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Rough sets models inspired by supra-topology structures

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Abstract
Our aim of writing this manuscript is to found novel rough-approximation operators inspired by an abstract structure called "supra-topology". This approach is more relaxed than topological ones and extends the scope of applications because an intersection condition of topology is dispensed. Firstly, we generate eight types of supra-topologies using N_k-neighborhood systems induced from any arbitrary relation. We elucidate the relationships between them and investigate the conditions under which some of them are identical. Then, we create new rough sets models from these supra-topologies and present the main characterizations of their lower and upper approximations. We apply these approximations to classify the regions of the subset and compute its accuracy measures. The master merits of the current approach are to produce the highest accuracy values compared with all approaches given in the published literature under a reflexive relation as well as preserve the monotonicity property of accuracy and roughness measures. Moreover, we demonstrate the good performance of the followed technique through analysis of some data of dengue fever disease. Ultimately, we debate the advantages and disadvantages of the followed approach and make a plan for some upcoming work.

Keywords N_k-neighborhood · Supra-topology · Supra upper and supra lower approximations · Accuracy and roughness measures · Dengue fever

1 Introduction

In recent years, rough set theory and its extended models have raised more and more scholars attention in various fields; especially, those who work in computer science and artificial intelligence. This theory was introduced by Pawlak (1982), as an effective and robust tool to cope with imperfect knowledge problems. It starts from an equivalence relation to classifying the objects and capture to what extent the information obtained from a set is complete. Two core principles in this theory are approximation operators and accuracy

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measures which supply the decision-makers with the required data regarding the structure and size of boundary region.

A strict stipulation of an equivalence relation limits the applications of conventional rough set theory, so several generalizations of rough set theory have been introduced under an arbitrary relation or a specific relation. Yao (1996, 1998) launched this line of research, in 1996, by defining two types of neighborhoods with respect to an arbitrary relation called "right neighborhood" and "left neighborhood". Then, some researchers assumed a specific relation to present various types of generalized rough set theory generated from right and left neighborhoods such as similarity (Abo-Tabl 2013; Slowinski and Vanderpooten 2000), tolerance Skowron and Stepaniuk (1996), quasiorder (Qin et al. 2008; Zhang et al. 2009) and dominance Zhang et al. (2016). In the light of this research trend, many authors and scholars made use of some operations between N_k-neighborhoods to explore new types of neighborhood systems; the recent ones of them are C_k-neighborhoods defined using superset relation Al-shami (2021a), S_k-neighborhoods defined using subset relation Al-shami and Ciucci (2022), E_k-neighborhoods defined using intersection relation Al-shami et al. (2021), maximal neighborhoods defined using union relation Dai et al. (2018), core neighborhoods defined using equality relation Mareay (2016), and remote neighborhood Sun et al. (2019). Admittedly, they were created with the goal of improving approximation operators, increasing accuracy measures, and handling some practical problems. In line with this trend, Abu-Donia (2008) displayed new generalized rough set models induced from a finite family of arbitrary relations. Syau et al. (2021) studied the characterization of incomplete decision tables using a variable precision generalized rough set approach. Campagner et al. (2022) reviewed the most relevant contributions studying the links between belief functions and rough sets.

Another interesting orientation of study rough sets is a topology. Skowron (1988) and Wiweger (1989) noted the similarity behaviours of topological and rough-set concepts, which implies the possibility of replacement of the rough-set concepts by their topological counterparts. This motivated many researchers to establish some topological structures and study rough-set notions and properties via them. Lashin et al. (2005) proposed a technique to initiate a topology from N_k-neighborhood systems. This technique is based on considering N_k-neighborhoods a subbase utilized to build a topology. Investigation of multi knowledge bases using rough approximations and topology was done by Abu-Donia (2012). Salama (2010) explored the solution of the missing attribute values problem from a topological view. Al-shami (2021b, 2022) benefited from two near open subsets of topological spaces called somewhere dense and somewhat open sets to introduce different types of lower and upper approximations and illustrated their merits compared to the past methods. A lot of published contributions were done to reformulate the rough set notions using topological ideas such as (Abo-Tabl 2014; Hosny 2018; Jin et al. 2021; Kondo and Dudek 2006; Li et al. 2012; Singh and Tiwari 2020; Zhu 2007). Topological structures were used in many applications such as those presented in El-Bably and Abo-Tabl (2021), El-Bably and El-Sayed (2022).

In recent years, it has been exploited some topological generalizations such as infra topology Al-shami and Mhemdi (2022), minimal structure (Azzam et al. 2020; El-Sharkasy 2021) and bitopology Salama (2020) to deal with rough set concepts and address some medical problems. Following this line, we suggest novel kinds of rough set models inspired by another abstract structure called "supra-topology". This concept was introduced by Mashhour (1983), in 1983, as an extension of topology. Afterward, many authors discussed the topological concepts and examined the validity of their characterizations via supra-topological structures. We draw attention to that some topological properties such as

$Int(X \cap Z) = Int(X) \cap Int(Z)$ and $Cl(X \cup Z) = Cl(X) \cup Cl(Z)$ are evaporated via supra-topology. However, a supra topology frame offers a convenient environment to cope with some real-life problems as illustrated in Kozae et al. (2016).

The major inducements to debate rough set models using a "supra-topology" standpoint are, first, to relieve some conditions imposed on topological rough set models, which make us in a position to dispense with some conditions that limit applications. Second, the followed approach preserves most of Pawlak properties of approximation operators, which are evaporated in some previous approaches induced from topological structures such as (Abd El-Monsef et al. 2014; Abu-Donia 2008; Dai et al. 2018; Yao 1996, 1998). Thirdly, the values of accuracy and roughness given herein satisfy the monotonic property. Fourthly, the best approximations and accuracy values produced by our approach are obtained in cases of union and minimal union, whereas they are obtained in cases of intersection and minimal intersection in the previous approaches. This implies our approach is more suitable to analyze and describe the large samples. Finally, the approximation operators and accuracy measures obtained from our approach under a reflexive relation are better than all preceding methods defined by topological structures (Abd El-Monsef et al. 2014; Abo-Tabl 2013; Allam et al. 2006; Al-shami 2021b; Amer et al. 2017; Hosny 2018; Kondo and Dudek 2006; Kozae et al. 2007) and their generalizations (Azzam et al. 2020; El-Sharkasy 2021; Salama 2020) and all the preceding methods directly defined by neighborhood systems (Abu-Donia 2008; Al-shami 2021a, 2022; Dai et al. 2018; Lashin et al. 2005).

The rest of this paper is designed as follows. Section 2 mentions the basic principles and results of rough sets and supra-topology required to understand this context as well as elaborates the motivations that led to these developments. In Sect. 3, we show how to construct supra-topology spaces utilizing N_k-neighborhood systems induced from any arbitrary relation. Then, we make use of these spaces to establish new rough set models and scrutinize their fundamental characterizations in Sect. 4. Also, we build an algorithm to illustrate how supra k-exact sets are determined. In Sect. 5, we investigate the effectiveness and robustness of the followed approach to analyze the data of dengue fever disease. In Sect. 6, we present the pros and cons of the followed approach compared to the past ones. In the end, in Sect. 7, we summarize the main contributions and give some thoughts that can be applied to expand the scope of this manuscript.

2 Basic concepts and results

We recall, in this section, the principles and results regarding rough sets and supra-topological structures that are required to understand the manuscript context. Also, we tackle the historical development of these concepts as well as the motivations of their study. Moreover, we give proof for equality of accuracy measures induced from N_k-neighborhoods and their counterpart topologies under a quasiorder relation.

2.1 Rough approximation operators and neighborhood systems

Through this manuscript, an approximation space is the ordered pair (E, δ) such that E is a nonempty finite set and δ is an arbitrary relation on E. (E, δ) is called Pawlak approximation space if δ is an equivalent relation, i.e. reflexive, symmetric and transitive.

The following definition, introduced by Pawlak (1982), is the cornerstone of this research scope.

Definition 1 We associate each subset X of Pawlak approximation space (E, δ) with two sets defined with respect to the equivalences classes E/δ by the next formulas.

$$\underline{\delta}(X) = \cup\{U \in E/\delta : U \subseteq X\}, \text{and}$$
$$\overline{\delta}(X) = \cup\{U \in E/\delta : U \cap X \neq \emptyset\}$$

The sets $\underline{\delta}(X)$ and $\overline{\delta}(X)$ are respectively recognized as lower and upper approximations of X. The core properties of these approximations are listed in the next proposition which is the key point of rough set theory.

Proposition 1 (Pawlak 1982) *Let X and Z be subsets of Pawlak approximation space (E, δ). The next properties are satisfied.*

$$\begin{array}{ll}
(L1)\ \underline{\delta}(X) \subseteq X & (U1)\ X \subseteq \overline{\delta}(X) \\
(L2)\ \underline{\delta}(\emptyset) = \emptyset & (U2)\ \overline{\delta}(\emptyset) = \emptyset \\
(L3)\ \underline{\delta}(E) = E & (U3)\ \overline{\delta}(E) = E \\
(L4)\ \mathit{If} X \subseteq Z,\ \mathit{then}\ \underline{\delta}(X) \subseteq \underline{\delta}(Z) & (U4)\ \mathit{If} X \subseteq Z,\ \mathit{then}\ \overline{\delta}(X) \subseteq \overline{\delta}(Z) \\
(L5)\ \underline{\delta}(X \cap Z) = \underline{\delta}(X) \cap \underline{\delta}(Z) & (U5)\ \overline{\delta}(X \cap Z) \subseteq \overline{\delta}(X) \cap \overline{\delta}(Z) \\
(L6)\ \underline{\delta}(X) \cup \underline{\delta}(Z) \subseteq \underline{\delta}(X \cup Z) & (U6)\ \overline{\delta}(X \cup Z) = \overline{\delta}(X) \cup \overline{\delta}(Z) \\
(L7)\ \underline{\delta}(X^c) = (\overline{\delta}(X))^c & (U7)\ \overline{\delta}(X^c) = (\underline{\delta}(X))^c \\
(L8)\ \underline{\delta}(\underline{\delta}(X)) = \underline{\delta}(X) & (U8)\ \overline{\delta}(\overline{\delta}(X)) = \overline{\delta}(X) \\
(L9)\ \underline{\delta}((\underline{\delta}(X))^c) = (\overline{\delta}(X))^c & (U9)\ \overline{\delta}((\overline{\delta}(X))^c) = (\overline{\delta}(X))^c \\
(L10)\ \mathit{If} X \in E/\delta,\ \mathit{then}\ \underline{\delta}(X) = X & (U10)\ \mathit{If} X \in E/\delta\ \mathit{then}\ \overline{\delta}(X) = X
\end{array}$$

The approximation operators are exploited to divide every subset into three regions helping us to specify the knowledge induced from a subset and discover its structure.

Definition 2 (Pawlak 1982) Every subset X of Pawlak approximation space (E, δ) is associated with three regions called positive, boundary, and negative. They are respectively given by the following formulas.

$$P^+(X) = \underline{\delta}(X),$$
$$B(X) = \overline{\delta}(X) \setminus \underline{\delta}(X),$$
$$P^-(X) = E \setminus \overline{\delta}(X)$$

To capture the degree of completeness and incompleteness of knowledge obtained from a subset, the next measures were familiarized.

Definition 3 (Pawlak 1982) Every subset X of Pawlak approximation space (E, δ) is associated with two measures (or values) called accuracy and roughness measures. They are respectively defined as follows.

$$M(X) = \frac{|\underline{\delta}(X)|}{|\overline{\delta}(X)|}, \text{ where } X \text{ is nonempty.}$$
$$R(X) = 1 - M(X).$$

As it is well known the equivalence relation limits the application scope of rough sets, this motivated Yao (1996, 1998) to come up with a brilliant idea called "right and left neighborhoods" which are formulated under any arbitrary relation as follows.

Definition 4 (Yao 1996, 1998) Let (E, δ) be an approximation space (herein, δ is an arbitrary relation need not be an equivalence relation). Then

The right neighborhood of $w \in E$, denoted by $N_r(w)$, is defined by
$$N_r(w) = \{x \in E : (w, x) \in \delta\}, \text{ and}$$
The left neighborhood of $w \in E$, denoted by $N_l(w)$, is defined by
$$N_l(w) = \{x \in E : (x, w) \in \delta\}.$$

Then, Yao formulated the approximation operators with respect to right and left neighborhoods as follows.

Definition 5 (Yao 1996, 1998) For $k \in \{r, l\}$, the k-lower and k-upper approximations induced from an approximation space (E, δ) are defined as follows.

$$\underline{\delta}_k(X) = \{w \in E : N_k(w) \subseteq X\}, \text{ and}$$
$$\overline{\delta}_k(X) = \{w \in E : N_k(w) \cap X \neq \emptyset\}$$

Remark 1 It should be noted that some features of Pawlak approximation space, displayed in Proposition 1, are lost, for instance, the properties report that "the k-lower approximation of the empty set is empty" and "the k-upper approximation of the universal set is the universal set" are generally false. Also, the distributive properties of intersection and union under k-lower and k-upper approximations, respectively, are evaporated.

Posteriorly, the researchers investigated different types of generalized rough sets with aim of increasing the accuracy measures and improving the approximations of rough subsets. These efforts produced several types of neighborhood systems listed in the following.

Definition 6 (Abd El-Monsef et al. 2014; Allam et al. 2005, 2006) For $k \in \{\langle r\rangle, \langle l\rangle, i, u, \langle i\rangle, \langle u\rangle\}$, the k-neighborhoods of each $w \in E$, denoted by $N_k(w)$, induced from an approximation space (E, δ) are formulated as follows.

(i)

$$N_{\langle r\rangle}(w) = \begin{cases} \bigcap_{w \in N_r(x)} N_r(x) & \text{there exists } N_r(x) \text{ including } w \\ \emptyset & \text{Otherwise} \end{cases}$$

(ii)

$$N_{\langle l\rangle}(w) = \begin{cases} \bigcap_{w\in N_l(x)} N_l(x) & \text{there exists } N_l(x) \text{ including } w \\ \emptyset & \text{Otherwise} \end{cases}$$

(iii) $N_i(w) = N_r(w) \cap N_l(w)$.
(iv) $N_u(w) = N_r(w) \cup N_l(w)$.
(v) $N_{\langle i\rangle}(w) = N_{\langle r\rangle}(w) \cap N_{\langle l\rangle}(w)$.
(vi) $N_{\langle u\rangle}(w) = N_{\langle r\rangle}(w) \cup N_{\langle l\rangle}(w)$.

The neighborhoods above were applied to introduce novel kinds of approximation operators following similar technique given in Definition 5. We draw attention to the shortcoming caused by using Pawlak accuracy measures when δ is not a reflexive relation, that is, we sometimes obtain accuracy measures greater than one or undefined as illustrated in the next example.

Example 1 Consider $\delta = \{(w,w),(w,x),(x,y)\}$ is a relation on $E = \{w,x,y,z\}$. It is clear that $N_r(w) = \{w,x\}$, $N_r(x) = \{y\}$ and $N_r(y) = N_r(z) = \emptyset$. It follows from Definition 5 that $\underline{\delta}_k(\{x,y\}) = \{x,y,z\}$ and $\overline{\delta}_k(\{x,y\}) = \{w,x\}$. According to Definition 3 we find $M(\{x,y\}) = \frac{3}{2} > 1$ which is a contradiction. Also, note that $\underline{\delta}_k(\{z\}) = \{y,z\}$ and $\overline{\delta}_k(\{z\}) = \emptyset$, which means that $M(\{z\})$ is undefined.

To get rid of these failures, it was proposed a slight modification for accuracy measure definition as given below.

Definition 7 (Abd El-Monsef et al. 2014; Allam et al. 2005, 2006; Yao 1996, 1998) For each k, the accuracy measures of a set X in an approximation space (E,δ) is given by

$$M_k(X) = \frac{|\ \underline{\delta}_k(X) \cap X\ |}{|\ \overline{\delta}_k(X) \cup X\ |}, \text{ where } X \text{ is nonempty.}$$

Remark 2

(i) If δ is a quasiorder (reflexive and transitive), then $N_k = N_{\langle k\rangle}$ for each $k \in \{r,l,i,u\}$
(ii) If δ is reflexive, then the formula given in Definition 7 is written as follows $M_k(X) = \frac{|\underline{\delta}_k(X)|}{|\overline{\delta}_k(X)|}$.

To investigate the monotonicity property of our accuracy and roughness measures, the next result will be helpful.

Proposition 2 (Al-shami 2022) *Let (E,δ_1) and (E,δ_2) be approximation spaces such that $\delta_1 \subseteq \delta_2$. Then $N_{1k}(w) \subseteq N_{2k}(w)$ for each $w \in E$ and $k \in \{r,l,i,u\}$.*

Definition 8 (see, Al-shami 2022) We call the approximations spaces (E,δ_1) and (E,δ_2) have the property of monotonicity accuracy (resp. monotonicity roughness) if $M_{\delta_1}(X) \geq M_{\delta_2}(X)$ (resp., $M_{\delta_1}(X) \leq M_{\delta_2}(X)$) whenever $\delta_1 \subseteq \delta_2$.

2.2 Rough set concepts via topological structures

A subcollection τ of the power set of a nonempty set E is called a topology on E provided that it is closed under finite intersection and arbitrary union as well as $\emptyset$ and E are members of τ.

Pawlak (1982) noted that the equivalences classes form a base for a specified type of topology (known as a quasi-discrete topology), which means there is a similarity between the behaviors of some topological and rough set concepts, for example, interior topological operator and lower approximation, and closure topological operator and upper approximation. Then, Skowron (1988) and Wiweger (1989) studied topological structures of rough sets. These pioneering works paved to conducting deep investigations concerning rough set concepts from a topological standpoint. Later on, N_k-neighborhood systems were used to establish new sorts of rough approximations inspired by topological structures. One of the suggested manners to do that is demonstrated by the next interesting result.

Theorem 1 *Abd El-Monsef et al.* (2014) *The topology on E generated from an approximation space* (E, δ) *given by* $\tau_k = \{U \subseteq E : N_k(w) \subseteq U \text{ for each } w \in U\}$ *for each k.*

The approximation rough operators were familiarized topologically depending on Theorem 1 as follows.

Definition 9 (Abd El-Monsef et al. 2014) The k-lower and k-upper approximations and accuracy measure of a set X induced from a topological space (E, τ_k) are respectively given by

$$\underline{O}_k(X) = \cup\{U \in \tau_k : U \subseteq X\},$$
$$\overline{O}_k(X) = \cap\{F : F^c \in \tau_k \text{ and } X \subseteq F\}, \text{ and}$$
$$\mathcal{T}_k(X) = \frac{\mid \underline{O}_k(X) \mid}{\mid \overline{O}_k(X) \mid}, \text{ where } X \text{ is nonempty.}$$

The regions of a subset were formulated using $\underline{O}_k(X)$ and $\overline{O}_k(X)$ following a similar manner to their counterparts given in Definition 2.

Now, we prove that accuracy measures produced by N_k-neighborhood systems are better than accuracy measures produced by topological approaches under an arbitrary relation, then we demonstrate that they are identical under a quasiorder relation.

Proposition 3 *For every subset X of an approximation space* (E, δ)*, we have* $\mathcal{T}_k(X) \leq M_k(X)$.

Proof Suppose that $w \in \underline{O}_k(X)$. Then, there exists $U \in \tau_k$ such that $w \in U \subseteq X$ and $N_k(U) \subseteq U$. This implies that $N_k(w) \subseteq N_k(U) \subseteq X$, so $w \in \underline{\delta}_k(X)$. This means that

$$\mid \underline{O}_k(X) \mid \leq \mid \underline{\delta}_k(X) \cap X \mid \tag{1}$$

Now, let $w \notin \overline{O}_k(X)$. Then, there exists $U \in \tau_k$ such that $w \in U$ and $U \cap X = \emptyset$, so $U \subseteq X^c$. Therefore, $N_k(w) \subseteq N_k(U) \subseteq X^c$. This means that $N_k(w) \cap X = \emptyset$, thus $w \notin \overline{\delta}_k(X)$. Hence, we get

$$|\ \overline{\delta}_k(X) \cup X\ | \leq |\ \overline{O}_k(X)\ | \tag{2}$$

It follows from (1) and (2) that $\frac{|\underline{O}_k(X)|}{|\overline{O}_k(X)|} \leq \frac{|\underline{\delta}_k(X) \cap X|}{|\overline{\delta}_k(X) \cup X|}$. This completes the proof that $\mathcal{T}_k(X) \leq M_k(X)$. □

To illustrate that the converse need not be true, consider a subset $\{y, z\}$ of an approximation space (E, δ) given in Example 1. Now, $\tau_r = \{\emptyset, E, \{y\}, \{z\}, \{y, z\}, \{x, y\}, \{x, y, z\}, \{w, x, y\}\}$. By calculation, we find that $M_r(\{y, z\}) = \frac{2}{3}$ whereas $\mathcal{T}_r(\{y, z\}) = \frac{1}{2}$.

Proposition 4 *Let* (E, δ) *be an approximation space such that* δ *is quasiorder. Then* $\mathcal{T}_k(X) = M_k(X)$ *for every* $X \subseteq E$.

Proof By proposition 3, we have $\mathcal{T}_k(X) \leq M_k(X)$. To prove that $M_k(X) \leq \mathcal{T}_k(X)$, it suffices to show that $N_k(w) \in \tau_k$ for each $w \in E$. In other words, $N_k(w) = N_k(N_k(w))$. It follows from the reflexivity of δ that $N_k(w) \subseteq N_k(N_k(w))$. Conversely, without lose of generality, we consider $k = r$. let $x \in N_r(N_r(w))$. Then there is $z \in N_r(w)$ such that $(z, x) \in \delta$. Now, we have $(w, z) \in \delta$. By transitivity of δ we obtain $(w, x) \in \delta$, which means that $x \in N_r(w)$. Thus, $N_r(N_r(w)) \subseteq N_r(w)$. Hence, $\mathcal{T}_k(X) = M_k(X)$, as required. □

Remark 3 There are different methods to form a topological structure from N_k-neighborhood systems such as studied by Lashin et al. (2005). Its methodology depends on considering the collection $\{N_k(w) : w \in E\}$ as a subbase for a topology on E.

In 1983, Mashhour (1983) extended the concept of topology to "supra-topology" by neglecting the intersection condition. That is, a supra-topology is defined on a nonempty set E as a subcollection $\mathcal{U}$ of the power set of E satisfying two axioms 1)$\emptyset, E \in \mathcal{U}$, and 2) $\mathcal{U}$ is closed under arbitrary union.

Definition 10 (Mashhour 1983) Let $\mathcal{U}$ be a supra-topology on E and $X \subseteq E$. We call X a supra-open (resp. supra-closed) set if it is a member of $\mathcal{U}$ (resp., its complement belongs to $\mathcal{U}$). We define the supra-interior points of a set X, denoted by $Int(X)$, as a union of all supra-open subsets of this set, and we define the supra-closure points of a set, denoted by $Cl(X)$, as the intersection of all supra-closed supersets of this set.

3 Generating supra-topologies from N_k-neighborhoods induced by an arbitrary relation

We dedicate this part to introducing new techniques for initiating supra-topologies from N_k -neighborhoods under any arbitrary relation. With the help of an illustrative example, we make some comparisons between these supra-topologies and determine under which relations we get equivalences between some of them. As we note from the techniques in the published literature that the largest structures are obtained in cases of intersection and minimal intersection, whereas our techniques produce the largest structures in cases of union and minimal union, which is convenient to model some phenomena.

Let us start with the next lemma which assists us to prove that N_k-neighborhood systems are closed under a union operator.

Lemma 1 *If N_k-neighborhood systems are induced from an approximation space (E, δ), then $N_k(X \cup Z) = N_k(X) \cup N_k(Z)$ for each $X, Z \subseteq E$.*

Proof It is apparent that $N_k(X) \subseteq N_k(X \cup Z)$ and $N_k(Z) \subseteq N_k(X \cup Z)$, so $N_k(X) \cup N_k(Z) \subseteq N_k(X \cup Z)$. Conversely, let $w \in N_k(X \cup Z)$. Then there exists $x \in X \cup Z$ such that $w \in N_k(x)$. This implies that $N_k(x) \subseteq N_k(X)$ or $N_k(x) \subseteq N_k(Z)$. Accordingly, $w \in N_k(X) \cup N_k(Z)$, which means that $N_k(X \cup Z) \subseteq N_k(X) \cup N_k(Z)$. Hence, the proof is complete. □

The next result presents a method of generating supra-topology structures from N_k -neighborhood systems.

Theorem 2 *Assume (E, δ) is an approximation space. Then, the collection $\mathcal{U} = \{E\} \cup \{U \subseteq E : U \subseteq N_k(U)\}$ forms a k-supra topology on E.*

Proof According to the collection definition, $E \in \mathcal{U}$, also, $N_k(\emptyset) = \emptyset$ which means that $\emptyset \in \mathcal{U}$. To prove that $\mathcal{U}$ is closed under union operator, let $U_1, U_2 \in \mathcal{U}$. Then $U_1 \subseteq N_k(U_1)$ and $U_2 \subseteq N_k(U_2)$. We automatically obtain $U_1 \cup U_2 \subseteq N_k(U_1) \cup N_k(U_2)$. By Lemma 1, we get $U_1 \cup U_2 \subseteq N_k(U_1 \cup U_2)$. Thus, $U_1 \cup U_2 \in \mathcal{U}$. Hence, $\mathcal{U}$ is a supra-topology on E, as required. □

Definition 11 The triple system $(E, \delta, \mathcal{U}_k)$ is said to be a k-supra topological space (briefly, *kSTS*), where $\mathcal{U}_k$ is a k-supra topology on E generated by Theorem 2.

We call a subset of E a k-supra open set if it is a member of $\mathcal{U}_k$, and we call a subset of E a k-supra closed set if its complement is a member of $\mathcal{U}_k$. The family of all k-supra closed subsets of E will be denoted by $\mathcal{U}_k^c$.

It is very important to note that the collection given in Theorem 2 need not be a topology, which makes this method is completely different than a method given in Theorem 1. To validate this note, consider $\{w, x\}$ and $\{w, y\}$ which are members of $\mathcal{U}_u$ given in Example 2; obviously, their intersection $\{w\}$ is not a member of $\mathcal{U}_u$.

To investigate these structures topologically, we need to put forward the counterparts of interior and closure topological operators.

Definition 12 The k-supra interior and k-supra closure points of a subset X of a *kSTS* $(E, \delta, \mathcal{U}_k)$ are defined respectively by

$$Int_k(X) = \cup\{G \in \mathcal{U}_k : G \subseteq X\}, \text{ and}$$
$$Cl_k(X) = \cap\{Y \in \mathcal{U}_k^c : X \subseteq Y\}$$

The next example demonstrates how to produce k-supra topologies from an approximation space.

Example 2 Consider $\delta = \{(y, y), (w, x), (w, y), (z, x)\}$ is a binary relation on $E = \{w, x, y, z\}$. Then, we first compute a neighborhood of each point in E in the Table 1.

Table 1 N_k-neighborhoods

	w	x	y	z
N_r	$\{x,y\}$	$\emptyset$	$\{y\}$	$\{x\}$
N_l	$\emptyset$	$\{w,z\}$	$\{w,y\}$	$\emptyset$
N_i	$\emptyset$	$\emptyset$	$\{y\}$	$\emptyset$
N_u	$\{x,y\}$	$\{w,z\}$	$\{w,y\}$	$\{x\}$
$N_{\langle r\rangle}$	$\emptyset$	$\{x\}$	$\{y\}$	$\emptyset$
$N_{\langle l\rangle}$	$\{w\}$	$\emptyset$	$\{w,y\}$	$\{w,z\}$
$N_{\langle i\rangle}$	$\emptyset$	$\emptyset$	$\{y\}$	$\emptyset$
$N_{\langle u\rangle}$	$\{w\}$	$\{x\}$	$\{w,y\}$	$\{w,z\}$

According to Definition 11, the k-supra topologies $\mathcal{U}_k$ generated from these neighborhoods are the following.

$$\begin{cases}\mathcal{U}_r = & \{\emptyset, E, \{y\}\};\\ \mathcal{U}_l = & \{\emptyset, E, \{y\}, \{w,y\}\};\\ \mathcal{U}_i = & \{\emptyset, E, \{y\}\};\\ \mathcal{U}_u = & \{\emptyset, E, \{y\}, \{w,x\}, \{w,y\}, \{x,z\}, \{w,x,y\}, \{w,x,z\}, \{x,y,z\}\};\\ \mathcal{U}_{\langle r\rangle} = & \{\emptyset, E, \{x\}, \{y\}, \{x,y\}\};\\ \mathcal{U}_{\langle l\rangle} = & \{\emptyset, E, \{w\}, \{y\}, \{z\}, \{w,y\}, \{w,z\}, \{y,z\}, \{w,y,z\}\};\\ \mathcal{U}_{\langle i\rangle} = & \{\emptyset, E, \{y\}\};\\ \mathcal{U}_{\langle u\rangle} = & P(E).\end{cases} \tag{3}$$

Now, we reveal the relationships between these structures and study the conditions under which some of these structures are identical.

Proposition 5 *Let $(E, \delta, \mathcal{U}_k)$ be a kSTS. Then*

(i) $\mathcal{U}_i \subseteq \mathcal{U}_r \subseteq \mathcal{U}_u$.
(ii) $\mathcal{U}_i \subseteq \mathcal{U}_l \subseteq \mathcal{U}_u$.
(iii) $\mathcal{U}_{\langle i\rangle} \subseteq \mathcal{U}_{\langle r\rangle} \subseteq \mathcal{U}_{\langle u\rangle}$.
(iv) $\mathcal{U}_{\langle i\rangle} \subseteq \mathcal{U}_{\langle l\rangle} \subseteq \mathcal{U}_{\langle u\rangle}$.

Proof To prove (i) and (ii), let X be a set in $\mathcal{U}_i$. Then $X \subseteq N_i(X)$. It is well recognized that $N_i(X) \subseteq N_r(X)$ and $N_i(X) \subseteq N_l(X)$, so $X \subseteq N_r(X)$ and $X \subseteq N_r(X)$. This means that $X \in \mathcal{U}_r$ and $X \in \mathcal{U}_l$. Thus, $\mathcal{U}_i \subseteq \mathcal{U}_r$ and $\mathcal{U}_i \subseteq \mathcal{U}_l$. Similarly, we prove that $\mathcal{U}_r \subseteq \mathcal{U}_u$ and $\mathcal{U}_i \subseteq \mathcal{U}_u$.

Following similar arguments, (iii) and (iv) are proved. □

Corollary 1 *Let $(E, \delta, \mathcal{U}_k)$ be a kSTS and $X \subseteq E$. Then*

(i) $Int_i(X) \subseteq Int_r(X) \subseteq Int_u(X)$ *and* $Cl_u(X) \subseteq Cl_r(X) \subseteq Cl_i(X)$.
(ii) $Int_i(X) \subseteq Int_l(X) \subseteq Int_u(X)$ *and* $Cl_u(X) \subseteq Cl_l(X) \subseteq Cl_i(X)$.
(iii) $Int_{\langle i\rangle}(X) \subseteq Int_{\langle r\rangle}(X) \subseteq Int_{\langle u\rangle}(X)$ *and* $Cl_{\langle u\rangle}(X) \subseteq Cl_{\langle r\rangle}(X) \subseteq Cl_{\langle i\rangle}(X)$.
(iv) $Int_{\langle i\rangle}(X) \subseteq Int_{\langle l\rangle}(X) \subseteq Int_{\langle u\rangle}(X)$ *and* $Cl_{\langle u\rangle}(X) \subseteq Cl_{\langle l\rangle}(X) \subseteq Cl_{\langle i\rangle}(X)$.

It follows from Example 2 and Example 3 that relations given in the four items of Proposition 5 are proper. Also, these examples show that $\mathcal{U}_r$ and $\mathcal{U}_l$ ($\mathcal{U}_{\langle r\rangle}$ and $\mathcal{U}_{\langle l\rangle}$) are independent of each other. Moreover, these examples demonstrate the converses of the four items of Corollary 1 are false in general.

Proposition 6 *Let* $(E, \delta, \mathcal{U}_k)$ *be a kSTS such that* δ *is symmetric. Then*

(i) $\mathcal{U}_r = \mathcal{U}_l = \mathcal{U}_i = \mathcal{U}_u$.
(ii) $\mathcal{U}_{\langle r\rangle} = \mathcal{U}_{\langle l\rangle} = \mathcal{U}_{\langle i\rangle} = \mathcal{U}_{\langle u\rangle}$.

Proof Follows from the fact that $N_r(w) = N_l(w)$ and $N_{\langle r\rangle}(w) = N_{\langle l\rangle}(w)$ under a symmetric relation. □

Corollary 2 *Let X be a subset of a kSTS* $(E, \delta, \mathcal{U}_k)$. *If* δ *is symmetric, then*

(i) $Int_u(X) = Int_r(X) = Int_l(X) = Int_i(X)$ *and* $Cl_u(X) = Cl_r(X) = Cl_l(X) = Cl_i(X)$.
(ii) $Int_{\langle u\rangle}(X) = Int_{\langle r\rangle}(X) = Int_{\langle l\rangle}(X) = Int_{\langle i\rangle}(X)$ *and* $Cl_{\langle u\rangle}(X) = Cl_{\langle r\rangle}(X) = Cl_{\langle l\rangle}(X) = Cl_{\langle i\rangle}(X)$.

Example 2 emphasizes that the symmetry condition of Proposition 6 and Corollary 2 is indispensable.

Recall that a relation δ is called serial if every element has a nonempty N_r-neighborhood. And it is called inverse serial (or surjective) if every element has a nonempty N_l -neighborhood.

Proposition 7 *If* δ *is an inverse serial relation on E, then* $\mathcal{U}_{\langle r\rangle}$ *and* $\mathcal{U}_{\langle u\rangle}$ *are identical; moreover, they are discrete topologies.*

Proof Since δ is an inverse serial relation, we have $\bigcup_{w\in E} N_r(w) = E$. This means that $N_{\langle r\rangle}(w) \neq \emptyset$. In this case we have $w \in N_{\langle r\rangle}(w)$ for each $w \in E$. This implies that any singleton subset of E is an r-supra open. Hence, $\mathcal{U}_{\langle r\rangle}$ is a discrete topology. Since $\mathcal{U}_{\langle r\rangle} \subseteq \mathcal{U}_{\langle u\rangle}$, we obtain $\mathcal{U}_{\langle u\rangle}$ is also a discrete topology. □

Proposition 8 *If* δ *is a serial relation on E, then* $\mathcal{U}_{\langle l\rangle}$ *and* $\mathcal{U}_{\langle u\rangle}$ *are identical; moreover, they are discrete topologies.*

Proof Similar to the proof of Proposition 7. □

Corollary 3 *If a relation* δ *is serial and inverse serial on E, then all* $\mathcal{U}_{\langle k\rangle}$ *are identical; moreover, they are discrete topologies.*

In Example 2, note that $N_r(x) = N_l(z) = \emptyset$, which means that a relation δ is neither serial nor inverse serial. On the other hand, $\mathcal{U}_{\langle u\rangle}$ is a discrete topology. So that, the converses of Proposition 7, Proposition 8 and Corollary 3 are false in general.

The significance of the following result is that we will rely on it to prove that the method introduced in the next section is better than all previous ones to produce approximations and accuracy values under a reflexive relation.

Proposition 9 *If δ is a reflexive relation on E, then all $\mathcal{U}_k$ are discrete topologies.*

Proof Let U be an arbitrary subset of E. Since δ is a reflexive relation, we obtain $U \subseteq N_k(U)$ for each k. Therefore, $U \in \mathcal{U}_k$, which means that every subset of E belongs to $\mathcal{U}_k$. Hence, $\mathcal{U}_k$ is the discrete topology on E, as required. □

Again, Example 2 serves as a fantastic tool to illustrate the invalidity of some results. It illuminates that the converse of Proposition 9 is generally not true. Note that $\mathcal{U}_{\langle u \rangle}$ is a discrete topology in spite of δ is not reflexive.

Proposition 10 *A relation δ on E is reflexive iff $\mathcal{U}_k$ is a discrete topology, where $k \in \{r, l, i\}$.*

Proof The condition of necessity comes from Proposition 9. We prove the sufficient part for $k = r$ and the other two cases are proved similarly. Since $\mathcal{U}_r$ is discrete, we have $\{w\} \in \mathcal{U}_r$ for each $w \in E$. This means that $\{w\} \subseteq N_r(w)$, i.e. $(w, w) \in \delta$ for each $w \in E$. Hence, δ is reflexive. □

The following result will help us to prove the monotonicity and roughness properties of rough sets models presented in the next section.

Proposition 11 *Let $(E, \delta_1, \mathcal{U}_{1k})$ and $(E, \delta_2, \mathcal{U}_{2k})$ be kSTSs such that $\delta_1 \subseteq \delta_2$. Then $\mathcal{U}_{1k} \subseteq \mathcal{U}_{2k}$ for each $k \in \{r, l, i, u\}$.*

Proof Let X be a set in $\mathcal{U}_{1k}$, where $k \in \{r, l, i, u\}$. Then $X \subseteq N_{1k}(X)$. Since $\delta_1 \subseteq \delta_2$ we get $N_{1k}(X) \subseteq N_{2k}(X)$. So that, $X \subseteq N_{2k}(X)$, which means that $X \in \mathcal{U}_{2k}$. Hence, we obtain the desired result. □

4 New rough models generated by supra-topology

In this section, we will establish novel rough models depending on k-supra topologies induced from N_k-neighborhood systems. We investigate their main properties and give an algorithm to illustrate how the supra accuracy values are calculated. To show the importance of these models, we elucidate that they improve the approximations and produce accuracy values which are better than all previous ones if the relation is reflexive.

4.1 Supra *k*-lower and supra *k*-upper approximations

Definition 13 We define supra k-lower approximation $\underline{\lambda}_k$ and supra k-upper approximation $\overline{\lambda}_k$ of a subset X of a *kSTS* $(E, \delta, \mathcal{U}_k)$ as follows.

$$\underline{\lambda}_k(X) = \cup\{U \in \mathcal{U}_k : U \subseteq X\}, \text{ and}$$
$$\overline{\lambda}_k(X) = \cap\{F \in \mathcal{U}_k^c : X \subseteq F\}.$$

Note that $\underline{\lambda}_k(X)$ and $\overline{\lambda}_k(X)$ represent the supra-interior and supra-closure points of X, respectively. Accordingly, we obtain $w \in \overline{\lambda}_k(X)$ iff $U \cap X \neq \emptyset$ for each $U \in \mathcal{U}_k$ containing w. As a special case, if δ is an equivalence relation, then $\underline{\lambda}_k(X)$ and $\overline{\lambda}_k(X)$ represent the lower and upper approximations in sense of Pawlak.

Foremost, we prove the first advantage of the current approximations which is to preserve most of properties of Pawlak approximations.

Proposition 12 *Let X and Z be subsets of a kSTS* $(E, \delta, \mathcal{U}_k)$. *Then the next properties are satisfied.*

(i) $\underline{\lambda}_k(X) \subseteq X$.
(ii) $\underline{\lambda}_k(\emptyset) = \emptyset$.
(iii) $\underline{\lambda}_k(E) = E$.
(iv) *If* $X \subseteq Z$, *then* $\underline{\lambda}_k(X) \subseteq \underline{\lambda}_k(Z)$.
(v) $\underline{\lambda}_k(X^c) = (\overline{\lambda}_k(X))^c$.
(vi) $\underline{\lambda}_k(\underline{\lambda}_k(X)) = \underline{\lambda}_k(X)$.

Proof The proofs of (i) and (ii) come from Definition 13.

The proof of (iii) comes from the fact that E is the largest supra-open subset of a *kSTS* $(E, \delta, \mathcal{U}_k)$.

(iv): Let $X \subseteq Z$. Then $\cup\{U \in \mathcal{U}_k : U \subseteq X\} \subseteq \cup\{U \in \mathcal{U}_k : U \subseteq Z\}$ and so $\underline{\lambda}_k(X) \subseteq \underline{\lambda}_k(Z)$.

(v): Let $w \in \underline{\lambda}_k(X^c)$. Then there is a supra-open set U satisfying $w \in U \subseteq X^c$, so $U \cap X = \emptyset$, which means that $w \notin \overline{\lambda}_k(X)$. Thus, $w \in (\overline{\lambda}_k(X))^c$. On the other hand, let $w \in (\overline{\lambda}_k(X))^c$. Then $w \notin \overline{\lambda}_k(X)$, which means there is a supra-open set U satisfying $w \in U$ and $U \cap X = \emptyset$. So $w \in U \subseteq X^c$. Hence $w \in \underline{\lambda}_k(X^c)$.

(vi): From (i) we get $\underline{\lambda}_k(\underline{\lambda}_k(X)) \subseteq \underline{\lambda}_k(X)$. Conversely, let $w \in \underline{\lambda}_k(X)$. Then there is a supra-open set U such that $w \in U \subseteq X$. It follows from (iv) that $\underline{\lambda}_k(U) \subseteq \underline{\lambda}_k(X)$. According to Definition 13 we have $U = \underline{\lambda}_k(U)$, so $w \in \underline{\lambda}_k(U) \subseteq \underline{\lambda}_k(\underline{\lambda}_k(X))$. Thus, $\underline{\lambda}_k(X) \subseteq \underline{\lambda}_k(\underline{\lambda}_k(X))$. Hence, we get the wished result. □

Corollary 4 *Let X and Z be subsets of a kSTS* $(E, \delta, \mathcal{U}_k)$. *Then* $\underline{\lambda}_k(X \cap Z) \subseteq \underline{\lambda}_k(X) \cap \underline{\lambda}_k(Z)$ *and* $\underline{\lambda}_k(X) \cup \underline{\lambda}_k(Z) \subseteq \underline{\lambda}_k(X \cup Z)$.

Proof Directly follows from (iv) of Proposition 12. □

In a *uSTS* $(E, \delta, \mathcal{U}_u)$ given in eq. (3) consider $X = \{w, x\}$, $Y = \{x, y\}$ and $Z = \{w, y, z\}$. Then

1. $\underline{\lambda}_u(Z) = \{w, y\} \subset Z$.
2. $\underline{\lambda}_u(Y) = \{y\} \subset \underline{\lambda}_l(Z) = \{w, y\}$ whereas $Y \nsubseteq Z$.
3. $\underline{\lambda}_u(X \cap Z) = \emptyset \subset \underline{\lambda}_u(X) \cap \underline{\lambda}_u(Z) = \{w\}$.
4. $\underline{\lambda}_u(X) \cup \underline{\lambda}_u(Z) = \{w, x, y\} \subset \underline{\lambda}_u(X \cup Z) = E$.

It follows from these computations that the inclusion relations of (i) and (iv) of Proposition 12 as well as Corollary 4 are proper.

Proposition 13 *Let X and Z be subsets of a kSTS* $(E, \delta, \mathcal{U}_k)$. *Then the next properties are satisfied.*

(i) $X \subseteq \overline{\lambda}_k(X)$.
(ii) $\overline{\lambda}_k(\emptyset) = \emptyset$.
(iii) $\overline{\lambda}_k(E) = E$.
(iv) *If* $X \subseteq Z$, *then* $\overline{\lambda}_k(X) \subseteq \overline{\lambda}_k(Z)$.
(v) $\overline{\lambda}_k(X^c) = (\underline{\lambda}_k(X))^c$.
(vi) $\overline{\lambda}_k(\overline{\lambda}_k(X)) = \overline{\lambda}_k(X)$.

Proof Similar to the proof of Proposition 12. □

Corollary 5 *Let X and Z be subsets of a kSTS* $(E, \delta, \mathcal{U}_k)$. *Then* $\overline{\lambda}_k(X \cap Z) \subseteq \overline{\lambda}_k(X) \cap \overline{\lambda}_k(Z)$ *and* $\overline{\lambda}_k(X) \cup \overline{\lambda}_k(Z) \subseteq \overline{\lambda}_k(X \cup Z)$.

Proof Directly follows from (iv) of Proposition 13. □

In a *uSTS* $(E, \delta, \mathcal{U}_u)$ given in equation (3) consider $X = \{w, x\}$, $Y = \{x, z\}$ and $Z = \{w, z\}$. Then

1. $X \subset \overline{\lambda}_u(X) = \{w, x, z\}$.
2. $\overline{\lambda}_u(Y) = Y \subset \overline{\lambda}_u(X)$ whereas $Y \nsubseteq X$.
3. $\overline{\lambda}_u(X \cap Z) = \{w\} \subset \overline{\lambda}_u(X) \cap \overline{\lambda}_u(Z) = \{w, x, z\}$.
4. $\overline{\lambda}_u(\{w\}) \cup \overline{\lambda}_u(\{z\}) = \{w, z\} \subset \overline{\lambda}_u(\{w, z\}) = \{w, x, z\}$.

It follows from these computations that the relations of inclusion given in (i) and (iv) of Proposition 13 as well as Corollary 5 are proper.

Definition 14 The supra k-accuracy and supra k-roughness measures (or values) of a set X in a *kSTS* $(E, \delta, \mathcal{U}_k)$ are defined respectively by

$$\mathcal{A}_k(X) = \frac{|\ \underline{\lambda}_k(X)\ |}{|\ \overline{\lambda}_k(X)\ |}, \text{ where } X \neq \emptyset.$$

$$\mathcal{R}_k(X) = 1 - \mathcal{A}_k(X).$$

Note that for every set $X \subseteq E$ the two values $\mathcal{A}_k(X)$ and $\mathcal{R}_k(X)$ lie in the closed interval [0, 1].

Definition 15 We call the *kSTSs* $(E, \delta_1, \mathcal{U}_{1k})$ and $(E, \delta_2, \mathcal{U}_{2k})$ have the property of monotonicity accuracy (resp. monotonicity roughness) if $\mathcal{A}_{1k}(X) \leq \mathcal{A}_{2k}(X)$ (resp., $\mathcal{R}_{1k}(X) \geq \mathcal{R}_{2k}(X)$) whenever δ_1 is a subset of δ_2.

The next two result illustrate that the supra k-accuracy and supra k-roughness measures satisfy the monotonicity property.

Proposition 14 *Let $(E, \delta_1, \mathcal{U}_{1k})$ and $(E, \delta_2, \mathcal{U}_{2k})$ be two kSTSs such that δ_1 is a subset of δ_2. Then for $k \in \{r, l, i, u\}$ and any set $X \subseteq E$ we have $\mathcal{A}_{1k}(X) \leq \mathcal{A}_{2k}(X)$.*

Proof Since δ_1 is a subset of δ_2, it follows from Proposition 11 that $\mathcal{U}_{1k}$ is a subset of $\mathcal{U}_{2k}$ for each $k \in \{r, l, i, u\}$. This automatically means that $| \underline{\lambda}_{1k}(X) | \leq | \underline{\lambda}_{2k}(X) |$ and $\frac{1}{|\overline{\lambda}_{1k}(X)|} \leq \frac{1}{|\overline{\lambda}_{2k}(X)|}$. Therefore, $\frac{|\underline{\lambda}_{1k}(X)|}{|\overline{\lambda}_{1k}(X)|} \leq \frac{|\underline{\lambda}_{2k}(X)|}{|\overline{\lambda}_{2k}(X)|}$. Hence, $\mathcal{A}_{1k}(X) \leq \mathcal{A}_{2k}(X)$, as required. □

Corollary 6 *Let $(E, \delta_1, \mathcal{U}_{1k})$ and $(E, \delta_2, \mathcal{U}_{2k})$ be two kSTSs such that δ_1 is a subset of δ_2. Then for $k \in \{r, l, i, u\}$ and any set $X \subseteq E$ we have $\mathcal{R}_{1k}(X) \geq \mathcal{R}_{2k}(X)$.*

Definition 16 The supra k-positive, supra k-boundary, and supra k-negative regions of a set X in a *kSTS* $(E, \delta, \mathcal{U}_k)$ are defined respectively by

$$\begin{aligned} \delta_k^+(X) &= \underline{\lambda}_k(X), \\ \mathcal{B}_k(X) &= \overline{\lambda}_k(X) \setminus \underline{\lambda}_k(X), \text{ and} \\ \delta_k^-(X) &= E \setminus \overline{\lambda}_k(X). \end{aligned}$$

Proposition 15 *Let $(E, \delta_1, \mathcal{U}_{1k})$ and $(E, \delta_2, \mathcal{U}_{2k})$ be kSTSs such that δ_1 is a subset of δ_2. Then for $k \in \{r, l, i, u\}$ and any set $X \subseteq E$ the following results hold true.*

(i) $\mathcal{B}_{2k}(X) \subseteq \mathcal{B}_{1k}(X)$.
(ii) $\delta_{1k}^-(X) \subseteq \delta_{2k}^-(X)$.

Proof Follows from Proposition 11 and Proposition 14. □

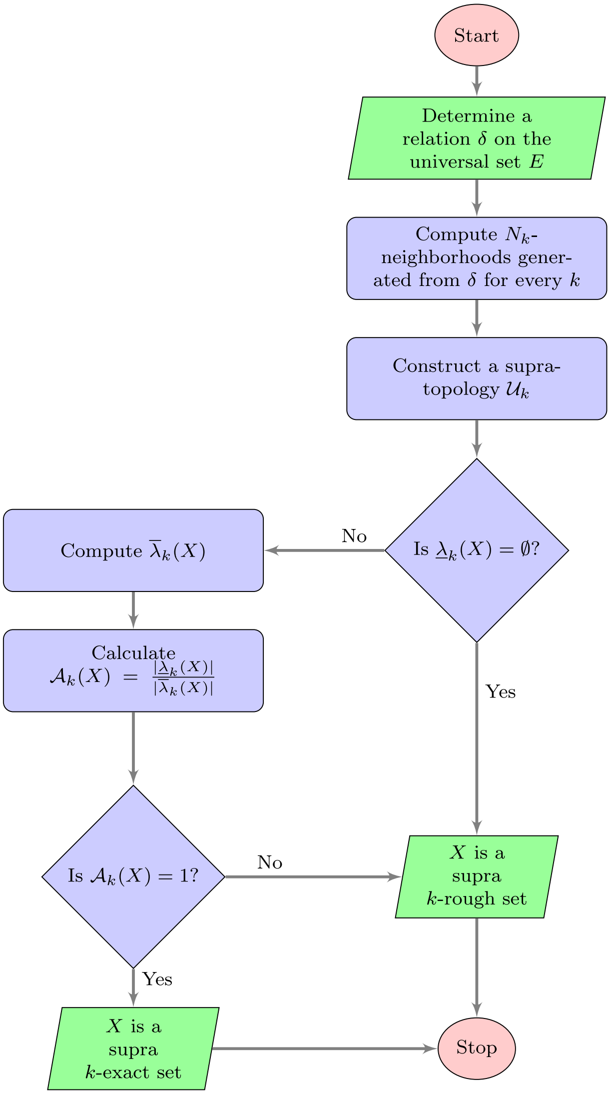
Definition 17 A subset X of a *kSTS* $(E, \delta, \mathcal{U}_k)$ is called supra k-exact if $\underline{\lambda}_k(X) = \overline{\lambda}_k(X) = X$. Otherwise, it is called a supra k-rough set.

Proposition 16 *A subset X of a kSTS $(E, \delta, \mathcal{U}_k)$ is supra k-exact iff $\mathcal{B}_k(X) = \emptyset$.*

Proof Assume that X is a supra k-exact set. Then $\mathcal{B}_k(X) = \overline{\lambda}_k(X) \setminus \underline{\lambda}_k(X) = \overline{\lambda}_k(X) \setminus \overline{\lambda}_k(X) = \emptyset$. Conversely, $\mathcal{B}_k(X) = \emptyset$ implies that $\overline{\lambda}_k(X) \setminus \underline{\lambda}_k(X) = \emptyset$; therefore, $\overline{\lambda}_k(X) \subseteq \underline{\lambda}_k(X)$. But it is well recognized that $\underline{\lambda}_k(X) \subseteq \overline{\lambda}_k(X)$. Thus, $\overline{\lambda}_k(X) = \underline{\lambda}_k(X)$. Hence, X is supra k-exact. □

In Algorithm 1 and Flow chart (in Fig. 1), we elaborate how we can determine whether a subset of a k-supra topology is supra k-exact or supra k-rough.

Fig. 1 Flow chart of determining supra k-exact and supra k-rough subsets of k-supra topologies



Input : An approximation space (E, δ).
Output: Determination whether a subset of a k-supra topology is supra k-exact or supra k-rough.

Determine a relation δ that associated the members of the whole set E;
for *each* k **do**
  Compute N_k-neighborhoods generated from δ of each $w \in E$;
  Initiate a supra-topology $\mathcal{U}_k$ using Theorem 2
end
for *each nonempty subset* X *of* E **do**
  Compute $\underline{\lambda}_k(X)$ utilizing formula given in Definition 13;
  if $\underline{\lambda}_k(X) = \emptyset$ **then**
    return X is a supra k-rough set
  else
    Compute $\overline{\lambda}_k(X)$ using formula given in Definition 13;
    Calculate $\mathcal{A}_k(X) = \frac{|\underline{\lambda}_k(X)|}{|\overline{\lambda}_k(X)|}$;
    if $\mathcal{A}_k(X) = 1$ **then**
      return X is a supra k-exact set
    else
      return X is a supra k-rough set
    end
  end
end

Algorithm 1: The algorithm of determining supra k-exact and supra k-rough subsets of k-supra topologies.

4.2 Comparison of our approach with the previous ones

In the following results, we explain some unique characteristics of our approximations and accuracy measures. As we will see they produce best approximations and highest accuracy measures in cases of union neighbourhood N_u and minimal union neighbourhood $N_{\langle u\rangle}$, which is more different than topological approaches. In fact, this behaviour is attributed to the way of creating our neighbourhood systems given in Theorem 2. Then, we prove that the current method produces higher accuracy and better approximations than all approaches defined in the published literatures under a reflexive relation such as those given in Abd El-Monsef et al. (2014); Abo-Tabl (2013); Abu-Donia (2008); Allam et al. (2006), Al-shami (2021a, 2021b, 2022), Amer et al. (2017), Azzam et al. (2020), El-Sharkasy (2021), Dai et al. (2018), Kondo and Dudek (2006), Kozae et al. (2007), Lashin et al. (2005), Salama (2020).

Proposition 17 *Let X be a subset of a kSTS* $(E, \delta, \mathcal{U}_k)$. *Then*

(i) $\underline{\lambda}_i(X) \subseteq \underline{\lambda}_r(X) \subseteq \underline{\lambda}_u(X)$.
(ii) $\underline{\lambda}_i(X) \subseteq \underline{\lambda}_l(X) \subseteq \underline{\lambda}_u(X)$.
(iii) $\underline{\lambda}_{\langle i\rangle}(X) \subseteq \underline{\lambda}_{\langle r\rangle}(X) \subseteq \underline{\lambda}_{\langle u\rangle}(X)$.
(iv) $\underline{\lambda}_{\langle i\rangle}(X) \subseteq \underline{\lambda}_{\langle l\rangle}(X) \subseteq \underline{\lambda}_{\langle u\rangle}(X)$.
(v) $\overline{\lambda}_u(X) \subseteq \overline{\lambda}_r(X) \subseteq \overline{\lambda}_i(X)$.
(vi) $\overline{\lambda}_u(X) \subseteq \overline{\lambda}_l(X) \subseteq \overline{\lambda}_i(X)$.
(vii) $\overline{\lambda}_{\langle u\rangle}(X) \subseteq \overline{\lambda}_{\langle r\rangle}(X) \subseteq \overline{\lambda}_{\langle i\rangle}(X)$.
(viii) $\overline{\lambda}_{\langle u\rangle}(X) \subseteq \overline{\lambda}_{\langle l\rangle}(X) \subseteq \overline{\lambda}_{\langle i\rangle}(X)$.

Proof To prove (i) and (ii), let $w \in \underline{\lambda}_i(X)$. Then there is $U \in \mathcal{U}_i$ such that $w \in U \subseteq X$. By Proposition 5, $\mathcal{U}_i$ is a subfamily of $\mathcal{U}_r$ and $\mathcal{U}_l$, so $U \in \mathcal{U}_r$ and $U \in \mathcal{U}_l$. Thus, $w \in Int_r(X) = \underline{\lambda}_r(X)$ and $w \in Int_l(X) = \underline{\lambda}_l(X)$. Hence, $\underline{\lambda}_i(X) \subseteq \underline{\lambda}_r(X)$ and $\underline{\lambda}_i(X) \subseteq \underline{\lambda}_l(X)$. Similarly, the relations $\underline{\lambda}_r(X) \subseteq \underline{\lambda}_u(X)$ and $\underline{\lambda}_l(X) \subseteq \underline{\lambda}_u(X)$ are proved.

Following similar arguments, one can prove the other cases. □

Corollary 7 *Let X be a subset of a kSTS* $(E, \delta, \mathcal{U}_k)$. *Then*

(i) $\mathcal{A}_i(X) \leq \mathcal{A}_r(X) \leq \mathcal{A}_u(X)$.
(ii) $\mathcal{A}_i(X) \leq \mathcal{A}_l(X) \leq \mathcal{A}_u(X)$.
(iii) $\mathcal{A}_{\langle i\rangle}(X) \leq \mathcal{A}_{\langle r\rangle}(X) \leq \mathcal{A}_{\langle u\rangle}(X)$.
(iv) $\mathcal{A}_{\langle i\rangle}(X) \leq \mathcal{A}_{\langle l\rangle}(X) \leq \mathcal{A}_{\langle u\rangle}(X)$.

Proof (i): It follows from Proposition 17 that $\underline{\lambda}_i(X) \subseteq \underline{\lambda}_r(X) \subseteq \underline{\lambda}_u(X)$ and $\overline{\lambda}_u(X) \subseteq \overline{\lambda}_r(X) \subseteq \overline{\lambda}_i(X)$, so we get

$$| \underline{\lambda}_i(X) | \leq | \underline{\lambda}_r(X) | \leq | \underline{\lambda}_u(X) | \tag{4}$$

and

$$\frac{1}{| \overline{\lambda}_i(X) |} \leq \frac{1}{| \overline{\lambda}_r(X) |} \leq \frac{1}{| \overline{\lambda}_u(X)(X) |} \tag{5}$$

By (4) and (5) we get

$$\frac{| \underline{\lambda}_i(X) |}{| \overline{\lambda}_i(X) |} \leq \frac{| \underline{\lambda}_r(X) |}{| \overline{\lambda}_r(X) |} \leq \frac{| \underline{\lambda}_u(X) |}{| \overline{\lambda}_u(X) |} \text{which is equivalent to } \mathcal{A}_i(X) \leq \mathcal{A}_r(X) \leq \mathcal{A}_u(X).$$

In a similar way, we prove the other cases. □

The data given in Tables 2 and 3 are computed for a different kinds of a *kSTSs* $(E, \delta, \mathcal{U}_k)$ displayed in Example 2. These computations emphasize the validity of the results presented in Proposition 17 and corollary 7.

The following two results demonstrate the conditions under which we obtain some equivalences.

Proposition 18 *Let X be a subset of a kSTS* $(E, \delta, \mathcal{U}_k)$ *such that* δ *is symmetric. Then*

(i) $\underline{\lambda}_u(X) = \underline{\lambda}_r(X) = \underline{\lambda}_l(X) = \underline{\lambda}_i(X)$ *and* $\overline{\lambda}_u(X) = \overline{\lambda}_r(X) = \overline{\lambda}_l(X) = \overline{\lambda}_i(X)$.
(ii) $\underline{\lambda}_{\langle u\rangle}(X) = \underline{\lambda}_{\langle r\rangle}(X) = \underline{\lambda}_{\langle l\rangle}(X) = \underline{\lambda}_{\langle i\rangle}(X)$ *and* $\overline{\lambda}_{\langle u\rangle}(X) = \overline{\lambda}_{\langle r\rangle}(X) = \overline{\lambda}_{\langle l\rangle}(X) = \overline{\lambda}_{\langle i\rangle}(X)$.

Proof From the equalities $\underline{\lambda}_k(X) = Int_k(X)$ and $\overline{\lambda}_k(X) = Cl_k(X)$ as well as Corollary 2, the proof follows. □

Corollary 8 *Let X be a subset of a kSTS* $(E, \delta, \mathcal{U}_k)$ *such that* δ *is symmetric. Then*

Table 2 The operators of approximations and values of accuracy obtained from k-supra topology, where $k \in \{i, r, l, u\}$

X	$\underline{\lambda}_i(X)$	$\overline{\lambda}_i(X)$	$\mathcal{A}_i(X)$	$\underline{\lambda}_r(X)$	$\overline{\lambda}_r(X)$	$\mathcal{A}_r(X)$	$\underline{\lambda}_l(X)$	$\overline{\lambda}_l(X)$	$\mathcal{A}_l(X)$	$\underline{\lambda}_u(X)$	$\overline{\lambda}_u(X)$	$\mathcal{A}_u(X)$
$\{w\}$	$\emptyset$	$\{w,x,z\}$	0	$\emptyset$	$\{w,x,z\}$	0	$\emptyset$	$\{w,x,z\}$	0	$\emptyset$	$\{w\}$	0
$\{x\}$	$\emptyset$	$\{w,x,z\}$	0	$\emptyset$	$\{w,x,z\}$	0	$\emptyset$	$\{x,z\}$	0	$\emptyset$	$\{x,z\}$	0
$\{y\}$	$\{y\}$	E	$\frac{1}{4}$	$\{y\}$	E	$\frac{1}{4}$	$\{y\}$	E	$\frac{1}{4}$	$\{y\}$	$\{y\}$	1
$\{z\}$	$\emptyset$	$\{w,x,z\}$	0	$\emptyset$	$\{w,x,z\}$	0	$\emptyset$	$\{x,z\}$	0	$\emptyset$	$\{z\}$	0
$\{w,x\}$	$\emptyset$	$\{w,x,z\}$	0	$\emptyset$	$\{w,x,z\}$	0	$\emptyset$	$\{w,x,z\}$	0	$\{w,x\}$	$\{w,x,z\}$	$\frac{2}{3}$
$\{w,y\}$	$\{y\}$	E	$\frac{1}{4}$	$\{y\}$	E	$\frac{1}{4}$	$\{w,y\}$	E	$\frac{1}{2}$	$\{w,y\}$	$\{w,y\}$	1
$\{w,z\}$	$\emptyset$	$\{w,x,z\}$	0	$\emptyset$	$\{w,x,z\}$	0	$\emptyset$	$\{w,x,z\}$	0	$\emptyset$	$\{w,x,z\}$	0
$\{x,y\}$	$\{y\}$	E	$\frac{1}{4}$	$\{y\}$	E	$\frac{1}{4}$	$\{y\}$	E	$\frac{1}{4}$	$\{y\}$	E	$\frac{1}{4}$
$\{x,z\}$	$\emptyset$	$\{w,x,z\}$	0	$\emptyset$	$\{w,x,z\}$	0	$\emptyset$	$\{x,z\}$	0	$\{x,z\}$	$\{x,z\}$	1
$\{y,z\}$	$\{y\}$	E	$\frac{1}{4}$	$\{y\}$	E	$\frac{1}{4}$	$\{y\}$	E	$\frac{1}{4}$	$\{y\}$	$\{y,z\}$	$\frac{1}{2}$
$\{w,x,y\}$	$\{y\}$	E	$\frac{1}{4}$	$\{y\}$	E	$\frac{1}{4}$	$\{w,y\}$	E	$\frac{1}{2}$	$\{w,x,y\}$	E	$\frac{3}{4}$
$\{w,x,z\}$	$\emptyset$	$\{w,x,z\}$	0	$\emptyset$	$\{w,x,z\}$	0	$\emptyset$	$\{w,x,z\}$	0	$\{w,x,z\}$	$\{w,y,z\}$	1
$\{w,y,z\}$	$\{y\}$	E	$\frac{1}{4}$	$\{y\}$	E	$\frac{1}{4}$	$\{w,y\}$	E	$\frac{1}{2}$	$\{w,y\}$	E	$\frac{1}{2}$
$\{x,y,z\}$	$\{y\}$	E	$\frac{1}{4}$	$\{y\}$	E	$\frac{1}{4}$	$\{y\}$	E	$\frac{1}{4}$	$\{x,y,z\}$	E	$\frac{3}{4}$

Table 3 The operators of approximations and values of accuracy obtained from k-supra topology, where $k \in \{\langle i\rangle, \langle r\rangle, \langle l\rangle, \langle u\rangle\}$

X	$\underline{\lambda}_{\langle i\rangle}(X)$	$\overline{\lambda}_{\langle i\rangle}(X)$	$\mathcal{A}_{\langle i\rangle}(X)$	$\underline{\lambda}_{\langle r\rangle}(X)$	$\overline{\lambda}_{\langle r\rangle}(X)$	$\mathcal{A}_{\langle r\rangle}(X)$	$\underline{\lambda}_{\langle l\rangle}(X)$	$\overline{\lambda}_{\langle l\rangle}(X)$	$\mathcal{A}_{\langle l\rangle}(X)$	$\underline{\lambda}_{\langle u\rangle}(X)$	$\overline{\lambda}_{\langle u\rangle}(X)$	$\mathcal{A}_{\langle u\rangle}(X)$
$\{w\}$	$\emptyset$	$\{w,x,z\}$	0	$\emptyset$	$\{w,z\}$	0	$\{w\}$	$\{w,x\}$	$\frac{1}{2}$	$\{w\}$	$\{w\}$	1
$\{x\}$	$\emptyset$	$\{w,x,z\}$	0	$\{x\}$	$\{w,x,z\}$	$\frac{1}{3}$	$\emptyset$	$\{w,x\}$	0	$\{x\}$	$\{x\}$	1
$\{y\}$	$\{y\}$	E	$\frac{1}{4}$	$\{y\}$	$\{w,y,z\}$	$\frac{1}{3}$	$\{y\}$	$\{x,y\}$	$\frac{1}{2}$	$\{y\}$	$\{y\}$	1
$\{z\}$	$\emptyset$	$\{w,x,z\}$	0	$\emptyset$	$\{w,z\}$	0	$\{z\}$	$\{x,z\}$	$\frac{1}{2}$	$\{z\}$	$\{z\}$	1
$\{w,x\}$	$\emptyset$	$\{w,x,z\}$	0	$\{x\}$	$\{w,x,z\}$	$\frac{1}{3}$	$\{w\}$	$\{w,x\}$	$\frac{1}{2}$	$\{w,x\}$	$\{w,x\}$	1
$\{w,y\}$	$\{y\}$	E	$\frac{1}{4}$	$\{y\}$	$\{w,y,z\}$	$\frac{1}{3}$	$\{w,y\}$	$\{w,x,y\}$	$\frac{2}{3}$	$\{w,y\}$	$\{w,y\}$	1
$\{w,z\}$	$\emptyset$	$\{w,x,z\}$	0	$\emptyset$	$\{w,z\}$	0	$\{w,z\}$	$\{w,x,z\}$	$\frac{2}{3}$	$\{w,z\}$	$\{w,z\}$	1
$\{x,y\}$	$\{y\}$	E	$\frac{1}{4}$	$\{x,y\}$	E	$\frac{1}{2}$	$\{y\}$	$\{x,y\}$	$\frac{1}{2}$	$\{x,y\}$	$\{x,y\}$	1
$\{x,z\}$	$\emptyset$	$\{w,x,z\}$	0	$\{x\}$	$\{w,x,z\}$	$\frac{1}{3}$	$\{z\}$	$\{x,z\}$	$\frac{1}{2}$	$\{x,z\}$	$\{x,z\}$	1
$\{y,z\}$	$\{y\}$	E	$\frac{1}{4}$	$\{y\}$	$\{w,y,z\}$	$\frac{1}{3}$	$\{y,z\}$	$\{x,y,z\}$	$\frac{2}{3}$	$\{y,z\}$	$\{y,z\}$	1
$\{w,x,y\}$	$\{y\}$	E	$\frac{1}{4}$	$\{x,y\}$	E	$\frac{1}{2}$	$\{w,y\}$	$\{w,x,y\}$	$\frac{2}{3}$	$\{w,x,y\}$	$\{w,x,y\}$	1
$\{w,x,z\}$	$\emptyset$	$\{w,x,z\}$	0	$\{x\}$	$\{w,x,z\}$	$\frac{1}{3}$	$\{w,z\}$	$\{w,x,z\}$	$\frac{2}{3}$	$\{w,x,z\}$	$\{w,x,z\}$	1
$\{w,y,z\}$	$\{y\}$	E	$\frac{1}{4}$	$\{y\}$	$\{w,y,z\}$	$\frac{1}{3}$	$\{w,y,z\}$	E	$\frac{3}{4}$	$\{w,y,z\}$	$\{w,y,z\}$	1
$\{x,y,z\}$	$\{y\}$	E	$\frac{1}{4}$	$\{x,y\}$	E	$\frac{1}{2}$	$\{y,z\}$	$\{x,y,z\}$	$\frac{2}{3}$	$\{x,y,z\}$	$\{x,y,z\}$	1

(i) $\mathcal{A}_u(X) = \mathcal{A}_r(X) = \mathcal{A}_l(X) = \mathcal{A}_i(X)$.
(ii) $\mathcal{A}_{\langle u\rangle}(X) = \mathcal{A}_{\langle r\rangle}(X) = \mathcal{A}_{\langle l\rangle}(X) = \mathcal{A}_{\langle i\rangle}(X)$.

Proposition 19 *Let (E, δ) be an approximation space such that δ is reflexive. Then $\underline{\lambda}_k(X) = \overline{\lambda}_k(X) = X$ for each $X \subseteq E$.*

Proof Since δ is a reflexive relation, it follows from Proposition 9 that $\mathcal{U}_k$ is a discrete topology for each k. This implies that $\underline{\lambda}_k(X) = Int_k(X) = X$ and $\overline{\lambda}_k(X) = Cl_k(X) = X$. Hence, we obtain the desired result. □

Corollary 9 *Let (E, δ) be an approximation space such that δ is reflexive. Then $\mathcal{M}_k(X) = 1$ for each nonempty subset X of E.*

Proposition 19 and Corollary 9 give an important characteristic of our method under a reflexive relation is that it is better than all previous methods defined by topological structures (Abd El-Monsef et al. 2014; Abo-Tabl 2013; Allam et al. 2006; Al-shami 2021b; Amer et al. 2017; Hosny 2018; Kondo and Dudek 2006; Kozae et al. 2007) and their generalizations such as minimal structures (Azzam et al. 2020; El-Sharkasy 2021) and bitopological spaces (Salama 2020). Also, it is better than all previous methods which were directly defined by some neighborhood systems such as (Abu-Donia 2008; Allam et al. 2005, 2006; Yao 1996, 1998) and those methods introduced depending on neighborhood systems and ideal structures (Hosny 2020; Hosny et al. 2022, 2021; Kandil et al. 2020; Nawar et al. 2022).

Proposition 20 *Let (E, δ) be an approximation space and $X \subseteq E$. If δ is reflexive, then*

(i) $\underline{\delta}_k(X) \subseteq \underline{\lambda}_k(X)$.
(ii) $\overline{\lambda}_k(X) \subseteq \overline{\delta}_k(X)$.

Proof Let $w \in \underline{\delta}_k(X)$. According to Definition 5, $N_k(w) \subseteq X$. Since δ is reflexive, $w \in N_k(w) \subseteq X$, and $N_k(w)$ is a supra-open set in $\mathcal{U}_k$. This means that $w \in Int_k(X) = \underline{\lambda}_k(X)$. Hence, the proof is complete.

Following similar arguments, (ii) is proved. □

Corollary 10 *Let (E, δ) be an approximation space such that δ is reflexive. Then $M_k(X) \leq \mathcal{A}_k(X)$ for each $X \subseteq E$.*

To validate that the current method produces higher accuracy and better approximations than those given (Abd El-Monsef et al. 2014; Allam et al. 2005, 2006; Yao 1996, 1998), we provide the following example.

Example 3 Consider $\delta = \{(w, w), (x, x), (y, y), (w, y), (y, x)\}$ is a binary relation on $E = \{w, x, y\}$. Then, we suffice by computing N_r-neighborhood of each point in E as follows: $N_r(w) = \{w, y\}$, $N_r(x) = \{x\}$ and $N_r(y) = \{x, y\}$. According to Theorem 1, the r-topology τ_k generated from N_r-neighborhoods is $\tau_r = \{\emptyset, E, \{x\}, \{x, y\}\}$. Since δ is reflexive, the generated supra-topology $\mathcal{U}_k$ is the discrete topology for each k. Now, we calculate,

Table 4 The approximations and accuracy values induced from N_r-neighborhood, r-topology and r-supra topology

X	$\underline{\delta}_r(X)$	$\overline{\delta}_r(X)$	$M_r(X)$	$\underline{O}_r(X)$	$\overline{O}_r(X)$	$\mathcal{T}_r(X)$	$\underline{\lambda}_r(X)$	$\overline{\lambda}_r(X)$	$\mathcal{A}_r(X)$
$\{w\}$	$\emptyset$	$\{w\}$	0	$\emptyset$	$\{w\}$	0	$\{w\}$	$\{w\}$	1
$\{x\}$	$\{x\}$	$\{x,y\}$	$\frac{1}{2}$	$\{x\}$	E	$\frac{1}{3}$	$\{x\}$	$\{x\}$	1
$\{y\}$	$\emptyset$	$\{w,y\}$	0	$\emptyset$	$\{w,y\}$	0	$\{y\}$	$\{y\}$	1
$\{w,x\}$	$\{x\}$	E	$\frac{1}{3}$	$\{x\}$	E	$\frac{1}{3}$	$\{w,x\}$	$\{w,x\}$	1
$\{w,y\}$	$\{w\}$	$\{w,y\}$	$\frac{1}{2}$	$\emptyset$	$\{w,y\}$	0	$\{w,y\}$	$\{w,y\}$	1
$\{x,y\}$	$\{x,y\}$	E	$\frac{2}{3}$	$\{x,y\}$	E	$\frac{2}{3}$	$\{x,y\}$	$\{x,y\}$	1

Table 5 Information system of dengue fever

E	O_1	O_2	O_3	O_4	Dengue fever
w_1	✓	✓	✓	vh	✓
w_2	✓	✗	✗	h	✗
w_3	✓	✗	✗	h	✓
w_4	✗	✗	✗	vh	✗
w_5	✗	✓	✓	h	✗
w_6	✓	✓	✗	vh	✓
w_7	✓	✓	✗	n	✓
w_8	✓	✓	✗	vh	✓

in Table 4, the approximations and accuracy values of each subset induced from N_r-neighborhood, rTS (E, δ, τ_r), and $rSTS$ $(E, \delta, \mathcal{U}_r)$.

5 Analysis of dengue fever using the supra-topology approach

In this section, we examine the performance of our method to analyze the data of dengue fever disease and prove it is better than the previous ones given in Abd El-Monsef et al. (2014), Allam et al. (2005, 2006), Yao (1996, 1998).

Dengue fever disease which is a global problem. It is transmitting to humans by virus-carrying Dengue mosquitoes Prabhat (2019). The symptoms of this disease mostly start from the third day of infection. The period of recovery takes a few days; usually, 2-7 days Prabhat (2019). According to the statistics of the World Health Organization (EHO), it spreads in more than 120 nations and causes a huge number of deaths around the world; in particular, Asia and South America World Health Organization (2016). Accordingly, this disease occupies an important place worldwide, which motivates us to analyze it by the approach introduced in this manuscript.

The data displayed in Table 5 decide this disease such that the columns give the symptoms of dengue fever as follows joint and muscle aches O_1, headache with puke O_2, skin rashes O_3, a temperature O_4 with three levels (normal (n), high (h), very high (vh)), and finally the decision D of infected or not. In contrast, the rows represents the patients under

Table 6 Similarity degrees between patients' symptoms

	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
w_1	1	0.25	0.25	0.25	0.5	0.75	0.5	0.75
w_2	0.25	1	1	0.5	0.25	0.5	0.5	0.5
w_3	0.25	1	1	0.5	0.25	0.5	0.5	0.5
w_4	0.25	0.5	0.5	1	0.25	0.5	0.25	0.5
w_5	0.5	0.25	0.25	0.25	1	0.25	0.25	0.25
w_6	0.75	0.5	0.5	0.5	0.25	1	0.75	1
w_7	0.5	0.5	0.5	0.25	0.25	0.75	1	0.75
w_8	0.75	0.5	0.5	0.5	0.25	1	0.75	1

Table 7 N_k and $N_{\langle k\rangle}$ of each $w_i \in W$

	N_k	$N_{\langle k\rangle}$
w_1	$\{w_1, w_6, w_8\}$	$\{w_1\}$
w_2	$\{w_2, w_3\}$	$\{w_2, w_3\}$
w_3	$\{w_2, w_3\}$	$\{w_2, w_3\}$
w_4	$\{w_4\}$	$\{w_4\}$
w_5	$\{w_1, w_5\}$	$\{w_1, w_5\}$
w_6	$\{w_1, w_6, w_7, w_8\}$	$\{w_6, w_8\}$
w_7	$\{w_6, w_7, w_8\}$	$\{w_6, w_7, w_8\}$
w_8	$\{w_1, w_6, w_7, w_8\}$	$\{w_6, w_8\}$

study $E = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8\}$. The mark '✓' (resp., '✗') denotes the patient has a symptom (resp., the patient has no symptom).

Now, the descriptions of attributes $\{O_i : i = 1, 2, 3, 4\}$ will be transmitted into quantity values showing the degree of similarities among the patients' symptoms; see, Table 6. We calculate similarity degree function between the patients a, b, denoted by $s(a, b)$, with respect to m conditions attributes by the next formula.

$$s(w, x) = \frac{\sum_{j=1}^{m} (X_k(w) = A_k(x))}{m} \tag{6}$$

The next procedure is proposing a relation, it is given according to the requirements of the standpoint of system's experts. In this example, we propose the following relation

$$(w, x) \in \delta \iff s(w, x) \geq 0.5.$$

Note that the given relation $\geq$ and number 0.5 can be replaced according to the conceptions of system's experts. It is clear that the suggested relation δ is reflexive and symmetric, so it produces two types of N_k-neighborhood systems. But δ is not transitive. This means that the Pawlak approximations space fails to describe this system.

In Table 7, we compute the two types of N_k and $N_{\langle k\rangle}$ neighborhoods for each patient w_i.

All supra-topologies produced from Table 7 are the discrete topologies because δ is a reflexive relation. To confirm the performance of our approach, we consider two subsets $U = \{w_1, w_2, w_3, w_4, w_7\}$ and $V = \{w_5, w_6, w_8\}$. Then, the approximations and accuracy measures of these two sets are computed with respect to N_k-neighborhoods and k-supra topology.

According to Table 8, the approximations and accuracy measures induced from k-supra topology are better than those induced from N_k-neighborhoods and $N_{\langle k \rangle}$-neighborhoods.

6 Discussions: strengths and limitations

This section demonstrates the main advantages of technique followed herein as well as shows its limitations.

– Strengths

1. The approach of supra-topological structures that we rely on to initiate new models of rough set theory in this manuscript is more relaxed than topological structures. This gives us a large scope for describing many phenomena because we get rid of an intersection condition that is unsuitable to them.
2. This approach also enables us to deal with some practical problems under any arbitrary relation, where as Pawlak approach stipulates an equivalent relation to model the problems under study.
3. It can be compared the different types of approximations and accuracy values generated from our approach as it is proved in the obtained results; see Proposition 17 and Corollary 7. But this characteristic does not hold for some previous approaches such as those produce approximations and accuracy values from near open subsets of topological spaces like α-open, pre-open, semi-open, b-open, and β-open sets.
4. The accuracy and roughness measures induced from the current approach are monotonic; whereas, this property is lost in some preceding topological approaches like those investigated in Abd El-Monsef et al. (2014), Abo-Tabl (2013), Al-shami (2021b).
5. The current method preserves all Pawlak properties of lower approximation operator except for the distributive property of intersection; see Proposition 12. Also, it preserves all Pawlak properties of upper approximation operator except for the distributive property of union; see Proposition 13. In contrast, some previous approaches such as Yao (1996, 1998) lose most of these properties.
6. The current approach is more suitable to handle the large samples because the best approximations and accuracy measures are obtained in cases of $k = u, \langle u \rangle$ which represent the largest N_k-neighbourhoods as we elaborated in Proposition 17 and Corollary 7. The importance of this matter is that we obtain a more accurate decision for the problems in which these cases are the appropriate frame to describe them; for instance, infectious diseases like COVID-19, flu, etc., in which the infection is proportional to the sample size. That is, the decision made in these two cases is more accurate. On the other hand, the performance of these cases via rough set models induced from topology is the weakest in terms of approximation operators and accuracy measures; hence, lack of confidence in the made decision.
7. The present method is more accurate than all foregoing methods with respect to the approximations and accuracy values obtained under a reflexive relation. More precisely, it represents an ideal case under reflexivity, so it is better than all previous methods defined by topological structures (Abd El-Monsef et al. 2014; Abo-Tabl

Table 8 Accuracy measures induced from N_k-neighborhoods and k-supra topology of the sets U and V

X	N_k-neighborhoods			$\mathcal{N}_{\langle k\rangle}$-neighborhoods			k-supra topology		
	$\underline{\delta}_k$	$\overline{\delta}_k$	M_k	$\underline{\delta}_{\langle k\rangle}$	$\overline{\delta}_{\langle k\rangle}$	$M_{\langle k\rangle}$	$\underline{\lambda}_k$	$\overline{\lambda}_k$	$\mathcal{A}_k$
U	$\{w_2, w_3, w_4\}$	$E \setminus \{w_5\}$	$\frac{3}{7}$	$\{w_1, w_2, w_3, w_4\}$	$E \setminus \{w_6, w_8\}$	$\frac{2}{3}$	$\{w_1, w_2, w_3, w_4, w_7\}$	$\{w_1, w_2, w_3, w_4, w_7\}$	1
V	$\emptyset$	$E \setminus \{w_2, w_3, w_4\}$	0	$\{w_6, w_8\}$	$\{w_5, w_6, w_7, w_8\}$	$\frac{1}{2}$	$\{w_5, w_6, w_8\}$	$\{w_5, w_6, w_8\}$	1

2013; Allam et al. 2006; Al-shami 2021b; Amer et al. 2017; Hosny 2018; Kondo and Dudek 2006; Kozae et al. 2007) and their generalizations such as minimal structures (Azzam et al. 2020; El-Sharkasy 2021) and bitopological spaces Salama (2020). Also, it is better than all previous methods which were directly defined by some neighborhood systems such as Abu-Donia (2008), Allam et al. (2005, 2006), Yao (1996), Yao (1998) and those methods introduced depending on neighborhood systems and ideal structures (Hosny 2020; Hosny et al. 2022, 2021; Kandil et al. 2020; Nawar et al. 2022).

– limitations

1. The present approach is generally incomparable with the topological approach introduced in Abd El-Monsef et al. (2014) when the relation is not reflexive. To illustrate this point, consider the neighborhoods system displayed in Example 1. It is clear that $\tau_r = \{\emptyset, E, \{y\}, \{z\}, \{y, z\}, \{x, y\}, \{x, y, z\}, \{w, x, y\}\}$ is an r-topology on E induced by Theorem 1, and $\mathcal{U}_r = \{\emptyset, E, \{w\}, \{w, x\}, \{w, x, y\}\}$ is a supra r-topology on E induced by Theorem 2. By calculations, we obtain $\mathcal{T}_r(\{w\}) = 0 < \mathcal{A}_r(\{w\}) = \frac{1}{4}$ whereas $\mathcal{T}_r(\{y\}) = \frac{1}{3} > \mathcal{A}_r(\{y\}) = 0$.
2. The distributive property of intersection and union operators are respectively lost by the supra k-lower and supra k-upper approximations introduced herein, whereas this property holds under rough models induced from topological or infra-topological structures.

7 Conclusion

Rough approximation operators and values of accuracy are the most significant characteristic of rough set theory. In practice, they provide a conception of the data contained in a subset and determine to what extent this subset is complete. Improvement of these operators and increase their values of accuracy lead to an accurate prediction. There are two main techniques to do that, one is to define new types of neighborhood systems such as those introduced in Abu-Donia (2008), Allam et al. (2005, 2006), Al-shami (2021a, 2022), Al-shami and Ciucci (2022), Dai et al. (2018), Hosny (2018), and the second is obtained by studying rough-sets concepts using their counterparts via topological spaces and their related structures as those given in Abd El-Monsef et al. (2014), Abo-Tabl (2013), Amer et al. (2017), Azzam et al. (2020), El-Sharkasy (2021), Hosny (2020).

Through this manuscript, we have followed the second technique to generate rough sets models. We have applied the concept of "supra-topology" to create new models which are more relaxed than topological models because a finite intersection stipulation is removed. We have begun our work by forming supra-topology spaces from an approximation space. Then, we have established novel rough set models by these spaces and investigated their master properties. We have explained their main advantages to improve approximation operators and accuracy values better than all previous models existing in the literature under a reflexivity condition. As an application, we have discussed the followed technique to describe dengue fever disease. Finally, we have demonstrated the merits of our approach and its failures compared with the foregoing ones.

In the future, we are going to study the next themes.

(i) Discuss the rough models introduced herein with respect to the recent neighborhood systems such as C_k-neighborhoods Al-shami (2021a) and S_k-neighborhoods Al-shami and Ciucci (2022).
(ii) Form a new frame consisting of ideal structure and supra-topology to improve the approximation operators and accuracy values given herein similarly to the combination of classical topology and ideal (Hosny 2020; Kandil et al. 2020; Nawar et al. 2022).
(iii) Investigate the manuscript thoughts with respect to some celebrated extensions of supra-open sets.
(iv) Reformulate the concepts studied herein in some frames such as soft rough set and fuzzy rough set.

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Declarations

Conflict of interest The authors declare that there is no conflict of interests regarding the publication of this article.

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