Ann Inst Stat Math (2011) 63:101–115
DOI 10.1007/s10463-008-0211-3

A new perspective to stress–strength models

Serkan Eryılmaz

Received: 10 December 2007 / Revised: 27 June 2008 / Published online: 11 October 2008

Abstract The stress–strength models have been widely used for reliability design of systems. In these models the reliability is defined as the probability that the strength is larger than the stress. The analysis is then based on the binary reliability theory since there are two possible states for the system. In this paper, we study the stress–strength reliability in a different framework assigning more than two states to the system depending on the difference between strength and stress values. In other words, the stress–strength reliability is studied under multi-state system modeling. System state probabilities are evaluated and estimated under various assumptions on the system. The multicomponent form is also studied and some results are provided for large systems.

Keywords Minimum variance unbiased estimator · Multi-state system · Stress–strength reliability

1 Introduction

In manufacturing, having an information about the mechanical reliability of designs through stress–strength models prior to production can significantly decrease the costs of production. It is a well accepted fact that the strength of a manufactured unit is a random variable. This fact is the basis of statistical reliability modeling.

In the traditional reliability theory, the system and each component are assumed to be only in one of two possible states being either working or failed. This theory fails in modeling the situations when the systems or components can be in intermediate states meaning that they might have more than two levels of performance.

S. Eryılmaz (✉)
Department of Mathematics, Izmir University of Economics, Balcova, Izmir, Turkey
e-mail: serkan.eryilmaz@ieu.edu.tr

Multi-state system approach is then used to present a system with some level of working and failing efficiency. In a discrete multi-state system model, it is assumed that the system and its components may each experience $M + 1$ $(M \geq 1)$ possible states: $0, 1, \ldots, M$, where the extreme states 0 and M represent the completely failed and completely working states respectively and the others are intermediate states. The reader is referred to Lisnianski and Levitin (2003) as well as Kuo and Zuo (2003) for the details of this theory. In a continuous multi-state system model, the states of the system and its components can be described by continuously many values. That is the system and its components may each be represented by a continuous random variable defined in the interval $[0, 1]$. For the details of continuous multi-state systems, readers are referred to Ross (1979); Block and Savits (1982).

There has been considerable interest about stress–strength reliability in the literature. In the simplest stress–strength model, a unit or system functions if its strength exceeds the stress imposed upon it. In this case the reliability is defined as $R = P\{X < Y\}$, where Y is the random strength of the unit and X is the random stress placed on the unit. This reliability and its estimation have been widely investigated under various distributional assumptions on X and Y. See, e.g. Johnson (1988) and Kotz et al. (2003) for an extensive and lucid review of the topic. Recent discussions on the topic lie in the works of Mokhlis (2005); Krishnamoorthy et al. (2007); An et al. (2008).

It should be noted that R is also of great interest in various fields since it provides a general measure of the difference between two populations. For instance, R may be used in treatment comparisons. In this context X is the response for a control group, and Y refers to a treatment group and R measures the effect of the treatment.

Stress–strength models have also been studied in a multi-component setup. Bhattacharyya and Johnson (1974) studied the situation where a system, consisting of n components, functions when at least k $(1 \leq k \leq n)$ of the components survive a common random stress X. This situation corresponds to the k-out-of-n : G system and the reliability under this setup can be formulated as $P\{Y_{n-k+1:n} > X\}$, where $Y_1, Y_2, \ldots, Y_n$ are the strengths of the components and $Y_{r:n}$ is the rth smallest among $Y_1, Y_2, \ldots, Y_n$. Recently, Eryılmaz (2008a) studied the reliability and its estimation for consecutive k -out-of-n : G system which functions if and only if at least k consecutive components survive a common random stress. Eryılmaz (2008b) studied the multivariate case whenever n independent components (or subsystems) each consisting of m dependent elements are subjected to a common random stress consists of m components. See also Turkkan and Gia (2007) for the multivariate case. Under these considerations, a system and its components may only be in two possible states: either working or failed and the reliability analysis mainly based on the evaluation of the probabilities of these two extreme events. However, assigning different levels to the state of the system and its components depending on the size of Y–X might be much more realistic in real life reliability studies. The latter case, of course, needs to be considered in the context of multi-state system modeling.

In this paper, we extend the definition of stress–strength reliability for the case of multi-state systems. In this case, the state of the component or system can take more than two values depending on the size of Y–X and a function which we call *Kernel*. The paper is organized as follows: in Sect. 2 we present our approach as well as some examples under various assumptions on the system. Section 3 contains the estimation of system state probabilities and some numerical results. In Sect. 4, we study the stress–strength reliability in a multicomponent form.

2 Setup of stress–strength models

In traditional setup of stress–strength interference, the state of the system, to be denoted by ξ, is defined as $\xi = 1$ if $Y > X$ and $\xi = 0$ if $Y \leq X$. That is the state of the system or component depends on the size of $Y - X$ in the binary stress–strength reliability. This suggest a way for defining the state of a system or component in a multi-state context. Let $K : \Re \rightarrow S$ be an increasing function. Then the state of system can be defined as

$$\xi = K(Y - X). \tag{1}$$

According to (1) the state of the system is assigned w.r.t. the difference between strength and stress values and the kernel function K. The function K is chosen to be increasing because the larger $Y - X$ must provide a larger state for the system. Statistically, if $Y_1 - X_1 \overset{\text{st}}{\leq} Y_2 - X_2$ and K is increasing then $\xi_1 = K(Y_1 - X_1) \overset{\text{st}}{\leq} K(Y_2 - X_2) = \xi_2$, where $\overset{\text{st}}{\leq}$ represents the usual stochastic ordering. The range S of the function K determines the type of a multi-state system to be either continuous or discrete. This kind of modeling might be useful when the values of the difference $Y - X$ is explanatory for the system/process. Although we are using the term "stress–strength" for the concept, the proposed procedure can also be used in real life problems not only in reliability but also in other fields.

2.1 Continuous multi-state system

Let K be a continuous function having an inverse K^{-1} and $S = [0, 1]$. In this case there are continuously many values for the state of a system. The event $\{\xi > K(0)\} \equiv \{Y > X\}$ implies the survival of the system and the values on the interval $(K(0), 1]$ determine the levels of the survival of the system. Similarly, the event $\{\xi \leq K(0)\} \equiv \{Y \leq X\}$ represents the failure of the system and the values on the interval $[0, K(0)]$ determine the levels of the failure. In a binary system reliability we consider the probabilities of the events $\{Y > X\}$ and $\{Y \leq X\}$ which respectively defines the reliability and unreliability of a system. In multi-state system modeling the state distribution of the system, that is the probabilities of intermediate states are evaluated. In this framework, it will be appropriate to evaluate the probabilities $P\{\xi > s\}$ and $P\{\xi \leq s\}$ for $s > K(0)$ and $s \leq K(0)$, respectively.

If X and Y are independent random variables having continuous cumulative distribution functions F and G respectively, then the probability that the system is above state s and the probability that the system is in state s or below can be computed respectively as

$$P\{\xi > s\} = \iint_{K(y-x)>s} \mathrm{d}F(x)\mathrm{d}G(y) = \int F(y - K^{-1}(s))\mathrm{d}G(y), \tag{2}$$

$$P\{\xi \le s\} = \iint_{K(y-x)\le s} \mathrm{d}F(x)\mathrm{d}G(y) = \int G(x + K^{-1}(s))\mathrm{d}F(x). \tag{3}$$

In our developments we only evaluate $P\{\xi > s\}$ for both cases $s > K(0)$ and $s \le K(0)$ instead of evaluating (2) and (3) separately. This evaluation includes $P\{\xi \le s\} = 1 - P\{\xi > s\}$ whenever $s \le K(0)$.

This kind of modeling might be necessary for the following situation.

Example 1 Consider the situation where a system may fail but it can be repaired depending on the level of failure. Suppose that a system cannot be repaired if the state of the system $\xi = K(Y - X)$ is below s for $s \le K(0)$. In this case $P\{\xi \le s\}$ denotes the probability that system cannot be repaired and $s \le K(0)$ determines the admissible level for the repair of the system.

In the following we compute (2) for various stress–strength distributions.

Example 2 Let $F(x) = 1 - \exp\{-\theta x\}$ and $G(x) = 1 - \exp\{-\lambda x\}$, $x \ge 0$. Then for $0 < s < 1$

$$R_{\theta,\lambda}(s) = P\{\xi > s\} = \begin{cases} 1 - \frac{\lambda}{\lambda+\theta} \exp\left\{\theta K^{-1}(s)\right\} & \text{if } s \le K(0) \\ \frac{\theta}{\lambda+\theta} \exp\left\{-\lambda K^{-1}(s)\right\} & \text{if } s > K(0), \end{cases}$$

and $P\{\xi > 0\} = 1$ and $P\{\xi > 1\} = 0$.

Example 3 Let X and Y have normal distributions with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2, respectively. Then

$$R(s) = P\{\xi > s\} = \Phi\left(\frac{(\mu_2 - \mu_1) - K^{-1}(s)}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right),$$

where $\Phi(.)$ denotes the cumulative distribution function (c.d.f.) of standard normal variable.

The choice of an appropriate K may be of special importance. Obviously any continuous distribution function whose support is $\Re$ can be used for $K(x)$. This is because one needs to assign a state for the system on the interval [0, 1] which is the main assumption in continuous multi-state system modeling. Consider the c.d.f.s of standard normal and logistic distributions which are given respectively, by

$$K_1(x) = \Phi(x), \quad x \in \Re \tag{4}$$

$$K_2(x) = \frac{1}{1 + \exp\{-x/\beta\}}, \quad x \in \Re, \beta > 0. \tag{5}$$

Under the assumptions of Example 2, the state probabilities of the system for these selections can be computed respectively as

$$R^{(1)}_{\theta,\lambda}(s) = \begin{cases} 1 - \frac{\lambda}{\lambda+\theta} \exp\left\{\theta \Phi^{-1}(s)\right\} & \text{if } 0 < s \leq \frac{1}{2} \\ \frac{\theta}{\lambda+\theta} \exp\left\{-\lambda \Phi^{-1}(s)\right\} & \text{if } \frac{1}{2} < s < 1 \end{cases},$$

$$R^{(2)}_{\theta,\lambda}(s) = \begin{cases} 1 - \frac{\lambda}{\lambda+\theta} \left(\frac{s}{1-s}\right)^{\theta\beta} & \text{if } 0 < s \leq \frac{1}{2} \\ \frac{\theta}{\lambda+\theta} \left(\frac{s}{1-s}\right)^{-\lambda\beta} & \text{if } \frac{1}{2} < s < 1 \end{cases},$$

with $R^{(i)}_{\theta,\lambda}(0) = 1$, $R^{(i)}_{\theta,\lambda}(1) = 0, i = 1, 2$.

It is clear that the c.d.f.s given by (4) and (5) are symmetric. With the use of a symmetric c.d.f. as a kernel the distances to the completely failed ("0") and completely working ("1") states are obtained to be same for the same value but opposite sign of $Y - X$. Because in this case $\xi = K(Y - X)$ and $1 - \xi = K(X - Y)$. This helps to measure the absolute closeness of the actual state to the well defined states such as completely failed and completely working.

An additional criteria can be taken into account for the choice of $K(x)$ to increase the sensitivity when assigning the state of the system. For example the usage of the principle of maximum entropy can be considered as a criteria. According to the principle of maximum entropy, if nothing is known about a distribution except that it belongs to a certain class, then the maximum entropy distribution for that class can be chosen as a default.

To compare the entropies of (4) and (5) we fix the means and variances of these distributions. The scale parameter of the logistic distribution is set to be $\beta = \sqrt{\frac{3}{\pi^2}}$ to achieve a mean of 0 and a variance of 1. In this case the entropies for standard normal and logistic are found to be $\ln\left(\sqrt{2\pi e}\right) = 1.4189$ and $\ln\left(\sqrt{\frac{3}{\pi^2}}\right) + 2 = 1.4046$. This suggests the use of $K_1(x) = \Phi(x)$ when our criteria is based on the principle of maximum entropy.

2.2 Discrete multi-state system

The range of the function K is chosen to be $S = \{0, 1, \ldots, M\}$ for the case of discrete multi-state system. Thus the system has $M + 1$ states and these states are assigned to the system considering pre-defined intervals for $Y - X$, that is when the value of Y–X falls into a pre-defined interval, then its corresponding state will be determined. Mathematically, let $S_1 = \{0, 1, \ldots, M_1\} \subset S$ denote the set of states representing the levels of failure and $S \backslash S_1$ be the set of states representing the levels of operation. Define the intervals $I_0 = (a_0, a_1]$, $I_1 = (a_1, a_2]$, $\ldots$, $I_{M_1} = \left(a_{M_1}, 0\right]$, $I_{M_1+1} = \left(0, a_{M_1+1}\right]$, $I_{M_1+2} = \left(a_{M_1+1}, a_{M_1+2}\right]$, $\ldots$, $I_M = (a_{M-1}, a_M)$, where $-\infty \equiv a_0 < a_1 < a_2 < \cdots < a_{M_1} < 0$ and $0 < a_{M_1+1} < a_{M_1+2} < \cdots < a_{M-1}, a_M \equiv \infty$.

The function K can now be constructed in the following way:

$$\begin{aligned}
&K(x) = i \Leftrightarrow x \in I_i,\ i = 0, 1, \ldots, M_1 - 1\\
&K(x) = M_1 \Leftrightarrow x \in I_{M_1},\\
&K(x) = M_1 + 1 \Leftrightarrow x \in I_{M_1+1},\\
&K(x) = i \Leftrightarrow x \in I_i,\ i = M_1 + 2, \ldots, M - 1, M.
\end{aligned}$$

For discrete consideration, the choice of K or equivalently $a_1, a_2, \ldots, a_{M-1}$ depends on the formulation of the problem and/or experimenter's prior information.

We give the following examples to illustrate the possible application of discrete multi-state setup with the selection of a certain kernel.

Example 4 A consignment is transferred from the vehicle A to the vehicle B through a certain store. Let X denote the arrival time of the vehicle A to the store and Y denote the departure time of the vehicle B from the store. If $Y \le X$ then consignment cannot be transferred to B from A. If $0 < Y - X \le a_1$ then a part of consignment can be transferred to B and the consignment is fully transferred to B if $Y - X > a_1$, that is $a_1 > 0$ is the minimum required time for transferring whole consignment from A to B. In this case the kernel is defined to be

$$K(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } 0 < x \le a_1 \\ 2 & \text{if } x > a_1. \end{cases}$$

Example 5 Heart rate variability (HRV) is concerned with the analysis of variations in the heart rate data over extended periods of time. A high degree of HRV is observed in persons with normal hearts, while low HRV can be an indicator of some diseases such as coronary artery disease, congestive heart failure, and diabetic neuropathy. One of the most useful indicators of HRV is the normal to normal (NN or RR) intervals in a continuous electrocardiography (ECG) record. NN interval is defined to be an interval from one QRS complex to the next. Let X and Y denote the successive lengths (in milliseconds) between the NN intervals. More explicitly, X and Y can be considered as successive interarrival times in a renewal process. One of the quantity for measuring HRV is based on the comparison of $|Y - X|$ with a fixed quantity (usually 50 ms). In this case, it will be appropriate to use the following kernel.

$$K(x) = \begin{cases} 0 & \text{if } x \le -50 \\ 1 & \text{if } -50 < x \le 0 \\ 2 & \text{if } 0 < x \le 50 \\ 3 & \text{if } x > 50. \end{cases}$$

An empirical estimate of $P\{\xi = 0\} + P\{\xi = 3\}$, denoted by $pNN50$ (proportion of consecutive NN intervals differing by more than 50 ms), is one of the widely used statistical measure of HRV (see, e.g. Ewing et al. 1984). Evaluation of $P\{\xi = 0\} + P\{\xi = 3\}$ in a parametric way is the main interest of the present paper.

In the following we compute the probability of being in a specific state under the discrete multi-state setup assuming X and Y are independent random variables having exponential distributions with means $1/\theta$ and $1/\lambda$, respectively.

Example 6 Let $F(x) = 1 - \exp\{-\theta x\}$ and $G(x) = 1 - \exp\{-\lambda x\}$, $x \geq 0$. Then

$$P\{\xi = i\} = \begin{cases} \frac{\lambda}{\lambda+\theta}\left(\exp\{\theta a_{i+1}\} - \exp\{\theta a_i\}\right) & \text{if } i = 0, 1, \ldots, M_1 - 1 \\ \frac{\theta}{\lambda+\theta}\left(\exp\{-\lambda a_{i-1}\} - \exp\{-\lambda a_i\}\right) & \text{if } i = M_1 + 2, \ldots, M - 1, M, \end{cases} \tag{6}$$

and

$$P\{\xi = M_1\} = \frac{\lambda}{\lambda+\theta}\left(1 - \exp\left\{\theta a_{M_1}\right\}\right),$$
$$P\{\xi = M_1 + 1\} = \frac{\theta}{\lambda+\theta}\left(1 - \exp\left\{-\lambda a_{M_1+1}\right\}\right).$$

The Eq. 6 can be easily obtained observing $P\{\xi = i\} = P\{c_1(i) < Y - X \leq c_2(i)\}$ and

$$P\{Y - X \geq c(i)\} = \begin{cases} \frac{\theta}{\lambda+\theta}\exp\{-\lambda c(i)\} & \text{if } c(i) > 0 \\ 1 - \frac{\lambda}{\lambda+\theta}\exp\{\theta c(i)\} & \text{if } c(i) \leq 0. \end{cases} \tag{7}$$

3 Estimation of state probabilities

Undoubtedly, the most flexible and widely used method for estimating stress–strength reliability is the method of maximum likelihood. Because the maximum likelihood estimator (MLE) of a function of parameters is the same function of the maximum likelihood estimator of the parameters.

Let the data consist of a random sample of size n_1 of stresses $X_1, X_2, \ldots, X_{n_1}$ and an independent random sample of size n_2 of strengths $Y_1, Y_2, \ldots, Y_{n_2}$. For the normal stress–strength model considered in Example 3 the MLE of the probability of being state above s is

$$\widehat{R(s)} = \Phi\left(\frac{\left(\bar{Y} - \bar{X}\right) - K^{-1}(s)}{\sqrt{S_X^2 + S_Y^2}}\right),$$

where $\bar{X} = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i$, $\bar{Y} = \frac{1}{n_2}\sum_{i=1}^{n_2} Y_i$, $S_X^2 = \frac{1}{n_1-1}\sum_{i=1}^{n_1}(X_i - \bar{X})^2$, $S_Y^2 = \frac{1}{n_2-1}\sum_{i=1}^{n_2}(Y_i - \bar{Y})^2$.

Because of its simple form the MLE can be used for setting confidence limits for $R(s)$. Consider the case when X and Y are independent normal variables with known variances σ_1^2 and σ_2^2 and unknown means μ_1 and μ_2, respectively. Then $\left[\left(\bar{Y}-\bar{X}\right)-(\mu_2-\mu_1)\right]/\sqrt{\sigma_1^2/n_1+\sigma_2^2/n_2}$ has standard normal distribution and hence an exact 100 $(1-\alpha)$ % lower confidence bound for $R(s)$ can be obtained as

$$\Phi\left(\frac{\left(\bar{Y}-\bar{X}\right)-K^{-1}(s)}{\sqrt{\sigma_1^2+\sigma_2^2}}-\frac{z_\alpha}{\sqrt{a}}\right),$$

where $a = n_1 n_2\left(\sigma_1^2+\sigma_2^2\right)/n_2\sigma_1^2+n_1\sigma_2^2$ and $1-\alpha=\Phi(z_\alpha)$.

Although it is sometimes difficult, the minimum variance unbiased (MVU) estimate of stress–strength reliability has also been studied for various situations. In the following we provide the MVU estimator of $R_{\theta,\lambda}(s)$ derived in Example 1. The result of the following theorem readily accommodates all distributions (F, G) having a structural relation of the form $(1-F)^{1/\theta}=(1-G)^{1/\lambda}=(1-H)$, where H is a known distribution and θ and λ are unknown parameters. Making an initial transformation $X'=-\log\left[1-H(X)\right]$, $Y'=-\log\left[1-H(Y)\right]$ to the data the problem can be reduced to find the MVU estimate under exponential stress–strength model. The proof of the following result is presented in Appendix.

Theorem 1 *Let $X_1, X_2, \ldots, X_{n_1}$ and $Y_1, Y_2, \ldots, Y_{n_2}$ be independent samples from $F(x)=1-\exp\{-\theta x\}$ and $G(x)=1-\exp\{-\lambda x\}$, respectively. Then the unique MVU estimator of $R_{\theta,\lambda}(s)$:*

For $s \le K(0)$

$$\widetilde{R_{\theta,\lambda}(s)}=\begin{cases}1-\int_0^{V+U\cdot K^{-1}(s)} h_{n_1,n_2}(x,U,V,s)\mathrm{d}x & \text{if } -\frac{V}{U}<K^{-1}(s)<\frac{1-V}{U}\\ 1-\int_0^1 h_{n_1,n_2}(x,U,V,s)\mathrm{d}x & \text{if } K^{-1}(s)>\max\left(-\frac{V}{U},\frac{1-V}{U}\right)\\ 1 & \text{if } K^{-1}(s)<-\frac{V}{U},\end{cases}$$

for $s > K(0)$

$$\widetilde{R_{\theta,\lambda}(s)}=\begin{cases}(1-U\cdot K^{-1}(s))^{n_2-1}-\int_{U\cdot K^{-1}(s)}^1 h_{n_1,n_2}(x,U,V,s)\mathrm{d}x & \text{if } K^{-1}(s)<\frac{1}{U}\\ 0 & \text{if } K^{-1}(s)>\frac{1}{U},\end{cases}$$

where $V=T_1/T_2$, $U=1/T_2$, $T_1=\sum_{i=1}^{n_1}X_i$, $T_2=\sum_{i=1}^{n_2}Y_i$ and

$$h_{n_1,n_2}(x,U,V,s)=(n_2-1)(1-x)^{n_2-2}\left[1-\frac{1}{V}\left(x-U\cdot K^{-1}(s)\right)\right]^{n_1-1}.$$

For the MVU of (6) it is enough to find the MVU of (7). This can be readily obtained using Theorem 1 with $K^{-1}(s)=c(i)$ because the simple unbiased estimator of (7) is

$$W(X_1,Y_1)=\begin{cases}1 & \text{if } X_1\le Y_1-c(i)\\ 0 & \text{otherwise}\end{cases}.$$

Table 1 True and estimated values of the probability of being in state s or below, $s = 0.3$

θ	λ	$\bar{R}^{(1)}_{\theta,\lambda}(s)$	$\bar{R}^{(2)}_{\theta,\lambda}(s)$	$\widetilde{\bar{R}^{(1)}_{\theta,\lambda}}(s)$	$\widetilde{\bar{R}^{(2)}_{\theta,\lambda}}(s)$	MSE_1	MSE_2
2	2	0.1752	0.1964	0.1754	0.1967	0.00402	0.00440
3	2	0.0830	0.0985	0.0836	0.0992	0.00176	0.00213
4	2	0.0409	0.0514	0.0407	0.0512	0.00069	0.00092
3	3	0.1037	0.1231	0.1029	0.1223	0.00244	0.00293
5	3	0.0272	0.0363	0.0269	0.0359	0.00042	0.00062

Table 2 True and estimated values for the probability of being state above s, $s = 0.6$

θ	λ	$R^{(1)}_{\theta,\lambda}(s)$	$R^{(2)}_{\theta,\lambda}(s)$	$\widetilde{R^{(1)}_{\theta,\lambda}}(s)$	$\widetilde{R^{(2)}_{\theta,\lambda}}(s)$	MSE_1	MSE_2
2	2	0.3012	0.3197	0.3009	0.3194	0.00601	0.00616
3	2	0.3615	0.3837	0.3607	0.3829	0.00654	0.00659
4	2	0.4017	0.4263	0.4034	0.4283	0.03400	0.03900
3	3	0.2338	0.2557	0.2335	0.2554	0.00522	0.00552
5	3	0.2923	0.3196	0.2916	0.3189	0.00621	0.00644

In the following we provide some numerical results for system state probabilities based on the functions given by (4) and (5).

A simulation study is performed to compute the estimates and their mean squared errors. Denote by $\widetilde{\bar{R}^{(i)}_{\theta,\lambda}}(s)$ and $\widetilde{R^{(i)}_{\theta,\lambda}}(s)$ MVU estimates of $P\{\xi \le s\}$ and $P\{\xi > s\}$ based on the function $K_i(x)$, $i = 1, 2$. We simulated 5,000 realizations of $\widetilde{\bar{R}^{(i)}_{\theta,\lambda}}(s)$ and $\widetilde{R^{(i)}_{\theta,\lambda}}(s)$ for $n_1 = n_2 = 20$. The mean squared errors of $\widetilde{\bar{R}^{(i)}_{\theta,\lambda}}(s)$ and $\widetilde{R^{(i)}_{\theta,\lambda}}(s)$ as well as the estimated reliabilities are computed and the results are presented in Tables 1 and 2. The true values of system state probabilities are also presented. It is seen that the mean squared errors for $K_1(x)$ are slightly less than the mean squared errors for $K_2(x)$.

4 Multicomponent setup

As it was stated before, the stress–strength interference can also be defined in a more general context where a system consists of n identical components whose random strengths are denoted by $Y_1, Y_2, \ldots, Y_n$ which are mutually independent having common c.d.f. G. If there are component level stresses (no common stress) then the calculation of system reliability is routine because in this case the states of the components, to be denoted by $\xi_i = K(Y_i - X_i)$, $i = 1, \ldots, n$, are independent. This case of component level stress is, therefore, not interesting. Suppose that the components are subjected to a common random stress X and the state of the ith component is defined by

$$\xi_i = K(Y_i - X), \quad i = 1, \ldots, n.$$

Define

$$N_1(s)=\sum_{i=1}^{n} I\left(\xi_i>s\right), \quad N_2(s)=\sum_{i=1}^{n} I\left(\xi_i\leq s\right),$$

where $I(A)=1$ if A occurs and $I(A)=0$ otherwise. It is clear that the random quantity $N_1(s)$ ($N_2(s)$) represent the total number of components whose states above (below) s. These quantities are of course important for the analysis of the whole system and hence their probability distributions should be derived for getting the state probabilities associated with the whole system. The proof of the following Theorems are given in Appendix section.

Theorem 2 *Let X and Y_i's be independent and $F(x)=P\{X\leq x\}$ and $G(x)=P\{Y_i\leq x\}, i=1,\ldots,n$. For $K:\Re\to[0,1]$*

$$P\{N_1(s)=k\}=\binom{n}{k}\int \bar{G}^k(x+K^{-1}(s))G^{n-k}(x+K^{-1}(s))\mathrm{d}F(x), \tag{8}$$

$$P\{N_2(s)=k\}=\binom{n}{k}\int G^k(x+K^{-1}(s))\bar{G}^{n-k}(x+K^{-1}(s))\mathrm{d}F(x), \tag{9}$$

where $\bar{G}=1-G$.

The state probabilities of series and parallel systems in this setup can be easily evaluated. The state of a series system is equal to the state of the worst component in the system while the state of a parallel system is equal to the state of the best component in the system (see, e.g. Barlow and Wu 1978). Using Theorem 2 the probabilities of series and parallel system being state above s can be computed by the following equations:

$$P\left\{\xi^S>s\right\}=P\{N_1(s)=n\}=\int \bar{G}^n(x+K^{-1}(s))\mathrm{d}F(x),$$

$$P\left\{\xi^P>s\right\}=P\{N_1(s)\geq 1\}=1-\int G^n(x+K^{-1}(s))\mathrm{d}F(x).$$

In the following Theorem we present the asymptotic behavior of the proportions of the components whose states above (below) s. This result might be useful for understanding the behavior of large systems.

Theorem 3 *For $K:\Re\to[0,1]$ as $n\to\infty$ we have*

$$\frac{N_1(s)}{n}\overset{d}{\to}\bar{G}(X+K^{-1}(s)),$$

$$\frac{N_2(s)}{n}\overset{d}{\to}G(X+K^{-1}(s)),$$

where $\overset{d}{\to}$ represents the convergence in distribution.

The following example illustrates the finite and asymptotic behaviour of the total number of components whose states above s.

Example 7 Let $F(x) = 1 - \exp\{-\theta x\}$ and $G(x) = 1 - \exp\{-\lambda x\}$, $x \geq 0$. Then for $k < n$ and $0 < s < 1$

$$P\{N_1(s)=k\}=\begin{cases}\binom{n}{k}\sum_{i=0}^{n-k}(-1)^i\binom{n-k}{i}\frac{\theta}{\lambda(k+i)+\theta}\exp\left\{\theta K^{-1}(s)\right\} & \text{if } s \leq K(0)\\ \binom{n}{k}\sum_{i=0}^{n-k}(-1)^i\binom{n-k}{i}\frac{\theta}{\lambda(k+i)+\theta}\exp\left\{-\lambda(k+i)K^{-1}(s)\right\} & \text{if } s > K(0),\end{cases}$$

and the asymptotic distribution of $\frac{N_1(s)}{n}$ as $n \to \infty$ is

$$P\left\{\bar{G}(X + K^{-1}(s)) \leq x\right\} = x^{\theta/\lambda}\exp\left\{\theta K^{-1}(s)\right\}, x < \exp\left\{-\lambda K^{-1}(s)\right\}.$$

Similar discussions can also be done on discrete multi-state modeling.

Define the following random quantities considering the setup given in the Sect. 2.2.

$$S_n^{(i)} = \sum_{j=1}^{n} I\left\{\xi_j = i\right\},$$

$i = 0, 1, .., M$. It is clear that $S_n^{(i)}$ denotes the total number of components whose states are exactly i. It is readily seen that

$$\begin{aligned}&P\left\{S_n^{(0)} = n_0, S_n^{(1)} = n_1, \ldots, S_n^{(M)} = n_M\right\}\\ &= \frac{n!}{n_0!n_1!\ldots n_M!}\int G^{n_0}(x+a_1)\left(G(x+a_2) - G(x+a_1)\right)^{n_1}\\ &\quad\times\left(G(x+a_3) - G(x+a_2)\right)^{n_2}\ldots\left(G(x) - G(x+a_{M_1})\right)^{n_{M_1}}\\ &\quad\times\left(G(x+a_{M_1+1}) - G(x)\right)^{n_{M_1+1}}\ldots\left(1 - G(x+a_{M-1})\right)^{n_M}\mathrm{d}F(x),\end{aligned} \tag{10}$$

where $n_0 + n_1 + \cdots + n_M = n$.

Theorem 4 *For* $K : \Re \to \{0, 1, \ldots, M\}$ *as* $n \to \infty$ *we have*

$$\begin{aligned}\left(\frac{S_n^{(0)}}{n}, \frac{S_n^{(1)}}{n}, \ldots, \frac{S_n^{(M)}}{n}\right) \xrightarrow{d} &\left(G(X+a_1), G(X+a_2)\right.\\ &\left.-G(X+a_1), \ldots, (1 - G(X+a_{M-1}))\right).\end{aligned}$$

Appendix

Proof of Theorem 1 The proof is based on Rao–Blackwell method. An unbiased estimate of $R_{\theta,\lambda}(s)$ is

$$W(X_1, Y_1) = \begin{cases}1 & \text{if } X_1 \leq Y_1 - K^{-1}(s)\\ 0 & \text{otherwise}\end{cases}.$$

Because $\mathbf{T} = (T_1, T_2)$ is complete sufficient statistic the unique MVU estimate of $R_{\theta,\lambda}(s)$ is

$$\widetilde{R_{\theta,\lambda}(s)} = E\left[W(X_1, Y_1) \mid \mathbf{T}\right] = P\left\{X_1 \leq Y_1 - K^{-1}(s) \mid \mathbf{T}\right\}.$$

Letting $W_1 = X_1/T_1$, $W_2 = Y_1/T_2$ and $V = T_1/T_2$, $U = 1/T_2$ one obtains

$$\widetilde{R_{\theta,\lambda}(s)} = P\left\{W_1 V \leq W_2 - U \cdot K^{-1}(s) \mid \mathbf{T}\right\} = \iint_G f(\omega_1, \omega_2 \mid \mathbf{t}) \mathrm{d}\omega_1 \mathrm{d}\omega_2,$$

where $G \equiv \left\{(\omega_1, \omega_2) : \omega_1 V \leq \omega_2 - U \cdot K^{-1}(s), 0 < \omega_1 < 1, 0 < \omega_2 < 1\right\}$. (W_1, W_2) is independent of $\mathbf{T}$ and the p.d.f. of (W_1, W_2) is

$$\begin{aligned} f(\omega_1, \omega_2 \mid \mathbf{t}) &= (n_1 - 1)(n_2 - 1)(1 - \omega_1)^{n_1 - 2}(1 - \omega_2)^{n_2 - 2}, \\ &\quad 0 < \omega_i < 1\ ,\ i = 1, 2. \end{aligned}$$

If $K^{-1}(s) \leq 0$ then $-\frac{U}{V} K^{-1}(s)$ is either (a) below 1 or (b) above 1. For the case (a) : $V + U \cdot K^{-1}(s)$ is either (a^0) below 1 or (b^0) above 1. Therefore for $(a) \wedge (a^0)$ we have

$$\begin{aligned} \widetilde{R_{\theta,\lambda}(s)} &= \int_0^{V + U \cdot K^{-1}(s)} \int_0^{\frac{\omega_2}{V} - \frac{U}{V} K^{-1}(s)} f(\omega_1, \omega_2 \mid \mathbf{t}) \mathrm{d}\omega_1 \mathrm{d}\omega_2 \\ &\quad + \int_{V + U \cdot K^{-1}(s)}^1 \int_0^1 f(\omega_1, \omega_2 \mid \mathbf{t}) \mathrm{d}\omega_1 \mathrm{d}\omega_2, \end{aligned}$$

for $(a) \wedge (b^0)$

$$\widetilde{R_{\theta,\lambda}(s)} = \int_0^1 \int_0^{\frac{\omega_2}{V} - \frac{U}{V} K^{-1}(s)} f(\omega_1, \omega_2 \mid \mathbf{t}) \mathrm{d}\omega_1 \mathrm{d}\omega_2.$$

For the case (b) integration is taken over $0 < \omega_i < 1, i = 1, 2$ and hence $\widetilde{R_{\theta,\lambda}(s)} = 1$.

If $K^{-1}(s) > 0$ then $U \cdot K^{-1}(s)$ is either below 1 or above 1. For the first case

$$\widetilde{R_{\theta,\lambda}(s)} = \int_{U \cdot K^{-1}(s)}^1 \int_0^{\frac{\omega_2}{V} - \frac{U}{V} K^{-1}(s)} f(\omega_1, \omega_2 \mid \mathbf{t}) \mathrm{d}\omega_1 \mathrm{d}\omega_2,$$

and if $U \cdot K^{-1}(s) > 1$ then the integration region is empty and hence $\widetilde{R_{\theta,\lambda}(s)} = 0$. □

Proof of Theorem 2 Because the random variables $Y_1, Y_2, \ldots, Y_n$ are i.i.d.

$$\begin{aligned} P\left\{N_1(s) = k\right\} &= \binom{n}{k} P\left\{\xi_1 > s, \ldots, \xi_k > s, \xi_{k+1} \leq s, \ldots, \xi_n \leq s\right\} \\ &= \binom{n}{k} P\Big\{Y_1 - X > K^{-1}(s), \ldots, Y_k - X > K^{-1}(s), \\ &\qquad Y_{k+1} - X \leq K^{-1}(s), \ldots, Y_n - X \leq K^{-1}(s)\Big\} \end{aligned}$$

The proof now follows conditioning on X. The proof of (9) is similar and hence it is omitted. □

Proof of Theorem 3 One can write

$$\frac{N_1(s)}{n} = \frac{1}{n}\sum_{i=1}^{n} I(Y_i > X + K^{-1}(s)) = \bar{G}_n(X + K^{-1}(s)),$$

$$\frac{N_2(s)}{n} = \frac{1}{n}\sum_{i=1}^{n} I(Y_i \le X + K^{-1}(s)) = G_n(X + K^{-1}(s)),$$

where G_n denotes the empirical distribution function of $Y_1, Y_2, \ldots, Y_n$. The proof now follows from Glivenko–Cantelli theorem. □

Proof of Theorem 4 Consider the joint characteristic function of $\frac{S_n^{(0)}}{n}, \frac{S_n^{(1)}}{n}, \ldots, \frac{S_n^{(M)}}{n}$

$$\begin{aligned}
&\varphi_{\frac{S_n^{(0)}}{n}, \frac{S_n^{(1)}}{n}, \ldots, \frac{S_n^{(M)}}{n}}(t_0, t_1, \ldots, t_M) \\
&\quad = \sum_{n_0+n_1+\cdots+n_M=n} e^{it_0\frac{n_0}{n}} e^{it_1\frac{n_1}{n}} \ldots e^{it_M\frac{n_M}{n}} P\left\{S_n^{(0)}=n_0, S_n^{(1)}=n_1, \ldots, S_n^{(M)}=n_M\right\}.
\end{aligned} \tag{11}$$

Via (10) the p.m.f. of $\left(S_n^{(0)}, S_n^{(1)}, \ldots, S_n^{(M)}\right)$ can be represented as

$$\begin{aligned}
&P\left\{S_n^{(0)} = n_0, S_n^{(1)} = n_1, \ldots, S_n^{(M)} = n_M\right\} \\
&\quad = \frac{n!}{n_0!n_1!\ldots n_M!} E\left(G^{n_0}(X + a_1)(G(X + a_2)\right. \\
&\qquad \left. - G(X + a_1))^{n_1} \ldots (1 - G(X + a_{M-1}))^{n_M}\right).
\end{aligned} \tag{12}$$

Using (12) in (11) one can write

$$\begin{aligned}
&\varphi_{\frac{S_n^{(0)}}{n}, \frac{S_n^{(1)}}{n}, \ldots, \frac{S_n^{(M)}}{n}}(t_0, t_1, \ldots, t_M) \\
&\quad = E\left(\sum_{n_0+n_1+\cdots+n_M=n} \frac{n!}{n_0!n_1!\ldots n_M!} e^{it_0\frac{n_0}{n}} e^{it_1\frac{n_1}{n}} \ldots e^{it_M\frac{n_M}{n}}\right. \\
&\qquad \left. \times\, G^{n_0}(X + a_1)(G(X + a_2) - G(X + a_1))^{n_1} \ldots (1 - G(X + a_{M-1}))^{n_M}\right)
\end{aligned}$$

$$= E\Bigg(\sum_{n_0+n_1+\cdots+n_M=n} \frac{n!}{n_0!n_1!\ldots n_M!}(G(X+a_1)e^{it_0/n})^{n_0}$$

$$\times ((G(X+a_2)-G(X+a_1))e^{it_1/n})^{n_1} \ldots ((1-G(X+a_{M-1}))e^{it_M/n})^{n_M}\Bigg)$$

$$= E\Big[G(X+a_1)e^{it_0/n}$$

$$+(G(X+a_2)-G(X+a_1))e^{it_1/n} + \cdots + (1-G(X+a_{M-1}))e^{it_M/n}\Big]^n \tag{13}$$

Now taking limit of (13) as $n \to \infty$, one obtains

$$\varphi_{\frac{S_n^{(0)}}{n},\frac{S_n^{(1)}}{n},\ldots,\frac{S_n^{(M)}}{n}}(t_0, t_1, \ldots, t_M) \underset{n\to\infty}{\to} \psi(t_0, t_1, \ldots, t_M),$$

where

$$\psi(t_0, t_1, \ldots, t_M) = E\Big(e^{it_0 G(X+a_1)} + e^{it_1(G(X+a_2)-G(X+a_1))} + \cdots + e^{it_M(1-G(X+a_{M-1}))}\Big)$$

and this completes the proof. □

5 Summary and conclusions

In the simple traditional setup of stress–strength models a failure occurs if the stress exceeds the strength and the analysis is based on the binary reliability theory. This setup does not provide further information about the level of failure if system fails and the level of operation if system functions. Having an information about the level of failure might be necessary to determine whether repair is possible or not, as stated in Example 1.

In this paper we studied the stress–strength reliability assigning more than two states to the system depending on the difference between strength and stress random variables. The new perspective presented in the paper actually includes the consideration of stress–strength interference in the framework of multi-state system modeling. By this approach we can have a knowledge not only about the survival and failure probabilities but also the probabilities of the levels of survival and failure. Therefore, the consideration of stress–strength interference in a setup of multi-state system framework enables us to make more sensitive inferences about the system.

As illustrated in the examples, the proposed procedure is also useful in real life problems not only in reliability but also in other fields.

Acknowledgments The author would like to thank the anonymous referees for their useful comments and suggestions.

References

An, Z.-W., Huang, H.-Z., Liu, Y. (2008). A discrete stress–strength interference model based on universal generating function. *Reliability Engineering & System Safety, 93,* 1485–1490.

Barlow, R. E., Wu, A. S. (1978). Coherent systems with multi-state components. *Mathematics of Operation Research, 3*, 275–281.

Bhattacharyya, G. K., Johnson, R. A. (1974). Estimation of reliability in a multicomponent stress–strength model. *Journal of the American Statistical Association, 69*, 966–970.

Block, H. W., Savits, T. H. (1982). A decomposition for multi-state monotone systems. *Journal of Applied Probability, 19*, 391–402.

Eryılmaz, S. (2008a). Consecutive k-out-of-n: G system in stress–strength setup. *Communications in Statistics: Simulation and Computation, 37*, 579–589.

Eryılmaz, S. (2008b). Multivariate stress–strength reliability model and its evaluation for coherent structures. *Journal of Multivariate Analysis*. doi:10.1016/j.jmva.2008.01.015.

Ewing, D. J., Neilson, J. M. M., Travis, P. (1984). New method for assessing cardiac parasympathetic activity using 24 hour elctrocardiograms. *British Heart Journal, 52*, 396–402.

Johnson, R. A. (1988). Stress–strength models for reliability. In P. R. Krishnaiah, C. R. Rao (Eds.) In handbook of statistics (pp. 27–54). Vol. 7, North-Holland: Elsevier.

Kotz, S., Lumelskii, Y., Pensky, M. (2003). *The stress-stength model and its generalizations*. Singapore: World Scientific.

Krishnamoorthy, K., Mukherjee S., Guo, H. Z. (2007). Inference on reliability in two-parameter exponential stress–strength model. *Metrika, 65*, 261–273.

Kuo, W., Zuo, M. J. (2003). *Optimal reliability modeling, principles and applications*. Wiley: New York.

Lisnianski, A., Levitin, G. (2003). *Multi-state system reliability: Assessment, optimization and applications*. Singapore: World Scientific.

Mokhlis, N. A. (2005). Reliability of a stress–strength model with Burr type III distributions. *Communications in Statistics: Theory and Methods, 34*, 1643–1657.

Ross, S. M. (1979). Multivalued state component systems. *The Annals of Probability, 7*, 379–383.

Turkkan, N., Gia, T. P. (2007). System stress–strength reliability: The multivariate case. *IEEE Transactions on Reliability, 56*, 115–124.