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τ-Categories I: Ladders

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Abstract. In this series of papers, we introduce τ-categories, which are additive categories with some kind of Auslander–Reiten sequences. We apply them to study the category of lattices over orders. In this first paper, we study minimal projective resolutions in functor categories over τ-categories. Then we give a structure theorem of completely graded τ-categories using mesh categories.

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In this series of papers, we study the category $\mathrm{lat}\,\Lambda$ of lattices over an order Λ over a complete regular local ring of dimension $d \leqslant 2$ (Section 2.2). In [1, 2] and so on, Auslander studied the Abelian category $\mathrm{Mod}(\mathrm{lat}\,\Lambda)$ of additive functors $(\mathrm{lat}\,\Lambda)^{\mathrm{op}} \to \mathcal{A}b$. He obtained the *Existence Theorem of Auslander–Reiten sequences*, which are good complexes in $\mathrm{lat}\,\Lambda$ derived from the minimal projective resolutions of simple objects in $\mathrm{Mod}(\mathrm{lat}\,\Lambda)$. As an application, one obtains an invariant $\mathbb{A}(\mathrm{lat}\,\Lambda)$ called the *Auslander–Reiten quiver* [5, 20], which displays terms of Auslander–Reiten sequences. It is a directed graph with a special combinatorial structure, called a *translation quiver* (Section 2.4). Since $\mathbb{A}(\mathrm{lat}\,\Lambda)$ is much simpler than Λ, it is important to consider the relationship between representation theoretic properties of Λ and combinatorial properties of $\mathbb{A}(\mathrm{lat}\,\Lambda)$. In this series of papers, we develop a basic theory to study such problems. Especially, we solve the following problem (P_d) for $d = 1$ in [11].

(P_d) Give a combinatorial characterization of finite translation quivers which are realized as an Auslander–Reiten quiver $\mathbb{A}(\mathrm{lat}\,\Lambda)$ of an order Λ over a complete regular local ring of dimension d.

Roughly speaking, our method is to compare the following things for a Krull–Schmidt category $\mathcal{C}$:

(Re) Representation theoretic realization of $\mathcal{C}$, namely an equivalence between $\mathcal{C}$ and the category $\mathrm{lat}\,\Lambda$ of some order Λ;

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(Ca) Categorical properties for $\mathcal{C}$, especially minimal projective resolutions of simple objects in Mod $\mathcal{C}$;
(Co) Combinatorial properties for $\mathbb{A}(\mathcal{C})$.

In particular, we introduce *τ-categories* (Section 2.1), which are Krull–Schmidt categories with some kind of Auslander–Reiten sequences, and define an invariant $\mathbb{A}(\mathcal{C})$ called the *Auslander–Reiten quiver* of a τ-category $\mathcal{C}$ (Section 2.4). In this paper, we study τ-categories and the relationship between (Ca) and (Co). Then, in [10] and [11], we mainly study the relationship between (Re) and (Ca) using homological conditions for Noetherian rings which are related to Auslander–Gorenstein rings [8] and Auslander orders [4].

Auslander's theory shows that lat Λ is a τ-category for any R-order Λ which is an isolated singularity with $\dim R \leqslant 2$ (Section 2.2). But, our introduction of τ-categories is motivated by another important example more strongly. In their solution of (P_0) [13], Igusa and Todorov introduced *τ-species* (= modulated translation quiver in [13]), which form an algebraic realization of translation quivers by division rings and their bimodules (Section 8.3). They defined an additive category $\widehat{\mathbb{M}}(\mathcal{Q})$ called the *mesh category* of a τ-species $\mathcal{Q}$ (Section 8.4), which is a generalization of the mesh categories of translation quivers introduced by Riedtmann [15] and Bongartz and Gabriel [7]. We show that $\widehat{\mathbb{M}}(\mathcal{Q})$ forms a τ-category (Section 8.4). Although some partial results for (P_1) were given by Wiedemann [17–19], one shall meet a sheer difficulty to get a general solution for (P_1), which was not existent in (P_0). In fact, we shall need some finiteness condition for mesh categories of some τ-species, so we study not only lat Λ but also general τ-categories in these papers. Thanks to our general treatment, we give an answer to (P_1) in [11] as an application of *Rejection theory* for τ-categories in [10], which generalizes results in [9].

Our study in this first paper is divided into two parts. In Part I, we develop a *ladder theory* of a τ-category $\mathcal{C}$, which is a study of minimal projective resolutions of some kind of $\mathcal{C}$-modules. First of all, we prove an *Existence Theorem of Ladders* (Section 3.3), which is a fundamental of our theory of τ-categories, and will be used frequently in [10, 11]. Then we obtain a *Radical Layers Theorem* (Section 4.2) [12], which immediately implies that the associated completely graded category $\widehat{\mathbb{G}}(\mathcal{C})$ is a τ-category whenever $\mathcal{C}$ itself is a τ-category (Section 5.2). In Section 6, we introduce a concept of *invertible ladders* and *invertible pairs* of $\mathcal{C}$-modules and $\mathcal{C}^{\mathrm{op}}$-modules, and study some homological properties. As an application, we obtain a *Recursion Formula* (Section 7.1), which allows to build ladders in a mere combinatorial fashion. We use it to study the relationship between above (Ca) and (Co).

In Part II, as an application of theorems in Part I, we develop a *structure theory* of τ-categories, which connect τ-categories and τ-species. Let $\mathcal{T}_{\mathrm{sp}}$ be the category of τ-species and $\mathcal{T}_{\mathrm{ca}}$ the category of skeletal τ-categories (Section 10.1). There are functors

$$\widehat{\mathbb{M}}\colon \mathcal{T}_{\mathrm{sp}} \longrightarrow \mathcal{T}_{\mathrm{ca}} \quad (\textit{mesh categories}) \text{ (Section 8.4)},$$

$$\widehat{\mathbb{A}}\colon \mathcal{T}_{\mathrm{ca}} \longrightarrow \mathcal{T}_{\mathrm{sp}} \quad (\textit{Auslander–Reiten species}) \text{ (Section 9.1)}$$

such that $\widehat{\mathbb{A}} \circ \widehat{\mathbb{M}} \simeq 1_{\mathcal{T}_{\mathrm{sp}}}$ and $\widehat{\mathbb{M}} \circ \widehat{\mathbb{A}} \simeq \widehat{\mathbb{G}}$ (Section 10.2). In particular, the category $\mathcal{T}_{\mathrm{sp}}$ is equivalent to $\mathcal{T}_{\mathrm{gca}}$ the category of completely graded skeletal τ-categories. This means that τ-categories are the natural domain to study mesh categories of τ-species.

Our writtings were inspired by the work of Igusa and Todorov [12, 13], which extends ring theoretic aspects of the work of Riedtmann [15] and Bongartz and Gabriel [7]. We have borrowed terminology (such as Ladders and Radical Layers) from [12]. Once Radical Layers Theorem is established in our general context, we define τ-species and their mesh categories (Section 8) in entirely similar manner as in [13]. Now we feel that our τ-category is rather a natural domain of consideration. For one thing, our proof of the Radical Layers Theorem is not only more general, but also simpler, even when restricted to the case of $d = 0$, than that of [12] where very restrictive assumptions for valuations of arrows of Auslander–Reiten quivers are imposed and essentially used in the proof. For another thing a translation quiver has a two-dimensional geometric realization in the sense of [7] §4, so that a τ-species can be regarded as a two-dimensional analogue of species. Therefore our Structure Theorem of τ-categories can be regarded as a two-dimensional analogue of the well known relationship between species and hereditary categories. We hope there should exist higher-dimensional τ-categories as well as τ-species corresponding to each other.

PART I. LADDER THEORY OF τ-CATEGORIES

1. Modules over Krull–Schmidt Categories

An additive category $\mathcal{C}$ is called *skeletally small* if the isomorphism classes of objects form a set, and called *Krull–Schmidt* if any object is isomorphic to a finite direct sum of objects whose endomorphism rings are local. By [14], $\mathcal{C}$ is Krull–Schmidt if and only if a ring of endomorphism of any object is semiperfect and idempotents split in $\mathcal{C}$. Recall that H. Bass introduced a definition of semiperfect rings, and this notion deals with a theory of projective covers developed by Eilenberg, Nakayama and Bass.

Throughout this paper, any additive category $\mathcal{C}$ is assumed to be skeletally small and Krull–Schmidt. We denote by $\mathcal{C}(X, Y)$ the set of morphisms from X to Y, and by $fg \in \mathcal{C}(X, Z)$ the composition of $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$. A *$\mathcal{C}$-module* is a contravariant additive functor from $\mathcal{C}$ to the category $\mathcal{A}b$ of Abelian groups. For $\mathcal{C}$-modules M and M', we denote by $\mathrm{Hom}(M, M')$ the set of natural transformations from M to M'. Thus we obtain the Abelian category $\mathrm{Mod}\,\mathcal{C}$ of $\mathcal{C}$-modules [1]. We review several basic facts and introduce some (nonstandard) notations.

(1) *Krull–Schmidt theorem* holds in $\mathcal{C}$, namely any object is *uniquely isomorphic* to a finite direct sum of indecomposable objects. We denote by $\mathrm{ind}\,\mathcal{C}$ the set of isomorphism classes of indecomposable objects in $\mathcal{C}$.

(2) Define functors $\mathrm{H}^{\mathcal{C}}\colon \mathcal{C} \to \operatorname{Mod}\mathcal{C}$ and $\mathrm{H}_{\mathcal{C}}\colon \mathcal{C} \to \operatorname{Mod}\mathcal{C}^{\mathrm{op}}$ by $\mathrm{H}_X^{\mathcal{C}} := \mathcal{C}(\ ,X)$ and $\mathrm{H}_{\mathcal{C}}^X := \mathcal{C}(X,\ )$. We say that a $\mathcal{C}$-module M is *finitely generated* if there exists an epimorphism $\mathrm{H}_X^{\mathcal{C}} \to M$ for some $X \in \mathrm{Ob}(\mathcal{C})$. Then Yoneda's lemma shows that $\mathrm{H}^{\mathcal{C}}$ (respectively, $\mathrm{H}_{\mathcal{C}}$) induces an equivalence from $\mathcal{C}$ (respectively, $\mathcal{C}^{\mathrm{op}}$) to the category of finitely generated projective $\mathcal{C}$-modules (respectively, $\mathcal{C}^{\mathrm{op}}$-modules).
$\mathcal{C}$ is called *left Artinian* (respectively, *right Artinian*) if $\mathrm{H}_X^{\mathcal{C}}$ (respectively, $\mathrm{H}_{\mathcal{C}}^X$) has finite length for any $X \in \mathrm{Ob}(\mathcal{C})$, and called *Artinian* if it is left Artinian and right Artinian.

(3) We denote by $\mathcal{J}_{\mathcal{C}}$ (or $\mathcal{J}$) the *Jacobson radical* of $\mathcal{C}$, which is an ideal of $\mathcal{C}$ such that $\mathcal{J}(\ ,X)$ (respectively, $\mathcal{J}(X,\ )$) is the *radical* (= intersection of all maximal submodules) of $\mathrm{H}_X^{\mathcal{C}}$ (respectively, $\mathrm{H}_{\mathcal{C}}^X$) for any $X \in \mathrm{Ob}(\mathcal{C})$. Define functors $\mathrm{S}^{\mathcal{C}}\colon \mathcal{C} \to \operatorname{Mod}\mathcal{C}$ and $\mathrm{S}_{\mathcal{C}}\colon \mathcal{C} \to \operatorname{Mod}\mathcal{C}^{\mathrm{op}}$ by $\mathrm{S}_X^{\mathcal{C}} := \mathrm{H}_X^{\mathcal{C}} / \mathcal{J}(\ ,X)$ and $\mathrm{S}_{\mathcal{C}}^X := \mathrm{H}_X^{\mathcal{C}} / \mathcal{J}(X,\ )$. Then $\mathrm{S}^{\mathcal{C}}$ (respectively, $\mathrm{S}_{\mathcal{C}}$) induces a bijection from $\operatorname{ind}\mathcal{C}$ to the set of isomorphism classes of simple $\mathcal{C}$-modules (respectively, $\mathcal{C}^{\mathrm{op}}$-modules).
We simply denote $\mathrm{H}_X^{\mathcal{C}}$ (respectively, $\mathrm{H}_{\mathcal{C}}^X, \mathrm{S}_X^{\mathcal{C}}, \mathrm{S}_{\mathcal{C}}^X$) by H_X (respectively, $\mathrm{H}^X, \mathrm{S}_X, \mathrm{S}^X$). We denote by $\operatorname{pd} L$ the projective dimension of a $\mathcal{C}$-module L, and by $\operatorname{ind}_n^+ \mathcal{C}$ (respectively, $\operatorname{ind}_n^- \mathcal{C}$) the subset of $\operatorname{ind}\mathcal{C}$ consisting of all X such that $\operatorname{pd}\mathrm{S}_X \leqslant n$ (respectively, $\operatorname{pd}\mathrm{S}^X \leqslant n$).

(4) Recall that an epimorphism $\mathrm{H}_X \overset{\phi}{\to} L$ is called a *projective cover* of a $\mathcal{C}$-module L if ϕ is essential, or equivalently, $\operatorname{Ker}\phi \subseteq \mathcal{J}(\ ,X)$ holds. In this sense $\mathcal{C}$ is semiperfect, i.e. any finitely generated $\mathcal{C}$-module has a projective cover. Recall that an exact sequence $\mathrm{H}_{X_n} \overset{\mathrm{H}_{f_n}}{\to} \cdots \overset{\mathrm{H}_{f_2}}{\to} \mathrm{H}_{X_1} \overset{\mathrm{H}_{f_1}}{\to} \mathrm{H}_{X_0} \to L_0 \to 0$ is called a *minimal* projective resolution of a $\mathcal{C}$-module L_0 if $\mathrm{H}_{X_i} \to L_i \to 0$ is a projective cover for any i $(0 \leqslant i \leqslant n)$, where we put $L_i := \operatorname{Im}\mathrm{H}_{f_i}$ $(1 \leqslant i \leqslant n)$.

(5) Let I be an ideal of $\mathcal{C}$. We write $f \in I$ if $f \in I(X,Y)$ for some $X, Y \in \mathrm{Ob}(\mathcal{C})$. For a $\mathcal{C}$-module L, define a subobject IL of L by $(IL)(X) := \sum_{Y \in \mathrm{Ob}(\mathcal{C})} I(X,Y)L(Y)$. Then the radical of a finitely generated $\mathcal{C}$-module L is given by $\mathcal{J} L$.

(6) For two complexes $\mathbf{A}$ and $\mathbf{A}'$ over $\mathcal{C}$, we write $\mathbf{A} \approx \mathbf{A}'$ if $\mathbf{A}$ is isomorphic to $\mathbf{A}'$ as a complex. Any morphism in $\mathcal{C}$ is regarded as a two-termed complex, and any object in $\mathcal{C}$ is regarded as a one-termed complex. Thus, for objects X and Y in $\mathcal{C}$, we write $X \approx Y$ if X is isomorphic to Y.
It is often used that, for any $f \in \mathcal{C}$, there exists $g \in \mathcal{J}$ such that $f \approx \left(\begin{smallmatrix}1 & 0\\ 0 & g\end{smallmatrix}\right)$.

(7) For a set Q, we will denote by $\mathbb{Z}Q$ (respectively, $\mathbb{N}Q$) the free Abelian group (respectively, free Abelian monoid) generated by Q, and regard Q as a subset of $\mathbb{N}Q$. Introduce an inner product $\langle\ ,\ \rangle$ on $\mathbb{Z}Q$ by taking Q as an orthonormal base, and for any $X \in \mathbb{Z}Q$, define $X_+, X_- \in \mathbb{N}Q$ by $X = X_+ - X_-$ and $\langle X_+, X_-\rangle = 0$. For a subset S of Q, we denote by $|_S\colon \mathbb{Z}Q \to \mathbb{Z}S$ the natural projection.
For example, we can identify $\mathrm{Ob}(\mathcal{C})/\approx$ with $\mathbb{N}\operatorname{ind}\mathcal{C}$.

2. τ-Categories

2.1. Let $\mathcal{C}$ be a Krull–Schmidt category and $n \geqslant 0$. Let α: $\operatorname{Mod}\mathcal{C} \to \operatorname{Mod}\mathcal{C}^{\mathrm{op}}$ be the left exact functor defined by $(\alpha(L))(X) := \operatorname{Hom}(L, \mathrm{H}_X^{\mathcal{C}})$, and $\mathrm{R}^n\alpha$: $\operatorname{Mod}\mathcal{C} \to \operatorname{Mod}\mathcal{C}^{\mathrm{op}}$ the nth derived functor of α, namely $(\mathrm{R}^n\alpha(L))(X) = \operatorname{Ext}^n(L, \mathrm{H}_X^{\mathcal{C}})$. Dually, $\mathrm{R}^n\alpha$: $\operatorname{Mod}\mathcal{C}^{\mathrm{op}} \to \operatorname{Mod}\mathcal{C}$ is defined [8].

We denote by $\operatorname{Mod}_n \mathcal{C}$ the full subcategory of $\operatorname{Mod}\mathcal{C}$ consisting of L which has a minimal projective resolution (Section 1(4)) $\mathrm{H}_{X_n} \xrightarrow{\mathrm{H}_{f_n}} \cdots \xrightarrow{\mathrm{H}_{f_2}} \mathrm{H}_{X_1} \xrightarrow{\mathrm{H}_{f_1}} \mathrm{H}_{X_0} \to L \to 0$ $(X_i \in \operatorname{Ob}(\mathcal{C}))$. For such L, put $\operatorname{Tr}_{n-1}(L) := \operatorname{Cok}\mathrm{H}^{f_n}$ and $\operatorname{Tr}(L) := \operatorname{Tr}_0(L)$. Notice that $\operatorname{Tr}_n(L)$ gives a minimal lifting of $J_n(L)$ in [3].

(1) $\mathcal{C}$ is called a *right τ-category* if $\mathrm{S}_X \in \operatorname{Ob}(\operatorname{Mod}_2\mathcal{C})$ and $\operatorname{Tr}_1(\mathrm{S}_X)$ is semisimple for any $X \in \operatorname{Ob}(\mathcal{C})$. Dually, $\mathcal{C}$ is called a *left τ-category* if $\mathrm{S}^X \in \operatorname{Ob}(\operatorname{Mod}_2\mathcal{C}^{\mathrm{op}})$ and $\operatorname{Tr}_1(\mathrm{S}^X)$ is semisimple for any $X \in \operatorname{Ob}(\mathcal{C})$.

(2) A right (respectively, left) τ-category $\mathcal{C}$ is called *right strict* (respectively, *left strict*) if $\operatorname{pd}\mathrm{S}_X \leqslant 2$ (respectively, $\operatorname{pd}\mathrm{S}^X \leqslant 2$) for any $X \in \operatorname{Ob}(\mathcal{C})$. A τ-category $\mathcal{C}$ is called *strict* if $\mathcal{C}$ is right strict and left strict.

(3) Assume that $X \in \operatorname{Ob}(\mathcal{C})$ satisfies $\mathrm{S}_X \in \operatorname{Ob}(\operatorname{Mod}_2\mathcal{C})$ (respectively, $\mathrm{S}^X \in \operatorname{Ob}(\operatorname{Mod}_2\mathcal{C}^{\mathrm{op}})$). Then we denote by $(X]_{\mathcal{C}} = (\tau^+X \xrightarrow{\nu_X^+} \theta^+X \xrightarrow{\mu_X^+} X)$ (respectively, $[X)_{\mathcal{C}} = (X \xrightarrow{\mu_X^-} \theta^-X \xrightarrow{\nu_X^-} \tau^-X)$) a complex such that $\mathrm{H}_{\tau^+X} \xrightarrow{\mathrm{H}_{\nu_X^+}} \mathrm{H}_{\theta^+X} \xrightarrow{\mathrm{H}_{\mu_X^+}} \mathrm{H}_X \to \mathrm{S}_X \to 0$ (respectively, $\mathrm{H}^{\tau^-X} \xrightarrow{\mathrm{H}^{\nu_X^-}} \mathrm{H}^{\theta^-X} \xrightarrow{\mathrm{H}^{\mu_X^-}} \mathrm{H}^X \to \mathrm{S}^X \to 0$) gives a minimal projective resolution. We omit the index $\mathcal{C}$ for simplicity.
Immediately, $(X]$ (respectively, $[X)$) is unique up to isomorphism, and $(X \oplus Y] \approx (X] \oplus (Y]$ holds for any $X, Y \in \operatorname{Ob}(\mathcal{C})$. Moreover, X is an indecomposable object if and only if $(X]$ (respectively, $[X)$) is indecomposable as a complex.

Moreover, $\operatorname{ind}_1^{\pm}\mathcal{C} = \{x \in \operatorname{ind}\mathcal{C} \mid \tau^{\pm}x = 0\}$ and $\operatorname{ind}_0^{\pm}\mathcal{C} = \{x \in \operatorname{ind}\mathcal{C} \mid \theta^{\pm}x = 0\}$ hold ((3) of Section 1).

2.2. EXAMPLES. (1) Let Λ be a Noetherian semiperfect ring and $\mathcal{C} := \operatorname{pr}\Lambda$ the category of finitely generated projective Λ-modules. Then $\operatorname{Mod}\mathcal{C}$ is equivalent to the category of left Λ-modules. In particular, $\mathcal{C}$ is a strict τ-category if and only if $\operatorname{gl.dim}\Lambda \leqslant 2$ and $\operatorname{Ext}_\Lambda^2(S, \Lambda)$ is semisimple for any simple left or right Λ-module S. By 2.3 below, this is also equivalent to $\operatorname{gl.dim}\Lambda \leqslant 2$, $\operatorname{Ext}_\Lambda^i(S, \Lambda) = 0$ $(i = 0, 1)$ and $\operatorname{Ext}_\Lambda^2(S, \Lambda)$ is simple for any simple left or right Λ-module S with $\operatorname{pd}S = 2$.

(2) Let R be a complete regular local ring of dimension $d \geqslant 0$. An R-algebra Λ is called an *R-order* if it is finitely generated free as an R-module. Assume that Λ is an R-order. A left Λ-module L is called a *Λ-lattice* if it is finitely generated free as an R-module. We denote by $\operatorname{lat}\Lambda$ the category of Λ-lattices, which forms a Krull–Schmidt category [6]. Then $(\ )^* = \operatorname{Hom}_R(\ , R)$ gives a duality between $\operatorname{lat}\Lambda$ and $\operatorname{lat}\Lambda^{\mathrm{op}}$. Let $\operatorname{rin}\Lambda := (\operatorname{pr}\Lambda^{\mathrm{op}})^*$ be the category of *relative injective* Λ-lattices.

Assume that Λ is an R-order which is an *isolated singularity*, namely $\mathrm{gl.dim}\,\Lambda > d$ and $\mathrm{gl.dim}\, R_{\wp} \otimes_R \Lambda = \mathrm{ht}\,\wp$ for any nonmaximal prime ideal $\wp$ of R. For any $X \in \mathrm{ind}(\mathrm{lat}\,\Lambda) - \mathrm{ind}(\mathrm{pr}\,\Lambda)$ (respectively, $Y \in \mathrm{ind}(\mathrm{lat}\,\Lambda) - \mathrm{ind}(\mathrm{rin}\,\Lambda)$), the Auslander–Reiten sequence [2] gives $(X] \approx [\tau^+ X)$ (respectively, $[Y) \approx (\tau^- Y]$).

In particular, if $d \leqslant 2$ and Λ is an R-order which is an isolated singularity, then $\mathrm{lat}\,\Lambda$ is a strict τ-category. Moreover, $\mathrm{ind}_1^+(\mathrm{lat}\,\Lambda) = \mathrm{ind}(\mathrm{pr}\,\Lambda)$ and $\mathrm{ind}_1^-(\mathrm{lat}\,\Lambda) = \mathrm{ind}(\mathrm{rin}\,\Lambda)$ if $d = 0$ or 1, and $\mathrm{ind}_1^+(\mathrm{lat}\,\Lambda) = \emptyset = \mathrm{ind}_1^-(\mathrm{lat}\,\Lambda)$ if $d = 2$.

(3) (Section 8.4) If $\mathcal{C}$ is the mesh category of a τ-species, then $\mathcal{C}$ is a τ-category.

Proof. (2) The former assertion is immediate from the definition. Let J_Λ be the Jacobson radical of Λ, $P \in \mathrm{ind}(\mathrm{pr}\,\Lambda)$ and $I \in \mathrm{ind}(\mathrm{rin}\,\Lambda)$. If $d \leqslant 1$, then $(P] \approx (0 \to J_\Lambda P \to P)$ and $[I) \approx (I \to (I^* J_\Lambda)^* \to 0)$. If $d = 2$, then the almost split sequence in the sence of [16] 2.1 gives $(P] \approx [\tau^+ P)$ (respectively, $[I) \approx (\tau^- I]$). □

2.3. THEOREM. *Let* $\mathcal{C}$ *be a* τ*-category. Then* $\mathrm{Tr}_1(\mathrm{S}_X) = \mathrm{S}^{\tau^+ X}$, $\mathrm{R}^1\alpha(\mathrm{S}_X) = 0$ *and* $(X] \approx [\tau^+ X)$ *hold for any* $X \in \mathrm{ind}\,\mathcal{C} - \mathrm{ind}_1^+\,\mathcal{C}$. *Dually,* $\mathrm{Tr}_1(\mathrm{S}^Y) = \mathrm{S}_{\tau^- Y}$, $\mathrm{R}^1\alpha(\mathrm{S}^Y) = 0$ *and* $[Y) \approx (\tau^- Y]$ *hold for any* $Y \in \mathrm{ind}\,\mathcal{C} - \mathrm{ind}_1^-\,\mathcal{C}$. *Hence* τ^+ *and* τ^- *give mutually inverse bijections between* $\mathrm{ind}\,\mathcal{C} - \mathrm{ind}_1^+\,\mathcal{C}$ *and* $\mathrm{ind}\,\mathcal{C} - \mathrm{ind}_1^-\,\mathcal{C}$.

2.3.1. (1) If $L \in \mathrm{Ob}(\mathrm{Mod}_2\,\mathcal{C})$ satisfies $\mathrm{Tr}_1(L) \in \mathrm{Ob}(\mathrm{Mod}_2\,\mathcal{C}^{\mathrm{op}})$, then $\mathrm{R}^1\alpha(\mathrm{Tr}_1(L)) = 0$ holds.

(2) Let I be the ideal of $\mathrm{Mod}\,\mathcal{C}$ consisting of morphisms which factor through a projective object, and $\underline{\mathrm{Mod}}\,\mathcal{C} := (\mathrm{Mod}\,\mathcal{C})/I$ the *stable category* [3]. Then there exists a bijection $(\underline{\mathrm{Mod}}\,\mathcal{C}^{\mathrm{op}})(\mathrm{Tr}_n(L), M) \to (\underline{\mathrm{Mod}}\,\mathcal{C})(\mathrm{Tr}_n(M), L)$ for any $n \geqslant 0$, $L \in \mathrm{Ob}(\mathrm{Mod}_{n+1}\,\mathcal{C})$ and $M \in \mathrm{Ob}(\mathrm{Mod}_{n+1}\,\mathcal{C}^{\mathrm{op}})$.

Proof. (1) Take minimal projective resolutions $\mathrm{H}_{X_2} \xrightarrow{\mathrm{H}_g} \mathrm{H}_{X_1} \xrightarrow{\mathrm{H}_f} \mathrm{H}_{X_0} \to L \to 0$ and $\mathrm{H}^Y \xrightarrow{\mathrm{H}^h} \mathrm{H}^{X_1} \xrightarrow{\mathrm{H}^g} \mathrm{H}^{X_2} \to \mathrm{Tr}_1(L) \to 0$. Then there exists f' such that $f = hf'$. For any $a \in \mathrm{Ker}\,\mathrm{H}_h$, we obtain $af = ahf' = 0$. Hence $a \in \mathrm{Im}\,\mathrm{H}_g$ holds. Thus $\mathrm{R}^1\alpha(\mathrm{Tr}_1(L)) = 0$ holds.

(2) The assertion for $n = 0$ is immediate since Tr gives a duality between $\underline{\mathrm{Mod}}_1\,\mathcal{C}$ and $\underline{\mathrm{Mod}}_1\,\mathcal{C}^{\mathrm{op}}$. Let $n > 0$ and take minimal projective resolutions $\mathrm{H}_{X_{n+1}} \xrightarrow{\mathrm{H}_{f_{n+1}}} \cdots \xrightarrow{\mathrm{H}_{f_1}} \mathrm{H}_{X_0} \to L \to 0$ and $\mathrm{H}^{Y_{n+1}} \xrightarrow{\mathrm{H}^{g_{n+1}}} \cdots \xrightarrow{\mathrm{H}^{g_1}} \mathrm{H}^{Y_0} \to M \to 0$. For $\phi \in (\mathrm{Mod}\,\mathcal{C}^{\mathrm{op}})(\mathrm{Tr}_n(L), M)$, take the following commutative diagram.

$$\begin{array}{ccccccccccc}
\mathrm{H}^{X_0} & \xrightarrow{\mathrm{H}^{f_1}} & \mathrm{H}^{X_1} & \xrightarrow{\mathrm{H}^{f_2}} & \cdots & \xrightarrow{\mathrm{H}^{f_{n+1}}} & \mathrm{H}^{X_{n+1}} & \longrightarrow & \mathrm{Tr}_n(L) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \mathrm{H}^{a_{n+1}}} & & \downarrow{\scriptstyle \mathrm{H}^{a_n}} & & & & \downarrow{\scriptstyle \mathrm{H}^{a_0}} & & \downarrow{\scriptstyle \phi} & & \\
\mathrm{H}^{Y_{n+1}} & \xrightarrow{\mathrm{H}^{g_{n+1}}} & \mathrm{H}^{Y_n} & \xrightarrow{\mathrm{H}^{g_n}} & \cdots & \xrightarrow{\mathrm{H}^{g_1}} & \mathrm{H}^{Y_0} & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

Define ψ by the following commutative diagram.

$$\begin{array}{ccccccccc} 0 \longleftarrow & L & \longleftarrow & \mathrm{H}_{X_0} & \xleftarrow{\mathrm{H}_{f_1}} & \mathrm{H}_{X_1} & \xleftarrow{\mathrm{H}_{f_2}} \cdots \xleftarrow{\mathrm{H}_{f_{n+1}}} & \mathrm{H}_{X_{n+1}} \\ & \uparrow{\psi} & & \uparrow{\mathrm{H}_{a_{n+1}}} & & \uparrow{\mathrm{H}_{a_n}} & & \uparrow{\mathrm{H}_{a_0}} \\ 0 \longleftarrow & \mathrm{Tr}_n(M) & \longleftarrow & \mathrm{H}_{Y_{n+1}} & \xleftarrow{\mathrm{H}_{g_{n+1}}} & \mathrm{H}_{Y_n} & \xleftarrow{\mathrm{H}_{g_n}} \cdots \xleftarrow{\mathrm{H}_{g_1}} & \mathrm{H}_{Y_0} \end{array}$$

It is easily checked that ϕ is zero in $\underline{\mathrm{Mod}}\,\mathcal{C}^{\mathrm{op}}$ if and only if there exists $b_i \in \mathcal{C}(Y_i, X_{n-i+2})$ $(1 \leqslant i \leqslant n+1)$ such that $a_i = b_i f_{n-i+2} + g_{i+1} b_{i+1}$ holds for any i $(1 \leqslant i \leqslant n)$ if and only if ψ is zero in $\underline{\mathrm{Mod}}\,\mathcal{C}$. Thus we obtain a well defined injection $(\underline{\mathrm{Mod}}\,\mathcal{C}^{\mathrm{op}})(\mathrm{Tr}_n(L), M) \to (\underline{\mathrm{Mod}}\,\mathcal{C})(\mathrm{Tr}_n(M), L)$. This is surjective since we obtain the inverse map by the dual argument. □

2.3.2. *Proof of 2.3.* Put $M := \mathrm{Tr}_1(\mathrm{S}_X)$. Then we have an epimorphism $M = \mathrm{Cok}\,\mathrm{H}^{\nu_X^+} \to \mathrm{S}^{\tau^+X}$. Since $\mathcal{C}$ is a τ-category, M is semisimple. Hence $\mathrm{Tr}_1(M)$ is semisimple again, and S^{τ^+X} is a direct summand of M. Since M is not projective, we obtain $(\underline{\mathrm{Mod}}\,\mathcal{C}^{\mathrm{op}})(M, M) \neq 0$. Hence $(\underline{\mathrm{Mod}}\,\mathcal{C})(\mathrm{Tr}_1(M), \mathrm{S}_X) \neq 0$ holds by 2.3.1(2). Thus S_X is a direct summand of $\mathrm{Tr}_1(M)$ since $\mathrm{Tr}_1(M)$ is semisimple and S_X is simple. Since $\mathrm{R}^1\alpha(\mathrm{Tr}_1(M)) = 0$ holds by 2.3.1(1), we obtain $\mathrm{R}^1\alpha(\mathrm{S}_X) = 0$. Hence $\mathrm{H}^X \overset{\mathrm{H}^{\mu_X^+}}{\to} \mathrm{H}^{\theta^+X} \overset{\mathrm{H}^{\nu_X^+}}{\to} \mathrm{H}^{\tau^+X} \to M \to 0$ is a projective resolution. Since $(X]$ is indecomposable by 2.1(3), M is indecomposable and this resolution is minimal. Thus $M = \mathrm{S}^{\tau^+X}$ and $(X] \approx [\tau^+X)$ hold. In particular, τ^+X is indecomposable and $X \approx \tau^-\tau^+X$ holds. Hence $\tau^+X \in \mathrm{ind}\,\mathcal{C} - \mathrm{ind}_1^-\,\mathcal{C}$ holds. □

2.4. DEFINITION. (1) $Q = (Q, Q^p, Q^i, \tau^+, d, d')$ is called a *translation quiver* if Q is a set, Q^p and Q^i are subsets of Q, τ^+ is a bijection $Q - Q^p \to Q - Q^i$, and d and d' are maps $Q \times Q \to \mathbb{N}_{\geqslant 0}$ such that $d(Y, X) = d'(\tau^+X, Y)$ holds for any $X \in Q - Q^p$ and $Y \in Q$, and $d(\ , X) = 0$ implies $X \in Q^p$.

Usually, we draw Q as a directed graph: Q is the set of vertices, and we draw valued arrows $X \xrightarrow{(d(X,Y),\ d'(X,Y))} Y$ for any $X, Y \in Q$ such that $d(X, Y) \neq 0$, and dotted arrows from X to τ^+X for any $X \in Q - Q^p$.

(2) For a τ-category $\mathcal{C}$, a translation quiver $\mathbb{A}(\mathcal{C}) = (Q, Q^p, Q^i, \tau^+, d, d')$ called the *Auslander–Reiten quiver* of $\mathcal{C}$ is defined by $Q := \mathrm{ind}\,\mathcal{C}$, $Q^p := \mathrm{ind}_1^+\,\mathcal{C}$, $Q^i := \mathrm{ind}_1^-\,\mathcal{C}$, $d(X, Y) := \langle \theta^+Y, X\rangle$ and $d'(X, Y) := \langle \theta^-X, Y\rangle$ ((7) of Section 1).

$\mathbb{A}(\mathcal{C})$ indicates terms of each $(X]$ and $[X)$ $(X \in \mathrm{ind}\,\mathcal{C})$ diagrammatically.

3. Existence Theorem of Ladders

In this section, we show basic results 3.1, 3.2 and 3.3, which are proved in 3.5 and 3.6.

3.1. (1) For a Krull–Schmidt category $\mathcal{C}$, we denote by $\mathcal{C}^\bullet$ (respectively, ${}^\bullet\mathcal{C}$, $\mathcal{C}^\times$) the collection of split monomorphisms (respectively, split epimorphisms, isomorphisms) in $\mathcal{C}$. Then it is easily shown that $\mathcal{C}^{\bullet}+\mathcal{J} \subseteq \mathcal{C}^{\bullet}$, ${}^{\bullet}\mathcal{C}+\mathcal{J} \subseteq {}^{\bullet}\mathcal{C}$ and $\mathcal{C}^{\times}+\mathcal{J} \subseteq \mathcal{C}^{\times}$ hold. We have mutually inverse bijections cok: $\mathcal{C}^{\bullet}/\approx\ \to {}^{\bullet}\mathcal{C}/\approx$ and ker: ${}^{\bullet}\mathcal{C}/\approx\ \to \mathcal{C}^{\bullet}/\approx$.

(2) For a right τ-category $\mathcal{C}$, we denote by $\mathcal{C}^{\bullet}\mu^{+}$ (respectively, $\nu^{+}\,{}^{\bullet}\mathcal{C}$) the collection of a morphism a such that $a = f\mu_X^{+}$ (respectively, $a = \nu_X^{+}f$) for some $X \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathcal{C}^{\bullet}$ (respectively, $f \in {}^{\bullet}\mathcal{C}$).

Similarly, for a left τ-category $\mathcal{C}$, we denote by $\mu^{-}\,{}^{\bullet}\mathcal{C}$ (respectively, $\mathcal{C}^{\bullet}\nu^{-}$) the collection of a morphism a such that $a = \mu_X^{-}f$ (respectively, $a = f\nu_X^{-}$) for some $X \in \mathrm{Ob}(\mathcal{C})$ and $f \in {}^{\bullet}\mathcal{C}$ (respectively, $f \in \mathcal{C}^{\bullet}$).

THEOREM. *Let $\mathcal{C}$ be a right* (*respectively, left*) *τ-category. Putting $l^{+}(f\mu_X^{+}) := \nu_X^{+}\operatorname{cok} f$* (*respectively, $l^{-}(\mu_X^{-}f) := (\ker f)\nu_X^{-}$*) *for any $X \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathcal{C}^{\bullet}$* (*respectively, $f \in {}^{\bullet}\mathcal{C}$*), *we obtain a well defined surjection $l^{+}\colon \mathcal{C}^{\bullet}\mu^{+}/\approx\ \to \nu^{+}\,{}^{\bullet}\mathcal{C}/\approx$* (*respectively, $l^{-}\colon \mu^{-}\,{}^{\bullet}\mathcal{C}/\approx\ \to \mathcal{C}^{\bullet}\nu^{-}/\approx$*), *which preserves direct sums.*

3.2. Let $\mathcal{C}$ be a right τ-category and $a_0 \in \mathcal{J}(X,Y)$. We say that a_0 *has a right ladder* $(a_n)_{0\leqslant n}$ if there exists $b_n \in \mathcal{C}^{\bullet}\mu^{+}$ such that $a_n \approx \binom{b_n}{0}$ and $l^{+}(b_n) = a_{n+1}$ for any $n \geqslant 0$. In other words, there exist a commutative diagram

$$\begin{array}{ccccccccc} Y_0 & \xleftarrow{f_1} & Y_1 & \xleftarrow{f_2} & Y_2 & \xleftarrow{f_3} & Y_3 & \xleftarrow{f_4} & \cdots \\ \uparrow{\scriptstyle b_0} & & \uparrow{\scriptstyle b_1} & & \uparrow{\scriptstyle b_2} & & \uparrow{\scriptstyle b_3} & & \cdots \\ Z_0 & \xleftarrow{g_1} & Z_1 & \xleftarrow{g_2} & Z_2 & \xleftarrow{g_3} & Z_3 & \xleftarrow{g_4} & \cdots, \end{array}$$

$U_{n+1} \in \mathrm{Ob}(\mathcal{C})$ and $h_{n+1} \in \mathcal{C}(U_{n+1}, Z_n)$ such that $a_0 \approx \binom{b_0}{0} \in \mathcal{C}(Z_0 \oplus U_0, Y_0)$ and

$$(Y_n] \approx (Z_{n+1} \oplus U_{n+1} \xrightarrow{\begin{pmatrix} b_{n+1} & -g_{n+1} \\ 0 & h_{n+1}\end{pmatrix}} Y_{n+1} \oplus Z_n \xrightarrow{\begin{pmatrix} f_{n+1} \\ b_n\end{pmatrix}} Y_n) \text{ for any } n \geqslant 0.$$

COROLLARY. *Each of a_n, b_n and U_n above is uniquely determined up to isomorphism.*

Put $l_n^{+}(a_0) := a_n$, $l_n^{+,e}(a_0) := b_n$ and $u_n^{+}(a_0) := U_n$. We call $(a_n)_{0\leqslant n\leqslant m}$ $(m \geqslant 0)$ a *right ladder of length m*. Dually, define $l_n^{-}(a_0)$, $l_n^{-,e}(a_0)$, $u_n^{-}(a_0)$ and a *left ladder* of a_0.

3.3. We call $a \in \mathcal{J}(X,Y)$ *special* if $a + f \approx a$ holds for any $f \in \mathcal{J}^2(X,Y)$.

THEOREM. *Let $\mathcal{C}$ be a right τ-category and $a_0 \in \mathcal{J}(X,Y)$.*

(1) *If a_0 is special, then a_0 has a right ladder.*
(2) *In particular, a_0 has a right ladder if one of $X = 0$, $a_0 = \nu_Z^{+}$ or* ($\mathcal{C}$ *is a τ-category and $a_0 = \mu_Z^{-}$*) ($Z \in \mathrm{Ob}(\mathcal{C})$) *holds.*
(3) *If $\mathcal{C}$ is right strict and a_0 is a special monomorphism, then $l_n^{+}(a_0) \approx l_n^{+,e}(a_0)$ is a monomorphism for any $n \geqslant 0$.*

3.4. Let $a, b \in \mathcal{J}$. We write $a \overset{l}{>} b$ (respectively, $a \overset{l}{\gg} b$, $b \overset{r}{>} a$, $b \overset{r}{\gg} a$) if $sa = bt$ holds for some s and t which satisfy $s \in \mathcal{C}^{\bullet}$ (respectively, $s \in \mathcal{C}^{\bullet}$ and $t \in \mathcal{C}^{\bullet}$, $t \in {}^{\bullet}\mathcal{C}$, $s \in {}^{\bullet}\mathcal{C}$ and $t \in {}^{\bullet}\mathcal{C}$).

$$\begin{array}{ccc} X & \xrightarrow{a} & Y \\ {\scriptstyle s}\uparrow & & \uparrow{\scriptstyle t} \\ X' & \xrightarrow{b} & Y' \end{array}$$

Immediately, $a \overset{l}{>} b$ (respectively, $b \overset{r}{>} a$) holds if and only if there exists an epimorphism $\operatorname{Cok} \mathrm{H}^a \to \operatorname{Cok} \mathrm{H}^b$ (respectively, $\operatorname{Cok} \mathrm{H}_b \to \operatorname{Cok} \mathrm{H}_a$).

3.5. *Proof of 3.1 and 3.2.* Assume that $\mathcal{C}$ is a right τ-category.

3.5.1. DEFINITION. Let a and a' be morphisms in a right (respectively, left) τ-category $\mathcal{C}$. We write $a \overset{+}{\succ} a'$ (respectively, $a' \overset{-}{\succ} a$) if there exist f and f' such that $(X' \xrightarrow{(a'\ f')} Y' \oplus X \xrightarrow{\binom{f}{a}} Y) \approx (Y]$ (respectively, $\approx [X')$).

3.5.2. (1) Assume that ϕ: $\mathbf{A} \to (Y]$ is the chain morphism below satisfying $f_3 \in {}^{\bullet}\mathcal{C}$. If either (i) $\mathbf{A} = (X_3]$ or (ii) ($\mathcal{C}$ is a τ-category and $\mathbf{A} = [X_1)$) hold, then f_1 and f_2 are in ${}^{\bullet}\mathcal{C}$.

$$\begin{array}{ccccc} \mathbf{A}: X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 \\ \downarrow^{\phi}\ \downarrow^{f_1} & & \downarrow^{f_2} & & \downarrow^{f_3} \\ (Y]: \tau^+ Y & \longrightarrow & \theta^+ Y & \longrightarrow & Y \end{array}$$

(2) Assume $b \overset{r}{>} a \overset{+}{\succ} a'$. If either (i) $b \overset{+}{\succ} b'$ or (ii) ($\mathcal{C}$ is a τ-category and $b' \overset{-}{\succ} b$) hold, then $b' \overset{r}{\gg} a'$.

(3) If $\binom{b}{0} \approx \binom{b'}{0}$ holds for $b, b' \in \mathcal{C}^{\bullet}\mu^+$, then $b \approx b'$ holds.

Proof. (1)(i) $(X_3]$ and $(Y]$ induce minimal projective resolutions of S_{X_3} and S_Y respectively, and ϕ induces a split epimorphism $\mathrm{S}_{X_3} \to \mathrm{S}_Y$. Hence, H_{f_2} and H_{f_3} are epimorphisms.

(ii) By 2.3, there exists $I \in \mathbb{N}\operatorname{ind}_1^- \mathcal{C}$ (Section 1(7)) such that $[X_1) \approx [I) \oplus (X_3]$. Put $\phi \approx \binom{\phi'}{\phi''} : [I) \oplus (X_3] \to (Y]$. Since ϕ'' satisfies the assumption of (i), we obtain the assertion.

(2) Put $sa = bt$ ($t \in {}^{\bullet}\mathcal{C}$), and $(X' \xrightarrow{(a'\ f')} Y' \oplus X \xrightarrow{\binom{f}{a}} Y) \approx (Y]$. Moreover, put $\mathbf{A} = (Z' \xrightarrow{(b'\ g')} W' \oplus Z \xrightarrow{\binom{g}{b}} W)$, where $\mathbf{A} \approx (W]$ for (i) and $\mathbf{A} \approx [Z')$ for (ii). Since $gt \in \mathcal{J}$, there exists $(t'\ u)$ such that $(t'\ u)\binom{f}{a} = gt$. Then $\begin{pmatrix} t' & u \\ 0 & s \end{pmatrix}\binom{f}{a} = \binom{g}{b}t$ shows that there exists s' such that $s'(a'\ f') = (b'\ g')\begin{pmatrix} t' & u \\ 0 & s \end{pmatrix}$. Thus we obtain a chain

morphism $\mathbf{A} \to (Y]$. Since $t \in {}^{\bullet}\mathcal{C}$, (1) shows that s', $\binom{t'\ u}{0\ s} \in {}^{\bullet}\mathcal{C}$. Thus $t' \in {}^{\bullet}\mathcal{C}$. Hence $s'a' = b't'$ shows $b' \overset{r}{\gg} a'$.

(3) Since $b, b' \in \mathcal{C}^{\bullet}\,\mu^+$, H_b (respectively, $\mathrm{H}_{b'}$) gives a minimal projective resolution of $\mathrm{Cok}\,\mathrm{H}_b$ (respectively, $\mathrm{Cok}\,\mathrm{H}_{b'}$). Since $\mathrm{Cok}\,\mathrm{H}_{\binom{b}{0}} = \mathrm{Cok}\,\mathrm{H}_b$ is isomorphic to $\mathrm{Cok}\,\mathrm{H}_{\binom{b'}{0}} = \mathrm{Cok}\,\mathrm{H}_{b'}$, we obtain $b \approx b'$. □

3.5.3. We will show 3.1. Assume $a \approx b$, $a \overset{+}{\succ} a'$ and $b \overset{+}{\succ} b'$. Since $b \overset{r}{\gg} a$ and $a \overset{r}{\gg} b$, we obtain $b' \overset{r}{\gg} a'$ and $a' \overset{r}{\gg} b'$ by 3.5.2(2)(i). Hence $a' \approx b'$ holds. Thus l^+ is well defined. Moreover, l^+ is surjective since $l^+((\ker f)\mu_X^+) = \nu_X^+ f$ holds for any $f \in {}^{\bullet}\mathcal{C}$. Now 3.2 follows immediately from 3.1 and 3.5.2(3). □

3.6. *Proof of 3.3.*

3.6.1. (1) For any special a, there exists a special $b \in \mathcal{C}^{\bullet}\,\mu^+$ such that $a \approx \binom{b}{0}$.

(2) Assume $l^+(a) = a'$ and $n \geqslant 2$.

(i) For any $b' \in a' + \mathcal{J}^n$, there exists $b \in a + \mathcal{J}^n$ such that $l^+(b) = b'$.
(ii) If a is special, then so is a'.
(iii) Assume that $\mathcal{C}$ is right strict. If a is a monomorphism, then so is a'.

Proof. (1) Put $a = f\mu_X^+$. By (6) of Section 1, there exist $g \in \mathcal{C}^{\bullet}$ and $h \in \mathcal{J}$ such that $f \approx \binom{g}{h}$. Then $a \approx \binom{g\mu_X^+}{h\mu_X^+} \approx \binom{g\mu_X^+}{0}$ holds since a is special and $\binom{0}{h\mu_X^+} \in \mathcal{J}^2$. Put $b := g\mu_X^+ \in \mathcal{C}^{\bullet}\,\mu^+$. For any $r \in \mathcal{J}^2$, $\binom{b+r}{0} = \binom{b}{0} + \binom{r}{0} \approx \binom{b}{0}$ holds since a is special. Hence $b + r \approx b$ holds by 3.5.2(3).

(2) Put $(X' \xrightarrow{(a'\ f')} Y' \oplus X \xrightarrow{\binom{f}{a}} Y) \approx (Y]$.

(i) Take $\binom{r}{r'} \in \mathcal{J}^{n-1}$ such that $b' - a' = (a'\ f')\binom{r}{r'}$. Put $\binom{g}{b} := \begin{pmatrix}1+r & 0\\ r' & 1\end{pmatrix}^{-1}\binom{f}{a}$. Then $b \in a + \mathcal{J}^n$ and $l^+(b) = b'$ hold by the following commutative diagram.

$$\begin{array}{ccccc} (Y]: X' & \xrightarrow{(a'\ f')} & Y' \oplus X & \xrightarrow{\binom{f}{a}} & Y \\ \| & & \downarrow{\scriptstyle \begin{pmatrix}1+r & 0\\ r' & 1\end{pmatrix}} & & \| \\ X' & \xrightarrow{(b'\ f')} & Y' \oplus X & \xrightarrow{\binom{g}{b}} & Y \end{array}$$

(ii) For any $b' \in a' + \mathcal{J}^2$, there exists $b \in a + \mathcal{J}^2$ such that $l^+(b) = b'$ by (i). Since a is special, we obtain $b \approx a$. Hence $b' \approx a'$ holds by 3.1.
(iii) Assume $ga' = 0$. Since $gf'a = -ga'f = 0$ implies $gf' = 0$, we obtain $g(a'\ f') = 0$. Since $\mathcal{C}$ is right strict, we obtain $g = 0$. □

3.6.2. We will show 3.3. There exists a unique special morphism $b_0 \in \mathcal{C}^{\bullet}\,\mu^+$ such that $a_0 \approx \binom{b_0}{0}$ by 3.6.1(1). Then $a_1 := l^+(b_0)$ is special by 3.6.1(2)(ii). Thus (1) follows inductively, and (3) follows from 3.6.1(2)(iii). For (2), we will show that

$a_0 := \nu_Z^+$ is special. For any $f \in \mathcal{J}^2$, there exists $f' \in \mathcal{J}$ such that $f = a_0 f'$. Hence $a_0 + f = a_0(1 + f') \approx a_0$ holds by 3.1(1). A similar argument shows that μ_Z^- is special. □

4. Minimal Projective Resolutions

In the rest of this paper, put $\mathcal{J}^n := \mathcal{C}$ for any $n \leqslant 0$ and $\mathcal{J}^\infty := \bigcap_{n\geqslant 0} \mathcal{J}^n$. For a $\mathcal{C}$-module L, put $\mathcal{J}^{(n)} L := \mathcal{J}^n L / \mathcal{J}^{n+1} L$ and $\mathcal{J}^{(n,m)} L := \mathcal{J}^n L / \mathcal{J}^m L$ for any $n \leqslant m$.

4.1. THEOREM. *Let $\mathcal{C}$ be a right τ-category, a_0 a morphism with the right ladder, $l_n^{+,e}(a_0) = b_n$ and $L :=$ Cok H_{a_0}. Then the diagram in* 3.2 *induces the following commutative diagram, where each vertical complex gives a minimal projective resolution of $\mathcal{J}^n L$ and each ψ_n is the natural inclusion. If $\mathcal{C}$ is right strict and a_0 is a monomorphism, then* pd $\mathcal{J}^n L \leqslant 1$ *holds for any $n \geqslant 0$.*

$$\begin{array}{ccccccccc}
L & \xleftarrow{\psi_1} & \mathcal{J} L & \xleftarrow{\psi_2} & \mathcal{J}^2 L & \xleftarrow{\psi_3} & \mathcal{J}^3 L & \xleftarrow{\psi_4} & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \cdots \\
\mathrm{H}_{Y_0} & \xleftarrow{\mathrm{H}_{f_1}} & \mathrm{H}_{Y_1} & \xleftarrow{\mathrm{H}_{f_2}} & \mathrm{H}_{Y_2} & \xleftarrow{\mathrm{H}_{f_3}} & \mathrm{H}_{Y_3} & \xleftarrow{\mathrm{H}_{f_4}} & \cdots \\
\uparrow{\scriptstyle \mathrm{H}_{b_0}} & & \uparrow{\scriptstyle \mathrm{H}_{b_1}} & & \uparrow{\scriptstyle \mathrm{H}_{b_2}} & & \uparrow{\scriptstyle \mathrm{H}_{b_3}} & & \cdots \\
\mathrm{H}_{Z_0} & \xleftarrow{\mathrm{H}_{g_1}} & \mathrm{H}_{Z_1} & \xleftarrow{\mathrm{H}_{g_2}} & \mathrm{H}_{Z_2} & \xleftarrow{\mathrm{H}_{g_3}} & \mathrm{H}_{Z_3} & \xleftarrow{\mathrm{H}_{g_4}} & \cdots
\end{array}$$

Proof. Let $\phi_n \colon \mathrm{H}_{Y_n} \to \mathrm{Cok}\,\mathrm{H}_{b_n}$ be the natural epimorphism and $\psi_n' \colon \mathrm{Cok}\,\mathrm{H}_{b_n} \to \mathrm{Cok}\,\mathrm{H}_{b_{n-1}}$ the morphism induced by f_n. Then $\mathcal{J}\,\mathrm{Cok}\,\mathrm{H}_{b_{n-1}} = (\mathcal{J}\,\mathrm{H}_{Y_{n-1}})\phi_{n-1} = (\mathrm{Im}\,\mathrm{H}_{\binom{f_n}{b_{n-1}}})\phi_{n-1} = (\mathrm{Im}\,\mathrm{H}_{f_n})\phi_{n-1} = \mathrm{Im}\,\psi_n'$ holds. Thus we only have to show that ψ_n' is a monomorphism. Assume $s \in \mathrm{H}_{Y_n}$ satisfies $(s)\phi_n'\psi_n = 0$. Since $(sf_n)\phi_{n-1} = 0$, there exists t such that $sf_n = tb_{n-1}$. Then $(s\ \ -t)\binom{f_n}{b_{n-1}} = 0$ shows there exists $(u\ v) \in \mathrm{H}_{Z_n \oplus U_n}$ such that $(s\ \ -t) = (u\ v)\begin{pmatrix} b_n & -g_n \\ 0 & h_n \end{pmatrix}$. Hence $s = ub_n$ shows $(s)\phi_n = 0$. □

4.2. By 3.3, the following theorem implies that the main theorem 4.3 of [12] holds for any Artin algebra Λ without any restriction. (Put $\mathcal{C} :=$ mod Λ and $a_0 := \nu_X^+$.)

RADICAL LAYERS THEOREM. *Let $\mathcal{C}$ be a right τ-category, a_0 a morphism with the right ladder, $b_n := l_n^{+,e}(a_0)$ and $L :=$* Cok H_{a_0}. *Then we have the following exact sequences for any $i, j, n \geqslant 0$, where $(0 \to)$ is added if $\mathcal{C}$ is right strict and a_0 is a monomorphism.*

$$(0 \to)\ \mathcal{J}^{i-1}\,\mathrm{H}_{Z_n} \xrightarrow{\mathrm{H}_{b_n}} \mathcal{J}^i\,\mathrm{H}_{Y_n} \to \mathcal{J}^{n+i} L \to 0$$

$$(0 \to)\ \mathcal{J}^{(i-1,j-1)}\,\mathrm{H}_{Z_n} \xrightarrow{\mathrm{H}_{b_n}} \mathcal{J}^{(i,j)}\,\mathrm{H}_{Y_n} \to \mathcal{J}^{(n+i,n+j)} L \to 0$$

Proof. We only have to show the exactness of the upper sequence. We will use induction on i. Our assertion for $i = 0$ follows from 4.1. Assume that the assertion is true for i. Put $M := \mathrm{Cok}(\mathcal{J}^i \mathrm{H}_{Z_n} \xrightarrow{\mathrm{H}_{b_n}} \mathcal{J}^{i+1} \mathrm{H}_{Y_n})$. Consider the following commutative diagram of exact sequences, where the lower sequence is exact by our assumption for i.

$$\begin{array}{ccccccc}
\mathcal{J}^i \mathrm{H}_{Z_n} & \xrightarrow{\mathrm{H}_{b_n}} & \mathcal{J}^{i+1} \mathrm{H}_{Y_n} & \longrightarrow & M & \longrightarrow & 0 \\
\uparrow{\scriptstyle \mathrm{H}_{g_{n+1}}} & & \uparrow{\scriptstyle \mathrm{H}_{f_{n+1}}} & & \uparrow{\scriptstyle \psi} & & \\
\mathcal{J}^{i-1} \mathrm{H}_{Z_{n+1}} & \xrightarrow{\mathrm{H}_{b_{n+1}}} & \mathcal{J}^i \mathrm{H}_{Y_{n+1}} & \longrightarrow & \mathcal{J}^{n+i+1} L & \longrightarrow & 0
\end{array}$$

Since $(Z_{n+1} \oplus U_{n+1} \xrightarrow{\left(\begin{smallmatrix} b_{n+1} & -g_{n+1} \\ 0 & h_{n+1} \end{smallmatrix}\right)} Y_{n+1} \oplus Z_n \xrightarrow{\left(\begin{smallmatrix} f_{n+1} \\ b_n \end{smallmatrix}\right)} Y_n) \approx (Y_n]$, we obtain the following exact sequence by applying our assumption for i to $a_0 := \nu^+_{Y_n}$ and $L := \mathcal{J} \mathrm{H}_{Y_n}$.

$$(0 \to)\ \mathcal{J}^{i-1} \mathrm{H}_{Z_{n+1} \oplus U_{n+1}} \xrightarrow{\mathrm{H}_{\left(\begin{smallmatrix} b_{n+1} & -g_{n+1} \\ 0 & h_{n+1} \end{smallmatrix}\right)}} \mathcal{J}^i \mathrm{H}_{Y_{n+1} \oplus Z_n} \xrightarrow{\mathrm{H}_{\left(\begin{smallmatrix} f_{n+1} \\ b_n \end{smallmatrix}\right)}} \mathcal{J}^{i+1} \mathrm{H}_{Y_n} \to 0 \qquad (*)$$

Hence we can easily check that ψ above is an isomorphism. Thus our assertion is true for $i + 1$. The assertion for $(0 \to)$ is immediate by 4.1. □

4.3. PROPOSITION. *In* 4.2*, put* $U_n := u^+_n(a_0)$ *and assume* $\mathcal{J}^N L = 0$ *for some* $N \geqslant 0$*. Then we have the following exact sequence for any* $i, n \geqslant 0$*, where* $k_{l,n} := h_l(g_{l-1} \cdots g_{n+1}) \in \mathcal{C}(U_l, Z_n)$ *and* $(0 \to)$ *is added if* $\mathcal{C}$ *is right strict.*

$$(0 \to)\ \bigoplus_{l=n+1}^{N} \mathcal{J}^{n+i-l-1} \mathrm{H}_{U_l} \xrightarrow{(\mathrm{H}_{k_{l,n}})_l} \mathcal{J}^{i-1} \mathrm{H}_{Z_n} \xrightarrow{\mathrm{H}_{b_n}} \mathcal{J}^i \mathrm{H}_{Y_n} \to \mathcal{J}^{n+i} L \to 0$$

Proof. If $n \geqslant N$, then the assertion is immediate since $Z_n = Y_n = 0$ holds by 4.1. Assume that our assertion is true for i. Consider the following commutative diagram, where the lower sequence is exact from our assumption.

$$\begin{array}{ccccccc}
(0\to) & \bigoplus_{l=n+1}^{N} \mathcal{J}^{n+i-l} \mathrm{H}_{U_l} & \xrightarrow{(\mathrm{H}_{k_{l,n}})_l} & \mathcal{J}^i \mathrm{H}_{Z_n} & \xrightarrow{\mathrm{H}_{b_n}} & \mathcal{J}^{i+1} \mathrm{H}_{Y_n} \\
& \| & & \uparrow{\scriptstyle \mathrm{H}_{\left(\begin{smallmatrix} g_{n+1} \\ h_{n+1} \end{smallmatrix}\right)}} & & \uparrow{\scriptstyle \mathrm{H}_{f_{n+1}}} \\
(0\to) & \left(\bigoplus_{l=n+2}^{N} \mathcal{J}^{n+i-l} \mathrm{H}_{U_l}\right) \oplus \mathcal{J}^{i-1} \mathrm{H}_{U_{n+1}} & \xrightarrow{\left(\begin{smallmatrix} (\mathrm{H}_{k_{l,n+1}})_l & 0 \\ 0 & 1 \end{smallmatrix}\right)} & \mathcal{J}^{i-1} \mathrm{H}_{Z_{n+1} \oplus U_{n+1}} & \xrightarrow{\mathrm{H}_{\left(\begin{smallmatrix} b_{n+1} \\ 0 \end{smallmatrix}\right)}} & \mathcal{J}^i \mathrm{H}_{Y_{n+1}}
\end{array}$$

Using the sequence $(*)$ in the proof of 4.2, we can easily show that the upper sequence is also exact. Thus our assertion is true for $i + 1$. □

5. Completion and the Associated Graded Category

5.1. Let $\mathcal{C}$ be a Krull–Schmidt category.

(1) Define the *associated completely graded category* $\widehat{\mathbb{G}}(\mathcal{C})$ of $\mathcal{C}$ by $\mathrm{Ob}(\widehat{\mathbb{G}}(\mathcal{C})) := \mathrm{Ob}(\mathcal{C})$ and $\widehat{\mathbb{G}}(\mathcal{C})(X,Y) := \prod_{n\geqslant 0}\mathcal{J}^{(n)}(X,Y)$ for $X, Y \in \mathrm{Ob}(\mathcal{C})$, where the composition is given by $(f_n)_{n\geqslant 0}\cdot(g_n)_{n\geqslant 0} := (\sum_{i=0}^{n} f_i g_{n-i})_{n\geqslant 0}$. Define the *completion* $\widehat{\mathcal{C}}$ of $\mathcal{C}$ by $\mathrm{Ob}(\widehat{\mathcal{C}}) := \mathrm{Ob}(\mathcal{C})$ and $\widehat{\mathcal{C}}(X,Y) := \varprojlim_{n\geqslant 0}\mathcal{J}^{(0,n)}(X,Y)$ for $X, Y \in \mathrm{Ob}(\mathcal{C})$.

Then $\widehat{\mathcal{C}}$ and $\widehat{\mathbb{G}}(\mathcal{C})$ are Krull–Schmidt categories with $\mathcal{J}_{\widehat{\mathcal{C}}} = \varprojlim_{n\geqslant 1}\mathcal{J}^{(1,n)}$ and $\mathcal{J}_{\widehat{\mathbb{G}}(\mathcal{C})} = \prod_{n\geqslant 1}\mathcal{J}^{(n)}$. Put $a[1] := (0, a, 0, 0, \ldots) \in \mathcal{J}_{\widehat{\mathbb{G}}(\mathcal{C})}(X,Y)$ for $a \in \mathcal{J}_{\mathcal{C}}(X,Y)$.

(2) $\mathcal{C}$ is called *complete* if the natural functor $\mathcal{C} \to \widehat{\mathcal{C}}$ yields the categorical equivalence, and called *completely graded* if there exists a Krull–Schmidt category $\mathcal{C}'$ such that $\mathcal{C}$ is equivalent to $\widehat{\mathbb{G}}(\mathcal{C}')$.

(3) Assume that $\mathrm{S}_X \in \mathrm{Ob}(\mathrm{Mod}_1\,\mathcal{C})$ holds for any $X \in \mathrm{Ob}(\mathcal{C})$. Then $\mathcal{J}^i_{\widehat{\mathcal{C}}} = \varprojlim_{n\geqslant i}\mathcal{J}^{(i,n)}$ and $\mathcal{J}^i_{\widehat{\mathbb{G}}(\mathcal{C})} = \prod_{n\geqslant i}\mathcal{J}^{(n)}$ hold for any $i \geqslant 0$. Thus $\widehat{\mathcal{C}}$ is equivalent to $\widehat{\widehat{\mathcal{C}}}$, and $\widehat{\mathbb{G}}(\mathcal{C})$ is complete and equivalent to $\widehat{\mathbb{G}}(\widehat{\mathbb{G}}(\mathcal{C}))$.

Proof. We only show the assertions for $\widehat{\mathbb{G}}(\mathcal{C})$.

(1) Fix $X \in \mathrm{ind}\,\mathcal{C}$. We only have to show that $f = (f_n)_{n\geqslant 0} \in \widehat{\mathbb{G}}(\mathcal{C})(X,X)$ is an isomorphism if and only if $f_0 \neq 0$. If $g = (g_n)_{n\geqslant 0}$ satisfies $fg = 1_X$, then $f_0 g_0 = 1$ implies $f_0 \neq 0$. Conversely, If $f_0 \neq 0$, then $g = (g_n)_{n\geqslant 0}$ satisfies $fg = 1_X$, where $g_0 := f_0^{-1}$ and $g_n := -f_0^{-1}\sum_{i=1}^{n} f_i g_{n-i}$ for $n > 0$.

(3) We only have to show $(\prod_{n\geqslant i}\mathcal{J}^{(n)})(\prod_{n\geqslant 1}\mathcal{J}^{(n)}) \supseteq \prod_{n\geqslant i+1}\mathcal{J}^{(n)}$ for any $i \geqslant 1$. For any $f = (f_n)_{n\geqslant i+1} \in \prod_{n\geqslant i+1}\mathcal{J}^{(n)}\,\mathrm{H}_X$, take a projective resolution $\mathrm{H}_Y \xrightarrow{\mathrm{H}_a} \mathcal{J}\,\mathrm{H}_X \to 0$. For any $n \geqslant i$, we can take $g_n \in \mathcal{J}^n$ such that $f_{n+1} = g_n a$. Then $g := (g_n)_{n\geqslant i} \in \prod_{n\geqslant i}\mathcal{J}^{(n)}\,\mathrm{H}_Y$ satisfies $g\cdot a[1] = f$. □

5.2. THEOREM. *Let $\mathcal{C}$ be a right τ-category (respectively, left τ-category, τ-category). Then so are $\widehat{\mathbb{G}}(\mathcal{C})$ and $\widehat{\mathcal{C}}$. If $\mathcal{C}$ is right strict, then so are $\widehat{\mathbb{G}}(\mathcal{C})$ and $\widehat{\mathcal{C}}$. Moreover, $(X]_{\widehat{\mathcal{C}}} \approx (X]_{\mathcal{C}}$ and $(X]_{\widehat{\mathbb{G}}(\mathcal{C})} \approx (\tau^+X \xrightarrow{\nu_X^+[1]} \theta^+X \xrightarrow{\mu_X^+[1]} X)$ for any $X \in \mathrm{Ob}(\mathcal{C})$.*

Proof. Applying 4.2 to $a_0 := \nu_X^+$, we obtain an exact sequence $(0 \to)\mathcal{J}^{(i-1)}\,\mathrm{H}_{\tau^+X} \overset{\mathrm{H}_{\nu_X^+}}{\to} \mathcal{J}^{(i)}\,\mathrm{H}_{\theta^+X} \overset{\mathrm{H}_{\mu_X^+}}{\to} \mathcal{J}^{(i+1)}\,\mathrm{H}_X \to 0$ for any $i \geqslant 0$. Since it is easily shown that $\mathcal{J}^{(i)}\,\mathrm{H}^{\theta^+X} \overset{\mathrm{H}^{\nu_X^+}}{\to} \mathcal{J}^{(i+1)}\,\mathrm{H}^{\tau^+X} \to 0$ is exact, the assertion for $\widehat{\mathbb{G}}(\mathcal{C})$ follows from 5.1(3). Similarly, since $(0 \to)\varprojlim\mathcal{J}^{(0,i)}\,\mathrm{H}_{\tau^+X} \overset{\mathrm{H}_{\nu_X^+}}{\to} \varprojlim\mathcal{J}^{(0,i+1)}\,\mathrm{H}_{\theta^+X} \overset{\mathrm{H}_{\mu_X^+}}{\to} \varprojlim\mathcal{J}^{(0,i+2)}\,\mathrm{H}_X \to 0$ is exact for any $i \geqslant 0$ by 4.2 again, the assertion for $\widehat{\mathcal{C}}$ follows. □

5.3. LEMMA. *Let $\mathcal{C}$ and $\mathcal{C}'$ be complete right τ-categories and $F\colon \mathcal{C} \to \mathcal{C}'$ be an additive functor such that $F(X]_{\mathcal{C}} \approx (FX]_{\mathcal{C}'}$ for any $X \in \mathrm{Ob}(\mathcal{C})$. If F induces an isomorphism $\mathcal{J}^{(i)}_{\mathcal{C}}(X, Y) \to \mathcal{J}^{(i)}_{\mathcal{C}'}(FX, FY)$ for any $X, Y \in \mathrm{Ob}(\mathcal{C})$ and $i = 0, 1$, then F is full faithful.*

Proof. By 4.2, we obtain the following commutative diagram of exact sequences.

$$\begin{array}{ccccccc}
\mathcal{J}^{(i-1)}_{\mathcal{C}}(X, \tau^+Y) & \xrightarrow{\mathrm{H}_{\nu^+_Y}} & \mathcal{J}^{(i)}_{\mathcal{C}}(X, \theta^+Y) & \xrightarrow{\mathrm{H}_{\mu^+_Y}} & \mathcal{J}^{(i+1)}_{\mathcal{C}}(X, Y) & \longrightarrow 0 \\
\downarrow^F & & \downarrow^F & & \downarrow^F & \\
\mathcal{J}^{(i-1)}_{\mathcal{C}'}(FX, F\tau^+Y) & \xrightarrow{\mathrm{H}_{F\nu^+_Y}} & \mathcal{J}^{(i)}_{\mathcal{C}'}(FX, F\theta^+Y) & \xrightarrow{\mathrm{H}_{F\nu^+_Y}} & \mathcal{J}^{(i+1)}_{\mathcal{C}'}(FX, FY) & \longrightarrow 0
\end{array}$$

Inductively, $F\colon \mathcal{J}^{(i)}_{\mathcal{C}}(X, Y) \to \mathcal{J}^{(i)}_{\mathcal{C}'}(FX, FY)$ is an isomorphism for any $i \geqslant 0$. Hence $F\colon \mathcal{J}^{(0,i)}_{\mathcal{C}}(X, Y) \to \mathcal{J}^{(0,i)}_{\mathcal{C}'}(FX, FY)$ is an isomorphism for any $i \geqslant 0$. Thus $F\colon \mathcal{C}(X, Y) = \varprojlim \mathcal{J}^{(0,i)}_{\mathcal{C}}(X, Y) \to \varprojlim \mathcal{J}^{(0,i)}_{\mathcal{C}'}(FX, FY) = \mathcal{C}'(FX, FY)$ is an isomorphism. □

6. Invertible Ladders

6.1. Let $\mathcal{C}$ be a τ-category.

(1) Let a_0 be a morphism with the right ladder, $a_i := l^+_i(a_0) \in \mathcal{C}(X_i, Y_i)$ and $0 \leqslant n \leqslant \infty$. Then $X_i|_{\mathrm{ind}^-_1 \mathcal{C}} = 0$ holds for any $i > 0$ since $X_i \approx \tau^+ Y_{i-1}$.

A right ladder $(a_i)_{0 \leqslant i \leqslant n}$ of length n is called *essential* if $u^+_i(a_0) = 0$ holds for any i $(0 \leqslant i \leqslant n)$. For example, if $\mathcal{C}$ is strict and a_0 is a monomorphism, then $(a_i)_{0 \leqslant i}$ is essential and any a_i is also a monomorphism by 3.3.

A right ladder $(a_i)_{0 \leqslant i \leqslant n}$ of length n $(n < \infty)$ is called *invertible* if $a_{i+1} \in \mu^- {}^\bullet\mathcal{C}$ and $a_i = l^-(a_{i+1})$ hold for any i $(0 \leqslant i < n)$. In this case, $(a_i)_{0 \leqslant i < n}$ is an essential ladder and $(a_{n-i})_{0 \leqslant i \leqslant n}$ gives a left ladder of a_n.

(2) Assume that $\mathrm{H}_X \overset{\mathrm{H}_{a_0}}{\to} \mathrm{H}_Y \to L \to 0$ and $\mathrm{H}^B \overset{\mathrm{H}^{c_0}}{\to} \mathrm{H}^A \to M \to 0$ are minimal projective resolutions of $L \in \mathrm{Ob}(\mathrm{Mod}\,\mathcal{C})$ and $M \in \mathrm{Ob}(\mathrm{Mod}\,\mathcal{C}^{\mathrm{op}})$ respectively. Clearly, L (respectively, M) is indecomposable if and only if a_0 (respectively, c_0) is indecomposable as a complex.

We say that L *has a right ladder* (respectively, M *has a left ladder*) if so does a_0 (respectively, c_0). Then $\mathcal{J}^n L$ (respectively, $\mathcal{J}^n M$) has a right ladder for any $n \geqslant 0$ by 4.1. Moreover, we call (L, M) an *invertible pair* of *distance n* $(n \geqslant 0)$ if a_0 has a right ladder, c_0 has a left ladder and $(l^+_i(a_0))_{0 \leqslant i \leqslant n}$ is an invertible ladder with $l^+_n(a_0) \approx c_0$.

6.2. PROPOSITION. *Let $\mathcal{C}$ be a τ-category, L a $\mathcal{C}$-module with right ladder, M a $\mathcal{C}^{\mathrm{op}}$-module with left ladder, $\mathrm{H}_X \overset{\mathrm{H}_{a_0}}{\to} \mathrm{H}_Y \to L \to 0$ a minimal projective resolution and $a_i := l^+_i(a_0) \in \mathcal{C}(X_i, Y_i)$.*

(1) *If (L, M) is an invertible pair of distance n $(n \geqslant 0)$, then the following hold.*

(i) $\mathcal{J}^{(i)} L = \mathrm{S}_{Y_i}$ *and* $\mathcal{J}^{(n-i)} M = \mathrm{S}^{X_i}$ *hold for any* i $(0 \leqslant i \leqslant n)$. *Hence any composition factor* S *of* $\mathcal{J}^{(0,n)} L$ *satisfies* $\operatorname{pd} S \geqslant 2$, *and any composition factor* T *of* $\mathcal{J}^{(0,n)} M$ *satisfies* $\operatorname{pd} T \geqslant 2$.

(ii) $\operatorname{Tr}(\mathcal{J}^i L) = \mathcal{J}^{n-i} M$ *holds for any* i $(0 \leqslant i < n)$ *and* $\operatorname{Tr}(\mathcal{J}^{n-i} M) = \mathcal{J}^i L$ *holds for any* i $(0 < i \leqslant n)$. *If* L *is indecomposable, then* $\mathcal{J}^i L$ *and* $\mathcal{J}^{n-i} M$ *are indecomposable* (*or zero*) *for any* i $(0 \leqslant i \leqslant n)$.

(iii) *There exists a complex* $(X_n \xrightarrow{(a_n\ e)} Y_n \oplus X_0 \xrightarrow{\binom{f}{a_0}} Y_0)$ *such that* $\mathrm{H}_{X_n} \xrightarrow{\mathrm{H}_{(a_n\ e)}} \mathrm{H}_{Y_n \oplus X_0} \xrightarrow{\mathrm{H}_{\binom{f}{a_0}}} \mathrm{H}_{Y_0} \to \mathcal{J}^{(0,n)} L \to 0$ *and* $\mathrm{H}^{Y_0} \xrightarrow{\mathrm{H}^{\binom{f}{a_0}}} \mathrm{H}^{Y_n \oplus X_0} \xrightarrow{\mathrm{H}^{(a_n\ e)}} \mathrm{H}^{X_n} \to \mathcal{J}^{(0,n)} M \to 0$ *are exact and* $f, g \in \mathcal{J}^n$.

(2) *Assume that L is indecomposable. If* $\operatorname{Tr}(\mathcal{J}^n L) = M \neq 0$ *and any composition factor* S *of* $\mathcal{J}^{(0,n)} L$ *satisfies* $\operatorname{pd} S \geqslant 2$, *then* (L, M) *is an invertible pair of distance* n.

6.2.1. Let $\mathcal{C}$ be a τ-category, a_0 a morphism with right ladder, $a_i := l_i^+(a_0) \in \mathcal{C}(X_i, Y_i)$, $L := \operatorname{Cok} \mathrm{H}_{a_0}$ and $n \geqslant 0$. Then the condition (1) below implies (2) and (3) below. Moreover, assume that a_0 is indecomposable as a complex and $X_n \oplus Y_n \neq 0$. Then the conditions (1)–(3) below are equivalent. In this case, a_i is indecomposable as a complex for any i $(0 \leqslant i \leqslant n)$.

(1) $(a_i)_{0 \leqslant i \leqslant n}$ is an invertible ladder.
(2) $Y_i|_{\operatorname{ind}_1^+ \mathcal{C}} = 0$ holds for any i $(0 \leqslant i < n)$.
(3) Any composition factor S of $\mathcal{J}^{(0,n)} L$ satisfies $\operatorname{pd} S \geqslant 2$.

Proof. Since $\mathcal{J}^{(i)} L = \mathrm{S}_{Y_i}$ holds by 4.1, (2) is equivalent to (3). Thus the former assertion follows from the dual of 6.1(1). We will show the latter assertion inductively. Assume that a_i is indecomposable and $Y_i|_{\operatorname{ind}_1^+ \mathcal{C}} = 0$. Then $a_i \approx b_i$ or $a_i \in \mathcal{C}(U_i, 0)$ holds. The latter case implies $n = i$ since $X_{i+1} \oplus Y_{i+1} = 0$. The former case implies $l^-(a_{i+1}) = a_i$ since $(Y_i] = [X_{i+1})$. Hence a_{i+1} is also indecomposable since l^- preserves direct sums. □

6.2.2. *Proof of 6.2*. (1) $l_i^{+,e}(a_0) = a_i$ holds for any i $(0 \leqslant i < n)$ and $l_{n-i}^{-,e}(c_0) = a_i$ holds for any i $(0 < i \leqslant n)$. Thus (i) and (ii) follow immediately from 4.1 and 6.2.1. We will show (iii).

Without loss of generality, a_0 is indecomposable. We use the notations in 3.2. Notice that $a_n = l_n^{+,e}(a_0)$ or $Y_n = 0$ holds since a_n is indecomposable. Put $e_i := g_i$ if $a_i = l_i^{+,e}(a_0)$, and $e_i := h_i$ if $Y_i = 0$. By 4.1, we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & \operatorname{Ker} \mathrm{H}_{a_{i-1}} & \longrightarrow & \mathrm{H}_{X_{i-1}} & \xrightarrow{\mathrm{H}_{a_{i-1}}} & \mathrm{H}_{Y_{i-1}} & \longrightarrow & \mathcal{J}^{i-1} L & \longrightarrow 0 \\ & \uparrow{\scriptstyle \phi_i} & & \uparrow{\scriptstyle \mathrm{H}_{e_i}} & & \uparrow{\scriptstyle \mathrm{H}_{f_i}} & & \cup & \\ 0 \longrightarrow & \operatorname{Ker} \mathrm{H}_{a_i} & \longrightarrow & \mathrm{H}_{X_i} & \xrightarrow{\mathrm{H}_{a_i}} & \mathrm{H}_{Y_i} & \longrightarrow & \mathcal{J}^i L & \longrightarrow 0. \end{array}$$

Since $(X_i \overset{(a_i\ -e_i)}{\longrightarrow} Y_i \oplus X_{i-1} \overset{\binom{f_i}{a_{i-1}}}{\longrightarrow} Y_{i-1}) \approx (Y_{i-1}]$ for any i $(0 < i \leqslant n)$, we can check that ϕ_i is an epimorphism. Put $f := f_n \cdots f_1 \in \mathcal{J}^n$, $e := -e_n \cdots e_1 \in \mathcal{J}^n$ and $\phi := \phi_n \cdots \phi_1$. Then the commutative diagram below shows the exactness of the former sequence since ϕ is an epimorphism.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ker} \mathrm{H}_{a_0} & \longrightarrow & \mathrm{H}_{X_0} & \xrightarrow{\mathrm{H}_{a_0}} & \mathrm{H}_{Y_0} & \longrightarrow & L & \longrightarrow & 0 \\ & & \uparrow^{\phi} & & \uparrow^{\mathrm{H}_{-e}} & & \uparrow^{\mathrm{H}_f} & & \cup & & \\ 0 & \longrightarrow & \operatorname{Ker} \mathrm{H}_{a_n} & \longrightarrow & \mathrm{H}_{X_n} & \xrightarrow{\mathrm{H}_{a_n}} & \mathrm{H}_{Y_n} & \longrightarrow & \mathcal{J}^n L & \longrightarrow & 0 \end{array}$$

Dually, we can show the exactness of the latter sequence.

(2) Since $\operatorname{Cok} \mathrm{H}^{a_n} = \operatorname{Tr}(\mathcal{J}^n L) = M \neq 0$, $(a_i)_{0 \leqslant i \leqslant n}$ is an invertible ladder and a_n is indecomposable by 6.2.1. Hence H^{a_n} gives a minimal projective resolution of M. Thus the assertion follows. □

6.3. THEOREM. *Let $\mathcal{C}$ be a τ-category, L a $\mathcal{C}$-module with right ladder, M a $\mathcal{C}^{\mathrm{op}}$-module with left ladder, $n \geqslant 0$ and ϕ: $\operatorname{Tr}(\mathcal{J}^n L) \to M$ an epimorphism.*

(1) *Any composition factor S of $\mathcal{J}^{(0,n)} M$ satisfies* $\operatorname{pd} S \geqslant 2$.
(2) *If $L = \operatorname{Tr}(\mathrm{S}^X)$ ($X \in \operatorname{ind} \mathcal{C}$) and one of* (i) *or* (ii) *holds, then ϕ is an isomorphism and (L, M) is an invertible pair of distance n:*

   (i) *$\mathcal{J}^n M \neq 0$;*
   (ii) *$\mathcal{C}$ is strict, $M \neq 0$ and* $\operatorname{pd} M \leqslant 1$.

6.3.1. LEMMA. *Let $\mathcal{C}$ be a τ-category, a_0 a morphism with right ladder, c_0 a morphism with left ladder, $a_i := l_i^+(a_0)$, $b_i := l_i^{+,e}(a_0)$, $c_i := l_i^-(c_0)$ and $d_i := l_i^{-,e}(c_0)$. Assume $a_n \overset{l}{>} c_0$ for some $n \geqslant 0$.*

(1) *$a_{n-i} \overset{l}{\gg} b_{n-i} \overset{l}{\gg} c_i \overset{l}{\gg} d_i$ holds for any i $(0 < i \leqslant n)$.*
(2) *Assume that $a_0 = \mu_X^-$ ($X \in \operatorname{ind} \mathcal{C}$) and one of* (i) *or* (ii) *holds. Then $(a_i)_{0 \leqslant i \leqslant n}$ is an invertible ladder such that $a_n \approx d_0$ and $a_{n-i} \approx b_{n-i} \approx c_i \approx d_i$ for any i $(0 < i \leqslant n)$.*

   (i) *The domain of c_n is nonzero.*
   (ii) *$\mathcal{C}$ is strict and c_0 is an epimorphism with domain $\neq 0$.*

Proof. (1) We use the induction. Since $b_{n-1} \overset{+}{\succ} a_n \overset{l}{>} c_0 \overset{l}{\gg} d_0 \overset{-}{\succ} c_1$, we obtain $b_{n-1} \overset{l}{\gg} c_1$ by the dual of 3.5.2(2)(ii). Thus $a_{n-1} \overset{l}{\gg} b_{n-1} \overset{l}{\gg} c_1 \overset{l}{\gg} d_1$ holds.

(2) Put $c_i \in \mathcal{C}(A_i, B_i)$.

(i) Since $a_0 \overset{l}{>} d_n$ holds by (1), we have an epimorphism ϕ: $\operatorname{Cok} \mathrm{H}^{a_0} \to \operatorname{Cok} \mathrm{H}^{d_n}$ by 3.4. Our assumption $a_0 = \mu_X^-$ with $X \in \operatorname{ind} \mathcal{C}$ imply that $\operatorname{Cok} \mathrm{H}^{a_0}$ is simple. Our assumption $A_n \neq 0$ implies $\operatorname{Cok} \mathrm{H}^{d_n} \neq 0$. Hence ϕ is an isomorphism. Thus $d_n \approx a_0$ holds since H^{d_n} gives a minimal projective resolution of $\operatorname{Cok} \mathrm{H}^{d_n}$. Hence we may assume $n > 0$. Then $a_0 \approx b_0 \approx c_n \approx d_n$ holds by (1). The dual of (1) shows that $c_{n-i} \overset{r}{\gg} d_{n-i} \overset{r}{\gg} a_i \overset{r}{\gg} b_i$ holds for any i $(0 < i \leqslant n)$. Thus $a_{n-i} \approx b_{n-i} \approx c_i \approx d_i$ holds for any i $(0 < i \leqslant n)$. Moreover, $l^-(d_0) = c_1$ and $A_0|_{\mathrm{ind}_1^- \mathcal{C}} = 0$ $(n > 0)$ imply $l^+(c_1) = d_0$. Thus $a_n = l^+(b_{n-1}) \approx l^+(c_1) = d_0$ holds.

(ii) By the dual of 6.1(1), $(c_i)_{0\leqslant i\leqslant n}$ is an essential ladder and each c_i is an epimorphism. By (i), we only have to show $A_n \neq 0$. Take a minimal i $(0 \leqslant i \leqslant n)$ such that $A_i = 0$. Then $B_i = 0$ holds since c_i is an epimorphism. Thus $i > 0$ holds since $A_0 \neq 0$. By $B_i = \tau^- A_{i-1}$ and $A_{i-1}|_{\mathrm{ind}_1^- \mathcal{C}} = 0$ by (1), we obtain $A_{i-1} = 0$, a contradiction. □

6.3.2. *Proof of 6.3*. Take minimal projective resolutions $\mathrm{H}_X \overset{\mathrm{H}_{a_0}}{\to} \mathrm{H}_Y \to L \to 0$ and $\mathrm{H}^B \overset{\mathrm{H}^{c_0}}{\to} \mathrm{H}^A \to M \to 0$, and use notations in 6.3.1. Then $b_n \overset{l}{>} c_0$ holds by 4.1 and 3.4. Thus 6.3 follows immediately from 6.3.1. □

6.4. The following theorem follows from 6.4.1(1).

THEOREM. *Let $\mathcal{C}$ be a τ-category satisfying $\mathcal{J}^\infty = 0$. Assume that $X \in \mathrm{ind}\,\mathcal{C} - \mathrm{ind}_0^- \mathcal{C}$ satisfies $\alpha(\mathrm{S}^X) \neq 0$. Then there exists $U \in \mathrm{ind}\,\mathcal{C}$ such that* $(\mathrm{Tr}(\mathrm{S}^X), \mathrm{H}^U)$ *is an invertible pair. Hence* $\mathrm{Tr}(\mathrm{S}^X)$ *has finite length and any composition factor S of* $\mathrm{Tr}(\mathrm{S}^X)$ *satisfies* $\mathrm{pd}\, S \geqslant 2$.

6.4.1. LEMMA. *Let $\mathcal{C}$ be a τ-category, $X \in \mathrm{ind}\,\mathcal{C}$, $a_i := l_i^+(\mu_X^-)$ and $U_i := u_i^+(\mu_X^-)$.*

(1) *If* $\mathrm{Ker}\,\mathrm{H}_{a_0} \not\subseteq \mathcal{J}^\infty \mathrm{H}_X$*, then there exists $n \geqslant 0$ such that $\mathcal{J}^n \mathrm{H}^{U_n} \neq 0$ and $(a_i)_{0\leqslant i\leqslant n}$ is an invertible ladder with $a_n \in \mathcal{C}(U_n, 0)$ and $U_n \in \mathrm{ind}\,\mathcal{C}$.*

(2) *Assume that $\mathcal{C}$ is strict and $\mathcal{J}^\infty = 0$. Then μ_X^- is a monomorphism if and only if $(a_i)_{0\leqslant i}$ is essential.*

Proof. (1) (i) Let a_0 be a morphism with right ladder and $U_i := u_i^+(a_0)$. We will show $\mathrm{Ker}\,\mathrm{H}_{a_0} \subseteq \mathcal{J}^\infty \mathrm{H}_X$ if $\mathcal{J}^i \mathrm{H}^{U_i} = 0$ holds for any i.

Put $b_i := l_i^{+,e}(a_0)$, then $(Y_{i-1}] \approx (Z_i \oplus U_i \overset{\left(\begin{smallmatrix} b_i & -g_i \\ 0 & h_i \end{smallmatrix}\right)}{\longrightarrow} Y_i \oplus Z_{i-1} \overset{\left(\begin{smallmatrix} f_i \\ b_{i-1} \end{smallmatrix}\right)}{\longrightarrow} Y_{i-1})$ holds for any $i \geqslant 1$ by 3.2. Assume $s_0 a_0 = 0$. Inductively, we will show that there exists s_{i-1} such that $s_{i-1} b_{i-1} = 0$ and $s_0 = s_{i-1}(g_{i-1} \cdots g_1) \in \mathcal{J}^{i-1}$ for any i.

$(0\ s_{i-1})\left(\begin{smallmatrix} f_i \\ b_{i-1} \end{smallmatrix}\right) = 0$ implies that there exists $(-s_i\ t_i)$ such that $(0\ s_{i-1}) = (-s_i\ t_i)\left(\begin{smallmatrix} b_i & -g_i \\ 0 & h_i \end{smallmatrix}\right)$. Hence $s_i b_i = 0$ and $s_{i-1} = s_i g_i + t_i h_i$ hold. Since $h_i(g_{i-1} \cdots g_1) \in \mathcal{J}^i(U_i, X) = 0$, we obtain $s_0 = s_i(g_i \cdots g_1)$.

(ii) We will show the assertion. There exists $n \geqslant 0$ such that $\mathcal{J}^n \mathrm{H}^{U_n} \neq 0$ by (i). Put $c_0 := 0 \in \mathcal{C}(U_n, 0)$ and $c_i := l_i^-(c_0)$. Then $a_n \overset{l}{\gg} c_0$ holds. By $\mathcal{J}^n \mathrm{H}^{U_n} \neq 0$ and the dual of 4.1, the domain of c_n is nonzero. Hence $(a_i)_{0\leqslant i\leqslant n}$ is an invertible ladder with $a_n \approx c_0$ by 6.3.1(2)(i). By 6.2.1, $U_n \in \mathrm{ind}\,\mathcal{C}$ holds.

(2) The 'if' part follows from (1), and the 'only if' part follows from 6.1(1). □

7. Recursion Formula and Some Properties of τ-categories

7.1. THEOREM. *Let $\mathcal{C}$ be a τ-category and a_0 a morphism with right ladder, $a_n := l_n^+(a_0) \in \mathcal{C}(X_n, Y_n)$, $b_n := l_n^{+,e}(a_0) \in \mathcal{C}(Z_n, Y_n)$ and $U_n := u_n^+(a_0)$. Assume that* $\operatorname{Cok} \mathrm{H}^{a_0}$ *is semisimple. Then the following equations hold, where we use notations in* (7) *of Section* 1:

$$Y_1 = (\theta^+ Y_0 - X_0)_+, \qquad Y_n = (\theta^+ Y_{n-1} - \tau^+ Y_{n-2})_+ \quad (n \geqslant 2),$$
$$X_{n+1} = \tau^+ Y_n, \qquad Z_n = \theta^+ Y_n - Y_{n+1}, \qquad U_n = X_n - Z_n \quad (n \geqslant 0).$$

In particular, we can compute the terms X_n, Y_n, Z_n and U_n from (X_0, Y_0) and the Auslander–Reiten quiver $\mathbb{A}(\mathcal{C})$. Moreover, $(a_n)_{0 \leqslant n}$ is essential if and only if the following equations hold:

$$Y_1 = \theta^+ Y_0 - X_0, \qquad Y_n = \theta^+ Y_{n-1} - \tau^+ Y_{n-2} \quad (n \geqslant 2).$$

Proof. (i) We will show $\mathcal{J}^{n+1} \mathrm{H}^{U_n} = 0$ and $\langle Y_{n+1}, U_n \rangle = 0$ for any $n \geqslant 0$.

Put $c_0 := 0 \in \mathcal{C}(U_n, 0)$ and use notations in 6.3.1. Since $a_n \overset{l}{\gg} c_0$ holds, we obtain $a_0 \overset{l}{\gg} d_n$ by 6.3.1(1). Thus we have an epimorphism $\operatorname{Cok} \mathrm{H}^{a_0} \to \operatorname{Cok} \mathrm{H}^{d_n}$ by 3.4. Hence $\operatorname{Cok} \mathrm{H}^{d_n}$ is semisimple. Since $\operatorname{Cok} \mathrm{H}^{d_n} = \mathcal{J}^n \mathrm{H}^{U_n}$ holds by the dual of 4.1, we obtain $\mathcal{J}^{n+1} \mathrm{H}^{U_n} = 0$. By 4.1, we have a natural epimorphism $\mathcal{J}^{n+1} \mathrm{H}_{Y_0} \to \mathcal{J}^{n+1} \operatorname{Cok} \mathrm{H}_{a_0} = \operatorname{Cok} \mathrm{H}_{b_{n+1}} \to \mathrm{S}_{Y_{n+1}}$. Hence $\mathcal{J}^{(0)}(U_n, Y_{n+1}) = 0$ holds.

(ii) $X_{n+1} = \tau^+ Y_n$, $Y_{n+1} + Z_n = \theta^+ Y_n$ and $Z_n + U_n = X_n$ hold for any $n \geqslant 0$ by 3.2. Put $W_n := \theta^+ Y_n - X_n \in \mathbb{Z} \operatorname{ind} \mathcal{C}$, then $Y_{n+1} = \theta^+ Y_n - X_n + U_n = (W_n)_+ - (W_n)_- + U_n$. Since $Y_{n+1} \geqslant 0$, we obtain $U_n \geqslant (W_n)_-$. Since $\langle Y_{n+1}, U_n \rangle = 0$ by (i), we obtain $Y_{n+1} = (W_n)_+$. □

7.2. Let $\mathcal{C}$ be a τ-category. For $n \geqslant 0$, define a map $\theta_n^+ \colon \mathbb{N} \operatorname{ind} \mathcal{C} \to \mathbb{N} \operatorname{ind} \mathcal{C}$ by $\theta_0^+ := 1_{\mathbb{N} \operatorname{ind} \mathcal{C}}$, $\theta_1^+ := \theta^+$ and $\theta_n^+ X := (\theta^+ \theta_{n-1}^+ X - \tau^+ \theta_{n-2}^+ X)_+$ for $n \geqslant 2$.

(1) θ_n^+ is a monoid morphism. Thus we can regard θ_n^+ as elements of $\operatorname{End}_{\mathbb{Z}}(\mathbb{Z} \operatorname{ind} \mathcal{C})$.

(2) Put $\tau_n^+ := \theta^+ \circ \theta_n^+ - \theta_{n+1}^+$. Then, for any $X \in \operatorname{Ob}(\mathcal{C})$ and $a_0 \in \mathcal{C}(0, X)$, $l_n^{+,e}(a_0) \in \mathcal{C}(\tau_n^+ X, \theta_n^+ X)$ and $u_n^+(a_0) = \tau^+ \theta_{n-1}^+ X - \tau_n^+ X$ hold. In particular, a minimal projective resolution of $\mathcal{J}^n \mathrm{H}_X$ is given by $\mathrm{H}_{\tau_n^+ X} \to \mathrm{H}_{\theta_n^+ X} \to \mathcal{J}^n \mathrm{H}_X \to 0$ for any $n \geqslant 0$.

Proof. (2) Immediate from 7.1.

(1) Take $X, Y \in \operatorname{Ob}(\mathcal{C})$. Since $\mathcal{J}^n \mathrm{H}_{X \oplus Y} = \mathcal{J}^n \mathrm{H}_X \oplus \mathcal{J}^n \mathrm{H}_Y$ holds, we obtain $\theta_n^+(X \oplus Y) \approx \theta_n^+ X \oplus \theta_n^+ Y$ by (2). □

7.3 (Artinian τ-categories). *Let $\mathcal{C}$ be a τ-category. Then $\mathcal{C}$ is left Artinian* ((2) *of Section* 1) *if and only if, for any $X \in \operatorname{ind} \mathcal{C}$, there exists $n \geqslant 0$ such that $\theta_n^+ X = 0$. In particular, under the assumption* $\operatorname{ind} \mathcal{C} < \infty$*, $\mathcal{C}$ is left Artinian if and only if $\mathcal{J}^n = 0$ for some $n \geqslant 0$ if and only if $\mathcal{C}$ is Artinian.*

Proof. We have a projective cover $\mathrm{H}_{\theta_i^+ X} \to \mathcal{J}^i\,\mathrm{H}_X$ by 7.2(2). Hence $\theta_i^+ X = 0$ is equivalent to $\mathcal{J}^i\,\mathrm{H}_X = 0$. Since $\mathcal{J}^{(i)}\,\mathrm{H}_X$ has finite length, the former assertion follows from the decreasing sequence $\mathrm{H}_X \supset \mathcal{J}\,\mathrm{H}_X \supset \mathcal{J}^2\,\mathrm{H}_X \supset \cdots$ and Nakayama's lemma. The latter assertion is immediate. □

7.4 (Strict τ-categories). *Let $\mathcal{C}$ be a τ-category with $\mathcal{J}^\infty = 0$.*

(1) *$\mathcal{C}$ is strict if and only if $\mathcal{C}$ is right strict if and only if the right ladder $(l_n^+(\nu_X^+))_{0 \leqslant n}$ is essential for any $X \in \operatorname{ind}\mathcal{C}$ if and only if $\tau^+ \circ \theta_{n-1}^+ = \tau_n^+$ holds for any $n \geqslant 1$.*

(2) *If $\mathrm{H}^X|_{\operatorname{ind}_1^- \mathcal{C}} \neq 0$ for any $X \in \operatorname{ind}\mathcal{C}$, then $\mathcal{C}$ is a strict τ-category, and the converse holds if $\mathcal{C}$ is right Artinian.*

Proof. (1) By definition, $\mathcal{C}$ is right strict if and only if ν_X^+ is a monomorphism for any $X \in \operatorname{ind}\mathcal{C}$. Hence the second equivalence follows from 6.1(1) and 6.4.1(1). The third equivalence follows from 7.2(2). We will show that right strictness implies left strictness, namely $\alpha(\mathrm{S}_{\tau^- X}) = 0$ holds for any $X \in \mathrm{Ob}(\mathcal{C})$. Take $\phi \in (\operatorname{Mod}\mathcal{C})(\mathrm{S}_{\tau^- X}, \mathrm{H}_Y)$. Since $\mathcal{J}^\infty = 0$, we only have to show that $\operatorname{Im}\phi \subseteq \mathcal{J}^n\,\mathrm{H}_Y$ holds for any $n \geqslant 0$. This is true for $n = 1$ since $\mathrm{S}_{\tau^- X} = \mathrm{Tr}_1(\mathrm{S}^X)$ does not have a non-zero projective direct summand. Moreover, since $\operatorname{pd}\mathcal{J}^n\,\mathrm{H}_Y \leqslant 1$ holds by 4.1, we obtain $(\operatorname{Mod}\mathcal{C}^{\mathrm{op}})(\mathrm{Tr}_1(\mathcal{J}^n\,\mathrm{H}_Y), \mathrm{S}^X) = 0$. Hence $(\underline{\operatorname{Mod}}\,\mathcal{C})(\mathrm{S}_{\tau^- X}, \mathcal{J}^n\,\mathrm{H}_Y) = 0$ holds by 2.3.1(2). Thus ϕ factors through a projective cover $\mathrm{H}_{\theta_n^+ Y} \to \mathcal{J}^n\,\mathrm{H}_Y \to 0$. Since any $\phi' \in (\operatorname{Mod}\mathcal{C})(\mathrm{S}_{\tau^- X}, \mathrm{H}_{\theta_n^+ Y})$ satisfies $\operatorname{Im}\phi' \subseteq \mathcal{J}\,\mathrm{H}_{\theta_n^+ Y}$ again, we obtain $\operatorname{Im}\phi \subseteq \mathcal{J}^{n+1}\,\mathrm{H}_Y$.

(2) To show the former assertion, fix $X \in \operatorname{ind}\mathcal{C}$ and put $a_0 := 0 \in \mathcal{C}(0, X)$ and $U_n := u_n^+(a_0)$. By (1), we only have to show that $U_n = 0$ holds for any $n \geqslant 0$. Put $c_0 := 0 \in \mathcal{C}(U_n, 0)$ and $d_i := l_i^{-,e}(c_0) \in \mathcal{C}(A_i, C_i)$. Since $a_n \overset{l}{\gg} c_0$ holds, we obtain $a_{n-i} \overset{l}{\gg} d_i$ by 6.3.1(1). Since $a_0 \in \mathcal{C}(0, X)$, we obtain $A_i|_{\operatorname{ind}_1^- \mathcal{C}} = 0$ for any i by 6.1. Since $\mathcal{J}^{(i)}\,\mathrm{H}^{U_n} = \mathrm{S}^{A_i}$ holds by the dual of 4.1, we obtain $\mathrm{H}^{U_n}|_{\operatorname{ind}_1^- \mathcal{C}} = 0$ by $\mathcal{J}^\infty = 0$. Thus $U_n = 0$ holds.

To show the latter assertion, assume that $\mathrm{H}^X|_{\operatorname{ind}_1^- \mathcal{C}} = 0$ holds for some $X \in \operatorname{ind}\mathcal{C}$. Since $\mathcal{C}$ is strict, any composition factor S of H^X satisfies $\alpha(S) = 0$. Since H^X has finite length, we obtain $\mathrm{H}_X = \alpha(\mathrm{H}^X) = 0$, a contradiction. □

PART II. STRUCTURE THEORY OF τ-CATEGORIES

8. τ-Species and Mesh Categories

In this section, first, we review species and their tensor categories. To make tensor categories Krull–Schmidt, we have to use direct product instead of direct sum. We use notations in Section 1(7).

8.1. DEFINITION. $\mathcal{Q} = (Q, D_X, {}_X M_Y)$ is called a *species* if Q is a set, D_X is a skew field for any $X \in Q$ and ${}_X M_Y$ is a (D_X, D_Y)-bimodule for any $X, Y \in Q$.

Put $d(X,Y) := \dim_{D_X} {}_X M_Y$ and $d'(X,Y) := \dim_{D_Y} {}_X M_Y$. A species is called *right finite* (respectively, *left finite*) if $\sum_{X\in Q} d(X,Y) < \infty$ (respectively, $\sum_{X\in Q} d'(Y,X) < \infty$) for any $Y \in Q$.

8.1.1. Let $\mathcal{Q} = (Q, D_X, {}_X M_Y)$ be a species and $X, Y \in Q$. Put $P_0(X,Y) := 0$ if $X \neq Y$, and D_X if $X = Y$. Put

$$P_n(X,Y) := \bigoplus_{Z_1,\dots,Z_{n-1}\in Q} {}_X M_{Z_1} \otimes_{D_{Z_1}} \cdots \otimes_{D_{Z_{n-1}}} {}_{Z_{n-1}} M_Y \quad \text{for } n > 0,$$

$$P_n(A,B) := \prod_{(X,Y)\in Q\times Q} \mathrm{Mat}_{\langle A,X\rangle,\langle B,Y\rangle}(P_n(X,Y)) \quad \text{for } A, B \in \mathbb{N}Q \text{ and } n \geqslant 0.$$

We have a natural map $P_n(X,Y) \times P_m(Y,Z) \to P_{n+m}(X,Z)$, $(f,g) \mapsto fg := f \otimes g$ for any $X, Y, Z \in Q$. Using matrix multiplication, we have a natural map $P_n(A,B) \times P_m(B,C) \to P_{n+m}(A,C)$ for any $A, B, C \in \mathbb{N}Q$.

Define additive categories $\widehat{\mathbb{P}}(\mathcal{Q})$ and $\mathbb{P}(\mathcal{Q})$ by $\mathrm{Ob}(\widehat{\mathbb{P}}(\mathcal{Q})) = \mathrm{Ob}(\mathbb{P}(\mathcal{Q})) := \mathbb{N}Q$, and $\widehat{\mathbb{P}}(\mathcal{Q})(A,B) := \prod_{n\geqslant 0} P_n(A,B)$ and $\mathbb{P}(\mathcal{Q})(A,B) := \bigoplus_{n\geqslant 0} P_n(A,B)$ for $A, B \in \mathbb{N}Q$, where the composition is given by $(f_n)_{n\geqslant 0} \cdot (g_n)_{n\geqslant 0} := (\sum_{i=0}^{n} f_i g_{n-i})_{n\geqslant 0}$.

8.2. PROPOSITION. *Let $\mathcal{Q}$ be a species. Then $\widehat{\mathbb{P}}(\mathcal{Q})$ is a Krull–Schmidt category called the* tensor category *of $\mathcal{Q}$. Moreover, if $\mathcal{Q}$ is left finite, then $\widehat{\mathbb{P}}(\mathcal{Q})$ is a completely graded category with $\mathcal{J}^i_{\widehat{\mathbb{P}}(\mathcal{Q})} = \prod_{n\geqslant i} P_n$ for any $i \geqslant 0$.*

Proof. By a similar argument as in the proof of 5.1(1), $\widehat{\mathbb{P}}(\mathcal{Q})(X,X)$ is a local ring whose maximal ideal is $\prod_{n\geqslant 1} P_n(X,X)$ for any $X \in Q$. Thus the former assertion holds. The latter assertion follows from a similar argument as in the proof of 5.1(3). □

8.3. DEFINITION. For a species $\mathcal{Q} = (Q, D_X, {}_X M_Y)$, we put $D_A := P_0(A,A)$ and ${}_A M_B := P_1(A,B)$ for $A, B \in \mathbb{N}Q$. Then ${}_A M_B$ becomes a (D_A, D_B)-bimodule.

(1) $\mathcal{Q} = (Q, D_X, {}_X M_Y, \tau^+, a, b)$ is called a *right τ-species* if $(Q, D_X, {}_X M_Y)$ is a right finite species, $\tau^+\colon \mathbb{N}Q \to \mathbb{N}Q$ is a monoid morphism, $a_X\colon D_X \to D_{\tau^+X}$ is a unital ring morphism for any $X \in Q$, and $b_{X,Y}\colon \mathrm{Hom}_{D_Y}({}_{\tau^+X}M_Y, D_Y) \to {}_Y M_X$ is a (D_Y, D_X)-monomorphism for any $X, Y \in Q$ where ${}_{\tau^+X}M_Y$ is regarded as a left D_X-module through a_X.

(2) $\mathcal{Q} = (Q, D_X, {}_X M_Y, \tau^-, a, b)$ is called a *left τ-species* if $(Q, D_X, {}_X M_Y)$ is a left finite species, $\tau^-\colon \mathbb{N}Q \to \mathbb{N}Q$ is a monoid morphism, $a_X\colon D_X \to D_{\tau^-X}$ is a unital ring morphism for any $X \in Q$, and $b_{X,Y}\colon \mathrm{Hom}_{D_Y}({}_Y M_{\tau^-X}, D_Y) \to {}_X M_Y$ is a (D_X, D_Y)-monomorphism for any $X, Y \in Q$.

(3) For a right τ-species $\mathcal{Q} = (Q, D_X, {}_X M_Y, \tau^+, a, b)$, put $Q^p := \{X \in Q \mid \tau^+X = 0\}$. Then $\mathcal{Q}$ is called a *τ-species* if it is left finite, τ^+ gives an injection $Q - Q^p \to Q$, and a_X and $b_{X,Y}$ are isomorphisms for any $X \in Q - Q^p$ and $Y \in Q$. In this case, put $Q^i := Q - \tau^+(Q)$, then we have a translation quiver

$|\mathcal{Q}| = (Q, Q^p, Q^i, \tau^+, d, d')$ called the *underlying quiver* of $\mathcal{Q}$, where d and d' are defined by 8.1.

Notice that any τ-species $\mathcal{Q} = (Q, D_X, {}_XM_Y, \tau^+, a, b)$ can be regarded as a left τ-species $(Q, D_X, {}_XM_Y, \tau^-, a', b')$ as follows: Put $\tau^-X := (\tau^+)^{-1}X$ for any $X \in Q - Q^i$ and $\tau^-X := 0$ for any $X \in Q^i$. For any $X \in Q - Q^i$ and $Y \in Q$, put $a'_X := (a_{\tau^-X})^{-1}$ and $b'_{X,Y}\colon \mathrm{Hom}_{D_Y}({}_YM_{\tau^-X}, D_Y) \to {}_XM_Y$ is induced by $b_{\tau^-X,Y}$ naturally.

8.3.1. Since $\mathrm{Hom}_{D_Y}(\mathrm{Hom}_{D_Y}({}_{\tau^+X}M_Y, D_Y), {}_YM_X) = {}_{\tau^+X}M_Y \otimes_{D_Y} {}_YM_X$, we can regard $b_{X,Y}$ as an element of ${}_{\tau^+X}M_Y \otimes_{D_Y} {}_YM_X$. Put $\gamma(X) := \sum_{Y\in Q} b_{X,Y} \in P_2(\tau^+X, X)$. Then $\gamma(X)f = a_X(f)\gamma(X)$ holds for any $f \in D_X$ since $b_{X,Y}$ is a D_X-morphism.

Let I be the ideal of $\mathbb{P}(\mathcal{Q})$ generated by $\{\gamma(X) \mid X \in Q\}$. Then we can write $I = \bigoplus_{n\geqslant 0} I_n$ $(I_n \subseteq P_n)$. Put $\widehat{\mathbb{M}}(\mathcal{Q}) := \widehat{\mathbb{P}}(\mathcal{Q})/\widehat{I}$, where $\widehat{I} := \prod_{n\geqslant 0} I_n$ is an ideal of $\widehat{\mathbb{P}}(\mathcal{Q})$.

8.4. PROPOSITION. *Let $\mathcal{Q}$ be a right τ-species (respectively, left τ-species, τ-species). Then $\widehat{\mathbb{M}}(\mathcal{Q})$ is a completely graded right τ-category (respectively, left τ-category, τ-category) called the* mesh category *of $\mathcal{Q}$.*

8.4.1. Fix $X, Y \in Q$. Put ${}_{\tau^+X}M_Y = \bigoplus_{1\leqslant i\leqslant e(Y,X)} v_Y^i D_Y$ as a right D_Y-module. Then we can put ${}_YM_X = \bigoplus_{1\leqslant i\leqslant d(Y,X)} D_Y u_Y^i$ as a left D_Y-module, where $u_Y^i \in \mathrm{Im}\, b_{X,Y}$ and $(b_{X,Y}^{-1}(u_Y^i))(v_Y^j) = \delta_{ij} 1_{D_Y}$ hold for any i and j $(1 \leqslant i, j \leqslant e(Y, X))$.

Put $\theta^+X := \sum_{Y\in Q} d(Y,X)Y \in \mathrm{Ob}(\widehat{\mathbb{P}}(\mathcal{Q}))$, $\mu_X^+ := (u_Y^i)_{Y\in Q, 1\leqslant i\leqslant d(Y,X)} \in P_1(\theta^+X, X)$ and $\nu_X^+ := (v_Y^i)_{Y\in Q, 1\leqslant i\leqslant d(Y,X)} \in P_1(\tau^+X, \theta^+X)$, where $v_Y^i := 0 \in {}_{\tau^+X}M_Y$ for any i with $e(X,Y) < i \leqslant d(X,Y)$. Then $b_{X,Y} = \sum_{1\leqslant i\leqslant d(X,Y)} v_Y^i \otimes u_Y^i$ and $\gamma(X) = \nu_X^+\mu_X^+$ hold in $P_2(\tau^+X, X)$.

Proof. By definition, $b_{X,Y}(f) = \sum_{1\leqslant i\leqslant d(Y,X)} f(v_Y^i)u_Y^i \in {}_YM_X$ holds for any $f \in \mathrm{Hom}_{D_Y}({}_{\tau^+X}M_Y, D_Y)$. This means $b_{X,Y} = \sum_{1\leqslant i\leqslant d(Y,X)} v_Y^i \otimes u_Y^i$. □

8.4.2. Put $\mathcal{P} := \widehat{\mathbb{P}}(\mathcal{Q})$ and $\mathcal{M} := \widehat{\mathbb{M}}(\mathcal{Q})$. Fix $X \in Q$.

(1) $0 \to \mathrm{H}^{\mathcal{P}}_{\theta^+X} \xrightarrow{\mathrm{H}_{\mu_X^+}} \mathcal{J}\,\mathrm{H}^{\mathcal{P}}_X \to 0$ and $\mathrm{H}_{\mathcal{P}}^{\theta^+X} \xrightarrow{\mathrm{H}^{\nu_X^+}} \mathcal{J}\,\mathrm{H}_{\mathcal{P}}^{\tau^+X} \to 0$ are exact.

(2) $I_0 = I_1 = 0$ and $I_{n+2}(\ , X) = P_n(\ , \tau^+X)\gamma(X) + I_{n+1}(\ , \theta^+X)\mu_X^+$ holds for $n \geqslant 0$. Hence $\widehat{I}(\ , X) = (\mathrm{H}^{\mathcal{P}}_{\tau^+X}\, \nu_X^+ + \widehat{I}(\ , \theta^+X))\mu_X^+$.

(3) $\mathrm{H}^{\mathcal{M}}_{\tau^+X} \xrightarrow{\mathrm{H}_{\nu_X^+}} \mathrm{H}^{\mathcal{M}}_{\theta^+X} \xrightarrow{\mathrm{H}_{\mu_X^+}} \mathcal{J}\,\mathrm{H}^{\mathcal{M}}_X \to 0$ and $\mathrm{H}_{\mathcal{M}}^{\theta^+X} \xrightarrow{\mathrm{H}^{\nu_X^+}} \mathcal{J}\,\mathrm{H}_{\mathcal{M}}^{\tau^+X} \to 0$ are exact.

Proof. (1) $P_n(\ , \theta^+X) \xrightarrow{\cdot\mu_X^+} P_{n+1}(\ , X)$ is a bijection for any $n \geqslant 0$ since $P_{n+1}(\ , X) = \bigoplus_{Y\in Q} P_n(\ , Y) \otimes_{D_Y} {}_YM_X = \bigoplus_Y P_n(\ , Y) \otimes_{D_Y} (\bigoplus_{1\leqslant i\leqslant d(Y,X)} D_Y u_Y^i) = \bigoplus_{Y,i} P_n(\ , Y)u_Y^i$. Thus the former sequence is exact. A similar argument shows the exactness of the latter sequence.

(2) $I_0 = I_1 = 0$ and $I_{n+2}(\ , X) = \sum_{Y\in Q, i\geqslant 0} P_{n-i}(\ , \tau^+Y)\gamma(Y)P_i(Y, X)$ hold by definition. $P_n(\ , \tau^+X)\gamma(X)D_X = P_n(\ , \tau^+X)\gamma(X)$ holds by 8.3.1. $P_{n-i}(\ , \tau^+Y) \cdot\gamma(Y)P_i(Y, X) = P_{n-i}(\ , \tau^+Y)\gamma(Y)P_{i-1}(Y, \theta^+X)\mu_X^+ \subseteq I_{n+1}(\ , \theta^+X)\mu_X^+$ holds for any $i > 0$ by (1).

(3) By (1), we only have to show $\operatorname{Ker} \mathrm{H}^{\mathcal{M}}_{\mu_X^+} = \operatorname{Im} \mathrm{H}^{\mathcal{M}}_{\nu_X^+}$. If $\overline{f} \in \operatorname{Ker} \mathrm{H}^{\mathcal{M}}_{\mu_X^+}$, then $f\mu_X^+ \in \widehat{I}(\ , X) = (\mathrm{H}^{\mathcal{P}}_{\tau^+X}\, \nu_X^+ + \widehat{I}(\ , \theta^+X))\mu_X^+$ holds by (2). Hence $f \in \mathrm{H}^{\mathcal{P}}_{\tau^+X}\, \nu_X^+ + \widehat{I}(\ , \theta^+X)$ by (1). Thus $\overline{f} \in \operatorname{Im} \mathrm{H}^{\mathcal{M}}_{\nu_X^+}$. □

8.4.3. *Proof of 8.4.* $\mathcal{M}$ is a right τ-category by 8.4.2(3). Moreover, $\mathcal{J}^i_{\mathcal{M}} = \prod_{n\geqslant i} P_n/I_n$ holds for $i \geqslant 0$ since $\mathcal{J}^i_{\mathcal{P}} = \prod_{n\geqslant i} P_n$ holds for $i \geqslant 0$ by 8.2. Hence $\mathcal{M}$ is completely graded. □

9. Auslander–Reiten Species of τ-Categories

9.1. DEFINITION. Let $\mathcal{C}$ be a right τ-category. We will define a right τ-species $\widehat{\mathbb{A}}(\mathcal{C})$ called the *Auslander–Reiten species* of $\mathcal{C}$. Then, it is easily shown that $|\widehat{\mathbb{A}}(\mathcal{C})| = \mathbb{A}(\mathcal{C})$ holds.

Put $\mathcal{Q} = (Q, D_X, {}_XM_Y, \tau^+, a, b)$, where $Q := \operatorname{ind} \mathcal{C}$, $D_X := \mathcal{J}^{(0)}(X, X)$, ${}_YM_X := \mathcal{J}^{(1)}(Y, X)$, and a_X and $b_{X,Y}$ are defined by

(1) For any $f \in \mathcal{C}(X, X)$, take $f' \in \mathcal{C}(\theta^+X, \theta^+X)$ and $f'' \in \mathcal{C}(\tau^+X, \tau^+X)$ such that $\mu_X^+ f = f'\mu_X^+$ and $\nu_X^+ f' = f''\nu_X^+$. Put $a_X(\overline{f}) := \overline{f''}$.
(2) Let $b_{X,Y}$ be the following composition, where $(\ )^* := \operatorname{Hom}_{D_Y}(\ , D_Y)$ and the isomorphism $i_{X,Y} : \mathcal{J}^{(0)}(Y, \theta^+X) \to \mathcal{J}^{(0)}(\theta^+X, Y)^*$ is defined by $(i_{X,Y}(f))(g) := fg$.

$$({}_{\tau^+X}M_Y)^* \xrightarrow{(\mathrm{H}^{\nu_X^+})^*} \mathcal{J}^{(0)}(\theta^+X, Y)^* \xrightarrow{i_{X,Y}^{-1}} \mathcal{J}^{(0)}(Y, \theta^+X) \xrightarrow{\mathrm{H}_{\mu_X^+}} {}_YM_X.$$

9.2. THEOREM. (1) *If $\mathcal{C}$ is a right τ-category, then $\widehat{\mathbb{G}}(\mathcal{C})$ is equivalent to $\widehat{\mathbb{M}}(\widehat{\mathbb{A}}(\mathcal{C}))$.*

(2) *An additive category $\mathcal{C}$ is a completely graded right τ-category* (*respectively, left τ-category, τ-category*) *if and only if $\mathcal{C}$ is equivalent to $\widehat{\mathbb{M}}(\mathcal{Q})$ for some right τ-species* (*respectively, left τ-species, τ-species*) $\mathcal{Q}$.

9.2.1. For any $X \in \operatorname{ind} \mathcal{C}$, write $\theta^+X \approx \bigoplus_{Y\in Q} Y^{d(Y,X)}$, $\mu_X^+ \approx (s_Y^i)_{Y\in Q, 1\leqslant i\leqslant d(Y,X)}$ and $\nu_X^+ \approx (t_Y^i)_{Y\in Q, 1\leqslant i\leqslant d(Y,X)}$. Then $b_{X,Y} = \sum_{1\leqslant i\leqslant d(Y,X)} \overline{t_Y^i} \otimes \overline{s_Y^i}$ holds in $\widehat{\mathbb{P}}(\widehat{\mathbb{A}}(\mathcal{C}))$.

Proof. Take $\phi \in ({}_{\tau^+X}M_Y)^*$ and put $d := d(Y, X)$. We will compute $b_{X,Y}(\phi)$ from the definition 9.1(2). First, ϕ is mapped to $((f_i)_{1\leqslant i\leqslant d} \mapsto \sum_{1\leqslant i\leqslant d} \phi(\overline{t_Y^i}) f_i) \in \mathcal{J}^{(0)}(\theta^+X, Y)^*$. Then it is mapped to $(\phi(\overline{t_Y^i}))_{1\leqslant i\leqslant d} \in \mathcal{J}^{(0)}(Y, \theta^+X)$. Hence $b_{X,Y}(\phi) = \sum_{1\leqslant i\leqslant d} \phi(\overline{t_Y^i})\overline{s_Y^i}$ holds. Thus the assertion follows. □

9.2.2. *Proof of 9.2.* By 5.1(3), $\mathcal{C}$ is completely graded if and only if $\mathcal{C}$ is equivalent to $\widehat{\mathbb{G}}(\mathcal{C})$. By 8.4, we only have to show (1). Let $\widehat{\mathbb{P}}(\mathcal{Q}) = \prod_{n\geqslant 0} P_n$ be the tensor category of $\mathcal{Q} := \widehat{\mathbb{A}}(\mathcal{C})$. Then we have natural identifications $P_0(X, X) = D_X = \mathcal{J}_{\mathcal{C}}^{(0)}(X, X)$ and $P_1(X, Y) = {}_X M_Y = \mathcal{J}_{\mathcal{C}}^{(1)}(X, Y)$ for any $X, Y \in \operatorname{ind}\mathcal{C}$. By our definition 8.1.1, $P_n(X, Y) \to \mathcal{J}_{\mathcal{C}}^{(n)}(X, Y)$ is induced for any $X, Y \in \operatorname{ind}\mathcal{C}$ and $n \geqslant 2$. Thus we obtain a dense functor $F\colon \widehat{\mathbb{P}}(\mathcal{Q}) \to \widehat{\mathbb{G}}(\mathcal{C})$. By 9.2.1, we obtain $F(\gamma(X)) = F(\overline{\nu_X^+})F(\overline{\mu_X^+}) = \nu_X^+[1] \cdot \mu_X^+[1] = 0$. Thus F induces a dense functor $F'\colon \widehat{\mathbb{M}}(\mathcal{Q}) \to \widehat{\mathbb{G}}(\mathcal{C})$. Since F' induces an isomorphism $\mathcal{J}^{(i)}_{\widehat{\mathbb{M}}(\mathcal{Q})} = P_i \to \mathcal{J}^{(i)}_{\mathcal{C}} = \mathcal{J}^{(i)}_{\widehat{\mathbb{G}}(\mathcal{C})}$ for $i = 0, 1$ by 8.4.2(2), F' is full faithful by 5.3. $\square$

10. The Category of τ-Categories and the Category of τ-Species

10.1. DEFINITION. (1) Denote by $\mathcal{T}^r_{\text{ca}}$ (respectively, $\mathcal{T}^l_{\text{ca}}$, $\mathcal{T}_{\text{ca}}$) the category of *skeletal* right τ-categories (respectively, left τ-categories, τ-categories) whose morphism sets consist of equivalences of categories. Denote by $\mathcal{T}^r_{\text{gca}}$ (respectively, $\mathcal{T}^l_{\text{gca}}$, $\mathcal{T}_{\text{gca}}$) the subcategory of $\mathcal{T}^r_{\text{ca}}$ (respectively, $\mathcal{T}^l_{\text{ca}}$, $\mathcal{T}_{\text{ca}}$) consisting of completely graded categories.

(2) Denote by $\mathcal{T}^r_{\text{sp}}$ the category of right τ-species, where $\mathcal{T}^r_{\text{sp}}(\mathcal{Q}, \mathcal{Q}')$ ($\mathcal{Q} = (Q, D_X, {}_X M_Y, \tau^+, a, b)$, $\mathcal{Q}' = (Q', D'_X, {}_X M'_Y, \tau^+, a', b')$) consists of $F = (F, F_X, F_{Y,X})$ satisfying the following conditions (i) and (ii).

(i) $F\colon \mathbb{N}Q \to \mathbb{N}Q'$ is a monoid isomorphism satisfying $F\tau^+ = \tau^+ F$, $F_X\colon D_X \to D'_{FX}$ ($X \in Q$) is a ring isomorphism and $F_{Y,X}\colon {}_Y M_X \to {}_{FY}M'_{FX}$ ($X, Y \in Q$) is a (D_Y, D_X)-isomorphism.

(ii) There exists $0 \neq d_X \in D'_{F\tau^+X}$ ($X \in Q$) which makes the following diagrams commutative, where $G_X(f) := d_X F_{\tau^+X}(f) d_X^{-1}$ and $G_{Y,X}(\phi)(g) := F_Y \circ \phi \circ F^{-1}_{\tau^+X,Y}(d_X^{-1} g)$ for $f \in D_{\tau^+X}$, $g \in {}_{\tau^+FX}M'_{FY}$ and $\phi \in \operatorname{Hom}_{D_Y}({}_{\tau^+X}M_Y, D_Y)$.

$$\begin{array}{ccccccc}
D_X & \xrightarrow{a_X} & D_{\tau^+X} & \quad & \operatorname{Hom}_{D_Y}({}_{\tau^+X}M_Y, D_Y) & \xrightarrow{b_{X,Y}} & {}_Y M_X \\
\downarrow F_X & & \downarrow G_X & & \downarrow G_{Y,X} & & \downarrow F_{Y,X} \\
D'_{FX} & \xrightarrow{a'_{FX}} & D'_{\tau^+FX} & & \operatorname{Hom}_{D_{FY}}({}_{\tau^+FX}M'_{FY}, D'_{FY}) & \xrightarrow{b'_{FX,FY}} & {}_{FY}M'_{FX}
\end{array}$$

For $F \in \mathcal{T}^r_{\text{sp}}(\mathcal{Q}, \mathcal{Q}')$ and $F' \in \mathcal{T}^r_{\text{sp}}(\mathcal{Q}', \mathcal{Q}'')$, define their composition FF' naturally. It is well defined since $d''_X := d'_{FX} F'_{\tau^+FX}(d_X) \in D''_{F'F\tau^+X}$ gives datum for FF' if d_X and d'_X give datum for F and F' respectively.

(3) Let $\mathcal{T}_{\text{sp}}$ be the full subcategory of $\mathcal{T}^r_{\text{sp}}$ consisting of τ-species. Define the category $\mathcal{T}^l_{\text{sp}}$ of left τ-species by the dual of (2). Then $\mathcal{T}_{\text{sp}}$ is regarded as a full subcategory of $\mathcal{T}^l_{\text{sp}}$ (cf. 8.3(3)).

10.2. THEOREM. *Let $* = r, l$ or nothing. Then $\widehat{\mathbb{M}}$, $\widehat{\mathbb{A}}$ and $\widehat{\mathbb{G}}$ define functors $\widehat{\mathbb{M}}\colon \mathcal{T}^*_{\text{sp}} \to \mathcal{T}^*_{\text{ca}}$, $\widehat{\mathbb{A}}\colon \mathcal{T}^*_{\text{ca}} \to \mathcal{T}^*_{\text{sp}}$ and $\widehat{\mathbb{G}}\colon \mathcal{T}^*_{\text{ca}} \to \mathcal{T}^*_{\text{ca}}$ such that $\widehat{\mathbb{A}} \circ \widehat{\mathbb{M}}$ is isomorphic* *to* $1_{\mathcal{T}^*_{\rm sp}}$ *and* $\widehat{\mathbb{M}} \circ \widehat{\mathbb{A}}$ *is isomorphic to* $\widehat{\mathbb{G}}$*. In particular,* $\widehat{\mathbb{M}}$ *and* $\widehat{\mathbb{A}}$ *give equivalences between* $\mathcal{T}^*_{\rm gca}$ *and* $\mathcal{T}^*_{\rm sp}$*.*

Proof. $\widehat{\mathbb{G}}$ gives a functor $\widehat{\mathbb{G}}\colon \mathcal{T}^*_{\rm ca} \to \mathcal{T}^*_{\rm ca}$ by 5.2.

(i) We will show that $\widehat{\mathbb{A}}$ defines a functor.

For $F \in \mathcal{T}^*_{\rm ca}(\mathcal{C}, \mathcal{C}')$, put $\mathcal{Q} := \widehat{\mathbb{A}}(\mathcal{C})$ and $\mathcal{Q}' := \widehat{\mathbb{A}}(\mathcal{C}')$. Define $F\colon \mathbb{N}Q \to \mathbb{N}Q'$, F_X and $F_{Y,X}$ from F naturally. Since F is an equivalence, $F(X]_{\mathcal{C}} \approx (FX]_{\mathcal{C}'}$ holds for any $X \in Q$. Thus we obtain the following isomorphism of complexes.

$$\begin{array}{lcccc}
(FX]_{\mathcal{C}'}: \tau^+FX & \xrightarrow{\nu^+_{FX}} & \theta^+FX & \xrightarrow{\mu^+_{FX}} & FX \\
\quad\downarrow^{d_X} & & \downarrow^{e_X} & & \| \\
F(X]_{\mathcal{C}}: F\tau^+X & \xrightarrow{F\nu^+_X} & F\theta^+X & \xrightarrow{F\mu^+_X} & FX
\end{array}$$

Since $\mathcal{C}$ is skeletal, we can regard d_X as an element of $D'_{F\tau^+X}$ by $\tau^+FX = F\tau^+X$. Immediately, the left diagram in 10.1(2)(ii) is commutative. The following diagram shows that the right diagram in 10.1(2)(ii) is commutative.

$$\begin{array}{ccccc}
\mathrm{Hom}_{D_Y}({}_{\tau^+X}M_Y, D_Y) & \xrightarrow{(F^{-1}_{\tau^+X,Y}, F_Y)} & \mathrm{Hom}_{D'_{FY}}({}_{F\tau^+X}M'_{FY}, D'_{FY}) & \xrightarrow{(d_X^{-1},1)} & \mathrm{Hom}_{D'_{FY}}({}_{\tau^+FX}M'_{FY}, D'_{FY}) \\
\downarrow^{(\nu^+_X,1)} & & \downarrow^{(F\nu^+_X,1)} & & \downarrow^{(\nu^+_{FX},1)} \\
\mathrm{Hom}_{D_Y}(\mathcal{J}^{(0)}(\theta^+X, Y), D_Y) & \xrightarrow{(F,F_Y)} & \mathrm{Hom}_{D'_{FY}}(\mathcal{J}^{(0)}(F\theta^+X, FY), D'_{FY}) & \xrightarrow{(e_X^{-1},1)} & \mathrm{Hom}_{D'_{FY}}(\mathcal{J}^{(0)}(\theta^+FX, FY), D'_{FY}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{J}^{(0)}(Y, \theta^+X) & \xrightarrow{F} & \mathcal{J}^{(0)}(FY, F\theta^+X) & \xrightarrow{(1,e_X^{-1})} & \mathcal{J}^{(0)}(FY, \theta^+FX) \\
\downarrow^{\mu^+_X} & & \downarrow^{F\mu^+_X} & & \downarrow^{\mu^+_{FX}} \\
{}_YM_X & \xrightarrow{F_{Y,X}} & {}_{FY}M'_{FX} & \xrightarrow{1} & {}_{FX}M'_{FX}
\end{array}$$

(ii) We will show that $\widehat{\mathbb{M}}$ defines a functor.

For $F \in \mathcal{T}^*_{\rm sp}(\mathcal{Q}, \mathcal{Q}')$, take d_X in 10.1(2)(ii). Clearly, F defines an equivalence $F\colon \widehat{\mathbb{P}}(\mathcal{Q}) \to \widehat{\mathbb{P}}(\mathcal{Q}')$. The diagram in 10.1(2)(ii) implies $F(b_{X,Y}) = d_X^{-1} b'_{FX,FY}$. Thus $F(\gamma(X)) = d_X^{-1}\gamma(FX)$ holds. Hence we have an equivalence $F\colon \widehat{\mathbb{M}}(\mathcal{Q}) \to \widehat{\mathbb{M}}(\mathcal{Q}')$.

(iii) It is easily shown that $\widehat{\mathbb{A}} \circ \widehat{\mathbb{M}}$ is isomorphic to $1_{\mathcal{T}^*_{\rm sp}}$. By a similar argument as in the proof of 9.2, $\widehat{\mathbb{M}} \circ \widehat{\mathbb{A}}$ is isomorphic to $\widehat{\mathbb{G}}$. Since the restriction of $\widehat{\mathbb{G}}$ to $\mathcal{T}^*_{\rm gca}$ is isomorphic to $1_{\mathcal{T}^*_{\rm gca}}$, $\widehat{\mathbb{M}}$ and $\widehat{\mathbb{A}}$ give equivalences between $\mathcal{T}^*_{\rm gca}$ and $\mathcal{T}^*_{\rm sp}$. □

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