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Confidence intervals for quantiles and tolerance intervals based on ordered ranked set samples

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Abstract Confidence intervals for quantiles and tolerance intervals based on ordered ranked set samples (ORSS) are discussed in this paper. For this purpose, we first derive the cdf of ORSS and the joint pdf of any two ORSS. In addition, we obtain the pdf and cdf of the difference of two ORSS, viz. $X_{s:N}^{\mathrm{ORSS}} - X_{r:N}^{\mathrm{ORSS}}$, $1 \leq r < s \leq N$. Then, confidence intervals for quantiles based on ORSS are derived and their properties are discussed. We compare with approximate confidence intervals for quantiles given by Chen (*Journal of Statistical Planning and Inference*, **83**, 125–135; 2000), and show that these approximate confidence intervals are not very accurate. However, when the number of cycles in the RSS increases, these approximate confidence intervals become accurate even for small sample sizes. We also compare with intervals based on usual order statistics and find that the confidence interval based on ORSS becomes considerably narrower than the one based on usual order statistics when n becomes large. By using the cdf of $X_{s:N}^{\mathrm{ORSS}} - X_{r:N}^{\mathrm{ORSS}}$, we then obtain tolerance intervals, discuss their properties, and present some tables for two-sided tolerance intervals.

Keywords Order statistics · Confidence interval · Expected width · Quantile · Percentage reduction

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1 Introduction

The basic procedure of obtaining a ranked set sample is as follows. First, we draw a random sample of size n from the population and order them (without actual measurement, for example, visually). Then, the smallest observation is measured and denoted as $X_{(1)}$, and the remaining are not measured. Next, another sample of size n is drawn and ordered, and only the second smallest observation is measured and denoted as $X_{(2)}$. This procedure is continued until the largest observation of the nth sample of size n is measured. The collection $\{X_{(1)}, \ldots, X_{(n)}\}$ is called as a *one-cycle ranked set sample* of size n. If we replicate the above procedure m times, we finally get a ranked set sample of total size $N = mn$. The data thus collected in this case is denoted by $\boldsymbol{X}_{\text{RSS}} = \left\{X_{1(1)}, X_{2(1)}, \ldots, X_{m(1)}, \ldots, X_{1(n)}, X_{2(n)}, \ldots, X_{m(n)}\right\}$.

The ranked set sampling was first proposed by McIntyre (1952) in order to find a more efficient method to estimate the average yield of pasture. Since then, numerous parametric and nonparametric inferential procedures based on ranked set samples have been developed in the literature. The reader is referred to, among others, Takahasi and Wakimoto (1968), Dell and Clutter (1972), Stokes (1977, 1980a,b, 1995), Chuiv and Sinha (1998), Stokes and Sager (1988), and Chen (1999, 2000a,b). For a comprehensive review of various developments on ranked set sampling, we refer the reader to Patil et al. (1999) and the monograph by Chen et al. (2004).

Distribution-free confidence intervals for quantiles and tolerance intervals based on the usual order statistics of simple random sample (OSRS) are well known in the literature; see David and Nagaraja (2003). In this paper, we extend these ideas to ordered ranked set samples (ORSS). In Sect. 2, we present the pdf, the cdf and the joint pdf of ORSS, as well as the corresponding formulas for the uniform distribution. Section 3 focuses on confidence intervals for quantiles based on RSS and their properties. Chen (2000b) gave approximate confidence intervals for quantiles by using the central limit theorem and we show that these approximate confidence intervals are not very accurate by computing the corresponding exact confidence levels and showing that they are significantly different from the nominal level. However, when the number of cycles in the RSS increases, these approximate confidence intervals become accurate even for small sample sizes. In Sect. 4, we derive tolerance intervals and discuss their properties. Finally, some tables for the two-sided tolerance intervals are given in this section.

2 Ordered ranked set samples

We first note that all $X_{i(j)}$'s $(1 \le i \le m,\ 1 \le j \le n)$ are independent. Moreover, for a fixed j, $X_{i(j)}$'s $(1 \le i \le m)$ are identically distributed with pdf $f_{j:n}(x)$. It is easy to see that if the ranking in RSS is perfect, $f_{j:n}(x)$ is actually the pdf of the jth order statistic from a SRS of size n, and is given by (see Arnold, Balakrishnan, & Nagaraja, 1992; David & Nagaraja, 2003)

$$f_{j:n}(x) = \frac{n!}{(j-1)!(n-j)!}\,[F(x)]^{j-1}\,[1-F(x)]^{n-j}\,f(x), \quad -\infty < x < \infty. \tag{1}$$

Let $\boldsymbol{X}_{\text{ORSS}} = \{X_{1:N}^{\text{ORSS}} \le X_{2:N}^{\text{ORSS}} \le \cdots \le X_{N:N}^{\text{ORSS}}\}$ denote the ORSS obtained by arranging $X_{i(j)}$'s in increasing order of magnitude. Then, using the results for order statistics from independent and non-identically distributed random variables (Vaughan and Venables, 1972; Balakrishnan, 1988, 1989), the distribution function of $X_{r:N}^{\text{ORSS}}$ $(1 \le r \le N)$ can be written as

$$F_{r:N}^{\text{ORSS}}(x) = \sum_{i=r}^{N} \sum_{S_i^{[N]}} \left\{ \prod_{l=1}^{i} F_{k_l:n}(x) \prod_{l=i+1}^{N} [1 - F_{k_l:n}(x)] \right\}$$
$$= \sum_{i=r}^{N} \sum_{S_i^{[N]}} \left\{ \prod_{l=1}^{i} I_{F(x)}(k_l, n-k_l+1) \prod_{l=i+1}^{N} [1 - I_{F(x)}(k_l, n-k_l+1)] \right\}, \tag{2}$$

where $\sum_{S_i^{[N]}}$ denotes the summation over all permutations $(j_1, j_2, \ldots, j_N)$ of $(1, 2, \ldots, N)$ for which $j_1 < \cdots < j_i$ and $j_{i+1} < \cdots < j_N$, $I_p(a, b)$, called as incomplete beta function, is defined by

$$I_p(a, b) = \frac{1}{B(a, b)} \int_0^p t^{a-1}(1-t)^{b-1} \mathrm{d}t,$$

$B(a, b)$ is complete beta function, and

$$k_l = \begin{cases} [j_l/m] & \text{if } j_l/m = [j_l/m],\, 1 \le l \le N, \\ [j_l/m] + 1 & \text{if } j_l/m > [j_l/m],\, 1 \le l \le N. \end{cases}$$

Moreover, the joint density function of $X_{r:N}^{\text{ORSS}}$ and $X_{s:N}^{\text{ORSS}}$ $(1 \le r < s \le N)$ can be expressed as

$$f_{r,s:N}^{\text{ORSS}}(x, y) = \frac{1}{(r-1)!(s-r-1)!(N-s)!} \sum_{P^{[N]}} \left\{ \prod_{k=1}^{r-1} \left[F_{i_k:N}(x)\right] f_{i_r:N}(x) \right.$$
$$\left. \times \prod_{k=r+1}^{s-1} \left[F_{i_k:N}(y) - F_{i_k:N}(x)\right] f_{i_s:N}(y) \prod_{k=s+1}^{N} \left[1 - F_{i_k:N}(y)\right] \right\}$$
$$= \sum_{P^{[N]}} \sum_{k_1=j_1}^{n} \cdots \sum_{k_{s-1}=j_{s-1}}^{n} \sum_{k_s=0}^{j_s-1} \cdots \sum_{k_N=0}^{j_N-1} \sum_{l_1=0}^{n-k_1} \cdots \sum_{l_{r-1}=0}^{n-k_{r-1}} \sum_{l_{r+1}=k_{r+1}+1-j_{r+1}}^{k_{r+1}}$$
$$\cdots \sum_{l_{s-1}=k_{s-1}+1-j_{s-1}}^{k_{s-1}} \sum_{l_{s+1}=0}^{k_{s+1}} \cdots \sum_{l_N=0}^{k_N} D_{j,k,l}^{*}(r, s) f_{r',s':nN}(x, y), \quad x < y, \tag{3}$$

where $f_{r',s':nN}(x, y)$ denotes the joint density function of r'th and s'th order statistic from a SRS of size nN, $\sum_{P^{[N]}}$ denotes the summation over all $N!$ permutations $(i_1, i_2, \ldots, i_N)$ of $(1, 2, \ldots, N)$, and

$$j_a = \begin{cases} [i_a/m] & \text{if } i_a/m = [i_a/m]\,, 1 \le a \le N, \\ [i_a/m] + 1 & \text{if } i_a/m > [i_a/m]\,, 1 \le a \le N, \end{cases}$$

$$D^*_{j,k,l}(r,s) = D_{j,k,l}\frac{(r'-1)!(s'-r'-1)!(nN-s')!}{(r-1)!(s-r-1)!(N-s)!(nN)!},$$

$$\begin{aligned} D_{j,k,l} = &\left\{\prod_{a=1}^{r-1}\binom{n}{k_a}\binom{n-k_a}{l_a}\right\}\left\{j_r\binom{n}{j_r}\binom{n-j_r}{k_r-j_r}\right\}\left\{\prod_{a=r+1}^{s-1}\binom{n}{k_a}\right. \\ &\left.\times\binom{k_a}{l_a}\right\}\left\{j_s\binom{n}{j_s}\binom{j_s-1}{k_s}\right\}\left\{\prod_{a=s+1}^{N}\binom{n}{k_a}\binom{k_a}{l_a}\right\}, \\ r' = &\sum_{\substack{a=1\\a\ne r,s}}^{N} k_a + j_r + j_s - \sum_{\substack{a=r+1\\a\ne s}}^{N} l_a - k_s - 1, \\ s' = &\sum_{\substack{a=1\\a\ne s}}^{N} k_a + j_s + \sum_{a=1}^{r-1} l_a. \end{aligned}$$

From the joint pdf of $X^{\text{ORSS}}_{r:N}$ and $X^{\text{ORSS}}_{s:N}$ in Eq. (3), we can use the Jacobian method to derive the pdf of some systematic statistics from ORSS. For example, the statistic $W^{\text{ORSS}}_{rs} = U^{\text{ORSS}}_{s:N} - U^{\text{ORSS}}_{r:N}$ $(1 \le r < s \le N)$, where $U^{\text{ORSS}}_{i:N}$ is the ith ORSS from the uniform [0,1] distribution, is very important in our discussion of tolerance intervals based on ORSS. For this specific reason, in Example 1, we derive the pdf and cdf of W^{ORSS}_{rs}.

Example 1 Let $W^{\text{ORSS}}_{rs} = U^{\text{ORSS}}_{s:N} - U^{\text{ORSS}}_{r:N}$ $(1 \le r < s \le N)$, where $U^{\text{ORSS}}_{i:N}$ is the ith ORSS of size $N = mn$ from the uniform [0,1] distribution.

From Eq. (3), we have the joint pdf of $U^{\text{ORSS}}_{r:N}$ and $U^{\text{ORSS}}_{s:N}$ as

$$\begin{aligned} f^{\text{ORSS}}_{r,s:N}(u_r,u_s) = &\sum_{P^{[N]}}\sum_{k_1=j_1}^{n}\cdots\sum_{k_{s-1}=j_{s-1}}^{n}\sum_{k_s=0}^{j_s-1}\cdots\sum_{k_N=0}^{j_N-1}\sum_{l_1=0}^{n-k_1}\cdots\sum_{l_{r-1}=0}^{n-k_{r-1}} \\ &\times\sum_{l_{r+1}=k_{r+1}+1-j_{r+1}}^{k_{r+1}}\cdots\sum_{l_{s-1}=k_{s-1}+1-j_{s-1}}^{k_{s-1}}\sum_{l_{s+1}=0}^{k_{s+1}}\cdots\sum_{l_N=0}^{k_N} D^*_{j,k,l}(r,s) \\ &\times\frac{(nN)!\ u_r^{r'-1}(u_s-u_r)^{s'-r'-1}(1-u_s)^{nN-s'}}{(r'-1)!(s'-r'-1)!(nN-s')!},\quad 0 \le u_r < u_s \le 1. \end{aligned}$$

By transforming u_r, u_s to $u_r, w_{rs} = u_s - u_r$, using the fact that $0 \le u_r \le 1 - w_{rs}$, and integrating out u_r, we can obtain the pdf and cdf of W_{rs}^{ORSS} as

$$f_{W_{rs}^{\text{ORSS}}}(w) = \sum_{P^{[N]}} \sum_{k_1=j_1}^{n} \cdots \sum_{k_{s-1}=j_{s-1}}^{n} \sum_{k_s=0}^{j_s-1} \cdots \sum_{k_N=0}^{j_N-1} \sum_{l_1=0}^{n-k_1} \cdots \sum_{l_{r-1}=0}^{n-k_{r-1}} \times \sum_{l_{r+1}=k_{r+1}+1-j_{r+1}}^{k_{r+1}} \cdots \sum_{l_{s-1}=k_{s-1}+1-j_{s-1}}^{k_{s-1}} \sum_{l_{s+1}=0}^{k_{s+1}} \cdots \sum_{l_N=0}^{k_N} D^*_{k,j,l}(r,s) \times \frac{w^{s'-r'-1}(1-w)^{nN-s'+r'}}{B(s'-r', nN-s'+r'+1)}, \quad 0 \le w \le 1,$$

and

$$F_{W_{rs}^{\text{ORSS}}}(w) = \sum_{P^{[N]}} \sum_{k_1=j_1}^{n} \cdots \sum_{k_{s-1}=j_{s-1}}^{n} \sum_{k_s=0}^{j_s-1} \cdots \sum_{k_N=0}^{j_N-1} \sum_{l_1=0}^{n-k_1} \cdots \sum_{l_{r-1}=0}^{n-k_{r-1}} \times \sum_{l_{r+1}=k_{r+1}+1-j_{r+1}}^{k_{r+1}} \cdots \sum_{l_{s-1}=k_{s-1}+1-j_{s-1}}^{k_{s-1}} \sum_{l_{s+1}=0}^{k_{s+1}} \cdots \sum_{l_N=0}^{k_N} D^*_{k,j,l}(r,s) \times I_w(s'-r', nN-s'+r'+1), \quad 0 \le w \le 1. \tag{4}$$

Remark 1 In contrast to the case of the ordered simple random sample (OSRS) in which the pdf and cdf of $W_{rs}^{\text{OSRS}} = U_{s:N}^{\text{OSRS}} - U_{r:N}^{\text{OSRS}}$ just depend on $s - r$ and not individually on r and s, the pdf and cdf of W_{rs}^{ORSS} depend on both r and s. Moreover, since $U_{i:N}^{\text{ORSS}} \stackrel{d}{=} 1 - U_{N-i+1:N}^{\text{ORSS}}$, we readily have

$$W_{rs}^{\text{ORSS}} \stackrel{d}{=} W_{N-s+1,N-r+1}^{\text{ORSS}}. \tag{5}$$

Example 2 Let $m = 1$, $n = 4$, $r_1 = 1$, $s_1 = 2$, $r_2 = 2$ and $s_2 = 3$. Then, $s_1 - r_1 = s_2 - r_2$, and

$$\begin{aligned} F_{W_{12}^{\text{ORSS}}}(x) &= 54.8571 I_x(1,7) - 192.5714 I_x(1,8) + 305.9048 I_x(1,9) \\ &\quad -284.9905 I_x(1,10) + 169.3091 I_x(1,11) - 65.6000 I_x(1,12) \\ &\quad +16.5594 I_x(1,13) - 2.7652 I_x(1,14) + 0.3165 I_x(1,15) \\ &\quad -0.0198 I_x(1,16) \\ &= 1 - 54.8571(1-x)^7 + 192.5714(1-x)^8 - 305.9048(1-x)^9 \\ &\quad +284.9905(1-x)^{10} - 169.3091(1-x)^{11} + 65.6000(1-x)^{12} \\ &\quad -16.5594(1-x)^{13} + 2.7652(1-x)^{14} - 0.3165(1-x)^{15} \\ &\quad +0.0198(1-x)^{16}, \\ F_{W_{23}^{\text{ORSS}}}(x) &= 1 - 28.8000(1-x)^6 + 93.2571(1-x)^7 - 141.7143(1-x)^8 \\ &\quad +131.8095(1-x)^9 - 83.3524(1-x)^{10} + 37.8182(1-x)^{11} \\ &\quad -12.6545(1-x)^{12} + 3.1329(1-x)^{13} - 0.5594(1-x)^{14} \\ &\quad +0.0671(1-x)^{15} - 0.0042(1-x)^{16}. \end{aligned}$$

It is obvious that $F_{W_{12}^{\text{ORSS}}}(x)$ and $F_{W_{23}^{\text{ORSS}}}(x)$ are polynomials in $(1-x)$. We know that these two polynomials are equal if and only if the coefficients of each $(1-x)^i$ are the same. Since this is not the case, we can conclude that the cdf of W_{rs}^{ORSS} depends on both r and s, and not just on $s-r$ as in the case of OSRS.

3 Distribution-free confidence intervals for quantiles

3.1 Confidence intervals for quantiles and their properties

Suppose X is a continuous variate with cdf $F(x)$, then the pth quantile is defined as

$$\xi_p = F^{-1}(p),$$

where $0 < p < 1$.

In order to construct a confidence interval for ξ_p based on ORSS, we first have to know the probability with which the random interval $\left[X_{r:N}^{\text{ORSS}}, X_{s:N}^{\text{ORSS}}\right]$ covers ξ_p. By using the fact that $F(X_{i:N}^{\text{ORSS}}) \overset{d}{=} U_{i:N}^{\text{ORSS}}$, we immediately have

$$\begin{aligned} Pr\left(X_{r:N}^{\text{ORSS}} \le \xi_p \le X_{s:N}^{\text{ORSS}}\right) &= Pr\left(X_{r:N}^{\text{ORSS}} \le F^{-1}(p) \le X_{s:N}^{\text{ORSS}}\right) \\ &= Pr\left\{F(X_{r:N}^{\text{ORSS}}) \le p \le F(X_{s:N}^{\text{ORSS}})\right\} \\ &= Pr\left(U_{r:N}^{\text{ORSS}} \le p \le U_{s:N}^{\text{ORSS}}\right) \\ &= Pr\left(U_{r:N}^{\text{ORSS}} \le p\right) - \Pr\left(U_{s:N}^{\text{ORSS}} \le p\right). \end{aligned} \tag{6}$$

By using the expression in Eq. (2) for the uniform [0, 1] case, we readily have the probability that the random interval $\left[X_{r:N}^{\text{ORSS}}, X_{s:N}^{\text{ORSS}}\right]$ covers ξ_p as

$$\pi(r, s, n, N, p) = \sum_{i=r}^{s-1} \sum_{S_i^{[N]}} \left\{ \prod_{l=1}^{i} I_p(k_l, n-k_l+1) \prod_{l=i+1}^{N} [1 - I_p(k_l, n-k_l+1)] \right\}. \tag{7}$$

Remark 2 It is obvious from Eq. (7) that the probability $\pi(r, s, n, N, p)$ depends only on r, s, n, N, and p, and not on $F(x)$. This means that the interval $\left[X_{r:N}^{\text{ORSS}}, X_{s:N}^{\text{ORSS}}\right]$ is indeed a distribution-free confidence interval for the unknown quantile ξ_p.

It should be noted first that for small sample size N, the exact confidence coefficient $1-\alpha$ may not be achieved due to the discreteness of the probability $\pi(r, s, n, N, p)$ in Eq. (7). Also since there may be more than one choice of r and s to construct such a confidence interval for ξ_p with confidence coefficient $\ge 1-\alpha$, we use the following two rules:

(1) Choose r and s such that $s-r$ is as small as possible;

(2) For different (r, s) with the same value of $s-r$, choose r and s such that the expected width of the interval $\left[U_{r:N}^{\text{ORSS}}, U_{s:N}^{\text{ORSS}}\right]$, viz. $\text{E}\left\{U_{s:N}^{\text{ORSS}} - U_{r:N}^{\text{ORSS}}\right\}$, is as small as possible.

Besides the confidence interval with confidence coefficient $\geq 1-\alpha$, we are also interested in the upper confidence limit $X_{s_u}^{\text{ORSS}}$ for ξ_p, where

$$s_u = \inf\left\{s : Pr\left(\xi_p \leq X_{s:N}^{\text{ORSS}}\right) \geq 1-\alpha\right\},$$

and the lower confidence limit $X_{s_l}^{\text{ORSS}}$ for ξ_p, where

$$s_l = \sup\left\{s : Pr\left(X_{s:N}^{\text{ORSS}} \leq \xi_p\right) \geq 1-\alpha\right\}.$$

Theorem 3.1 presents some properties of these confidence intervals and confidence limits for the unknown population quantiles.

Theorem 3.1 *Suppose* $0 < p < 1$, *and* ξ_p *is the pth quantile such that* $F(\xi_p) = p$. *Then:*

(1) $\left[X_{r:N}^{\text{ORSS}}, X_{s:N}^{\text{ORSS}}\right]$ *is the confidence interval for* ξ_p *with confidence coefficient* $\geq 1-\alpha$ *if and only if* $\left[X_{N-s+1:N}^{\text{ORSS}}, X_{N-r+1:N}^{\text{ORSS}}\right]$ *is the confidence interval for* ξ_{1-p} *with confidence coefficient* $\geq 1-\alpha$*, i.e.,*

$$Pr\left(X_{r:N}^{\text{ORSS}} \leq \xi_p \leq X_{s:N}^{\text{ORSS}}\right) \geq 1-\alpha \\ \Leftrightarrow\ Pr\left(X_{N-s+1:N}^{\text{ORSS}} \leq \xi_{1-p} \leq X_{N-r+1:N}^{\text{ORSS}}\right) \geq 1-\alpha;$$

(2) $X_{s:N}^{\text{ORSS}}$ *is the upper confidence limit for* ξ_p *with confidence coefficient* $\geq 1-\alpha$ *if and only if* $X_{N-s+1:N}^{\text{ORSS}}$ *is the lower confidence limit for* ξ_{1-p} *with confidence coefficient* $\geq 1-\alpha$*, i.e.,*

$$Pr\left(\xi_p \leq X_{s:N}^{\text{ORSS}}\right) \geq 1-\alpha \Leftrightarrow Pr\left(X_{N-s+1:N}^{\text{ORSS}} \leq \xi_{1-p}\right) \geq 1-\alpha.$$

Proof Since $U_{r:N}^{\text{ORSS}} \overset{d}{=} 1 - U_{N-r+1:N}^{\text{ORSS}}$, we readily have $\Pr(U_{r:N}^{\text{ORSS}} \leq p) = \Pr(1 - U_{N-r+1:N}^{\text{ORSS}} \leq p)$. Then by using Eq. (6), the result in (1) can be established as follows:

$$\begin{aligned} & Pr\left(X_{r:N}^{\text{ORSS}} \leq \xi_p \leq X_{s:N}^{\text{ORSS}}\right) \geq 1-\alpha \\ \Leftrightarrow\ & Pr\left(U_{r:N}^{\text{ORSS}} \leq p \leq U_{s:N}^{\text{ORSS}}\right) \geq 1-\alpha \\ \Leftrightarrow\ & Pr\left(1 - U_{N-r+1:N}^{\text{ORSS}} \leq p \leq 1 - U_{N-s+1:N}^{\text{ORSS}}\right) \geq 1-\alpha \\ \Leftrightarrow\ & Pr\left(U_{N-s+1:N}^{\text{ORSS}} \leq 1-p \leq 1 - U_{N-r+1:N}^{\text{ORSS}}\right) \geq 1-\alpha \\ \Leftrightarrow\ & Pr\left(X_{N-s+1:N}^{\text{ORSS}} \leq \xi_{1-p} \leq X_{N-r+1:N}^{\text{ORSS}}\right) \geq 1-\alpha. \end{aligned}$$

Using similar arguments, the result in (2) can also be established. □

Tables 1, 2, 3 and 4 present 90 and 95% confidence intervals as well as upper and lower confidence limits for the pth quantile, where $p = 0.1(0.1)0.9$. Here, we have chosen one-cycle ORSS, that is, $m = 1$ and $N = n$. It is important to observe that the numerical results presented in these tables are consistent with the theoretical properties established in Theorem 3.1.

Table 1 ORSS 90% confidence intervals for the pth quantile based on one cycle

n	$p = 0.1$	$p = 0.2$	$p = 0.3$	$p = 0.4$	$p = 0.5$	$p = 0.6$	$p = 0.7$	$p = 0.8$	$p = 0.9$
2									
3					[1, 3]				
4				[1, 4]	[1, 4]	[1, 4]			
5			[1, 4]	[1, 4]	[1, 4]*	[2, 5]	[2, 5]		
					[2, 5]				
6			[1, 4]	[1, 4]	[2, 5]	[3, 6]	[3, 6]		
7		[1, 4]	[1, 4]	[2, 5]	[2, 6]	[3, 6]	[4, 7]	[4, 7]	
8		[1, 4]	[1, 4]	[2, 5]	[3, 6]	[4, 7]	[5, 8]	[5, 8]	
9		[1, 4]	[2, 5]	[2, 6]**	[3, 7]	[4, 8]**	[5, 8]	[6, 9]	
				[3, 7]		[3, 7]			
10		[1, 4]	[2, 5]	[3, 6]	[4, 7]	[5, 8]	[6, 9]	[7, 10]	

$*$ The expected widths of [1, 4] and [2, 5] are the same
$**$ The confidence interval that is chosen based on the minimum expected width

Table 2 ORSS 95% confidence interval for the pth quantile based on one cycle

n	$p = 0.1$	$p = 0.2$	$p = 0.3$	$p = 0.4$	$p = 0.5$	$p = 0.6$	$p = 0.7$	$p = 0.8$	$p = 0.9$
2									
3									
4				[1, 4]	[1, 4]	[1, 4]			
5				[1, 4]	[1, 5]	[2, 5]			
6			[1, 4]	[1, 5]	[2, 5]	[2, 6]	[3, 6]		
7			[1, 4]	[1, 5]**	[2, 6]	[3, 7]**	[4, 7]		
				[2, 6]		[2, 6]			
8			[1, 5]	[2, 6]	[3, 7]	[3, 7]	[4, 8]		
9		[1, 4]	[1, 5]	[2, 6]	[3, 7]	[4, 8]	[5, 9]	[6, 9]	
10		[1, 4]	[1, 5]**	[2, 6]**	[3, 7]*	[5, 9]**	[6, 10]**	[7, 10]	
			[2, 6]	[3, 7]	[4, 8]	[4, 8]	[5, 9]		

$*$ The expected widths of [3, 7] and [4, 8] are the same
$**$ The confidence interval that is chosen based on the minimum expected width

3.2 Comparison with approximate confidence intervals for quantiles

Chen (2000b) presented approximate confidence intervals $\left[X_{l_1}^{\text{ORSS}}, X_{l_2}^{\text{ORSS}}\right]$ with confidence coefficient $1 - \alpha$, and equal tail probabilities, i.e., intervals satisfying

$$Pr\left(X_{l_1:N}^{\text{ORSS}} \leq \xi_p \leq X_{l_2:N}^{\text{ORSS}}\right) = 1 - \alpha$$

and

$$Pr\left(\xi_p \leq X_{l_1:N}^{\text{ORSS}}\right) = Pr\left(X_{l_2:N}^{\text{ORSS}} \leq \xi_p\right) = \alpha/2.$$

Table 3 ORSS $(100(1-\alpha)\%)$ upper confidence limit for the pth quantile based on one cycle

n	$p=0.1$ 90%	$p=0.1$ 95%	$p=0.2$ 90%	$p=0.2$ 95%	$p=0.3$ 90%	$p=0.3$ 95%	$p=0.4$ 90%	$p=0.4$ 95%	$p=0.5$ 90%	$p=0.5$ 95%	$p=0.6$ 90%	$p=0.6$ 95%	$p=0.7$ 90%	$p=0.7$ 95%	$p=0.8$ 90%	$p=0.8$ 95%	$p=0.9$ 90%	$p=0.9$ 95%
2	2	2	2	2	2	2	2											
3	2	2	2	2	3	3	3	3	3	3								
4	2	2	3	3	3	3	3	4	4	4	4	4						
5	2	2	3	3	3	4	4	4	4	5	5	5	5					
6	2	3	3	3	4	4	4	5	5	5	6	6	6	6				
7	2	3	3	4	4	4	5	5	6	6	6	6	7	7	7			
8	3	3	4	4	4	5	5	6	6	6	7	7	8	8	8			
9	3	3	4	4	5	5	6	6	7	7	7	8	8	9	9	9		
10	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10		

Table 4 ORSS $(100(1-\alpha)\%)$ lower confidence limit for the pth quantile based on one cycle

n	$p=0.1$ 90%	$p=0.1$ 95%	$p=0.2$ 90%	$p=0.2$ 95%	$p=0.3$ 90%	$p=0.3$ 95%	$p=0.4$ 90%	$p=0.4$ 95%	$p=0.5$ 90%	$p=0.5$ 95%	$p=0.6$ 90%	$p=0.6$ 95%	$p=0.7$ 90%	$p=0.7$ 95%	$p=0.8$ 90%	$p=0.8$ 95%	$p=0.9$ 90%	$p=0.9$ 95%
2											1		1	1	1	1	1	1
3									1	1	1	1	1	1	2	2	2	2
4							1	1	1	1	2	1	2	2	2	2	3	3
5					1		1	1	2	1	2	2	3	2	3	3	4	4
6					1	1	1	1	2	2	3	2	3	3	4	4	5	4
7			1		1	1	2	2	2	2	3	3	4	4	5	4	6	5
8			1		1	1	2	2	3	3	4	3	5	4	5	5	6	6
9			1	1	2	1	3	2	3	3	4	4	5	5	6	6	7	7
10			1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8

By using the central limit theorem, Chen (2000b) showed that

$$\begin{cases} l_1 \approx Np - Z_{1-\alpha/2}\sqrt{m\sum_{r=1}^{n} I_p(r,n-r+1)\left[1-I_p(r,n-r+1)\right]}, \\ l_2 \approx Np + Z_{1-\alpha/2}\sqrt{m\sum_{r=1}^{n} I_p(r,n-r+1)\left[1-I_p(r,n-r+1)\right]}, \end{cases} \tag{8}$$

where Z_a denotes the ath quantile of the standard normal distribution.

Moreover, the approximate upper confidence limit $X_{L_u:N}^{\mathrm{ORSS}}$ and the approximate lower confidence limit $X_{L_l:N}^{\mathrm{ORSS}}$ can also be expressed as

$$\begin{cases} L_l \approx Np - Z_{1-\alpha}\sqrt{m\sum_{r=1}^{n} I_p(r,n-r+1)\left[1-I_p(r,n-r+1)\right]}, \\ L_u \approx Np + Z_{1-\alpha}\sqrt{m\sum_{r=1}^{n} I_p(r,n-r+1)\left[1-I_p(r,n-r+1)\right]}. \end{cases} \tag{9}$$

Table 5 Approximate (90%) ORSS confidence interval for the pth quantile, based on one and two cycles, with exact level of confidence

m	n	$p = 0.1$	$p = 0.2$	$p = 0.3$	$p = 0.4$	$p = 0.5$	$p = 0.6$	$p = 0.7$	$p = 0.8$	$p = 0.9$
1	2	[0, 1]	[0, 1]	[0, 2]	[0, 2]	[0, 2]	[0, 2]	[0, 2]	[1, 2]	[1, 2]
		80%	61%	95%	90%	81%	70%	55%	37%	20%
	3	[0, 1]	[0, 2]	[0, 2]	[0, 2]	[0, 3]	[1, 3]	[1, 3]	[1, 3]	[2, 3]
		71%	95%	84%	69%	95%	85%	73%	54%	28%
	4	[0, 1]	[0, 2]	[0, 2]	[0, 3]	[1, 3]	[1, 4]	[2, 4]	[2, 4]	[3, 4]
		62%	88%	68%	91%	75%	95%	83%	67%	36%
	5	[0, 1]	[0, 2]	[0, 3]	[1, 3]	[1, 4]	[2, 4]	[2, 5]	[3, 5]	[4, 5]
		54%	79%	92%	74%	91%	73%	93%	76%	43%
	6	[0, 2]	[0, 2]	[1, 3]	[1, 4]	[2, 4]	[2, 5]	[3, 5]	[4, 6]	[4, 6]
		94%	68%	80%	92%	71%	87%	64%	82%	54%
	7	[0, 2]	[0, 3]	[1, 3]	[1, 4]	[2, 5]	[3, 6]	[4, 6]	[4, 7]	[5, 7]
		91%	94%	69%	81%	88%	93%	74%	90%	60%
	8	[0, 2]	[0, 3]	[1, 4]	[2, 5]	[3, 5]	[3, 6]	[4, 7]	[5, 8]	[6, 8]
		87%	89%	90%	92%	69%	80%	86%	93%	66%
	9	[0, 2]	[0, 3]	[1, 4]	[2, 5]	[3, 6]	[4, 7]	[5, 8]	[6, 9]	[7, 9]
		83%	82%	83%	85%	86%	89%	91%	95%	71%
	10	[0, 2]	[1, 3]	[2, 4]	[2, 6]	[3, 7]	[4, 8]	[6, 8]	[7, 9]	[8, 10]
		78%	72%	69%	95%	95%	95%	68%	70%	75%
2	2		[0, 2]	[0, 3]	[0, 3]	[1, 3]	[1, 4]	[1, 4]	[2, 4]	
			83%	95%	86%	70%	90%	80%	61%	
	3		[0, 3]	[0, 3]	[1, 4]	[1, 5]	[2, 5]	[3, 6]	[3, 6]	
			94%	79%	87%	95%	83%	90%	79%	
	4	[0, 2]	[0, 3]	[1, 4]	[1, 5]	[2, 6]	[3, 7]	[4, 7]	[5, 8]	[6, 8]
		83%	85%	85%	90%	93%	95%	81%	87%	60%
	5	[0, 2]	[0, 4]	[1, 5]	[2, 6]	[3, 7]	[4, 8]	[5, 9]	[6, 10]	[6, 10]
		75%	94%	92%	91%	90%	91%	92%	94%	71%

Tables 5 and 6 present 90 and 95% approximate confidence intervals for the pth quantile, respectively, with exact levels of confidence which were computed by using Eq. (7). In these tables, we present the results for one-cycle ORSS for size n up to 10, ie., $m = 1$, $N = n$, and for two-cycle ORSS for size n up to 5, ie., $m = 2$, $N = 2n$. The corresponding exact one-cycle ORSS confidence intervals can be found in Tables 1 and 2, while the exact 90 and 95% two-cycle ORSS confidence intervals are presented in Tables 7 and 8, respectively. We note from Tables 5 and 6 that the approximate confidence intervals are not accurate enough even for large N based on one cycle, and particularly worse when p is away from 0.5. For example, when $m = 1$ and $n = 10$, the 90% approximate confidence interval for $\xi_{0.7}$ is $[X_{6:10}^{\text{ORSS}}, X_{8:10}^{\text{ORSS}}]$, with exact confidence level just 68%, when the nominal level is supposed to be 90%. The approximate upper and lower confidence limits in Eq. (9) are not accurate either in this case. However, when the number of cycles increases, the approximate confidence intervals of Chen (2000b) become more accurate even for small n. For example, when $m = 2$ and $n = 5$, the approximate 90% confidence interval for $\xi_{0.7}$ is $[X_{5:10}^{\text{ORSS}}, X_{9:10}^{\text{ORSS}}]$, with exact confidence level being 92%.

Table 6 Approximate (95%) ORSS confidence interval for the pth quantile, based on one and two cycles, with exact level of confidence

m	n	$p=0.1$	$p=0.2$	$p=0.3$	$p=0.4$	$p=0.5$	$p=0.6$	$p=0.7$	$p=0.8$	$p=0.9$
1	3			[0, 2] 84%	[0, 3] 98%	[0, 3] 95%	[0, 3] 87%	[1, 3] 73%		
	4		[0, 2] 88%	[0, 3] 97%	[0, 3] 91%	[1, 3] 75%	[1, 4] 95%	[1, 4] 86%	[2, 4] 67%	
	5		[0, 2] 79%	[0, 3] 92%	[0, 4] 98%	[1, 4] 91%	[1, 5] 98%	[2, 5] 93%	[3, 5] 76%	
	6		[0, 3] 97%	[0, 3] 83%	[1, 4] 92%	[1, 5] 97%	[2, 5] 87%	[3, 6] 96%	[3, 6] 85%	
	7		[0, 3] 94%	[0, 4] 96%	[1, 4] 81%	[2, 5] 88%	[3, 6] 93%	[3, 7] 98%	[4, 7] 90%	
	8	[0, 2] 87%	[0, 3] 89%	[1, 4] 90%	[1, 5] 94%	[2, 6] 93%	[3, 7] 98%	[4, 7] 86%	[5, 8] 93%	[6, 8] 66%
	9	[0, 2] 83%	[0, 3] 82%	[1, 4] 83%	[2, 5] 85%	[3, 6] 86%	[4, 7] 89%	[5, 8] 91%	[6, 9] 95%	[7, 9] 71%
	10	[0, 2] 78%	[0, 4] 97%	[1, 5] 96%	[2, 6] 95%	[3, 7] 95%	[4, 8] 95%	[5, 9] 96%	[6, 10] 98%	[6, 10] 76%
2	2			[0, 3] 95%	[0, 3] 87%	[0, 4] 96%	[1, 4] 90%	[1, 4] 80%	[2, 5] 99%	[2, 5] 100%
	3		[0, 3] 94%	[0, 4] 79%	[1, 4] 87%	[1, 5] 95%	[2, 5] 83%	[2, 6] 93%	[3, 6] 79%	[4, 7] 100%
	4		[0, 3] 85%	[0, 4] 87%	[1, 5] 90%	[2, 6] 93%	[3, 7] 95%	[4, 8] 96%	[5, 8] 88%	[6, 9] 99%
	5		[0, 4] 94%	[1, 5] 92%	[2, 6] 91%	[3, 7] 91%	[4, 8] 91%	[5, 9] 92%	[6, 10] 94%	[7, 11] 100%

Table 7 ORSS (90%) confidence interval for the pth quantile based on two cycles

m	n	$p=0.1$	$p=0.2$	$p=0.3$	$p=0.4$	$p=0.5$	$p=0.6$	$p=0.7$	$p=0.8$	$p=0.9$
2	2				[1, 4]	[1, 4]	[1, 4]			
	3			[1, 4]	[1, 5]	[2, 5]	[2, 6]	[3, 6]		
	4			[1, 5]	[1, 5]	[2, 6] [3, 7]	[4, 8]	[4, 8]		
	5		[1, 5]	[1, 6]	[2, 6]	[3, 7]	[4, 8]	[6, 10]	[6, 10]	

Table 8 ORSS (95%) confidence interval for the pth quantile based on two cycles

m	n	$p=0.1$	$p=0.2$	$p=0.3$	$p=0.4$	$p=0.5$	$p=0.6$	$p=0.7$	$p=0.8$	$p=0.9$
2	3				[1, 5]	[1, 5] [2, 6]	[2, 6]			
	4			[1, 5]	[2, 6]	[2, 7]	[3, 7]	[4, 8]		
	5			[1, 6]	[1, 6]	[3, 8]	[4, 9]			

Table 9 Approximate $(100(1-\alpha)\%)$ ORSS upper confidence limit for the pth quantile with exact level of confidence

	$p=0.1$		$p=0.2$		$p=0.3$		$p=0.4$		$p=0.5$		$p=0.6$		$p=0.7$		$p=0.8$		$p=0.9$	
n	90%	95%	90%	95%	90%	95%	90%	95%	90%	95%	90%	95%	90%	95%	90%	95%	90%	95%
2	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2	2	2
		80%		61%	45%	95%		90%		81%		70%		55%		39%		20%
3	1	1	1	2	2	2	2	2	2	3	3	3	3	3	3	3	3	3
		71%	46%	95%		84%		69%	50%	95%		87%		74%		54%		29%
4	1	1	2	2	2	2	3	3	3	3	3	4	4	4	4	4	4	4
		62%		88%		68%		91%		76%	55%	95%		86%		67%		38%
5	1	1	2	2	2	3	3	3	4	4	4	4	4	5	5	5	5	5
		54%		79%	51%	92%		75%		91%		75%	49%	93%		77%		46%
6	1	2	2	2	3	3	3	4	4	4	5	5	5	5	6	6	6	6
	46%	94%		68%		83%	55%	92%		74%		88%		65%		85%		54%
7	2	2	2	3	3	3	4	4	5	5	5	6	6	6	7	7	7	7
		91%	57%	94%		70%		81%		89%	64%	95%		78%		90%		60%
8	2	2	3	3	3	4	4	5	5	5	6	6	7	7	7	8	8	8
		87%		89%	55%	91%	64%	94%		72%		80%		87%	54%	94%		67%
9	2	2	3	3	4	4	5	5	6	6	7	7	7	8	8	9	9	9
		83%		82%		83%		85%		87%		90%	59%	93%	64%	96%		72%
10	2	2	3	3	4	4	5	6	6	7	7	8	8	8	9	9	10	10
		78%		74%		72%	71%	95%	71%	95%	71%	96%		72%		73%		76%

Remark 3 Even though the approximate ORSS confidence intervals proposed by Chen (2000b) are applicable when $N = mn$ is large (which holds true even when $m = 1$ and n is large), we see for the case $m = 1$ big discrepancy between the exact coverage probability and the nominal level, particularly when p is away from 0.5. However, the coverage probability of the approximate ORSS confidence interval gets closer to the nominal level even for small n when the number of cycles is more than one.

Tables 9 and 10 present 90 and 95% approximate upper confidence limits and lower confidence limits for the pth quantile, respectively. In these tables, we present the results for one-cycle ORSS of size n up to 10 $(m = 1, N = n)$ and $p = 0.1(0.1)0.9$. For comparison, we present the exact confidence level corresponding to each approximate confidence limit. Table 9 shows that for the same interval $(-\infty, X_{i:N}^{\text{ORSS}})$, the exact confidence level gets smaller as p gets larger. Similarly, in Table 10, for the same interval $(X_{i:N}^{\text{ORSS}}, \infty)$, the exact confidence level gets larger as p gets smaller. However, we observe from Tables 9 and 10 that the exact confidence levels are too low compared to the nominal levels (when p is away from 0.5) even for large N. But, as in Tables 5 and 6, the exact confidence levels become close to the nominal level in these cases as well even for small n when the number of cycles is more than one.

3.3 Comparison with intervals based on usual order statistics

David and Nagaraja (2003) discuss non-parametric confidence intervals for quantiles based on order statistics from a simple random sample. Following a method

Table 10 Approximate (100(1 − α)%) ORSS lower confidence limit for the pth quantile with exact level of confidence

	$p = 0.1$		$p = 0.2$		$p = 0.3$		$p = 0.4$		$p = 0.5$		$p = 0.6$		$p = 0.7$		$p = 0.8$		$p = 0.9$	
n	90%	95%	90%	95%	90%	95%	90%	95%	90%	95%	90%	95%	90%	95%	90%	95%	90%	95%
2													1 (95%)		1	1 (99%)	1	1 (100%)
3									1 (95%)		1	1 (98%)	1	1 (100%)	2 (95%)	1 (100%)	2	2 (99%)
4							1 (95%)		1	1 (99%)	1	1 (100%)	2	2 (98%)	2	2 (100%)	3	3 (98%)
5					1 (93%)		1	1 (98%)	1	1 (100%)	2	2 (98%)	3 (92%)	2 (100%)	3	3 (99%)	4	4 (96%)
6					1	1 (100%)	1	1 (100%)	2	2 (97%)	3 (92%)	2 (100%)	3	3 (99%)	4	4 (97%)	5 (94%)	4 (100%)
7					1	1 (99%)	2 (95%)	1 (100%)	2	2 (99%)	3	3 (98%)	4	4 (96%)	5 (94%)	4 (100%)	5	5 (100%)
8			1 (94%)		1	1 (99%)	2	2 (98%)	3	3 (96%)	4 (94%)	3 (100%)	5 (91%)	4 (100%)	5	5 (99%)	6	6 (100%)
9			1 (96%)		2 (93%)	1 (100%)	2	2 (99%)	3	3 (99%)	4	4 (99%)	5	5 (98%)	6	6 (99%)	7	7 (99%)
10			1	1 (98%)	2	2 (96%)	3 (96%)	2 (100%)	4 (95%)	3 (100%)	5 (95%)	4 (100%)	6	6 (96%)	7	7 (97%)	8	8 (99%)

similar to the one in Sect. 3.1 based on ORSS, confidence intervals for the pth quantile can be obtained based on the usual order statistics. Now, let $\tilde{I}_p$ and $\tilde{L}_p$ denote the confidence interval for the pth quantile based on the usual order statistics and the expected length of this interval, respectively. Similarly, let I_p^* and L_p^* denote the corresponding quantities based on ORSS. Then, the percentage reduction in L_p^* compared to $\tilde{L}_p$ can be defined as

$$PR = \frac{\tilde{L}_p - L_p^*}{\tilde{L}_p}.$$

Table 11 presents the 90% confidence interval $\tilde{I}_p$ for the pth quantile, its expected length $\tilde{L}_p$, the expected length L_p^* of the ORSS confidence interval, and the percentage reduction of L_p^* compared to $\tilde{L}_p$. The results are for one-cycle for n up to 10. From Table 11, it is clear that confidence intervals based on ORSS are more efficient than the corresponding ones based on OS. Moreover, the PR gets larger as n increases which means that the confidence interval based on ORSS becomes considerably narrower than the one based on ordinary OS when n becomes large.

4 Distribution-free tolerance intervals

To construct a tolerance interval that covers at least a fixed proportion γ of the population with tolerance level β, we seek $X_{r:N}^{\text{ORSS}}$ and $X_{s:N}^{\text{ORSS}}$ $(1 \leq r < s \leq N)$ such that

Table 11 OS confidence interval (90%) $\tilde{I}_p$ for the pth quantile, its expected length $\tilde{L}_p$, the expected length L_p^* of the ORSS confidence interval, and the percentage reduction of L_p^* compared to $\tilde{L}_p$

n		$p=0.1$	$p=0.2$	$p=0.3$	$p=0.4$	$p=0.5$	$p=0.6$	$p=0.7$	$p=0.8$	$p=0.9$
5	$\tilde{I}_p$				[1, 5]	[1, 5]	[1, 5]			
	$\tilde{L}_p$				0.6667	0.6667	0.6667			
	L_p^*				0.5574	0.5574	0.5574			
	PR				16.39%	16.39%	16.39%			
6	$\tilde{I}_p$				[1, 5]	[1, 6]	[2, 6]			
	$\tilde{L}_p$				0.5714	0.7143	0.5714			
	L_p^*				0.4697	0.4771	0.4697			
	PR				17.80%	33.21%	17.80%			
7	$\tilde{I}_p$			[1, 6]	[1, 6]	[1, 6]/[2, 7]	[2, 7]	[2, 7]		
	$\tilde{L}_p$			0.6250	0.6250	0.6250	0.6250	0.6250		
	L_p^*			0.4049	0.4129	0.5495	0.4129	0.4049		
	PR			35.21%	33.94%	12.08%	33.94%	35.21%		
8	$\tilde{I}_p$			[1, 6]	[1, 6]	[2, 7]	[3, 8]	[3, 8]		
	$\tilde{L}_p$			0.5556	0.5556	0.5556	0.5556	0.5556		
	L_p^*			0.3556	0.3631	0.3646	0.3631	0.3556		
	PR			36.00%	34.65%	34.38%	34.65%	36.00%		
9	$\tilde{I}_p$			[1, 6]	[2, 7]	[1, 7]/[3, 9]	[3, 8]	[4, 9]		
	$\tilde{L}_p$			0.5000	0.5000	0.6000	0.5000	0.5000		
	L_p^*			0.3229	0.4315	0.4339	0.4315	0.3229		
	PR			35.42%	13.70%	27.68%	13.70%	35.42%		
10	$\tilde{I}_p$			[1, 6]	[2, 7]	[2, 8]/[3, 9]	[4, 9]	[5, 10]		
	$\tilde{L}_p$			0.4545	0.4545	0.5455	0.4545	0.4545		
	L_p^*			0.2918	0.2938	0.2944	0.2938	0.2918		
	PR			35.80%	35.37%	46.03%	35.37%	35.80%		

$$Pr\left\{\int_{X_{r:N}^{\mathrm{ORSS}}}^{X_{s:N}^{\mathrm{ORSS}}} f(x)\mathrm{d}x \geq \gamma\right\} = \beta. \tag{10}$$

Upon setting $X_{r:N}^{\mathrm{ORSS}} = -\infty$ or $X_{s:N}^{\mathrm{ORSS}} = \infty$, we get one-sided tolerance intervals.

By using Eq. (4), the left hand side of Eq. (10) can be rewritten as

$$Pr\left\{\int_{X_{r:N}^{\mathrm{ORSS}}}^{X_{s:N}^{\mathrm{ORSS}}} f(x)\mathrm{d}x \geq \gamma\right\} = Pr\left\{F\left(X_{s:N}^{\mathrm{ORSS}}\right) - F\left(X_{r:N}^{\mathrm{ORSS}}\right) \geq \gamma\right\}$$

$$= Pr\left\{U_{s:N}^{\mathrm{ORSS}} - U_{r:N}^{\mathrm{ORSS}} \geq \gamma\right\} = 1 - F_{W_{rs}^{\mathrm{ORSS}}}(\gamma).$$

It is obvious that Eq. (10) can't be satisfied exactly, but we can choose r and s making $s - r + 1$ as small as possible and satisfying that

Table 12 Two-sided tolerance interval (90%) that covers γ proportion of the population

n	$\gamma = 0.1$	$\gamma = 0.2$	$\gamma = 0.3$	$\gamma = 0.4$	$\gamma = 0.5$	$\gamma = 0.6$	$\gamma = 0.7$	$\gamma = 0.8$	$\gamma = 0.9$
2									
3	[1, 3]	[1, 3]							
4	[1, 3]	[1, 3]	[1, 4]	[1, 4]					
	[2, 4]	[2, 4]							
5	[1, 3]*	[1, 4]	[1, 4]	[1, 5]	[1, 5]				
	[3, 5]*	[2, 5]	[2, 5]						
	[2, 4]								
6	[1, 3]*	[1, 4]*	[1, 5]	[1, 5]	[1, 6]	[1, 6]			
	[4, 6]*	[3, 6]*	[2, 6]	[2, 6]					
	[2, 4]	[2, 5]							
	[3, 5]								
7	[1, 3]*	[1, 4]*	[1, 5]*	[1, 6]	[1, 6]	[1, 7]			
	[5, 7]*	[4, 7]*	[3, 7]*	[2, 7]	[2, 7]				
	[2, 4]	[2, 6]	[2, 5]						
	[4, 6]	[3, 6]							
	[3, 5]								
8	[1, 4]*	[1, 4]	[1, 5]*	[1, 6]	[1, 7]	[1, 8]	[1, 8]		
	[5, 8]*	[5, 8]	[4, 8]*	[3, 8]	[2, 8]				
	[2, 5]		[2, 6]	[2, 7]					
	[4, 7]		[3, 7]						
	[3, 6]								
9	[1, 4]*	[1, 5]*	[1, 6]*	[1, 7]*	[1, 7]*	[1, 8]	[1, 9]		
	[6, 9]*	[5, 9]*	[4, 9]*	[3, 9]*	[3, 9]*	[2, 9]			
	[2, 5]	[2, 6]	[2, 7]	[2, 8]	[2, 8]				
	[5, 8]	[4, 8]	[3, 8]						
	[3, 6]	[3, 7]							
	[4, 7]								
10	[1, 4]*	[1, 5]*	[1, 6]*	[1, 7]*	[1, 8]*	[1, 9]	[1, 10]		
	[7, 10]*	[6, 10]*	[5, 10]*	[4, 10]*	[3, 10]*	[2, 10]			
	[2, 5]	[2, 6]	[2, 7]	[2, 8]	[2, 9]				
	[6, 9]	[5, 9]	[4, 9]	[3, 9]					
	[3, 6]	[3, 7]	[3, 8]						
	[5, 8]	[4, 8]							
	[4, 7]								

$*$ Intervals with the shortest expected width.

$$Pr\left\{ \int_{X_{r:N}^{\text{ORSS}}}^{X_{s:N}^{\text{ORSS}}} f(x)\mathrm{d}x \geq \gamma \right\} \geq \beta. \tag{11}$$

From Eq. (5), we can easily prove the symmetry property of tolerance intervals, which is formally stated in Theorem 4.1.

Theorem 4.1 *Suppose* $0 < \gamma, \beta < 1$*, then:*

(1) $\left[X_{r:N}^{\text{ORSS}}, X_{s:N}^{\text{ORSS}}\right]$ *is the tolerance interval that covers* γ *proportion of the population with confidence coefficient* β *if and only if* $\left[X_{N-s+1:N}^{\text{ORSS}}, X_{N-r+1:N}^{\text{ORSS}}\right]$ *is the tolerance interval that covers* γ *proportion of the population with confidence coefficient* β*, i.e.,*

Table 13 Two-sided tolerance interval (95%) that covers γ proportion of the population

n	$\gamma = 0.1$	$\gamma = 0.2$	$\gamma = 0.3$	$\gamma = 0.4$	$\gamma = 0.5$	$\gamma = 0.6$	$\gamma = 0.7$	$\gamma = 0.8$	$\gamma = 0.9$
2									
3	[1, 3]	[1, 3]							
4	[1, 3]	[1, 4]	[1, 4]						
	[2, 4]								
5	[1, 3]*	[1, 4]	[1, 5]	[1, 5]					
	[3, 5]*	[2, 5]							
	[2, 4]								
6	[1, 4]*	[1, 4]*	[1, 5]	[1, 6]	[1, 6]				
	[3, 6]*	[3, 6]*	[2, 6]						
	[2, 5]	[2, 5]							
7	[1, 4]*	[1, 5]*	[1, 5]*	[1, 6]	[1, 7]				
	[4, 7]*	[3, 7]*	[3, 7]*	[2, 7]					
	[2, 5]	[2, 6]	[2, 6]						
	[3, 6]								
8	[1, 4]*	[1, 5]*	[1, 6]*	[1, 6]	[1, 7]	[1, 8]			
	[5, 8]*	[4, 8]*	[3, 8]*	[3, 8]	[2, 8]				
	[2, 5]	[2, 6]	[2, 7]						
	[4, 7]	[3, 7]							
	[3, 6]								
9	[1, 4]*	[1, 5]*	[1, 6]*	[1, 7]*	[1, 8]	[1, 9]	[1, 9]		
	[6, 9]*	[5, 9]*	[4, 9]*	[3, 9]*	[2, 9]				
	[2, 5]	[2, 6]	[2, 7]	[2, 8]					
	[5, 8]	[4, 8]	[3, 8]						
	[3, 6]	[3, 7]							
	[4, 7]								
10	[1, 4]*	[1, 5]*	[1, 6]*	[1, 7]*	[1, 8]*	[1, 9]	[1, 10]		
	[7, 10]*	[6, 10]*	[5, 10]*	[4, 10]*	[3, 10]*	[2, 10]			
	[2, 5]	[2, 6]	[2, 7]	[2, 8]	[2, 9]				
	[6, 9]	[5, 9]	[4, 9]	[3, 9]					
	[3, 6]	[3, 7]	[3, 8]						
	[5, 8]	[4, 8]							
	[4, 7]								

∗ Intervals with the shortest expected width.

$$Pr\left\{\int_{X_{r:N}^{\mathrm{ORSS}}}^{X_{s:N}^{\mathrm{ORSS}}} f(x)\mathrm{d}x \geq \gamma\right\} = \beta \Leftrightarrow Pr\left\{\int_{X_{N-s+1:N}^{\mathrm{ORSS}}}^{X_{N-r+1:N}^{\mathrm{ORSS}}} f(x)\mathrm{d}x \geq \gamma\right\} = \beta;$$

(2) $[X_{r:N}^{\mathrm{ORSS}}, \infty)$ *is the one-sided tolerance interval that covers* γ *proportion of the population with the confidence coefficient* β *if and only if* $(-\infty, X_{N-r+1:N}^{\mathrm{ORSS}}]$ *is the one-sided tolerance interval that covers* γ *proportion of the population with the confidence coefficient* β*, i.e.*

$$Pr\left\{\int_{X_{r:N}^{\mathrm{ORSS}}}^{\infty} f(x)\mathrm{d}x \geq \gamma\right\} = \beta \Leftrightarrow Pr\left\{\int_{-\infty}^{X_{N-r+1:N}^{\mathrm{ORSS}}} f(x)\mathrm{d}x \geq \gamma\right\} = \beta.$$

Tables 12 and 13 present 90 and 95% two-sided tolerance intervals that cover γ proportion of the population, where $\gamma = 0.1(0.1)0.9$. Once again, we use one-cycle ORSS ($m = 1$, $N = n$), with n up to 10. These two tables show that for the same n and γ, there may be various intervals satisfying Eq. (11). In this case, we will choose the one with the shortest expected width as we did earlier in Sect. 3.

Appendix A: Fortran program for Table 1

```
**************************************************************
******THIS IS TO COMPUTE THE CI FOR P-TH QUANTILE BASED ON ORSS
**************************************************************
*******THIS IS TO CALCULATE 'N!'*******************************
    SUBROUTINE  FACFAC(JIECHENG,JD)
*******OUTPUT:JIECHENG; JIECHENG(N)=(N-1)!*********************
    DOUBLE PRECISION JIECHENG(JD+1)
    INTEGER JD,I

    JIECHENG(1)=1.0
    IF (JD.GE.2) THEN
        DO I=2,JD+1
            JIECHENG(I)=JIECHENG(I-1)*(I-1.0)
        END DO
    END IF
    END

*******THIS IS TO GET N! PERMUTATIONS OF (1,2,...,N)************
    SUBROUTINE PAI(P,N,M,JIECHENG)
    INTEGER P(M,N),J,I,JJ,ICOUNT,ITEMP,TEMP
    DOUBLE PRECISION JIECHENG(N+1)

    P(1,1)=1
    P(1,2)=2
    P(2,1)=2
    P(2,2)=1
    IF (N.GE.3) THEN
    DO J=3,N
        DO I=1,JIECHENG(J)
            P(I,J)=J
        END DO
        DO JJ=1,J
            DO I=JIECHENG(J)+1,JIECHENG(J+1)
                P(I,JJ)=P(I-JIECHENG(J),JJ)
            END DO
        END DO
        DO JJ=1,J-1
            DO ICOUNT=1,JIECHENG(J)
                ITEMP=JJ*JIECHENG(J)+ICOUNT
                TEMP=P(ITEMP,JJ)
                P(ITEMP,JJ)=P(ITEMP,J)
                P(ITEMP,J)=TEMP
            END DO
        END DO
    END DO
    END IF
    END

****** THIS IS TO COMPUTE THE COMBINATION*****************
```

```
    SUBROUTINE COMBINATION(COMB,N,JIECHENG)
****** COMB(I,J) IS THE COMBINATION OF (I-1,J-1) ***********
****** EG:COMB(1,1)IS C(0,0), COMB(2,1)IS C(1,0)... ********
    INTEGER N,I,J
    DOUBLE PRECISION JIECHENG(N+1),COMB(N+1,N+1)

    DO I=1,N+1
    DO J=1,I
    COMB(I,J)=JIECHENG(I)/(JIECHENG(J)*JIECHENG(I-J+1)*1.0)
    END DO
    END DO
    END

******* THIS IS TO READ THE MU_ORSS OF UNIFORM DISTRIBUTION***
    SUBROUTINE READ_UNIFMU(UNIMMU,N,MUIODATA)
    INTEGER I,J,MUIODATA
    DOUBLE PRECISION UNIMMU(N),TEMP(N,N)

    DO I=1,N
        DO J=1,I
            READ (MUIODATA,*) TEMP(I,J)
        END DO
    END DO
    UNIMMU=TEMP(N,:)
    END

***************************************************************
    DOUBLE PRECISION FUNCTION BIFUNC(KINST,N,COMB,PLOCAL)
    DOUBLE PRECISION COMB(N+1,N+1),PLOCAL
    INTEGER N,II,KINST(N)

    BIFUNC=1.D0
    DO II=1,N
    BIFUNC=BIFUNC*COMB(N+1,KINST(II)+1)*PLOCAL**KINST(II)
     C          *(1-PLOCAL)**(N-KINST(II))
    END DO
    END

****************************************************************
****************************************************************
    PROGRAM MAIN
    PARAMETER (N=5,M=120,CC=0.90,NUMP=9)
****************************************************************
******* THIS PROGRAM IS FOR ONE-CYCLE ORSS *******************
******* N IS # THE SAMPLE SIZE, M IS THE FACTORIAL OF N ********
******* NUMP IS THE NUMBER OF P-th QUANTILES (eg: 0.1(0.1)0.9)**
******* CC IS PREFIXED CONFIDENCE COEFFICIENT *****************
****************************************************************
   INTEGER II,R,S,IJ,RUNTIME,COUNT,IJCOUNT
   INTEGER IODATA,I,CILEP(NUMP),CIREP(NUMP)
   INTEGER J(N), PERT(M,N),KTOP(N),KBOTTEM(N),KINST(N),DRS(NUMP)
   DOUBLE PRECISION CPORSS(NUMP,N,N),TEMP(NUMP),UNIMMU(N),P(NUMP)
   DOUBLE PRECISION JIECHENG(N+1),COMB(N+1,N+1),BIFUNC
   DOUBLE PRECISION EL(NUMP),ELNEW(NUMP)
   LOGICAL JUDG
***************************OUTPUT*****************************
******CPORSS(I,J,K): COVERAGE PROB OF [X_J,X_K] FOR P(I)-TH
******QUANTILE************************************************
******EL(I): EXPECTED LENGTH OF CI FOR P(I)-TH QUANTILE ********
```

```
******CILEP(I): INDEX OF LEFT END POINT OF CI FOR P(I)TH QUANTILE
******CIREP(I): INDEX OF RIGHT END POINT OF CI FOR P(I)TH QUANTILE
*****************************************************************
    CALL FACFAC(JIECHENG,N)
    CALL COMBINATION(COMB,N,JIECHENG)
    CALL PAI(PERT,N,M,JIECHENG)

    IODATA=75
    OPEN(UNIT=IODATA,FILE='UNIMORSS.TXT')
    CALL READ_UNIFMU (UNIMMU,N,IODATA)
    CLOSE (IODATA)
    IODATA=76
    OPEN (UNIT=IODATA,FILE='PQCIOUT.TXT')

    P(1)=DBLE(1.0)/DBLE(10.0)
    DO II=2,NUMP
        P(II)=P(II-1)+0.1D0
    END DO
****** LOOP 200 IS TO COMPUTE THE COVERAGE PROB OF [X_R, X_S]****
    DO 200 R=1,N-1
        DO II=1,NUMP
            CPORSS(II,R,R)=0
        END DO
    DO 300 S=R+1,N
      I=S-1
    DO II=1,NUMP
        TEMP(II)=0
    END DO
    DO 500 II=1,M
        DO IJ=1,N
            J(IJ)=PERT(II,IJ)
        END DO
        IF (N.GE.3) THEN
            IF (I.EQ.1) THEN
                DO IJ=2,N-1
                    IF (J(IJ).GT.J(IJ+1)) GOTO 500
                END DO
            ELSE IF (I.EQ.N-1) THEN
                DO IJ=1,N-2
                    IF (J(IJ).GT.J(IJ+1)) GOTO 500
                END DO
            ELSE
                DO IJ=1,I-1
                    IF (J(IJ).GT.J(IJ+1)) GOTO 500
                END DO
                DO IJ=I+1,N-1
                    IF (J(IJ).GT.J(IJ+1)) GOTO 500
                END DO
            END IF
        END IF
    DO 510 IJ=1,N
        IF (IJ.LE.I) THEN
            KINST(IJ)=J(IJ)
            KTOP(IJ)=N
            KBOTTEM(IJ)=J(IJ)
        ELSE
            KINST(IJ)=0
            KTOP(IJ)=J(IJ)-1
            KBOTTEM(IJ)=0
```

```
        END IF
510 CONTINUE

    RUNTIME=1
    DO IJ=1,N
        RUNTIME=RUNTIME*(KTOP(IJ)+1-KBOTTEM(IJ))
    END DO
      KINST(1)=KINST(1)-1
    DO 530 COUNT=1,RUNTIME
        KINST(1)=KINST(1)+1
        JUDG=.FALSE.
        DO 550 IJCOUNT=1,N-1
            IF (JUDG) GOTO 501
                IF (KINST(IJCOUNT).GT.KTOP(IJCOUNT)) THEN
                KINST(IJCOUNT)=KBOTTEM(IJCOUNT)
                KINST(IJCOUNT+1)=KINST(IJCOUNT+1)+1
                IF (KINST(IJCOUNT+1).LE.KTOP(IJCOUNT+1)) THEN
                    JUDG=.TRUE.
                END IF
            ELSE
                JUDG=.TRUE.
            END IF
550     CONTINUE 501 DO IJ=1,NUMP
        TEMP(IJ)=TEMP(IJ)+BIFUNC(KINST,N,COMB,P(IJ))
    END DO
530 CONTINUE 500 CONTINUE
    DO II=1,NUMP
        CPORSS(II,R,S)=CPORSS(II,R,S-1)+TEMP(II)
    END DO
300   CONTINUE 200   CONTINUE
    DO R=1,N-1
        DO S=R+1,N
            WRITE(IODATA,*) R,S,CPORSS(:,R,S)
        END DO
    END DO

******* THIS IS TO OBTAIN THE CI_ORSS *********************
******* EL: EXPECTED LENGTH OF CI FOR P-TH QUANTILE*********
******* DRS: VALUE OF ‘S-R’*********************************
******* CILEP(CIREP): INDEX OF ENDPOINT OF CI**************
    DO 540 II=1,NUMP
        EL(II)=DBLE(1)
        DRS(II)=N
        DO R=1,N-1
            DO S=R+1,N
             IF (ANINT(CPORSS(II,R,S)*100.0) .GE. REAL(CC*100)
     C               .AND. (S-R).LT.DRS(II)) THEN
                DRS(II)=S-R
                ELNEW(II)=UNIMMU(S)-UNIMMU(R)
                IF (ELNEW(II).LT. EL(II)) THEN
                    CILEP(II)=R
                    CIREP(II)=S
                    EL(II)=ELNEW(II)
                END IF
            END IF
        END DO
    END DO
******* THE FOLLOWING IS TO PRINT OUT THE RESULT*************
   IF (CILEP(II).GE.1) THEN
```

```
   WRITE(*,600) 'P=', P(II), ',' , 'CI_ORSS=[' , CILEP(II), ',',
     C  CIREP(II),'],','CPORSS=',CPORSS(II,CILEP(II),CIREP(II)),
     C  'EL_ORSS=',EL(II)
600 FORMAT (1X,A3,F3.1,A1,A11,I2,A2,I2,A3,A10,F15.13,A10,F15.13)
WRITE(IODATA,*) P(II), CILEP(II), CIREP(II), CPORSS(II,CILEP(II),
  C  CIREP(II)),EL(II)
    END IF
540 END DO

    END
```

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