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Asymptotically distribution free test for parameter change in a diffusion process model

Ilia Negri · Yoichi Nishiyama

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Abstract A test procedure to detect a change in the value of the parameter in the drift of a diffusion process is proposed. The test statistic is asymptotically distribution free under the null hypothesis that the true parameter does not change. Also, the test is shown to be consistent under the alternative that there exists a change point.

Keywords Asymptotically distribution free test · Consistent test · Change point problem · Martingale

1 Introduction

Testing on structural change problems has been an important issue in statistics. It originally starts in quality control context, where one is concerned about the output of a production line and wants to find any departure from an acceptable standard of the products. Rapidly the problem of abrupt changes moved to various fields such as economics, finance, biology and environmental sciences. From the statistical point of view, the problem consists in testing whether there is a statistically significant change point in a sequence of chronologically ordered data. The problem for an i.i.d. sample was first considered in the paper of Page (1955); see also Hinkley (1970), and for a general survey of the change point detection and estimation, see Chen and Gupta (2001). The parameter change point problem became very popular in regression and

I. Negri (✉)
Department of Information Technology and Mathematical Methods, University of Bergamo, Viale Marconi, 5, 24044 Dalmine, BG, Italy
e-mail: ilia.negri@unibg.it

Y. Nishiyama
The Institute of Statistical Mathematics, 10-3 Midori-cho, Tachikawa, Tokyo 190-8562, Japan

time series models. This is because these models can be used to describe structural changes that often occur in financial and economic phenomena (due, for example, to a change of political situation or to a change of economic policy) or in environmental phenomena (due to sudden changes in weather situation or the occurrence of catastrophic natural events). In such kind of phenomena the first problem one has to deal with is to test if a change of parameter has occurred in the factor of interest. For regression models see for example, Hinkley (1969); Quandt (1960); Brown et al. (1975) and Chen (1998). Refer to Brodsky and Darkhovsky (2000) for a complete review on methods to identify change points for sequential random sequence. For time series models, Picard (1985) considers the problem of detecting a change-point occurring in the mean or in the covariance of an autoregressive process. Ling (2007) deals with detecting structural changes in a general time series framework that includes ARMA and GARCH models among others. For a general review we refer also to Csörgő and Horváth (1997) and to Chen and Gupta (2000) for parametric methods and analysis.

Diffusion process can be considered as the most popular continuous time stochastic process and it has been playing a central role in modeling phenomena in many fields, not only in finance and more generally in economics science, but also in other fields such as biology, medicine, physics and engineering. Despite the fact of their importance in applications, few works are devoted to testing change point in parameter for diffusions. For a complete reference on statistical problems for ergodic diffusions based on continuous time observations see Kutoyants (2004). In Lee et al. (2006), the cusum test based on one-step estimator is considered and up to our knowledge there are no existing literature on this subject and on this framework.

In this work a test for detecting if a change of the parameter in the drift of a diffusion process takes place is proposed. The test is based on the continuous observation of the process up to time T. Our idea to construct the test is based on the *Fisher-score process* introduced in the independent data case by Horváth and Parzen (1994). The interest for this test is that it may be used for the most common family of diffusion process because the conditions on the coefficients of the diffusion process are very general. Moreover the asymptotic distribution of the test statistics does not depend on the unknown parameter, so the test is asymptotically distribution free. It is also proved that the test is consistent against any alternative where the alternative means that at a certain instant the parameter specifying the drift coefficient changes.

The rest of the paper is organized as follows. In Sect. 2 the model, the conditions and some preliminary results needed later are presented. The main result, consisting of the asymptotic distribution of the test statistic and the construction of an asymptotically distribution free and consistent test, are given in Sect. 3. The proofs are given in Sect. 4. Finally, a necessary lemma needed to prove one of our lemmas in the main text is stated in the Appendix.

To close this section, we make some conventions. We denote by $A^\top$ the transpose of the vector or matrix A. The finite-dimensional Euclidean norm is denoted by $||\cdot||$. The notations $\to^p$ and $\to^d$ mean the convergence in probability and the convergence in distribution (i.e., the weak convergence), respectively, as $T \to \infty$. We refer to van der Vaart and Wellner (1996) for the weak convergence theory in the space $\ell^\infty(T)$, the space of bounded functions on T with the uniform metric.

2 Preliminaries

Consider the stochastic differential equation (SDE) with the state space $I = (l, r)$, where $-\infty \le l < r \le \infty$, given by:

$$X_t = X_0 + \int_0^t S(X_s, \theta)ds + \int_0^t \sigma(X_s)dW_s, \tag{1}$$

where $s \rightsquigarrow W_s$ is a standard Wiener process and X_0 is an I-valued random variable which is independent of $s \rightsquigarrow W_s$. We suppose that the functions $S(\cdot, \theta) : I \to \mathbb{R}$ and $\sigma(\cdot) : I \to (0, \infty)$ are such that a solution X^θ to this SDE exists. We also suppose that, for every fixed θ, the solution X^θ is ergodic with the invariant measure μ_θ, that is, it holds for every μ_θ-integrable function f that

$$\frac{1}{T}\int_0^T f(X_s^\theta)ds \to^p \int_I f(x)\mu_\theta(dx), \quad \text{as } T \to \infty.$$

We suppose that the parameter space Θ to which θ is belonging is a bounded open subset of $\mathbb{R}^d$. We consider the SDE (1) where θ may change at a certain point $s \in [0, T]$. We wish to test:

H_0: there exists a certain $\theta_0 \in \Theta$ such that $\theta = \theta_0$ for all $s \in [0, T]$;

H_1: there exist two different values $\theta_0 \ne \theta_1$ both belonging to Θ, and a certain $u_* \in (0, 1)$, such that $\theta = \theta_0$ for $s \in [0, Tu_*]$ and $\theta = \theta_1$ for $s \in (Tu_*, T]$.

We introduce the following regularity conditions on the functions S and σ. We suppose that $\theta \mapsto S(x, \theta)$ is two times continuously differentiable, and denote by $\dot{S}$ and $\ddot{S}$ the first and the second derivatives. We suppose the following:

$$\begin{aligned}
&\int_I \frac{S(x,\theta)^2}{\sigma(x)^2}\mu_{\theta_0}(dx) < \infty, \quad \forall\theta, \theta_0 \in \Theta;\\
&\int_I \frac{||\dot{S}(x,\theta)||^2}{\sigma(x)^2}\mu_{\theta_0}(dx) < \infty, \quad \forall\theta, \theta_0 \in \Theta;\\
&\int_I \frac{\sup_{\theta\in\Theta}||\ddot{S}(x,\theta)||^2}{\sigma(x)^2}\mu_{\theta_0}(dx) < \infty, \quad \forall\theta_0 \in \Theta;
\end{aligned} \tag{2}$$

The matrix

$$I_{\theta_0}(\theta) = \int_I \frac{\dot{S}(x,\theta)\dot{S}(x,\theta)^\top}{\sigma(x)^2}\mu_{\theta_0}(dx)$$

is positive definite for every $\theta, \theta_0 \in \Theta$; For every $\theta_0 \in \Theta$, the function

$$\theta \mapsto g(\theta, \theta_0) = \frac{1}{2}\int_I \frac{|S(x,\theta) - S(x,\theta_0)|^2}{\sigma(x)^2}\mu_{\theta_0}(dx)$$

attains the local and global minimum at θ_0 (actually, we suppose that the function

$$\theta \mapsto \dot{g}(\theta, \theta_0) = \int_I \frac{(S(x,\theta) - S(x,\theta_0))\dot{S}(x,\theta)}{\sigma(x)^2} \mu_{\theta_0}(dx)$$

is zero if and only if $\theta = \theta_0$, assuming hereafter that the order of integration and differentiation is exchangeable). The function

$$\theta \mapsto u_* g(\theta, \theta_0) + (1 - u_*) g(\theta, \theta_1)$$

attains the local and global minimum at θ_*, where θ_0, θ_1 and u_* are the ones given in H_1.

Let $\widehat{\theta}_T$ be the solution to the equation

$$\frac{1}{T} \int_0^T \frac{\dot{S}(X_s, \theta)}{\sigma(X_s)^2} (dX_s - S(X_s, \theta) ds) = 0.$$

The following lemma is easy to show, so the proof is omitted.

Lemma 1 (i) *Under H_0, it holds that $\sqrt{T}(\widehat{\theta}_T - \theta_0) \to^d N(0, I_{\theta_0}(\theta_0)^{-1})$.*
(ii) *Under H_1, it holds that $\widehat{\theta}_T \to^p \theta_*$.*

3 Main result

In this section, we construct a test statistic which is asymptotically distribution free under H_0, and consistent under H_1. Hereafter, we suppose that all the conditions stated in Sect. 2 hold. In order to construct a test statistic for the testing problem, we introduce the stochastic process $t \rightsquigarrow U_{T,t}$ given by:

$$U_{T,t} = \frac{1}{\sqrt{T}} \int_0^t \frac{\dot{S}(X_s, \theta)}{\sigma(X_s)^2} (dX_s - S(X_s, \theta) ds) \bigg|_{\theta = \widehat{\theta}_T}.$$

We also introduce the random field $\{V_T(u, \theta) : (u, \theta) \in [0, 1] \times \Theta\}$ given by

$$V_T(u, \theta) = \frac{1}{\sqrt{T}} \int_0^T (1\{s \le Tu\} - u) \frac{\dot{S}(X_s, \theta)}{\sigma(X_s)^2} (dX_s - S(X_s, \theta) ds).$$

Then, it holds that $U_{T,Tu} = V_T(u, \widehat{\theta}_T)$.

The following lemma plays a key role in our context.

Lemma 2 *Under H_1 without the assumption $\theta_0 \neq \theta_1$, the random field $(u, \theta) \rightsquigarrow M_T(u, \theta)$ given by*

$$M_T(u, \theta) = \frac{1}{\sqrt{T}} \int_0^T (1\{s \le Tu\} - u) \frac{\dot{S}(X_s, \theta)}{\sigma(X_s)} dW_s$$

converges weakly to a centered Gaussian random field $(u, \theta) \rightsquigarrow G(u, \theta)$ *in* $\ell^\infty([0, 1] \times \Theta)$. *Moreover, almost all paths of the limit are continuous with respect to the Euclidean metric on* $[0, 1] \times \Theta$. *In particular, under* H_0, *the limit can be written as* $G(u, \theta) = I_{\theta_0}(\theta)^{1/2} B^\circ(u)$ *where* $B^\circ(u) = (B^{\circ,1}(u), ..., B^{\circ,d}(u))^\top$ *are* d*-vector of independent standard Brownian bridges.*

There are a few ways to prove this claim, in particular to check the asymptotic tightness. One possible way is to check the Kolmogorov–Chentsov type criterion [see, e.g., Corollary 16.9 of Kallenberg (2002)], however, for this purpose we need to strengthen the condition (2) to

$$\int_I \frac{||\dot{S}(x, \theta)||^p}{\sigma(x)^2} \mu_{\theta_0}(dx) < \infty,$$

for $p > d + \gamma$ with some $\gamma > 0$. Another way is to check the tightness criterion for ℓ^∞-valued continuous martingales developed by Nishiyama (1999), where the condition for $p = 2$ is sufficient. See also Nishiyama (2000) and van der Vaart and van Zanten (2005). Since it is easy to check those conditions, we omit the proof. See for example Negri and Nishiyama (2009).

By combining the above lemma and Lemma 5 in the Appendix, we obtain the following lemma.

Lemma 3 *Under* H_0, *it holds that the random field* $u \rightsquigarrow M_T(u, \widehat{\theta}_T)$ *converges weakly to* $u \rightsquigarrow I_{\theta_0}(\theta_0)^{1/2} B^\circ(u)$ *in* $\ell^\infty([0, 1])$.

The following lemma is a crucial point in our paper.

Lemma 4 *Under* H_0, *it holds that*

$$\sup_{u \in [0,1]} |V_T(u, \widehat{\theta}_T) - M_T(u, \widehat{\theta}_T)| \to^p 0.$$

By combining Lemmas 3 and 4, we have that, under H_0, $u \rightsquigarrow V_T(u, \widehat{\theta}_T)$ converges weakly to $u \rightsquigarrow I_{\theta_0}(\theta_0)^{1/2} B^\circ(u)$ in $\ell^\infty([0, 1])$. This leads to our idea to propose the test statistic

$$\begin{aligned} \mathcal{S}_T &= \sup_{t \in [0,T]} |U_{T,t}^\top \widehat{I}_T^{-1} U_{T,t}| \\ &= \sup_{u \in [0,1]} |V_T(u, \widehat{\theta}_T)^\top \widehat{I}_T^{-1} V_T(u, \widehat{\theta}_T)|, \end{aligned}$$

where

$$\widehat{I}_T = \frac{1}{T} \int_0^T \frac{\dot{S}(X_s, \widehat{\theta}_T) \dot{S}(X_s, \widehat{\theta}_T)^\top}{\sigma(X_s)^2} ds$$

which is a consistent estimator for $I_{\theta_0}(\theta_0)$ under H_0, and for $I_* := u_* I_{\theta_0}(\theta_*) + (1 - u_*) I_{\theta_1}(\theta_*)$ under H_1. It is clear that $\widehat{I}_T$ is symmetric and positive definite with probability tending to 1, under both H_0 and H_1.

Theorem 1 (i) *Under H_0, it holds that*

$$\mathcal{S}_T \to^d \sup_{u\in[0,1]} \sum_{i=1}^{d} |B^{\circ,i}(u)|^2,$$

where $B^{\circ,i}(u)$'s are independent standard Brownian bridges. Hence the test is asymptotically distribution free.

(ii) *Under H_1, it holds that*

$$2\mathcal{S}_T \geq T(v_*^\top I_*^{-1} v_* - o_P(1)) - O_P(1),$$

where $v_* = -u_*(1-u_*)(\dot{g}(\theta_*,\theta_0) - \dot{g}(\theta_*,\theta_1))$. Moreover, v_* is not zero. Hence the test is consistent.

4 Proofs

Proof of Lemma 4. Notice that

$$\begin{aligned}
V_T(u,\widehat{\theta}_T) &- M_T(u,\widehat{\theta}_T)\\
&= \frac{1}{\sqrt{T}}\int_0^T (1\{s\leq Tu\}-u)\frac{\dot{S}(X_s,\widehat{\theta}_T)}{\sigma(X_s)^2}(S(X_s,\theta_0)-S(X_s,\widehat{\theta}_T))ds\\
&= \frac{1}{T}\int_0^T (1\{s\leq Tu\}-u)\frac{\dot{S}(X_s,\widehat{\theta}_T)\dot{S}(X_s,\widetilde{\theta}_T)^\top}{\sigma(X_s)^2}ds\sqrt{T}(\theta_0-\widehat{\theta}_T),
\end{aligned}$$

where $\widetilde{\theta}_T$ is a point between $\widehat{\theta}_T$ and θ_0. Since $\sqrt{T}(\widehat{\theta}_T-\theta_0) = O_P(1)$, we deal with

$$\begin{aligned}
&\frac{1}{T}\int_0^T (1\{s\leq Tu\}-u)\frac{\dot{S}(X_s,\widehat{\theta}_T)\dot{S}(X_s,\widetilde{\theta}_T)^\top}{\sigma(X_s)^2}ds\\
&\quad= \frac{1}{T}\int_0^T (1\{s\leq Tu\}-u)\frac{\dot{S}(X_s,\theta_0)\dot{S}(X_s,\theta_0))^\top}{\sigma(X_s)^2}ds + O_P(||\widehat{\theta}_T-\theta_0||),
\end{aligned}$$

uniformly in u. Showing that the first term on the right hand side converges in probability to zero uniformly in u is easy (consider the positive and negative parts of each component of $\dot{S}(\cdot,\theta_0)\dot{S}(\cdot,\theta_0)^\top$ separately). The proof is finished. □

Proof of Theorem 1. The part (i) has already been proved (use the continuous mapping theorem). Let us prove the part (ii). In general, for any non-negative definite matrix Σ and vectors v, w it holds that $2v^\top\Sigma v + 2w^\top\Sigma w = (v+w)^\top\Sigma(v+w) + (v-w)^\top\Sigma(v-w) \geq (v-w)^\top\Sigma(v-w)$. Thus we have

$$2\mathcal{S}_T \geq TA_T^\top\widehat{I}_T^{-1}A_T - 2M_T(u_*,\widehat{\theta}_T)^\top\widehat{I}_T^{-1}M_T(u_*,\widehat{\theta}_T),$$

where

$$A_T = \frac{1}{\sqrt{T}}(V_T(u_*, \widehat{\theta}_T) - M_T(u_*, \widehat{\theta}_T))$$
$$= \frac{1}{T}\int_0^T (1\{s \le Tu_*\} - u_*)\frac{\dot{S}(X_s, \widehat{\theta}_T)}{\sigma(X_s)^2}(S(X_s, \theta_{\text{true}}) - S(X_s, \widehat{\theta}_T))ds$$

with the convention that θ_{true} should be read as θ_0 for $s \in [0, Tu_*]$ and as θ_1 for $s \in (Tu_*, T]$. It is easy to see that

$$A_T = \frac{1}{T}\int_0^T (1\{s \le Tu_*\} - u_*)\frac{\dot{S}(X_s, \theta_*)}{\sigma(X_s)^2}(S(X_s, \theta_{\text{true}}) - S(X_s, \theta_*))ds + o_P(1) \to^p v_*.$$

Since $\widehat{I}_T \to^p I_*$ and $M_T(u_*, \widehat{\theta}_T)$ is asymptotically tight, we obtain the first claim of (ii).

To show the last claim, notice that it follows from our assumption that $u_*\dot{g}(\theta_*, \theta_0) + (1 - u_*)\dot{g}(\theta_*, \theta_1) = 0$. If v_* were zero, it should hold that $\dot{g}(\theta_*, \theta_0) = \dot{g}(\theta_*, \theta_1) = 0$, but this contradicts the assumption that θ_i is the unique solution of the equation $\dot{g}(\theta, \theta_i) = 0$ for $i = 0, 1$ and the assumption that $\theta_0 \neq \theta_1$. Hence we have proved all the assertions. □

Appendix

The following lemma, which was used in the main text, can be proved in the same way as Lemma 2.2 of Nishiyama (2009).

Lemma 5 *Let (S, ρ_S) and (T, ρ_T) be semimetric spaces. Suppose that $Z_n \to^d Z$ in $\ell^\infty(S \times T)$ and that almost all paths of Z are continuous with respect to $\rho = \rho_S \vee \rho_T$. If T-valued random sequence $\widehat{t}_n$ satisfies $\rho_T(\widehat{t}_n, t_0) \to^p 0$ for some nonrandom $t_0 \in T$, then it holds that $Z_n(\cdot, \widehat{t}_n) - Z_n(\cdot, t_0)$ converges in probability to zero in $\ell^\infty(S)$.*

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