal{A}^*)^R$ and any element $a \in A$ the identities

$$\sum_k S(a)x_k^{(1)} \otimes x_k^{(2)} \otimes \lambda_k^* = \sum_k x_k^{(1)} \otimes a x_k^{(2)} \otimes \lambda_k^* \quad \text{and}$$
$$\sum_k a x_k^{(1)} \otimes S(x_k^{(2)}) \otimes \lambda_k^* = \sum_k x_k^{(1)} \otimes S(x_k^{(2)}) a \otimes \lambda_k^*$$

hold true as identities in ${}^RA^{\mathbf{R}} \otimes {}^{\mathbf{R}}A \otimes \mathcal{L}(\mathcal{A}^*)^R$ and in ${}^RA^{\mathbf{R}} \otimes {}_{\mathbf{R}}A \otimes \mathcal{L}(\mathcal{A}^*)^R$, respectively.

(2) $\Rightarrow$ (1) In terms of the invariant $\sum_k x_k \otimes \lambda_k^*$ the inverse of the antipode is constructed explicitly as

$$S^{-1}\colon A \to A, \quad a \mapsto \sum_k (\lambda_k^* \leftharpoonup a) \rightharpoonup x_k.$$

The dual bases $\{b_i\} \subset A$ and $\{^*\beta_i\} \subset {}^*\mathcal{A}$ for the module $^RA$ are introduced by the requirement that

$$\sum_i {}^*\beta_i \otimes b_i = \sum_k \lambda_k^*(S(\_)x_k^{(1)}) \otimes x_k^{(2)}$$

as elements of $^*\mathcal{A}_R \otimes {}^RA$. Together with Lemma 2.6 this proves the implication (2) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2) If $S$ is bijective then in the case of the Hopf algebroid $\mathcal{A}_{\mathrm{cop}}$ the isomorphism (4.14) takes the form

$$\alpha_L^{\mathrm{cop}}\colon\ A^L \otimes {}^L\mathcal{L}(^*\mathcal{A}) \to {}^*\mathcal{A}, \quad a \otimes {}^*\lambda \mapsto {}^*\lambda \leftharpoondown S^{-1}(a),$$

where the left $L$-module structure on $\mathcal{L}(^*\mathcal{A})$ is defined by $l \cdot {}^*\lambda := {}^*\lambda \leftharpoondown t_L(l)$.

In terms of $\sum_k x_k \otimes {}^*\lambda_k := (\alpha_L^{\mathrm{cop}})^{-1}(\pi_R)$ the required invariant of $^RA \otimes \mathcal{L}(\mathcal{A}^*)^R$ is given by $\sum_k x_k \otimes {}^*\lambda_k \circ S^{-1}$. □

In any Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$, in which the module $A_L$ is finitely generated and projective, the extensions $s_R\colon R \to A$ and $t_L\colon L^{\mathrm{op}} \to A$ satisfy the left depth two (or D2, for short) condition and the extensions $t_R\colon R^{\mathrm{op}} \to A$ and $s_L\colon L \to A$ satisfy the right D2 condition of [18]. If furthermore $S$ is bijective then all the four extensions satisfy both the left and the right D2 conditions. This means [18, Lemma 3.7] in the case of $s_R\colon R \to A$, for example, the existence of finite sets (the so called D2 quasi-bases) $\{d_k\} \subset A^R \otimes {}_RA$, $\{\delta_k\} \subset {}_R\mathrm{End}_R({}_RA^R)$, $\{f_l\} \subset A^R \otimes {}_RA$ and $\{\phi_l\} \subset {}_R\mathrm{End}_R({}_RA^R)$ satisfying

$$\sum_k d_k \cdot m_A \circ (\delta_k \otimes {}_RA)(u) = u$$

and

$$\sum_l m_A \circ (A^R \otimes \phi_l)(u) \cdot f_l = u$$

for all elements $u$ in $A^R \otimes {}_RA$, where the $A$–$A$ bimodule structure on $A^R \otimes {}_RA$ is defined by left multiplication in the first factor and right multiplication in the second factor.

The D2 quasi-bases for the extension $s_R\colon R \to A$ can be constructed in terms of the invariants $\sum_i x_i \otimes \lambda_i^* := \alpha_L^{-1}(\pi_R)$ and $\sum_j x'_j \otimes {}^*\lambda'_j := (\alpha_L^{\mathrm{cop}})^{-1}(\pi_R)$ via the requirements that

$$\sum_k d_k \otimes \delta_k = \sum_i x_{i(1)}{}^{(1)} \otimes S(x_{i(1)}{}^{(2)}) \otimes [\lambda_i^* \leftharpoonup S(x_{i(2)})] \rightharpoonup \_$$

and

$$\sum_l \phi_l \otimes f_l = \sum_j \_ \leftharpoonup [x'_{j(1)} \rightharpoonup \pi_L \circ s_R \circ {}^*\lambda'_j \circ S^{-1}] \otimes \otimes x'_{j(2)}{}^{(1)} \otimes S(x'_{j(2)}{}^{(2)})$$

as elements of $A^R \otimes_R A^L \otimes_L [{}_R\mathrm{End}_R({}_RA^R)]$ and of $[{}_R\mathrm{End}_R({}_RA^R)]_L \otimes_L A^R \otimes_R A$, respectively. (The $L$–$L$ bimodule structure on ${}_R\mathrm{End}_R({}_RA^R)$ is given by

$$l_1 \cdot \Psi \cdot l_2 = s_L(l_1)\Psi(\_)s_L(l_2) \quad \text{for } l_1, l_2 \in L,\ \Psi \in {}_R\mathrm{End}_R({}_RA^R).)$$

The D2 property of the extensions $t_R\colon R^{\mathrm{op}} \to A$, $s_L\colon L \to A$ and $t_L\colon L^{\mathrm{op}} \to A$ follows by applying these formulae to the Hopf algebroids $\mathcal{A}_{\mathrm{cop}}$, $\mathcal{A}_{\mathrm{cop}}^{\mathrm{op}}$ and $\mathcal{A}^{\mathrm{op}}$, respectively.

The following theorem, characterizing Frobenius Hopf algebroids $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ – that is, Hopf algebroids such that the extensions, given by the source and target maps of the bialgebroids $\mathcal{A}_L$ and $\mathcal{A}_R$, are Frobenius extensions – is the main result of this section.

Recall that for a homomorphism $s\colon R \to A$ of $k$-algebras the canonical $R$–$A$ bimodule ${}_RA_A$ is a 1-cell in the additive bicategory of [$k$-algebras, bimodules, bimodule maps], possessing a right dual, the bimodule ${}_AA_R$. If $A$ is finitely generated and projective as a left $R$-module, then ${}_RA_A$ possesses also a left dual, the bimodule ${}_A[{}_R\mathrm{Hom}(A, R)]_R$ defined as

$$a \cdot \phi \cdot r = \phi(\_\,a)r \quad \text{for } r \in R,\ a \in A,\ \phi \in {}_R\mathrm{Hom}(A, R).$$

A monomorphism of $k$-algebras $s\colon R \to A$ is called a *Frobenius extension* if the module ${}_RA$ is finitely generated and projective and the left and right duals

$${}_AA_R \quad \text{and} \quad {}_A[{}_R\mathrm{Hom}(A, R)]_R$$

of the bimodule ${}_RA_A$ are isomorphic. Equivalently, if $A_R$ is finitely generated and projective and the left and right duals

$${}_RA_A \quad \text{and} \quad {}_R[\mathrm{Hom}_R(A, R)]_A$$

of the bimodule ${}_AA_R$ are isomorphic. This property holds if and only if there exists a *Frobenius system* $(\psi, \sum_i u_i \otimes v_i)$, where $\psi\colon A \to R$ is an $R$–$R$ bimodule map and $\sum_i u_i \otimes v_i$ is an element of $A \underset{R}{\otimes} A$ such that

$$\sum_i s \circ \psi(au_i)\, v_i = a = \sum_i u_i\, s \circ \psi(v_i a) \quad \text{for all } a \in A.$$

THEOREM 4.7. *The following statements on a Hopf algebroid* $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ *are equivalent*:

(1.a) *The map* $s_R\colon R \to A$ *is a Frobenius extension of* $k$*-algebras.*
(1.b) *The map* $t_R\colon R^{\mathrm{op}} \to A$ *is a Frobenius extension of* $k$*-algebras.*
(1.c) *The map* $s_L\colon L \to A$ *is a Frobenius extension of* $k$*-algebras.*
(1.d) *The map* $t_L\colon L^{\mathrm{op}} \to A$ *is a Frobenius extension of* $k$*-algebras.*
(2.a) *The module* $A^R$ *is finitely generated and projective and the module* $\mathcal{L}(\mathcal{A}^*)^L$, *defined by* $\lambda^* \cdot l := \lambda^* \leftharpoonup s_L(l)$, *is free of rank* 1.
(2.b) $S$ *is bijective, the module* ${}^RA$ *is finitely generated and projective and the module* ${}^L\mathcal{L}({}^*\mathcal{A})$, *defined by* $l \cdot {}^*\lambda := {}^*\lambda \leftharpoondown t_L(l)$, *is free of rank* 1.

(2.c) *The module* $_LA$ *is finitely generated and projective and the module* $_R\mathcal{R}(_*\mathcal{A})$, *defined by* $r \cdot {}_*\rho := s_R(r) \rightharpoondown {}_*\rho$, *is free of rank* 1.
(2.d) *S is bijective, the module* $A_L$ *is finitely generated and projective and the module* $\mathcal{R}(\mathcal{A}_*)_R$, *defined by* $\rho_* \cdot r := t_R(r) \rightharpoonup \rho_*$, *is free of rank* 1.
(3.a) *The module* $A^R$ *is finitely generated and projective and there exists an element* $\lambda^* \in \mathcal{L}(\mathcal{A}^*)$ *such that the map*

$$\mathcal{F}\colon A \to \mathcal{A}^*, \quad a \mapsto \lambda^* \leftharpoonup a \tag{4.16}$$

*is bijective.*
(3.b) *S is bijective, the module* $^RA$ *is finitely generated and projective and there exists an element* ${}^*\lambda \in \mathcal{L}({}^*\mathcal{A})$ *such that the map* $A \to {}^*\mathcal{A}$, $a \mapsto {}^*\lambda \leftharpoondown a$ *is bijective.*
(3.c) *The module* $_LA$ *is finitely generated and projective and there exists an element* ${}_*\rho \in \mathcal{R}({}_*\mathcal{A})$ *such that the map* $A \to {}_*\mathcal{A}$, $a \mapsto a \rightharpoondown {}_*\rho$ *is bijective.*
(3.d) *S is bijective, the module* $A_L$ *is finitely generated and projective and there exists an element* $\rho_* \in \mathcal{R}(\mathcal{A}_*)$ *such that the map* $A \to \mathcal{A}_*$, $a \mapsto a \rightharpoonup \rho_*$ *is bijective.*
(4.a) *There exists a left integral* $\ell \in \mathcal{L}(\mathcal{A})$ *such that the map*

$$\mathcal{F}^*\colon \mathcal{A}^* \to A, \quad \phi^* \mapsto \phi^* \rightharpoonup \ell \tag{4.17}$$

*is bijective.*
(4.b) *S is bijective and there exists a left integral* $\ell \in \mathcal{L}(\mathcal{A})$ *such that the map*

$${}^*\mathcal{F}\colon {}^*\mathcal{A} \to A, \quad {}^*\phi \mapsto {}^*\phi \rightharpoondown \ell \tag{4.18}$$

*is bijective.*
(4.c) *There exists a right integral* $\wp \in \mathcal{R}(\mathcal{A})$ *such that the map* ${}_*\mathcal{A} \to A$, ${}_*\phi \mapsto \wp \leftharpoondown {}_*\phi$ *is bijective.*
(4.d) *S is bijective and there exists a right integral* $\wp \in \mathcal{R}(\mathcal{A})$ *such that the map* $\mathcal{A}_* \to A$, $\phi_* \mapsto \wp \leftharpoonup \phi_*$ *is bijective.*

*In particular, the integrals* $\lambda^*$, ${}^*\lambda$, ${}_*\rho$ *and* $\rho_*$ *on* $\mathcal{A}$ *satisfying the condition in* (3.a), (3.b), (3.c) *and* (3.d), *respectively, are Frobenius functionals themselves for the extensions* $s_R\colon R \to A$, $t_R\colon R^{\rm op} \to A$, $s_L\colon L \to A$ *and* $t_L\colon L^{\rm op} \to A$, *respectively.*

*What is more, under the equivalent conditions of the theorem the left integrals* $\ell \in \mathcal{L}(\mathcal{A})$ *satisfying the conditions in* (4.a) *and* (4.b) *can be chosen to be equal, that is, to be a* nondegenerate *left integral in* $\mathcal{A}$. *Similarly, the right integrals* $\wp \in \mathcal{R}(\mathcal{A})$ *satisfying the conditions in* (4.c) *and* (4.d) *can be chosen to be equal, that is to be a* nondegenerate *right integral in* $\mathcal{A}$.

*Proof.* (4.a) $\Rightarrow$ (1.a) In terms of the left integral $\ell$ in (4.a) define $\lambda^* := \mathcal{F}^{*-1}(1_A) \in \mathcal{A}^*$. We claim that $\lambda^*$ is a left $s$-integral on $\mathcal{A}$. The element $\ell \otimes \lambda^* \in {}^R\mathcal{L}(A) \otimes \mathcal{L}(\mathcal{A}^*)^R$ is an invariant of the left $A$-module ${}^RA \otimes \mathcal{L}(\mathcal{A}^*)^R$, hence by Proposition 4.6 the antipode is bijective and the modules $A^R$ and ${}^R A$ are finitely generated and projective. Since for all $\phi^* \in \mathcal{A}^*$,

$$\phi^* \lambda^* = \mathcal{F}^{*-1}(\phi^* \rightharpoonup 1_A) = \mathcal{F}^{*-1}(s^* \circ \phi^*(1_A) \rightharpoonup 1_A) = s^* \circ \pi^*(\phi^*)\, \lambda^*,$$

$\lambda^*$ is an $s$-integral on $\mathcal{A}_R$, so in particular an $R$–$R$ bimodule map ${}_R A^R \to R$.

Since for all $a \in A$,

$$\ell^{(2)}\, t_R \circ \lambda^*(S(a)\ell^{(1)}) = a,$$

we have $\mathcal{F}^{*-1}(a) = \lambda^* \leftharpoonup S(a)$ hence $\ell^{(1)}\, s_R \circ \lambda^* \circ S(\ell^{(2)}) = 1_A$. A Frobenius system for the extension $s_R\colon R \to A$ is provided by $(\lambda^*, \ell^{(1)} \otimes S(\ell^{(2)}))$.

(1.a) $\Rightarrow$ (2.a) The module $A^R$ is finitely generated and projective by assumption. In terms of a Frobenius system $(\psi, \sum_i u_i \otimes v_i)$ for the extension $s_R\colon R \to A$ one constructs an isomorphism of right $L$-modules as

$$\kappa\colon \mathcal{L}(\mathcal{A}^*) \to L, \quad \lambda^* \mapsto \pi_L\Big[\sum_i s_R \circ \lambda^*(u_i)v_i\Big] \tag{4.19}$$

with inverse

$$\kappa^{-1}\colon L \to \mathcal{L}(\mathcal{A}^*), \quad l \mapsto E_{\mathcal{A}^*}(\psi \leftharpoonup s_L(l)), \tag{4.20}$$

where $E_{\mathcal{A}^*}$ is the map (4.12). The right $L$-linearity of $\kappa$ follows from the property of the Frobenius system $(\psi, \sum_i u_i \otimes v_i)$ that $\sum_i a u_i \otimes v_i = \sum_i u_i \otimes v_i a$ for all $a \in A$, the bialgebroid axiom (2.5), and left $R$-linearity of the map $\lambda^*\colon {}_R A \to R$ and the right $L$-linearity of $\pi_L\colon {}_L A \to L$.

The maps $\kappa$ and $\kappa^{-1}$ are mutual inverses as

$$\begin{aligned}
&\kappa^{-1} \circ \kappa(\lambda^*) \\
&\quad = \sum_{i,j} [\chi^{-1}(\beta_*^j)\psi] \leftharpoonup s_L \circ \pi_L(s_R \circ \lambda^*(u_i)v_i)\, S^2(b_j) \\
&\quad = \sum_{i,j} [\chi^{-1}(\beta_*^j)\psi] \leftharpoonup S^2(b_j^{(2)})\, t_R \circ \pi_R[t_R \circ \pi_R \circ S(s_R \circ \lambda^*(u_i)v_i)\, S^2(b_j^{(1)})] \\
&\quad = \sum_{i,j} [\chi^{-1}(\beta_*^j)\psi] \leftharpoonup S^2(b_j^{(2)})\, s_L \circ \pi_L[S(b_j^{(1)})\, s_R \circ \lambda^*(u_i)v_i] = \lambda^*,
\end{aligned} \tag{4.21}$$

where in the first step we used (4.9), in the second step the fact that by Proposition 2.3 we have $s_L \circ \pi_L = t_R \circ \pi_R \circ S$, then the right analogue of (2.5) and finally in the last step the identity in ${}^R\mathcal{L}(\mathcal{A}^*) \otimes A_R$:

$$\begin{aligned}
&\sum_{i,j} [\chi^{-1}(\beta_*^j)\psi] \leftharpoonup S^2(b_j^{(2)}) \otimes S(b_j^{(1)})\, s_R \circ \lambda^*(u_i)v_i \\
&\quad = \alpha_R^{-1}\Big(\sum_i \psi \leftharpoonup s_R \circ \lambda^*(u_i)v_i\Big) = \lambda^* \otimes 1_A,
\end{aligned}$$

which follows from the explicit form of the inverse of the map (4.15). In a similar way, also

$$\begin{aligned}\kappa\circ\kappa^{-1}(l) &= \sum_{i,j}\pi_L[s_R\circ(\chi^{-1}(\beta_*^j)\psi)(s_L(l)S^2(b_j)u_i)\ v_i]\\ &= \sum_{i,j}\pi_L[s_R\circ(\chi^{-1}(\beta_*^j)\ \psi)(s_L(l)u_i)\ v_i\ S^2(b_j)]\\ &= \sum_{i,j}\pi_L[s_R\circ(\chi^{-1}(\beta_*^j)\ \psi)(s_L(l)u_i)\ v_i\ t_L\circ\pi_L\circ S^2(b_j)]\\ &= \sum_{i,j}\pi_L[s_R\circ(\chi^{-1}(\beta_*^j)\ \psi)(s_L(l)\ t_L\circ\pi_L\circ S^2(b_j)\ u_i)\ v_i]\\ &= \sum_{i,j}\pi_L\{s_R\circ[(\chi^{-1}(\beta_*^j)\leftharpoonup t_L\circ\pi_L\circ S^2(b_j))\psi](s_L(l)u_i)\ v_i\}\\ &= l,\end{aligned}$$

where in the last step we used that $\sum_j \chi^{-1}(\beta_*^j)\leftharpoonup t_L\circ\pi_L\circ S^2(b_j) = \chi^{-1}(\sum_j \beta_*^j\ t_*\circ\pi_L(b_j)) = \pi_R$.

(2.a) $\Rightarrow$ (3.a) If $\kappa\colon\ \mathcal{L}(\mathcal{A}^*)^L\to L$ is an isomorphism of $L$-modules then $\pi_R\circ s_L\circ\kappa\colon\ {}^R\mathcal{L}(\mathcal{A}^*)\to R$ is an isomorphism of $R$-modules. Introduce the cyclic and separating generator $\lambda^* := \kappa^{-1}(1_L)$ for the module $\mathcal{L}(\mathcal{A}^*)^L$. The map $\mathcal{F}$ in (4.16) is equal to $\alpha_R\circ(\kappa^{-1}\circ\pi_L\circ t_R\otimes A_R)$ – where $\alpha_R$ is the isomorphism (4.15) – hence bijective.

(3.a) $\Rightarrow$ (4.a), (4.b) A Frobenius system for the extension $s_R\colon\ R\to A$ is given in terms of the dual bases $\{b_i\}\subset A$ and $\{\beta_i^*\}\subset\mathcal{A}^*$ for the module $A^R$ as $(\lambda^*, \sum_i b_i\otimes\mathcal{F}^{-1}(\beta_i^*))$.

The element $\ell := \sum_i b_i\ t_L\circ\pi_L\circ\mathcal{F}^{-1}(\beta_i^*)$ is a left integral in $\mathcal{A}$. Using the identities

$$\begin{aligned}\lambda^*\rightharpoonup\ell &= s_R\circ\lambda^*\left[\sum_i b_i\ t_L\circ\pi_L\circ\mathcal{F}^{-1}(\beta_i^*)\right]\\ &= t_L\circ\pi_L\left[\sum_i s_R\circ\lambda^*(b_i)\ \mathcal{F}^{-1}(\beta_i^*)\right] = 1_A,\\ \ell^{(1)}\otimes S(\ell^{(2)}) &= \sum_i b_i\ s_R\circ\lambda^*[\mathcal{F}^{-1}(\beta_i^*)\ell^{(1)}]\otimes S(\ell^{(2)})\\ &= \sum_i b_i\otimes S[\ell^{(2)}\ t_R\circ\lambda^*(\ell^{(1)})]\mathcal{F}^{-1}(\beta_i^*)\\ &= \sum_i b_i\otimes\mathcal{F}^{-1}(\beta_i^*)\end{aligned}$$

one checks that the inverse of the map $\mathcal{F}^*$ in (4.17) is given by $\mathcal{F}\circ S$. This implies, in particular, that $S$ is bijective.

The inverse of the map ${}^*\mathcal{F}$ in (4.18) – defined in terms of the same left integral $\ell$ – is the map

$$A \to {}^*\mathcal{A}, \quad a \mapsto \lambda^* \circ S \leftharpoondown S^{-1}(a).$$

(1.a) $\Leftrightarrow$ (1.d) The datum $(\psi, \sum_i u_i \otimes v_i)$ is a Frobenius system for the extension $s_R\colon R \to A$ if and only if $(\pi_L \circ s_R \circ \psi, \sum_i u_i \otimes v_i)$ is a Frobenius system for $t_L\colon L^{\mathrm{op}} \to A$, where $\pi_L \circ s_R\colon R \to L^{\mathrm{op}}$ was claimed to be an isomorphism of $k$-algebras in part (1) of Proposition 2.3.

(1.a) $\Rightarrow$ (1.c) We have already seen that (1.a) $\Rightarrow$ (3.a) $\Rightarrow$ $S$ is bijective. If the datum $(\psi, \sum_i u_i \otimes v_i)$ is a Frobenius system for the extension $s_R\colon R \to A$ then $(\pi_L \circ s_R \circ \psi \circ S^{-1}, S(v_i) \otimes S(u_i))$ is a Frobenius system for $s_L\colon L \to A$.

(4.c) $\Rightarrow$ (1.c) $\Rightarrow$ (2.c) $\Rightarrow$ (3.c) $\Rightarrow$ (4.c), (1.c) $\Leftrightarrow$ (1.b) and (1.c) $\Rightarrow$ (1.a) follow by applying (4.a) $\Rightarrow$ (1.a) $\Rightarrow$ (2.a) $\Rightarrow$ (3.a) $\Rightarrow$ (4.a), (1.a) $\Leftrightarrow$ (1.d) and (1.a) $\Rightarrow$ (1.c) to the Hopf algebroid $\mathcal{A}^{\mathrm{op}}_{\mathrm{cop}}$.

(1.b) $\Rightarrow$ (2.b) $\Rightarrow$ (3.b) $\Rightarrow$ (4.b) $\Rightarrow$ (1.b) We have seen that (1.b) $\Leftrightarrow$ (1.c) $\Rightarrow$ $S$ is bijective. Hence we can apply (1.a) $\Rightarrow$ (2.a) $\Rightarrow$ (3.a) $\Rightarrow$ (4.a) $\Rightarrow$ (1.a) to the Hopf algebroid $\mathcal{A}_{\mathrm{cop}}$.

(1.d) $\Rightarrow$ (2.d) $\Rightarrow$ (3.d) $\Rightarrow$ (4.d) $\Rightarrow$ (1.d) follows by applying (1.b) $\Rightarrow$ (2.b) $\Rightarrow$ (3.b) $\Rightarrow$ (4.b) $\Rightarrow$ (1.b) to the Hopf algebroid $\mathcal{A}^{\mathrm{op}}_{\mathrm{cop}}$. □

It is proven in [5, Theorem 5.17] that under the equivalent conditions of Theorem 4.7 the duals, $\mathcal{A}^*$, ${}^*\mathcal{A}$, ${}_*\mathcal{A}$ and $\mathcal{A}_*$ of the Hopf algebroid $\mathcal{A}$, possess (anti-) isomorphic Hopf algebroid structures.

The Hopf algebroids, satisfying the equivalent conditions of Theorem 4.7, provide examples of distributive Frobenius double algebras [37]. (Notice that the integrals, which we call nondegenerate, are called Frobenius integrals in [37].)

Our result naturally raises the question, under what conditions on the base algebra the equivalent conditions of Theorem 4.7 hold true. That is, what is the generalization of Pareigis' condition – the triviality of the Picard group of the commutative base ring of a Hopf algebra – to the noncommutative base algebra of a Hopf algebroid. We are going to return to this problem in a different publication.

## 5. The Quasi-Frobenius Property

It is known [27, Theorem added in proof], that any finitely generated projective Hopf algebra over a commutative ring $k$ is (both a left and a right) quasi-Frobenius extension of $k$ in the sense of [23]. In this section we examine in what Hopf algebroids the total algebra is (a left or a right) quasi-Frobenius extension of the base algebra.

The quasi-Frobenius property of an extension $s\colon R \to A$ of $k$-algebras has been introduced by Müller [23] as a weakening of the Frobenius property (see the paragraph preceding Theorem 4.7). The extension $s\colon R \to A$ is *left quasi-Frobenius* (or left QF, for short) if the module ${}_RA$ is finitely generated and projective (hence the bimodule ${}_R A_A$ possesses both a right dual ${}_A A_R$ and a left dual ${}_A[{}_R\mathrm{Hom}(A, R)]_R$) and the bimodule ${}_A A_R$ is a direct summand in a finite direct sum of copies of ${}_A[{}_R\mathrm{Hom}(A, R)]_R$.

The extension $s\colon R \to A$ is *right QF* if $s$, considered as a map $R^{\mathrm{op}} \to A^{\mathrm{op}}$, is a left QF extension. That is, if the module $A_R$ is finitely generated and projective and the left dual bimodule ${}_R A_A$ is a direct summand in a finite direct sum of copies of the right dual bimodule ${}_R[\mathrm{Hom}_R(A, R)]_A$.

To our knowledge it is not known whether the notions of left and right QF extensions are equivalent (except in particular cases, such as central extensions, where the answer turns out to be affirmative [29]; and Frobenius extensions, which are also both left and right QF [23]).

A powerful characterization of a Frobenius extension $s\colon R \to A$ is the existence of a Frobenius system – see the paragraph preceding Theorem 4.7. In the following lemma a generalization to quasi-Frobenius extensions is introduced:

LEMMA 5.1.

(1) *An algebra extension $s\colon R \to A$ is left QF if and only if the module ${}_R A$ is finitely generated and projective and there exist finite sets $\{\psi_k\} \subset {}_R\mathrm{Hom}_R(A, R)$ and $\{\sum_i u_i^k \otimes v_i^k\} \subset A \otimes_R A$ satisfying*

$$\sum_{i,k} u_i^k\ s \circ \psi_k(v_i^k) = 1_A$$

*and*

$$\sum_{i,k} a u_i^k \otimes v_i^k = u_i^k \otimes v_i^k a \quad \textit{for all } a \in A.$$

*The datum $\{\psi_k, \sum_i u_i^k \otimes v_i^k\}$ is called a* left QF-system *for the extension $s\colon R \to A$.*

(2) *An algebra extension $s\colon R \to A$ is right QF if and only if the module $A_R$ is finitely generated and projective and there exist finite sets $\{\psi_k\} \subset {}_R\mathrm{Hom}_R(A, R)$ and $\{\sum_i u_i^k \otimes v_i^k\} \subset A \otimes_R A$ satisfying*

$$\sum_{i,k} s \circ \psi_k(u_i^k)\ v_i^k = 1_A$$

*and*

$$\sum_{i,k} a u_i^k \otimes v_i^k = u_i^k \otimes v_i^k a \quad \textit{for all } a \in A.$$

*The datum $\{\psi_k, \sum_i u_i^k \otimes v_i^k\}$ is called a* right QF-system *for the extension $s\colon R \to A$.*

*Proof.* Let us spell out the proof in the case (1). Suppose that there exists a left QF system $\{\psi_k, \sum_i u_i^k \otimes v_i^k\}$ for the extension $s\colon R \to A$. The bimodule ${}_A A_R$

is a direct summand in a finite direct sum of copies of ${}_A[{}_R\mathrm{Hom}(A, R)]_R$ by the existence of $A$–$R$ bimodule maps

$$\Phi_k\colon\ {}_R\mathrm{Hom}(A, R) \to A, \quad \phi \mapsto \sum_i u_i^k\, s \circ \phi(v_i^k) \quad \text{and}$$

$$\Phi'_k\colon\ A \to {}_R\mathrm{Hom}(A, R), \quad a \mapsto \psi_k(\,\_\,a)$$

satisfying $\sum_k \Phi_k \circ \Phi'_k = A$.

Conversely, in terms of the $A$–$R$ bimodule maps $\{\Phi_k\colon\ {}_R\mathrm{Hom}(A, R) \to A\}$ and $\{\Phi'_k\colon\ A \to {}_R\mathrm{Hom}(A, R)\}$, satisfying $\sum_k \Phi_k \circ \Phi'_k = A$, and the dual bases, $\{b_j\} \subset A$ and $\{\beta_j\} \subset {}_R\mathrm{Hom}(A, R)$ for the module ${}_R A$, a left QF system can be constructed as

$$\psi_k := \Phi'_k(1_A) \in {}_R\mathrm{Hom}_R(A, R)$$

and

$$\sum_i u_i^k \otimes v_i^k := \sum_j \Phi_k(\beta_j) \otimes b_j \in A \underset{R}{\otimes} A. \qquad \square$$

Lemma 5.1 implies, in particular, that for a left/right QF extension $R \to A$, $A$ is finitely generated and projective also as a right/left $R$-module.

THEOREM 5.2. *The following properties of a Hopf algebroid* $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ *are equivalent*:

(1.a) $s_R\colon\ R \to A$ *is a left QF extension.*
(1.b) $t_L\colon\ L^{\mathrm{op}} \to A$ *is a left QF extension.*
(1.c) *The modules* $A^R$ *and* $\mathcal{L}(\mathcal{A}^*)^L$ *– defined by* $\lambda^* \cdot l := \lambda^* \leftharpoonup s_L(l)$ *– are finitely generated and projective.*
(1.d) *The module* $A^R$ *is finitely generated and projective and the module* $\mathcal{L}(\mathcal{A}^*)^L$ *is flat.*
(1.e) *The module* $A^R$ *is finitely generated and projective and the invariants of the left $A$-module* ${}^L A \otimes \mathcal{L}(\mathcal{A}^*)^L$ *– defined via left multiplication in the first factor – with respect to* $\mathcal{A}_L$ *are the elements of* ${}^L\mathcal{L}(\mathcal{A}) \otimes \mathcal{L}(\mathcal{A}^*)^L$.
(1.f) *There exist finite sets* $\{\ell_k\} \subset \mathcal{L}(\mathcal{A})$ *and* $\{\lambda_k^*\} \subset \mathcal{L}(\mathcal{A}^*)$ *satisfying* $\sum_k \lambda_k^* \circ S(\ell_k) = 1_R$.
(1.g) *The left $A$-module* ${}_A\mathcal{A}^*$ *– defined by* $a \cdot \phi^* := \phi^* \leftharpoonup S(a)$ *– is finitely generated and projective with generator set* $\{\lambda_k^*\} \subset \mathcal{L}(\mathcal{A}^*)$.

*The following properties of* $\mathcal{A}$ *are also equivalent*:

(2.a) $s_L\colon\ L \to A$ *is a right QF extension.*
(2.b) $t_R\colon\ R^{\mathrm{op}} \to A$ *is a right QF extension.*
(2.c) *The modules* ${}_L A$ *and* ${}_R\mathcal{R}({}_*\mathcal{A})$ *– defined by* $r \cdot {}_*\rho := s_R(r) \rightharpoonup {}_*\rho$ *– are finitely generated and projective.*
(2.d) *The module* ${}_L A$ *is finitely generated and projective and the module* ${}_R\mathcal{R}({}_*\mathcal{A})$ *is flat.*

(2.e) *The module* ${}_L A$ *is finitely generated and projective and the invariants of the right A-module* ${}_R\mathcal{R}({}_*\mathcal{A}) \otimes A_R$ *– defined via right multiplication in the second factor – with respect to* $\mathcal{A}_R$ *are the elements of* ${}_R\mathcal{R}({}_*\mathcal{A}) \otimes \mathcal{R}(\mathcal{A})_R$.

(2.f) *There exist finite sets* $\{\wp_k\} \subset \mathcal{R}(\mathcal{A})$ *and* $\{{}_*\rho_k\} \subset \mathcal{R}({}_*\mathcal{A})$ *satisfying* $\sum_k {}_*\rho_k \circ S(\wp_k) = 1_L$.

(2.g) *The right A-module* ${}_*\mathcal{A}_A$ *– defined by* ${}_*\phi \cdot a := S(a) \rightharpoondown {}_*\phi$ *– is finitely generated and projective with generator set* $\{{}_*\rho_k\} \subset \mathcal{R}({}_*\mathcal{A})$.

*If furthermore the antipode is bijective, then the conditions* (1.a)–(1.g) *and* (2.a)–(2.g) *are equivalent to each other and also to*

(1.h) *The left* ${}^*\mathcal{A}$*-module on A – defined by* ${}^*\phi \cdot a := {}^*\phi \rightharpoondown a$ *– is finitely generated and projective with generator set* $\{\ell_k\} \in \mathcal{L}(\mathcal{A})$.

(2.h) *The right* $\mathcal{A}_*$*-module on A – defined by* $a \cdot \phi_* := a \leftharpoonup \phi_*$ *– is finitely generated and projective with generator set* $\{\wp_k\} \in \mathcal{R}(\mathcal{A})$.

*Proof.* (1.a) $\Leftrightarrow$ (1.b) It follows from part (1) of Proposition 2.3 that the module $A_L$ is finitely generated and projective if and only if ${}_R A$ is, and the datum $\{\psi_k, \sum_i u_i^k \otimes v_i^k\}$ is a left QF system for the extension $s_R\colon R \to A$ if and only if $\{\pi_L \circ s_R \circ \psi_k, \sum_i u_i^k \otimes v_i^k\}$ is a left QF system for $t_L\colon L^{\mathrm{op}} \to A$.

(1.a) $\Rightarrow$ (1.c) The module $A^R$ is finitely generated and projective by Lemma 5.1. In terms of the left QF system, $\{\psi_k, \sum_i u_i^k \otimes v_i^k\}$ for the extension $s_R\colon R \to A$, the dual bases for the module $\mathcal{L}(\mathcal{A}^*)^L$ are given with the help of the map (4.12) as $\{E_{\mathcal{A}^*}(\psi_k)\} \subset \mathcal{L}(\mathcal{A}^*)$ and $\{\kappa_k := \pi_L[\sum_i s_R \circ {}_{\_}(u_i^k)\, v_i^k]\} \subset \mathrm{Hom}_L(\mathcal{L}(\mathcal{A}^*)^L, L)$.

The right $L$-linearity of the maps $\kappa_k\colon \mathcal{L}(\mathcal{A}^*) \to L$ is checked similarly to the right $L$-linearity of the map (4.19). Notice that for any $R$–$R$ bimodule map $\psi\colon {}_R A^R \to R$ we have

$$\begin{aligned}
E_{\mathcal{A}^*}(\psi) \leftharpoonup s_L(l) &= \sum_j [\chi^{-1}(\beta_*^j)\psi] \leftharpoonup s_L(l) S^2(b_j) \\
&= \sum_j [\chi^{-1}(t_* \circ \pi_L \circ t_R \circ \pi_R \circ t_L(l)\ \beta_*^j)\psi] \leftharpoonup S^2(b_j) \\
&= \sum_j [\chi^{-1}(\beta_*^j)\ t^* \circ \pi_R \circ t_L(l)\ \psi] \leftharpoonup S^2(b_j) \\
&= \sum_j [\chi^{-1}(\beta_*^j)\ s^* \circ \pi_R \circ t_L(l)\ \psi] \leftharpoonup S^2(b_j) \\
&= \sum_j [\chi^{-1}(s_* \circ \pi_L \circ t_R \circ \pi_R \circ t_L(l)\ \beta_*^j)\ \psi] \leftharpoonup S^2(b_j) \\
&= E_{\mathcal{A}^*}(\psi \leftharpoonup s_L(l))
\end{aligned}$$

for all $l \in L$, where in the first step we used (4.12) and (4.9), in the second step the property of the dual bases $\{b_j\} \subset A$ and $\{\beta_*^j\} \subset \mathcal{A}_*$ that $\sum_j \beta_*^j \otimes s_L(l) b_j = \sum_j t_*(l)\beta_*^j \otimes b_j$ for all $l \in L$ as elements of ${}^L\mathcal{A}_* \otimes A_L$, in the third step the identity

$\chi^{-1} \circ t_* = t^* \circ \pi_R \circ s_L$, in the fourth step the fact that by the left $R$-linearity of $\psi$ we have $t^*(r)\psi = s^*(r)\psi$ for all $r \in R$, in the fifth step $\chi^{-1} \circ s_* = s^* \circ \pi_R \circ s_L$, and finally $\sum_j \beta_*^j \otimes b_j s_L(l) = \sum_j s_*(l)\beta_*^j \otimes b_j$, holding true for all $l \in L$ as an identity in ${}^L\mathcal{A}_* \otimes A_L$.

The dual basis property of the sets $\{E_{\mathcal{A}^*}(\psi_k)\}$ and $\{\kappa_k\}$ is verified by the property that $\sum_{i,k} E_{\mathcal{A}^*}(\psi_k \leftharpoondown s_L \circ \kappa_k(\lambda^*)) = \lambda^*$ for all $\lambda^* \in \mathcal{L}(\mathcal{A}^*)$, which is checked similarly to (4.21).

(1.c) $\Rightarrow$ (1.d) is a standard result.

(1.d) $\Rightarrow$ (1.e) If the module $A^R$ – equivalently, by Lemma 2.6 the module $A_L$ – is finitely generated and projective then the invariants of any left $A$-module $M$ with respect to $\mathcal{A}_L$ are the elements of the kernel of the map

$$\zeta_M\colon\ M \to {}^L\mathcal{A}_* \otimes M_L, \quad m \mapsto \left(\sum_i \beta_*^i \otimes b_i \cdot m\right) - \pi_L \otimes m,$$

where the right $L$ module $M_L$ is defined via $t_L$, and the sets $\{b_i\} \subset A$ and $\{\beta_*^i\} \subset \mathcal{A}_*$ are dual bases for the module $A_L$.

The map $\zeta_A$, corresponding to the left regular $A$-module, is a left $L$-module map ${}^LA \to {}^{\mathbf{L}}\mathcal{A}^* \otimes {}^LA_{\mathbf{L}}$ and $\zeta_{{}^LA \otimes \mathcal{L}(\mathcal{A}^*)^L} = \zeta_A \otimes \mathcal{L}(\mathcal{A}^*)^L$. Since tensoring with $\mathcal{L}(\mathcal{A}^*)^L$ is an exact functor by assumption, it preserves the kernels, that is the invariants in this case.

(1.e) $\Rightarrow$ (1.f) With the help of the map (4.14) introduce

$$\sum_k \ell_k \otimes \lambda_k^* := \alpha_L^{-1}(\pi_R) \in \mathrm{Inv}({}^LA \otimes \mathcal{L}(\mathcal{A}^*)^L) \equiv {}^L\mathcal{L}(\mathcal{A}) \otimes \mathcal{L}(\mathcal{A}^*)^L.$$

It satisfies $\sum_k \lambda_k^* \circ S(\ell_k) = \alpha_L \circ \alpha_L^{-1}(\pi_R)(1_A) = 1_R$.

(1.f) $\Rightarrow$ (1.a) In terms of the sets $\{\ell_k\} \subset \mathcal{L}(\mathcal{A})$ and $\{\lambda_k^*\} \subset \mathcal{L}(\mathcal{A}^*)$ a left QF system for the extension $s_R\colon R \to A$ can be constructed as $\{\lambda_k^*, \ell_k^{(1)} \otimes S(\ell_k^{(2)})\}$.

The module $A^R$ is finitely generated and projective since there exist dual bases $\{b_i\} \subset A$ and $\{\beta_i^*\} \subset \mathcal{A}^*$ defined by $\sum_i b_i \otimes \beta_i^* = \sum_k \ell_k^{(1)} \otimes \lambda_k^*[S(\ell_k^{(2)})\ \_\ ]$, as elements of $A^R \otimes {}_R\mathcal{A}^*$. The module $A_L$ is finitely generated and projective by Lemma 2.6, hence so is ${}_RA$.

(1.f) $\Rightarrow$ (1.g) In terms of the sets $\{\ell_k\} \subset \mathcal{L}(\mathcal{A})$ and $\{\lambda_k^*\} \subset \mathcal{L}(\mathcal{A}^*)$ the dual bases for the module ${}_A\mathcal{A}^*$ are given by $\{\lambda_k^*\} \subset \mathcal{L}(\mathcal{A}^*)$ and $\{\ \_ \rightharpoonup \ell_k\} \subset {}_A\mathrm{Hom}({}_A\mathcal{A}^*, A)$.

(1.g) $\Rightarrow$ (1.f) In terms of the dual bases $\{\lambda_k^*\} \subset \mathcal{L}(\mathcal{A}^*)$ and $\{\Xi_k\} \subset {}_A\mathrm{Hom}({}_A\mathcal{A}^*, A)$ one defines the required left integrals $\ell_k := \Xi_k(\pi_R)$ in $\mathcal{A}$.

The equivalence of the conditions (2.a)–(2.g) follows by applying the above results to the Hopf algebroid $\mathcal{A}_{\mathrm{cop}}^{\mathrm{op}}$.

Now assume that $S$ is bijective. Then

(1.f) $\Leftrightarrow$ (2.f) follows from Scholium 2.10.

(1.f) $\Rightarrow$ (1.h) Scholium 2.8, (1.b) and Scholium 2.10, (3.c) can be used to show that in terms of the sets $\{\ell_k\} \subset \mathcal{L}(\mathcal{A})$ and $\{\lambda_k^*\} \subset \mathcal{L}(\mathcal{A}^*)$ the dual bases for the left ${}^*\mathcal{A}$-module on $A$ are given by $\{\ell_k\} \subset \mathcal{L}(\mathcal{A})$ and $\{\lambda_k^* \circ S \leftharpoondown S^{-1}(\_)\} \subset {}_{*\mathcal{A}}\mathrm{Hom}(A, {}^*\mathcal{A})$.

(1.h) $\Rightarrow$ (1.f) Let $\{\ell_k\} \subset \mathcal{L}(A)$ and $\{\chi_k\} \subset {}_{*\mathcal{A}}\mathrm{Hom}(A, {}^*\mathcal{A})$ be dual bases for the left ${}^*\mathcal{A}$-module $A$. Since for all $a \in A$ we have $\sum_k \ell_k^{(1)} s_R \circ \chi_k(a)(\ell^{(2)}) = a$, the module $A^R$, and hence by Proposition 2.6 also ${}^R A$, is finitely generated and projective. For any value of the index $k$ the element $\chi_k(1_A)$ is an invariant of the left regular ${}^*\mathcal{A}$-module, hence a $t$-integral on $\mathcal{A}_R$. By Scholium 2.10 the elements $\lambda_k^* := \chi_k(1_A) \circ S^{-1}$ are $s$-integrals on $\mathcal{A}_R$, satisfying

$$\sum_k \lambda_k^* \circ S(\ell_k) = \pi_R\left[\sum_k \chi_k(1_A) \rightharpoonup \ell_k\right] = 1_R.$$

(2.f) $\Leftrightarrow$ (2.h) follows by applying (1.f) $\Leftrightarrow$ (1.h) to the Hopf algebroid $\mathcal{A}^{\mathrm{op}}_{\mathrm{cop}}$. $\square$

If the antipode of a Hopf algebroid $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ is bijective then the application of Theorem 5.2 to the Hopf algebroid $\mathcal{A}^{\mathrm{op}}$ results in equivalent conditions under which the extensions $s_R\colon R \to A$ and $t_L\colon L^{\mathrm{op}} \to A$ are right QF, and $s_L\colon L \to A$ and $t_R\colon R^{\mathrm{op}} \to A$ are left QF.

In order to show that – in contrast to Hopf algebras over commutative rings – not any finitely generated projective Hopf algebroid is quasi-Frobenius, let us give here an example (with bijective antipode) such that the total algebra is finitely generated and projective as a module over the base algebra (in all the four senses listed in (2.16)) and the total algebra is neither a left nor a right QF extension of the base algebra.

The example is taken from [21, Example 3.1] where it is shown that for any algebra $B$ over a commutative ring $k$ the $k$-algebra $A := B \underset{k}{\otimes} B^{\mathrm{op}}$ has a left bialgebroid structure, $\mathcal{A}_L$, over the base $B$ with structural maps

$$\begin{aligned}
&s_L\colon B \to A, && b \mapsto b \otimes 1_B,\\
&t_L\colon B^{\mathrm{op}} \to A, && b \mapsto 1_B \otimes b,\\
&\gamma_L\colon A \to A_B \otimes {}_B A, && b_1 \otimes b_2 \mapsto (b_1 \otimes 1_B) \otimes (1_B \otimes b_2),\\
&\pi_L\colon A \to B, && b_1 \otimes b_2 \mapsto b_1 b_2.
\end{aligned} \tag{5.1}$$

The bialgebroid $\mathcal{A}_L$ satisfies the Hopf algebroid axioms of [21] with the involutive antipode $S$, equal to the flip map

$$S\colon B \underset{k}{\otimes} B^{\mathrm{op}} \to B^{\mathrm{op}} \underset{k}{\otimes} B, \quad b_1 \otimes b_2 \mapsto b_2 \otimes b_1. \tag{5.2}$$

The reader may check that $A$ has a Hopf algebroid structure also in the sense of this paper with left bialgebroid structure (5.1), antipode (5.2) and right bialgebroid structure $\mathcal{A}_R = (A, B^{\mathrm{op}}, S \circ s_L, S \circ t_L, (S \otimes S) \circ \gamma_L^{\mathrm{op}} \circ S, \pi_L \circ S)$.

If $B$ is finitely generated and projective as a $k$-module then all modules $A^{B^{\mathrm{op}}}$, ${}^{B^{\mathrm{op}}}A$, $A_B$ and ${}_B A$ are finitely generated and projective, and vice versa. What is more, we have

LEMMA 5.3. *Let $B$ be an algebra over the commutative ring $k$ with trivial center. The following statements are equivalent*:

(1) *The extension $k \to B$ is left QF.*
(2) *The extension $k \to B$ is right QF.*
(3) *The extension $B \to B \underset{k}{\otimes} B^{\mathrm{op}}$, $b \mapsto b \otimes 1_B$ is left QF.*
(4) *The extension $B \to B \underset{k}{\otimes} B^{\mathrm{op}}$, $b \mapsto b \otimes 1_B$ is right QF.*

The equivalence (1) $\Leftrightarrow$ (2) is proven in [29] and the rest can be proven using the techniques of quasi-Frobenius systems.

In view of Lemma 5.3 it is easy to construct a finitely generated projective Hopf algebroid which is not QF. Let us choose, for example, $B$ to be the algebra of $n \times n$ upper triangle matrices with entries in the commutative ring $k$. Then $B$ has trivial center and it is neither a left nor a right QF extension of $k$, hence $A = B \underset{k}{\otimes} B^{\mathrm{op}}$ is neither a left nor a right QF extension of $B$.

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