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Decision support modeling for multiple criteria assessments using a likelihood-based consensus ranking method under Pythagorean fuzzy uncertainty

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Abstract

This paper intends to exploit point operator-oriented likelihood measures to constitute a likelihood-based consensus ranking model aimed at conducting multiple criteria decision making encompassing complex uncertain evaluations with Pythagorean fuzzy sets. This paper takes advantage of Pythagorean fuzzy point operators and the scalar functions of upper and lower estimations to formulate a point operator-oriented likelihood measure for preference intensity. On this basis, this paper propounds the notion of penalty weights to characterize dominated relations for acquiring the measurement of comprehensive disagreement and constituting a likelihood-based consensus ranking model. The primary contributions of this study are fourfold. Firstly, two useful point operators are initiated for upper and lower estimations towards Pythagorean membership grades. Secondly, an effective likelihood measure is exploited for determining outranking relations of Pythagorean fuzzy information. Thirdly, a pragmatic concept of penalty weights is proposed for characterizing the dominated relations among alternatives and measuring degrees of comprehensive disagreement. Fourthly, a functional likelihood-based consensus ranking model is constructed for implementing a multiple criteria evaluation with Pythagorean fuzzy uncertainty. Furthermore, a real-life application relating to a financing problem is presented to provide a justification for the practicability of the proposed methodology. This paper executes an analysis of parameters sensitivity and comparative studies for showing more theoretical insights.

Keywords Point operator-oriented likelihood measure · Consensus ranking model · Multiple criteria decision making · Pythagorean fuzzy set · Measurement of comprehensive disagreement

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1 Introduction

Multiple criteria decision analysis (MCDA) predominantly concentrates on a prioritizing problem where the decision maker aspires to determine the preference ranking concerning candidate choice options from the most advantageous to the most disadvantageous on grounds of multiple evaluative criteria (Farrokhizadeh et al. 2021; Fei and Feng 2021; Wang and Chen 2020). Various MCDA methods and techniques have been developed successfully in a wide-ranging fashion, and thus, they provide a frequently used tool for tackling human cognitive and decision-making activities (Chen 2021; Farid and Riaz 2021; Tsao and Chen 2021). Nonetheless, decision makers are faced with increasingly convoluted circumstances because of fierce market competition, changeable socioeconomic environments, almost real-time response speed, insufficient expertise and knowledge, huge hybrid data, and burden of information overload (Farid and Riaz 2021; Oztaysi et al. 2021; Phillips-Wren et al. 2006). Accordingly, uncertainty and vagueness are frequently encountered in real-world decision circumstances, making it more challenging to decision-making works (Jan et al. 2021; Li et al. 2021; Tang et al. 2020; Wan et al. 2020; Zhang et al. 2021). Making allowance for such difficulties, theoretical modeling and manipulation of highly intricate uncertainties in MCDA techniques are crucial to manage real-world decision-making affairs in the modern scientific age (Farid and Riaz 2021; Iampan et al. 2021).

The sets of intuitionistic fuzziness and Pythagorean fuzziness play a particularly important role in computational intelligence, neural network, machine learning, and artificial intelligence (Farhadinia 2021; Farid and Riaz 2021; Iampan et al. 2021). For convenience, let the notations μ and ν symbolize a membership grade and a nonmembership grade, respectively, in the unit interval [0, 1] throughout this paper. In decision-making practices, decision makers can employ the membership grade μ to deliver their subjective appraisal outcome to which an alternative satisfies a specific criterion; on the flip side, the nonmembership grade ν can be utilized to convey the subjective appraisal outcome to which an alternative does not satisfy a specific criterion (Tsao and Chen 2021; Wang and Chen 2020). Atanassov (1986) initiated a general notion of intuitionistic fuzzy sets that are symbolized through μ and ν fulfilling the prerequisite $\mu+\nu\leq 1$. Pythagorean fuzzy (PF) sets, initially propounded by Yager (2013), are expounded as a generalized configuration of intuitionistic fuzzy sets (Jan et al. 2021; Siddique et al. 2021; Zhou and Chen 2021). In a more general way, PF sets are characterized by μ and ν that meet the prerequisite $\mu^2+\nu^2\leq 1$. The main difference between the two prerequisites is a much broader space possessed by PF sets, which brings about an appealing capability of modelling more uncertainties (Garg 2021; Siddique et al. 2021; Wang and Chen 2020). Moreover, PF sets enjoy greater flexibility in accommodating highly complicated ambiguous and equivocal information (Munir et al. 2021; Peng and Luo 2021; Tsao and Chen 2021). Figure 1 portrays a geometrical comparison of the intuitionistic fuzzy and PF configurations. As sketched in this figure, the space of a Pythagorean membership grade is evidently broader than the space of an intuitionistic membership grade. As an illustration, suppose that the grade to which an alternative satisfies a criterion is 0.8, while the grade to which this alternative dissatisfies such a criterion is 0.4. Because $0.8+0.5=1.3>1$, this example cannot be expounded by use of the intuitionistic fuzzy theory. On the contrary, this difficulty can be overcome effectually using the PF theory for the cause that $0.8^2+0.5^2=0.89\leq 1$. Obviously, PF sets possess a more powerful ability than intuitionistic fuzzy sets in terms of modeling uncertain information

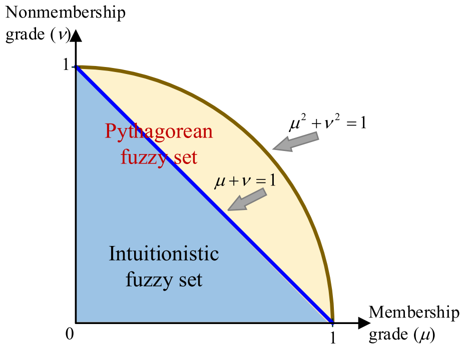


Fig. 1 A geometrical contrast of intuitionistic fuzzy and PF sets

in empirical MCDA affairs. In consideration of the aforesaid advantages enjoyed by PF sets, this study strives to exploit the PF theory to manage equivocal and nebulous assessments in realistic decision situations.

Academics have perceived the PF theory as a forceful tool to support intelligent decision-making processes under intricate uncertainties (Munir et al. 2021; Peng and Luo 2021; Siddique et al. 2021; Tsao and Chen 2021). The evolution and enrichment in the PF theory have become very active in realistic MCDA fields because the PF configuration can manage inexact and vague information more effectively and efficiently. There have been technological developments in PF MCDA approaches that facilitate decision-making activities in diverse practical areas such as multiple criteria analysis (Akram and Shahzadi 2021; Chen 2021; Siddique et al. 2021), group decision making (Biswas and Sarkar 2018, 2019; Garg 2021; Wan et al. 2020; Wang and Chen 2020), clustering analysis (Xian and Cheng 2021), risk assessment and evaluation (Akram et al. 2021; Rodriguez 2021; Zhao et al. 2021), medical diagnosis (Sun et al. 2021), investment project and strategy (Li et al. 2021; Riaz et al. 2021), and waste disposal location selection (Oztaysi et al. 2021).

However, there are some motivational considerations that are needed to be resolved in the existing literature. The motivations for this study are threefold: (1) the necessity of using PF point operators for measuring dominance relationships, (2) the inappropriate or unjustifiable delineations of current PF likelihood measures, and (3) the lack of developing the consensus ranking methodology in PF circumstances. Firstly, the uncertainty in conjunction with PF sets can be lessened under the influence of some recently propounded point operators, such as PF point operators in Biswas and Sarkar (2018), Chen (2021), Peng and Yuan (2016), Wan et al. (2020), Zhou and Chen (2021), and Zhu et al. (2018), on Pythagorean membership grades. Previous studies have made progress towards averaging operators and aggregation operations for PF sets through the utility of PF point operators (Biswas and Sarkar 2018; Peng and Yuan 2016; Wan et al. 2020; Zhu et al. 2018). However, few studies have hitherto concentrated on the usage of PF point operators to ascertain dominance relationships to render rankings of PF evaluation values by predominance, which creates the first research motivation.

Secondly, there was considerable diversity in current PF MCDA methods and techniques; but the likelihood measures for a comparison between Pythagorean membership grades have been limited investigated (Fei et al. 2019; Garg 2018). Garg (2018) exploited a likelihood measure that was elucidated in ordinary interval numbers, not in the PF context. Fei et al. (2019) put forward a soft likelihood function that aimed to set forth an aggregation operation, not to determine the likelihood of PF preference relations. Liang et al. (2020) took advantage of a uniform distribution to randomly choose the approval and disapproval for allocating the indeterminacy part to the belongingness portion and the non-belongingness portion, respectively. However, the designation of uniform distribution-based likelihood suffered from certain limitations, such as the appositeness of probability density functions and implemented complexity in distinct surroundings. Tsao and Chen (2021) employed a beta distribution to delineate a fresh PF likelihood function. By the same token, their assumption about the probability density function of beta distributions was justified difficultly. As a matter of course, the aforesaid considerations establish the second motivation for performing this study.

The third motivation concerns the necessity of developing an appropriate consensus ranking model within PF environments. The consensus ranking techniques are simple and easy-to-understand models for solving ordinal consensus ranking problems (Beck and Lin 1983; Cook and Kress 1985). By measuring the distance between two ranking orders, the consensus ranking method is sufficiently competent for handling the issue by merging various rankings into one overall ranking in a computationally simple manner (Chen 2018a; Tavana et al. 2007). Giving consideration to the effectiveness and user-friendliness of the consensus ranking method, it is practicable and favorable to utilize relevant consensus ranking techniques for tackling decision-making activities with multiple criteria (Yu et al. 2021; Zhang et al. 2020; Zhang et al. 2021). The consensus ranking methodology has been used for the determination of consensus ranking from various individual rankings, such as a consensus ranking technique (Teng and Tzeng 1994), an ideal-seeking consensus ranking approach (Tavana et al. 2007), a weighted sum-based ordinal consensus ranking approach (Tavana et al. 2008), a consensus ranking model predicated on a mixed choice strategy (Chen 2018a), and a three-stage distance-based consensus ranking procedure (Aghayi and Tavana 2019). However, except Chen (2018a), the previous consensus ranking methods cannot manipulate PF information and process MCDA tasks in PF settings. The enrichment of the consensus ranking methodology within PF environments has been encouraging enough to merit further investigation. This topic highlights the third motivation for performing this research.

The research objective of this study is to set forth a point operator-oriented likelihood measure and formulate a likelihood-based consensus ranking model for effectively managing an MCDA problem containing complex uncertain information developed on PF sets. In contrast to existing PF point operators, this paper initiates two easy-to-use point operators to conveniently transmute uncertain PF information into another adapted outcome for the construction of rational upper and lower estimations in PF contexts. In particular, the use of allocation parameters can facilitate the ascertainment of rational expectations in an optimistic or pessimistic attitude. For example, the PF point operator with optimism exerts an effective influence over positive (or favorable) perceptions and outcome expectations, which leads to rational upper estimations of PF information. In contrast, the PF point operator with pessimism exerts potent influence over negative (or unfavorable) perceptions and outcome expectations, which yields rational lower estimations about PF information.

Instead of the existing approach in accordance with probability distributions, this paper makes use of scalar functions to advocate a beneficial PF likelihood measure through the medium of the upper and lower estimations towards Pythagorean membership grades under PF surroundings. Regarding the strength of the favorable and advantageous properties, the point operator-oriented likelihood measure can be used to appraise the possibility of an outranking relation separating two PF evaluative ratings in decision contexts involving Pythagorean fuzziness. Using the new PF likelihood measure, this paper proposes a useful concept of penalty weights as a substantial basis to ascertain measurements of comprehensive disagreement. Specifically, these penalty weights can facilitate the identification of an overall predominate ranking of an alternative over all others with respect to each criterion. By integrating penalty weights and PF importance weights, this paper develops advantageous measures of comprehensive disagreement indicators and indices to establish a PF likelihood-based consensus ranking model. This consensus ranking model is represented by the agency of a zero–one linear programming format that is straightforward and simple to implement.

To scrutinize the appropriateness and achievability of the initiated methodology, this paper investigates a down-to-earth application relating to a financing problem involving working capital policies. In addition, this paper conducts comparison studies that consist of an effectiveness analysis, two sensitivity studies, and a comparative study to highlight beneficial theoretical insights contained in the proposed methods. Specifically, this paper executes an effectiveness analysis to describe the application results through the agency of the degrees of appropriateness. Furthermore, this paper performs two sensitivity analyses and scrutinizes the effects of diverse settings concerning the allocation parameters. Also, this paper administers a comparative study with other well-developed decision-making techniques to corroborate the usefulness and strengths of the evolved PF likelihood-based consensus ranking method.

Consider the practical motivation and implications linked to artificial intelligence in this paper. Although numerous MCDA models and techniques have been investigated comprehensively for promoting an intelligent decision support system in the past few years, few, if any, researches have manipulated a likelihood-based architecture of the consensus ranking methodology for the manipulation of decision information involving Pythagorean fuzziness. Traditional consensus ranking methods have often been conducted within the decision environment on the basis of precise information. To handle ambiguity and vagueness in MCDA processes, the consensus ranking methodology has been extended to uncertain conditions, such as Yu et al. (2021), Zhang et al. (2020), and Zhang et al. (2021). Previous research has demonstrated the potential of the consensus ranking model in fuzzy circumstances. However, the specific advancement of the consensus ranking model in the PF framework has not been explored, and rare studies have exclusively devoted effort to such issues in an intelligent decision support system. Considering needs on an appropriate consensus ranking methodology for decision informatics, this paper intends to make an effort at establishing an evolved likelihood-based consensus ranking model predicated on Pythagorean fuzziness. By developing a point operator-oriented likelihood measure and some relevant useful notions, this paper advances the systematic thinking of smart and intelligent decision making in PF uncertain circumstances. More importantly, the propounded model and techniques are expected to create grounds for intelligent decision support in business and management. The potential usefulness of the propounded methodology can be anticipated in the realm of artificial intelligence, including the enrichment of an intelligent decision support system and the construction of a sophisticated decision-aiding tool under unpredictable and highly uncertain conditions.

The remainder of this research is arranged along these lines. Section 2 reviews the relevant works concerning MCDA methods in PF decision contexts and identifies some research gaps. Section 3 exhibits essential notions about PF sets to provide an essential foundation. Section 4 first highlights the theoretical bases of the proposed methodology; next, it initiates two new PF point operators to exploit the upper and lower estimations of PF information and critically examines several helpful and appealing properties. Section 5 constructs an effective PF likelihood measure and reveals an ingenious likelihood-based consensus ranking model for tackling MCDA tasks involving Pythagorean fuzziness. Section 6 presents an investigation of evaluating financing policies for working capitals to demonstrate the propounded algorithmic procedure. Also, sensitivity analyses and comparisons are put into practice to justify the effectuality and major strengths of the initiated methodology. At last, Sect. 7 provides a conclusion and several prospective research suggestions.

2 Relevant literature and research gaps

PF sets, as a generalized format of intuitionistic fuzzy sets, have the clear ability to accommodate imprecise and equivocal information during human decision-making processes. PF sets provide forceful tools necessary to address the ambiguity and vagueness that arise in realistic and pragmatic situations and guide the multiple criteria evaluation procedure from conception through completion (Munir et al. 2021; Peng and Luo 2021; Wang and Chen 2020). Accordingly, the theory of PF sets is becoming widespread in the MCDA field, and many studies have been launched to develop systematically quantitative methods and procedures and provide decision-aiding assistance in PF decision contexts (Akram and Shahzadi 2021; Akram et al. 2021; Fei and Feng 2021; Garg 2021; Riaz et al. 2021; Zhao et al. 2021).

PF sets have been broadly employed in diverse MCDA problems in actual decision situations. Moreover, the methods and applications of exploiting PF sets to investigate MCDA issues have also received widespread attention. By way of illustration, Deng et al. (2021) investigated Muirhead mean operators in the context of 2-tuple linguistic PF information for decision support with multiple criteria analysis. Farhadinia (2021) advanced several diverse types of PF similarity measures and put forward a PF similarity-based MCDA technique. Garg (2021) brought forward sine trigonometric operational laws and then developed the PF aggregation operators for treating group decision-making matters. Oztaysi et al. (2021) propounded a PF regime method to appraise qualitative evaluations and then exploited it to solve the selection issue of waste disposal locations. Rodriguez (2021) propounded a risk-assessing method for decision making via an artificial-neuron-like evaluation node predicated on PF contexts. Siddique et al. (2021) launched PF hypersoft weighted average and weighted geometric operators and established a useful MCDA method with PF hypersoft sets. Sun et al. (2021) exploited grey relational analyses to pose a group decision-making method of roughly approximating uncertainty information with PF sets in two multi-granular spaces of the universe. Tsao and Chen (2021) exploited a beta distribution to work out a PF likelihood function and brough forward a dominance ordering model aimed at processing MCDA tasks in PF circumstances. Xian and Cheng (2021) advanced a n-PF time series model by virtue of PF c-means and an evolved Markov prediction approach for better forecasting accuracy. Zulqarnain et al. (2021) explored PF soft information contained in MCDA problems to unfold PF soft interaction weighted average and weighted geometric operators.

Notably, point operators are beneficial and efficient tools that can be used to regulate the uncertainty of evaluation data by virtue of administrative parameters that depend on the decision maker's attitude and thus lead to the acquisition of more comprehensive information during the MCDA process (Chen 2021). Because existing fuzzy point operators are not available for adapting PF uncertain environments, several scholars have proposed appropriate point operators to enable them conform to Pythagorean membership grades under PF state of affairs. By way of explanation, Biswas and Sarkar (2018), Chen (2021), Peng and Yuan (2016), Wan et al. (2020), and Zhu et al. (2018), inspired by intuitionistic fuzzy point operators, proposed new point operators for PF sets. To resolve MCDA problems, Peng and Yuan (2016) brought forward certain point operators in PF settings and advanced PF point-weighted averaging operators to regulate the degree of aggregated arguments by controlling the parameters. Biswas and Sarkar (2018) exploited PF point operators to reveal several helpful similarity measures for PF sets and further defined new aggregation operators, i.e., PF-dependent averaging operators and geometric operators, for conducting collective decision analysis. Zhu et al. (2018) employed PF point operators with the goal of diminishing uncertainties in the evaluation data and enhancing the accuracy of the evaluated information; they also combined the analytic hierarchy process (AHP) to launch a fresh AHP-based MCDA technique. Wan et al. (2020) devised two PF point operators to provide a representation about points in the space associated with a Pythagorean membership grade and then showed the relative distance along with reliability information to acquire a valuable order with respect to PF information. Further, Zhou and Chen (2021) employed Wan et al.'s PF point operators to stand for the decision maker's risk preference that reveals an attitude about emerging science and technology. Chen (2021) made use of the notion of PF scalar functions to generalize dual point operators and applied them to formulate a fresh PF preference ranking organization method (PROMETHEE) for enriching evaluations and supporting decisions.

In addition to PF environments, Biswas and Sarkar (2019) and Peng and Yang (2016) promoted the growth of fuzzy point operators for use under interval-valued PF circumstances. Strictly speaking, Peng and Yang (2016) conceived of new point operators with interval-valued PF sets and generated weighted averaging operators with an interval-valued PF format that would regulate the grade of aggregated arguments. Biswas and Sarkar (2019) advanced beneficial similarity measures supported by point operators and interval-valued PF sets and proposed an interactive MCDA method. As a recapitulation of PF sets, q-rung orthopair fuzzy sets possess great flexibility and adjustability due to the dynamic adaptability of changing information via parameter q (Peng and Luo 2021; Tang et al. 2020; Zeng et al. 2021). Xing et al. (2019) proposed new point operators concerning q-rung orthopair fuzzy numbers and procured a category of point-weighted aggregation operators with the goal of synthesizing uncertain information via q-rung orthopair fuzziness. In a general sense, the point operators appropriate for interval-valued PF contexts or for q-rung orthopair fuzzy environments are particularly apposite compared to PF information.

On the flip side, the fundamental propositions and broad applicability of PF sets can be managed to convoluted and complex uncertainties involved in realistic decision information. In particular, the likelihood measure differentiating between two Pythagorean membership grades can be exploited to facilitate paired comparisons for PF assessments and evaluations. Garg (2018) suggested new exponential operational laws along with the corresponding aggregation operators to tackle MCDA problems with interval-valued PF sets. Garg used the likelihood between two interval numbers for the construction of the possibility degree matrix, while this likelihood was available in ordinary interval numbers instead of Pythagorean membership grades. Fei et al. (2019) developed the soft likelihood function of PF sets to aggregate multiple pieces of probabilistic evidence. Basically, the soft likelihood function belongs to a type of logical "anding" operation of criteria for a given alternative. Thus, Fei et al. proposed the notion of ordered weighted averaging soft likelihood functions for managing decision-making affairs. Nonetheless, their proposed soft likelihood function was exploited to set up an aggregation method, not to assess the likelihood of PF preference relations between PF evaluation values. In contrast with the approaches of

Fei et al. (2019) and Garg (2018), Liang et al. (2020) studied the probability information of PF evaluation values and proposed a new likelihood measure for managing PF preference relations. They made a simplified assumption that Pythagorean membership grades generate an associated uniform distribution. Working on this assumption, the probability density that aligns with a uniform distribution was used to ascertain a likelihood measurement for PF evaluation values. Liang et al. formulated a generalized linear assignment method on the grounds of their proposed likelihood measure and partitioned fuzzy measures for finding a solution to PF MCDA problems. Furthermore, Tsao and Chen (2021) utilized a symmetric beta distribution for ascertaining the possibilities of outranking/outranked relationships about PF information and put forward a PF likelihood function for formulating the dominance ordering model. However, in the PF likelihood measure developed by Liang et al. (2020) and the PF likelihood function by Tsao and Chen (2021), the assumptions concerning the probability density functions, i.e., uniform distribution or symmetric beta distribution, remain unsubstantiated in practice.

The foregoing discussions of the relevant literature can help identify three research gaps. Firstly, the uncertainty with respect to PF sets (or more generally, interval-valued PF formats and the q-rung orthopair fuzzy framework) can be reduced under the influence of these newly developed point operators on Pythagorean membership grades. The PF point operators also demonstrate proficiency with the change in the degree of the aggregated argument by virtue of certain administrative parameters. The emphasis of most previous studies has been incorporating PF point operators into any aggregation process for exploitation of new averaging operators in PF contexts, as illustrated in Biswas and Sarkar (2018), Peng and Yuan (2016), Wan et al. (2020) and Zhu et al. (2018). Thus, grounded in these previous studies, it is understood that PF point operators play key roles for modeling aggregation operations with respect to PF information. Nonetheless, the range of applicability of PF point operators for decision support and exposure of influential information remains limited. So far, few researches have concerned the exploration of employing PF point operators in ascertaining dominance relationships for PF evaluation values. Therefore, the necessity of using PF point operators for rendering rankings of PF evaluation values by predominance forms the first research gap.

Secondly, due to the sophistication and imprecision of PF information, ways to perform trustworthy paired comparisons for PF assessments and evaluations remain unclear. In particular, there was limited research on the possibility of comparing Pythagorean membership grades using likelihood measures (Fei et al. 2019; Garg 2018). The likelihood measure adopted by Garg (2018) was delineated in ordinary interval numbers, not in PF settings. The soft likelihood function developed by Fei et al. (2019) aimed to set up an aggregation approach, not to determine the likelihood of PF preference relations. Over and above that, the likelihood measure and the PF likelihood function launched by Liang et al. (2020) and Tsao and Chen (2021), respectively, were based on assumptions, not evidence. The assumptions concerning the probability density functions (i.e., uniform and beta distributions based on Liang et al.'s and Tsao and Chen's proposals, respectively) remain unproven, which brings about the second research gap.

Thirdly, making allowance for the usefulness and user-friendliness of the consensus ranking method, it is more advantageous to use relevant consensus ranking techniques for coping with MCDA issues (Yu et al. 2021; Zhang et al. 2020; Zhang et al. 2021). For example, Teng and Tzeng (1994) exploited a consensus ranking technique to generate an overall ranking due to minimum recognition differences among all decision makers. Tavana et al. (2007) built a useful ideal-seeking consensus ranking technique by use of hybrid distances for decision-aiding analysis. Considering the weights generated by a sigmoid function, Tavana et al. (2008) developed a weighted sum-based ordinal consensus ranking approach for synthesizing individual rankings to yield a representative consensus ranking result. By determining an amalgamation of criterion-specific and category-based schemes, Chen (2018a) constructed a pragmatic consensus ranking technique with a mixed choice strategy to produce an overall ranking of competing alternatives under the uncertainty of Pythagorean fuzziness. Aghayi and Tavana (2019) proposed a consensus ranking method using three-stage distances in an attempt to yield group ranks of alternatives. However, relevant studies for the advancement and promotion of the consensus ranking methodology have not been investigated in detail. The existing consensus ranking models and techniques, excluding the mixed-choice-strategy-based approach developed by Chen (2018a), seem incapable of manipulating PF information and managing MCDA issues in PF circumstances. Thus, the extension or enrichment of PF sets in uncertain circumstances is inevitably necessary to theoretically develop and apply consensus ranking methodology, which gives rise to the third research gap.

From a conspectus of the need to resolve the three research gaps in relevant literature, this paper attempts to provide an effectual approach to surmount the previously described difficulties and limitations. In subsequent contents, this paper will propose an advanced methodology that is beneficial and possesses certain special points in theory and practice compared to existing techniques. This paper makes concrete features in theoretical models and relevant techniques, as shown:

(1) Given the effectiveness of likelihood measures and point operators, this paper would like to conceive the conception of a PF point operator-oriented likelihood measure with an eye towards a beneficial MCDA method within uncertain environments with Pythagorean fuzziness.
(2) Rather than using existing PF point operators and their corresponding averaging/aggregation operations, this paper proposes two simple and straightforward PF point operators to generate upper and lower estimations for the rational determination of the appropriate measurements of dominant relationships within PF environments.
(3) Through fusing the notion of scalar functions, this paper puts forward a workable likelihood measure using the proposed PF point operators. The initiated PF point operator-oriented likelihood measure provides decision makers not only an estimate of rational upper and lower adapted outcomes for this likelihood but also the possibility of finding an outranking relation between PF evaluative ratings in uncertain PF circumstances.
(4) Most importantly, this paper advances a likelihood-based consensus ranking model for enriching the current consensus ranking methodology and addressing MCDA issues involving PF uncertainties.

3 General background for PF sets

This section presents certain elementary notions concerning PF sets, including the characterization parameters, Pythagorean membership grades, arithmetic operations, and scalar functions. PF theory was initially developed by Yager (2013). Following pioneering works (e.g., Yager 2013; Yager and Abbasov 2013), Chen (2019) put forward a comprehensive mathematical expression for depicting a PF set, as presented in the subsequent relevant definitions.

Definition 1 (Chen 2019, Yager 2014) Let X signify a finite universe of discourse. To have established on X, a PF set P is an object possessing the subsequent representation:

$$P = \left\{ \left\langle x, \left(\mu_P(x), \nu_P(x); r_P(x), d_P(x)\right) \right\rangle \middle| x \in X \right\}, \tag{1}$$

which is delineated using the membership grade $\mu_P(x) : X \to [0, 1]$, nonmembership grade $\nu_P(x) : X \to [0, 1]$, strength of commitment $r_P(x) : X \to [0, 1]$, and the direction of commitment $d_P(x) : X \to [0, 1]$ of an element $x \in X$ to P.

Definition 2 (Yager 2016; Yager and Abbasov 2013) In the matter of a PF set P defined in X, a Pythagorean membership grade p of an element $x \in X$ affiliated with P is represented via four characterization parameters:

$$p = \left(\mu_P(x), \nu_P(x); r_P(x), d_P(x)\right) \tag{2}$$

that is constrained by $0 \leq (\mu_P(x))^2 + (\nu_P(x))^2 \leq 1$. Moreover,

$$\mu_P(x) = r_P(x) \cdot \cos\left(\theta_P(x)\right), \tag{3}$$

$$\nu_P(x) = r_P(x) \cdot \sin\left(\theta_P(x)\right), \tag{4}$$

where $\theta_P(x)$ is indicated as radians in the interval $[0, \pi/2]$. Furthermore, the strength and direction characterization parameters relative to p are given by:

$$r_P(x) = \sqrt{(\mu_P(x))^2 + (\nu_P(x))^2}, \tag{5}$$

$$d_P(x) = \frac{\pi - 2 \cdot \theta_P(x)}{\pi}. \tag{6}$$

Definition 3 (Chen 2018b, Yager 2014, 2016) The indeterminacy grade $\tau_P(x) : X \to [0, 1]$ associated with p for each $x \in X$ to P is derived in this way:

$$\tau_P(x) = \sqrt{1 - (\mu_P(x))^2 - (\nu_P(x))^2}. \tag{7}$$

The relationship between $\tau_P(x)$ and $r_P(x)$ accommodates the property of duality:

$$(\tau_P(x))^2 + (r_P(x))^2 = 1. \tag{8}$$

Moreover, the standard complement corresponding to p is derived as shown:

$$p^c = \left(\mu_{P^c}(x), \nu_{P^c}(x); r_{P^c}(x), d_{P^c}(x)\right) = \left(\nu_P(x), \mu_P(x); r_P(x), 1 - d_P(x)\right). \tag{9}$$

Definition 4 (Yager 2013, 2014) On grounds of the Takagi–Sugeno approach grounded in fuzzy rule foundations, the scalar function $V(p) \in [0, 1]$ of p is computed like this:

$$V(p) = \frac{1}{2} + r_P(x) \cdot \left(d_P(x) - \frac{1}{2}\right) = \frac{1}{2} + r_P(x) \cdot \left(\frac{1}{2} - \frac{2 \cdot \theta_P(x)}{\pi}\right). \tag{10}$$

Definition 5 (Yager 2014, Yager and Abbasov 2013) Place two Pythagorean membership grades $p_1=(\mu_P(x_1),\nu_P(x_1);r_P(x_1),d_P(x_1))$ and $p_2=(\mu_P(x_2),\nu_P(x_2);r_P(x_2),d_P(x_2))$. Let $\succeq_Q$ delineate a natural quasi-ordering in PF contexts; herein, $p_1\succeq_Q p_2$ if and only if $\mu_P(x_1)\geq\mu_P(x_2)$ and $\nu_P(x_1)\leq\nu_P(x_2)$.

Definition 6 (Zhang and Xu 2014) Consider three Pythagorean membership grades p, p_1, and p_2 in P. Let $p'=(\mu_P(x),\nu_P(x))$, $p_1'=(\mu_P(x_1),\nu_P(x_1))$, and $p_2'=(\mu_P(x_2),\nu_P(x_2))$ denote the two-dimensional representations of p, p_1, and p_2, respectively, for notational convenience. Let $\lambda>0$. Some arithmetic operations are given by:

$$p_1'\oplus p_2'=\left(\sqrt{(\mu_P(x_1))^2+(\mu_P(x_2))^2-(\mu_P(x_1))^2\cdot(\mu_P(x_2))^2},\nu_P(x_1)\cdot\nu_P(x_2)\right),\quad(11)$$

$$p_1'\otimes p_2'=\left(\mu_P(x_1)\cdot\mu_P(x_2),\sqrt{(\nu_P(x_1))^2+(\nu_P(x_2))^2-(\nu_P(x_1))^2\cdot(\nu_P(x_2))^2}\right),\quad(12)$$

$$\lambda\odot p'=\left(\sqrt{1-\left(1-(\mu_P(x))^2\right)^{\lambda}},(\nu_P(x))^{\lambda}\right),\quad(13)$$

$$p'^{\lambda}=\left((\mu_P(x))^{\lambda},\sqrt{1-\left(1-(\nu_P(x))^2\right)^{\lambda}}\right).\quad(14)$$

4 Theoretical highlights and PF point operators

This section first highlights and summarizes the theoretical basis about the evolved PF likelihood-based consensus ranking methodology. Next, this section unfolds the notions of PF point operators that are the solid foundation of the propounded techniques.

4.1 Overview of the theoretical framework

The PF likelihood-based consensus ranking model is constructed on a solid foundation on theory and conceptions. Figure 2 manifests the theoretical bases of the propounded methodology, involving the theoretical development processes of PF point operators, PF likelihood measures, and the PF likelihood-based consensus ranking model.

As exhibited in this figure, on the subject of the theoretical bases of PF point operators, this study takes advantage of allocation parameters to reveal two functional PF point operators that can be used to rationally determine upper and lower estimations relating to Pythagorean membership grades. Several valuable properties, such as reformation outcomes of recurrent upper and lower estimations, are inspected in detail to validate the applicability and significance towards new point operators. Next, give consideration to the theoretical bases of PF likelihood measures. This paper exploits the scalar functions of upper and lower estimations to formulate a point operator-oriented likelihood measure for determining the preference intensity. The proposed likelihood measure possesses certain beneficial and attractive properties, which facilitate the assertion of the possibility of an outranking relation. In what follows, consider the theoretical foundation of the PF likelihood-based

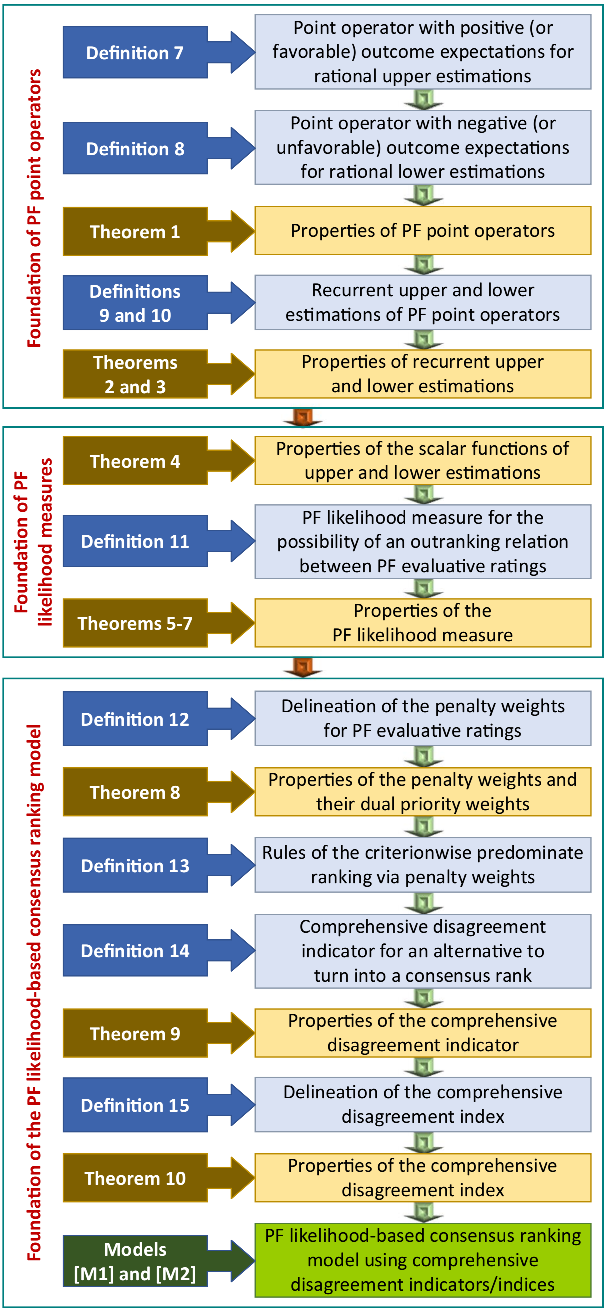


Fig. 2 Theoretical foundation of the PF likelihood-based consensus ranking methodology

consensus ranking model. To have established on the PF likelihood measure, this study delivers the idea of penalty weights for characterizing dominated relations and acquiring the measurement of comprehensive disagreement. Penalty weights can be used to identify an overall predominate ranking of an alternative for each criterion. By integrating the concepts of penalty weights and PF importance weights, this paper propounds a comprehensive disagreement indicator and index for constituting a likelihood-based consensus ranking model. In the end, an efficient computational algorithm is launched to use the initiated consensus ranking model aimed at resolving multiple-criteria evaluation issues on the grounds of Pythagorean fuzziness.

4.2 Upper and lower estimations via PF point operators

This section develops two useful point operators to estimate the adapted outcomes of PF information and investigate several critical and attractive properties. The advocated PF point operators provide effective ways of transforming Pythagorean membership grades into other grades for the establishment of upper and lower estimations in PF contexts.

Different from Chen's (2021) dual point operators, this study propounds two simple-to-use PF point operators M_α in Definition 7 and N_β in Definition 8 that can be utilized conveniently to regulate the uncertainty of evaluation data by dint of allocation parameters α and β. As mentioned by Definition 7, the PF point operator M_α embodies positive (or favorable) outcome expectations, which generates rational upper estimations. Conversely, as stated by Definition 8, the PF point operator N_β embodies negative (or unfavorable) outcome expectations, which produces rational lower estimations.

Definition 7 To have established on a finite universe of discourse X, place a PF set $P = \{\langle x, (\mu_P(x), \nu_P(x); r_P(x), d_P(x))\rangle | x \in X\}$. Let $\alpha \in [0, 1]$ represent an allocation parameter. The PF point operator M_α of P is expressed by:

$$M_\alpha(P) = \left\{ \left\langle x, \left(\mu_{M_\alpha(P)}(x), \nu_{M_\alpha(P)}(x); r_{M_\alpha(P)}(x), d_{M_\alpha(P)}(x)\right) \right\rangle \middle| x \in X \right\}, \tag{15}$$

which is elucidated using a Pythagorean membership grade $M_\alpha(p)$ like this:

$$M_\alpha(p) = \left(\mu_{M_\alpha(P)}(x), \nu_{M_\alpha(P)}(x); r_{M_\alpha(P)}(x), d_{M_\alpha(P)}(x)\right), \tag{16}$$

where $\mu_{M_\alpha(P)}(x)$ and $\nu_{M_\alpha(P)}(x)$ are calculated on this wise:

$$\mu_{M_\alpha(P)}(x) = \sqrt{(\mu_P(x))^2 + \alpha(\tau_P(x))^2}, \tag{17}$$

$$\nu_{M_\alpha(P)}(x) = \nu_P(x). \tag{18}$$

Definition 8 To have established on X, place a PF set $P = \{\langle x, (\mu_P(x), \nu_P(x); r_P(x), d_P(x))\rangle | x \in X\}$. Let $\beta \in [0, 1]$ indicate an allocation parameter. The PF point operator N_β of P is depicted like this:

$$N_\beta(P) = \left\{ \left\langle x, \left(\mu_{N_\beta(P)}(x), \nu_{N_\beta(P)}(x); r_{N_\beta(P)}(x), d_{N_\beta(P)}(x)\right) \right\rangle \middle| x \in X \right\}; \tag{19}$$

it is elucidated by a Pythagorean membership grade $N_\beta(p)$ on this wise:

$$N_\beta(p) = \left(\mu_{N_\beta(P)}(x), \nu_{N_\beta(P)}(x); r_{N_\beta(P)}(x), d_{N_\beta(P)}(x)\right), \tag{20}$$

where $\mu_{N_\beta(P)}(x)$ and $\nu_{N_\beta(P)}(x)$ are calculated in such manner:

$$\mu_{N_\beta(P)}(x) = \mu_P(x), \tag{21}$$

$$\nu_{N_\beta(P)}(x) = \sqrt{(\nu_P(x))^2 + \beta(\tau_P(x))^2}. \tag{22}$$

From the basis in Definitions 7 and 8, using the proposed M_α and N_β can facilitate the transformation of a PF set into another PF set to estimate the adaptational outcomes of Pythagorean membership grades. More importantly, these two PF point operators enjoy certain desirable features, as demonstrated in the upcoming theorem.

Theorem 1 *Concerning an element $x \in X$ affiliated with a* PF *set P, place a Pythagorean membership grade $p = (\mu_P(x), \nu_P(x); r_P(x), d_P(x))$. By employing the* PF *point operators M_α and N_β, the upper estimation $M_\alpha(p) = (\mu_{M_\alpha(P)}(x), \nu_{M_\alpha(P)}(x); r_{M_\alpha(P)}(x), d_{M_\alpha(P)}(x))$ and the lower estimation $N_\beta(p) = (\mu_{N_\beta(P)}(x), \nu_{N_\beta(P)}(x); r_{N_\beta(P)}(x), d_{N_\beta(P)}(x))$ of p possess the subsequent properties*:

(T1.1) $0 \le (\mu_{M_\alpha(P)}(x))^2 + (\nu_{M_\alpha(P)}(x))^2 \le 1$ *and* $0 \le (\mu_{N_\beta(P)}(x))^2 + (\nu_{N_\beta(P)}(x))^2 \le 1$;
(T1.2) $\mu_{N_\beta(P)}(x) = \mu_P(x) \le \mu_{M_\alpha(P)}(x)$ *and* $\nu_{M_\alpha(P)}(x) = \nu_P(x) \le \nu_{N_\beta(P)}(x)$;
(T1.3) $M_\alpha(p) \succeq_Q p \succeq_Q N_\beta(p)$;
(T1.4) $\max\left\{\tau_{M_\alpha(P)}(x), \tau_{N_\beta(P)}(x)\right\} \le \tau_P(x)$ *and* $\min\left\{r_{M_\alpha(P)}(x), r_{N_\beta(P)}(x)\right\} \ge r_P(x)$;
(T1.5) $d_{N_\beta(P)}(x) \le d_P(x) \le d_{M_\alpha(P)}(x)$ *and* $\theta_{M_\alpha(P)}(x) \le \theta_P(x) \le \theta_{N_\beta(P)}(x)$; and
(T1.6) $(M_\alpha(p^c))^c = N_\alpha(p)$;
(T1.7) $(N_\beta(p^c))^c = M_\beta(p)$.

Proof See "Appendix A.1".

If the decision maker performs multiple M_α operations on a Pythagorean membership grade p, the recurrent upper estimation corresponding to $M_\alpha(p)$ can be generated, as precisely delineated in Definition 9. Moreover, the relevant properties of such a recurrent upper estimation are investigated in Theorem 2.

Definition 9 Consider the upper estimation $M_\alpha(p)$ of $p = (\mu_P(x), \nu_P(x); r_P(x), d_P(x))$, where $\alpha \in [0, 1]$. Let η be a nonnegative integer. Denote $M_\alpha^0(p) = p$ and $M_\alpha^\eta(p) = M_\alpha^\eta(M_\alpha^{\eta-1}(p))$. The recurrent upper estimation after η reformations is shown like this:

$$M_\alpha^\eta(p) = \left(\mu_{M_\alpha^\eta(P)}(x), \nu_{M_\alpha^\eta(P)}(x); r_{M_\alpha^\eta(P)}(x), d_{M_\alpha^\eta(P)}(x)\right). \tag{23}$$

Theorem 2 *Employing the* PF *point operator M_α on p after η reformations, the recurrent upper estimation $M_\alpha^\eta(p)$ accommodates the succeeding properties*:

(T2.1) $\mu_{M_\alpha^\eta(P)}(x) = \sqrt{(\mu_P(x))^2 + \left(1 - (\mu_P(x))^2\right)(1 - (1-\alpha)^\eta) - \alpha(\nu_P(x))^2\left(\sum_{k=0}^{\eta-1}(1-\alpha)^k\right)}$;
(T2.2) $\nu_{M_\alpha^\eta(P)}(x) = \nu_P(x)$;
(T2.3) $r_{M_\alpha^\eta(P)}(x) = \sqrt{(\mu_P(x))^2 + (\nu_P(x))^2 + \left(1 - (\mu_P(x))^2\right)(1 - (1-\alpha)^\eta) - \alpha(\nu_P(x))^2\left(\sum_{k=0}^{\eta-1}(1-\alpha)^k\right)}$;
(T2.4) $M_\alpha^\eta(p) \succeq_Q M_\alpha^{\eta-1}(p)$;

(T2.5) $\lim_{\eta\to\infty} M_\alpha^\eta(p) = \left(\sqrt{1-(\nu_P(x))^2}, \nu_P(x);1, \frac{\pi-2\cdot sin^{-1}(\nu_P(x))}{\pi}\right)$.

Proof See "Appendix A.2".

In an analogous way, when the decision maker performs multiple N_β operations on p, the recurrent lower estimation in connection with $N_\beta(P)$ can be rendered, as manifested in Definition 10. Furthermore, the relevant properties of such a recurrent lower estimation are explored in Theorem 3.

Definition 10 Consider the lower estimation $N_\beta(P)$ of $p = (\mu_P(x), \nu_P(x);r_P(x), d_P(x))$, where $\beta \in [0, 1]$. Let η be a nonnegative integer. Denote $N_\beta^0(p) = p$ and $N_\beta^\eta(p) = N_\beta^\eta(N_\beta^{\eta-1}(p))$. The recurrent lower estimation after η reformations is shown like this:

$$N_\beta^\eta(p) = \left(\mu_{N_\beta^\eta(P)}(x), \nu_{N_\beta^\eta(P)}(x);r_{N_\beta^\eta(P)}(x), d_{N_\beta^\eta(P)}(x)\right). \tag{24}$$

Theorem 3 *Employing the* PF *point operator N_β on p after η reformations, the recurrent lower estimation $N_\beta^\eta(p)$ accommodates the following properties*:

(T3.1) $\mu_{N_\beta^\eta(P)}(x) = \mu_P(x)$;

(T3.2) $\nu_{N_\beta^\eta(P)}(x) = \sqrt{(\nu_P(x))^2 + \left(1-(\nu_P(x))^2\right)(1-(1-\beta)^\eta) - \beta(\mu_P(x))^2\left(\sum_{k=0}^{\eta-1}(1-\beta)^k\right)}$;

(T3.3) $r_{N_\beta^\eta(P)}(x) = \sqrt{(\nu_P(x))^2 + (\mu_P(x))^2 + \left(1-(\nu_P(x))^2\right)(1-(1-\beta)^\eta) - \beta(\mu_P(x))^2\left(\sum_{k=0}^{\eta-1}(1-\beta)^k\right)}$;

(T3.4) $N_\beta^{\eta-1}(p) \succeq_Q N_\beta^\eta(p)$;

(T3.5) $\lim_{\eta\to\infty} N_\beta^\eta(p) = \left(\mu_P(x), \sqrt{1-(\mu_P(x))^2};1, \left(\pi - 2\cdot\cos^{-1}(\mu_P(x))\right)/\pi\right)$.

Proof The proofs of (T3.1) – (T3.5) are analogously to the proving process of Theorem 2.

This paper exploits two allocation parameters, α and β, to identify beneficial PF point operators M_α and N_β, respectively, to determine the upper estimation $M_\alpha(p)$ and lower estimation $N_\beta(p)$ of Pythagorean membership grade p. To enhance the understanding from a geometric perspective, Fig. 3 portrays a two-dimensional space representation

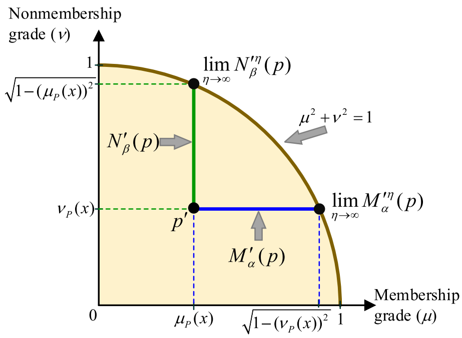


Fig. 3 Two-dimensional space representations of the upper and lower estimations

concerning the upper and lower estimations in PF contexts. The possible range of $M_\alpha(p)$ is concisely described through the agency of a two-dimensional representation $M'_\alpha(p)$ (i.e., by virtue of $\mu_{M_\alpha(P)}(x)$ and $\nu_{M_\alpha(P)}(x)$), where $M'_\alpha(p)=\left(\sqrt{(\mu_P(x))^2+\alpha(\tau_P(x))^2}, \nu_P(x)\right)$. The limit of the recurrent upper estimation $M^\eta_\alpha(p)$ is shown by the representation $\lim_{\eta\to\infty} M'^\eta_\alpha(p)=\left(\sqrt{1-(\nu_P(x))^2}, \nu_P(x)\right)$. Specifically, the membership grade $\mu_{M_\alpha(P)}(x)$ of the upper estimation $M_\alpha(p)$ is acquired from the square root of the sum of the squared membership grade $(\mu_P(x))^2$ and part of the squared indeterminacy grade $(\tau_P(x))^2$. In particular, the repeated use of the PF point operator M_α on p would yield the highest possible membership grade (i.e., $\sqrt{1-(\nu_P(x))^2}=\sqrt{(\mu_P(x))^2+(\tau_P(x))^2}$) when the time to redistribute the indeterminacy grade is sufficiently large. The use of the PF point operator M_α encompasses positive (or favorable) perceptions and outcome expectations but does not deny negative (or unfavorable) outcomes by maintaining identical nonmembership grades. Thus, the result of $M_\alpha(p)$ represents the adapted outcome under rational expectation in an optimistic attitude about information on the decision environment. It is reasonable and desirable to use the PF point operator M_α to determine a rational upper estimation of PF information.

The possible range of $N_\beta(p)$ is briefly expressed using $\mu_{N_\beta(P)}(x)$ and $\nu_{N_\beta(P)}(x)$, as demonstrated in the two-dimensional representation $N'_\beta(p)$ in Fig. 3, in which $N'_\beta(p)=\left(\mu_P(x), \sqrt{(\nu_P(x))^2+\beta(\tau_P(x))^2}\right)$. The limit of the recurrent lower estimation $N^\eta_\beta(p)$ is exhibited with the aid of the representation $\lim_{\eta\to\infty} N'^\eta_\beta(p)=\left(\mu_P(x), \sqrt{1-(\mu_P(x))^2}\right)$. The nonmembership grade $\nu_{N_\beta(P)}(x)$ of the lower estimation $N_\beta(p)$ is determined from the square root of the sum of the squared nonmembership grade $(\nu_P(x))^2$ and part of the squared indeterminacy grade $(\tau_P(x))^2$. The repeated use of the PF point operator N_β on p leads to the highest possible nonmembership grade, $\sqrt{1-(\mu_P(x))^2}=\sqrt{(\nu_P(x))^2+(\tau_P(x))^2}$, when the time to redistribute the indeterminacy grade is sufficiently large. The employment of the PF point operator N_β encompasses negative (or unfavorable) perceptions and outcome expectations but does not deny positive (or favorable) outcomes by maintaining the same membership grades. Accordingly, the result of $N_\beta(p)$ expresses the adapted outcome under a rational expectation based on a pessimistic attitude about the information in the decision environment. It is appropriate to use the PF point operator N_β to determine a rational lower estimation $N_\beta(p)$ for Pythagorean membership grades in the surroundings with PF sets.

5 The PF likelihood-based consensus ranking method

This section describes an uncertain decision-making problem via the agency of a PF representation, develops a new likelihood measure for PF evaluation information, launches a workable PF likelihood-based consensus ranking model, and provides an effective algorithmic procedure for tackling MCDA problems in PF uncertain circumstances.

5.1 Presentation of MCDA problems with PF sets

This subsection develops a configuration regarding an MCDA problem under uncertain circumstances predicated on PF sets. We construct an MCDA representation involving m

candidate alternatives and n evaluative criteria, where $m, n \geq 2$. Let $A = \{a_1, a_2, \ldots, a_m\}$ to state a discrete set of candidate alternatives; moreover, let $C = \{c_1, c_2, \ldots, c_n\}$ describe a finite set of evaluative criteria.

Place a Pythagorean membership grade p_{ij} to signify the PF evaluative rating concerning an alternative $a_i \in A$ (for $i = 1, 2, \ldots, m$) relevant to criterion $c_j \in C$ (for $j = 1, 2, \ldots, n$):

$$p_{ij} = (\mu_{ij}, \nu_{ij}; r_{ij}, d_{ij}) \tag{25}$$

with the prerequisite $0 \leq (\mu_{ij})^2 + (\nu_{ij})^2 \leq 1$, in which $\mu_{ij} = r_{ij} \cdot \cos(\theta_{ij})$, $\nu_{ij} = r_{ij} \cdot sin(\theta_{ij})$, $r_{ij} = \sqrt{(\mu_{ij})^2 + (\nu_{ij})^2}$, and $d_{ij} = (\pi - 2 \cdot \theta_{ij})\big/\pi$ for $\theta_{ij} \in [0, \pi/2]$. Herein, μ_{ij} and ν_{ij} represent the satisfaction grade and dissatisfaction grade, respectively, of a_i in connection with c_j derived from subjective appraisals and discernments. The indeterminacy grade with relevance to the PF evaluative rating p_{ij} is produced by $\tau_{ij} = \sqrt{1 - (\mu_{ij})^2 - (\nu_{ij})^2}$.

Let a PF set P_i delineate the PF characteristic for each $a_i \in A$; it is defined as a collection of all p_{ij} of a_i over the n criteria:

$$P_i = \left\{ \left\langle c_j, p_{ij} \right\rangle \middle| c_j \in C \right\} = \left\{ \left\langle c_j, \left(\mu_{ij}, \nu_{ij}; r_{ij}, d_{ij}\right) \right\rangle \middle| c_j \in C \right\}. \tag{26}$$

The PF evaluation matrix P can be succinctly represented by collecting all PF characteristics as follows:

$$P = \left[p_{ij}\right]_{m \times n} = \begin{array}{c} \\ a_1 \\ a_2 \\ \vdots \\ a_m \end{array} \begin{array}{c} \begin{array}{cccc} c_1 & c_2 & \cdots & c_n \end{array} \\ \begin{bmatrix} (\mu_{11}, \nu_{11}; r_{11}, d_{11}) & (\mu_{12}, \nu_{12}; r_{12}, d_{12}) & \cdots & (\mu_{1n}, \nu_{1n}; r_{1n}, d_{1n}) \\ (\mu_{21}, \nu_{21}; r_{21}, d_{21}) & (\mu_{22}, \nu_{22}; r_{22}, d_{22}) & \cdots & (\mu_{2n}, \nu_{2n}; r_{2n}, d_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (\mu_{m1}, \nu_{m1}; r_{m1}, d_{m1}) & (\mu_{m2}, \nu_{m2}; r_{m2}, d_{m2}) & \cdots & (\mu_{mn}, \nu_{mn}; r_{mn}, d_{mn}) \end{bmatrix} \end{array}. \tag{27}$$

Let a Pythagorean membership grade w_j indicate the PF importance weight of criterion c_j:

$$w_j = (\omega_j, \varpi_j; r_j^w, d_j^w) \tag{28}$$

such that $0 \leq (\omega_j)^2 + (\varpi_j)^2 \leq 1$ for each $j \in \{1, 2, \ldots, n\}$, in which $\omega_j = r_j^w \cdot \cos(\theta_j^w)$, $\varpi_j = r_j^w \cdot sin(\theta_j^w)$, $r_j^w = \sqrt{(\omega_j)^2 + (\varpi_j)^2}$, and $d_j^w = (\pi - 2 \cdot \theta_j^w)\big/\pi$ for $\theta_j^w \in [0, \pi/2]$. The indeterminacy grade associated with each PF importance weight w_j is derived as $\tau_j^w = \sqrt{1 - (\omega_j)^2 - (\varpi_j)^2}$. The two-dimensional representation of w_j is denoted as $w_j' = (\omega_j, \varpi_j)$ for notational convenience.

5.2 New likelihood measure within PF environments

This subsection exploits the notion of scalar functions to bring forward a functional PF likelihood measure aimed at ascertaining the possibility of an outranking relation towards Pythagorean membership grades in a PF setting effectively. For concrete cases, an alternative a_i outranks a_k in connection with c_j (denoted as $p_{ij} \succeq p_{kj}$) if and only if one can find substantial evidence that a_i is superior to a_k or at least that a_i is as favorable as a_k in regards to c_j. To identify a preference intensity between p_{ij} and p_{kj}, one can utilize the scalar functions of the upper and lower estimations corresponding to p_{ij} and p_{kj}, and develop a PF likelihood measure to exploit the outranking relation.

Let α_{ij} and β_{ij} denote the allocation parameters compared to each PF evaluative rating p_{ij}, where $\alpha_{ij}, \beta_{ij} \in [0, 1]$. The designation of α_{ij} and β_{ij} depends on the decision maker's requirements. For example, it is reasonable and acceptable to designate the values of α_{ij} and β_{ij} in the following four ways: (1) $\alpha_{ij} = \mu_{ij}$ and $\beta_{ij} = \nu_{ij}$; (2) $\alpha_{ij} = (\mu_{ij})^2$ and $\beta_{ij} = (\nu_{ij})^2$, where $(\alpha_{ij})^2 + (\beta_{ij})^2 \leq 1$; (3) $\alpha_{ij} = \mu_{ij}\big/(\mu_{ij} + \nu_{ij})$ and $\beta_{ij} = \nu_{ij}\big/(\mu_{ij} + \nu_{ij})$, where $\alpha_{ij} + \beta_{ij} = 1$; and (4) $\alpha_{ij} = (\mu_{ij})^2\big/((\mu_{ij})^2 + (\nu_{ij})^2)$ and $\beta_{ij} = (\nu_{ij})^2\big/((\mu_{ij})^2 + (\nu_{ij})^2)$, where $(\alpha_{ij})^2 + (\beta_{ij})^2 = 1$. Alternately, the most convenient way of setting fixed values of α_{ij} and β_{ij} is illustrated as follows: let $\alpha_{ij} = \alpha$ and $\beta_{ij} = \beta$ for each $a_i \in A$ and $c_j \in C$. It is suggested that the assignment mechanism can take either designation $\alpha_{ij} = \beta_{ij}$ or $\alpha_{ij} + \beta_{ij} = 1$ for practical applications. By the agency of the PF point operators $M_{\alpha_{ij}}$ and $N_{\beta_{ij}}$, which are denoted as M_α and N_β, respectively, for brevity, the upper estimation $M_{\alpha_{ij}}(p_{ij})$ and the lower estimation $N_{\beta_{ij}}(p_{ij})$ with reference to Definitions 7 and 8, respectively, are represented as follows:

$$M_{\alpha_{ij}}(p_{ij}) = \left(\mu_{ij}^M, \nu_{ij}^M; r_{ij}^M, d_{ij}^M\right) = \left(\sqrt{(\mu_{ij})^2 + \alpha_{ij}(\tau_{ij})^2}, \nu_{ij}; \sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \alpha_{ij}(\tau_{ij})^2}, \frac{\pi - 2 \cdot \theta_{ij}^M}{\pi}\right), \tag{29}$$

$$N_{\beta_{ij}}(p_{ij}) = \left(\mu_{ij}^N, \nu_{ij}^N; r_{ij}^N, d_{ij}^N\right) = \left(\mu_{ij}, \sqrt{(\nu_{ij})^2 + \beta_{ij}(\tau_{ij})^2}; \sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \beta_{ij}(\tau_{ij})^2}, \frac{\pi - 2 \cdot \theta_{ij}^N}{\pi}\right) \tag{30}$$

for $\theta_{ij}^M, \theta_{ij}^N \in [0, \pi/2]$, in which:

$$\theta_{ij}^M = \arccos\left(\frac{\sqrt{(\mu_{ij})^2 + \alpha_{ij}(\tau_{ij})^2}}{\sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \alpha_{ij}(\tau_{ij})^2}}\right) = \arcsin\left(\frac{\nu_{ij}}{\sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \alpha_{ij}(\tau_{ij})^2}}\right), \tag{31}$$

$$\theta_{ij}^N = \arccos\left(\frac{\mu_{ij}}{\sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \beta_{ij}(\tau_{ij})^2}}\right) = \arcsin\left(\frac{\sqrt{(\nu_{ij})^2 + \beta_{ij}(\tau_{ij})^2}}{\sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \beta_{ij}(\tau_{ij})^2}}\right) \tag{32}$$

The scalar functions $V(p_{ij})$, $V(M_{\alpha_{ij}}(p_{ij}))$, and $V(N_{\beta_{ij}}(p_{ij}))$ are computed in this way:

$$V(p_{ij}) = \frac{1}{2} + r_{ij} \cdot \left(d_{ij} - \frac{1}{2}\right) = \frac{1}{2} + r_{ij} \cdot \left(\frac{1}{2} - \frac{2 \cdot \theta_{ij}}{\pi}\right), \tag{33}$$

$$V(M_{\alpha_{ij}}(p_{ij})) = \frac{1}{2} + r_{ij}^M \cdot \left(d_{ij}^M - \frac{1}{2}\right) = \frac{1}{2} + r_{ij}^M \cdot \left(\frac{1}{2} - \frac{2 \cdot \theta_{ij}^M}{\pi}\right), \tag{34}$$

$$V(N_{\beta_{ij}}(p_{ij})) = \frac{1}{2} + r_{ij}^N \cdot \left(d_{ij}^N - \frac{1}{2}\right) = \frac{1}{2} + r_{ij}^N \cdot \left(\frac{1}{2} - \frac{2 \cdot \theta_{ij}^N}{\pi}\right). \tag{35}$$

These scalar functions possess several advantageous properties, as discussed in the upcoming theorem.

Theorem 4 *Consider the upper estimation* $M_{\alpha_{ij}}(p_{ij})= (\mu_{ij}^M, \nu_{ij}^M; r_{ij}^M, d_{ij}^M)$ *and the lower estimation* $N_{\beta_{ij}}(p_{ij})= (\mu_{ij}^N, \nu_{ij}^N; r_{ij}^N, d_{ij}^N)$ *of* $p_{ij} = (\mu_{ij}, \nu_{ij}; r_{ij}, d_{ij})$ *for all* $a_i \in A$ *and* $c_j \in C$. *Their corresponding scalar functions* ($V(M_{\alpha_{ij}}(p_{ij}))$ *and* $V(N_{\beta_{ij}}(p_{ij}))$) *exhibit the subsequent features*:

$0 \le V(N_{\beta_{ij}}(p_{ij})) \le V(p_{ij}) \le V(M_{\alpha_{ij}}(p_{ij})) \le 1$;
$V(M_{\alpha_{ij}}(p_{ij}))$ *is monotonically nondecreasing with the allocation parameter* α_{ij};
$V(N_{\beta_{ij}}(p_{ij}))$ *is monotonically nonincreasing with the allocation parameter* β_{ij}.

Proof See "Appendix A.3".

Making allowances for the usefulness of $V(M_{\alpha_{ij}}(p_{ij}))$ and $V(N_{\beta_{ij}}(p_{ij}))$, this paper develops a PF likelihood measure in Definition 11 to produce the possibility of outranking relations differentiating between two PF evaluative ratings in an MCDA problem. To clarify the distinction between p_{ij} and p_{kj}, this study presents a fresh definition of the PF likelihood measure in Definition 11. Specifically, for two PF evaluative ratings p_{ij} and p_{kj}, the advanced PF likelihood measure aims to yield the possibility of an outranking relation "$p_{ij} \succeq p_{kj}$," which indicates that p_{ij} is not inferior to p_{kj} from a scalar function-centered perspective in conjunction with the associated upper and lower estimations (i.e., through the medium of $V(M_{\alpha_{ij}}(p_{ij}))$ and $V(N_{\beta_{ij}}(p_{ij}))$ relevant to p_{ij} as well as $V(M_{\alpha_{ij}}(p_{kj}))$ and $V(N_{\beta_{ij}}(p_{kj}))$ relevant to p_{kj}. It is noted that this study assumes that $V(M_{\alpha_{ij}}(p_{ij})) = V(N_{\beta_{ij}}(p_{ij}))$ and $V(M_{\alpha_{kj}}(p_{kj})) = V(N_{\beta_{kj}}(p_{kj}))$ do not exist concurrently to avoid a meaningless denominator in Definition 11.

Definition 11 Let p_{ij} and p_{kj} manifest two PF evaluative ratings of a_i and a_k, respectively, on the subject of c_j. On the grounds of the PF point operators M_α and N_β, the PF likelihood measure $Lik(p_{ij} \succeq p_{kj})$ of an outranking relation "$p_{ij} \succeq p_{kj}$" is determined as follows:

$$Lik(p_{ij} \succeq p_{kj}) = \max\left\{1 - \max\left\{\frac{V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{ij}}(p_{ij}))}{\left(V(M_{\alpha_{ij}}(p_{ij})) - V(N_{\beta_{ij}}(p_{ij}))\right) + \left(V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{kj}}(p_{kj}))\right)}, 0\right\}, 0\right\}, \tag{36}$$

in which without loss of generality, the differences between $V(M_{\alpha_{ij}}(p_{ij}))$ and $V(N_{\beta_{ij}}(p_{ij}))$ and between $V(M_{\alpha_{kj}}(p_{kj}))$ and $V(N_{\beta_{kj}}(p_{kj}))$ cannot be equal to zero simultaneously.

Using the PF likelihood measure can assist in determining the intensity of the outranking relationships for PF information and render the criterion-wise rankings among candidate alternatives. More importantly, the proposed PF likelihood measure $Lik(p_{ij} \succeq p_{kj})$ by virtue of the PF point operators M_α and N_β has certain relevant advantageous properties, as investigated in Theorems 5–7. More precisely, Theorem 5 demonstrates some essential features possessed by $Lik(p_{ij} \succeq p_{kj})$. Next, Theorem 6 focuses on the necessity and sufficiency with respect to an implicational relationship $Lik(p_{ij} \succeq p_{kj}) \ge 0.5$. Finally, Theorem 7 corroborates the truth of the property of weak transitivity.

Theorem 5 *The* PF *likelihood measure* $Lik(p_{ij} \succeq p_{kj})$ *based on the* PF *point operators* M_α *and* N_β *possesses the subsequent features*:

(T5.1) $0 \le Lik(p_{ij} \succeq p_{kj}) \le 1$;
(T5.2) $Lik(p_{ij} \succeq p_{kj}) = 0$ *if and only if* $V(M_{\alpha_{ij}}(p_{ij})) \le V(N_{\beta_{kj}}(p_{kj}))$;
(T5.3) $Lik(p_{ij} \succeq p_{kj}) = 1$ *if and only if* $V(N_{\beta_{ij}}(p_{ij})) \ge V(M_{\alpha_{kj}}(p_{kj}))$;
(T5.4) $Lik(p_{ij} \succeq p_{kj}) + Lik(p_{kj} \succeq p_{ij}) = 1$;
(T5.5) $Lik(p_{ij} \succeq p_{kj}) = Lik(p_{kj} \succeq p_{ij}) = 0.5$ *if* $Lik(p_{ij} \succeq p_{kj}) = Lik(p_{kj} \succeq p_{ij})$;
(T5.6) $Lik(p_{ij} \succeq p_{ij}) = 0.5$;
(T5.7) $\sum_{i=1}^{m} \sum_{k=1}^{m} Lik(p_{ij} \succeq p_{kj}) = m^2/2$.

Proof See "Appendix A.4".

Theorem 6 *We consider p_{ij} and p_{kj} as the alternatives a_i and a_k, respectively, on the subject of criterion c_j. Through the utility of the* PF *point operators M_α and N_β, the* PF *likelihood measure $Lik(p_{ij} \succeq p_{kj}) \ge 0.5$ if and only if $V(M_{\alpha_{ij}}(p_{ij})) + V(N_{\beta_{ij}}(p_{ij})) \ge V(M_{\alpha_{kj}}(p_{kj})) + V(N_{\beta_{kj}}(p_{kj}))$.*

Proof For necessity, if $Lik(p_{ij} \succeq p_{kj}) = \max\{1 - \max\{\Lambda(p_{ij}, p_{kj}), 0\}, 0\} \ge 0.5$, $1 - \max\{\Lambda(p_{ij}, p_{kj}), 0\} \ge 0.5$ and $\max\{\Lambda(p_{ij}, p_{kj}), 0\} \le 0.5$ are obtained. Thus, it follows that $\Lambda(p_{ij}, p_{kj}) \le 0.5$, namely $2(V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{ij}}(p_{ij}))) \le (V(M_{\alpha_{ij}}(p_{ij})) - V(N_{\beta_{ij}}(p_{ij}))) + (V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{kj}}(p_{kj})))$. Thus, $V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{ij}}(p_{ij})) \le V(M_{\alpha_{ij}}(p_{ij})) - V(N_{\beta_{kj}}(p_{kj}))$, which indicates that $V(M_{\alpha_{ij}}(p_{ij})) + V(N_{\beta_{ij}}(p_{ij})) \ge V(M_{\alpha_{kj}}(p_{kj})) + V(N_{\beta_{kj}}(p_{kj}))$. For sufficiency, it suffices that assumption $V(M_{\alpha_{ij}}(p_{ij})) + V(N_{\beta_{ij}}(p_{ij})) \ge V(M_{\alpha_{kj}}(p_{kj})) + V(N_{\beta_{kj}}(p_{kj}))$ leads to the outcome $V(M_{\alpha_{ij}}(p_{ij})) - V(N_{\beta_{kj}}(p_{kj})) + (V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{ij}}(p_{ij}))) \ge 2(V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{ij}}(p_{ij})))$, which brings about $\Lambda(p_{ij}, p_{kj}) \le 0.5$. It is concluded that $Lik(p_{ij} \succeq p_{kj}) = \max\{1 - \max\{\Lambda(p_{ij}, p_{kj}), 0\}, 0\} \ge 0.5$, which completes the proof.

Theorem 7 *For p_{ij}, p_{kj}, and p_{lj} of a_i, a_k, and a_l, respectively, on the subject of c_j, the* PF *likelihood measures supported by the* PF *point operators M_α and N_β satisfy the property of weak transitivity; thus, $Lik(p_{ij} \succeq p_{kj}) \ge 0.5$ if $Lik(p_{ij} \succeq p_{lj}) \ge 0.5$ and $Lik(p_{lj} \succeq p_{kj}) \ge 0.5$.*

Proof See "Appendix A.5".

5.3 Proposed PF likelihood-based consensus ranking model

This subsection establishes several useful concepts, such as priority weights, penalty weights, and comprehensive disagreement indicators/indices, and formulates an advantageous PF likelihood-based consensus ranking model to manage ambiguity and imprecision, and conduct a multiple criteria evaluation task with high uncertainty and Pythagorean fuzziness.

This paper proposes a helpful concept of penalty weights as solid bases to measure the degree of comprehensive disagreement and form a comprehensive disagreement matrix. Firstly, all values of the PF likelihood measure $Lik(p_{ij} \succeq p_{kj})$ (for $i, k = 1, 2, \cdots, m$) in connection with a specific criterion c_j are precisely shown in a matrix configuration. Let $LIK_j = [Lik(p_{ij} \succeq p_{kj})]_{m \times m}$ denote the PF likelihood matrix with relevance to criterion c_j, in which $Lik(p_{ij} \succeq p_{ij}) = 0.5$ from (T5.6), as follows:

$$
LIK_j = \begin{array}{c} \\ a_1 \\ a_2 \\ \vdots \\ a_m \end{array}
\begin{array}{c}
\begin{array}{cccc} \quad a_1 \quad & \quad a_2 \quad & \cdots & \quad a_m \end{array} \\
\begin{bmatrix}
0.5 & Lik(p_{1j} \succeq p_{2j}) & \cdots & Lik(p_{1j} \succeq p_{mj}) \\
Lik(p_{2j} \succeq p_{1j}) & 0.5 & \cdots & Lik(p_{2j} \succeq p_{mj}) \\
\vdots & \vdots & \ddots & \vdots \\
Lik(p_{mj} \succeq p_{1j}) & Lik(p_{mj} \succeq p_{2j}) & \cdots & 0.5
\end{bmatrix}
\end{array}. \tag{37}
$$

It is known that $0 \le Lik(p_{ij} \succeq p_{kj}) \le 1$ and $Lik(p_{ij} \succeq p_{kj}) + Lik(p_{kj} \succeq p_{ij}) = 1$ based on (T5.1) and (T5.4), respectively, which implies that the PF likelihood matrix LIK_j is a fuzzy complementary judgment matrix for all $c_j \in C$. For each entry in LIK_j, a linear transformation can be made with the assistance of a pair of $\sum_{l=1}^{m} Lik(p_{ij} \succeq p_{lj})$ and $\sum_{l=1}^{m} Lik(p_{kj} \succeq p_{lj})$ $(i, k = 1, 2, \cdots, m)$, as follows:

$$
\Xi(p_{ij} \succeq p_{kj}) = \frac{\sum_{l=1}^{m} Lik(p_{ij} \succeq p_{lj}) - \sum_{l=1}^{m} Lik(p_{kj} \succeq p_{lj})}{2(m-1)} + \frac{1}{2}. \tag{38}
$$

The transformed matrix $[\Xi(p_{ij} \succeq p_{kj})]_{m \times m}$ is fuzzy complementary and consistent as a consequence of the properties of fuzziness, complementarity, and additive transitivity (Li 2011). Thus, a priority weight $Pri(p_{ij})$ is derived by using the normalized outcome of $\sum_{l=1}^{m} \Xi(p_{ij} \succeq p_{lj})$ as follows:

$$
\Pr i(p_{ij}) = \sum_{l=1}^{m} \Xi(p_{ij} \succeq p_{lj}) \bigg/ \sum_{i'=1}^{m} \sum_{l=1}^{m} \Xi(p_{i'j} \succeq p_{lj}). \tag{39}
$$

Because of the property in (T5.7), we know that $\sum_{i=1}^{m} \sum_{k=1}^{m} Lik(p_{ij} \succeq p_{kj}) = m^2 \big/ 2$. It follows that:

$$
\begin{aligned}
\Pr i(p_{ij}) &= \sum_{k=1}^{m} \left(\frac{\sum_{l=1}^{m} Lik(p_{ij} \succeq p_{lj}) - \sum_{l=1}^{m} Lik(p_{kj} \succeq p_{lj})}{2(m-1)} + \frac{1}{2} \right) \Bigg/ \sum_{i'=1}^{m} \sum_{k=1}^{m} \left(\frac{\sum_{i=1}^{m} Lik(p_{i'j} \succeq p_{lj}) - \sum_{l=1}^{m} Lik(p_{kj} \succeq p_{lj})}{2(m-1)} + \frac{1}{2} \right) \\
&= \frac{\sum_{k=1}^{m} \left(\frac{\sum_{l=1}^{m} Lik(p_{ij} \succeq p_{lj}) - \sum_{l=1}^{m} Lik(p_{kj} \succeq p_{lj})}{2(m-1)} + \frac{1}{2} \right)}{\sum_{\substack{i'=1, \\ 1 \le i' < k \le m}}^{m} \left(\begin{array}{c} \frac{\sum_{l=1}^{m} Lik(p_{i'j} \succeq p_{lj}) - \sum_{l=1}^{m} Lik(p_{kj} \succeq p_{lj})}{2(m-1)} + \frac{1}{2} \\ + \frac{\sum_{l=1}^{m} Lik(p_{kj} \succeq p_{lj}) - \sum_{l=1}^{m} Lik(p_{i'j} \succeq p_{lj})}{2(m-1)} + \frac{1}{2} \end{array} \right) + \frac{m}{2}} = \frac{\sum_{k=1}^{m} \left(\frac{\sum_{l=1}^{m} Lik(p_{ij} \succeq p_{lj}) - \sum_{l=1}^{m} Lik(p_{kj} \succeq p_{lj})}{2(m-1)} + \frac{1}{2} \right)}{\frac{m(m-1)}{2} + \frac{m}{2}} \\
&= \frac{m \cdot \sum_{l=1}^{m} Lik(p_{ij} \succeq p_{lj}) - \sum_{k=1}^{m} \sum_{l=1}^{m} Lik(p_{kj} \succeq p_{lj}) + m(m-1) +}{m^2(m-1)} = \frac{\sum_{k=1}^{m} Lik(p_{ij} \succeq p_{kj}) + \frac{m}{2} - 1}{m(m-1)}.
\end{aligned}
$$

Next, regarding the strength of the dual concept of the priority weight $Pri(p_{ij})$, this paper exploits the penalty weight $Pen(p_{ij})$ in Definition 12 and explores its desirable properties in Theorem 8. On the grounds of these beneficial features, the penalty weights defined by PF likelihood measures contribute a realistic approach to establishing a PF likelihood-based consensus ranking method via an easy-to-use linear assignment model.

Definition 12 Let a PF set $P_i = \{\langle c_j, p_{ij} \rangle | c_j \in C\}$ portray the PF characteristic of a_i. Based on the PF likelihood matrix LIK_j, the penalty weight $Pen(p_{ij})$ for each $p_{ij} \in P_i$ is defined by:

$$
Pen(p_{ij}) = \frac{1}{m(m-1)} \left(\sum_{k=1}^{m} Lik(p_{kj} \succeq p_{ij}) + \frac{m}{2} - 1 \right). \tag{40}
$$

Theorem 8 *The penalty weight $Pen(p_{ij})$ for each $p_{ij} \in P_i$ meets the following features*:

(T8.1) $1/2m \leq Pen(p_{ij}) \leq 3/2m$;
(T8.2) $Pen(p_{ij}) + Pri(p_{ij}) = 2/m$;
(T8.3) $\sum_{i=1}^{m} Pen(p_{ij}) = 1$ *for each* $c_j \in C$;
(T8.4) $\sum_{j=1}^{n} \sum_{i=1}^{m} Pen(p_{ij}) = n$.

Proof See "Appendix A.6".

The penalty weight $Pen(p_{ij})$ can describe the extent to which p_{ij} performs worse than all PF evaluative ratings pertaining to the same criterion; thus, the use of $Pen(p_{ij})$ can facilitate the identification of an overall predominate ranking of an alternative, a_i, over all m alternatives in conjunction with criterion c_j. The rules of identifying a predominating relationship between PF evaluative ratings are illustrated in Definition 13. More concretely, the condition $Pen(p_{ij}) < Pen(p_{kj})$ represents that p_{ij} is better than p_{kj} or that a_i is preferred to a_k with reference to c_j, and in contrast, $Pen(p_{ij}) > Pen(p_{kj})$ demonstrates that p_{ij} is less favorable than p_{kj} or that a_i is less preferred to a_k with regard to c_j. The equation $Pen(p_{ij}) = Pen(p_{kj})$ indicates no difference between p_{ij} and p_{kj} or a_i and a_k with relevance to c_j. The smaller the penalty weight $Pen(p_{ij})$ is, the lower the aggregated possibility that p_{ij} is not inferior to the ratings $p_{1j}, p_{2j}, \cdots, p_{mj}$, and thus the higher the predominance is for p_{ij}. Following the rationale in Definition 13, the criterion-wise predominate ranks of all m alternatives are efficiently developed by judging the increasing order of the penalty weight $Pen(p_{ij})$ in the PF context.

Definition 13 For two PF evaluative ratings p_{ij} and p_{kj}, three predominating relationships between a_i and a_k are stated in conformity with their penalty weights:

(D13.1) a_i is better than a_k on the subject of c_j if $Pen(p_{ij}) < Pen(p_{kj})$;
(D13.2) a_i is as good as a_k on the subject of c_j if $Pen(p_{ij}) = Pen(p_{kj})$;
(D13.3) a_i is worse than a_k on the subject of c_j if $Pen(p_{ij}) > Pen(p_{kj})$.

This paper integrates the concepts of penalty weights and PF importance weights to exploit a new measure of comprehensive disagreement indicators that provides a solid basis of the subsequent PF likelihood-based consensus ranking technique. By extending the practicality towards the distance between the ranking orders, the disagreement indicator for an alternative a_i to turn into the ϕth overall rank is derived by the total sum of distances between the criterion-wise predominate rank and the ϕth rank. Specifically using Definition 13, all alternatives in set A can be effectively ranked by judging the rising order of the $Pen(p_{ij})$ values for each $c_j \in C$. Place Φ_{ij} to signify the criterion-wise predominate ranking of a_i in connection with c_j. Notably, an average rank ought to be designated to the matched alternatives under the circumstance that a tie occurs in the criterion-wise predominate ranking. For example, when two alternatives are equally rank third, a precedence rank of 3.5 [i.e., (3 + 4)/2] must be designated. When the decision maker designates equivalent importance towards the n criteria, the disagreement indicator $\hat{\Delta}_i^{\phi}$ is defined as the total distance for a_i to be ranked ϕth:

$$\hat{\Delta}_i^{\phi} = \sum_{j=1}^{n} \left| \Phi_{ij} - \phi \right|. \tag{41}$$

However, this definition would not be valid when the assumption that each criterion has equal importance is not satisfied. Additionally, information accommodated in the PF evaluation matrix P cannot be fully expressed because the PF evaluative ratings are not incorporated into the determination of $\hat{\Delta}_i^{\phi}$. To contain influential decision information in the PF evaluation matrix, this paper proposes a more comprehensive model for thoroughly exploiting decision information covered by PF importance weights and PF evaluative ratings.

To be precise, this paper suggests using $Pen(p_{ij})$ to replace p_{ij} because the concept of penalty weights can sufficiently capture the criterion-wise aggregated outranking outcomes of candidate alternatives and can identify the precedence relationship among all alternatives. This result implies that the penalty weights can lead to the determination of a new disagreement indicator in a more appropriate and effective way. For these reasons, in contrast to the previous index $\hat{\Delta}_i^{\phi}$, this study amalgamates the PF importance weights and the penalty weights into the measurement of degrees of disagreement. With the subsequent definition, this paper constructs an advantageous concept of a comprehensive disagreement indicator in Definition 14, which embodies more persuasive and convincing information than the classical disagreement indicator. Furthermore, the property of Δ_i^{ϕ} is explored in Theorem 9.

Definition 14 For $a_i \in A$ and $c_j \in C$, we consider a PF importance weight w_j and a PF evaluative rating p_{ij}. Supported by an ascending order of the penalty weight $Pen(p_{ij})$, let Φ_{ij} denote the criterion-wise predominate ranking of an alternative a_i for criterion c_j, in which a mean rank is designated in the event of the occurrence of ties. The comprehensive disagreement indicator Δ_i^{ϕ} for a_i to turn into the ϕth consensus rank is determined like this:

$$\Delta_i^{\phi} = (\mu_{\Delta i}^{\phi}, \nu_{\Delta i}^{\phi}; r_{\Delta i}^{\phi}, d_{\Delta i}^{\phi}) = \bigoplus_{j=1}^{n} \left[\left(Pen(p_{ij}) \cdot \left| \Phi_{ij} - \phi \right| \right) \odot w_j \right]. \tag{42}$$

Theorem 9 *Let $Pen(p_{ij})$ and $w_j = (\omega_j, \varpi_j; r_j^w, d_j^w)$ be the penalty weight and the* PF *importance weight, respectively. The two-dimensional representation $\Delta_i'^{\phi} = (\mu_{\Delta i}^{\phi}, \nu_{\Delta i}^{\phi})$ pertaining to the comprehensive disagreement indicator Δ_i^{ϕ} is determined as follows*:

$$\Delta_i'^{\phi} = (\mu_{\Delta i}^{\phi}, \nu_{\Delta i}^{\phi}) = \left(\sqrt{1 - \prod_{j=1}^{n} \left(1 - (\omega_j)^2 \right)^{Pen(p_{ij}) \cdot \left| \Phi_{ij} - \phi \right|}}, \prod_{j=1}^{n} (\varpi_j)^{Pen(p_{ij}) \cdot \left| \Phi_{ij} - \phi \right|} \right) \quad \text{for } i, \phi = 1, 2, \ldots, m. \tag{43}$$

Proof See "Appendix A.7".

Within the above theorem, it is also worthy of mention that the strength of commitment relative to comprehensive disagreement indicator Δ_i^{ϕ} is derived as follows: $r_{\Delta i}^{\phi} = \sqrt{1 - \Pi_{j=1}^{n} (1 - (\omega_j)^2)^{Pen(p_{ij}) \cdot \left| \Phi_{ij} - \phi \right|} + \Pi_{j=1}^{n} (\varpi_j)^{2 \cdot Pen(p_{ij}) \cdot \left| \Phi_{ij} - \phi \right|}}$. It follows that the radians $\theta_{\Delta i}^{\phi} = \arccos(\mu_{\Delta i}^{\phi} / r_{\Delta i}^{\phi})$ (or equivalently, $\theta_{\Delta i}^{\phi} = \text{arc}sin(\nu_{\Delta i}^{\phi} / r_{\Delta i}^{\phi})$). Moreover, the direction of commitment $d_{\Delta i}^{\phi} = (\pi - 2 \cdot \theta_{\Delta i}^{\phi}) / \pi$.

The proposed PF likelihood-based consensus ranking method aims to receive a consensus ranking of candidate alternatives, i.e., the so-called median ranking, for the sake of creating a rank with as little differentiation from whole criterion-wise predominating ranks as possible. This can be effectively accomplished by constructing an assignment model. In general, the MCDA problem focuses on the collection of complete (or total) rankings of candidate alternatives that is precisely stated by any one of a number of linear programming formulations. Notably, if the decision maker restricts consideration to a complete ranking, an MCDA of this nature can be addressed much more effectively by representing it as an assignment problem. Let Ψ represent a permutation matrix whose entry ψ_i^ϕ is a zero–one binary variable, where $\psi_i^\phi = 1$ if a_i is designated as the consensus rank ϕ, and $\psi_i^\phi = 0$ otherwise. Notably, $\sum_{\phi=1}^m \psi_i^\phi = 1$ (i.e., a_i is obliged to be allocated to only one rank). Similarly, $\sum_{i=1}^m \psi_i^\phi = 1$ (i.e., a consensus rank ϕ must merely retain an alternative). The matrix Ψ is described in this manner:

$$\Psi = \left[\psi_i^\phi\right]_{m\times m} = \begin{array}{c} \\ a_1 \\ a_2 \\ \vdots \\ a_m \end{array} \begin{array}{c} \begin{array}{cccc} \text{1st} & \text{2nd} & \cdots & m\text{th} \end{array} \\ \begin{bmatrix} \psi_1^1 & \psi_1^2 & \cdots & \psi_1^m \\ \psi_2^1 & \psi_2^2 & \cdots & \psi_2^m \\ \vdots & \vdots & \ddots & \vdots \\ \psi_m^1 & \psi_m^2 & \cdots & \psi_m^m \end{bmatrix} \end{array}. \tag{44}$$

To determine an optimal consensus ranking towards feasible alternatives that possess the least disagreement with all criterion-wise predominate rankings, the proposed concept of the comprehensive disagreement indicator Δ_i^ϕ can be used to define the objective function in the assignment model. To solve for the optimal consensus ranking that generates the lowest $\oplus_{i=1}^m [\oplus_{\phi=1}^m (\psi_i^\phi \odot \Delta_i^\phi)]$, the subsequent formulation of an MCDA task under the uncertain PF environment can be used:

$$\begin{aligned} &\min \left\{ \bigoplus_{i=1}^m \left[\bigoplus_{\phi=1}^m \left(\psi_i^\phi \odot \Delta_i^\phi \right) \right] \right\} \\ &\text{subject to } \sum_{\phi=1}^m \psi_i^\phi = 1 (i = 1, 2, \ldots, m),\ \sum_{i=1}^m \psi_i^\phi = 1 (\phi = 1, 2, \ldots, m),\ \psi_i^\phi = 0 \text{ or } 1 \quad \text{for all } i \text{ and } \phi. \end{aligned} \tag{45}$$

The comprehensive disagreement indicator $\Delta_i^\phi = (\mu_{\Delta i}^\phi, \nu_{\Delta i}^\phi; r_{\Delta i}^\phi, d_{\Delta i}^\phi)$ is a Pythagorean membership grade, which highlights the critical issue of a difficult determination when resolving the model [M1]. Using comparable scalar functions, this study identifies the comprehensive disagreement index in Definition 15 to overcome the difficulty in resolving the model [M1] in a fairly straightforward way. Certain favorable properties of the comprehensive disagreement index are investigated in Theorem 10.

Definition 15 Let $\Delta_i^\phi = (\mu_{\Delta i}^\phi, \nu_{\Delta i}^\phi; r_{\Delta i}^\phi, d_{\Delta i}^\phi)$ delineate a comprehensive disagreement indicator of alternative a_i to be allocated to a consensus rank ϕ, $a_i \in A$ and $\phi = 1, 2, \cdots, m$. The comprehensive disagreement index $V(\Delta_i^\phi)$ of Δ_i^ϕ is computed like this:

$$V(\Delta_i^\phi) = \frac{1}{2} + r_{\Delta i}^\phi \cdot \left(d_{\Delta i}^\phi - \frac{1}{2} \right) = \frac{1}{2} + r_{\Delta i}^\phi \cdot \left(\frac{1}{2} - \frac{2 \cdot \theta_{\Delta i}^\phi}{\pi} \right). \tag{46}$$

Theorem 10 *The comprehensive disagreement index $V(\Delta_i^\phi)$ associated with each comprehensive disagreement indicator Δ_i^ϕ for all $i, \phi \in \{1, 2, \cdots, m\}$ fulfills the following features*:

(T10.1) $0 \le V(\Delta_i^\phi) \le 1$;
(T10.2) $V(\Delta_i^\phi) = 0$ *if and only if* $\Delta_i^\phi = (0, 1; 1, 0)$;
(T10.3) $V(\Delta_i^\phi) = 1$ *if and only if* $\Delta_i^\phi = (1, 0; 1, 1)$;

(T10.4) $V(\Delta_i^{\phi}) = 0.5$ *if* $\Delta_i^{\phi} = (0,0;0,0.5)$;

(T10.5) $V(\Delta_i^{\phi}) = d_{\Delta i}^{\phi}$ *if* $\tau_{\Delta i}^{\phi} = 0$ (*or equivalently*, $r_{\Delta i}^{\phi} = 1$).

Proof See "Appendix A.8".

The concept of the comprehensive disagreement index can be used to constitute a PF likelihood-based consensus ranking model. Concerning an alternative a_i in connection with each consensus rank ϕ, $V(\Delta_i^{\phi})$ can estimate the extent of the overall disagreement towards the criterion-wise predominate rankings. Let $V(\Delta)$ denote the comprehensive disagreement matrix whose entries $V(\Delta_i^{\phi})$ for all $i, \phi = 1, 2, \cdots, m$ are given by Definition 15, in this fashion:

$$V(\Delta) = \left[V(\Delta_i^{\phi})\right]_{m\times m} = \begin{array}{c} \\ a_1 \\ a_2 \\ \vdots \\ a_m \end{array}\begin{array}{c} \begin{array}{cccc} \text{1st} & \text{2nd} & \cdots & m\text{th} \end{array} \\ \begin{bmatrix} V(\Delta_1^1) & V(\Delta_1^2) & \cdots & V(\Delta_1^m) \\ V(\Delta_2^1) & V(\Delta_2^2) & \cdots & V(\Delta_2^m) \\ \vdots & \vdots & \ddots & \vdots \\ V(\Delta_m^1) & V(\Delta_m^2) & \cdots & V(\Delta_m^m) \end{bmatrix} \end{array}. \tag{47}$$

The lower the degree of disagreement exhibited by $V(\Delta_i^{\phi})$ is, the higher the conformity is from designating the alternative a_i to the ϕth consensus rank. Thus, the decision maker intends to arrange m elements in the comprehensive disagreement matrix $V(\Delta)$ to designate a well-suited consensus rank for each alternative. Specifically, one must choose m elements in different rows and columns whose sum is the minimum. This executable procedure can be efficiently achieved by constituting the following PF likelihood-based consensus ranking model:

$$\begin{aligned} &\min\left\{\sum_{i=1}^{m}\sum_{\phi=1}^{m}\left(\psi_i^{\phi}\cdot V(\Delta_i^{\phi})\right)\right\} \\ &\text{subject to } \sum_{\phi=1}^{m}\psi_i^{\phi} = 1\,(i=1,2,\ldots,m),\ \sum_{i=1}^{m}\psi_i^{\phi} = 1\,(\phi=1,2,\ldots,m),\ \psi_i^{\phi} = 0 \text{ or } 1 \quad \text{for all } i \text{ and } \phi. \end{aligned} \tag{48}$$

With the assistance of the Hungarian method, one can easily solve model [M2] to acquire the optimal consensus ranking that generates the lowest $\sum_{i=1}^{m}\sum_{\phi=1}^{m}(\psi_i^{\phi}\cdot V(\Delta_i^{\phi}))$. Finally, this paper exploits the optimal permutation matrix $\hat{\Psi} = [\hat{\psi}_i^{\phi}]_{m\times m}$ in an effort to render the optimal consensus ranks concerning all alternatives using the following method:

$$A\cdot\hat{\Psi} = \left(a_1, a_2, \ldots, a_m\right)\cdot\begin{bmatrix} \hat{\psi}_1^1 & \hat{\psi}_1^2 & \cdots & \hat{\psi}_1^m \\ \hat{\psi}_2^1 & \hat{\psi}_2^2 & \cdots & \hat{\psi}_2^m \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\psi}_m^1 & \hat{\psi}_m^2 & \cdots & \hat{\psi}_m^m \end{bmatrix}. \tag{49}$$

5.4 Proposed algorithmic procedure

The proposed PF likelihood-based consensus ranking method to manage MCDA issues comprising PF uncertain information (i.e., PF importance weights and evaluative ratings) consists of five phases: problem formulation (see Steps 1 and 2), ascertaining PF likelihood measures (see Steps 3–5), criterion-wise predominate rankings (see Steps 6 and 7), measurement of comprehensive disagreement (see Steps 8 and 9), and a linear assignment model supported by likelihood measures (see Steps 10 and 11). The five phases can be described using the algorithmic procedure below:

Steps 1 and 2: Representing an MCDA problem in PF contexts

Step 1: Define an MCDA task through the agency of the set of candidate alternatives $A = \{a_1, a_2, \dots, a_m\}$ and the set of evaluative criteria $C = \{c_1, c_2, \cdots, c_n\}$.

Step 2: Investigate a PF importance weight $w_j = (\omega_j, \varpi_j; r_j^w, d_j^w)$ of each $c_j \in C$ and a PF evaluative rating $p_{ij} = (\mu_{ij}, \nu_{ij}; r_{ij}, d_{ij})$ of $a_i \in A$ towards c_j. Construct the PF evaluation matrix $P = [p_{ij}]_{m\times n}$ in (27).

Steps 3–5: Ascertaining upper/lower estimations and PF likelihood measures

Step 3: Designate the settings of two allocation parameters α_{ij} and β_{ij} within the PF point operators M_α and N_β, respectively, for each p_{ij}, where $\alpha_{ij}, \beta_{ij} \in [0, 1]$.

Step 4: Use (29) and (30) to compute the upper estimation $M_{\alpha_{ij}}(p_{ij})$ and the lower estimation $N_{\beta_{ij}}(p_{ij})$, respectively, associated with each p_{ij} in the PF evaluation matrix P.

Step 5: Use (36) to identify the PF likelihood measure $Lik(p_{ij} \underline{\succ} p_{kj})$ of an outranking relation "$p_{ij} \underline{\succ} p_{kj}$" for $a_i, a_k \in A$ and $c_j \in C$. Form a PF likelihood matrix $LIK_j = [Lik(p_{ij} \underline{\succ} p_{kj})]_{m\times m}$ in (37).

Steps 6 and 7: Identifying criterion-wise predominating ranks via penalty weights

Step 6: Use (40) to calculate the penalty weight $Pen(p_{ij})$ for each p_{ij} belonging to the PF characteristic $P_i = \{\langle c_j, p_{ij}\rangle | c_j \in C\}$ in (26).

Step 7: Identify the criterion-wise predominate rank Φ_{ij} of all a_i following an ascending order of the $Pen(p_{ij})$ values in terms of each c_j, where an average rank is distributed in the event of the occurrence of ties.

Steps 8 and 9: Determining comprehensive disagreement indicators and indices

Step 8: Apply (42) and (43) to calculate the comprehensive disagreement indicator $\Delta_i^\phi = (\mu_{\Delta i}^\phi, \nu_{\Delta i}^\phi; r_{\Delta i}^\phi, d_{\Delta i}^\phi)$ for a_i to become the consensus rank ϕ.

Step 9: Compute the comprehensive disagreement index $V(\Delta_i^\phi)$ of each Δ_i^ϕ using (46) and construct the comprehensive disagreement matrix $V(\Delta) = [V(\Delta_i^\phi)]_{m\times m}$ in (47).

Steps 10 and 11: Solving a likelihood-based linear assignment model

Step 10: Construct a permutation matrix $\Psi = [\psi_i^\phi]_{m\times m}$ in (44), in which the entry ψ_i^ϕ is a binary variable for each $i, \phi \in \{1, 2, \dots, m\}$, and establish a PF likelihood-based consensus ranking model using [M2].

Step 11: Determine the optimal permutation matrix $\hat{\Psi} = [\hat{\psi}_i^\phi]_{m\times m}$. Use (49) to acquire the optimal consensus rank (in conformity with $\hat{\psi}_i^\phi = 1$) for each alternative.

6 Model application and comparative studies

This section presents a realistic application relating to a financing problem about working capital policies and performs sensitivity analyses and comparative discussions. As an illustrative demonstration of the developed approach, this section aims to describe beneficial theoretical insights contained in the proposed methodology and verify its appropriateness and usefulness to solve real problems. Certain comparative studies and explorations are also investigated to validate the strong points of the advanced PF likelihood-based consensus ranking method.

6.1 Practical application

This subsection uses a pragmatic MCDA issue about financing policies to demonstrate the computational algorithm of the PF likelihood-based consensus ranking method. The investigated financing decision-making problem, originally formulated by Wang and Chen (2018), focuses on a multiple criteria evaluation task to choose a fitting financing policy to assist with efficacious working capital management in the field of medicine.

The management of working capital plays a key role for an enterprise (an organization or other entity) to conduct effective financial management and business activities. Working capital, i.e., the subtraction of current liabilities from current assets, is a financial metric that stands for the operating liquidity behaved by an enterprise. The main goal of working capital management is to provide sufficient liquidity for the survival and operation of the enterprise and fulfill its financial obligations. Efficacious working capital management is important for companies of all sizes because it gives the enterprise financial flexibility and reduces reliance on external funding sources. Working capital management can protect the enterprise from potential financial problems, and is vital to the growth and development of the enterprise. In general, working capital financing policies can be compartmentalized into three basic policies: conservative, aggressive, and moderate policies, as shown in Fig. 4. The conservative financing policy can maintain relatively high current assets, hold large amounts of cash, securities, inventory, etc., and will also give customers more relaxed credit conditions. The aggressive financing policy is to maintain the minimum level of current assets, stricter credit conditions for customers, shorten the payback period of accounts receivable, and use funds for other high-return investments. The moderate financing policy is somewhere in between, maintaining an appropriate level of current assets, and matching the maturity dates of current liabilities to improve the efficiency in working capital utilization.

Chang Gung Memorial Hospital (CGMH) was founded in 1976, and has successively established large hospitals in Linkou, Keelung, Taipei, Taoyuan, Chiayi, Yunlin, Kaohsiung, etc., which treats more than 31,500 patients per day and has 9000 beds in Taiwan. Among them, Linkou CGMH is one of the general hospitals with the largest scale, the most complete equipment and the best operating performance in the Far East. In order to furnish highly professional and favorable medical services to young children, Linkou CGMH constructed a large-scale children's medical center in 1993. With the aim of making an efficacious usage of medical resources, Linkou CGMH established a nursing home and Taoyuan

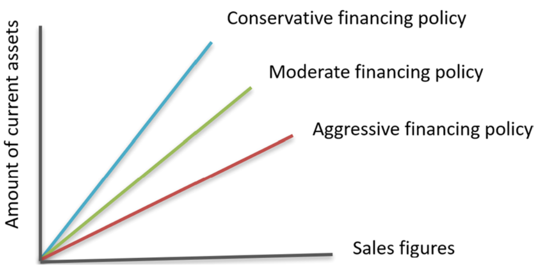


Fig. 4 Schematic diagram of three basic types of financing policies

branch in 2001 and 2003, respectively, predicated on acute and chronic medical treatments. Linkou CGMH is committed to developing subacute, chronic medical, and long-term care services. Moreover, it is vertically integrated into a complete medical system to provide the people with complete medical care. Furthermore, Linkou CGMH is the hospital with the largest number of foreign patients in Taiwan. Every year, more than tens of thousands of foreign patients from all over the world come here for medical treatments. For such a large-scale healthcare institution, the authority is faced with many complex issues when making financial decisions. Over and above that, there are many factors that lead to an increase in the cost of health care, such as the aging of the population, the application of high-tech care services, the cost of prescription drugs, chronic diseases, legal considerations, etc. Therefore, Linkou CGMH must have sufficient working capital to cover various expense items.

According to Taiwan's medical payment system, after billing patients or third-party payers, it often takes more than two months to receive the relevant payments. Healthcare providers usually depend on third-party payers for paying patient bills. The authority must evaluate the Linkou CGMH's financial status and conditions, and choose appropriate working capital strategies to ensure that they have sufficient working capital, while maximizing liquidity and profitability. This is a very difficult and complex task for Linkou CGMH. Based on the problem description in Wang and Chen (2018), the contour of a financing affair compared to working capital policies is shown in Fig. 5. This problem focuses on an MCDA task to determine a suitable financing policy to assist with working capital management for Linkou CGMH. The propounded methodology would be used subsequently to address this financing decision issue.

According to the problem representation proposed by Wang and Chen (2018), the authority of Linkou CGMH used six factors to appraise five candidate alternatives compared to working capital financing policies in a cautious manner. The data of

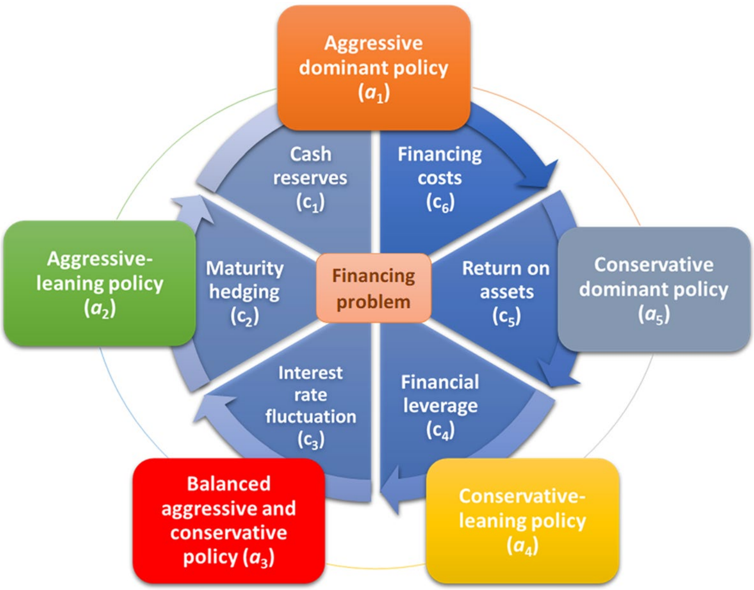


Fig. 5 Connotation of a financing issue relating to working capital policies

PF importance weights and evaluation values in this financing decision-making problem was established by Wang and Chen (2018). On the basis of the uncertain PF information presented in Wang and Chen (2018), the developed algorithmic procedure of the PF likelihood-based consensus ranking model was used to find out a proper choice for Linkou CGMH. In Step 1, the sets of candidate alternatives and evaluative criteria are stated as follows: $A = \{a_1, a_2, \ldots, a_5\}$ and $C = \{c_1, c_2, \ldots, c_6\}$, respectively.

In Step 2, aligned with the managers' expertise and past experiences at Linkou CGMH, the PF importance weights of the evaluative criteria were developed by: w_1 = (0.35, 0.65; 0.74, 0.31), w_2 = (0.55, 0.45; 0.71, 0.56), w_3 = (0.65, 0.35; 0.74, 0.69), w_4 = (0.25, 0.75; 0.79, 0.20), w_5 = (0.75, 0.25; 0.79, 0.80), and w_6 = (0.85, 0.15; 0.86, 0.89). The PF evaluation matrix $P = [p_{ij}]_{5\times6}$ can be formed of all $p_{ij} = (\mu_{ij}, \nu_{ij}; r_{ij}, d_{ij})$ for $a_i \in A$ and $c_j \in C$. Due to the PF evaluation values originating from Wang and Chen (2018), the matrix P was constructed as follows:

$$
P = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{array}
\begin{array}{c}
\begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\
\left[\begin{array}{ccc}
(0.21, 0.83; 0.86, 0.16) & (0.33, 0.78; 0.85, 0.25) & (0.49, 0.54; 0.73, 0.47) \\
(0.36, 0.74; 0.82, 0.29) & (0.55, 0.48; 0.73, 0.54) & (0.61, 0.45; 0.76, 0.60) \\
(0.63, 0.52; 0.82, 0.56) & (0.77, 0.25; 0.81, 0.80) & (0.88, 0.21; 0.90, 0.85) \\
(0.82, 0.21; 0.85, 0.84) & (0.83, 0.15; 0.84, 0.89) & (0.78, 0.27; 0.83, 0.79) \\
(0.94, 0.14; 0.95, 0.91) & (0.90, 0.10; 0.91, 0.93) & (0.61, 0.42; 0.74, 0.62)
\end{array}\right.
\end{array}
$$

$$
\begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{array}
\begin{array}{c}
\begin{array}{ccc} c_4 & c_5 & c_6 \end{array} \\
\left.\begin{array}{ccc}
(0.29, 0.70; 0.76, 0.25) & (0.91, 0.13; 0.92, 0.91) & (0.49, 0.40; 0.63, 0.56) \\
(0.41, 0.63; 0.75, 0.37) & (0.80, 0.17; 0.82, 0.87) & (0.73, 0.31; 0.79, 0.74) \\
(0.61, 0.47; 0.77, 0.58) & (0.71, 0.31; 0.77, 0.74) & (0.92, 0.11; 0.93, 0.92) \\
(0.74, 0.31; 0.80, 0.75) & (0.32, 0.79; 0.85, 0.25) & (0.84, 0.18; 0.86, 0.87) \\
(0.82, 0.26; 0.86, 0.80) & (0.11, 0.88; 0.89, 0.08) & (0.51, 0.59; 0.78, 0.45)
\end{array}\right]
\end{array}.
$$

In Step 3, via illustration, we take a case in which the settings of the allocation parameters can be arranged as $\alpha_{ij} = (\mu_{ij})^2 \big/ ((\mu_{ij})^2 + (\nu_{ij})^2)$ and $\beta_{ij} = (\nu_{ij})^2 \big/ ((\mu_{ij})^2 + (\nu_{ij})^2)$ in connection with each p_{ij}, which directly leads to $(\alpha_{ij})^2 + (\beta_{ij})^2 = 1$. Considering p_{53} = (0.61, 0.42; 0.74, 0.62) as an example, one obtains $\alpha_{53} = 0.61^2/(0.61^2 + 0.42^2) = 0.6874$ and $\beta_{53} = 0.42^2/(0.61^2 + 0.42^2) = 0.3216$. The relevant settings of α_{ij} and β_{ij} for all $a_i \in A$ and $c_j \in C$ are exhibited in Tables 1 and 2, respectively, as stated in Appendix B.

In Step 4, the upper estimation $M_{\alpha_{ij}}(p_{ij})$ and the lower estimation $N_{\beta_{ij}}(p_{ij})$ were derived through the medium of α_{ij} and β_{ij}, respectively. Taking $M_{\alpha_{53}}(p_{53})$ as an example, it is known that $\tau_{53} = 0.6719$ and $\theta_{53} = 0.6030$ with reference to p_{53}. By virtue of (29) and Definition 7, the following results can be acquired: $\mu_{53}^M = \sqrt{(\mu_{53})^2 + \alpha_{53}(\tau_{53})^2} = \sqrt{0.61^2 + 0.6784 \times 0.6719^2} = 0.8236$, $\nu_{53}^M = \nu_{52} = 0.4200$, and $r_{53}^M = \sqrt{(\mu_{53})^2 + (\nu_{53})^2 + \alpha_{53}(\tau_{53})^2} = \sqrt{0.61^2 + 0.42^2 + 0.6784 \times 0.6719^2} = 0.9246$. According to (31), it is easy to see that $\theta_{53}^M = \arccos(0.8236/0.9246) = 0.4716$, which follows that $d_{53}^M = (\pi - 2 \times \theta_{53}^M)/\pi = (3.1416 - 2 \times 0.4716)/3.1416 = 0.6998$. Hence, $M_{\alpha_{53}}(p_{53}) = (\mu_{53}^M, \nu_{53}^M; r_{53}^M, d_{53}^M)$ = (0.8236, 0.4200; 0.9246, 0.6998). Analogously, $N_{\beta_{53}}(p_{53})$ = (0.6100, 0.5671; 0.8329, 0.5232) by means of (30) and (32). The obtained results appertaining to $M_{\alpha_{ij}}(p_{ij})$ and $N_{\beta_{ij}}(p_{ij})$ are indicated in Tables 1 and 2, respectively.

In Step 5, to determine the possibility of an outranking relation towards PF evaluative ratings, this study used (34) and (35) to generate the scalar functions of $M_{\alpha_{ij}}(p_{ij})$ and $N_{\beta_{ij}}(p_{ij})$, respectively. The computation results of $V(M_{\alpha_{ij}}(p_{ij}))$ and $V(N_{\beta_{ij}}(p_{ij}))$ are manifested in the rightmost columns of Tables 1 and 2, respectively. On the grounds of the obtained scalar functions, this study used (36) to ascertain the PF likelihood measure $Lik(p_{ij}\succeq p_{kj})$ for $a_i, a_k \in A$ and $c_j \in C$. Consider $Lik(p_{14}\succeq p_{24})$ as an illustration. It is known that $V(M_{\alpha_{14}}(p_{14}))=0.3552$ and $V(M_{\alpha_{24}}(p_{24}))=0.4619$ from Table 1. Moreover, one has $V(N_{\beta_{14}}(p_{14}))=0.2033$ and $V(N_{\beta_{24}}(p_{24}))=0.3037$ from Table 2. By virtue of (36), the measure $Lik(p_{14}\succeq p_{24})$ was determined in the following manner:

$$Lik(p_{14}\succeq p_{24}) = \max\left\{1-\max\left\{\frac{0.4619-0.2033}{(0.3552-0.2033)+(0.4619-0.3037)},0\right\},0\right\} = 0.1661.$$

The PF likelihood matrix $LIK_j = [Lik(p_{ij}\succeq p_{kj})]_{5\times 5}$ can be constructed by collecting the obtained PF likelihood measures in connection with c_j. Specifically, the PF likelihood matrices with relevance to the six criteria were determined as follows:

$$LIK_1 = \begin{bmatrix} 0.5000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 1.0000 & 0.5000 & 0.0000 & 0.0000 & 0.0000 \\ 1.0000 & 1.0000 & 0.5000 & 0.0000 & 0.0000 \\ 1.0000 & 1.0000 & 1.0000 & 0.5000 & 0.0000 \\ 1.0000 & 1.0000 & 1.0000 & 1.0000 & 0.5000 \end{bmatrix}, \quad LIK_2 = \begin{bmatrix} 0.5000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 1.0000 & 0.5000 & 0.0000 & 0.0000 & 0.0000 \\ 1.0000 & 1.0000 & 0.5000 & 0.0978 & 0.0000 \\ 1.0000 & 1.0000 & 0.9022 & 0.5000 & 0.1314 \\ 1.0000 & 1.0000 & 1.0000 & 0.8686 & 0.5000 \end{bmatrix},$$

$$LIK_3 = \begin{bmatrix} 0.5000 & 0.1593 & 0.0000 & 0.0000 & 0.1188 \\ 0.8407 & 0.5000 & 0.0000 & 0.0000 & 0.4453 \\ 1.0000 & 1.0000 & 0.5000 & 0.9296 & 1.0000 \\ 1.0000 & 1.0000 & 0.0704 & 0.5000 & 1.0000 \\ 0.8812 & 0.5547 & 0.0000 & 0.0000 & 0.5000 \end{bmatrix}, \quad LIK_4 = \begin{bmatrix} 0.5000 & 0.1661 & 0.0000 & 0.0000 & 0.0000 \\ 0.8339 & 0.5000 & 0.0000 & 0.0000 & 0.0000 \\ 1.0000 & 1.0000 & 0.5000 & 0.0000 & 0.0000 \\ 1.0000 & 1.0000 & 1.0000 & 0.5000 & 0.2151 \\ 1.0000 & 1.0000 & 1.0000 & 0.7849 & 0.5000 \end{bmatrix},$$

$$LIK_5 = \begin{bmatrix} 0.5000 & 0.8861 & 1.0000 & 1.0000 & 1.0000 \\ 0.1139 & 0.5000 & 0.9867 & 1.0000 & 1.0000 \\ 0.0000 & 0.0133 & 0.5000 & 1.0000 & 1.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.5000 & 1.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.5000 \end{bmatrix}, \quad LIK_6 = \begin{bmatrix} 0.5000 & 0.0451 & 0.0000 & 0.0000 & 0.7526 \\ 0.9549 & 0.5000 & 0.0000 & 0.0000 & 1.0000 \\ 1.0000 & 1.0000 & 0.5000 & 1.0000 & 1.0000 \\ 1.0000 & 1.0000 & 0.0000 & 0.5000 & 1.0000 \\ 0.2474 & 0.0000 & 0.0000 & 0.0000 & 0.5000 \end{bmatrix}.$$

In Step 6, this study calculated the penalty weight $Pen(p_{ij})$ of each p_{ij} belonging to the PF characteristic P_i. Consider $p_{32} \in P_3$ $(=\{\langle c_1,p_{31}\rangle,\langle c_2,p_{32}\rangle,\cdots,\langle c_6,p_{36}\rangle\})$ as an example. By applying (40), one obtains $Pen(p_{32})=(Lik(p_{12}\succeq p_{32})+Lik(p_{22}\succeq p_{32})+Lik(p_{32}\succeq p_{32})+Lik(p_{42}\succeq p_{32})+Lik(p_{52}\succeq p_{32})+(5/2)-1)/(5\times 4)=(0.0000+0.0000+0.5000+0.9022+1.0000+2.5-1)/20=0.1951$. The results of the penalty weight $Pen(p_{ij})$ are revealed in the top bottom of Table 3 in Appendix B. In Step 7, the criterion-wise predominate ranks with respect to the alternatives were obtained in light of an increasing order of the $Pen(p_{ij})$ values related to a specific c_j, as displayed in the bottom part of Table 3.

In Step 8, the product of $Pen(p_{ij})$ and $|\Phi_{ij}-\phi|$ was first calculated for each $i,\phi=1,2,\cdots,5$ and $j=1,2,\cdots,6$. Next, the comprehensive disagreement indicator Δ_i^{ϕ} was determined using (42) and (43). The computed outcomes of $Pen(p_{ij})\cdot|\Phi_{ij}-\phi|$ and Δ_i^{ϕ} are shown in Table 4 of Appendix B. Regarding a_3 to be designated the fourth consensus rank for instance, the following results were acquired: $Pen(p_{31})\cdot|\Phi_{31}-4| = 0.2000\cdot|3-4| = 0.2000$, $Pen(p_{32})\cdot|\Phi_{32}-4| = 0.1951\cdot|3-4| = 0.1951$, $Pen(p_{33})\cdot|\Phi_{33}-4| = 0.1035\cdot|1-4| = 0.3106$, $Pen(p_{34})\cdot|\Phi_{34}-4| = 0.2000\cdot|3-4| = 0.2000$, $Pen(p_{35})\cdot|\Phi_{35}-4| = 0.1993\cdot|3-4| = 0.1993$, and $Pen(p_{36})\cdot|\Phi_{36}-4| = 0.1000\cdot|1-4| = 0.3000$. Next, consider the two-dimensional representation $w'_j=(\omega_j,\varpi_j)$ pertaining to the PF importance weight

w_j. This study would like to combine the obtained $Pen(p_{3j}) \cdot |\Phi_{3j} - 4|$ with the PF importance weights to determine the comprehensive disagreement indicator Δ_3^4. On the grounds of the PF importance weights, it is known that $w'_1 = (\omega_1, \varpi_1)$ =(0.35, 0.65), $w'_2 = (\omega_2, \varpi_2)$ =(0.55, 0.45), $w'_3 = (\omega_3, \varpi_3)$ =(0.65, 0.35), $w'_4 = (\omega_4, \varpi_4)$ =(0.25, 0.75), $w'_5 = (\omega_5, \varpi_5)$ =(0.75, 0.25), and $w'_6 = (\omega_6, \varpi_6)$ =(0.85, 0.15). With use of (43), the two-dimensional representation $\Delta_3^{\prime 4} = (\mu_{\Delta 3}^4, \nu_{\Delta 3}^4)$ relating to the indicator Δ_3^4 was ascertained as follows:

$$\Delta_3^{\prime 4} = \left(\sqrt{1 - \prod_{j=1}^{6} \left(1 - (\omega_j)^2\right)^{Pen(p_{3j}) \cdot |\Phi_{3j} - 4|}}, \prod_{j=1}^{6} (\varpi_j)^{Pen(p_{3j}) \cdot |\Phi_{3j} - 4|} \right)$$

$$= \left(\left[1 - (1 - 0.35^2)^{0.2000} \times (1 - 0.55^2)^{0.1951} \times (1 - 0.65^2)^{0.3106} \times (1 - 0.25^2)^{0.2000} \times (1 - 0.75^2)^{0.1993} \times (1 - 0.85^2)^{0.3000}\right]^{0.5}, 0.65^{0.2000} \times 0.45^{0.1951} \times 0.35^{0.3106} \times 0.75^{0.2000} \times 0.25^{0.1993} \times 0.15^{0.3000} \right) = (0.7507, 0.2297).$$

Accordingly, it can be acquired that $\Delta_3^4 = (\mu_{\Delta 3}^4, \nu_{\Delta 3}^4; r_{\Delta 3}^4, d_{\Delta 3}^4)$ =(0.7507, 0.2297; 0.7851, 0.8110). The computed results pertaining to the comprehensive disagreement indicator Δ_i^ϕ for each $i, \phi = 1, 2, \ldots, 5$ are demonstrated in Table 4.

In Step 9, this study used (46) to compute the comprehensive disagreement index $V(\Delta_i^\phi)$. Taking Δ_3^4 for example, it is acquired that $V(\Delta_3^4) = 0.5 + r_{\Delta 3}^4 \times (d_{\Delta 3}^4 - 0.5) = 0.5 + 0.7851 \times (0.8110 - 0.5) = 0.7441$. On the grounds of the obtained $V(\Delta_i^\phi)$, the comprehensive disagreement matrix $V(\Delta)$ was developed as follows:

$$V(\Delta) = \left[V(\Delta_i^\phi)\right]_{5 \times 5} = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{array} \begin{array}{c} \text{1st 2nd 3rd 4th 5th} \\ \begin{bmatrix} 0.9670 & 0.9321 & 0.8524 & 0.6666 & 0.6669 \\ 0.9014 & 0.7584 & 0.5593 & 0.5773 & 0.8320 \\ 0.6529 & 0.5858 & 0.5025 & 0.7441 & 0.8671 \\ 0.7878 & 0.5369 & 0.6325 & 0.7072 & 0.8649 \\ 0.9748 & 0.9427 & 0.8659 & 0.7789 & 0.6341 \end{bmatrix} \end{array}.$$

In Step 10, a permutation matrix Ψ involving 25 (=5×5) binary variables was represented as $\Psi = [\psi_i^\phi]_{5\times5}$. According to model [M2], this study constructed the following PF likelihood-based consensus ranking model for solving the financing problem:

$$\min \left\{ \begin{array}{l} 0.9670 \cdot \psi_1^1 + 0.9321 \cdot \psi_1^2 + 0.8524 \cdot \psi_1^3 + 0.6666 \cdot \psi_1^4 + 0.6669 \cdot \psi_1^5 \\ +0.9014 \cdot \psi_2^1 + 0.7584 \cdot \psi_2^2 + 0.5593 \cdot \psi_2^3 + 0.5773 \cdot \psi_2^4 + 0.8320 \cdot \psi_2^5 \\ +0.6529 \cdot \psi_3^1 + 0.5858 \cdot \psi_3^2 + 0.5025 \cdot \psi_3^3 + 0.7441 \cdot \psi_3^4 + 0.8671 \cdot \psi_3^5 \\ +0.7878 \cdot \psi_4^1 + 0.5369 \cdot \psi_4^2 + 0.6325 \cdot \psi_4^3 + 0.7072 \cdot \psi_4^4 + 0.8649 \cdot \psi_4^5 \\ +0.9748 \cdot \psi_5^1 + 0.9427 \cdot \psi_5^2 + 0.8659 \cdot \psi_5^3 + 0.7789 \cdot \psi_5^4 + 0.6341 \cdot \psi_5^5 \end{array} \right\}$$

$$\text{subject to} \quad \sum_{\phi=1}^{5} \psi_i^\phi = 1 \ (i = 1, 2, \ldots, 5), \ \sum_{i=1}^{5} \psi_i^\phi = 1 \ (\phi = 1, 2, \ldots, 5), \ \psi_i^\phi = 0 \text{ or } 1 \quad \text{for all } i \text{ and } \phi.$$

Regarding Step 11, by resolving the preceding model, we determined that the optimal objective value 3.0498, $\hat{\psi}_1^4 = \hat{\psi}_2^3 = \hat{\psi}_3^1 = \hat{\psi}_4^2 = \hat{\psi}_5^5 = 1$, and the remaining $\hat{\psi}_i^\phi = 0$. Using (49), this study multiplied A by $\hat{\Psi}$ to produce the optimal consensus ranking, that is:

$$A \cdot \hat{\Psi} = \left(a_1, a_2, a_3, a_4, a_5\right) \cdot \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \left(a_3, a_4, a_2, a_1, a_5\right).$$

Based on the above consequences, it straightly came to a conclusion of $a_3 \succ a_4 \succ a_2 \succ a_1 \succ a_5$, which yields the optimal consensus ranking of the available financing policies. Herein, such a consensus ranking enjoys the least disagreement regarding the outcomes of all criterion-wise predominating rankings. Additionally, the balanced policy (a_3) is the most proper financing policy for Linkou CGMH. Notably, outcome yield was concordant with the ranking result obtained by Wang and Chen (2018).

6.2 Exploration via comparative analyses

This subsection performs a sensitivity examination and implements a comparative investigation to verify the application consequences and manifest the helpfulness and strong points of the advanced PF likelihood-based consensus ranking methodology. Figure 6 offers a quick recap of the focal points in connection with an effectiveness analysis, two sensitivity analyses, and a comparative analysis. Firstly, in the effectiveness analysis, this subsection investigates the application results via degrees of appropriateness. Secondly, this subsection executes two sensitivity studies and conducts comparative analyses to scrutinize the effects of allocation parameters. Finally, this subsection makes comprehensive comparisons with the PF technique for order preference by similarity to ideal solution (TOPSIS) for justifying the efficacity and merits about the evolved methodology.

Firstly, this paper performs an effectiveness analysis to explore the results obtained in the financing problem. Thus, this paper introduces the degree of appropriateness $V^c(\Delta_i^{\phi})$ concerning an alternative $a_i \in A$ to be designated to a consensus rank ϕ, in which:

$$V^c(\Delta_i^{\phi}) = 1 - V(\Delta_i^{\phi}), \tag{50}$$

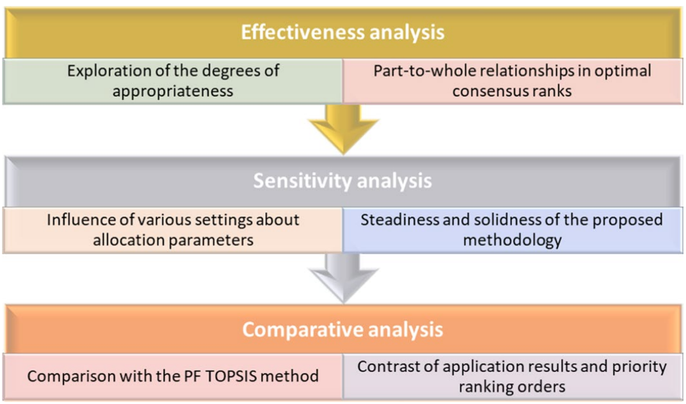


Fig. 6 Focal points of the effectiveness, sensitivity, and comparative analyses

where $i, \phi \in \{1, 2, \dots, 5\}$. The appropriateness degree $V^c(\Delta_i^\phi)$ is a complementary concept of the comprehensive disagreement index $V(\Delta_i^\phi)$. Considering (T10.1), it is easy to see that $0 \le V^c(\Delta_i^\phi) \le 1$.

Based on the obtained $V^c(\Delta_i^\phi)$ values, this paper presents a 100% stacked bar chart in Fig. 7 to show part-to-whole changes from first place (i.e., $\phi=1$) to fifth place (i.e., $\phi=5$) over the five alternatives. This chart was sketched to demonstrate the relative percentage of multiple obtained outcomes of $V^c(\Delta_i^\phi)$ in stacked bars, in which the total of each stacked bar (i.e., cumulative percentage) is always equal to 100%. Accordingly, this chart can represent a central tendency measure of relative fittingness that conforms to an accepted standard via the smallest value of comprehensive disagreement indices over all consensus ranks. In addition to the part-to-whole relationships, Fig. 7 shows how the proportions change over five priority places (for $\phi = 1, 2, \dots, 5$) in various consensus ranks occupied by the five financing policies.

As stated previously, the degree of appropriateness $V^c(\Delta_i^\phi)$ is the counter version of the comprehensive disagreement index $V(\Delta_i^\phi)$. In this regard, the larger the $V^c(\Delta_i^\phi)$ is, the better the concordance would be from designating the alternative a_i to the ϕth consensus rank. Figure 7 depicts the part-to-whole relationships of the degrees of appropriateness among the five financing policies. As shown in this chart, for example, the conservative-leaning policy (a_4) holds the highest relative proportion in the second priority place (i.e., $\phi=2$). That being the case, the alternative a_4 is the second ranked with respect to the optimal consensus ranking $a_3 \succ a_4 \succ a_2 \succ a_1 \succ a_5$. Nonetheless, it is worthwhile to mention that the balanced policy (a_3) enjoys the highest relative proportions in the first place (i.e., $\phi=1$) and the third place (i.e., $\phi=3$). One may wonder why it is the best choice instead of the third one in the light of the obtained results. The main reason is that the alternative a_3 occupies the largest relative proportion in the best ranked position. To be specific, it is easily known that $V^c(\Delta_3^3) > V^c(\Delta_3^1)$ because $V^c(\Delta_3^1) = 1 - V(\Delta_3^1) = 0.3471$ and $V^c(\Delta_3^3) = 1 - V(\Delta_3^3) = 0.4975$. Moreover, the other degrees of appropriateness were derived as follows: $V^c(\Delta_3^2) = 0.4142$, $V^c(\Delta_3^4) = 0.2559$, and $V^c(\Delta_3^5) = 0.1329$. This indicates that $\max_{\phi=1}^{5} V^c(\Delta_3^\phi) = V^c(\Delta_3^3)$. Based on the comparison of the degrees of appropriateness, it seems to reasonably figure out the third priority rank of a_3. From this perspective, why is it ranked the first place in the optimal consensus ranking? The 100 stacked bar chart in Fig. 7 can make this precise. This chart shows stacked bars normalized to 100%; thus, the relative

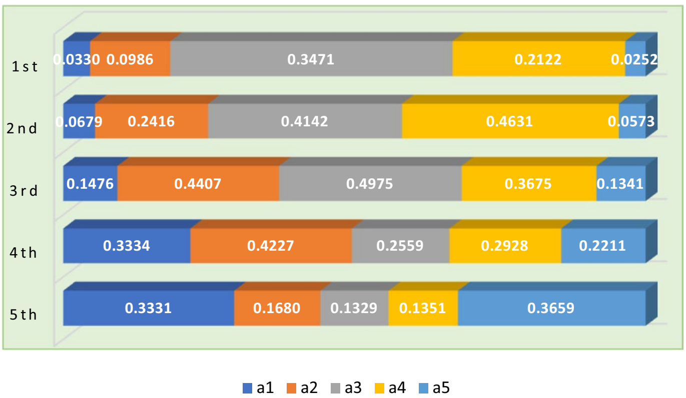


Fig. 7 Part-to-entire relationships about the degree of appropriateness $V^c(\Delta_i^\phi)$

dominance can be clearly differentiated through the agency of the relative proportions in the part-to-whole relationships of the $V^c(\Delta_i^{\phi})$ values. In first place, alternative a_3 occupies the significantly largest proportion compared to the other four financing policies; thus, it is plausible and convincing to conclude the best rank of a_3 is the optimal consensus ranking.

Next, this paper implements two sensitivity analyses and yields certain comparisons and discussions to explore the influences of distinct settings concerning the allocation parameters α_{ij} and β_{ij} on the results. As demonstrated in Theorems 1–4, the allocation parameters α_{ij} and β_{ij} have an impact on the adaptational consequences by the agency of the PF point operators M_α and N_β, respectively. On account of this, two arrangement schemes were devised for attaining some particular settings of α_{ij} and β_{ij} into effect, consisting of $\alpha_{ij}=\beta_{ij}$ and $\alpha_{ij}+\beta_{ij}=1$.

The first sensitivity analysis was applied to the subject of the arrangement scheme of $\alpha_{ij}=\beta_{ij}$. Identical values from 0.1 to 1.0 were assigned to the allocation parameters α_{ij} and β_{ij}. The pair of allocation parameters $(\alpha_{ij}, \beta_{ij})=(0.1, 0.1), (0.2, 0.2), \cdots, (1.0, 1.0)$ (i.e., ten instances) was examined. For each p_{ij}, the upper estimation $M_{\alpha_{ij}}(p_{ij})$ and the lower estimation $N_{\beta_{ij}}(p_{ij})$ were determined via the PF point operators M_α and N_β, respectively, under a specific setting of $(\alpha_{ij}, \beta_{ij})$. On the strength of the obtained $M_{\alpha_{ij}}(p_{ij})$ and $N_{\beta_{ij}}(p_{ij})$, this paper derived the PF likelihood measure $Lik(p_{ij} \succeq p_{kj})$ in each PF likelihood matrix LIK_j $(= [Lik(p_{ij} \succeq p_{kj})]_{5\times 5})$, wherein $i, k = 1, 2, \cdots, 5$ and $j = 1, 2, \cdots, 6$. By employing Definition 12, this paper ascertained the penalty weight $Pen(p_{ij})$ in various $(\alpha_{ij}, \beta_{ij})$ settings from (0.1,0.1) to (1.0,1.0) in conjunction with each criterion $c_j \in C$. A comparison of the consequences through the sensitivity analysis involving the penalty weights is shown in Fig. 8, with the results compared to criteria c_1–c_6 are presented in Fig. 8a–f. Based on the increasing order of $Pen(p_{ij})$ values, the criterion-wise predominate rank Φ_{ij} can be identified in connection with each c_j. As shown in Fig. 8a, b, d, consistent criterion-wise predominate ranking results were obtained in regard to criteria c_1, c_2, and c_4. The criterion-wise predominate ranks were as follows: $\Phi_{1j}=5$, $\Phi_{2j}=4$, $\Phi_{3j}=3$, $\Phi_{4j}=2$, and $\Phi_{5j}=1$ for $j=1$, 2, and 4 based on the ten settings of the $(\alpha_{ij}, \beta_{ij})$ pair. Considering Fig. 8c, it is known that $\Phi_{13}=5$, $\Phi_{23}=4$, $\Phi_{33}=1$, $\Phi_{43}=2$, and $\Phi_{53}=3$ are because of criterion c_3, which shows a difference between the criterion-wise predominate ranks Φ_{3j} and Φ_{5j}. In Fig. 8e, f, the following results were generated under the ten settings of the $(\alpha_{ij}, \beta_{ij})$ pair: $\Phi_{15}=1$, $\Phi_{25}=2$, $\Phi_{35}=3$, $\Phi_{45}=4$, and $\Phi_{55}=5$ with respect to c_5; $\Phi_{16}=4$, $\Phi_{26}=3$, $\Phi_{36}=1$, $\Phi_{46}=2$, and $\Phi_{56}=5$ with respect to c_6. Notably, identical criterion-wise predominate ranks were rendered for all $(\alpha_{ij}, \beta_{ij})$ pairs with respect to a specific criterion, which indicates that the contrasting results of the $Pen(p_{ij})$ values are consistent and stable. The developed approach can yield believable and conceivable criterion-wise comparison results and facilitate the determination of reliable comprehensive disagreement indicators.

Furthermore, this paper implemented a sensitivity analysis of the comprehensive disagreement index $V(\Delta_i^{\phi})$ with various $(\alpha_{ij}, \beta_{ij})$ settings from (0.1, 0.1) to (1.0, 1.0). Comparisons with respect to distinct priority ranks are shown in Fig. 9. By solving the PF likelihood-based consensus ranking model in each $(\alpha_{ij}, \beta_{ij})$ setting, a consistent optimal consensus ranking $a_3 \succ a_4 \succ a_2 \succ a_1 \succ a_5$ toward the five financing policies was determined, which indicates the optimal consensus ranks with trustworthy qualities. Considering that the balanced policy a_3 has the highest rank for all $(\alpha_{ij}, \beta_{ij})$ pairs, this paper uses a line graph to draw the outline of the obtained $V(\Delta_3^{\phi})$ values to highlight the comprehensive disagreement index $V(\Delta_3^{\phi})$ of a_3, while the other results of $V(\Delta_1^{\phi})$, $V(\Delta_2^{\phi})$, $V(\Delta_4^{\phi})$, and $V(\Delta_5^{\phi})$ were sketched as bar charts. The smaller the $V(\Delta_i^{\phi})$ is, the less discordance is generated from assigning the alternative a_i to the ϕ-th consensus rank. Figure 9a exhibits the contrast

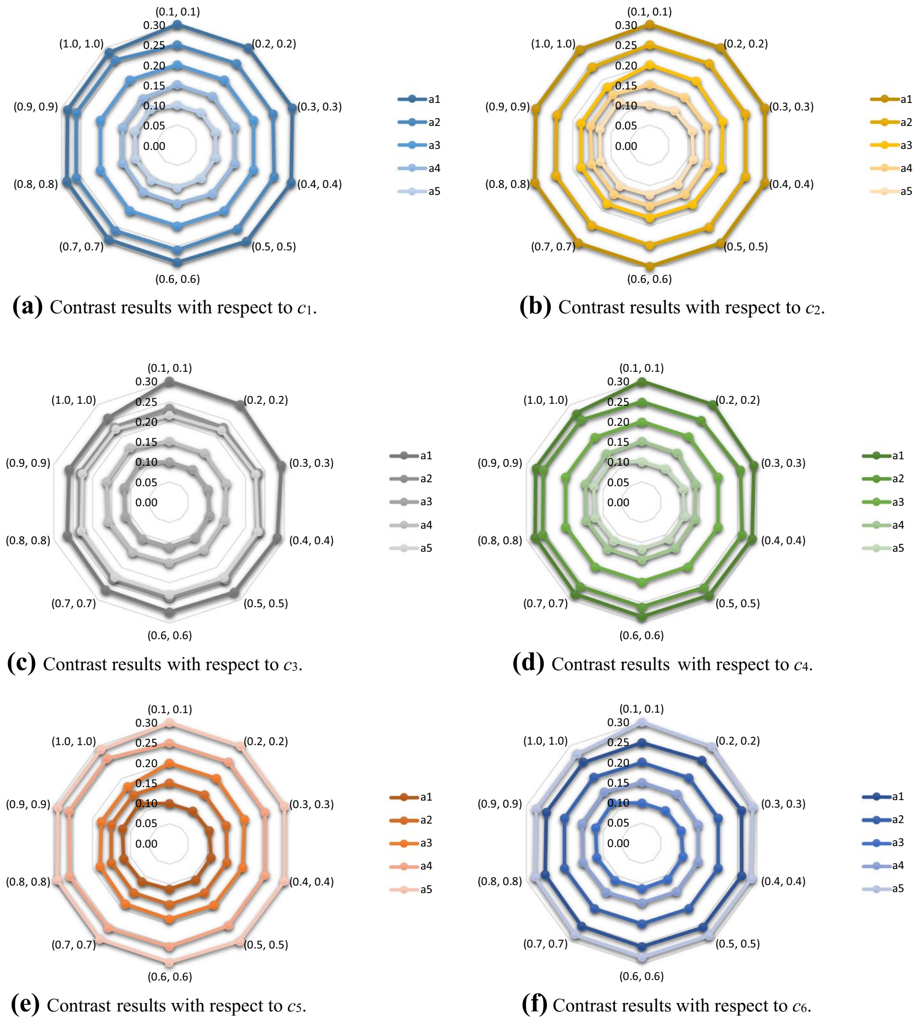
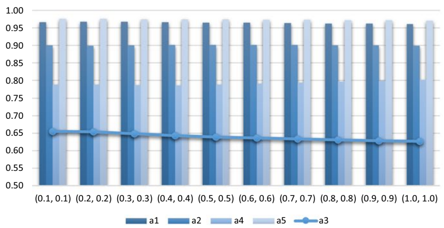
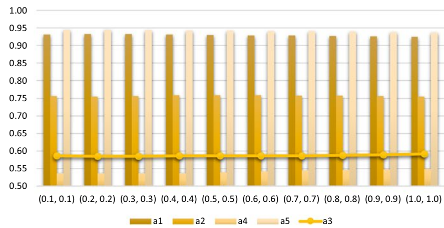


Fig. 8 Sensitivity analysis of the penalty weight $Pen(p_{ij})$ in various $(\alpha_{ij}, \beta_{ij})$ settings from (0.1, 0.1) to (1.0, 1.0)

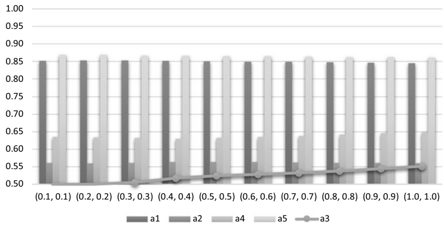
outcomes of the comprehensive disagreement indices of the five financing policies in connection with first place (i.e., $\phi = 1$). Notably, the $V(\Delta_3^1)$ obtained is significantly lower than the other indices $V(\Delta_1^1)$, $V(\Delta_2^1)$, $V(\Delta_4^1)$, and $V(\Delta_5^1)$, which implies the least discordance of a_3 with first place. As shown in Fig. 9e, alternative a_5 has the lowest comprehensive disagreement index because the obtained $V(\Delta_5^5)$ is the smallest among all the $V(\Delta_i^5)$ values in fifth place (i.e., $\phi = 5$). It is no wonder that a_3 is the top-ranked alternative, while a_5 is ranked last. Comparisons of the $V(\Delta_i^\phi)$ values from the first priority ranking to the fifth priority ranking are shown in Fig. 9a–e. The alternatives having the least disagreement in each priority ranking were identified as follows: a_3 in first place, a_4 in second place, a_3 in third place, a_2 in fourth place, and a_5 in fifth place in all $(\alpha_{ij}, \beta_{ij})$ settings from (0.1, 0.1) to (1.0, 1.0). The alternatives having the highest disagreement in each priority ranking were arranged as follows: a_5 in first place, a_5 in second place, a_5 in third place, a_5 in fourth



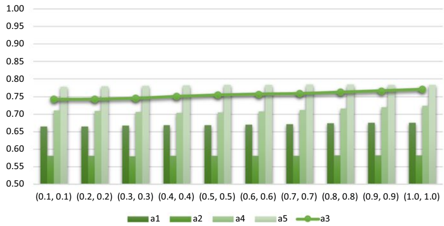
(a) Results in first place (i.e., ϕ=1).



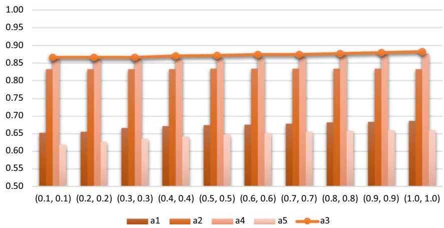
(b) Results in second place (i.e., ϕ=2).



(c) Results in third place (i.e., ϕ=3).



(d) Results in fourth place (i.e., ϕ=4).

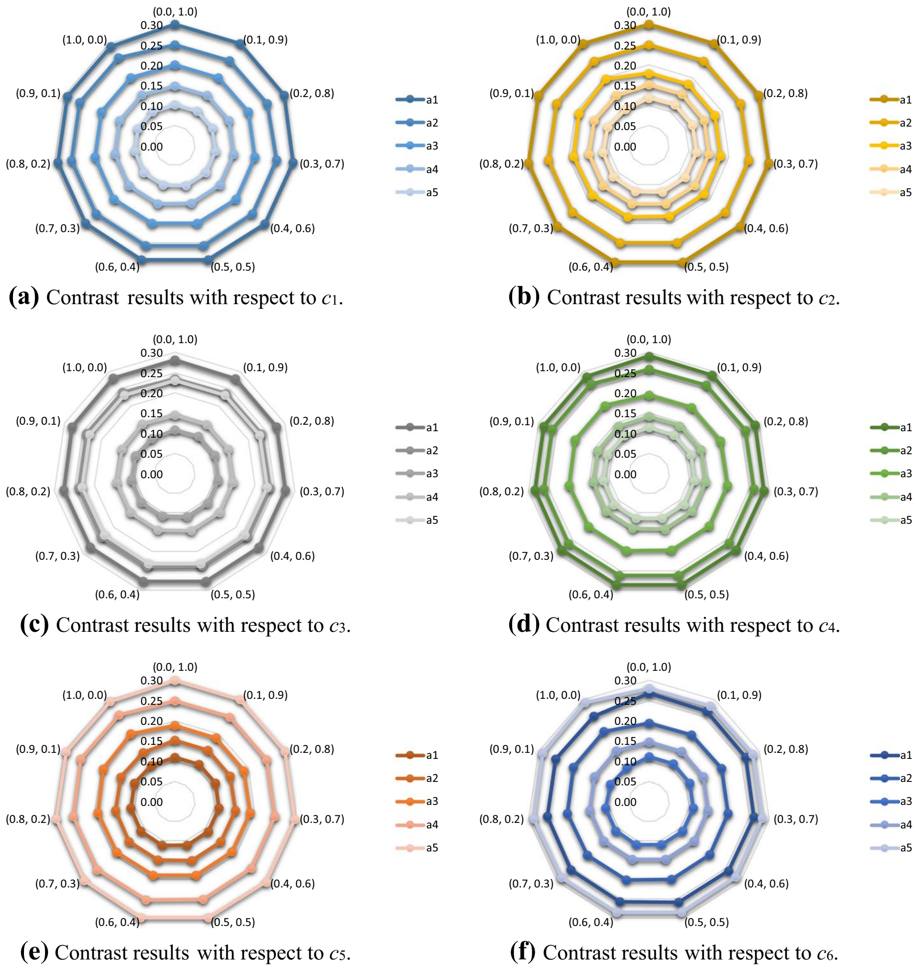


(e) Results in fifth place (i.e., ϕ=5).

Fig. 9 Sensitivity analysis of the comprehensive disagreement index $V(\Delta_i^{\phi})$ in various $(\alpha_{ij}, \beta_{ij})$ settings from (0.1, 0.1) to (1.0, 1.0)

place, and a_3 in fifth place for all $(\alpha_{ij}, \beta_{ij})$ pairs. Due to these consequences, the comprehensive disagreement indices produced by the proposed methodology are consistent with the optimal consensus ranking results. The first sensitivity analysis demonstrated the stability and solidness of the results obtained under various $(\alpha_{ij}, \beta_{ij})$ settings.

The second sensitivity analysis focused on the prerequisite of a fixed sum of the two allocation parameters α_{ij} and β_{ij}. Specifically, this paper examined the influences of various settings concerning the arrangement scheme $\alpha_{ij}+\beta_{ij}=1$ on the solution outcomes. This sensitivity analysis explored the solution results under the assigned values of α_{ij} and β_{ij} from 0.0 to 1.0 and from 1.0 to 0.0, respectively, when determining the upper estimation $M_{\alpha_{ij}}(p_{ij})$ and the lower estimation $N_{\beta_{ij}}(p_{ij})$ using the PF point operators M_α and N_β, respectively. That is, this study investigated eleven instances containing the pairs $(\alpha_{ij}, \beta_{ij})=(0.0, 1.0),(0.1, 0.9), \cdots, (1.0, 0.0)$ with the condition $\alpha_{ij}+\beta_{ij}=1$. Using the proposed methodology to tackle the financing problem with respect to working capital policies, the comparison of the results of the penalty weight $Pen(p_{ij})$ in various $(\alpha_{ij}, \beta_{ij})$ settings from (0.0, 1.0) to (1.0, 0.0) with reference to each criterion $c_j \in C$ is shown in Fig. 10. In particular, the results connected with criteria c_1–c_6 are shown in Fig. 10a–f, respectively. Following the increasing order of $Pen(p_{ij})$ values, this paper determined the criterion-wise predominate rank Φ_{ij} in terms of six criteria. Overall, the



(a) Contrast results with respect to c_1.

(b) Contrast results with respect to c_2.

(c) Contrast results with respect to c_3.

(d) Contrast results with respect to c_4.

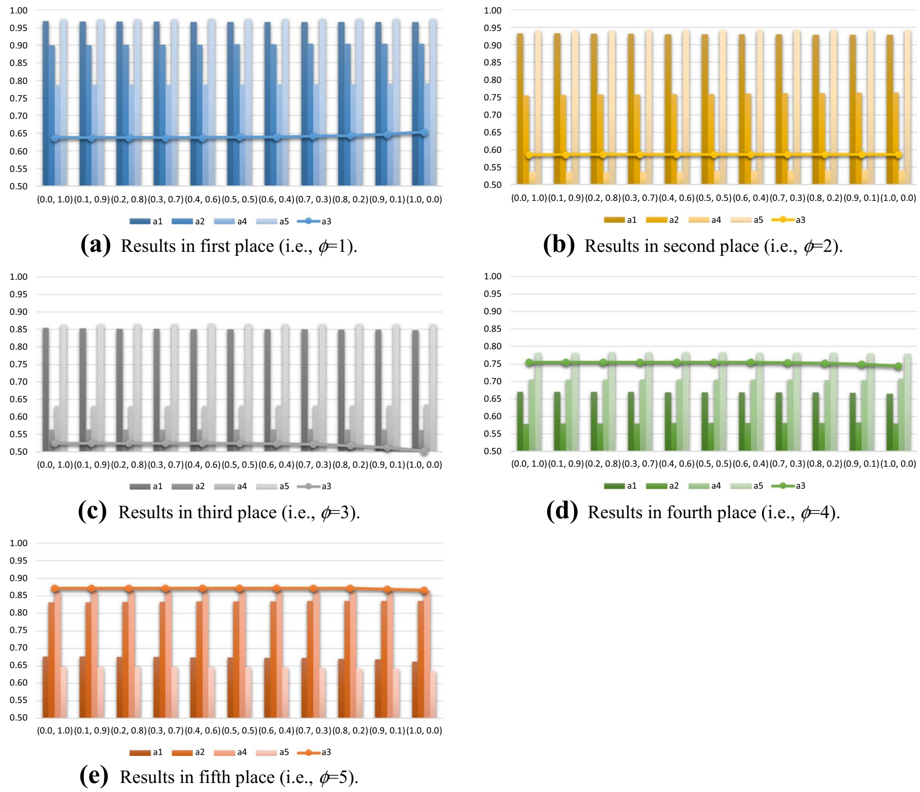
(e) Contrast results with respect to c_5.

(f) Contrast results with respect to c_6.

Fig. 10 Sensitivity analysis of the penalty weight $Pen(p_{ij})$ in various $(\alpha_{ij}, \beta_{ij})$ settings from (0.0, 1.0) to (1.0, 0.0)

consequences are similar to those of the first sensitivity analysis. A consistent consequence of the criterion-wise predominate rankings was rendered as shown in Fig. 10a, b, d (i.e., $\Phi_{1j}=5$, $\Phi_{2j}=4$, $\Phi_{3j}=3$, $\Phi_{4j}=2$, and $\Phi_{5j}=1$ for $c_j \in \{c_1, c_2, c_4\}$ under the eleven settings of the $(\alpha_{ij}, \beta_{ij})$ pair). Based on Fig. 10c, e, f, the following results were obtained: $\Phi_{13}=5$, $\Phi_{23}=4$, $\Phi_{33}=1$, $\Phi_{43}=2$, and $\Phi_{53}=3$ for criterion c_3, $\Phi_{15}=1$, $\Phi_{25}=2$, $\Phi_{35}=3$, $\Phi_{45}=4$, and $\Phi_{55}=5$ for c_5; $\Phi_{16}=4$, $\Phi_{26}=3$, $\Phi_{36}=1$, $\Phi_{46}=2$, and $\Phi_{56}=5$ for c_6. The same criterion-wise predominate rankings were generated for all $(\alpha_{ij}, \beta_{ij})$ pairs with regard to a specific criterion. Thus, the results of the $Pen(p_{ij})$ values are consistent and stable under the eleven settings of $(\alpha_{ij}, \beta_{ij})$ from (0.0, 1.0) to (1.0, 0.0).

With regard to the sensitivity analysis of the comprehensive disagreement index $V(\Delta_i^{\phi})$ in the eleven instances of the $(\alpha_{ij}, \beta_{ij})$ pair, Fig. 11 summarizes comparisons of the connections among the five priority rankings. An identical optimal consensus



(a) Results in first place (i.e., ϕ=1).

(b) Results in second place (i.e., ϕ=2).

(c) Results in third place (i.e., ϕ=3).

(d) Results in fourth place (i.e., ϕ=4).

(e) Results in fifth place (i.e., ϕ=5).

Fig. 11 Sensitivity analysis of the comprehensive disagreement index $V(\Delta_i^{\phi})$ in various $(\alpha_{ij}, \beta_{ij})$ settings from (0.0, 1.0) to (1.0, 0.0)

ranking $a_3 \succ a_4 \succ a_2 \succ a_1 \succ a_5$ toward the five alternatives was rendered by solving the PF likelihood-based consensus ranking model in each $(\alpha_{ij}, \beta_{ij})$ setting, which is analogous to that of the first sensitivity analysis. In a similar vein, the results of the comprehensive disagreement index $V(\Delta_3^{\phi})$ for optimal a_3 (i.e., a balanced policy) are highlighted with a line graph, while the other results of $V(\Delta_1^{\phi})$, $V(\Delta_2^{\phi})$, $V(\Delta_4^{\phi})$, and $V(\Delta_5^{\phi})$ are shown as bar charts. Figure 11a–e shows comparisons of the comprehensive disagreement indices $V(\Delta_1^{\phi}), V(\Delta_2^{\phi}), \cdots, V(\Delta_5^{\phi})$. Concerning first place in Fig. 11a, the $V(\Delta_3^1)$ values are the smallest comprehensive disagreement indices for each instance of the $(\alpha_{ij}, \beta_{ij})$ pair, which justifies assigning the first consensus rank to a_3. In contrast, alternative a_5, which ranked last in the consensus results, has the largest $V(\Delta_5^1)$ value. The smallest and largest comprehensive disagreement indices were $V(\Delta_4^2)$ and, in second place; $V(\Delta_3^3)$ and $V(\Delta_5^3)$, in third place; $V(\Delta_2^4)$ and $V(\Delta_5^4)$ in fourth place; and $V(\Delta_5^5)$ and $V(\Delta_3^5)$ in fifth place for all $(\alpha_{ij}, \beta_{ij})$ pairs. These findings can support the rationality of the consequences produced by the proposed model and techniques. The top-ranked alternative a_3 exhibited the smallest $V(\Delta_3^1)$ and the largest $V(\Delta_3^5)$; in contrast, the last-ranked alternative a_5 had the smallest $V(\Delta_5^5)$ and the largest $V(\Delta_5^1)$. The second sensitivity analysis also confirmed the applicability and robustness of the resolution results generated by the proposed methodology.

Finally, this paper performs comparative research with PF TOPSIS to justify the productiveness and correctness of the proposed methodology. The TOPSIS method, which is commonly utilized in a field of multiple criteria evaluation and decision analysis, is established on the rationale that the selected alternative ought to possess the shortest and longest distances towards the approach anchor point (i.e., positive-ideal choice) and the avoidance anchor point (i.e., negative-ideal choice), respectively. Using the agency of the kernel structure of TOPSIS, this study would like to advance a PF version of the TOPSIS approach for providing the facilitations of comparative analyses and demonstrating the justifiability of the current PF likelihood-based consensus ranking method through the agency of the likelihood-based linear assignment model. In order to adapt to the uncertainty with Pythagorean fuzziness, this paper presents a directly-extended PF TOPSIS procedure with the aim of tackling appropriately MCDA affairs in PF circumstances. Notice that the classical TOPSIS considers the proximity to the approach anchor point and the remoteness from the avoidance anchor point to define the closeness coefficient. The Hamming distances between PF information would be employed to determine the separation of differentiating an alternative from the ideal solutions. Following such a rationale behind the discussion above, the closeness coefficient can be acquired based on the obtained separation measures with the purpose of acquiring priority rankings among competing alternatives.

Place the positive-ideal choice a_* and the negative-ideal choice $a_\#$. Let two PF sets P_* and $P_\#$ signify the PF characteristics of a_* and $a_\#$, respectively, over all n criteria; these are in such manner:

$$P_* = \left\{ \langle c_j, p_{*j} \rangle \middle| c_j \in C \right\} = \left\{ \langle c_j, (\mu_{*j}, \nu_{*j}; r_{*j}, d_{*j}) \rangle \middle| c_j \in C \right\} = \left\{ \langle c_j, (1, 0; 1, 1) \rangle \middle| c_j \in C \right\}, \tag{51}$$

$$P_\# = \left\{ \langle c_j, p_{\#j} \rangle \middle| c_j \in C \right\} = \left\{ \langle c_j, (\mu_{\#j}, \nu_{\#j}; r_{\#j}, d_{\#j}) \rangle \middle| c_j \in C \right\} = \left\{ \langle c_j, (0, 1; 1, 0) \rangle \middle| c_j \in C \right\}. \tag{52}$$

In the PF TOPSIS method, one must determine the weighted PF evaluative rating $p_{ij}^w = (\mu_{ij}^w, \nu_{ij}^w; r_{ij}^w, d_{ij}^w)$ of $a_i \in A$ with regard to $c_j \in C$. The two-dimensional representation $p_{ij}^{\prime w} = (\mu_{ij}^w, \nu_{ij}^w)$ associated with each p_{ij}^w is calculated as follows:

$$p_{ij}^{\prime w} = w_j' \otimes p_{ij}' = (\omega_j, \varpi_j) \otimes (\mu_{ij}, \nu_{ij}) = \left(\omega_j \cdot \mu_{ij}, \sqrt{(\varpi_j)^2 + (\nu_{ij})^2 - (\varpi_j)^2 \cdot (\nu_{ij})^2} \right). \tag{53}$$

The indeterminacy grade pertaining to p_{ij}^w is derived as $\tau_{ij}^w = \sqrt{1 - (\mu_{ij}^w)^2 - (\nu_{ij}^w)^2}$. Let $p_{*j}^w = (\mu_{*j}^w, \nu_{*j}^w; r_{*j}^w, d_{*j}^w)$ and $p_{\#j}^w = (\mu_{\#j}^w, \nu_{\#j}^w; r_{\#j}^w, d_{\#j}^w)$ represent the weighted PF evaluative ratings of a_* and $a_\#$, respectively, pertaining to c_j. The two-dimensional representations $p_{*j}^{\prime w} = (\mu_{*j}^w, \nu_{*j}^w)$ and $p_{\#j}^{\prime w} = (\mu_{\#j}^w, \nu_{\#j}^w)$ are separately acquired as $p_{*j}^{\prime w} = w_j' \otimes p_{*j}' = (\omega_j, \varpi_j) \otimes (1, 0) = (\omega_j, \varpi_j)$ and $p_{\#j}^{\prime w} = w_j' \otimes p_{\#j}' = (\omega_j, \varpi_j) \otimes (0, 1) = (0, 1)$. The indeterminacy grades pertaining to p_{*j}^w and $p_{\#j}^w$ are separately identified using $\tau_{*j}^w = \sqrt{1 - (\mu_{*j}^w)^2 - (\nu_{*j}^w)^2}$ and $\tau_{\#j}^w = \sqrt{1 - (\mu_{\#j}^w)^2 - (\nu_{\#j}^w)^2}$.

A variety of distance measurement functions have been presented in the PF context; however, the Hamming distance model is an easy and effectual method that can calculate the distance for PF information. Let $D^w(P_i, P_*)$ and $D^w(P_i, P_\#)$ denote the separation measures towards the PF characteristics P_i and P_* and towards P_i and $P_\#$, respectively. By use of the weighted Hamming distance model, they are generated on this wise:

$$D^w(P_i, P_*) = \frac{1}{n}\sum_{j=1}^{n} D(p_{ij}^w, p_{*j}^w) = \frac{1}{2n}\sum_{j=1}^{n}\left(\left|(\mu_{ij}^w)^2 - (\mu_{*j}^w)^2\right| + \left|(\nu_{ij}^w)^2 - (\nu_{*j}^w)^2\right| + \left|(\tau_{ij}^w)^2 - (\tau_{*j}^w)^2\right|\right), \tag{54}$$

$$D^w(P_i, P_\#) = \frac{1}{n}\sum_{j=1}^{n} D(p_{ij}^w, p_{\#j}^w) = \frac{1}{2n}\sum_{j=1}^{n}\left(\left|(\mu_{ij}^w)^2 - (\mu_{\#j}^w)^2\right| + \left|(\nu_{ij}^w)^2 - (\nu_{\#j}^w)^2\right| + \left|(\tau_{ij}^w)^2 - (\tau_{\#j}^w)^2\right|\right). \tag{55}$$

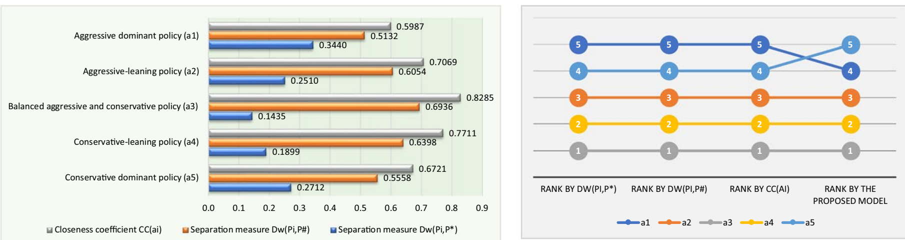
The closeness coefficient $CC(a_i)$ $(0 \le CC(a_i) \le 1)$ of each alternative a_i is calculated as follows:

$$CC(a_i) = \frac{D^w(P_i, P_\#)}{D^w(P_i, P_*) + D^w(P_i, P_\#)}, \tag{56}$$

where $CC(a_i) = 0$ when $P_i = P_\#$, and $CC(a_i) = 1$ when $P_i = P_*$. In the PF TOPSIS procedure, the priority ranking among all alternatives is acquired on the basis of the closeness coefficients in descending order. The alternative with the maximum $CC(a_i)$ is the most proper choice.

We now consider the same financing problem. Figure 12a shows the application results, which consist of the outcomes of $D^w(P_i, P_*)$, $D^w(P_i, P_\#)$, and $CC(a_i)$, that are produced by the PF TOPSIS method. As shown in the descending order of $CC(a_i)$, the priority ranking order is $a_3 \succ a_4 \succ a_2 \succ a_5 \succ a_1$, which shows that the most appropriate choice is the balanced policy a_3. The preference orders based on the separation measures $D^w(P_i, P_*)$ in ascending order and based on $D^w(P_i, P_\#)$ in descending order are the same as those obtained based on $CC(a_i)$. Figure 12b highlights the contrast of the ranking orders using PF TOPSIS and the developed PF likelihood-based consensus ranking method.

The priority ranking outcomes determined using the PF TOPSIS method are somewhat different from the optimal consensus ranking $a_3 \succ a_4 \succ a_2 \succ a_1 \succ a_5$ produced by the proposed method. The differentiation focuses on the dominance relationship between a_1 and a_5. Due to the PF importance weights of six criteria offered by the authority of Linkou CGMH, the two most important criteria are financing costs [PF weight $w_6 = (0.85, 0.15; 0.86, 0.89)$] and return on assets [PF weight $w_5 = (0.75, 0.25; 0.79, 0.80)$], which indicates that the authority attaches great importance to the criteria relevant to profitability. Considering that the target of this model is the largest medical center in Taiwan, a good financial condition is a necessity for Linkou CGMH. Compared to conservative financing policies (e.g., the conservative dominant and conservative-leaning policies), aggressive financing policies (e.g., the aggressive dominant and aggressive-leaning policies) is the most befitting



(a) Solution outcomes rendered by the PF TOPSIS. **(b)** Priority orders compared to the proposed model.

Fig. 12 Comparisons with PF TOPSIS in the financing problem for Linkou CGMH. As expected, the dominance relationship of $a_1 \succ a_5$ is acceptable. The proposed PF likelihood-based consensus ranking method rendered desirable results (i.e., $a_1 \succ a_5$) according to the optimal consensus ranking $a_3 \succ a_4 \succ a_2 \succ a_1 \succ a_5$. However, the PF TOPSIS method generated conflicting dominance relationships (i.e., $a_5 \succ a_1$) based on the preference ranking $a_3 \succ a_4 \succ a_2 \succ a_5 \succ a_1$. Based on the foregoing discussions, the comparative analysis furnishes a supportable consequence for confirming the appositeness and reasonability of the optimal consensus ranking. Comparisons provide clear evidence of the helpfulness and superiority of the proposed model and relevant techniques compared to the PF TOPSIS method.

The effectuality and superiority of the current PF likelihood-based consensus ranking method have been justified and corroborated through illustrated applications and comprehensive comparative studies that consist of effectiveness, sensitivity, and comparative analyses. Based on the illustrative consequences, the usefulness and reasonableness of the advanced techniques have been verified and supported by the real-world demonstrations of a working capital financing problem. A comparison of the consequences and explorations has highlighted the justifiability and appositeness of the advanced methodology. Also, the solidness and steadiness of the obtained consequences yielded by the initiated methodology have been examined via the utility of two sensitivity studies. Also, the proposed comprehensive disagreement indicators and indices, as well as the likelihood-based linear assignment model, have generated believable and convincing results for ranking dominance relationships among alternatives within the decision environment under PF uncertainty.

7 Conclusion and future studies

This paper proposed a functional likelihood-based consensus ranking methodology to solve uncertain MCDA issues involving PF importance weights of criteria and PF evaluative ratings for externalizing alternative performances. This paper proposed two useful point operators via the medium of flexible allocation parameters to transform Pythagorean membership grades to establish rational upper and lower estimations with uncertain information. The concept of scalar functions along with the point operator-oriented adaptational outcomes of PF evaluative ratings provides a foundation for an effectual likelihood measure. Thus, this paper has taken advantage of the scalar functions of upper and lower estimations to develop a beneficial PF likelihood measure to exploit outranking relations with uncertain information. Certain functional concepts of priority weights and penalty weights have been proposed to provide a solid basis to determine the degree of each disagreement for alternatives in connection with specific consensus ranks. Considering the developed concepts, a likelihood-based assignment technique was constructed to ascertain a consensus ranking of competing alternatives and manage MCDA problems in PF uncertain circumstances. Specifically, considering the strengths of the PF likelihood measures and the penalty weights, this paper proposed a practical conception of comprehensive disagreement indicators and then established a comprehensive disagreement matrix whose entries measure the degree of disagreement through which an alternative is designated by a specific rank based on the resulting criterion-wise predominate relations of the alternatives. A pragmatic likelihood-based consensus ranking method elicited from a zero–one integer linear programming model has been constructed to assist decision makers in treating complex PF uncertain information and acquire believable and desirable MCDA outcomes. Finally, the justifiability and efficiency of the presented methodology have been critically validated and supported via real-world applications and comparative analyses.

This paper has made certain contributions to the research of MCDA methodology in PF uncertain circumstances. Specifically, the merits and achievements of this study are highlighted in the four aspects. For the first merit, this paper has put forward two PF point operators to estimate the adapted outcomes with respect to Pythagorean membership grades and identify rational upper and lower estimations relating to PF information. For the second merit, this paper has exploited the propounded PF point operators to launch a powerful and efficacious likelihood measurement that is helpful to produce a desired result concerning outranking relations for PF evaluation ratings. For the third merit, this paper has brought forward the idea of penalty weights that can delineate the extent to which a PF evaluative rating performs worse than all others with respect to each evaluative criterion. The employment of penalty weights can characterize dominated relations among PF characteristics and render criterion-wise predomination rankings for candidate alternatives. For the fourth merit, this paper has revealed comprehensive disagreement indicators and indices to formulate a workable likelihood-based consensus ranking model. Moreover, this study has proposed an algorithmic procedure that can be implemented quickly and effectively to acquire the optimal consensus ranking and tackle MCDA problems consisting of PF uncertain information.

However, one must recognize a realistic limitation exists on the proposed PF likelihood-based consensus ranking model. Based on Definition 11, it was recognized that the denominator of the PF likelihood measure $Lik(p_{ij} \succeq p_{kj})$ cannot have the value zero. On this basis, this study assumed that the differences between $V(M_{\alpha_{ij}}(p_{ij}))$ and $V(N_{\beta_{ij}}(p_{ij}))$ and between $V(M_{\alpha_{kj}}(p_{kj}))$ and $V(N_{\beta_{kj}}(p_{kj}))$ should not be equal to zero at the same time. Such an assumption is often valid in real-world decision circumstances. Nevertheless, once an extreme situation occurs, that is, when $V(M_{\alpha_{ij}}(p_{ij})) = V(N_{\beta_{ij}}(p_{ij}))$ and $V(M_{\alpha_{kj}}(p_{kj})) = V(N_{\beta_{kj}}(p_{kj}))$ occurs concurrently, the formula in Definition 11 will be meaningless. Fortunately, this extreme situation almost never happens. Within the limitation in Definition 11, the decision maker and analysts are still free to determine the PF likelihood measure of an outranking relation as they need.

Our recommendations for future research directions are fourfold: (1) exploring the appropriateness of distinct point operator-oriented adapted outcomes for identifying upper and lower estimations with PF information; (2) extending the applicability range of the proposed PF likelihood measures to characterizing the dominate relations and predominate rankings; (3) extending the proposed point operators and the PF point operator-oriented likelihood measure to interval-valued PF, q-rung orthopair fuzzy, single valued spherical fuzzy, and T-spherical fuzzy environments; and (4) aligning other relevant outranking-based methodologies through the utility of new measurements of comprehensive disagreement by ascertaining outranking relationships.

In particular, the developed approach can be generalized to decision contexts supported by the spherical fuzzy framework (Mahmood et al. 2019) that are subject to certain modifications. Give consideration to a single valued spherical fuzzy set (Kutlu Gündoğdu and Kahraman 2019), S, as an illustration. Let $\mu_S(x), \tau_S(x), \nu_S(x) : X \to [0, 1]$ behave the membership, indeterminacy, and nonmembership grades, respectively, of $x \in X$ to S, where $S = \{\langle x, (\mu_S(x), \tau_S(x), \nu_S(x))\rangle | x \in X\}$ with the axiom $0 \le (\mu_S(x))^2 + (\tau_S(x))^2 + (\nu_S(x))^2 \le 1$. Additionally, the refusal grade of x to S is derived as $r_S(x) = \sqrt{1 - (\mu_S(x))^2 - (\tau_S(x))^2 - (\nu_S(x))^2}$. Based on the definition in Ju et al. (2021), the score function of a spherical fuzzy value s is given by

$Sc(s) = [1 + (\mu_S(x))^2 - (\tau_S(x))^2 - (\nu_S(x))^2]/2$, where $Sc(s) \in [0, 1]$. Using the PF point operators M_α, the membership and nonmembership grades embedded in the upper estimation $M_\alpha(s)$ can be separately ascertained in this way: $\mu_{M_\alpha(S)}(x) = \sqrt{(\mu_S(x))^2 + \alpha(\tau_S(x))^2}$ and $\nu_{M_\alpha(S)}(x) = \nu_S(x)$. With a constant refusal degree, $\tau_{M_\alpha(S)}(x)^2 = \sqrt{(1-\alpha)(\tau_S(x))^2}$ is obtained. Using the PF point operator N_β, it is recognized that $\mu_{N_\beta(S)}(x) = \mu_S(x)$, $\nu_{N_\beta(S)}(x) = \sqrt{(\nu_S(x))^2 + \beta(\tau_S(x))^2}$, and $\tau_{N_\beta(S)}(x) = \sqrt{(1-\beta)(\tau_S(x))^2}$. Being able to adapt to the method to spherical fuzzy environments, this paper suggests using score functions instead of scalar functions to determine the likelihood measure in spherical fuzzy settings. For example, let s_{ij} and s_{kj} represent two evaluative ratings on the basis of the single valued spherical fuzzy model. Using the score function proposed by Ju et al. (2021), the likelihood measure in spherical fuzzy circumstances is determined as follows: $Lik(s_{ij} \succeq s_{kj}) = \max\{1 - \max\{[Sc(M_{\alpha_{kj}}(s_{kj})) - Sc(N_{\beta_{ij}}(s_{ij}))]/[(Sc(M_{\alpha_{ij}}(s_{ij})) - Sc(N_{\beta_{ij}}(s_{ij}))) + (Sc(M_{\alpha_{kj}}(s_{kj})) - Sc(N_{\beta_{kj}}(s_{kj})))], 0\}, 0\}$. Then, the notions of penalty weights and comprehensive disagreement indicators/indices can be derived to establish a likelihood-based consensus ranking model in spherical fuzzy contexts. As stated above, this method provides decision makers with an appropriate flexible adjustment, while the proposed approach can be used to generate more potential applicable scenarios in distinct fuzzy environments.

Appendix A: Detailed proofs

A.1 Proof of Theorem 1

(T1.1) In line with Definition 7, one obtains $(\mu_{M_\alpha(P)}(x))^2 + (\nu_{M_\alpha(P)}(x))^2 = (\mu_P(x))^2 + \alpha(\tau_P(x))^2 + (\nu_P(x))^2$. Because $(\mu_P(x))^2 + (\nu_P(x))^2 + (\tau_P(x))^2 = 1$, it can be recognized that $0 \le (\mu_{M_\alpha(P)}(x))^2 + (\nu_{M_\alpha(P)}(x))^2 \le 1$. The inequality $0 \le (\mu_{N_\beta(P)}(x))^2 + (\nu_{N_\beta(P)}(x))^2 \le 1$ is satisfied because $(\mu_{N_\beta(P)}(x))^2 + (\nu_{N_\beta(P)}(x))^2 = (\mu_P(x))^2 + (\nu_P(x))^2 + \beta(\tau_P(x))^2$ through Definition 8. Thus, (T1.1) is correct.

(T1.2) It is obvious that $\mu_P(x) \le \sqrt{(\mu_P(x))^2 + \alpha(\tau_P(x))^2}$ from $0 \le \alpha, \tau_P(x) \le 1$; thus, $\mu_{N_\beta(P)}(x) = \mu_P(x) \le \mu_{M_\alpha(P)}(x)$. Analogously, $\nu_{M_\alpha(P)}(x) = \nu_P(x) \le \nu_{N_\beta(P)}(x)$ is satisfied because $\nu_P(x) \le \sqrt{(\nu_P(x))^2 + \beta(\tau_P(x))^2}$. Thus, the correctness of (T1.2) is confirmed.

(T1.3) As stated in Definition 5, the quasi-ordering $M_\alpha(p) \succeq_Q p \succeq_Q N_\beta(p)$ holds because $\mu_{M_\alpha(P)}(x) \ge \mu_P(x) \ge \mu_{N_\beta(P)}(x)$ and $\nu_{M_\alpha(P)}(x) \le \nu_P(x) \le \nu_{N_\beta(P)}(x)$.

(T1.4) Applying Definitions 3, 7, and 8, the following is true:

$$\tau_{M_\alpha(P)}(x) = \sqrt{1 - (\mu_P(x))^2 - \alpha(\tau_P(x))^2 - (\nu_P(x))^2} \le \sqrt{1 - (\mu_P(x))^2 - (\nu_P(x))^2} = \tau_P(x),$$

$$\tau_{N_\beta(P)}(x) = \sqrt{1 - (\mu_P(x))^2 - (\nu_P(x))^2 - \beta(\tau_P(x))^2} \le \sqrt{1 - (\mu_P(x))^2 - (\nu_P(x))^2} = \tau_P(x).$$

Accordingly, this equation indicates that $\max\{\tau_{M_\alpha(P)}(x), \tau_{N_\beta(P)}(x)\} \le \tau_P(x)$. Based on Definition 3, it is known that $r_P(x) = \sqrt{1 - (\tau_P(x))^2}$, $r_{M_\alpha(P)}(x) = \sqrt{1 - (\tau_{M_\alpha(P)}(x))^2}$, and $r_{N_\beta(P)}(x) = \sqrt{1 - (\tau_{N_\beta(P)}(x))^2}$. It indicates that $\min\{r_{M_\alpha(P)}(x), r_{N_\beta(P)}(x)\} \ge r_P(x)$; thus, the property of (T1.4) holds.

(T1.5) As demonstrated in Chen (2018b), the closer the directions of a commitment (i.e., $d_P(x)$, $d_{M_\alpha(P)}(x)$, and $d_{N_\beta(P)}(x)$) are to 1 (or 0), the closer the radians (i.e., $\theta_P(x)$, $\theta_{M_\alpha(P)}(x)$, and $\theta_{N_\beta(P)}(x)$) are to 0 (or $\pi/2$) and the greater the strength of commitment (i.e., $r_P(x)$, $r_{M_\alpha(P)}(x)$, and $r_{N_\beta(P)}(x)$) is to the supporting (or disapproving) belongingness of x in P. Therefore, the results shown in (T1.2) indicate that $d_{N_\beta(P)}(x) \le d_P(x) \le d_{M_\alpha(P)}(x)$ and $\theta_{M_\alpha(P)}(x) \le \theta_P(x) \le \theta_{N_\beta(P)}(x)$ hold. The correctness of (T1.5) is confirmed.

(T1.6) Based on Definition 3, it is recognized that $p^c = (\mu_{P^c}(x), \nu_{P^c}(x); r_{P^c}(x), d_{P^c}(x)) = (\nu_P(x), \mu_P(x); r_P(x), 1 - d_P(x))$, which also indicates that $\tau_{P^c}(x) = \tau_P(x)$ from $r_{P^c}(x) = r_P(x)$. Applying Definition 7, the upper estimation $M_\alpha(p^c)$ of the complement of p is given by: $M_\alpha(p^c) = (\mu_{M_\alpha(P^c)}(x), \nu_{M_\alpha(P^c)}(x); r_{M_\alpha(P^c)}(x), d_{M_\alpha(P^c)}(x))$. The membership grade and the nonmembership grade in $M_\alpha(p^c)$ are separately derived as follows: $\mu_{M_\alpha(P^c)}(x) = \sqrt{(\mu_{P^c}(x))^2 + \alpha(\tau_{P^c}(x))^2} = \sqrt{(\nu_P(x))^2 + \alpha(\tau_P(x))^2} = \nu_{N_\alpha(P)}(x)$ and $\nu_{M_\alpha(P^c)}(x) = \nu_{P^c}(x) = \mu_P(x) = \mu_{N_\alpha(P)}(x)$. For the complement $(M_\alpha(p^c))^c$ of the upper estimation $M_\alpha(p^c)$, $\mu_{(M_\alpha(P^c))^c}(x) = \nu_{M_\alpha(P^c)}(x) = \mu_{N_\alpha(P)}(x)$ and $\nu_{(M_\alpha(P^c))^c}(x) = \mu_{M_\alpha(P^c)}(x) = \nu_{N_\alpha(P)}(x)$. One can verify that $(M_\alpha(p^c))^c = N_\alpha(p)$.

(T1.7) Based on Definition 8, the lower estimation $N_\beta(p^c)$ of the complement of p is obtained by $N_\beta(p^c) = (\mu_{N_\beta(P^c)}(x), \nu_{N_\beta(P^c)}(x); r_{N_\beta(P^c)}(x), d_{N_\beta(P^c)}(x))$. The membership and nonmembership grades in $N_\beta(p^c)$ are separately calculated as follows: $\mu_{N_\beta(P^c)}(x) = \mu_{P^c}(x) = \nu_P(x) = \nu_{M_\beta(P)}(x)$ and $\nu_{N_\beta(P^c)}(x) = \sqrt{(\nu_{P^c}(x))^2 + \beta(\tau_{P^c}(x))^2} = \sqrt{(\mu_P(x))^2 + \beta(\tau_P(x))^2} = \mu_{M_\beta(P)}(x)$. For the complement $(N_\beta(p^c))^c$ of the lower estimation $N_\beta(p^c)$, it is apparent that $\mu_{(N_\beta(P^c))^c}(x) = \nu_{N_\beta(P^c)}(x) = \mu_{M_\beta(P)}(x)$ and $\nu_{(N_\beta(P^c))^c}(x) = \mu_{N_\beta(P^c)}(x) = \nu_{M_\beta(P)}(x)$, which yields $(N_\beta(p^c))^c = M_\beta(p)$, i.e., (T1.7) is valid. Notably, (T1.6) and (T1.7) demonstrate that the two PF point operators M_α and N_β are dualities. This confirms the truth of Theorem 1.

A.2 Proof of Theorem 2

In the first place, (T2.2) is trivial. (T2.1) and (T2.3) are demonstrated by virtue of mathematical induction on η. Using the agency of the PF point operator M_α on the recurrent upper estimation $M_\alpha^{\eta-1}(p)$, it is generated that:

$$\mu_{M_\alpha^\eta(P)}(x) = \sqrt{(\mu_{M_\alpha^{\eta-1}(P)}(x))^2 + \alpha(\tau_{M_\alpha^{\eta-1}(P)}(x))^2},$$

$$\nu_{M_\alpha^\eta(P)}(x) = \nu_{M_\alpha^{\eta-1}(P)}(x) = \nu_P(x),$$

$$r_{M_\alpha^\eta(P)}(x) = \sqrt{(\mu_{M_\alpha^\eta(P)}(x))^2 + (\nu_{M_\alpha^\eta(P)}(x))^2} = \sqrt{(\mu_{M_\alpha^{\eta-1}(P)}(x))^2 + \alpha(\tau_{M_\alpha^{\eta-1}(P)}(x))^2 + (\nu_P(x))^2}.$$

for $\eta = 0, 1, 2, \ldots$, in which $\mu_{M_\alpha^0(P)}(x) = \mu_P(x)$, $\nu_{M_\alpha^0(P)}(x) = \nu_P(x)$, and $\tau_{M_\alpha^0(P)}(x) = \tau_P(x)$. Let $\eta = 1$. Based on Definition 7 and the condition that $(\tau_P(x))^2 = 1 - (\mu_P(x))^2 - (\nu_P(x))^2$, one can render:

$$\mu_{M_\alpha^1(P)}(x) = \sqrt{(\mu_P(x))^2 + \alpha(\tau_P(x))^2} = \sqrt{(\mu_P(x))^2 + \alpha\left(1 - (\mu_P(x))^2\right) - \alpha(\nu_P(x))^2},$$

$$r_{M_\alpha^1(P)}(x)=\sqrt{(\mu_P(x))^2+\alpha\big(1-(\mu_P(x))^2-(\nu_P(x))^2\big)+(\nu_P(x))^2}$$
$$=\sqrt{(\mu_P(x))^2+(\nu_P(x))^2+\alpha\big(1-(\mu_P(x))^2\big)-\alpha(\nu_P(x))^2}.$$

The above results are concordant with the outcome of $\eta=1$ in (T2.1) and (T2.3); that is, the two properties hold for $\eta=1$. Next, let $\eta=2$. It is evident to deduce that $\nu_{M_\alpha^2(P)}(x)=\nu_{M_\alpha^1(P)}(x)=\nu_P(x)$. By applying Definitions 7 and 9, it is straightly gained that:

$$\mu_{M_\alpha^2(P)}(x)=\Big\{(\mu_P(x))^2+\alpha\big(1-(\mu_P(x))^2\big)-\alpha(\nu_P(x))^2+\alpha\big[1-(\mu_P(x))^2-(\nu_P(x))^2-\alpha\big(1-(\mu_P(x))^2\big)$$
$$+\alpha(\nu_P(x))^2\big]\Big\}^{0.5}=\sqrt{(\mu_P(x))^2+\big(1-(\mu_P(x))^2\big)\big(2\alpha-\alpha^2\big)-\alpha(\nu_P(x))^2(2-\alpha)},$$

$$r_{M_\alpha^2(P)}(x)=\Big\{(\mu_P(x))^2+\alpha\big(1-(\mu_P(x))^2\big)-\alpha(\nu_P(x))^2+\alpha\big[1-(\mu_P(x))^2-(\nu_P(x))^2-$$
$$\alpha\big(1-(\mu_P(x))^2\big)+\alpha(\nu_P(x))^2\big]+(\nu_P(x))^2\Big\}^{0.5}=\sqrt{(\mu_P(x))^2+(\nu_P(x))^2+\big(1-(\mu_P(x))^2\big)\big(2\alpha-\alpha^2\big)-\alpha(\nu_P(x))^2(2-\alpha)}.$$

The results are consistent with the outcome of $\eta=2$ in (T2.1) and (T2.3); thus, the determination equations are valid in the case of $\eta=2$. Next, assume that (T2.1) and (T2.3) hold for $\eta=\vartheta$. Then,

$$\mu_{M_\alpha^\vartheta(P)}(x)=\sqrt{(\mu_P(x))^2+\big(1-(\mu_P(x))^2\big)\big(1-(1-\alpha)^\vartheta\big)-\alpha(\nu_P(x))^2\left(\sum_{k=0}^{\vartheta-1}(1-\alpha)^k\right)},$$

$$r_{M_\alpha^\vartheta(P)}(x)=\sqrt{(\mu_P(x))^2+(\nu_P(x))^2+\big(1-(\mu_P(x))^2\big)\big(1-(1-\alpha)^\vartheta\big)-\alpha(\nu_P(x))^2\left(\sum_{k=0}^{\vartheta-1}(1-\alpha)^k\right)},$$

and $\nu_{M_\alpha^\vartheta(P)}(x)=\nu_P(x)$. When $\eta=\vartheta+1$, one has $\nu_{M_\alpha^{\vartheta+1}(P)}(x)=\nu_P(x)$; moreover, by way of Definitions 7 and 9, it is yielded that:

$$\mu_{M_\alpha^{\vartheta+1}(P)}(x)=\sqrt{(\mu_{M_\alpha^\vartheta(P)}(x))^2+\alpha(\tau_{M_\alpha^\vartheta(P)}(x))^2}=\sqrt{(\mu_{M_\alpha^\vartheta(P)}(x))^2+\alpha\Big(1-(r_{M_\alpha^\vartheta(P)}(x))^2\Big)}$$
$$=\left\{(\mu_P(x))^2+\big(1-(\mu_P(x))^2\big)\big(1-(1-\alpha)^\vartheta\big)-\alpha(\nu_P(x))^2\left(\sum_{k=0}^{\vartheta-1}(1-\alpha)^k\right)\right.$$
$$\left.+\alpha\big(1-(\mu_P(x))^2\big)-\alpha(\nu_P(x))^2-\alpha\big(1-(\mu_P(x))^2\big)\big(1-(1-\alpha)^\vartheta\big)+\alpha^2(\nu_P(x))^2\left(\sum_{k=0}^{\vartheta-1}(1-\alpha)^k\right)\right\}^{0.5}$$
$$=\Big\{(\mu_P(x))^2+\big(1-(\mu_P(x))^2\big)\big[1-(1-\alpha)^\vartheta+\alpha-\alpha\big(1-(1-\alpha)^\vartheta\big)\big]$$
$$\left.+\alpha(\nu_P(x))^2\left(-\sum_{k=0}^{\vartheta-1}(1-\alpha)^k-1+\alpha\right)\right\}^{0.5}$$
$$=\sqrt{(\mu_P(x))^2+\big(1-(\mu_P(x))^2\big)\big(1-(1-\alpha)(1-\alpha)^\vartheta\big)-\alpha(\nu_P(x))^2\left((1-\alpha)+\sum_{k=0}^{\vartheta-1}(1-\alpha)^k\right)}$$
$$=\sqrt{(\mu_P(x))^2+\big(1-(\mu_P(x))^2\big)\big(1-(1-\alpha)^{\vartheta+1}\big)-\alpha(\nu_P(x))^2\left(\sum_{k=0}^{\vartheta}(1-\alpha)^k\right)},$$

$$r_{M_\alpha^{\vartheta+1}(P)}(x)=\sqrt{(\mu_P(x))^2+(\nu_P(x))^2+\big(1-(\mu_P(x))^2\big)\big(1-(1-\alpha)^{\vartheta+1}\big)-\alpha(\nu_P(x))^2\left(\sum_{k=0}^{\vartheta}(1-\alpha)^k\right)}.$$

On this basis, (T2.1) and (T2.3) hold for $\eta = \vartheta + 1$. Accordingly, the properties of (T2.1) and (T2.3) are fulfilled corresponding to all η values.

(T2.4) Based on (T2.1), the discrepancy between squared membership grades in the recurrent upper estimations $M_\alpha^\eta(p)$ and $M_\alpha^{\eta-1}(p)$ for all $\alpha \in [0,1]$ is calculated in this way:

$$\begin{aligned}
&(\mu_{M_\alpha^\eta(P)}(x))^2 - (\mu_{M_\alpha^{\eta-1}(P)}(x))^2 = (\mu_P(x))^2 + \left(1-(\mu_P(x))^2\right)(1-(1-\alpha)^\eta) - \alpha(\nu_P(x))^2\left(\sum_{k=0}^{\eta-1}(1-\alpha)^k\right) - (\mu_P(x))^2 \\
&\quad - \left(1-(\mu_P(x))^2\right)\left(1-(1-\alpha)^{\eta-1}\right) + \alpha(\nu_P(x))^2\left(\sum_{k=0}^{\eta-2}(1-\alpha)^k\right) \\
&\quad = \left(1-(\mu_P(x))^2\right)\left((1-\alpha)^{\eta-1}-(1-\alpha)^\eta\right) - \alpha(\nu_P(x))^2\left(\sum_{k=0}^{\eta-1}(1-\alpha)^k - \sum_{k=0}^{\eta-2}(1-\alpha)^k\right) \\
&\quad = \left(1-(\mu_P(x))^2\right)\left((1-\alpha)^{\eta-1}-(1-\alpha)(1-\alpha)^{\eta-1}\right) - \alpha(\nu_P(x))^2\left(\sum_{k=0}^{\eta-2}(1-\alpha)^k + (1-\alpha)^{\eta-1} - \sum_{k=0}^{\eta-2}(1-\alpha)^k\right) \\
&\quad = \alpha(1-\alpha)^{\eta-1}\left(1-(\mu_P(x))^2-\alpha(\nu_P(x))^2\right).
\end{aligned}$$

It becomes clear that $\alpha \geq 0$, $(1-\alpha)^{\eta-1} \geq 0$, and $1-(\mu_P(x))^2-\alpha(\nu_P(x))^2 \geq 0$ from $\alpha \in [0,1]$ and $0 \leq (\mu_P(x))^2 + (\nu_P(x))^2 \leq 1$. This evidences that $(\mu_{M_\alpha^\eta(P)}(x))^2 - (\mu_{M_\alpha^{\eta-1}(P)}(x))^2 \geq 0$, which reveals that $\mu_{M_\alpha^\eta(P)}(x) \geq \mu_{M_\alpha^{\eta-1}(P)}(x)$. On the other side, based on (T2.2), the difference between the non-membership grades in the recurrent upper estimations $M_\alpha^\eta(p)$ and $M_\alpha^{\eta-1}(p)$ is always equal to zero because $\nu_{M_\alpha^\eta(P)}(x) = \nu_{M_\alpha^{\eta-1}(P)}(x) = \nu_P(x)$ for each $\alpha \in [0,1]$, which straightly infers that the inequality $\nu_{M_\alpha^\eta(P)}(x) \leq \nu_{M_\alpha^{\eta-1}(P)}(x)$ is satisfied. From Definition 5, the natural quasi-ordering $M_\alpha^\eta(p) \succeq_Q M_\alpha^{\eta-1}(p)$ holds because $\mu_{M_\alpha^\eta(P)}(x) \geq \mu_{M_\alpha^{\eta-1}(P)}(x)$ and $\nu_{M_\alpha^\eta(P)}(x) \leq \nu_{M_\alpha^{\eta-1}(P)}(x)$. As a result, (T2.4) is correct.

(T2.5) By using (T2.1), it is yielded that $\lim_{\eta\to\infty} \mu_{M_\alpha^\eta(P)}(x) = \lim_{\eta\to\infty}[(\mu_P(x))^2 + (1-(\mu_P(x))^2)\cdot (1-(1-\alpha)^\eta) - \alpha(\nu_P(x))^2(\sum_{k=0}^{\eta-1}(1-\alpha)^k)]^{0.5} = \sqrt{1-(\nu_P(x))^2}$. From (T2.2), it is revealed that $\lim_{\eta\to\infty} \nu_{M_\alpha^\eta(P)}(x) = \lim_{\eta\to\infty} \nu_P(x) = \nu_P(x)$. By applying Definition 2, it indicates that $\lim_{\eta\to\infty} M_\alpha^\eta(p) = (\sqrt{1-(\nu_P(x))^2}, \nu_P(x); 1, (\pi - 2\cdot sin^{-1}(\nu_P(x)))/\pi))$. Thus, (T2.5) is confirmed, which demonstrates the truth of Theorem 2.

A.3 Proof of Theorem 4

(T4.1) The values of scalar functions range from 0 to 1 on the basis of Definition 4; thus, $0 \leq V(p_{ij}), V(M_{\alpha_{ij}}(p_{ij})), V(N_{\beta_{ij}}(p_{ij})) \leq 1$. In accordance with the property in (T1.5), $d_{ij}^N \leq d_{ij} \leq d_{ij}^M$ and $\theta_{ij}^M \leq \theta_{ij} \leq \theta_{ij}^N$. Moreover, $r_{ij}^M \geq r_{ij}$ and $r_{ij}^N \geq r_{ij}$ because $\min\{r_{ij}^M, r_{ij}^N\} \geq r_{ij}$ in (T1.4). It can be deduced that the two inequalities $V(p_{ij}) \leq V(M_{\alpha_{ij}}(p_{ij}))$ and $V(N_{\beta_{ij}}(p_{ij})) \leq V(p_{ij})$ are satisfied because $r_{ij}(d_{ij}-0.5) \leq r_{ij}^M(d_{ij}^M - 0.5)$ and $r_{ij}^N(0.5 - 2\cdot\theta_{ij}^N\big/\pi) \leq r_{ij}(0.5-2\cdot\theta_{ij}\big/\pi)$, respectively. Thus, $0 \leq V(N_{\beta_{ij}}(p_{ij})) \leq V(p_{ij}) \leq V(M_{\alpha_{ij}}(p_{ij})) \leq 1$.

(T4.2) From (29) and (31), it is known that $r_{ij}^M = \sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \alpha_{ij}(\tau_{ij})^2}$ and $\theta_{ij}^M = \arccos\left(\sqrt{(\mu_{ij})^2 + \alpha_{ij}(\tau_{ij})^2}\Big/\sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \alpha_{ij}(\tau_{ij})^2}\right)$, respectively. From (34), the following consequence can be determined:

$$V(M_{\alpha_{ij}}(p_{ij})) = \frac{1}{2} + r_{ij}^{M} \cdot \left(\frac{1}{2} - \frac{2 \cdot \theta_{ij}^{M}}{\pi}\right) = \frac{1}{2} + \sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \alpha_{ij}(\tau_{ij})^2} \cdot \left[\frac{1}{2} - \left(\frac{2}{\pi}\right)\arccos\left(\frac{\sqrt{(\mu_{ij})^2 + \alpha_{ij}(\tau_{ij})^2}}{\sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \alpha_{ij}(\tau_{ij})^2}}\right)\right].$$

The partial derivative of $V(M_{\alpha_{ij}}(p_{ij}))$ in regard to α_{ij} is computed in this fashion:

$$\begin{aligned}
\frac{\partial V(M_{\alpha_{ij}}(p_{ij}))}{\partial \alpha_{ij}} &= \frac{\partial \sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \alpha_{ij}(\tau_{ij})^2}}{\partial \alpha_{ij}} \cdot \left[\frac{1}{2} - \left(\frac{2}{\pi}\right)\arccos\left(\frac{\sqrt{(\mu_{ij})^2 + \alpha_{ij}(\tau_{ij})^2}}{\sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \alpha_{ij}(\tau_{ij})^2}}\right)\right] \\
&+ \sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \alpha_{ij}(\tau_{ij})^2} \cdot \frac{\partial \left[\frac{1}{2} - \left(\frac{2}{\pi}\right)\arccos\left(\frac{\sqrt{(\mu_{ij})^2+\alpha_{ij}(\tau_{ij})^2}}{\sqrt{(\mu_{ij})^2+(\nu_{ij})^2+\alpha_{ij}(\tau_{ij})^2}}\right)\right]}{\partial \alpha_{ij}} \\
&= \frac{(\tau_{ij})^2}{2\sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \alpha_{ij}(\tau_{ij})^2}} \cdot \left[\frac{1}{2} - \left(\frac{2}{\pi}\right)\arccos\left(\frac{\sqrt{(\mu_{ij})^2 + \alpha_{ij}(\tau_{ij})^2}}{\sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \alpha_{ij}(\tau_{ij})^2}}\right)\right] \\
&+ \sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \alpha_{ij}(\tau_{ij})^2} \cdot \left\{-\left(\frac{2}{\pi}\right)\left[-1 \middle/ \sqrt{1 - \left(\frac{\sqrt{(\mu_{ij})^2 + \alpha_{ij}(\tau_{ij})^2}}{\sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \alpha_{ij}(\tau_{ij})^2}}\right)^2}\right]\right\} \\
&= \frac{(\tau_{ij})^2}{2\sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \alpha_{ij}(\tau_{ij})^2}} \cdot \left[\frac{1}{2} - \left(\frac{2}{\pi}\right)\arccos\left(\frac{\sqrt{(\mu_{ij})^2 + \alpha_{ij}(\tau_{ij})^2}}{\sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \alpha_{ij}(\tau_{ij})^2}}\right)\right] \\
&+ \sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \alpha_{ij}(\tau_{ij})^2} \cdot \left[\left(\frac{2}{\pi}\right) \middle/ \sqrt{1 - \frac{(\mu_{ij})^2 + \alpha_{ij}(\tau_{ij})^2}{(\mu_{ij})^2 + (\nu_{ij})^2 + \alpha_{ij}(\tau_{ij})^2}}\right].
\end{aligned}$$

Thus, $\partial V(M_{\alpha_{ij}}(p_{ij})) \big/ \partial \alpha_{ij} \geq 0$ because $\arccos\left(\sqrt{(\mu_{ij})^2 + \alpha_{ij}(\tau_{ij})^2} \big/ \sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \alpha_{ij}(\tau_{ij})^2}\right) \leq (\pi/4)$ and $(2/\pi) \Big/ \sqrt{1 - \left((\mu_{ij})^2 + \alpha_{ij}(\tau_{ij})^2\right) \big/ \left((\mu_{ij})^2 + (\nu_{ij})^2 + \alpha_{ij}(\tau_{ij})^2\right)} \geq 0$. Thus, $V(M_{\alpha_{ij}}(p_{ij}))$ is monotonically nondecreasing with the allocation parameter α_{ij}, thereby confirming the truth of (T4.2).

(T4.3) In accordance with (30) and (32), $r_{ij}^{N} = \sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \beta_{ij}(\tau_{ij})^2}$ and $\theta_{ij}^{N} = \arcsin\left(\sqrt{(\nu_{ij})^2 + \beta_{ij}(\tau_{ij})^2} \big/ \sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \beta_{ij}(\tau_{ij})^2}\right)$, respectively. Applying (35), the following is true:

$$V(N_{\beta_{ij}}(p_{ij})) = \frac{1}{2} + r_{ij}^{N} \cdot \left(\frac{1}{2} - \frac{2 \cdot \theta_{ij}^{N}}{\pi} \right) = \frac{1}{2} + \sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \beta_{ij}(\tau_{ij})^2} \cdot \left[\frac{1}{2} - \left(\frac{2}{\pi} \right) \arcsin \left(\frac{\sqrt{(\nu_{ij})^2 + \beta_{ij}(\tau_{ij})^2}}{\sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \beta_{ij}(\tau_{ij})^2}} \right) \right].$$

The partial derivative of $V(N_{\beta_{ij}}(p_{ij}))$ compared to β_{ij} is calculated in this manner:

$$\begin{aligned}
\frac{\partial V(N_{\beta_{ij}}(p_{ij}))}{\partial \beta_{ij}} &= \frac{\partial \sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \beta_{ij}(\tau_{ij})^2}}{\partial \beta_{ij}} \cdot \left[\frac{1}{2} - \left(\frac{2}{\pi} \right) \arcsin \left(\frac{\sqrt{(\nu_{ij})^2 + \beta_{ij}(\tau_{ij})^2}}{\sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \beta_{ij}(\tau_{ij})^2}} \right) \right] \\
&+ \sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \beta_{ij}(\tau_{ij})^2} \cdot \frac{\partial \left[\frac{1}{2} - \left(\frac{2}{\pi} \right) \arcsin \left(\frac{\sqrt{(\nu_{ij})^2 + \beta_{ij}(\tau_{ij})^2}}{\sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \beta_{ij}(\tau_{ij})^2}} \right) \right]}{\partial \beta_{ij}} \\
&= \frac{(\tau_{ij})^2}{2\sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \beta_{ij}(\tau_{ij})^2}} \cdot \left[\frac{1}{2} - \left(\frac{2}{\pi} \right) \arcsin \left(\frac{\sqrt{(\nu_{ij})^2 + \beta_{ij}(\tau_{ij})^2}}{\sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \beta_{ij}(\tau_{ij})^2}} \right) \right] \\
&+ \sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \beta_{ij}(\tau_{ij})^2} \cdot \left\{ -\left(\frac{2}{\pi} \right) \left[1 \Big/ \sqrt{1 - \left(\frac{\sqrt{(\nu_{ij})^2 + \beta_{ij}(\tau_{ij})^2}}{\sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \beta_{ij}(\tau_{ij})^2}} \right)^2} \right] \right\} \\
&= \frac{(\tau_{ij})^2}{2\sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \beta_{ij}(\tau_{ij})^2}} \cdot \left[\frac{1}{2} - \left(\frac{2}{\pi} \right) \arcsin \left(\frac{\sqrt{(\nu_{ij})^2 + \beta_{ij}(\tau_{ij})^2}}{\sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \beta_{ij}(\tau_{ij})^2}} \right) \right] \\
&+ \sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \beta_{ij}(\tau_{ij})^2} \cdot \left[-\left(\frac{2}{\pi} \right) \Big/ \sqrt{1 - \frac{(\nu_{ij})^2 + \beta_{ij}(\tau_{ij})^2}{(\mu_{ij})^2 + (\nu_{ij})^2 + \beta_{ij}(\tau_{ij})^2}} \right] \leq 0.
\end{aligned}$$

Because $\arcsin\left(\sqrt{(\nu_{ij})^2 + \beta_{ij}(\tau_{ij})^2} \Big/ \sqrt{(\mu_{ij})^2 + (\nu_{ij})^2 + \beta_{ij}(\tau_{ij})^2} \right) \geq (\pi/4)$ and $-(2/\pi)/\sqrt{1 - \left((\nu_{ij})^2 + \beta_{ij}(\tau_{ij})^2\right) \Big/ \left((\mu_{ij})^2 + (\nu_{ij})^2 + \beta_{ij}(\tau_{ij})^2\right)} \leq 0$, it is deduced that $\partial V(N_{\beta_{ij}}(p_{ij})) \Big/ \partial \beta_{ij} \leq 0$. Thus, $V(N_{\beta_{ij}}(p_{ij}))$ is monotonically nonincreasing with the allocation parameter β_{ij}. Thus, the correctness of (T4.3) is confirmed, which gives substance to the truth of Theorem 4.

A.4 Proof of Theorem 5

(T5.1) For notational convenience, we denote:

$$\Lambda(p_{ij},p_{kj}) = \frac{V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{ij}}(p_{ij}))}{\left(V(M_{\alpha_{ij}}(p_{ij})) - V(N_{\beta_{ij}}(p_{ij}))\right) + \left(V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{kj}}(p_{kj}))\right)},$$

from which $Lik(p_{ij}\succeq p_{kj}) = \max\{1 - \max\{\Lambda(p_{ij},p_{kj}),0\},0\}$. The fact that $\max\{\Lambda(p_{ij},p_{kj}),0\} \geq 0$ yields the outcomes $1 - \max\{\Lambda(p_{ij},p_{kj}),0\} \leq 1$ and $0 \leq \max\{1 - \max\{\Lambda(p_{ij},p_{kj}),0\},0\} \leq 1$. Therefore, the boundedness property of $0 \leq Lik(p_{ij}\succeq p_{kj}) \leq 1$ is satisfied, and (T5.1) is correct.

(T5.2) For necessity, $\max\{1 - \max\{\Lambda(p_{ij},p_{kj}),0\},0\} = 0$ from the given condition $Lik(p_{ij}\succeq p_{kj}) = 0$. Thus, $\max\{\Lambda(p_{ij},p_{kj}),0\} \geq 1$, which leads to $\Lambda(p_{ij},p_{kj}) \geq 1$. It can be demonstrated that $V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{ij}}(p_{ij})) \geq V(M_{\alpha_{ij}}(p_{ij})) - V(N_{\beta_{ij}}(p_{ij})) + V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{kj}}(p_{kj}))$, which leads to $V(N_{\beta_{kj}}(p_{kj})) \geq V(M_{\alpha_{ij}}(p_{ij}))$. For sufficiency, based on the property in (T4.1), $V(M_{\alpha_{ij}}(p_{ij}))-V(N_{\beta_{ij}}(p_{ij})) \geq 0$ and $V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{kj}}(p_{kj})) \geq 0$. Thus, $V(M_{\alpha_{ij}}(p_{ij})) - V(N_{\beta_{ij}}(p_{ij}))+ V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{kj}}(p_{kj})) \geq 0$, from which $V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{ij}}(p_{ij})) - (V(N_{\beta_{kj}}(p_{kj}))- V(M_{\alpha_{ij}}(p_{ij}))) \geq 0$. By combining the condition $V(N_{\beta_{kj}}(p_{kj})) \geq V(M_{\alpha_{ij}}(p_{ij}))$, $V(M_{\alpha_{kj}}(p_{kj}))- V(N_{\beta_{ij}}(p_{ij})) \geq (V(M_{\alpha_{ij}}(p_{ij})) - V(N_{\beta_{ij}}(p_{ij}))) + (V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{kj}}(p_{kj})))$, from which $\Lambda(p_{ij},p_{kj})\geq 1$. In addition, $\max\{\Lambda(p_{ij},p_{kj}),0\} = \Lambda(p_{ij},p_{kj}) \geq 1$ and $1 - \max\{\Lambda(p_{ij},p_{kj}),0\} \leq 0$. Thus, $\max\{1 - \max\{\Lambda(p_{ij},p_{kj}),0\},0\} = 0$ (i.e., $Lik(p_{ij}\succeq p_{kj}) = 0$), and (T5.2) is valid.

(T5.3) For necessity, the given assumption $Lik(p_{ij}\succeq p_{kj}) = 1$ leads to the deduction that $\max\{1 - \max\{\Lambda(p_{ij},p_{kj}),0\},0\} = 1$. Logically, $\Lambda(p_{ij},p_{kj}) \leq 0$ because $\max\{\Lambda(p_{ij},p_{kj}),0\} = 0$. Thus, $V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{ij}}(p_{ij})) \leq 0$, i.e., $V(N_{\beta_{ij}}(p_{ij})) \geq V(M_{\alpha_{kj}}(p_{kj}))$. For sufficiency, the given assumption $V(N_{\beta_{ij}}(p_{ij})) \geq V(M_{\alpha_{kj}}(p_{kj}))$ indicates that $V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{ij}}(p_{ij})) \leq 0$. It is verified that $V(M_{\alpha_{ij}}(p_{ij})) - V(N_{\beta_{ij}}(p_{ij})) \geq 0$ and $V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{kj}}(p_{kj})) \geq 0$ based on (T4.1). Because $V(M_{\alpha_{ij}}(p_{ij})) - V(N_{\beta_{ij}}(p_{ij})) = 0$ and $V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{kj}}(p_{kj})) = 0$ do not occur at the same time, the denominator in $\Lambda(p_{ij},p_{kj})$ is greater than zero on all occasions. Thus, $\Lambda(p_{ij},p_{kj}) \leq 0$, which yields $\max\{1 - \max\{\Lambda(p_{ij},p_{kj}),0\},0\}=1$, namely, $Lik(p_{ij}\succeq p_{kj}) = 1$). As a result, (T5.3) is satisfied.

(T5.4) To show the complementarity property, this paper makes the following four assumptions about the scalar functions presented in $\Lambda(p_{ij},p_{kj})$: (1) $V(N_{\beta_{ij}}(p_{ij})) \leq V(N_{\beta_{kj}}(p_{kj}))$ and $V(M_{\alpha_{ij}}(p_{ij})) \leq V(M_{\alpha_{kj}}(p_{kj}))$; (2) $V(N_{\beta_{ij}}(p_{ij})) \geq V(N_{\beta_{kj}}(p_{kj}))$ and $V(M_{\alpha_{ij}}(p_{ij})) \geq V(M_{\alpha_{kj}}(p_{kj}))$; (3) $V(N_{\beta_{ij}}(p_{ij})) \leq V(N_{\beta_{kj}}(p_{kj}))$ and $V(M_{\alpha_{ij}}(p_{ij})) \geq V(M_{\alpha_{kj}}(p_{kj}))$; and (4) $V(N_{\beta_{ij}}(p_{ij})) \geq V(N_{\beta_{kj}}(p_{kj}))$ and $V(M_{\alpha_{ij}}(p_{ij})) \leq V(M_{\alpha_{kj}}(p_{kj}))$. We denote $\Lambda(p_{kj},p_{ij}) = (V(M_{\alpha_{ij}}(p_{ij})) - V(N_{\beta_{kj}}(p_{kj})))/$ $(V(M_{\alpha_{ij}}(p_{ij})) - V(N_{\beta_{ij}}(p_{ij})) + V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{kj}}(p_{kj})))$. It is apparent that $\Lambda(p_{ij},p_{kj}) + \Lambda(p_{kj},p_{ij})= 1$. In Case (i), in the light of the assumption and the property in (T4.1), $V(N_{\beta_{ij}}(p_{ij})) \leq V(M_{\alpha_{ij}}(p_{ij}))\leq V(M_{\alpha_{kj}}(p_{kj}))$, which indicates that $V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{ij}}(p_{ij})) \geq 0$. In the event that $V(M_{\alpha_{ij}}(p_{ij})) \geq V(N_{\beta_{kj}}(p_{kj}))$, it is readily corroborated that $V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{ij}}(p_{ij})) \leq (V(M_{\alpha_{ij}}(p_{ij})) - V(N_{\beta_{ij}}(p_{ij})))$ $+(V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{kj}}(p_{kj})))$. Thus, $0 \leq \Lambda(p_{ij},p_{kj}) \leq 1$ and $Lik(p_{ij}\succeq p_{kj}) = 1 - \Lambda(p_{ij},p_{kj})$. Furthermore, the condition $V(M_{\alpha_{ij}}(p_{ij})) \geq V(N_{\beta_{kj}}(p_{kj}))$ implies that $0 \leq V(M_{\alpha_{ij}}(p_{ij})) - V(N_{\beta_{kj}}(p_{kj}))\leq 1$. It is apparent that $0 \leq V(M_{\alpha_{ij}}(p_{ij})) - V(N_{\beta_{ij}}(p_{ij})) \leq 1$ and

$0 \leq V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{kj}}(p_{kj})) \leq 1$, which yields $Lik(p_{kj} \succeq p_{ij}) = 1 - \Lambda(p_{kj}, p_{ij})$. Accordingly, it is found that:

$$Lik(p_{ij} \succeq p_{kj}) + Lik(p_{kj} \succeq p_{ij}) = 2 - \frac{V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{ij}}(p_{ij})) + V(M_{\alpha_{ij}}(p_{ij})) - V(N_{\beta_{kj}}(p_{kj}))}{\left(V(M_{\alpha_{ij}}(p_{ij})) - V(N_{\beta_{ij}}(p_{ij}))\right) + \left(V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{kj}}(p_{kj}))\right)} = 1.$$

Therefore, (T5.4) holds when $V(M_{\alpha_{ij}}(p_{ij})) \geq V(N_{\beta_{kj}}(p_{kj}))$ in Case (1). When $V(M_{\alpha_{ij}}(p_{ij})) \leq V(N_{\beta_{kj}}(p_{kj}))$, $V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{ij}}(p_{ij})) \geq (V(M_{\alpha_{ij}}(p_{ij})) - V(N_{\beta_{ij}}(p_{ij}))) + (V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{kj}}(p_{kj})))$. It follows that $\Lambda(p_{ij}, p_{kj}) \geq 1$ and $Lik(p_{ij} \succeq p_{kj}) = \max\{1 - \Lambda(p_{ij}, p_{kj}), 0\} = 0$. Thus, $V(M_{\alpha_{ij}}(p_{ij})) - V(N_{\beta_{kj}}(p_{kj})) \leq 0$ based on the condition $V(M_{\alpha_{ij}}(p_{ij})) \leq V(N_{\beta_{kj}}(p_{kj}))$ and $Lik(p_{kj} \succeq p_{ij}) = \max\{1 - 0, 0\} = 1$, which indicates that $Lik(p_{ij} \succeq p_{kj}) + Lik(p_{kj} \succeq p_{ij}) = 1$. Therefore, (T5.4) holds when $V(M_{\alpha_{ij}}(p_{ij})) \leq V(N_{\beta_{kj}}(p_{kj}))$ in Case (1). Case (2) can be corroborated in an analogous way. Next, it is known that $V(N_{\beta_{ij}}(p_{ij})) \leq V(N_{\beta_{kj}}(p_{kj}))$ and $V(M_{\alpha_{ij}}(p_{ij})) \geq V(M_{\alpha_{kj}}(p_{kj}))$ in Case (3). Based on (T4.1), $V(M_{\alpha_{kj}}(p_{kj})) \geq V(N_{\beta_{kj}}(p_{kj}))$, which shows that $V(M_{\alpha_{kj}}(p_{kj})) \geq V(N_{\beta_{kj}}(p_{kj})) \geq V(N_{\beta_{ij}}(p_{ij}))$ and $V(M_{\alpha_{ij}}(p_{ij})) \geq V(M_{\alpha_{kj}}(p_{kj})) \geq V(N_{\beta_{kj}}(p_{kj}))$. Thus, $V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{ij}}(p_{ij})) \geq 0$ and $V(M_{\alpha_{ij}}(p_{ij})) - V(N_{\beta_{kj}}(p_{kj})) \geq 0$ are obtained. It is verified that $(V(M_{\alpha_{ij}}(p_{ij})) - V(N_{\beta_{ij}}(p_{ij}))) + (V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{kj}}(p_{kj}))) = (V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{ij}}(p_{ij}))) + (V(M_{\alpha_{ij}}(p_{ij})) - V(N_{\beta_{kj}}(p_{kj}))) \geq V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{ij}}(p_{ij})) \geq 0$. Thus, $0 \leq \Lambda(p_{ij}, p_{kj}) \leq 1$, which implies that $Lik(p_{ij} \succeq p_{kj}) = \max\{1 - \max\{\Lambda(p_{ij}, p_{kj}), 0\}, 0\} = 1 - \Lambda(p_{ij}, p_{kj})$. Thereafter, the result $Lik(p_{kj} \succeq p_{ij}) = 1 - \Lambda(p_{kj}, p_{ij})$ can be acquired in a similar manner. Thus, $Lik(p_{ij} \succeq p_{kj}) + Lik(p_{kj} \succeq p_{ij}) = (1 - \Lambda(p_{ij}, p_{kj})) + (1 - \Lambda(p_{kj}, p_{ij})) = 1$. Accordingly, (T5.4) is fulfilled in Case (iii), and Case (iv) is corroborated in an analogous manner. Based on these results, the complementarity property is possessed by the proposed PF likelihood measure, and (T5.4) is fulfilled.

(T5.5) When $Lik(p_{ij} \succeq p_{kj}) = Lik(p_{kj} \succeq p_{ij})$, it is clearly understood that $Lik(p_{ij} \succeq p_{kj}) = Lik(p_{kj} \succeq p_{ij}) = 0.5$ through the agency of (T5.4), and thus, (T5.5) is valid. (T5.6) can be easily inferred from (T5.5).

(T5.7) Based on (T5.4) and (T5.6), $Lik(p_{ij} \succeq p_{kj}) + Lik(p_{kj} \succeq p_{ij}) = 1$ and $Lik(p_{ij} \succeq p_{ij}) = 0.5$, respectively. Accordingly, the following can be derived:

$$\sum_{i=1}^{m} \sum_{k=1}^{m} Lik(p_{ij} \succeq p_{kj}) = \sum_{i=1}^{m} Lik(p_{ij} \succeq p_{ij}) + \sum_{i=1, i<k}^{m} \left(Lik(p_{ij} \succeq p_{kj}) + Lik(p_{kj} \succeq p_{ij})\right) = \frac{m}{2} + \frac{m(m-1)}{2} = \frac{m^2}{2}.$$

This confirms the correctness of Theorem 5.

A.5 Proof of Theorem 7

Let us denote $\Lambda(p_{ij}, p_{lj}) = (V(M_{\alpha_{lj}}(p_{lj})) - V(N_{\beta_{ij}}(p_{ij})))/(V(M_{\alpha_{ij}}(p_{ij})) - V(N_{\beta_{ij}}(p_{ij})) + V(M_{\alpha_{lj}}(p_{lj})) - V(N_{\beta_{lj}}(p_{lj})))$ and $\Lambda(p_{lj}, p_{ij}) = (V(M_{\alpha_{ij}}(p_{ij})) - V(N_{\beta_{lj}}(p_{lj})))/(V(M_{\alpha_{ij}}(p_{ij})) - V(N_{\beta_{ij}}(p_{ij})) + V(M_{\alpha_{lj}}(p_{lj})) - V(N_{\beta_{lj}}(p_{lj})))$ for brevity. Thus, $\Lambda(p_{ij}, p_{lj}) + \Lambda(p_{lj}, p_{ij}) = 1$. Consider the supposition that $Lik(p_{ij} \succeq p_{lj}) \geq 0.5$ and $Lik(p_{lj} \succeq p_{kj}) \geq 0.5$. The assumption $Lik(p_{ij} \succeq p_{lj}) = \max\{1 - \max\{\Lambda(p_{ij}, p_{lj}), 0\}, 0\} \geq 0.5$ indicates that $V(M_{\alpha_{lj}}(p_{lj})) - V(N_{\beta_{ij}}(p_{ij})) \geq 0$ and $1 - \Lambda(p_{ij}, p_{lj}) \geq 0.5$. It is known that $\Lambda(p_{lj}, p_{ij}) \geq 0.5$ because $\Lambda(p_{ij}, p_{lj}) + \Lambda(p_{lj}, p_{ij}) = 1$. It is clear that

$$\Lambda(p_{lj}, p_{ij}) = \frac{V(M_{\alpha_{ij}}(p_{ij})) - V(N_{\beta_{lj}}(p_{lj}))}{\left(V(M_{\alpha_{lj}}(p_{lj})) - V(N_{\beta_{lj}}(p_{lj}))\right) + \left(V(M_{\alpha_{ij}}(p_{ij})) - V(N_{\beta_{ij}}(p_{ij}))\right)} \geq \frac{1}{2}.$$

Thus, $0 \leq V(M_{\alpha_{lj}}(p_{lj})) - V(N_{\beta_{ij}}(p_{ij})) \leq V(M_{\alpha_{ij}}(p_{ij})) - V(N_{\beta_{lj}}(p_{lj}))$. The assumption $0.5Lik(p_{lj} \succeq p_{kj}) \geq$ (i.e., $\max\{1 - \max\{\Lambda(p_{lj}, p_{kj}), 0\}, 0\} \geq 0.5$) shows that $V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{lj}}(p_{lj})) \geq 0$ and $1 - \Lambda(p_{lj}, p_{kj}) \geq 0.5$. From $\Lambda(p_{lj}, p_{kj}) + \Lambda(p_{kj}, p_{lj}) = 1$, one obtains $\Lambda(p_{kj}, p_{lj}) \geq 0.5$. To be specific,

$$\Lambda(p_{kj}, p_{lj}) = \frac{V(M_{\alpha_{lj}}(p_{lj})) - V(N_{\beta_{kj}}(p_{kj}))}{\left(V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{kj}}(p_{kj}))\right) + \left(V(M_{\alpha_{lj}}(p_{lj})) - V(N_{\beta_{lj}}(p_{lj}))\right)} \geq \frac{1}{2}.$$

It can be determined that $0 \leq V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{lj}}(p_{lj})) \leq V(M_{\alpha_{lj}}(p_{lj})) - V(N_{\beta_{kj}}(p_{kj}))$. By adding the obtained inequalities $0 \leq V(M_{\alpha_{lj}}(p_{lj})) - V(N_{\beta_{ij}}(p_{ij})) \leq V(M_{\alpha_{ij}}(p_{ij})) - V(N_{\beta_{lj}}(p_{lj}))$ and $0 \leq V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{lj}}(p_{lj})) \leq V(M_{\alpha_{lj}}(p_{lj})) - V(N_{\beta_{kj}}(p_{kj}))$, it is verified that $0 \leq V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{ij}}(p_{ij})) \leq V(M_{\alpha_{ij}}(p_{ij})) - V(N_{\beta_{kj}}(p_{kj}))$. By adding $V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{ij}}(p_{ij}))$ to this inequality, the following is obtained: $0 \leq V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{ij}}(p_{ij})) + V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{ij}}(p_{ij})) \leq V(M_{\alpha_{ij}}(p_{ij})) - V(N_{\beta_{kj}}(p_{kj})) + V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{ij}}(p_{ij}))$. This indicates that:

$$0 \leq 2\left(V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{ij}}(p_{ij}))\right) \leq \left(V(M_{\alpha_{ij}}(p_{ij})) - V(N_{\beta_{ij}}(p_{ij}))\right) + \left(V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{kj}}(p_{kj}))\right).$$

As a consequence, the following is obtained:

$$0 \leq \frac{V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{ij}}(p_{ij}))}{\left(V(M_{\alpha_{ij}}(p_{ij})) - V(N_{\beta_{ij}}(p_{ij}))\right) + \left(V(M_{\alpha_{kj}}(p_{kj})) - V(N_{\beta_{kj}}(p_{kj}))\right)} \leq \frac{1}{2}.$$

Namely, $0 \leq \Lambda(p_{ij}, p_{kj}) \leq 0.5$. Thus, it can be demonstrated that $0 \leq \max\{\Lambda(p_{ij}, p_{kj}), 0\} \leq 0.5$ and $0.5 \leq 1 - \max\{\Lambda(p_{ij}, p_{kj}), 0\} \leq 1$. Accordingly, it is shown that $0.5 \leq \max\{1 - \max\{\Lambda(p_{ij}, p_{kj}), 0\}, 0\} \leq 1$, which indicates that $Lik(p_{ij} \succeq p_{kj}) \geq 0.5$. Therefore, the PF likelihood measure possesses the property of weak transitivity, which demonstrates the truth of Theorem 7.

A.6 Proof of Theorem 8

(T8.1) According to (T5.1) and (T5.6), it is known that $0 \leq Lik(p_{kj} \succeq p_{ij}) \leq 1$ and $Lik(p_{kj} \succeq p_{kj}) = 0.5$, respectively. Thus, the subsequent outcomes can be determined:

$$Pen(p_{ij}) = \frac{1}{m(m-1)}\left(Lik(p_{kj} \succeq p_{kj}) + \sum_{i=1, i\neq k}^{m} Lik(p_{kj} \succeq p_{ij}) + \frac{m}{2} - 1 \right)$$
$$= \frac{1}{m(m-1)}\left(\frac{1}{2} + \sum_{i=1, i\neq k}^{m} Lik(p_{kj} \succeq p_{ij}) + \frac{m}{2} - 1 \right)$$
$$\geq \frac{1}{m(m-1)}\left(\frac{1}{2} + 0 + \frac{m}{2} - 1 \right) = \frac{1}{2m},$$

$$Pen(p_{ij}) \leq \frac{1}{m(m-1)}\left(\frac{1}{2} + \sum_{i=1, i\neq k}^{m} 1 + \frac{m}{2} - 1 \right) = \frac{1}{m(m-1)}\left(\frac{1}{2} + (m-1) + \frac{m}{2} - 1 \right) = \frac{3}{2m}.$$

Therefore, it follows that $1/2m \leq Pen(p_{ij}) \leq 3/2m$, and (T8.1) is correct.

(T8.2) With the help of the property $Lik(p_{kj} \succeq p_{ij}) + Lik(p_{ij} \succeq p_{kj}) = 1$ based on (T5.4), one can quickly obtain the following:

$$Pen(p_{ij}) + \Pr i(p_{ij}) = \frac{1}{m(m-1)}\left(\sum_{k=1}^{m} Lik(p_{kj} \succeq p_{ij}) + \frac{m}{2} - 1 \right) + \frac{1}{m(m-1)}\left(\sum_{k=1}^{m} Lik(p_{ij} \succeq p_{kj}) + \frac{m}{2} - 1 \right)$$
$$= \frac{1}{m(m-1)}\left(\sum_{k=1}^{m} \left(Lik(p_{kj} \succeq p_{ij}) + Lik(p_{ij} \succeq p_{kj}) \right) + m - 2 \right)$$
$$= \frac{1}{m(m-1)}(m + m - 2) = \frac{2}{m}.$$

(T8.3) Regarding an evaluative criterion c_j, it can be determined the following:

$$\sum_{i=1}^{m} Pen(p_{ij}) = \sum_{i=1}^{m} \frac{1}{m(m-1)}\left(\sum_{k=1}^{m} Lik(p_{kj} \succeq p_{ij}) + \frac{m}{2} - 1 \right)$$
$$= \frac{1}{m(m-1)}\left(\sum_{i=1}^{m}\sum_{k=1}^{m} Lik(p_{kj} \succeq p_{ij}) + \sum_{i=1}^{m}\left(\frac{m}{2} - 1 \right) \right)$$
$$= \frac{1}{m(m-1)}\left(\frac{m^2}{2} + \frac{m(m-2)}{2} \right) = \frac{1}{m(m-1)}\left(\frac{2m(m-1)}{2} \right) = 1.$$

(T8.4) can be trivially proven through the medium of (T8.3), which demonstrates the correctness of Theorem 8.

A.7 Proof of Theorem 9

Theorem 9 will be demonstrated under the aegis of mathematical induction on n. On the occasion that $n=2$, it can be received that $\Delta_i^{\phi} = \left[\left(Pen(p_{i1}) \cdot \left|\Phi_{i1} - \phi\right|\right) \odot w_1\right] \oplus \left[\left(Pen(p_{i2}) \cdot \left|\Phi_{i2} - \phi\right|\right) \odot w_2\right]$. Applying the arithmetic operations in Definition 6, the two-dimensional representation $\Delta_i^{\prime\phi}$ $(= (\mu_{\Delta i}^{\phi}, \nu_{\Delta i}^{\phi}))$ is calculated as follows:

$$
\begin{aligned}
\Delta_i'^{\phi} &= \left[\left(Pen(p_{i1}) \cdot \left|\Phi_{i1} - \phi\right|\right) \odot w_1'\right] \oplus \left[\left(Pen(p_{i2}) \cdot \left|\Phi_{i2} - \phi\right|\right) \odot w_2'\right] \\
&= \left(\sqrt{1 - \left(1 - (\omega_1)^2\right)^{Pen(p_{i1})\cdot|\Phi_{i1}-\phi|}}, (\varpi_1)^{Pen(p_{i1})\cdot|\Phi_{i1}-\phi|}\right) \\
&\oplus \left(\sqrt{1 - \left(1 - (\omega_2)^2\right)^{Pen(p_{i2})\cdot|\Phi_{i2}-\phi|}}, (\varpi_2)^{Pen(p_{i2})\cdot|\Phi_{i2}-\phi|}\right) \\
&= \left(\left(1 - \left(1 - (\omega_1)^2\right)^{Pen(p_{i1})\cdot|\Phi_{i1}-\phi|} + 1 - \left(1 - (\omega_2)^2\right)^{Pen(p_{i2})\cdot|\Phi_{i2}-\phi|}\right.\right. \\
&\left.\left. - \left(1 - \left(1 - (\omega_1)^2\right)^{Pen(p_{i1})\cdot|\Phi_{i1}-\phi|}\right) \cdot \left(1 - \left(1 - (\omega_2)^2\right)^{Pen(p_{i2})\cdot|\Phi_{i2}-\phi|}\right)\right)^{0.5}, \right. \\
&\left. (\varpi_1)^{Pen(p_{i1})\cdot|\Phi_{i1}-\phi|} \cdot (\varpi_2)^{Pen(p_{i2})\cdot|\Phi_{i2}-\phi|}\right) \\
&= \left(\sqrt{1 - \left(1 - \left(1 - \left(1 - (\omega_1)^2\right)^{Pen(p_{i1})\cdot|\Phi_{i1}-\phi|}\right)\right) \cdot \left(1 - \left(1 - \left(1 - (\omega_2)^2\right)^{Pen(p_{i2})\cdot|\Phi_{i2}-\phi|}\right)\right)}, \right. \\
&\left. (\varpi_1)^{Pen(p_{i1})\cdot|\Phi_{i1}-\phi|} \cdot (\varpi_2)^{Pen(p_{i2})\cdot|\Phi_{i2}-\phi|}\right) \\
&= \left(\sqrt{1 - \prod_{j=1}^{2}\left(1 - (\omega_j)^2\right)^{Pen(p_{ij})\cdot\left|\Phi_{ij}-\phi\right|}}, \prod_{j=1}^{2}(\varpi_j)^{Pen(p_{ij})\cdot\left|\Phi_{ij}-\phi\right|}\right).
\end{aligned}
$$

It can be observed that (43) is fulfilled for $n=2$. Suppose that (43) holds for $n = \varsigma$, namely,

$$
\Delta_i'^{\phi} = (\mu_{\Delta i}^{\phi}, \nu_{\Delta i}^{\phi}) = \left(\sqrt{1 - \prod_{j=1}^{\varsigma}\left(1 - (\omega_j)^2\right)^{Pen(p_{ij})\cdot\left|\Phi_{ij}-\phi\right|}}, \prod_{j=1}^{\varsigma}(\varpi_j)^{Pen(p_{ij})\cdot\left|\Phi_{ij}-\phi\right|}\right).
$$

When $n = \varsigma + 1$, based on Definition 14, it can be acquired that:

$$
\Delta_i^{\phi} = \left(\bigoplus_{j=1}^{\varsigma}\left[\left(Pen(p_{ij}) \cdot \left|\Phi_{ij} - \phi\right|\right) \odot w_j\right]\right) \oplus \left[\left(Pen(p_{i,\varsigma+1}) \cdot \left|\Phi_{i,\varsigma+1} - \phi\right|\right) \odot w_{\varsigma+1}\right].
$$

By use of the arithmetic operations with respect to the two-dimensional representation for Pythagorean membership grades, one obtains:

$$\Delta_i'^{\phi} = \left(\bigoplus_{j=1}^{\varsigma}\left[\left(Pen(p_{ij})\cdot\left|\Phi_{ij}-\phi\right|\right)\odot w_j'\right]\right)\oplus\left[\left(Pen(p_{i,\varsigma+1})\cdot\left|\Phi_{i,\varsigma+1}-\phi\right|\right)\odot w_{\varsigma+1}'\right]$$
$$=\left(\sqrt{1-\prod_{j=1}^{\varsigma}\left(1-(\omega_j)^2\right)^{Pen(p_{ij})\cdot\left|\Phi_{ij}-\phi\right|}},\prod_{j=1}^{\varsigma}(\varpi_j)^{Pen(p_{ij})\cdot\left|\Phi_{ij}-\phi\right|}\right)$$
$$\oplus\left(\sqrt{1-\left(1-(\omega_{\varsigma+1})^2\right)^{Pen(p_{i,\varsigma+1})\cdot\left|\Phi_{i,\varsigma+1}-\phi\right|}},(\varpi_{\varsigma+1})^{Pen(p_{i,\varsigma+1})\cdot\left|\Phi_{i,\varsigma+1}-\phi\right|}\right)$$
$$=\left(\left(\left(1-\prod_{j=1}^{\varsigma}\left(1-(\omega_j)^2\right)^{Pen(p_{ij})\cdot\left|\Phi_{ij}-\phi\right|}+1-\left(1-\left(1-\left(1-(\omega_{\varsigma+1})^2\right)^{Pen(p_{i,\varsigma+1})\cdot\left|\Phi_{i,\varsigma+1}-\phi\right|}\right)\right)\right.\right.\right.$$
$$\left.\left.-\left(1-\prod_{j=1}^{\varsigma}\left(1-(\omega_j)^2\right)^{Pen(p_{ij})\cdot\left|\Phi_{ij}-\phi\right|}\right)\cdot\left(1-\left(1-\left(1-\left(1-(\omega_{\varsigma+1})^2\right)^{Pen(p_{i,\varsigma+1})\cdot\left|\Phi_{i,\varsigma+1}-\phi\right|}\right)\right)\right)\right)^{0.5},$$
$$\left.\left(\prod_{j=1}^{\varsigma}(\varpi_j)^{Pen(p_{ij})\cdot\left|\Phi_{ij}-\phi\right|}\right)\cdot(\varpi_{\varsigma+1})^{Pen(p_{i,\varsigma+1})\cdot\left|\Phi_{i,\varsigma+1}-\phi\right|}\right)$$
$$=\left(\sqrt{1-\prod_{j=1}^{\varsigma+1}\left(1-(\omega_j)^2\right)^{Pen(p_{ij})\cdot\left|\Phi_{ij}-\phi\right|}},\prod_{j=1}^{\varsigma+1}(\varpi_j)^{Pen(p_{ij})\cdot\left|\Phi_{ij}-\phi\right|}\right).$$

It reveals that (43) is satisfied for $n=\varsigma+1$. As a result, (43) is valid in connection with all n, which confirms the truth of Theorem 9.

A.8 Proof of Theorem 10

(T10.1) Based on Definition 15 and $0\le r_{\Delta i}^{\phi}, d_{\Delta i}^{\phi}\le 1$, it is readily corroborated that $V(\Delta_i^{\phi})$ would be in the range of $0-1$ (i.e., $0\le V(\Delta_i^{\phi})\le 1$).

(T10.2) Concerning necessity, the precondition $V(\Delta_i^{\phi})=0$ indicates that $0.5+r_{\Delta i}^{\phi}\cdot(d_{\Delta i}^{\phi}-0.5)=0$ (or equivalently, $0.5+r_{\Delta i}^{\phi}\cdot(0.5-(2/\pi)\cdot\theta_{\Delta i}^{\phi})=0$). It is directly validated that $r_{\Delta i}^{\phi}=1$ and $d_{\Delta i}^{\phi}=0$ hold due to the conditions $r_{\Delta i}^{\phi}\cdot(d_{\Delta i}^{\phi}-0.5)=-0.5$ and $0\le r_{\Delta i}^{\phi}, d_{\Delta i}^{\phi}\le 1$. Alternately, the conditions $r_{\Delta i}^{\phi}\cdot(0.5-(2/\pi)\cdot\theta_{\Delta i}^{\phi})=-0.5$, $0\le r_{\Delta i}^{\phi}\le 1$, and $0\le\theta_{\Delta i}^{\phi}\le\pi/2$ demonstrate the truth of $r_{\Delta i}^{\phi}=1$ and $\theta_{\Delta i}^{\phi}=\pi/2$. In light of Definition 2, $\mu_{\Delta i}^{\phi}=r_{\Delta i}^{\phi}\cdot\cos(\theta_{\Delta i}^{\phi})=0$ and $\nu_{\Delta i}^{\phi}=r_{\Delta i}^{\phi}\cdot sin(\theta_{\Delta i}^{\phi})=1$. Thus, it can be confirmed that $\Delta_i^{\phi}=(0,1;1,0)$. For sufficiency, the condition $\Delta_i^{\phi}=(0,1;1,0)$ yields $V(\Delta_i^{\phi})=0$ by use of Definition 15. Based on these results, (T10.2) is valid.

(T10.3) For necessity, the precondition $V(\Delta_i^{\phi})=1$ indicates that $r_{\Delta i}^{\phi}\cdot(d_{\Delta i}^{\phi}-0.5)=0.5$ (or equivalently, $r_{\Delta i}^{\phi}\cdot(0.5-(2/\pi)\cdot\theta_{\Delta i}^{\phi})=0.5$). On the basis of Definitions 1 and 2, it is known that $r_{\Delta i}^{\phi}=d_{\Delta i}^{\phi}=1$ (or equivalently, $r_{\Delta i}^{\phi}=1$ and $\theta_{\Delta i}^{\phi}=0$). It is shown that $\Delta_i^{\phi}=(1,0;1,1)$ from $\mu_{\Delta i}^{\phi}=r_{\Delta i}^{\phi}\cdot\cos(\theta_{\Delta i}^{\phi})=1$ and $\nu_{\Delta i}^{\phi}=r_{\Delta i}^{\phi}\cdot sin(\theta_{\Delta i}^{\phi})=0$. For sufficiency, it is revealed that $V(\Delta_i^{\phi})=1$ when $\Delta_i^{\phi}=(1,0;1,1)$. Therefore, (T10.3) is valid.

(T10.4) is known based on Definition 15. In (T10.5), the precondition $\tau_{\Delta i}^{\phi}=0$ indicates that $r_{\Delta i}^{\phi}=1$. Thus, $V(\Delta_i^{\phi})=0.5+1\cdot(d_{\Delta i}^{\phi}-0.5)=d_{\Delta i}^{\phi}$, which completes the proof of Theorem 10.

Appendix B: Detailed result tables

Table 1 Computation outcomes related to the PF point operator-oriented upper estimation

a_i	c_j	α_{ij}	$M_{\alpha_{ij}}(p_{ij}) = (\mu_{ij}^M, \nu_{ij}^M; r_{ij}^M, d_{ij}^M)$	θ_{ij}^M	τ_{ij}^M	$V(M_{\alpha_{ij}}(p_{ij}))$
a_1	c_1	0.0602	(0.2453, 0.8300; 0.8655, 0.1829)	1.2835	0.5009	0.2256
	c_2	0.1518	(0.3896, 0.7800; 0.8719, 0.2949)	1.1075	0.4897	0.3212
	c_3	0.4516	(0.6720, 0.5400; 0.8621, 0.5691)	0.6769	0.5068	0.5595
	c_4	0.1465	(0.3827, 0.7000; 0.7978, 0.3185)	1.0704	0.6029	0.3552
	c_5	0.9800	(0.9899, 0.1300; 0.9984, 0.9169)	0.1306	0.0557	0.9162
	c_6	0.6001	(0.7747, 0.4000; 0.8718, 0.6966)	0.4766	0.4898	0.6714
a_2	c_1	0.1914	(0.4375, 0.7400; 0.8596, 0.3399)	1.0369	0.5109	0.3624
	c_2	0.5676	(0.7534, 0.4800; 0.8933, 0.6389)	0.5672	0.4494	0.6241
	c_3	0.6476	(0.8047, 0.4500; 0.9220, 0.6754)	0.5099	0.3872	0.6617
	c_4	0.2975	(0.5455, 0.6300; 0.8333, 0.4543)	0.8572	0.5528	0.4619
	c_5	0.9568	(0.9782, 0.1700; 0.9928, 0.8905)	0.1721	0.1196	0.8876
	c_6	0.8472	(0.9204, 0.3100; 0.9712, 0.7932)	0.3249	0.2381	0.7848
a_3	c_1	0.5948	(0.7712, 0.5200; 0.9302, 0.6223)	0.5932	0.3672	0.6138
	c_2	0.9046	(0.9511, 0.2500; 0.9834, 0.8364)	0.2570	0.1813	0.8308
	c_3	0.9461	(0.9727, 0.2100; 0.9951, 0.8646)	0.2126	0.0989	0.8628
	c_4	0.6275	(0.7921, 0.4700; 0.9211, 0.6591)	0.5355	0.3894	0.6465
	c_5	0.8399	(0.9165, 0.3100; 0.9675, 0.7923)	0.3262	0.2530	0.7828
	c_6	0.9859	(0.9929, 0.1100; 0.9990, 0.9298)	0.1103	0.0447	0.9293
a_4	c_1	0.9385	(0.9687, 0.2100; 0.9912, 0.8641)	0.2135	0.1321	0.8609
	c_2	0.9684	(0.9841, 0.1500; 0.9954, 0.9037)	0.1513	0.0955	0.9019
	c_3	0.8930	(0.9450, 0.2700; 0.9828, 0.8228)	0.2783	0.1847	0.8173
	c_4	0.8507	(0.9223, 0.3100; 0.9730, 0.7936)	0.3242	0.2306	0.7857
	c_5	0.1409	(0.3754, 0.7900; 0.8747, 0.2824)	1.1272	0.4847	0.3097
	c_6	0.9561	(0.9778, 0.1800; 0.9942, 0.8841)	0.1820	0.1072	0.8819
a_5	c_1	0.9783	(0.9891, 0.1400; 0.9989, 0.9105)	0.1406	0.0458	0.9101
	c_2	0.9878	(0.9939, 0.1000; 0.9989, 0.9362)	0.1003	0.0469	0.9357
	c_3	0.6784	(0.8236, 0.4200; 0.9246, 0.6998)	0.4716	0.3811	0.6847
	c_4	0.9086	(0.9532, 0.2600; 0.9881, 0.8305)	0.2663	0.1541	0.8265
	c_5	0.0154	(0.1240, 0.8800; 0.8887, 0.0891)	1.4308	0.4585	0.1349
	c_6	0.4277	(0.6540, 0.5900; 0.8808, 0.5327)	0.7340	0.4735	0.5288

Table 2 Computation outcomes related to the PF point operator-oriented lower estimation

a_i	c_j	β_{ij}	$N_{\beta_{ij}}(p_{ij}) = (\mu_{ij}^N, \nu_{ij}^N; r_{ij}^N, d_{ij}^N)$	θ_{ij}^N	τ_{ij}^N	$V(N_{\beta_{ij}}(p_{ij}))$
a_1	c_1	0.9398	(0.2100, 0.9695; 0.9919, 0.1358)	1.3575	0.1267	0.1387
	c_2	0.8482	(0.3300, 0.9210; 0.9783, 0.2190)	1.2267	0.2072	0.2251
	c_3	0.5484	(0.4900, 0.7406; 0.8880, 0.3721)	0.9863	0.4599	0.3864
	c_4	0.8535	(0.2900, 0.9239; 0.9683, 0.1936)	1.2666	0.2498	0.2033
	c_5	0.0200	(0.9100, 0.1414; 0.9209, 0.9018)	0.1542	0.3897	0.8701
	c_6	0.3999	(0.4900, 0.6324; 0.8000, 0.4197)	0.9116	0.6000	0.4357
a_2	c_1	0.8086	(0.3600, 0.8992; 0.9686, 0.2424)	1.1900	0.2485	0.2505
	c_2	0.4324	(0.5500, 0.6575; 0.8572, 0.4435)	0.8742	0.5149	0.4515
	c_3	0.3524	(0.6100, 0.5936; 0.8512, 0.5086)	0.7718	0.5249	0.5074
	c_4	0.7025	(0.4100, 0.8381; 0.9330, 0.2896)	1.1158	0.3598	0.3037
	c_5	0.0432	(0.8000, 0.2079; 0.8266, 0.8382)	0.2542	0.5628	0.7795
	c_6	0.1528	(0.7300, 0.3909; 0.8281, 0.6870)	0.4916	0.5606	0.6549
a_3	c_1	0.4052	(0.6300, 0.6366; 0.8956, 0.4967)	0.7906	0.4448	0.4970
	c_2	0.0954	(0.7700, 0.3088; 0.8296, 0.7572)	0.3814	0.5583	0.7134
	c_3	0.0539	(0.8800, 0.2321; 0.9101, 0.8358)	0.2579	0.4144	0.8056
	c_4	0.3725	(0.6100, 0.6103; 0.8629, 0.4998)	0.7857	0.5054	0.4998
	c_5	0.1601	(0.7100, 0.4001; 0.8150, 0.6733)	0.5132	0.5795	0.6412
	c_6	0.0141	(0.9200, 0.1187; 0.9276, 0.9183)	0.1283	0.3735	0.8880
a_4	c_1	0.0615	(0.8200, 0.2481; 0.8567, 0.8130)	0.2938	0.5158	0.7681
	c_2	0.0316	(0.8300, 0.1778; 0.8488, 0.8656)	0.2111	0.5287	0.8104
	c_3	0.1070	(0.7800, 0.3271; 0.8458, 0.7472)	0.3971	0.5335	0.7091
	c_4	0.1493	(0.7400, 0.3864; 0.8348, 0.6937)	0.4812	0.5506	0.6617
	c_5	0.8591	(0.3200, 0.9268; 0.9805, 0.2116)	1.2384	0.1963	0.2173
	c_6	0.0439	(0.8400, 0.2095; 0.8657, 0.8444)	0.2445	0.5005	0.7981
a_5	c_1	0.0217	(0.9400, 0.1473; 0.9515, 0.9010)	0.1554	0.3077	0.8816
	c_2	0.0122	(0.9000, 0.1104; 0.9067, 0.9223)	0.1221	0.4217	0.8829
	c_3	0.3216	(0.6100, 0.5671; 0.8329, 0.5232)	0.7490	0.5534	0.5193
	c_4	0.0914	(0.8200, 0.3022; 0.8739, 0.7752)	0.3531	0.4861	0.7405
	c_5	0.9846	(0.1100, 0.9923; 0.9984, 0.0703)	1.4604	0.0573	0.0710
	c_6	0.5723	(0.5100, 0.7565; 0.9124, 0.3776)	0.9776	0.4093	0.3883

Table 3 Results of the penalty weights and their criterion-wise predominating ranks

a_i	c_1	c_2	c_3	c_4	c_5	c_6
Results of the penalty weight $Pen(p_{ij})$						
a_1	0.3000	0.3000	0.2861	0.2917	0.1057	0.2601
a_2	0.2500	0.2500	0.2357	0.2583	0.1450	0.2023
a_3	0.2000	0.1951	0.1035	0.2000	0.1993	0.1000
a_4	0.1500	0.1483	0.1465	0.1392	0.2500	0.1500
a_5	0.1000	0.1066	0.2282	0.1108	0.3000	0.2876
Results of the criterion-wise predominating rank Φ_{ij}						
a_1	5th	5th	5th	5th	1st	4th
a_2	4th	4th	4th	4th	2nd	3rd
a_3	3rd	3rd	1st	3rd	3rd	1st
a_4	2nd	2nd	2nd	2nd	4th	2nd
a_5	1st	1st	3rd	1st	5th	5th

Table 4 Computation results related to the comprehensive disagreement indicator

a_i	c_1	c_2	c_3	c_4	c_5	c_6	Δ_i^{ϕ}
$\phi=1$	Results of $Pen(p_{ij}) \cdot \lvert \Phi_{ij}-1 \rvert$						Results of Δ_i^1
a_1	1.2000	1.2000	1.1444	1.1668	0.0000	0.7804	(0.9482, 0.0112; 0.9482, 0.9925)
a_2	0.7500	0.7500	0.7071	0.7749	0.1450	0.4045	(0.8742, 0.0575; 0.8761, 0.9582)
a_3	0.4000	0.3902	0.0000	0.4000	0.3987	0.0000	(0.6496, 0.3161; 0.7225, 0.7117)
a_4	0.1500	0.1483	0.1465	0.1392	0.7500	0.1500	(0.7891, 0.1825; 0.8099, 0.8553)
a_5	0.0000	0.0000	0.4564	0.0000	1.2000	1.1505	(0.9664, 0.0132; 0.9665, 0.9913)
$\phi=2$	Results of $Pen(p_{ij}) \cdot \lvert \Phi_{ij}-2 \rvert$						Results of Δ_i^2
a_1	0.9000	0.9000	0.8583	0.8751	0.1057	0.5202	(0.9064, 0.0336; 0.9071, 0.9764)
a_2	0.5000	0.5000	0.4714	0.5166	0.0000	0.2023	(0.7411, 0.1936; 0.7660, 0.8373)
a_3	0.2000	0.1951	0.1035	0.2000	0.1993	0.1000	(0.6068, 0.4172; 0.7364, 0.6166)
a_4	0.0000	0.0000	0.0000	0.0000	0.5000	0.0000	(0.5819, 0.5000; 0.7672, 0.5481)
a_5	0.1000	0.1066	0.2282	0.1108	0.9000	0.8629	(0.9323, 0.0375; 0.9331, 0.9744)
$\phi=3$	Results of $Pen(p_{ij}) \cdot \lvert \Phi_{ij}-3 \rvert$						Results of Δ_i^3
a_1	0.6000	0.6000	0.5722	0.5834	0.2114	0.2601	(0.8275, 0.1010; 0.8337, 0.9227)
a_2	0.2500	0.2500	0.2357	0.2583	0.1450	0.0000	(0.5675, 0.4360; 0.7156, 0.5829)
a_3	0.0000	0.0000	0.2070	0.0000	0.0000	0.2000	(0.5562, 0.5506; 0.7826, 0.5032)
a_4	0.1500	0.1483	0.1465	0.1392	0.2500	0.1500	(0.6554, 0.3649; 0.7502, 0.6766)
a_5	0.2000	0.2131	0.0000	0.2215	0.6000	0.5753	(0.8608, 0.1061; 0.8673, 0.9219)
$\phi=4$	Results of $Pen(p_{ij}) \cdot \lvert \Phi_{ij}-4 \rvert$						Results of Δ_i^4
a_1	0.3000	0.3000	0.2861	0.2917	0.3171	0.0000	(0.6656, 0.3034; 0.7315, 0.7277)
a_2	0.0000	0.0000	0.0000	0.0000	0.2899	0.2023	(0.6268, 0.4558; 0.7750, 0.5997)
a_3	0.2000	0.1951	0.3106	0.2000	0.1993	0.3000	(0.7507, 0.2297; 0.7851, 0.8110)
a_4	0.3000	0.2966	0.2930	0.2785	0.0000	0.3000	(0.7128, 0.2664; 0.7609, 0.7723)
a_5	0.3000	0.3197	0.2282	0.3323	0.3000	0.2876	(0.7750, 0.1861; 0.7970, 0.8499)
$\phi=5$	Results of $Pen(p_{ij}) \cdot \lvert \Phi_{ij}-5 \rvert$						Results of Δ_i^5
a_1	0.0000	0.0000	0.0000	0.0000	0.4228	0.2601	(0.7035, 0.3397; 0.7812, 0.7136)
a_2	0.2500	0.2500	0.2357	0.2583	0.4349	0.4045	(0.8261, 0.1354; 0.8371, 0.8966)
a_3	0.4000	0.3902	0.4141	0.4000	0.3987	0.4000	(0.8510, 0.0958; 0.8564, 0.9286)
a_4	0.4500	0.4450	0.4394	0.4177	0.2500	0.4500	(0.8482, 0.0972; 0.8537, 0.9274)
a_5	0.4000	0.4263	0.4564	0.4430	0.0000	0.0000	(0.6199, 0.3265; 0.7007, 0.6914)

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Data availability The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

Declarations

Conflict of interest The author declares that she has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Human and animal rights This article does not contain any studies with human participants or animals that were performed by the author.

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