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Sangyeol Lee · Yoichi Nishiyama · Nakahiro Yoshida

Test for parameter change in diffusion processes by cusum statistics based on one-step estimators

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Abstract In this paper, we consider the problem of testing for a parameter change using the cusum test based on one-step estimators in diffusion processes. It is shown that under regularity conditions the cusum test statistic has the limiting distribution of a functional of Brownian bridge.

Keywords Test for parameter change · Cusum test · One-step estimator · Diffusion process · Weak convergence · Brownian bridge

1 Introduction

The problem of testing for parameter change has long been a core issue in statistical inferences. It originally started in the quality control context and then rapidly moved to various areas such as economics, finance and medicine. Since the paper of Page (1955), the problem has generated much interest and a vast amount of literatures have been published in various fields. For a general review, we refer to Csörgő and Horváth (1997); Chen and Gupta (2000) and the articles therein. The change point problem was first dealt in i.i.d. samples but it became very popular in time series models since the structural change often occurs in economic models owing to a change of policy and critical social events. For relevant references in

S. Lee
Department of Statistics, Seoul National University, Seoul 151-742, Korea

Y. Nishiyama (✉)
The Institute of Statistical Mathematics, 4-6-7 Minami-Azabu, Minato-ku,
Tokyo 106-8569, Japan
E-mail: nisiyama@ism.ac.jp

N. Yoshida
Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba,
Meguro-ku, Tokyo 153-8914, Japan

i.i.d. samples and time series models, we refer to Hinkley (1971), Brown, Durbin, and Evans (1975), Winchern, Miller, and Hsu (1976), Zacks (1983), Picard (1985), Csörgő and Horváth (1988), Krishnaiah and Miao (1988), Bai (1994), Lee and Park (2001), and the references therein.

In handling the change point problem, one can consider either a parametric approach using the likelihood as in Chen and Gupta (2000) or a nonparametric approach like the cusum test as in Inclán and Tiao (1994). In the same spirit of Inclán and Tiao (1994) and Nyblom (1989), Lee, Ha, Na, and Na (2003) devised a cusum test widely applicable to time series models. The basic concept of the cusum test is the same as the one for the mean and variance change, but it can include a large number of other cases, such as the autoregressive coefficient in the random coefficient autoregressive models, the parameters in GARCH models and nonlinear autoregressive models. In particular, the GARCH model is one of the most popular models in the financial time series context. The cusum test has an advantage that it can test for the existence of change points and allocate their locations.

Although the issue of the change point analysis has drawn much attention from researchers, to our knowledge there are no existing literatures as to continuous time stochastic processes. The diffusion process, the most popular continuous time stochastic process, has been playing a central role in many applications such as finance, engineering and medicine. As with its probabilistic properties themselves, the statistical inference for the diffusion process model has been studied by many authors. For relevant literatures, see Prakasa Rao (1999) and Kutoyants (2004). In view that the ARCH process, which is a discrete version of the diffusion process, often suffers from parameter changes, one can reason that with high possibility the same might occur in modelling financial data set by a diffusion process in actual practice. It is therefore worthwhile to study the parameter change test for diffusion processes.

Conventionally, the cusum test is based on the estimators such as the least squares estimator and maximum likelihood estimator (cf. Lee and Lee 2004; Lee and Na 2004a,b). However, here we employ the cusum test based on a one-step estimator, which certainly has the merit that it includes the case that the maximum likelihood estimator is not given in an explicit form. Hence, our result covers a more general case than those in the literatures mentioned above.

The organization of this paper is as follows. In Sect. 2, we introduce the cusum test for diffusion processes and show that the test statistic converges weakly to the sup of a Brownian bridge. We outline the proof of the theorem in this section, and complete it in Sect. 3.

2 Main result and the outline of the proof

Let Θ be a parameter space which is a bounded, convex, open subset of $\mathbb{R}$. Let $V_0 : \mathbb{R} \times \Theta \to \mathbb{R}$ and $V : \mathbb{R} \to (0, \infty)$ be some functions. Let us consider the one-dimensional stationary diffusion process $t \rightsquigarrow X_t$ given by the stochastic differential equation (SDE)

$$\mathrm{d}X_t = V_0(X_t, \theta)\mathrm{d}t + V(X_t)\mathrm{d}W_t,$$

where $t \rightsquigarrow W_t$ is a one-dimensional standard Wiener process. Here, we implicitly assume that the solutions of the SDE exist. We denote by X^θ the solution under the probability measure P_θ for $\theta \in \Theta$, and by E_θ the expectation with respect to P_θ. The stationary distribution $\nu_\theta(x)dx$ is given by

$$\nu_\theta(x) = \frac{n_\theta(x)}{\int_{\mathbb{R}} n_\theta(y)\mathrm{d}y} \quad \text{with} \quad n_\theta(x) = \frac{1}{V(x)^2} \exp\left(2\int_0^x \frac{V_0(z,\theta)}{V(z)^2}\mathrm{d}z\right).$$

It holds that $E_\theta f(X_t^\theta) = \int_{\mathbb{R}} f(x)\nu_\theta(x)\mathrm{d}x \quad$ for all $t \in [0, \infty)$.

We wish to test the hypotheses

H_0 : The true value $\theta_0 \in \Theta$ of the model does not change. versus

H_1 : not H_0.

Below, we will introduce the cusum test statistic and derive its asymptotic distribution under H_0. Under H_0, the probability and the expectation below will be taken under the probability measure P_{θ_0} and E_{θ_0}, respectively.

Here are the assumptions to ensure the Brownian bridge result for the test statistic below. We denote $\delta^k = \frac{\mathrm{d}^k}{\mathrm{d}\theta^k}$ for an integer k.

Assumption 1 The stochastic process $t \rightsquigarrow X_t^{\theta_0}$ is stationary and α-mixing with exponential rate under P_{θ_0}.

We assume the α-mixing property only for applying a functional limit theorem for stationary processes, namely, Theorem VIII.3.79 of Jacod and Shiryaev (1987). Thus, we can replace the assumption by any other condition which implies the assumption 3.80 with $p = q = 2$ there; see also Lemma VIII.3.102. As for the mixing rate, we refer to Veretennikov (1987, 1997).

Assumption 2 For every $x \in \mathbb{R}$, the function $\theta \rightsquigarrow V_0(x, \theta)$ is three times continuously differentiable. Moreoever, $E_\theta[|\frac{\delta V_0}{V}(X_0^\theta, \theta)|^2] > 0$ for every $\theta \in \Theta$.

Assumption 3 $E_{\theta_0}[|\frac{\delta^k V_0}{V}(X_0^{\theta_0}, \theta_0)|^{2q}] < \infty \quad$ for some $q > 1$, for every $k = 1, 2, 3$.

Assumption 4 It holds that

$$E_{\theta_0}\left[\sup_{\theta\in\Theta}\left|\frac{\delta^k V_0}{V^2}(X_0, \theta)V_0(X_0, \theta_0)\right|\right] < \infty, \quad \text{for every } k = 1, 2, 3.$$

Also, it holds that

$$E_{\theta_0}\left[\sup_{\theta\in\Theta}\left|\frac{\delta^k V_0 \delta^l V_0}{V^2}(X_0, \theta)\right|\right] < \infty,$$
$$\text{for every } k, l = 0, 1, 2, 3 \text{ such that } 1 \le k + l \le 3.$$

Motivated by the fact that the log-likelihood is given by

$$\ell_T^{\theta_0}(\theta) = \int_0^T \frac{V_0}{V^2}(X_t^{\theta_0}, \theta)\mathrm{d}X_t^{\theta_0} - \frac{1}{2}\int_0^T \left(\frac{V_0}{V}\right)^2 (X_t^{\theta_0}, \theta)\mathrm{d}t,$$

we define $\psi_T^{\theta_0}(\theta)$ and $\eta(\theta)$ as follows:

$$\psi_T^{\theta_0}(\theta) := \int_0^T \frac{\delta V_0}{V^2}(X_t^{\theta_0}, \theta)\mathrm{d}X_t^{\theta_0} - \int_0^T \frac{V_0 \delta V_0}{V^2}(X_t^{\theta_0}, \theta)\mathrm{d}t;$$

$$\eta(\theta) := -E_\theta\left[\left(\frac{\delta V_0}{V}\right)^2 (X_0^\theta, \theta)\right].$$

As for the latter, we have kept the stationarity of X in mind. In fact, it holds for any T that

$$E_{\theta_0}[\delta\psi_T^{\theta_0}(\theta_0)] = T\eta(\theta_0), \quad \forall\theta_0 \in \Theta. \tag{1}$$

Now, we consider the cusum test procedure based on one-step estimators. We assume that an initial estimator $\widehat{\theta}_T^0$ satisfies the following.

Assumption 5 Fix $\theta_* \in \Theta$. For $T \in [0, 1]$, set $\widehat{\theta}_T^0 = \theta_*$. For $T \in (1, \infty)$, it holds that $\sup_{T\in(1,\infty)} r_T|\widehat{\theta}_T^0 - \theta_0| < \infty$, P_{θ_0}-almost surely, for some $r_T > 0$ with $T^{1/4}r_T^{-1} \to 0$ as $T \to \infty$.

The rate like $r_T = T^{1/4}\log\log(T + 2)$ would be useful. Also, if we set $r_T = 1$ for $T \in [0, 1]$, it is clear that for any $\eta > 0$, there exists $M_\eta > 0$, such that

$$P_{\theta_0}\left[\sup_{T\in[0,\infty)} r_T|\widehat{\theta}_T^0 - \theta_0| > M_\eta\right] \le \eta. \tag{2}$$

Taking account of (1), we define the one-step estimator $\widehat{\theta}_T$ by

$$\widehat{\theta}_T = \widehat{\theta}_T^0 - [T\eta(\widehat{\theta}_T^0)]^{-1}\psi_T^{\theta_0}(\widehat{\theta}_T^0), \tag{3}$$

and consider the stochastic process $u \rightsquigarrow S_T^u$ given by

$$S_T^u = u\sqrt{T}(\widehat{\theta}_{uT} - \widehat{\theta}_T), \quad \forall u \in [0, 1]. \tag{4}$$

The following is the main theorem of this paper.

Theorem 1 *Assume that Assumptions 1–5 hold. Then, under H_0, as $T \to \infty$, the sequence of stochastic processes $u \rightsquigarrow S_T^u$ converges weakly to a zero-mean Gaussian process $u \rightsquigarrow Y^u$ in $\ell^\infty([0, 1])$, such that $EY^uY^v = -\eta(\theta_0)^{-1}\{(u \wedge v) - uv\}$.*

Below, we present an outline of the proof of Theorem 1. Some relevant lemmas to complete the proof will be given in the next section.

From the above theorem, we can easily derive the asymptotic behavior of the test statistic

$$\widehat{S}_T = \sup_{u\in[0,1]} u^2T(-\eta(\widehat{\theta}_T))|\widehat{\theta}_{uT} - \widehat{\theta}_T|^2. \tag{5}$$

Corollary 2 *Assume Assumptions 1–5 hold. Then under H_0, the sequence of random variables $\widehat{S}_T$ converges weakly to* $\sup_{u\in[0,1]}|W_u^\circ|^2$ *as $T \to \infty$, where $u \rightsquigarrow W_u^\circ$ is a standard Brownian bridge.*

Now we address the outline of the proof of Thoerem 1. Recalling (3), it follows from the Taylor expansion that

$$\widehat{\theta}_T - \theta_0 = -\frac{1}{T}\eta(\theta_0)^{-1}\psi_T^{\theta_0}(\theta_0) + \Delta_T,$$

where $\Delta_T = \Delta_T^{(1)} + \Delta_T^{(2)} + \Delta_T^{(3)}$ and

$$\Delta_T^{(1)} = (\widehat{\theta}_T^0 - \theta_0) - \frac{1}{T}\eta(\theta_0)^{-1}\delta\psi_T^{\theta_0}(\theta_0)(\widehat{\theta}_T^0 - \theta_0) \tag{6}$$

$$\Delta_T^{(2)} = \frac{1}{T}\eta(\theta_0)^{-2}\delta\eta(\theta_0)\psi_T^{\theta_0}(\theta_0)(\widehat{\theta}_T^0 - \theta_0) \tag{7}$$

$$\Delta_T^{(3)} = -\frac{1}{T}\int_0^1 (1-\alpha)\delta^2(\eta^{-1}\psi_T^{\theta_0})(\theta_0 + \alpha(\widehat{\theta}_T^0 - \theta_0))\mathrm{d}\alpha \cdot (\widehat{\theta}_T^0 - \theta_0)^2. \tag{8}$$

So we have

$$\begin{aligned}\widehat{\theta}_{uT} - \widehat{\theta}_T &= (\widehat{\theta}_{uT} - \theta_0) - (\widehat{\theta}_T - \theta_0)\\ &= \frac{\eta(\theta_0)^{-1}}{uT}(-\psi_{uT}^{\theta_0}(\theta_0) + u\psi_T^{\theta_0}(\theta_0)) + (\Delta_{uT} - \Delta_T).\end{aligned}$$

Recall (4). In the next section, we will show the weak convergence of the process $u \rightsquigarrow Y_T^u$, where

$$Y_T^u = \frac{\eta(\theta_0)^{-1}}{\sqrt{T}}(-\psi_{uT}^{\theta_0}(\theta_0) + u\psi_T^{\theta_0}(\theta_0)), \tag{9}$$

to the process $u \rightsquigarrow Y^u$ that appeared in Theorem 1, and the assertion that

$$\sup_{u\in[0,1]} u\sqrt{T}|\Delta_{uT}^{(i)}| = o_{P_{\theta_0}}(1) \quad \text{for } i = 1, 2, 3.$$

These two facts ensure the result of our main theorem.

Remarks It is possible to generalize our result to the case where Θ is a subset of $\mathbb{R}^J$ for $J \geq 1$. In this case, the test statistic $\widehat{S}_T$ given by (5) is replaced by

$$\widehat{S}_T = \sup_{u\in[0,1]} u^2 T(\widehat{\theta}_{uT} - \widehat{\theta}_T)'(-\eta(\widehat{\theta}_T))(\widehat{\theta}_{uT} - \widehat{\theta}_T),$$

where $-\eta(\theta)$ is the Fisher information matrix. Then the limit in Corollary 2 is replaced by $S^J = \sup_{u\in[0,1]}\sum_{j=1}^J |W_u^{\circ,j}|^2$, where $u \rightsquigarrow (W_u^{\circ,1}, \ldots, W_u^{\circ,J})'$ is a J-dimensional standard Brownian bridge. Using our result, one can determine the critical region $(\widehat{S}_T \geq C_\alpha)$, given a level α, where C_α is the $(1-\alpha)$-quantile point of S^J. Although it is not easy to calculate the critical values analytically, Lee et al. (2003) give a table through a Monte Carlo simulation (see their Table 1).

3 Lemmas and their proofs

In this section, we will prove some lemmas to show Theorem 1. Since the true value θ_0 is fixed, we will mostly denote $P = P_{\theta_0}$, $E = E_{\theta_0}$, $X = X^{\theta_0}$, $\psi_T(\theta) = \psi_T^{\theta_0}(\theta)$, $\delta\psi_T(\theta) = \delta\psi_T^{\theta_0}(\theta)$ and $\delta^2\psi_T(\theta) = \delta^2\psi_T^{\theta_0}(\theta)$, without any confusion.

Lemma 3 *Under Assumptions 1–3, it holds that under $P = P_{\theta_0}$, the sequence of the processes $u \rightsquigarrow Y_T^u$ given by (9) converges weakly in $\ell^\infty([0,1])$ to the process $u \rightsquigarrow Y^u$ in Theorem 1.*

Proof Put

$$M_z^{u,T} = \frac{\eta(\theta_0)^{-1}}{\sqrt{T}} \int_0^{zT} \left\{-1_{\{s\le uT\}} + u\right\} \frac{\delta V_0}{V}(X_s, \theta_0)\mathrm{d}W_s, \quad \forall z \in [0,1].$$

Then, the process $z \rightsquigarrow M_z^{u,T}$ is a continuous martingale, and it holds that $M_1^{u,T} = Y_T^u$. Here, we have

$$\begin{aligned}
&\langle M^{u,T}, M^{v,T}\rangle_1 \\
&= \frac{\eta(\theta_0)^{-2}}{T} \int_0^T \left\{1_{\{s\le uT\}} - u\right\}\left\{1_{\{s\le vT\}} - v\right\}\left(\frac{\delta V_0}{V}(X_s,\theta_0)\right)^2 \mathrm{d}s \\
&= \frac{\eta(\theta_0)^{-2}}{T}\left\{\int_0^{(u\wedge v)T}\left(\frac{\delta V_0}{V}(X_s,\theta_0)\right)^2 \mathrm{d}s - v\int_0^{uT}\left(\frac{\delta V_0}{V}(X_s,\theta_0)\right)^2 \mathrm{d}s\right. \\
&\quad \left. - u\int_0^{vT}\left(\frac{\delta V_0}{V}(X_s,\theta_0)\right)^2 \mathrm{d}s + uv\int_0^{T}\left(\frac{\delta V_0}{V}(X_s,\theta_0)\right)^2 \mathrm{d}s\right\}.
\end{aligned}$$

So it holds that

$$\langle M^{u,T}, M^{v,T}\rangle_1 \xrightarrow{P} -\eta(\theta_0)^{-1}\{(u\wedge v) - uv\}.$$

Hence we have the finite-dimensional convergence. On the other hand, to show the uniform tightness, we will apply Theorem 3.4.2 of Nishiyama (2000), although a more classical tightness criterion might work well, too. For $q > 1$ which is in Assumption 3, choose $p > 1$ such that $(1/p) + (1/q) = 1$. Then we have

$$\begin{aligned}
&\langle M^{u,T} - M^{v,T}, M^{u,T} - M^{v,T}\rangle_1 \\
&= \frac{\eta(\theta_0)^{-2}}{T}\int_0^T \left\{(1_{\{s\le uT\}} - u) - (1_{\{s\le vT\}} - v)\right\}^2 \left(\frac{\delta V_0}{V}(X_s,\theta_0)\right)^2 \mathrm{d}s \\
&\le \frac{\eta(\theta_0)^{-2}}{T}\int_0^T \left\{(1_{\{s\le uT\}} - 1_{\{s\le vT\}}) - (u - v)\right\}^2 \left(\frac{\delta V_0}{V}(X_s,\theta_0)\right)^2 \mathrm{d}s \\
&\le \eta(\theta_0)^{-2}\left\{\frac{1}{T}\int_0^T \left\{(1_{\{s\le uT\}} - 1_{\{s\le vT\}}) - (u-v)\right\}^{2p}\mathrm{d}s\right\}^{\frac{1}{p}} Z_T^{(q)},
\end{aligned}$$

where

$$Z_T^{(q)} = \left\{ \frac{1}{T} \int_0^T \left(\frac{\delta V_0}{V}(X_s, \theta_0) \right)^{2q} \mathrm{d}s \right\}^{\frac{1}{q}}.$$

The right hand side of the above inequality is bounded by

$$\begin{aligned} &\eta(\theta_0)^{-2} \left\{ \frac{C_p}{T} \int_0^T \left\{ |1_{\{s \le uT\}} - 1_{\{s \le vT\}}|^{2p} + |u - v|^{2p} \right\} \mathrm{d}s \right\}^{\frac{1}{p}} Z_T^{(q)} \\ &\le \eta(\theta_0)^{-2} \left\{ C_p \left\{ |u - v| + |u - v|^{2p} \right\} \right\}^{\frac{1}{p}} Z_T^{(q)} \\ &\le \eta(\theta_0)^{-2} (2C_p)^{1/p} |u - v|^{\frac{1}{p}} Z_T^{(q)}, \end{aligned}$$

where $C_p > 0$ is a constant depending only on p. Since $Z_T^{(q)} = O_P(1)$ by assumption, we have

$$\sup_{u \neq v} \frac{\langle M^{u,T} - M^{v,T}, M^{u,T} - M^{v,T} \rangle_1}{|u - v|^{1/p}} = O_P(1).$$

This completes the proof. □

Lemma 4 *Under Assumptions 1–3 and 5, it holds that* $\sup_{u \in [0,1]} u\sqrt{T} |\Delta_{uT}^{(2)}| = o_{P_{\theta_0}}(1)$ *as* $T \to \infty$, *where* $\Delta_T^{(2)}$ *is given by (7).*

Proof Fix any $\varepsilon, \eta > 0$. Choose $M = M_\eta > 0$ for which (2) is satisfied. Choose any (small) $u_0 \in (0, 1)$, then we have

$$\begin{aligned} P\left[\sup_{u \in [0,1]} u\sqrt{T} |\Delta_{uT}^{(2)}| > \varepsilon \right] &\le P\left[\sup_{u \in [0,u_0]} u\sqrt{T} |\Delta_{uT}^{(2)}| > \varepsilon \right] \\ &\quad + P\left[\sup_{u \in (u_0,1]} u\sqrt{T} |\Delta_{uT}^{(2)}| > \varepsilon \right] \\ &=: (\mathrm{I}) + (\mathrm{II}). \end{aligned}$$

By (2), it holds that

$$\begin{aligned} (\mathrm{I}) &\le P\left[\sup_{u \in [0,u_0]} u\sqrt{T} |\Delta_{uT}^{(2)}| > \varepsilon, \sup_{u \in [0,u_0]} |\widehat{\theta}_{uT}^0 - \theta_0| \le M \right] + \eta \\ &\le P\left[\sup_{u \in [0,u_0]} \left| \frac{c}{\sqrt{T}} \psi_{uT}(\theta_0) \right| M > \varepsilon \right] + \eta, \end{aligned}$$

where $c = \eta(\theta_0)^{-2}\delta\eta(\theta_0)$. Since $\psi_t(\theta_0) = \int_0^t \frac{\delta V_0}{V}(X_s, \theta_0)\mathrm{d}W_s$, it follows from Doob's inequality that the right hand side is bounded by

$$\frac{M^2c^2}{\varepsilon^2 T} E\left[\int_0^{u_0T} \left(\frac{\delta V_0}{V}\right)^2 (X_s, \theta_0)\mathrm{d}s\right] + \eta$$
$$= \frac{M^2c^2}{\varepsilon^2 T} \int_0^{u_0T} E\left[\left(\frac{\delta V_0}{V}\right)^2 (X_0, \theta_0)\right] \mathrm{d}s + \eta$$
$$= \frac{M^2c^2}{\varepsilon^2} E\left[\left(\frac{\delta V_0}{V}\right)^2 (X_0, \theta_0)\right] \cdot u_0 + \eta.$$

Since u_0 can be taken to be arbitrary small, the right hand side can also be arbitrary small.

Next, fix any $\eta' > 0$. Choose any large $T > 0$ such that $1 < u_0T$ and $Mr_{u_0T}^{-1} \leq \eta'$. For such T, it follows from (2) that

$$(\mathrm{II}) \leq P\left[\sup_{u\in(u_0,1]} u\sqrt{T}|\Delta_{uT}^{(2)}| > \varepsilon,\ \sup_{u\in(u_0,1]} |\widehat{\theta}_{uT}^0 - \theta_0| \leq \eta'\right] + \eta$$
$$= P\left[\sup_{u\in(u_0,1]} \left|\frac{c}{\sqrt{T}}\psi_{uT}(\theta_0)(\widehat{\theta}_{uT}^0 - \theta_0)\right| > \varepsilon,\ \sup_{u\in(u_0,1]} |\widehat{\theta}_{uT}^0 - \theta_0| \leq \eta'\right] + \eta$$
$$\leq P\left[\sup_{u\in(u_0,1]} \left|\frac{c}{\sqrt{T}}\psi_{uT}(\theta_0)\right| \eta' > \varepsilon\right] + \eta$$

where $c = \eta(\theta_0)^{-2}\delta\eta(\theta_0)$. In the same way as above, we have that the right hand side is bounded by

$$\frac{|\eta'|^2c^2}{\varepsilon^2 T} E\left[\int_0^{T} \left(\frac{\delta V_0}{V}\right)^2 (X_s, \theta_0)\mathrm{d}s\right] + \eta$$
$$\leq \frac{|\eta'|^2c^2}{\varepsilon^2} E\left[\left(\frac{\delta V_0}{V}\right)^2 (X_0, \theta_0)\right] + \eta.$$

Since η' can be taken to be arbitrary small, the right hand side can also be arbitrary small. The proof is finished. □

Lemma 5 *Under Assumptions 1–3 and 5, it holds that* $\sup_{u\in[0,1]} u\sqrt{T}|\Delta_{uT}^{(1)}| = o_{P_{\theta_0}}(1)$ *as* $T \to \infty$, *where* $\Delta_T^{(1)}$ *is given by (6).*

Proof By definition it holds that

$$u\sqrt{T}\Delta_{uT}^{(1)} = -\frac{\delta\psi_{uT}(\theta_0) - uT\eta(\theta_0)}{\sqrt{T}\eta(\theta_0)}(\widehat{\theta}_{uT}^0 - \theta_0).$$

Notice that

$$\begin{aligned}
&|\delta\psi_t(\theta_0) - t\eta(\theta_0)| \\
&= \left| \int_0^t \frac{\delta^2 V_0}{V}(X_v, \theta_0)\mathrm{d}W_v \right. \\
&\quad \left. + \int_0^t \left\{ -\left(\frac{\delta V_0}{V}\right)^2 (X_v, \theta_0) + E\left(\frac{\delta V_0}{V}\right)^2 (X_0, \theta_0) \right\} \mathrm{d}v \right| \\
&\le \left| \int_0^t \frac{\delta^2 V_0}{V}(X_v, \theta_0)\mathrm{d}W_v \right| \\
&\quad + \left| \int_0^t \left\{ \left(\frac{\delta V_0}{V}\right)^2 (X_v, \theta_0) - E\left(\frac{\delta V_0}{V}\right)^2 (X_0, \theta_0) \right\} \mathrm{d}v \right| \\
&= \text{(I)} + \text{(II)}.
\end{aligned}$$

As for (I), we have

$$\begin{aligned}
E\left[\sup_{t \le s} \left| \int_0^t \frac{\delta^2 V_0}{V}(X_v, \theta_0)\mathrm{d}W_v \right|^2 \right] &\le E\left[\int_0^s \left| \frac{\delta^2 V_0}{V}(X_v, \theta_0) \right|^2 \mathrm{d}v \right] \\
&\le s \cdot \sup_{v \in [0,\infty)} E\left[\left| \frac{\delta^2 V_0}{V}(X_v, \theta_0) \right|^2 \right].
\end{aligned}$$

Thus the same argment as the proof of the previous lemma establishes the estimate of the term (I).

On the other hand, by virtue of Assumption 1, Theorem VIII.3.79 of Jacod and Shiryaev (1987) implies that $\frac{1}{\sqrt{t}}$(II) is uniformly tight as a stochastic process with parameter $t \in [0, \infty)$. So Assumption 5 yields the conclusion of the lemma. □

Lemma 6 *Under Assumptions 1–5, it holds that* $\sup_{u\in[0,1]} u\sqrt{T}|\Delta_{uT}^{(3)}| = o_{P_{\theta_0}}(1)$ *as* $T \to \infty$, *where* $\Delta_T^{(3)}$ *is given by (8).*

Proof We will show that

$$\left| \int_0^1 (1-\alpha)(\delta^2(\eta^{-1}\psi_t)(\theta_0 + \alpha(\widehat{\theta}_t^0 - \theta_0))\mathrm{d}\alpha \right| \le C\xi_t,$$

where $C > 0$ is a constant, and ξ_t is a stochastic process, such that

$$\sup_{t\le 1} \xi_t < \infty \quad \text{a.s.} \tag{10}$$

and that

$$\frac{1}{t}\xi_t = O_P(1) \quad \text{as } t \to \infty. \tag{11}$$

Since

$$\delta^2(\eta^{-1}\psi_t)(\theta) = \delta^2(\eta^{-1})(\theta)\psi_t(\theta) + 2\delta(\eta^{-1})(\theta)\delta\psi_t(\theta) + \eta^{-1}(\theta)\delta^2\psi_t(\theta),$$

we have

$$\sup_{\theta\in\Theta} |\delta^2(\eta^{-1}\psi_t)(\theta)| \le \text{const.} \sup_{\theta\in\Theta} \sum_{k=0}^{2} |\delta^k\psi_t(\theta)|. \tag{12}$$

Here, noticing that

$$\begin{aligned}
\psi_t(\theta) &= \int_0^t \frac{\delta V_0}{V}(X_s,\theta)\mathrm{d}W_s + \int_0^t \frac{\delta V_0}{V^2}(X_s,\theta)V_0(X_s,\theta_0)\mathrm{d}s \\
&\quad - \int_0^t \frac{V_0\delta V_0}{V^2}(X_s,\theta)\mathrm{d}s, \\
\delta\psi_t(\theta) &= \int_0^t \frac{\delta^2 V_0}{V}(X_s,\theta)\mathrm{d}W_s + \int_0^t \frac{\delta^2 V_0}{V^2}(X_s,\theta)V_0(X_s,\theta_0)\mathrm{d}s \\
&\quad - \int_0^t \frac{(\delta V_0)^2}{V^2}(X_s,\theta)\mathrm{d}s - \int_0^t \frac{V_0\delta^2 V_0}{V^2}(X_s,\theta)\mathrm{d}s, \\
\delta^2\psi_t(\theta) &= \int_0^t \frac{\delta^3 V_0}{V}(X_s,\theta)\mathrm{d}W_s + \int_0^t \frac{\delta^3 V_0}{V^2}(X_s,\theta)V_0(X_s,\theta_0)\mathrm{d}s \\
&\quad -3\int_0^t \frac{\delta V_0\delta^2 V_0}{V^2}(X_s,\theta)\mathrm{d}s - \int_0^t \frac{V_0\delta^3 V_0}{V^2}(X_s,\theta)\mathrm{d}s,
\end{aligned}$$

we can see that the right hand side of (12) is bounded by a constant times ξ_t, where

$$\begin{aligned}
\xi_t = \sup_{\theta\in\Theta} \Bigg\{ &\sum_{k=1}^{3} \left| \int_0^t \frac{\delta^k V_0}{V}(X_s,\theta)\mathrm{d}W_s \right| + \sum_{k=1}^{3} \left| \int_0^t \frac{\delta^k V_0}{V^2}(X_s,\theta)V_0(X_s,\theta_0)\mathrm{d}s \right| \\
&+ \left| \int_0^t \frac{V_0\delta V_0}{V^2}(X_s,\theta)\mathrm{d}s \right| + \left| \int_0^t \frac{(\delta V_0)^2}{V^2}(X_s,\theta)\mathrm{d}s \right| + \left| \int_0^t \frac{V_0\delta^2 V_0}{V^2}(X_s,\theta)\mathrm{d}s \right| \\
&+ \left| \int_0^t \frac{\delta V_0\delta^2 V_0}{V^2}(X_s,\theta)\mathrm{d}s \right| + \left| \int_0^t \frac{V_0\delta^3 V_0}{V^2}(X_s,\theta)\mathrm{d}s \right| \Bigg\}.
\end{aligned}$$

Since $t \rightsquigarrow \xi_t$ is continuous, (10) is trivial. On the other hand, notice that the right hand side contains two kinds of integrals, namely, the stochastic integrals with respect to the Brownian motion W and the (normal) integrals with respect to the Lebesgue measure. To get (11), we can use Nishiyama's (2000) theorem for the former by Assumption 3 in the same way as the proof of Lemma 3, and the law of large numbers for the latter by Assumption 4.

Now, notice that

$$\begin{aligned}\sup_{u\in[0,1]} u\sqrt{T}|\Delta_{uT}^{(3)}| &\le \sup_{u\in[0,1]} \frac{1}{\sqrt{T}}\xi_{uT}(\widehat{\theta}_{uT}^0-\theta_0)^2 \\ &\le \sup_{u\in[0,T^{-1}]} \frac{1}{\sqrt{T}}\xi_{uT}(\widehat{\theta}_{uT}^0-\theta_0)^2 + \sup_{u\in(T^{-1},1]} \frac{1}{\sqrt{T}}\xi_{uT}(\widehat{\theta}_{uT}^0-\theta_0)^2 \\ &=: \text{(I)} + \text{(II)}.\end{aligned}$$

Note that owing to (10),

$$\text{(I)} = \sup_{t\in[0,1]} \frac{1}{\sqrt{T}}\xi_t(\widehat{\theta}_t^0-\theta_0)^2 = O_P\left(\frac{1}{\sqrt{T}}\right) \quad \text{as } T\to\infty.$$

On the other hand, using (11), we have

$$\begin{aligned}\text{(II)} &= \sup_{u\in(T^{-1},1]} \frac{1}{uT}\xi_{uT}\cdot\frac{uT}{\sqrt{T}}(\widehat{\theta}_{uT}^0-\theta_0)^2 \\ &\le \sup_{t>1}\frac{1}{t}\xi_t \cdot \sup_{u\in(T^{-1},1]}\frac{uT}{\sqrt{T}r_{uT}^2}\cdot \sup_{u\in(T^{-1},1]} r_{uT}^2(\widehat{\theta}_{uT}^0-\theta_0)^2 \\ &= O_P(1)\cdot o(1)\cdot O_P(1) \quad \text{as } T\to\infty.\end{aligned}$$

This establishes the lemma. □

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