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Hybrid group decision-making technique under spherical fuzzy *N*-soft expert sets

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Abstract

This paper presents the concept of a new hybrid model called spherical fuzzy *N*-soft expert sets, which is an extension of spherical fuzzy soft expert sets. The proposed model is highly suitable to describe the multinary data evaluation in terms of spherical fuzzy soft information considering multiple experts' opinions. Some fundamental properties, including subset, weak complement, spherical fuzzy complement, spherical fuzzy weak complement, union, intersection, AND operation, and OR operation, are discussed. Our proposed concepts are explained with detailed examples. An efficient algorithm is developed to solve multi-attribute group decision-making (MAGDM) problems. Further, to guarantee the high applicability scope and flexibility of our initiated framework, two real-world MAGDM problems, that is, predicting local election results using survey ratings before the election and ranking credibility of the smartphones using customer feedback, are solved. Finally, to endorse the accuracy and advantages of the proposed technique, a comprehensive comparative analysis of the presented approach with existing models such as spherical fuzzy soft expert sets and *N*-soft sets is provided.

Keywords Soft expert set · *N*-soft set · Spherical fuzzy *N*-soft expert set · Algorithm · Group decision-making

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1 Introduction

Ranking alternatives, going for the best option available and making predictions based on available information are all concerned by decision-makers as important aspects of the decision sciences. But these scenarios become more problematic when dealing with uncertainties. Many researchers have developed numerous mathematical tools and algorithms to deal with such situations following the invention of modern probability theory in the 16th century. These models are used widely in different areas ranging from social sciences to medical sciences and engineering, from artificial intelligence to commerce and economics. As a revolution in decision sciences, Zadeh (1965) introduced the concept of fuzzy sets capable of dealing with situations not solvable by crisp set theory. A fuzzy set allowed membership degrees for elements ranging in the closed interval [0, 1], thus handling partial truth between absolute false and absolute truth. This concept was later used to solve many decision-making situations concerning uncertain and vague information (Alcantud 2016).

One limitation of fuzzy set theory is that it restricts the non-membership degree $\nu(u)$ of the element 'u' to the condition $\nu(u) = 1 - \mu(u)$ where $\mu(u)$ is the membership degree. To overcome this limitation, Atanassov (1986) introduced the idea of intuitionistic fuzzy sets (IFSs) as a generalization of fuzzy sets by considering two aspects, i.e., the membership degree $\mu(u)$ and the non-membership degree $\nu(u)$ of the element 'u' with the condition $0 \leq \mu(u) + \nu(u) \leq 1$. Later, extensions to this like interval-valued intuitionistic fuzzy sets (Atanassov and Gargov 1989) were developed. These models were found to be restricted by the limitation that they cannot solve problems, where the sum of membership and non-membership degrees exceeds 1. This limitation led Atanassov (1999) to develop an extended version of IFSs called 'IFSs of second type' with the condition $0 \leq \mu^2(u) + \nu^2(u) \leq 1$. Later, Yager (2013) presented the idea of Pythagorean fuzzy sets (PyFSs) which is equivalent to IFSs of second type. This generalization of IFSs proved to be more applicable in many problems as compared to the previous models (Peng and Selvachandran 2019; Zhang et al. 2020).

Even with the high applicability, IFSs and PyFSs are not applicable in situations concerning neutral membership degrees. For example, when considering a voter's opinion about a candidate, it can turn out to be satisfactory, dissatisfactory, or neutral (neither satisfactory nor dissatisfactory). Moreover, the public's views about the role of social media[1] in a country vary widely by political affiliation and ideology, that is, positive, negative, and neutral opinions (see Fig. 1). To overcome this limitation, the concept of picture fuzzy sets (PFSs) was developed by Cuong (2013a, 2013b) introducing three indices, i.e., positive membership degree $\mu(u)$, neutral membership degree $\tau(u)$, and negative membership degree $\nu(u)$ of an element 'u' with the condition $0 \leq \mu(u) + \tau(u) + \nu(u) \leq 1$. This concept proved to be very helpful in dealing situations concerning positive, negative and neutral aspects. Peng and Luo (2021) presented a decision-making model for China's stock market bubble warning under picture fuzzy information. Lin et al. (2021b) developed certain picture fuzzy aggregation operators based on interactional partitioned Heronian mean and applied them to solve a decision-making problem. Despite their high applicability, they fail to be used in situations where the sum of these three degrees exceeds 1. This situation led to the introduction of spherical fuzzy sets (SFSs) by Kahraman and Gündoğdu (2018) as a powerful extension of PFSs with the condition $0 \leq \mu^2(u) + \tau^2(u) + \nu^2(u) \leq 1$.

[1] https://www.pewresearch.org/fact-tank/2020/10/15/.

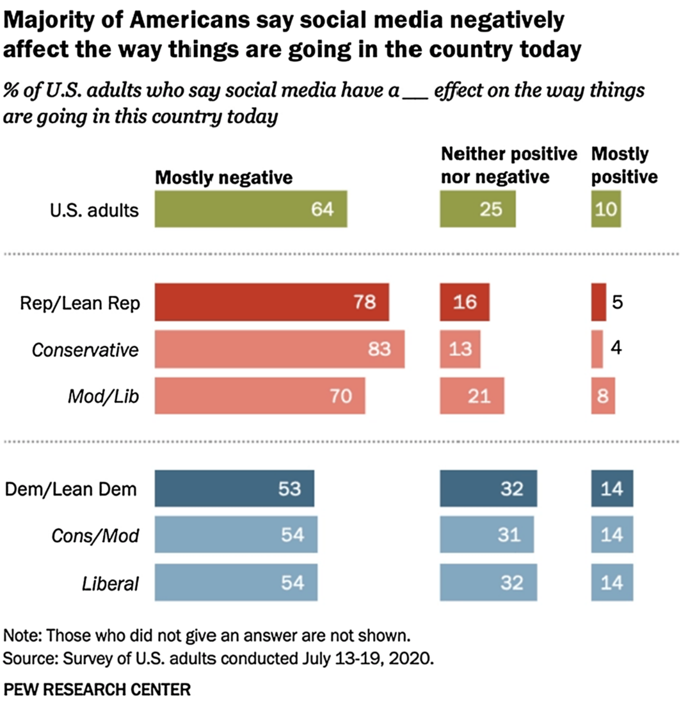

Fig. 1 Survey report result of United States (U.S.) public views about impact of social media by Pew research center

Later, Gündogdu and Kahraman (2019) presented generalized SFSs and TOPSIS method based on SFSs together with decision-making application. This concept allows us to deal with situations too complex to be dealt with the models discussed above (see Akram et al. 2021f, g).

A common limitation of all the above methods is their inefficiency in dealing with situations containing different parameters. To solve this lack of parameterization, Molodtsov (1999) was the first who initiated the concept of soft sets. His method is different from the pre-existing methods, which allows handling situations concerning different parameters. The soft set theory has proved to be applicable in numerous problems from various scientific domains like medicine, economics, engineering, computer sciences, etc. Maji et al. (2002) offered an application of soft sets in their work. Ali et al. (2009) investigated several properties of soft sets. The powerful parameterization capability led many researchers to find extensions of the model and the development of hybrid models combining the strength of soft sets with the already existing models. These include picture fuzzy soft sets (Yang et al. 2015), interval-valued m-polar fuzzy soft sets (Akram et al. 2021e), spherical fuzzy soft sets (Perveen et al. 2019), and many more.

These decision-making tools proved to be quite handy in their respective domains. However, there exist situations in many real-life problems and decisive scenarios where the opinion of more than one expert is needed. For example, in selecting the best candidate for a high-rank job, the selection committee needs to take multiple judges' opinions about the candidate's suitability. Similarly, in the circumstances where a questionnaire is to be distributed, it is better to have a model carrying all experts' opinions in one place, rather than applying different operations to combine results using single expert models. Considering this need, Alkhazaleh and Salleh (2011) introduced the idea of soft expert sets (SESs). This model allows users to handle multiple expert's opinions in one model without performing any operations. Later, the same authors combined their model with fuzzy set theory introducing fuzzy SESs (Alkhazaleh and Salleh 2014). Many other researchers also extended the previously existing models to the multiple expert approaches. Bashir and Salleh (2012a) introduced the concept of fuzzy parameterized SESs. Perveen et al. (2020) developed spherical fuzzy SESs. Many other similar hybrid models involving SESs as their component have been proposed (Akram et al. 2021a; Bashir and Salleh 2012b).

All the above models used binary grading procedures in dealing with uncertainties. However, in daily-life, we often encounter non-binary evaluations in many areas. For instance, Alcantud and Laruelle (2014) studied ternary voting situations in a social choice environment. We often encounter non-binary or multinary evaluations in ranking or rating systems. Some real-life examples include rating an app on the play-store or app-store, rating a hotel, or rating services of a telecommunication company using multinary evaluations. These ratings may adopt the form of the number of stars or hearts (like three stars, two stars, etc.) by numbers as labels (0 for bad, 1 for average, 2 for good). In such situations, a model is needed to deal efficiently with multinary information. For this, Fatimah et al. (2018) introduced the notion of *N*-soft sets (NSSs), which deals with situations concerning multinary information in the forms of ratings and grades. Many hybrid models have also been developed with NSSs to allow applicability to the already existing tools outside the binary evaluations. These include intuitionistic fuzzy *N*-soft rough sets (Akram et al. 2019), *N*-SESs and fuzzy *N*-SESs (Ali and Akram 2020), etc. For more useful terminologies and an overview of the recent advances, referred to Akram et al. (2021b, 2021c, 2021d), Ali et al. (2020), Ashraf et al. (2019), Aydogdu and Gül (2020), Fatimah and Alcantud (2021), Gündogdu (2020), Gündogdu and Kahraman (2020a, 2020b, 2021), Huang et al. (2020), Kahraman and Gündogdu (2020), Kamaci and Petchimuthu (2020), Lin et al. (2020, 2021a, 2021c), Liu et al. (2021a), Mahmood et al. (2019) and Cuong and Kreinovich (2014).

In this paper, we develop a novel hybrid model called spherical fuzzy *N*-soft expert sets by combining spherical fuzzy soft expert sets (SFSESs) (Perveen et al. 2020) and NSSs (Fatimah et al. 2018). The motivations of the paper are described as follows:

1. The inability of existing models like intuitionistic fuzzy NSSs (Akram et al. 2019), Pythagorean fuzzy NSSs (Zhang et al. 2020), and fuzzy *N*-SESs (Ali and Akram 2020) in dealing with situations concerning positive, negative, and neutral behavior of the experts.
2. The restriction of SFSESs (Perveen et al. 2020) to binary evaluations.
3. Further, the role of SFSs is very significant in sentiment analysis[2] (or opinion mining) to find whether data is positive, neutral, or negative. Sentiment analysis is often executed on textual data to support businesses monitor brand and product sentiment in customer response and understand customer requirements (see Fig. 2).

The contributions of this paper are described as follows:

[2] https://monkeylearn.com/sentiment-analysis/.

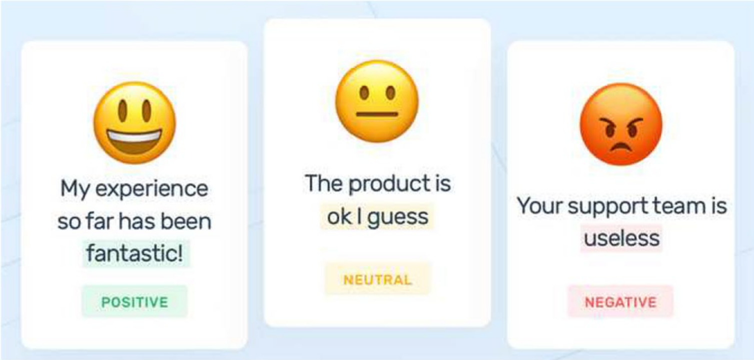

Fig. 2 Sentiment analysis

1. The highly applicable and powerful model of SFSESs is merged with the multinary evaluation skills of NSSs to develop a model with higher applicability and better handling of multinary information.
2. The operations, properties, and results of the proposed model are provided and supported with illustrative examples.
3. A real-world application based on the MAGDM scenario, i.e., prediction of local election results using survey reports before the election, is modeled and solved with the proposed model.
4. An efficient algorithm to solve MAGDM problems under SFNSESs is provided.
5. The advantages and limitations of the model; and comparison with the existing models are provided in the comparative analysis.

The remaining paper is organized in the following manner: Sect. 2 recalls some necessary definitions and results to be used afterward in the paper. Section 3 presents a new hybrid model, namely, SFNSESs, and discusses its operations and properties, including subset relation, complements, unions, intersections, agree-SFNSES, disagree-SFNSES, AND operation and OR operation. In Sect. 4, two real-world applications are modeled and solved using SFNSESs. An algorithm is also developed. In Sect. 5, a comparative analysis of the newly developed SFNSES model with pre-existing SFSES and NSS models is provided. Finally, in Sect. 6, some concluding remarks and future directions are provided.

2 Preliminaries

In this section, we recall some definitions and results that will be used throughout the paper. We first go through the notion of a SFS, which is a direct generalization of PFSs.

Definition 1 (Kahraman and Gündoğdu 2018) A *spherical fuzzy set* or SFS $\mathcal{S}$ on a universe $\mathcal{U}$ is an object of the form

$$\mathcal{S} = \{(u, \mu_{\mathcal{S}}(u), \tau_{\mathcal{S}}(u), \nu_{\mathcal{S}}(u)) | u \in \mathcal{U}\}$$

where $\mu_{\mathcal{S}}(u), \tau_{\mathcal{S}}(u), \nu_{\mathcal{S}}(u) \in [0,1]$ are called the positive membership degree, neutral membership degree and negative membership degree of $u \in \mathcal{U}$, respectively, following the condition,

$$\mu^2_{\mathcal{S}}(u) + \tau^2_{\mathcal{S}}(u) + \nu^2_{\mathcal{S}}(u) \leq 1, \ \forall\, u \in \mathcal{U}.$$

We denote the set of all SFSs on $\mathcal{U}$ by $SFS(\mathcal{U})$.

Since SFS model acts as a basic component in constructing the proposed model, most of the operations and properties of the proposed model are based on the properties of SFSs, which are given below:

Definition 2 (Kahraman and Gündoğdu 2018) For any two SFSs $\mathcal{A}$ and $\mathcal{B}$ over the universe $\mathcal{U}$, we have the following:

1. $\mathcal{A} \subseteq \mathcal{B}$ if $\forall u \in \mathcal{U}$, $\mu_{\mathcal{A}}(u) \leq \mu_{\mathcal{B}}(u), \tau_{\mathcal{A}}(u) \leq \tau_{\mathcal{B}}(u)$ and $\nu_{\mathcal{A}}(u) \geq \nu_{\mathcal{B}}(u)$.
2. $\mathcal{A} = \mathcal{B}$ iff $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{A}$.
3. $\mathcal{A} \cup \mathcal{B} = \{(u, \max\{\mu_{\mathcal{A}}(u), \mu_{\mathcal{B}}(u)\}, \min\{\tau_{\mathcal{A}}(u), \tau_{\mathcal{B}}(u)\}, \min\{\nu_{\mathcal{A}}(u), \nu_{\mathcal{B}}(u)\}) | u \in \mathcal{U}\}$.
4. $\mathcal{A} \cap \mathcal{B} = \{(u, \min\{\mu_{\mathcal{A}}(u), \mu_{\mathcal{B}}(u)\}, \min\{\tau_{\mathcal{A}}(u), \tau_{\mathcal{B}}(u)\}, \max\{\nu_{\mathcal{A}}(u), \nu_{\mathcal{B}}(u)\}) | u \in \mathcal{U}\}$.
5. $(\mathcal{A})^c = \{(u, \nu_{\mathcal{A}}(u), \tau_{\mathcal{A}}(u), \mu_{\mathcal{A}}(u)) | u \in \mathcal{U}\}$.

To deal with problems considering multiple parameters, the notion of SFSSs is given below.

Definition 3 (Perveen et al. 2019) Let $\mathcal{U}$ be the universe, P a set of parameters and $X \subseteq P$. A pair $(\mathcal{F}, X)$ is called a *spherical fuzzy soft set* (or SFSS) over $\mathcal{U}$, if $\mathcal{F} : X \rightarrow SFS(\mathcal{U})$ is a mapping defined as

$$\mathcal{F}(x) = \{(u, \mu_{\mathcal{F}(x)}(u), \tau_{\mathcal{F}(x)}(u), \nu_{\mathcal{F}(x)}(u)) | u \in \mathcal{U}, x \in X\}$$

where $\mu_{\mathcal{F}(x)}(u), \tau_{\mathcal{F}(x)}(u), \nu_{\mathcal{F}(x)}(u)$ are positive membership degree, neutral membership degree and negative membership degree, respectively, with the condition, $\mu^2_{\mathcal{F}(x)}(u) + \tau^2_{\mathcal{F}(x)}(u) + \nu^2_{\mathcal{F}(x)}(u) \leq 1$.

Some more useful notions like level sets and the threshold functions for SFSSs are provided below.

Definition 4 (Perveen et al. 2019) Suppose $\varpi = (\mathcal{F}, X)$ is a SFSS over the universe $\mathcal{U}$. Let $\lambda : X \rightarrow [0,1]^3$ be a function, such that $\lambda(x) = (\mathfrak{p}(x), \mathfrak{q}(x), \mathfrak{r}(x))\, \forall\, x \in X$, where $\mathfrak{p}(x), \mathfrak{q}(x), \mathfrak{r}(x) \in [0,1]$. Then, the level soft set of ϖ with respect to λ is a crisp set $\mathcal{L}(\varpi, \lambda) = (\mathcal{F}_{\lambda}, X)$, defined by

$$\mathcal{F}_{\lambda}(x) = \{u \in \mathcal{U} | \mu_{\mathcal{F}(x)}(u) \geq \mathfrak{p}(x), \tau_{\mathcal{F}(x)}(u) \leq \mathfrak{q}(x), \nu_{\mathcal{F}(x)}(u) \leq \mathfrak{r}(x)\}, \ \forall\, x \in X.$$

Definition 5 (Perveen et al. 2019) Let $\varpi = (\mathcal{F}, X)$ be a SFSS over $\mathcal{U}$, then following are the four well-known threshold functions for ϖ defined as:

1. *Mid-level Threshold Function* (mid_{ϖ}):
   For $\varpi = (\mathcal{F}, X)$, the function $mid_{\varpi} : X \to [0,1]^3$ is defined as

$$mid_{\varpi}(x) = \left(\mathfrak{p}_{mid_{\varpi}}(x), \mathfrak{q}_{mid_{\varpi}}(x), \mathfrak{r}_{mid_{\varpi}}(x)\right) \quad \forall\, x \in X,$$

   such that

$$\mathfrak{p}_{mid_{\varpi}}(x) = \frac{1}{|\mathcal{U}|}\sum_{u\in\mathcal{U}} \mu_{\mathcal{F}(x)}(u);\ \mathfrak{q}_{mid_{\varpi}}(x) = \frac{1}{|\mathcal{U}|}\sum_{u\in\mathcal{U}} \tau_{\mathcal{F}(x)}(u);\ \mathfrak{r}_{mid_{\varpi}}(x) = \frac{1}{|\mathcal{U}|}\sum_{u\in\mathcal{U}} \nu_{\mathcal{F}(x)}(u).$$

   The corresponding level soft set $\mathcal{L}(\varpi, mid_{\varpi}(x))$ is termed as the mid-level soft set of ϖ.
2. *Top-bottom-bottom-level Threshold Function* (tbb_{ϖ}):
   For $\varpi = (\mathcal{F}, X)$, the function $tbb_{\varpi} : X \to [0,1]^3$ is defined as

$$tbb_{\varpi}(x) = \left(\mathfrak{p}_{tbb_{\varpi}}(x), \mathfrak{q}_{tbb_{\varpi}}(x), \mathfrak{r}_{tbb_{\varpi}}(x)\right) \quad \forall\, x \in X,$$

   such that

$$\mathfrak{p}_{tbb_{\varpi}}(x) = \max_{u\in\mathcal{U}}\ \mu_{\mathcal{F}(x)}(u);\ \mathfrak{q}_{tbb_{\varpi}}(x) = \min_{u\in\mathcal{U}}\ \tau_{\mathcal{F}(x)}(u);\ \mathfrak{r}_{tbb_{\varpi}}(x) = \min_{u\in\mathcal{U}}\ \nu_{\mathcal{F}(x)}(u).$$

   The corresponding level soft set $\mathcal{L}(\varpi, tbb_{\varpi}(x))$ is termed as the tbb-level soft set of ϖ.
3. *Bottom-bottom-bottom-level Threshold Function* (bbb_{ϖ}):
   For $\varpi = (\mathcal{F}, X)$, the function $bbb_{\varpi} : X \to [0,1]^3$ is defined as

$$bbb_{\varpi}(x) = \left(\mathfrak{p}_{bbb_{\varpi}}(x), \mathfrak{q}_{bbb_{\varpi}}(x), \mathfrak{r}_{bbb_{\varpi}}(x)\right) \quad \forall\, x \in X,$$

   such that

$$\mathfrak{p}_{bbb_{\varpi}}(x) = \min_{u\in\mathcal{U}}\ \mu_{\mathcal{F}(x)}(u);\ \mathfrak{q}_{bbb_{\varpi}}(x) = \min_{u\in\mathcal{U}}\ \tau_{\mathcal{F}(x)}(u);\ \mathfrak{r}_{bbb_{\varpi}}(x) = \min_{u\in\mathcal{U}}\ \nu_{\mathcal{F}(x)}(u).$$

   The corresponding level soft set $\mathcal{L}(\varpi, bbb_{\varpi}(x))$ is termed as the bbb-level soft set of ϖ.
4. *Med Threshold Function* (med_{ϖ}):
   For $\varpi = (\mathcal{F}, X)$, the function $med_{\varpi} : X \to [0,1]^3$ is defined as

$$med_{\varpi}(x) = \left(\mathfrak{p}_{med_{\varpi}}(x), \mathfrak{q}_{med_{\varpi}}(x), \mathfrak{r}_{med_{\varpi}}(x)\right) \quad \forall\, x \in X,$$

   such that

$$\mathfrak{p}_{med_{\varpi}}(x) = \begin{cases} \mu_{\mathcal{F}(x)}\left(u_{\left(\frac{|\mathcal{U}|+1}{2}\right)}\right), & \text{if } |\mathcal{U}| \text{ is odd,} \\ \dfrac{\mu_{\mathcal{F}(x)}\left(u_{\left(\frac{|\mathcal{U}|}{2}\right)}\right) + \mu_{\mathcal{F}(x)}\left(u_{\left(\frac{|\mathcal{U}|}{2}+1\right)}\right)}{2}, & \text{if } |\mathcal{U}| \text{ is even.} \end{cases}$$

$$\mathfrak{q}_{med_{\varpi}}(x) = \begin{cases} \tau_{\mathcal{F}(x)}\left(u_{\left(\frac{|\mathcal{U}|+1}{2}\right)}\right), & \text{if } |\mathcal{U}| \text{ is odd,} \\ \dfrac{\tau_{\mathcal{F}(x)}\left(u_{\left(\frac{|\mathcal{U}|}{2}\right)}\right) + \tau_{\mathcal{F}(x)}\left(u_{\left(\frac{|\mathcal{U}|}{2}+1\right)}\right)}{2}, & \text{if } |\mathcal{U}| \text{ is even.} \end{cases}$$

$$\mathfrak{r}_{med_{\varpi}}(x) = \begin{cases} \nu_{\mathcal{F}(x)}\left(u_{\left(\frac{|\mathcal{U}|+1}{2}\right)}\right), & \text{if } |\mathcal{U}| \text{ is odd,} \\ \dfrac{\nu_{\mathcal{F}(x)}\left(u_{\left(\frac{|\mathcal{U}|}{2}\right)}\right) + \nu_{\mathcal{F}(x)}\left(u_{\left(\frac{|\mathcal{U}|}{2}+1\right)}\right)}{2}, & \text{if } |\mathcal{U}| \text{ is even.} \end{cases}$$

Here $\mathfrak{p}_{med_{\varpi}}(x)$, $\mathfrak{q}_{med_{\varpi}}(x)$, $\mathfrak{r}_{med_{\varpi}}(x)$ are the medians by ranking the positive, neutral and negative membership degrees, respectively, arranged from large to small (or small to large). The corresponding level soft set $\mathcal{L}(\varpi, med_{\varpi}(x))$ is termed as the med-level soft set of ϖ.

Following is the definition of SFSESs capable of dealing with situations in a soft expert framework.

Definition 6 (Perveen et al. 2020) Let $\mathcal{U}$ be the universal set, P be the set of parameters, E be the set of experts and $\mathcal{O} = \{1 = agree, 0 = disagree\}$ be the set of opinions. Let $Z = P \times E \times \mathcal{O}$ and $X \subseteq Z$. A pair $(\mathcal{F}, X)$ is said to be a *spherical fuzzy soft expert set* (or SFSES) over $\mathcal{U}$, if $\mathcal{F}$ is a mapping, given by $\mathcal{F} : X \to SFS(\mathcal{U})$.

We now recall the definition of NSS as follows.

Definition 7 (Fatimah et al. 2018) Let $\mathcal{U}$ be the universe of objects, P the set of parameters and $\mathcal{A} \subseteq P$. Let $G = \{0, 1, 2, \ldots, N-1\}$ be the set of grades where $N \in \{2, 3, \ldots\}$. Then $(\mathcal{F}, \mathcal{A}, N)$ is said to be an *N-soft set* or NSS if $\mathcal{F} : \mathcal{A} \to 2^{\mathcal{U} \times G}$, where for each $\alpha \in \mathcal{A}$, $\exists$ a unique $(u, g_{\alpha}) \in \mathcal{U} \times G$ such that $(u, g_{\alpha}) \in \mathcal{F}(\alpha)$, $u \in \mathcal{U}$, $g_{\alpha} \in G$.

3 Spherical fuzzy *N*-soft expert sets

In this section, we first present the main notion of this study and then investigate some of its basic properties and important results with numerical examples.

Let $\mathcal{U}$ be a non-empty universal set of objects, P a set of parameters and E a set of experts. Let $\mathcal{O} = \{1 = \text{agree}, 0 = \text{disagree}\}$ be the set of expert's opinions and $G = \{0, 1, 2, 3, \ldots, N-1\}$ be the set of evaluation grades where $N \in \{2, 3, \ldots\}$. Let $R = P \times E \times \mathcal{O}$ and $X \subseteq R$.

Table 1 Novel ratings

$\mathcal{U}/P$	p_1	p_2	p_3	p_4
$\mathfrak{u}_1$	★★	★★★	★★	★★★
$\mathfrak{u}_2$	★	★★★★	★★★	★★
$\mathfrak{u}_3$	★★★	★★	★★★	★★
$\mathfrak{u}_4$	★★	★★	★★★	★★★

Table 2 5-soft set corresponding to Table 1

$\mathcal{U}/P$	p_1	p_2	p_3	p_4
$\mathfrak{u}_1$	2	3	2	3
$\mathfrak{u}_2$	1	4	3	2
$\mathfrak{u}_3$	3	2	3	2
$\mathfrak{u}_4$	2	2	3	3

Definition 8 A triple (γ, X, N) is said to be a spherical fuzzy N-soft expert set or SFNSES, where γ is a function given as follows:

$$\gamma : X \to SFS(\mathcal{U} \times G),$$

where $SFS(\mathcal{U} \times G)$ represents the set of all spherical fuzzy subsets of $\mathcal{U} \times G$ in such a way that for each $x \in X$ and $\mathfrak{u} \in \mathcal{U}$, $\exists$ a unique pair $(\mathfrak{u}, g_x) \in \mathcal{U} \times G$, such that $\gamma(x) = \langle (\mathfrak{u}, g_x), \kappa(\mathfrak{u}, g_x) \rangle$ for $g_x \in G$ and $\kappa(\mathfrak{u}, g_x) \in SFS(\mathcal{U} \times G)$ where

$$\kappa(\mathfrak{u}, g_x) = \Big(\mu_{\gamma(x)}(\mathfrak{u}, g_x), \tau_{\gamma(x)}(\mathfrak{u}, g_x), \nu_{\gamma(x)}(\mathfrak{u}, g_x) \Big),$$

with the condition $\mu^2_{\gamma(x)}(\mathfrak{u}, g_x) + \tau^2_{\gamma(x)}(\mathfrak{u}, g_x) + \nu^2_{\gamma(x)}(\mathfrak{u}, g_x) \leq 1$.

Here g_x denotes the grade of the objects with respect to parameters, $\mu_{\gamma(x)}(\mathfrak{u}, g_x)$ denotes the positive membership degree, $\tau_{\gamma(x)}(\mathfrak{u}, g_x)$ denotes neutral membership degree, and $\nu_{\gamma(x)}(\mathfrak{u}, g_x)$ denotes the negative membership degree. The condition $\mu^2_{\gamma(x)}(\mathfrak{u}, g_x) + \tau^2_{\gamma(x)}(\mathfrak{u}, g_x) + \nu^2_{\gamma(x)}(\mathfrak{u}, g_x) \leq 1 \,\forall x \in X$ and $(\mathfrak{u}, g_x) \in \mathcal{U} \times G$ is kept as a consistency constraint for dealing with the spherical fuzzy soft data.

The following example illustrates our proposed idea of SFNSESs.

Example 1 In a book awards contest, multiple novels are nominated for the “Best Novel of the Year” title on the basis of reader’s ratings. The four most top-rated novels are shortlisted, comprising the set $\mathcal{U} = \{\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3, \mathfrak{u}_4\}$ for the selection of the best novel. The reader’s ratings (0–4 stars) on the basis of parameters $P = \{p_1 =$ innovative $, p_2 =$ strong storyline $, p_3 =$ lengthy $, p_4 =$ entertaining $\}$ are provided in Table 1.

These star ratings can be interpreted as natural numbers making a 5-soft set as shown in Table 2, where

Table 3 Rating criteria

Grade	Positive membership	Neutral membership	Negative membership
g_x	$\mu_{\gamma(x)}$	$\tau_{\gamma(x)}$	$\nu_{\gamma(x)}$
0	[0.0, 0.2)	[0, 0.182)	[0.8, 1.0]
1	[0.2, 0.4)	(0, 0.121]	[0.6, 0.8)
2	[0.4, 0.6)	[0, 0.101)	[0.4, 0.6)
3	[0.6, 0.8)	(0, 0.017)	[0.2, 0.4)
4	[0.8, 1.0]	[0, 0.017)	[0.0, 0.2)

'•': 0 stands for 'Bad',
'⋆': 1 stands for 'Below average',
'⋆ ⋆': 2 stands for 'Average',
'⋆ ⋆ ⋆': 3 stands for 'Above average',
'⋆ ⋆ ⋆ ⋆': 4 stands for 'Outstanding'

Consider there are three judges (experts) comprising the set $E = \{r, s, t\}$ to declare the final decision using the above star ratings. The extraction of data as grades is easier in this case, but due to the uncertainty in ratings by multiple users, the experts decide to interpret the data in terms of positive membership, neutral membership, and negative membership as defined in Definition 8. The following criteria are used for integrating the provided star ratings with spherical fuzzy data by the experts:

$$\begin{array}{ll} -1.0 \leq S_x < -0.6, & if\ g_x = 0, \\ -0.6 \leq S_x < -0.2, & if\ g_x = 1, \\ -0.2 \leq S_x < 0.2, & if\ g_x = 2, \\ 0.2 \leq S_x < 0.6, & if\ g_x = 3, \\ 0.6 \leq S_x \leq 1.0, & if\ g_x = 4, \end{array}$$

where $S_x = \mu^2_{\gamma(x)} - \tau^2_{\gamma(x)} - \nu^2_{\gamma(x)}$, such that $S_x \in [-1, 1]$.

Based upon the above criteria, we get Table 3 as below:

Finally, using Table 3 and Definition 8, the spherical fuzzy 5-soft expert set is provided as follows:

Table 4 General tabular representation of the SFNSES (γ, X, N)

(γ, X, N)	$\mathfrak{u}_1$	$\mathfrak{u}_2$	$\cdots$	$\mathfrak{u}_n$
x_1	$\langle g_{11}, (\mu_{11}, \tau_{11}, \nu_{11})\rangle$	$\langle g_{12}, (\mu_{12}, \tau_{12}, \nu_{12})\rangle$	$\cdots$	$\langle g_{1n}, (\mu_{1n}, \tau_{1n}, \nu_{1n})\rangle$
x_2	$\langle g_{21}, (\mu_{21}, \tau_{21}, \nu_{21})\rangle$	$\langle g_{22}, (\mu_{22}, \tau_{22}, \nu_{22})\rangle$	$\cdots$	$\langle g_{2n}, (\mu_{2n}, \tau_{2n}, \nu_{2n})\rangle$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
x_m	$\langle g_{m1}, (\mu_{m1}, \tau_{m1}, \nu_{m1})\rangle$	$\langle g_{m2}, (\mu_{m2}, \tau_{m2}, \nu_{m2})\rangle$	$\cdots$	$\langle g_{mn}, (\mu_{mn}, \tau_{mn}, \nu_{mn})\rangle$

$$\gamma(p_1, r, 1) = \left\{ \begin{array}{l} \langle(\mathfrak{u}_1, 2), (0.55, 0.09, 0.40)\rangle,\ \langle(\mathfrak{u}_2, 1), (0.33, 0.11, 0.75)\rangle, \\ \langle(\mathfrak{u}_3, 3), (0.75, 0.01, 0.30)\rangle,\ \langle(\mathfrak{u}_4, 2), (0.50, 0.10, 0.50)\rangle. \end{array} \right\},$$

$$\gamma(p_1, s, 1) = \left\{ \begin{array}{l} \langle(\mathfrak{u}_1, 2), (0.59, 0.10, 0.41)\rangle,\ \langle(\mathfrak{u}_2, 1), (0.20, 0.12, 0.60)\rangle, \\ \langle(\mathfrak{u}_3, 3), (0.7, 0.015, 0.25)\rangle,\ \langle(\mathfrak{u}_4, 2), (0.45, 0.08, 0.55)\rangle. \end{array} \right\},$$

$$\gamma(p_1, t, 1) = \left\{ \begin{array}{l} \langle(\mathfrak{u}_1, 2), (0.57, 0.05, 0.43)\rangle,\ \langle(\mathfrak{u}_2, 1), (0.35, 0.10, 0.75)\rangle, \\ \langle(\mathfrak{u}_3, 3), (0.75, 0.015, 0.2)\rangle,\ \langle(\mathfrak{u}_4, 2), (0.50, 0.09, 0.40)\rangle. \end{array} \right\},$$

$$\gamma(p_2, r, 1) = \left\{ \begin{array}{l} \langle(\mathfrak{u}_1, 3), (0.70, 0.01, 0.30)\rangle,\ \langle(\mathfrak{u}_2, 4), (0.90, 0.01, 0.10)\rangle, \\ \langle(\mathfrak{u}_3, 2), (0.55, 0.10, 0.45)\rangle,\ \langle(\mathfrak{u}_4, 2), (0.50, 0.10, 0.50)\rangle. \end{array} \right\},$$

$$\gamma(p_2, s, 1) = \left\{ \begin{array}{l} \langle(\mathfrak{u}_1, 3), (0.65, 0.015, 0.3)\rangle,\ \langle(\mathfrak{u}_2, 4), (0.95, 0.00, 0.09)\rangle, \\ \langle(\mathfrak{u}_3, 2), (0.50, 0.09, 0.50)\rangle,\ \langle(\mathfrak{u}_4, 2), (0.55, 0.01, 0.45)\rangle. \end{array} \right\},$$

$$\gamma(p_2, t, 1) = \left\{ \begin{array}{l} \langle(\mathfrak{u}_1, 3), (0.70, 0.01, 0.35)\rangle,\ \langle(\mathfrak{u}_2, 4), (0.80, 0.01, 0.15)\rangle, \\ \langle(\mathfrak{u}_3, 2), (0.50, 0.10, 0.55)\rangle,\ \langle(\mathfrak{u}_4, 2), (0.52, 0.05, 0.45)\rangle. \end{array} \right\},$$

$$\gamma(p_3, r, 1) = \left\{ \begin{array}{l} \langle(\mathfrak{u}_1, 2), (0.50, 0.10, 0.55)\rangle,\ \langle(\mathfrak{u}_2, 3), (0.65, 0.01, 0.35)\rangle, \\ \langle(\mathfrak{u}_3, 3), (0.79, 0.005, 0.3)\rangle,\ \langle(\mathfrak{u}_4, 3), (0.60, 0.01, 0.39)\rangle. \end{array} \right\},$$

$$\gamma(p_3, s, 1) = \left\{ \begin{array}{l} \langle(\mathfrak{u}_1, 2), (0.55, 0.05, 0.55)\rangle,\ \langle(\mathfrak{u}_2, 3), (0.75, 0.015, 0.2)\rangle, \\ \langle(\mathfrak{u}_3, 3), (0.60, 0.01, 0.39)\rangle,\ \langle(\mathfrak{u}_4, 3), (0.70, 0.01, 0.39)\rangle. \end{array} \right\},$$

$$\gamma(p_3, t, 1) = \left\{ \begin{array}{l} \langle(\mathfrak{u}_1, 2), (0.50, 0.10, 0.50)\rangle,\ \langle(\mathfrak{u}_2, 3), (0.70, 0.01, 0.30)\rangle, \\ \langle(\mathfrak{u}_3, 3), (0.65, 0.01, 0.35)\rangle,\ \langle(\mathfrak{u}_4, 3), (0.70, 0.01, 0.30)\rangle. \end{array} \right\},$$

$$\gamma(p_4, r, 1) = \left\{ \begin{array}{l} \langle(\mathfrak{u}_1, 3), (0.70, 0.01, 0.39)\rangle,\ \langle(\mathfrak{u}_2, 2), (0.55, 0.10, 0.45)\rangle, \\ \langle(\mathfrak{u}_3, 2), (0.45, 0.10, 0.55)\rangle,\ \langle(\mathfrak{u}_4, 3), (0.75, 0.01, 0.35)\rangle. \end{array} \right\},$$

$$\gamma(p_4, s, 1) = \left\{ \begin{array}{l} \langle(\mathfrak{u}_1, 3), (0.75, 0.001, 0.2)\rangle,\ \langle(\mathfrak{u}_2, 2), (0.45, 0.10, 0.59)\rangle, \\ \langle(\mathfrak{u}_3, 2), (0.50, 0.01, 0.59)\rangle,\ \langle(\mathfrak{u}_4, 3), (0.75, 0.01, 0.30)\rangle. \end{array} \right\},$$

$$\gamma(p_4, t, 1) = \left\{ \begin{array}{l} \langle(\mathfrak{u}_1, 3), (0.60, 0.01, 0.39)\rangle,\ \langle(\mathfrak{u}_2, 2), (0.59, 0.10, 0.55)\rangle, \\ \langle(\mathfrak{u}_3, 2), (0.40, 0.10, 0.55)\rangle,\ \langle(\mathfrak{u}_4, 3), (0.70, 0.01, 0.35)\rangle. \end{array} \right\},$$

Table 5 SF5SES $(\gamma, X, 5)$ in Example 1

$(\gamma, X, 5)$	$\mathfrak{u}_1$	$\mathfrak{u}_2$	$\mathfrak{u}_3$	$\mathfrak{u}_4$
$(p_1, r, 1)$	$\langle 2, (0.55, 0.09, 0.40)\rangle$	$\langle 1, (0.33, 0.11, 0.75)\rangle$	$\langle 3, (0.75, 0.01, 0.30)\rangle$	$\langle 2, (0.50, 0.10, 0.50)\rangle$
$(p_1, s, 1)$	$\langle 2, (0.59, 0.10, 0.41)\rangle$	$\langle 1, (0.20, 0.12, 0.60)\rangle$	$\langle 3, (0.7, 0.015, 0.25)\rangle$	$\langle 2, (0.45, 0.08, 0.55)\rangle$
$(p_1, t, 1)$	$\langle 2, (0.57, 0.05, 0.43)\rangle$	$\langle 1, (0.35, 0.10, 0.75)\rangle$	$\langle 3, (0.75, 0.015, 0.2)\rangle$	$\langle 2, (0.50, 0.09, 0.40)\rangle$
$(p_2, r, 1)$	$\langle 3, (0.70, 0.01, 0.30)\rangle$	$\langle 4, (0.90, 0.01, 0.10)\rangle$	$\langle 2, (0.55, 0.10, 0.45)\rangle$	$\langle 2, (0.50, 0.10, 0.50)\rangle$
$(p_2, s, 1)$	$\langle 3, (0.65, 0.015, 0.3)\rangle$	$\langle 4, (0.95, 0.00, 0.09)\rangle$	$\langle 2, (0.50, 0.09, 0.50)\rangle$	$\langle 2, (0.55, 0.01, 0.45)\rangle$
$(p_2, t, 1)$	$\langle 3, (0.70, 0.01, 0.35)\rangle$	$\langle 4, (0.80, 0.01, 0.15)\rangle$	$\langle 2, (0.50, 0.10, 0.55)\rangle$	$\langle 2, (0.52, 0.05, 0.45)\rangle$
$(p_3, r, 1)$	$\langle 2, (0.50, 0.10, 0.55)\rangle$	$\langle 3, (0.65, 0.01, 0.35)\rangle$	$\langle 3, (0.79, 0.005, 0.3)\rangle$	$\langle 3, (0.60, 0.01, 0.39)\rangle$
$(p_3, s, 1)$	$\langle 2, (0.55, 0.05, 0.55)\rangle$	$\langle 3, (0.75, 0.015, 0.2)\rangle$	$\langle 3, (0.60, 0.01, 0.39)\rangle$	$\langle 3, (0.70, 0.01, 0.39)\rangle$
$(p_3, t, 1)$	$\langle 2, (0.50, 0.10, 0.50)\rangle$	$\langle 3, (0.70, 0.01, 0.30)\rangle$	$\langle 3, (0.65, 0.01, 0.35)\rangle$	$\langle 3, (0.70, 0.01, 0.30)\rangle$
$(p_4, r, 1)$	$\langle 3, (0.70, 0.01, 0.39)\rangle$	$\langle 2, (0.55, 0.10, 0.45)\rangle$	$\langle 2, (0.45, 0.10, 0.55)\rangle$	$\langle 3, (0.75, 0.01, 0.35)\rangle$
$(p_4, s, 1)$	$\langle 3, (0.75, 0.001, 0.2)\rangle$	$\langle 2, (0.45, 0.10, 0.59)\rangle$	$\langle 2, (0.50, 0.01, 0.59)\rangle$	$\langle 3, (0.75, 0.01, 0.30)\rangle$
$(p_4, t, 1)$	$\langle 3, (0.60, 0.01, 0.39)\rangle$	$\langle 2, (0.59, 0.10, 0.55)\rangle$	$\langle 2, (0.40, 0.10, 0.55)\rangle$	$\langle 3, (0.70, 0.01, 0.35)\rangle$
$(p_1, r, 0)$	$\langle 2, (0.55, 0.10, 0.55)\rangle$	$\langle 1, (0.39, 0.12, 0.60)\rangle$	$\langle 3, (0.75, 0.01, 0.35)\rangle$	$\langle 2, (0.55, 0.10, 0.40)\rangle$
$(p_1, s, 0)$	$\langle 2, (0.50, 0.10, 0.55)\rangle$	$\langle 1, (0.00, 0.12, 0.75)\rangle$	$\langle 3, (0.60, 0.01, 0.35)\rangle$	$\langle 2, (0.55, 0.10, 0.50)\rangle$
$(p_1, t, 0)$	$\langle 2, (0.40, 0.10, 0.55)\rangle$	$\langle 1, (0.20, 0.10, 0.60)\rangle$	$\langle 3, (0.65, 0.01, 0.35)\rangle$	$\langle 2, (0.50, 0.10, 0.50)\rangle$
$(p_2, r, 0)$	$\langle 3, (0.60, 0.01, 0.39)\rangle$	$\langle 4, (1.0, 0.005, 0.20)\rangle$	$\langle 2, (0.45, 0.10, 0.55)\rangle$	$\langle 2, (0.50, 0.10, 0.50)\rangle$
$(p_2, s, 0)$	$\langle 3, (0.79, 0.01, 0.20)\rangle$	$\langle 4, (0.80, 0.00, 0.19)\rangle$	$\langle 2, (0.50, 0.10, 0.50)\rangle$	$\langle 2, (0.40, 0.01, 0.50)\rangle$
$(p_2, t, 0)$	$\langle 3, (0.60, 0.01, 0.25)\rangle$	$\langle 4, (1.0, 0.005, 0.05)\rangle$	$\langle 2, (0.50, 0.10, 0.50)\rangle$	$\langle 2, (0.40, 0.10, 0.55)\rangle$
$(p_3, r, 0)$	$\langle 2, (0.50, 0.10, 0.50)\rangle$	$\langle 3, (0.79, 0.01, 0.30)\rangle$	$\langle 3, (0.60, 0.01, 0.39)\rangle$	$\langle 3, (0.75, 0.01, 0.25)\rangle$
$(p_3, s, 0)$	$\langle 2, (0.45, 0.10, 0.40)\rangle$	$\langle 3, (0.65, 0.01, 0.39)\rangle$	$\langle 3, (0.75, 0.01, 0.20)\rangle$	$\langle 3, (0.65, 0.01, 0.39)\rangle$
$(p_3, t, 0)$	$\langle 2, (0.50, 0.10, 0.50)\rangle$	$\langle 3, (0.60, 0.01, 0.39)\rangle$	$\langle 3, (0.70, 0.01, 0.30)\rangle$	$\langle 3, (0.60, 0.01, 0.39)\rangle$
$(p_4, r, 0)$	$\langle 3, (0.75, 0.01, 0.30)\rangle$	$\langle 2, (0.45, 0.10, 0.55)\rangle$	$\langle 2, (0.55, 0.10, 0.40)\rangle$	$\langle 3, (0.79, 0.01, 0.30)\rangle$
$(p_4, s, 0)$	$\langle 3, (0.65, 0.01, 0.39)\rangle$	$\langle 2, (0.59, 0.10, 0.45)\rangle$	$\langle 2, (0.50, 0.10, 0.50)\rangle$	$\langle 3, (0.60, 0.01, 0.39)\rangle$
$(p_4, t, 0)$	$\langle 3, (0.70, 0.01, 0.30)\rangle$	$\langle 2, (0.45, 0.10, 0.55)\rangle$	$\langle 2, (0.55, 0.10, 0.45)\rangle$	$\langle 3, (0.60, 0.01, 0.25)\rangle$

$$\gamma(p_1,r,0)=\left\{\begin{array}{l}\langle(\mathfrak{u}_1,2),(0.55,0.10,0.55)\rangle,\ \langle(\mathfrak{u}_2,1),(0.39,0.12,0.60)\rangle,\\ \langle(\mathfrak{u}_3,3),(0.75,0.01,0.35)\rangle,\ \langle(\mathfrak{u}_4,2),(0.55,0.10,0.40)\rangle.\end{array}\right\},$$

$$\gamma(p_1,s,0)=\left\{\begin{array}{l}\langle(\mathfrak{u}_1,2),(0.50,0.10,0.55)\rangle,\ \langle(\mathfrak{u}_2,1),(0.00,0.12,0.75)\rangle,\\ \langle(\mathfrak{u}_3,3),(0.60,0.01,0.35)\rangle,\ \langle(\mathfrak{u}_4,2),(0.55,0.10,0.50)\rangle.\end{array}\right\},$$

$$\gamma(p_1,t,0)=\left\{\begin{array}{l}\langle(\mathfrak{u}_1,2),(0.40,0.10,0.55)\rangle,\ \langle(\mathfrak{u}_2,1),(0.20,0.10,0.60)\rangle,\\ \langle(\mathfrak{u}_3,3),(0.65,0.01,0.35)\rangle,\ \langle(\mathfrak{u}_4,2),(0.50,0.10,0.50)\rangle.\end{array}\right\},$$

$$\gamma(p_2,r,0)=\left\{\begin{array}{l}\langle(\mathfrak{u}_1,3),(0.60,0.01,0.39)\rangle,\ \langle(\mathfrak{u}_2,4),(1.0,0.005,0.20)\rangle,\\ \langle(\mathfrak{u}_3,2),(0.45,0.10,0.55)\rangle,\ \langle(\mathfrak{u}_4,2),(0.50,0.10,0.50)\rangle.\end{array}\right\},$$

$$\gamma(p_2,s,0)=\left\{\begin{array}{l}\langle(\mathfrak{u}_1,3),(0.79,0.01,0.20)\rangle,\ \langle(\mathfrak{u}_2,4),(0.80,0.00,0.19)\rangle,\\ \langle(\mathfrak{u}_3,2),(0.50,0.10,0.50)\rangle,\ \langle(\mathfrak{u}_4,2),(0.40,0.01,0.50)\rangle.\end{array}\right\},$$

$$\gamma(p_2,t,0)=\left\{\begin{array}{l}\langle(\mathfrak{u}_1,3),(0.60,0.01,0.25)\rangle,\ \langle(\mathfrak{u}_2,4),(1.0,0.005,0.05)\rangle,\\ \langle(\mathfrak{u}_3,2),(0.50,0.10,0.50)\rangle,\ \langle(\mathfrak{u}_4,2),(0.40,0.10,0.55)\rangle.\end{array}\right\},$$

$$\gamma(p_3,r,0)=\left\{\begin{array}{l}\langle(\mathfrak{u}_1,2),(0.50,0.10,0.50)\rangle,\ \langle(\mathfrak{u}_2,3),(0.79,0.01,0.30)\rangle,\\ \langle(\mathfrak{u}_3,3),(0.60,0.01,0.39)\rangle,\ \langle(\mathfrak{u}_4,3),(0.75,0.01,0.25)\rangle.\end{array}\right\},$$

$$\gamma(p_3,s,0)=\left\{\begin{array}{l}\langle(\mathfrak{u}_1,2),(0.45,0.10,0.40)\rangle,\ \langle(\mathfrak{u}_2,3),(0.65,0.01,0.39)\rangle,\\ \langle(\mathfrak{u}_3,3),(0.75,0.01,0.20)\rangle,\ \langle(\mathfrak{u}_4,3),(0.65,0.01,0.39)\rangle.\end{array}\right\},$$

$$\gamma(p_3,t,0)=\left\{\begin{array}{l}\langle(\mathfrak{u}_1,2),(0.50,0.10,0.50)\rangle,\ \langle(\mathfrak{u}_2,3),(0.60,0.01,0.39)\rangle,\\ \langle(\mathfrak{u}_3,3),(0.70,0.01,0.30)\rangle,\ \langle(\mathfrak{u}_4,3),(0.60,0.01,0.39)\rangle.\end{array}\right\},$$

$$\gamma(p_4,r,0)=\left\{\begin{array}{l}\langle(\mathfrak{u}_1,3),(0.75,0.01,0.30)\rangle,\ \langle(\mathfrak{u}_2,2),(0.45,0.10,0.55)\rangle,\\ \langle(\mathfrak{u}_3,2),(0.55,0.10,0.40)\rangle,\ \langle(\mathfrak{u}_4,3),(0.79,0.01,0.30)\rangle.\end{array}\right\},$$

$$\gamma(p_4,s,0)=\left\{\begin{array}{l}\langle(\mathfrak{u}_1,3),(0.65,0.01,0.39)\rangle,\ \langle(\mathfrak{u}_2,2),(0.59,0.10,0.45)\rangle,\\ \langle(\mathfrak{u}_3,2),(0.50,0.10,0.50)\rangle,\ \langle(\mathfrak{u}_4,3),(0.60,0.01,0.39)\rangle.\end{array}\right\},$$

$$\gamma(p_4,t,0)=\left\{\begin{array}{l}\langle(\mathfrak{u}_1,3),(0.70,0.01,0.30)\rangle,\ \langle(\mathfrak{u}_2,2),(0.45,0.10,0.55)\rangle,\\ \langle(\mathfrak{u}_3,2),(0.55,0.10,0.45)\rangle,\ \langle(\mathfrak{u}_4,3),(0.60,0.01,0.25)\rangle.\end{array}\right\}.$$

Here it can be seen how different experts have given different membership values for the same ratings. For example, expert 'r' gives positive membership 0.33, neutral membership 0.11, and negative membership 0.75 for the novel $\mathfrak{u}_2$ to be innovative; whereas for the same, expert 's' considers positive membership 0.20, neutral membership 0.12, and negative membership 0.60 more suitable for the novel $\mathfrak{u}_2$ to be innovative. This means that under the same ratings, experts' judgments about the novels vary. This provides a more efficient framework in dealing with uncertain situations as compared to dealing with the ratings directly.

For a finite number of objects $\mathfrak{u}_i \in \mathcal{U}$ and parameter-based opinions $x_i \in P \times E \times \mathcal{O}$, the general tabular representation of a SFNSES (γ, X, N) is shown in Table 4.

The tabular representation of the SF5SES in Example 1 is provided in Table 5.

Now we define the subset relation on SFNSESs as follows.

Definition 9 Consider two SFNSESs (γ, X, N) and (δ, Y, N) over the universe $\mathcal{U}$. Then (γ, X, N) is said to be a spherical fuzzy N-soft expert subset of (δ, Y, N) if

1. $X \subseteq Y$,

Table 6 SF5SES $(\gamma, Y, 5)$ in Example 2

$(\gamma, Y, 5)$	$\mathfrak{u}_1$	$\mathfrak{u}_2$	$\mathfrak{u}_3$	$\mathfrak{u}_4$
$(p_1, r, 1)$	$\langle 2, (0.55, 0.09, 0.40)\rangle$	$\langle 1, (0.33, 0.11, 0.75)\rangle$	$\langle 3, (0.75, 0.01, 0.30)\rangle$	$\langle 2, (0.50, 0.10, 0.50)\rangle$
$(p_2, s, 1)$	$\langle 3, (0.65, 0.015, 0.3)\rangle$	$\langle 4, (0.95, 0.00, 0.09)\rangle$	$\langle 2, (0.50, 0.09, 0.50)\rangle$	$\langle 2, (0.55, 0.01, 0.45)\rangle$
$(p_3, t, 1)$	$\langle 2, (0.50, 0.10, 0.50)\rangle$	$\langle 3, (0.70, 0.01, 0.30)\rangle$	$\langle 3, (0.65, 0.01, 0.35)\rangle$	$\langle 3, (0.70, 0.01, 0.30)\rangle$
$(p_4, r, 0)$	$\langle 3, (0.75, 0.01, 0.30)\rangle$	$\langle 2, (0.45, 0.10, 0.55)\rangle$	$\langle 2, (0.55, 0.10, 0.40)\rangle$	$\langle 3, (0.79, 0.01, 0.30)\rangle$
$(p_4, t, 0)$	$\langle 3, (0.70, 0.01, 0.30)\rangle$	$\langle 2, (0.45, 0.10, 0.55)\rangle$	$\langle 2, (0.55, 0.10, 0.45)\rangle$	$\langle 3, (0.60, 0.01, 0.25)\rangle$

Table 7 SF5SES $(\delta, Z, 5)$ in Example 2

$(\delta, Z, 5)$	$\mathfrak{u}_1$	$\mathfrak{u}_2$	$\mathfrak{u}_3$	$\mathfrak{u}_4$
$(p_1, r, 1)$	$\langle 2, (0.59, 0.09, 0.40)\rangle$	$\langle 1, (0.35, 0.12, 0.65)\rangle$	$\langle 3, (0.79, 0.01, 0.25)\rangle$	$\langle 2, (0.50, 0.10, 0.50)\rangle$
$(p_1, t, 1)$	$\langle 2, (0.55, 0.05, 0.45)\rangle$	$\langle 1, (0.30, 0.10, 0.70)\rangle$	$\langle 3, (0.70, 0.01, 0.20)\rangle$	$\langle 2, (0.50, 0.09, 0.50)\rangle$
$(p_2, s, 1)$	$\langle 3, (0.75, 0.015, 0.2)\rangle$	$\langle 4, (1.0, 0.011, 0.05)\rangle$	$\langle 2, (0.55, 0.09, 0.45)\rangle$	$\langle 2, (0.55, 0.01, 0.40)\rangle$
$(p_3, t, 1)$	$\langle 2, (0.55, 0.10, 0.45)\rangle$	$\langle 3, (0.75, 0.01, 0.25)\rangle$	$\langle 3, (0.70, 0.01, 0.30)\rangle$	$\langle 3, (0.75, 0.01, 0.30)\rangle$
$(p_3, t, 0)$	$\langle 2, (0.50, 0.05, 0.55)\rangle$	$\langle 3, (0.70, 0.01, 0.39)\rangle$	$\langle 3, (0.70, 0.01, 0.35)\rangle$	$\langle 3, (0.65, 0.01, 0.35)\rangle$
$(p_4, r, 0)$	$\langle 3, (0.75, 0.01, 0.25)\rangle$	$\langle 2, (0.55, 0.10, 0.45)\rangle$	$\langle 2, (0.55, 0.10, 0.40)\rangle$	$\langle 3, (0.79, 0.01, 0.25)\rangle$
$(p_4, t, 0)$	$\langle 3, (0.75, 0.01, 0.30)\rangle$	$\langle 2, (0.55, 0.10, 0.55)\rangle$	$\langle 2, (0.55, 0.10, 0.40)\rangle$	$\langle 3, (0.70, 0.01, 0.25)\rangle$

2. $\forall x \in X$, $\gamma(x) \subseteq \delta(x)$, that is $\forall\langle(\mathfrak{u}, g_x^1), \kappa(\mathfrak{u}, g_x^1)\rangle \in \gamma(x)$ and $\forall\langle(\mathfrak{u}, g_x^2), \kappa(\mathfrak{u}, g_x^2)\rangle \in \delta(x)$, we have $g_x^1 \leq g_x^2$ and $\kappa(\mathfrak{u}, g_x^1)$ as spherical fuzzy subset of $\kappa(\mathfrak{u}, g_x^2)$ for $\mathfrak{u} \in \mathcal{U}$, $g_x^1, g_x^2 \in G$.

We denote this subset relation by $(\gamma, X, N)\bar{\subseteq}(\delta, Y, N)$. Moreover, (δ, Y, N) is said to be a spherical fuzzy N-soft expert superset of (γ, X, N). This superset relation is denoted by $(\delta, Y, N)\bar{\supseteq}(\gamma, X, N)$.

The following example investigates the concept of spherical fuzzy N-soft expert subset.

Example 2 Reconsider Example 1, suppose that $Y = \{(p_1, r, 1), (p_2, s, 1), (p_3, t, 1), \}\{(p_4, r, 0), (p_4, t, 0)\}$, and $Z = \{(p_1, r, 1), (p_1, t, 1), (p_2, s, 1), (p_3, t, 1), (p_3, t, 0), (p_4, r, 0), (p_4, t, 0)\}$.

It is clearly visible, that $Y \subset Z$. Let $(\gamma, Y, 5)$ and $(\delta, Z, 5)$ be two SF5SESs as shown in Tables 6 and 7, respectively.

Hence, $(\gamma, Y, 5)\bar{\subseteq}(\delta, Z, 5)$.

Definition 10 Any two SFNSESs (γ, X, N) and (δ, Y, N) over the universe $\mathcal{U}$ are said to be equal if (γ, X, N) is a spherical fuzzy N-soft expert subset of (δ, Y, N), and (δ, Y, N) is a spherical fuzzy N-soft expert subset of (γ, X, N). It is denoted by $(\gamma, X, N) = (\delta, Y, N)$.

We now give some algebraic properties and operations, including complements, extended (and restricted) union, extended (and restricted) intersection, AND, OR operations, and illustrate them with respective examples.

The following definition gives the idea of weak complement of a SFNSES:

Definition 11 Let (γ, X, N) be a SFNSES on $\mathcal{U}$, where $\gamma(x) = \langle(\mathfrak{u}, g_x), \kappa(\mathfrak{u}, g_x)\rangle$. Then the SFNSES $(\gamma, X, N)_{\mathfrak{w}} = (\gamma_{\mathfrak{w}}, X, N)$ with $\gamma_{\mathfrak{w}}(x) = \langle(\mathfrak{u}, g_x^{\mathfrak{w}}), \kappa(\mathfrak{u}, g_x^{\mathfrak{w}})\rangle$ is said to be the weak complement of (γ, X, N), if and only if

$$g_x \neq g_x^{\mathfrak{w}} \quad \forall x \in X \text{ and } \mathfrak{u} \in \mathcal{U}.$$

As we know that the spherical fuzzy complement of $\kappa = (\mu_x(\mathfrak{u}), \tau_x(\mathfrak{u}), \nu_x(\mathfrak{u})) \in SFS(\mathcal{U})$ is the spherical fuzzy number $\kappa^c = (\nu_x(\mathfrak{u}), \tau_x(\mathfrak{u}), \mu_x(\mathfrak{u})) \in SFS(\mathcal{U})$ where $x \in X$.

We now define the spherical fuzzy complement of SFNSESs as follows:

Definition 12 Let (γ, X, N) be a SFNSES on $\mathcal{U}$, where $\gamma(x) = \langle(\mathfrak{u}, g_x), \kappa(\mathfrak{u}, g_x)\rangle$. Then the SFNSES $(\gamma, X, N)^c = (\gamma^c, X, N)$ with $\gamma^c(x) = \langle(\mathfrak{u}, g_x), \kappa^c(\mathfrak{u}, g_x)\rangle$ is said to be the spherical fuzzy complement of (γ, X, N), if $\kappa^c(\mathfrak{u}, g_x)$ is the complement of spherical fuzzy set $\kappa(\mathfrak{u}, g_x)$, where $\kappa, \kappa^c \in SFS(\mathcal{U} \times G)$.

Example 3 Reconsider the SF5SES $(\gamma, X, 5)$ in Example 1. Then, its spherical fuzzy complement $(\gamma, X, 5)^c$ is provided in Table 8.

Definition 13 A SFNSES $(\gamma, X, N)^c_{\mathfrak{w}} = (\gamma^c_{\mathfrak{w}}, X, N)$ is said to be the spherical fuzzy weak complement of the SFNSES (γ, X, N), if for each $\langle(\mathfrak{u}, g_x), \kappa(\mathfrak{u}, g_x)\rangle \in \gamma(x)$ and $\langle(\mathfrak{u}, g_x^{\mathfrak{w}}), \kappa^c(\mathfrak{u}, g_x^{\mathfrak{w}})\rangle \in \gamma^c_{\mathfrak{w}}(x)$, we have $g_x \neq g_x^{\mathfrak{w}}$ and κ^c is a spherical fuzzy complement of κ, $\forall x \in X$, $\mathfrak{u} \in \mathcal{U}$ and $\kappa^c, \kappa \in SFS(\mathcal{U} \times G)$.

Example 4 Reconsider the SF5SES $(\gamma, Y, 5)$ in Example 2. Then, its weak complement $(\gamma, Y, 5)_{\mathfrak{w}}$ and spherical fuzzy weak complement $(\gamma, Y, 5)^c_{\mathfrak{w}}$ are provided in Tables 9 and 10, respectively.

The following definitions give the idea of top (and bottom respectively) weak and spherical fuzzy weak complements of the SFNSESs.

Definition 14 For a SFNSES (γ, X, N), the top weak complement of (γ, X, N) is the SFNSES $(\bar{\gamma}_{\mathfrak{w}}, X, N)$ such that

$$\bar{g}_x^{\mathfrak{w}} = \begin{cases} 0, & \text{if } g_x = \mathrm{N} - 1, \\ \mathrm{N} - 1, & \text{if } g_x < \mathrm{N} - 1. \end{cases}$$

Table 8 Spherical fuzzy complement of the SF5SES $(\gamma, X, 5)$

$(\gamma, X, 5)^c$	$\mathfrak{u}_1$	$\mathfrak{u}_2$	$\mathfrak{u}_3$	$\mathfrak{u}_4$
$(p_1, r, 1)$	$\langle 2, (0.40, 0.09, 0.55)\rangle$	$\langle 1, (0.75, 0.11, 0.33)\rangle$	$\langle 3, (0.30, 0.01, 0.75)\rangle$	$\langle 2, (0.50, 0.10, 0.50)\rangle$
$(p_1, s, 1)$	$\langle 2, (0.41, 0.10, 0.59)\rangle$	$\langle 1, (0.60, 0.12, 0.20)\rangle$	$\langle 3, (0.25, 0.015, 0.7)\rangle$	$\langle 2, (0.55, 0.08, 0.45)\rangle$
$(p_1, t, 1)$	$\langle 2, (0.43, 0.05, 0.57)\rangle$	$\langle 1, (0.75, 0.10, 0.35)\rangle$	$\langle 3, (0.2, 0.015, 0.75)\rangle$	$\langle 2, (0.40, 0.09, 0.50)\rangle$
$(p_2, r, 1)$	$\langle 3, (0.30, 0.01, 0.70)\rangle$	$\langle 4, (0.10, 0.01, 0.90)\rangle$	$\langle 2, (0.45, 0.10, 0.55)\rangle$	$\langle 2, (0.50, 0.10, 0.50)\rangle$
$(p_2, s, 1)$	$\langle 3, (0.3, 0.015, 0.65)\rangle$	$\langle 4, (0.09, 0.00, 0.95)\rangle$	$\langle 2, (0.50, 0.09, 0.50)\rangle$	$\langle 2, (0.45, 0.01, 0.55)\rangle$
$(p_2, t, 1)$	$\langle 3, (0.35, 0.01, 0.70)\rangle$	$\langle 4, (0.15, 0.01, 0.80)\rangle$	$\langle 2, (0.55, 0.10, 0.50)\rangle$	$\langle 2, (0.45, 0.05, 0.52)\rangle$
$(p_3, r, 1)$	$\langle 2, (0.55, 0.10, 0.50)\rangle$	$\langle 3, (0.35, 0.01, 0.65)\rangle$	$\langle 3, (0.3, 0.005, 0.79)\rangle$	$\langle 3, (0.39, 0.01, 0.60)\rangle$
$(p_3, s, 1)$	$\langle 2, (0.55, 0.05, 0.55)\rangle$	$\langle 3, (0.2, 0.015, 0.75)\rangle$	$\langle 3, (0.39, 0.01, 0.60)\rangle$	$\langle 3, (0.39, 0.01, 0.70)\rangle$
$(p_3, t, 1)$	$\langle 2, (0.50, 0.10, 0.50)\rangle$	$\langle 3, (0.30, 0.01, 0.70)\rangle$	$\langle 3, (0.35, 0.01, 0.65)\rangle$	$\langle 3, (0.30, 0.01, 0.70)\rangle$
$(p_4, r, 1)$	$\langle 3, (0.39, 0.01, 0.70)\rangle$	$\langle 2, (0.45, 0.10, 0.55)\rangle$	$\langle 2, (0.55, 0.10, 0.45)\rangle$	$\langle 3, (0.35, 0.01, 0.75)\rangle$
$(p_4, s, 1)$	$\langle 3, (0.2, 0.001, 0.75)\rangle$	$\langle 2, (0.59, 0.10, 0.45)\rangle$	$\langle 2, (0.59, 0.01, 0.50)\rangle$	$\langle 3, (0.30, 0.01, 0.75)\rangle$
$(p_4, t, 1)$	$\langle 3, (0.39, 0.01, 0.60)\rangle$	$\langle 2, (0.55, 0.10, 0.59)\rangle$	$\langle 2, (0.55, 0.10, 0.40)\rangle$	$\langle 3, (0.35, 0.01, 0.70)\rangle$
$(p_1, r, 0)$	$\langle 2, (0.55, 0.10, 0.55)\rangle$	$\langle 1, (0.60, 0.12, 0.39)\rangle$	$\langle 3, (0.35, 0.01, 0.75)\rangle$	$\langle 2, (0.40, 0.10, 0.55)\rangle$
$(p_1, s, 0)$	$\langle 2, (0.55, 0.10, 0.50)\rangle$	$\langle 1, (0.75, 0.12, 0.00)\rangle$	$\langle 3, (0.35, 0.01, 0.60)\rangle$	$\langle 2, (0.50, 0.10, 0.55)\rangle$
$(p_1, t, 0)$	$\langle 2, (0.55, 0.10, 0.40)\rangle$	$\langle 1, (0.60, 0.10, 0.20)\rangle$	$\langle 3, (0.35, 0.01, 0.65)\rangle$	$\langle 2, (0.50, 0.10, 0.50)\rangle$
$(p_2, r, 0)$	$\langle 3, (0.39, 0.01, 0.60)\rangle$	$\langle 4, (0.20, 0.005, 1.0)\rangle$	$\langle 2, (0.55, 0.10, 0.45)\rangle$	$\langle 2, (0.50, 0.10, 0.50)\rangle$
$(p_2, s, 0)$	$\langle 3, (0.20, 0.01, 0.79)\rangle$	$\langle 4, (0.19, 0.00, 0.80)\rangle$	$\langle 2, (0.50, 0.10, 0.50)\rangle$	$\langle 2, (0.50, 0.01, 0.40)\rangle$
$(p_2, t, 0)$	$\langle 3, (0.25, 0.01, 0.60)\rangle$	$\langle 4, (0.05, 0.005, 1.0)\rangle$	$\langle 2, (0.50, 0.10, 0.50)\rangle$	$\langle 2, (0.55, 0.10, 0.40)\rangle$
$(p_3, r, 0)$	$\langle 2, (0.50, 0.10, 0.50)\rangle$	$\langle 3, (0.30, 0.01, 0.79)\rangle$	$\langle 3, (0.39, 0.01, 0.60)\rangle$	$\langle 3, (0.25, 0.01, 0.75)\rangle$
$(p_3, s, 0)$	$\langle 2, (0.40, 0.10, 0.45)\rangle$	$\langle 3, (0.39, 0.01, 0.65)\rangle$	$\langle 3, (0.20, 0.01, 0.75)\rangle$	$\langle 3, (0.39, 0.01, 0.65)\rangle$
$(p_3, t, 0)$	$\langle 2, (0.50, 0.10, 0.50)\rangle$	$\langle 3, (0.39, 0.01, 0.60)\rangle$	$\langle 3, (0.30, 0.01, 0.70)\rangle$	$\langle 3, (0.39, 0.01, 0.60)\rangle$
$(p_4, r, 0)$	$\langle 3, (0.30, 0.01, 0.75)\rangle$	$\langle 2, (0.55, 0.10, 0.45)\rangle$	$\langle 2, (0.40, 0.10, 0.55)\rangle$	$\langle 3, (0.30, 0.01, 0.79)\rangle$
$(p_4, s, 0)$	$\langle 3, (0.39, 0.01, 0.65)\rangle$	$\langle 2, (0.45, 0.10, 0.59)\rangle$	$\langle 2, (0.50, 0.10, 0.50)\rangle$	$\langle 3, (0.39, 0.01, 0.60)\rangle$
$(p_4, t, 0)$	$\langle 3, (0.30, 0.01, 0.70)\rangle$	$\langle 2, (0.55, 0.10, 0.45)\rangle$	$\langle 2, (0.45, 0.10, 0.55)\rangle$	$\langle 3, (0.25, 0.01, 0.60)\rangle$

Similarly, the top spherical fuzzy weak complement of (γ, X, N) is the SFNSES $(\bar{\gamma}^c_{\mathfrak{w}}, X, N)$ with $\kappa^c(\mathfrak{u}, \bar{g}^{\mathfrak{w}}_x)$ as the spherical fuzzy complement of $\kappa(\mathfrak{u}, g_x)$.

Example 5 Consider the SF5SES $(\gamma, Y, 5)$ as in Example 2. Its top weak complement $(\bar{\gamma}_{\mathfrak{w}}, Y, 5)$ and top spherical fuzzy weak complement $(\bar{\gamma}^c_{\mathfrak{w}}, Y, 5)$ are represented by Tables 11 and 12, respectively.

Definition 15 For a SFNSES (γ, X, N), the bottom weak complement of (γ, X, N) is the SFNSES $(\underline{\gamma}_{\mathfrak{w}}, X, N)$ such that

$$\underline{g}^{\mathfrak{w}}_x = \begin{cases} \mathrm{N}-1, & \textit{if } g_x = 0, \\ 0, & \textit{if } g_x > 0, \end{cases}$$

Similarly, the bottom spherical fuzzy weak complement of (γ, X, N) is the SFNSES $(\underline{\gamma}^c_{\mathfrak{w}}, X, N)$ with $\kappa^c(\mathfrak{u}, \underline{g}^{\mathfrak{w}}_x)$ as the spherical fuzzy complement of $\kappa(\mathfrak{u}, g_x)$.

Table 9 Weak complement of the SF5SES $(\gamma, Y, 5)$

$(\gamma, Y, 5)_{\mathfrak{w}}$	$\mathfrak{u}_1$	$\mathfrak{u}_2$	$\mathfrak{u}_3$	$\mathfrak{u}_4$
$(p_1, r, 1)$	$\langle 3, (0.55, 0.09, 0.40)\rangle$	$\langle 2, (0.33, 0.11, 0.75)\rangle$	$\langle 4, (0.75, 0.01, 0.30)\rangle$	$\langle 3, (0.50, 0.10, 0.50)\rangle$
$(p_2, s, 1)$	$\langle 4, (0.65, 0.015, 0.3)\rangle$	$\langle 0, (0.95, 0.00, 0.09)\rangle$	$\langle 3, (0.50, 0.09, 0.50)\rangle$	$\langle 3, (0.55, 0.01, 0.45)\rangle$
$(p_3, t, 1)$	$\langle 3, (0.50, 0.10, 0.50)\rangle$	$\langle 4, (0.70, 0.01, 0.30)\rangle$	$\langle 4, (0.65, 0.01, 0.35)\rangle$	$\langle 4, (0.70, 0.01, 0.30)\rangle$
$(p_4, r, 0)$	$\langle 4, (0.75, 0.01, 0.30)\rangle$	$\langle 3, (0.45, 0.10, 0.55)\rangle$	$\langle 3, (0.55, 0.10, 0.40)\rangle$	$\langle 4, (0.79, 0.01, 0.30)\rangle$
$(p_4, t, 0)$	$\langle 4, (0.70, 0.01, 0.30)\rangle$	$\langle 3, (0.45, 0.10, 0.55)\rangle$	$\langle 3, (0.55, 0.10, 0.45)\rangle$	$\langle 4, (0.60, 0.01, 0.25)\rangle$

Table 10 Spherical fuzzy weak complement of the SF5SES $(\gamma, Y, 5)$

$(\gamma, Y, 5)^c_{\mathfrak{w}}$	$\mathfrak{u}_1$	$\mathfrak{u}_2$	$\mathfrak{u}_3$	$\mathfrak{u}_4$
$(p_1, r, 1)$	$\langle 3, (0.40, 0.09, 0.55)\rangle$	$\langle 2, (0.75, 0.11, 0.33)\rangle$	$\langle 4, (0.30, 0.01, 0.75)\rangle$	$\langle 3, (0.50, 0.10, 0.50)\rangle$
$(p_2, s, 1)$	$\langle 4, (0.3, 0.015, 0.65)\rangle$	$\langle 0, (0.09, 0.00, 0.95)\rangle$	$\langle 3, (0.50, 0.09, 0.50)\rangle$	$\langle 3, (0.45, 0.01, 0.55)\rangle$
$(p_3, t, 1)$	$\langle 3, (0.50, 0.10, 0.50)\rangle$	$\langle 4, (0.30, 0.01, 0.70)\rangle$	$\langle 4, (0.35, 0.01, 0.65)\rangle$	$\langle 4, (0.30, 0.01, 0.70)\rangle$
$(p_4, r, 0)$	$\langle 4, (0.30, 0.01, 0.75)\rangle$	$\langle 3, (0.55, 0.10, 0.45)\rangle$	$\langle 3, (0.40, 0.10, 0.55)\rangle$	$\langle 4, (0.30, 0.01, 0.79)\rangle$
$(p_4, t, 0)$	$\langle 4, (0.30, 0.01, 0.70)\rangle$	$\langle 3, (0.55, 0.10, 0.45)\rangle$	$\langle 3, (0.45, 0.10, 0.55)\rangle$	$\langle 4, (0.25, 0.01, 0.60)\rangle$

Table 11 Top weak complement of the SF5SES $(\gamma, Y, 5)$

$(\bar{\gamma}_{\mathfrak{w}}, Y, 5)$	$\mathfrak{u}_1$	$\mathfrak{u}_2$	$\mathfrak{u}_3$	$\mathfrak{u}_4$
$(p_1, r, 1)$	$\langle 4, (0.55, 0.09, 0.40)\rangle$	$\langle 4, (0.33, 0.11, 0.75)\rangle$	$\langle 4, (0.75, 0.01, 0.30)\rangle$	$\langle 4, (0.50, 0.10, 0.50)\rangle$
$(p_2, s, 1)$	$\langle 4, (0.65, 0.015, 0.3)\rangle$	$\langle 0, (0.95, 0.00, 0.09)\rangle$	$\langle 4, (0.50, 0.09, 0.50)\rangle$	$\langle 4, (0.55, 0.01, 0.45)\rangle$
$(p_3, t, 1)$	$\langle 4, (0.50, 0.10, 0.50)\rangle$	$\langle 4, (0.70, 0.01, 0.30)\rangle$	$\langle 4, (0.65, 0.01, 0.35)\rangle$	$\langle 4, (0.70, 0.01, 0.30)\rangle$
$(p_4, r, 0)$	$\langle 4, (0.75, 0.01, 0.30)\rangle$	$\langle 4, (0.45, 0.10, 0.55)\rangle$	$\langle 4, (0.55, 0.10, 0.40)\rangle$	$\langle 4, (0.79, 0.01, 0.30)\rangle$
$(p_4, t, 0)$	$\langle 4, (0.70, 0.01, 0.30)\rangle$	$\langle 4, (0.45, 0.10, 0.55)\rangle$	$\langle 4, (0.55, 0.10, 0.45)\rangle$	$\langle 4, (0.60, 0.01, 0.25)\rangle$

Example 6 Consider the SF5SES $(\gamma, Y, 5)$ in Example 2. Its bottom weak complement $(\underline{\gamma}_{\mathfrak{w}}, Y, 5)$ and bottom spherical fuzzy weak complement $(\underline{\gamma}^c_{\mathfrak{w}}, Y, 5)$ are represented by Tables 13 and 14, respectively.

Proposition 1 *Let (γ, X, N) be a SFNSES over the universe $\mathcal{U}$, then*

1. $((\gamma, X, N)^c)^c = (\gamma, X, N)$

Table 12 Top spherical fuzzy weak complement of the SF5SES $(\gamma, Y, 5)$

$(\bar{\gamma}^c_{\mathfrak{w}}, Y, 5)$	$\mathfrak{u}_1$	$\mathfrak{u}_2$	$\mathfrak{u}_3$	$\mathfrak{u}_4$
$(p_1, r, 1)$	$\langle 4, (0.40, 0.09, 0.55)\rangle$	$\langle 4, (0.75, 0.11, 0.33)\rangle$	$\langle 4, (0.30, 0.01, 0.75)\rangle$	$\langle 4, (0.50, 0.10, 0.50)\rangle$
$(p_2, s, 1)$	$\langle 4, (0.3, 0.015, 0.65)\rangle$	$\langle 0, (0.09, 0.00, 0.95)\rangle$	$\langle 4, (0.50, 0.09, 0.50)\rangle$	$\langle 4, (0.45, 0.01, 0.55)\rangle$
$(p_3, t, 1)$	$\langle 4, (0.50, 0.10, 0.50)\rangle$	$\langle 4, (0.30, 0.01, 0.70)\rangle$	$\langle 4, (0.35, 0.01, 0.65)\rangle$	$\langle 4, (0.30, 0.01, 0.70)\rangle$
$(p_4, r, 0)$	$\langle 4, (0.30, 0.01, 0.75)\rangle$	$\langle 4, (0.55, 0.10, 0.45)\rangle$	$\langle 4, (0.40, 0.10, 0.55)\rangle$	$\langle 4, (0.30, 0.01, 0.79)\rangle$
$(p_4, t, 0)$	$\langle 4, (0.30, 0.01, 0.70)\rangle$	$\langle 4, (0.55, 0.10, 0.45)\rangle$	$\langle 4, (0.45, 0.10, 0.55)\rangle$	$\langle 4, (0.25, 0.01, 0.60)\rangle$

Table 13 Bottom weak complement of the SF5SES $(\gamma, Y, 5)$

$(\underline{\gamma}_{\mathfrak{w}}, Y, 5)$	$\mathfrak{u}_1$	$\mathfrak{u}_2$	$\mathfrak{u}_3$	$\mathfrak{u}_4$
$(p_1, r, 1)$	$\langle 0, (0.55, 0.09, 0.40)\rangle$	$\langle 0, (0.33, 0.11, 0.75)\rangle$	$\langle 0, (0.75, 0.01, 0.30)\rangle$	$\langle 0, (0.50, 0.10, 0.50)\rangle$
$(p_2, s, 1)$	$\langle 0, (0.65, 0.015, 0.3)\rangle$	$\langle 0, (0.95, 0.00, 0.09)\rangle$	$\langle 0, (0.50, 0.09, 0.50)\rangle$	$\langle 0, (0.55, 0.01, 0.45)\rangle$
$(p_3, t, 1)$	$\langle 0, (0.50, 0.10, 0.50)\rangle$	$\langle 0, (0.70, 0.01, 0.30)\rangle$	$\langle 0, (0.65, 0.01, 0.35)\rangle$	$\langle 0, (0.70, 0.01, 0.30)\rangle$
$(p_4, r, 0)$	$\langle 0, (0.75, 0.01, 0.30)\rangle$	$\langle 0, (0.45, 0.10, 0.55)\rangle$	$\langle 0, (0.55, 0.10, 0.40)\rangle$	$\langle 0, (0.79, 0.01, 0.30)\rangle$
$(p_4, t, 0)$	$\langle 0, (0.70, 0.01, 0.30)\rangle$	$\langle 0, (0.45, 0.10, 0.55)\rangle$	$\langle 0, (0.55, 0.10, 0.45)\rangle$	$\langle 0, (0.60, 0.01, 0.25)\rangle$

Table 14 Bottom spherical fuzzy weak complement of the SF5SES $(\gamma, Y, 5)$

$(\underline{\gamma}^c_{\mathfrak{w}}, Y, 5)$	$\mathfrak{u}_1$	$\mathfrak{u}_2$	$\mathfrak{u}_3$	$\mathfrak{u}_4$
$(p_1, r, 1)$	$\langle 0, (0.40, 0.09, 0.55)\rangle$	$\langle 0, (0.75, 0.11, 0.33)\rangle$	$\langle 0, (0.30, 0.01, 0.75)\rangle$	$\langle 0, (0.50, 0.10, 0.50)\rangle$
$(p_2, s, 1)$	$\langle 0, (0.3, 0.015, 0.65)\rangle$	$\langle 0, (0.09, 0.00, 0.95)\rangle$	$\langle 0, (0.50, 0.09, 0.50)\rangle$	$\langle 0, (0.45, 0.01, 0.55)\rangle$
$(p_3, t, 1)$	$\langle 0, (0.50, 0.10, 0.50)\rangle$	$\langle 0, (0.30, 0.01, 0.70)\rangle$	$\langle 0, (0.35, 0.01, 0.65)\rangle$	$\langle 0, (0.30, 0.01, 0.70)\rangle$
$(p_4, r, 0)$	$\langle 0, (0.30, 0.01, 0.75)\rangle$	$\langle 0, (0.55, 0.10, 0.45)\rangle$	$\langle 0, (0.40, 0.10, 0.55)\rangle$	$\langle 0, (0.30, 0.01, 0.79)\rangle$
$(p_4, t, 0)$	$\langle 0, (0.30, 0.01, 0.70)\rangle$	$\langle 0, (0.55, 0.10, 0.45)\rangle$	$\langle 0, (0.45, 0.10, 0.55)\rangle$	$\langle 0, (0.25, 0.01, 0.60)\rangle$

Proof

1. By Definition 12, we have $(\gamma, X, N)^c = (\gamma^c, X, N)$ with $\gamma^c = \langle(\mathfrak{u}, g_x), \kappa^c(\mathfrak{u}, g_x)\rangle$, such that $\forall \alpha \in \mathcal{U} \times G$,
$$\kappa^c(\alpha) = \Big(\mu(\alpha), \tau(\alpha), \nu(\alpha)\Big)^c = \Big(\nu(\alpha), \tau(\alpha), \mu(\alpha)\Big),$$
which implies that
$$\Big(\kappa^c(\alpha)\Big)^c = \Big(\nu(\alpha), \tau(\alpha), \mu(\alpha)\Big)^c = \Big(\mu(\alpha), \tau(\alpha), \nu(\alpha)\Big).$$
Thus $(\gamma^c)^c(x) = \gamma(x) = \langle(\mathfrak{u}, g_x), \kappa(\mathfrak{u}, g_x)\rangle\ \forall\, x \in X$, which proves that
$$\Big((\gamma, X, N)^c\Big)^c = (\gamma, X, N).$$

$\square$

Table 15 Agree-SF5SES of $(\gamma, X, 5)$ in Example 7

$(\gamma, X, 5)_1$	$\mathfrak{u}_1$	$\mathfrak{u}_2$	$\mathfrak{u}_3$	$\mathfrak{u}_4$
(p_1, r)	$\langle 2, (0.55, 0.09, 0.40)\rangle$	$\langle 1, (0.33, 0.11, 0.75)\rangle$	$\langle 3, (0.75, 0.01, 0.30)\rangle$	$\langle 2, (0.50, 0.10, 0.50)\rangle$
(p_1, s)	$\langle 2, (0.59, 0.10, 0.41)\rangle$	$\langle 1, (0.20, 0.12, 0.60)\rangle$	$\langle 3, (0.7, 0.015, 0.25)\rangle$	$\langle 2, (0.45, 0.08, 0.55)\rangle$
(p_1, t)	$\langle 2, (0.57, 0.05, 0.43)\rangle$	$\langle 1, (0.35, 0.10, 0.75)\rangle$	$\langle 3, (0.75, 0.015, 0.2)\rangle$	$\langle 2, (0.50, 0.09, 0.40)\rangle$
(p_2, r)	$\langle 3, (0.70, 0.01, 0.30)\rangle$	$\langle 4, (0.90, 0.01, 0.10)\rangle$	$\langle 2, (0.55, 0.10, 0.45)\rangle$	$\langle 2, (0.50, 0.10, 0.50)\rangle$
(p_2, s)	$\langle 3, (0.65, 0.015, 0.3)\rangle$	$\langle 4, (0.95, 0.00, 0.09)\rangle$	$\langle 2, (0.50, 0.09, 0.50)\rangle$	$\langle 2, (0.55, 0.01, 0.45)\rangle$
(p_2, t)	$\langle 3, (0.70, 0.01, 0.35)\rangle$	$\langle 4, (0.80, 0.01, 0.15)\rangle$	$\langle 2, (0.50, 0.10, 0.55)\rangle$	$\langle 2, (0.52, 0.05, 0.45)\rangle$
(p_3, r)	$\langle 2, (0.50, 0.10, 0.55)\rangle$	$\langle 3, (0.65, 0.01, 0.35)\rangle$	$\langle 3, (0.79, 0.005, 0.3)\rangle$	$\langle 3, (0.60, 0.01, 0.39)\rangle$
(p_3, s)	$\langle 2, (0.55, 0.05, 0.55)\rangle$	$\langle 3, (0.75, 0.015, 0.2)\rangle$	$\langle 3, (0.60, 0.01, 0.39)\rangle$	$\langle 3, (0.70, 0.01, 0.39)\rangle$
(p_3, t)	$\langle 2, (0.50, 0.10, 0.50)\rangle$	$\langle 3, (0.70, 0.01, 0.30)\rangle$	$\langle 3, (0.65, 0.01, 0.35)\rangle$	$\langle 3, (0.70, 0.01, 0.30)\rangle$
(p_4, r)	$\langle 3, (0.70, 0.01, 0.39)\rangle$	$\langle 2, (0.55, 0.10, 0.45)\rangle$	$\langle 2, (0.45, 0.10, 0.55)\rangle$	$\langle 3, (0.75, 0.01, 0.35)\rangle$
(p_4, s)	$\langle 3, (0.75, 0.001, 0.2)\rangle$	$\langle 2, (0.45, 0.10, 0.59)\rangle$	$\langle 2, (0.50, 0.01, 0.59)\rangle$	$\langle 3, (0.75, 0.01, 0.30)\rangle$
(p_4, t)	$\langle 3, (0.60, 0.01, 0.39)\rangle$	$\langle 2, (0.59, 0.10, 0.55)\rangle$	$\langle 2, (0.40, 0.10, 0.55)\rangle$	$\langle 3, (0.70, 0.01, 0.35)\rangle$

Definition 16 Consider a SFNSES (γ, X, N) over the universe $\mathcal{U}$. Then, the agree-SFNSES $(\gamma, X, N)_1$ is a spherical fuzzy N-soft expert subset of (γ, X, N) defined as

$$(\gamma, X, N)_1 = \{\gamma(x) : x \in P \times E \times \{1\}\}.$$

Definition 17 Consider a SFNSES (γ, X, N) over the universe $\mathcal{U}$. Then, the disagree-SFNSES $(\gamma, X, N)_0$ is a spherical fuzzy N-soft expert subset of (γ, X, N) defined as

$$(\gamma, X, N)_0 = \{\gamma(x) : x \in P \times E \times \{0\}\}.$$

Example 7 Reconsider Example 1. Then the agree SF5SES $(\gamma, X, 5)_1$ and the disagree SF5SES $(\gamma, X, 5)_0$ of the SF5SES $(\gamma, X, 5)$ over $\mathcal{U}$ are represented by Tables 15 and 16, respectively.

Definition 18 The restricted intersection of two SFNSESs (γ, X, N) and (δ, Y, N) over the universe $\mathcal{U}$ is again a SFNSES (H_R, A, N) over $\mathcal{U}$ denoted by $(\gamma, X, N)\bar{\cap}_R(\delta, Y, N)$, where $A = X \cap Y \neq \emptyset$ and $H_R(x) = \gamma(x) \cap \delta(x) \forall x \in A$ is defined as

$$H_R(x) = \left\{\langle(\mathfrak{u}, \min(g_x^1, g_x^2)), \kappa(\mathfrak{u}, g_x^1) \cap_S \kappa(\mathfrak{u}, g_x^2)\rangle : (\mathfrak{u}, g_x^1) \in \gamma(x), (\mathfrak{u}, g_x^2) \in \delta(x)\right\},$$

here $\kappa(\mathfrak{u}, g_x^1) \cap_S \kappa(\mathfrak{u}, g_x^2)$ represents spherical fuzzy intersection of $\kappa(\mathfrak{u}, g_x^1)$ and $\kappa(\mathfrak{u}, g_x^2)$.

Definition 19 The restricted union of two SFNSESs (γ, X, N) and (δ, Y, N) over the universe $\mathcal{U}$ is again a SFNSES (J_R, A, N) over $\mathcal{U}$ denoted by $(\gamma, X, N)\bar{\cup}_R(\delta, Y, N)$, where $A = X \cap Y \neq \emptyset$ and $J_R(x) = \gamma(x) \cup \delta(x) \forall x \in A$ is defined as

$$J_R(x) = \left\{\langle(\mathfrak{u}, \max(g_x^1, g_x^2)), \kappa(\mathfrak{u}, g_x^1) \cup_S \kappa(\mathfrak{u}, g_x^2)\rangle : (\mathfrak{u}, g_x^1) \in \gamma(x), (\mathfrak{u}, g_x^2) \in \delta(x)\right\},$$

here $\kappa(\mathfrak{u}, g_x^1) \cup_S \kappa(\mathfrak{u}, g_x^2)$ represents spherical fuzzy union of $\kappa(\mathfrak{u}, g_x^1)$ and $\kappa(\mathfrak{u}, g_x^2)$.

Example 8 Considering Example 1 again and suppose that $Y = \{(p_1, r, 1), (p_2, r, 1), (p_3, s, 1), \}$ $\{(p_4, t, 1), (p_1, s, 0), (p_3, t, 0)\}$, and $Z = \{(p_1, r, 1), (p_2, s, 1), (p_3, s, 1), (p_4, t, 1), (p_2, r, 0), (p_3, t, 0)\}$.

Table 16 Disagree-SF5SES of $(\gamma, X, 5)$ in Example 7

$(\gamma, X, 5)_0$	$\mathfrak{u}_1$	$\mathfrak{u}_2$	$\mathfrak{u}_3$	$\mathfrak{u}_4$
(p_1, r)	$\langle 2, (0.55, 0.10, 0.55)\rangle$	$\langle 1, (0.39, 0.12, 0.60)\rangle$	$\langle 3, (0.75, 0.01, 0.35)\rangle$	$\langle 2, (0.55, 0.10, 0.40)\rangle$
(p_1, s)	$\langle 2, (0.50, 0.10, 0.55)\rangle$	$\langle 1, (0.00, 0.12, 0.75)\rangle$	$\langle 3, (0.60, 0.01, 0.35)\rangle$	$\langle 2, (0.55, 0.10, 0.50)\rangle$
(p_1, t)	$\langle 2, (0.40, 0.10, 0.55)\rangle$	$\langle 1, (0.20, 0.10, 0.60)\rangle$	$\langle 3, (0.65, 0.01, 0.35)\rangle$	$\langle 2, (0.50, 0.10, 0.50)\rangle$
(p_2, r)	$\langle 3, (0.60, 0.01, 0.39)\rangle$	$\langle 4, (1.0, 0.005, 0.20)\rangle$	$\langle 2, (0.45, 0.10, 0.55)\rangle$	$\langle 2, (0.50, 0.10, 0.50)\rangle$
(p_2, s)	$\langle 3, (0.79, 0.01, 0.20)\rangle$	$\langle 4, (0.80, 0.00, 0.19)\rangle$	$\langle 2, (0.50, 0.10, 0.50)\rangle$	$\langle 2, (0.40, 0.01, 0.50)\rangle$
(p_2, t)	$\langle 3, (0.60, 0.01, 0.25)\rangle$	$\langle 4, (1.0, 0.005, 0.05)\rangle$	$\langle 2, (0.50, 0.10, 0.50)\rangle$	$\langle 2, (0.40, 0.10, 0.55)\rangle$
(p_3, r)	$\langle 2, (0.50, 0.10, 0.50)\rangle$	$\langle 3, (0.79, 0.01, 0.30)\rangle$	$\langle 3, (0.60, 0.01, 0.39)\rangle$	$\langle 3, (0.75, 0.01, 0.25)\rangle$
(p_3, s)	$\langle 2, (0.45, 0.10, 0.40)\rangle$	$\langle 3, (0.65, 0.01, 0.39)\rangle$	$\langle 3, (0.75, 0.01, 0.20)\rangle$	$\langle 3, (0.65, 0.01, 0.39)\rangle$
(p_3, t)	$\langle 2, (0.50, 0.10, 0.50)\rangle$	$\langle 3, (0.60, 0.01, 0.39)\rangle$	$\langle 3, (0.70, 0.01, 0.30)\rangle$	$\langle 3, (0.60, 0.01, 0.39)\rangle$
(p_4, r)	$\langle 3, (0.75, 0.01, 0.30)\rangle$	$\langle 2, (0.45, 0.10, 0.55)\rangle$	$\langle 2, (0.55, 0.10, 0.40)\rangle$	$\langle 3, (0.79, 0.01, 0.30)\rangle$
(p_4, s)	$\langle 3, (0.65, 0.01, 0.39)\rangle$	$\langle 2, (0.59, 0.10, 0.45)\rangle$	$\langle 2, (0.50, 0.10, 0.50)\rangle$	$\langle 3, (0.60, 0.01, 0.39)\rangle$
(p_4, t)	$\langle 3, (0.70, 0.01, 0.30)\rangle$	$\langle 2, (0.45, 0.10, 0.55)\rangle$	$\langle 2, (0.55, 0.10, 0.45)\rangle$	$\langle 3, (0.60, 0.01, 0.25)\rangle$

Consider Tables 17 and 18 represent the SF5SESs $(\gamma, Y, 5)$ and $(\delta, Z, 5)$ corresponding to the sets Y and Z respectively. Let $A = Y \cap Z$. Then using Definition 18, their restricted intersection $(H_R, A, 5)$ is given in the Table 19.

Similarly, using Definition 19, Table 20 represents the restricted union $(J_R, A, 5)$ of the SF5SESs $(\gamma, Y, 5)$ and $(\delta, Z, 5)$.

Proposition 2 *If (γ, X, N), (δ, Y, N) and (ψ, Z, N) are three SFNSESs over a common universe $\mathcal{U}$, then*

1. $(\gamma, X, N) \bar{\cup}_R (\delta, Y, N) = (\delta, Y, N) \bar{\cup}_R (\gamma, X, N)$.
2. $(\gamma, X, N) \bar{\cap}_R (\delta, Y, N) = (\delta, Y, N) \bar{\cap}_R (\gamma, X, N)$.
3. $(\gamma, X, N) \bar{\cup}_R ((\delta, Y, N) \bar{\cup}_R (\psi, Z, N)) = ((\gamma, X, N) \bar{\cup}_R (\delta, Y, N)) \bar{\cup}_R (\psi, Z, N)$.
4. $(\gamma, X, N) \bar{\cap}_R ((\delta, Y, N) \bar{\cap}_R (\psi, Z, N)) = ((\gamma, X, N) \bar{\cap}_R (\delta, Y, N)) \bar{\cap}_R (\psi, Z, N)$.

Proof

1. Let $(\mathfrak{u}, g_x^1) \in \gamma(x)$ and $(\mathfrak{u}, g_x^2) \in \delta(x)\, \forall\, x \in X \cap Y$, then by Definition 19,
$$\begin{aligned}(\gamma, X, N) \bar{\cup}_R (\delta, Y, N) &= \big\{\langle\big(\mathfrak{u}, \max(g_x^1, g_x^2)\big), \kappa(\mathfrak{u}, g_x^1) \cup_S \kappa(\mathfrak{u}, g_x^2)\rangle | \mathfrak{u} \in \mathcal{U}\big\},\\ &= \big\{\langle\big(\mathfrak{u}, \max(g_x^2, g_x^1)\big), \kappa(\mathfrak{u}, g_x^2) \cup_S \kappa(\mathfrak{u}, g_x^1)\rangle | \mathfrak{u} \in \mathcal{U}\big\},\\ &= (\delta, Y, N) \bar{\cup}_R (\gamma, X, N),\end{aligned}$$

where $\kappa(\mathfrak{u}, g_x^1) \cup_S \kappa(\mathfrak{u}, g_x^2) = \kappa(\mathfrak{u}, g_x^2) \cup_S \kappa(\mathfrak{u}, g_x^1)$. Hence $(\gamma, X, N) \bar{\cup}_R (\delta, Y, N) = (\delta, Y, N) \bar{\cup}_R (\gamma, X, N)$. The remaining parts can be proved similarly. □

Definition 20 The extended intersection of two SFNSESs (γ, X, N) and (δ, Y, N) over the universe $\mathcal{U}$ is the SFNSES $(H_\mathcal{E}, A, N)$ over $\mathcal{U}$, where $A = X \cup Y$ and $\forall\, x \in A$,

Table 17 SF5SES $(\gamma, Y, 5)$ in Example 8

$(\gamma, Y, 5)$	$\mathfrak{u}_1$	$\mathfrak{u}_2$	$\mathfrak{u}_3$	$\mathfrak{u}_4$
$(p_1, r, 1)$	$\langle 2, (0.55, 0.09, 0.40)\rangle$	$\langle 1, (0.33, 0.11, 0.75)\rangle$	$\langle 3, (0.75, 0.01, 0.30)\rangle$	$\langle 2, (0.50, 0.10, 0.50)\rangle$
$(p_2, r, 1)$	$\langle 3, (0.70, 0.01, 0.30)\rangle$	$\langle 4, (0.90, 0.01, 0.10)\rangle$	$\langle 2, (0.55, 0.10, 0.45)\rangle$	$\langle 2, (0.50, 0.10, 0.50)\rangle$
$(p_3, s, 1)$	$\langle 2, (0.55, 0.05, 0.55)\rangle$	$\langle 3, (0.75, 0.015, 0.2)\rangle$	$\langle 3, (0.60, 0.01, 0.39)\rangle$	$\langle 3, (0.70, 0.01, 0.39)\rangle$
$(p_4, t, 1)$	$\langle 3, (0.60, 0.01, 0.39)\rangle$	$\langle 2, (0.59, 0.10, 0.55)\rangle$	$\langle 2, (0.40, 0.10, 0.55)\rangle$	$\langle 3, (0.70, 0.01, 0.35)\rangle$
$(p_1, s, 0)$	$\langle 2, (0.50, 0.10, 0.55)\rangle$	$\langle 1, (0.00, 0.12, 0.75)\rangle$	$\langle 3, (0.60, 0.01, 0.35)\rangle$	$\langle 2, (0.55, 0.10, 0.50)\rangle$
$(p_3, t, 0)$	$\langle 2, (0.50, 0.10, 0.50)\rangle$	$\langle 3, (0.60, 0.01, 0.39)\rangle$	$\langle 3, (0.70, 0.01, 0.30)\rangle$	$\langle 3, (0.60, 0.01, 0.39)\rangle$

Table 18 SF5SES $(\delta, Z, 5)$ in Example 8

$(\delta, Z, 5)$	$\mathfrak{u}_1$	$\mathfrak{u}_2$	$\mathfrak{u}_3$	$\mathfrak{u}_4$
$(p_1, r, 1)$	$\langle 2, (0.45, 0.09, 0.50)\rangle$	$\langle 1, (0.35, 0.11, 0.70)\rangle$	$\langle 3, (0.70, 0.01, 0.39)\rangle$	$\langle 2, (0.55, 0.10, 0.50)\rangle$
$(p_2, s, 1)$	$\langle 3, (0.65, 0.01, 0.35)\rangle$	$\langle 4, (0.90, 0.00, 0.10)\rangle$	$\langle 2, (0.45, 0.09, 0.55)\rangle$	$\langle 2, (0.55, 0.01, 0.40)\rangle$
$(p_3, s, 1)$	$\langle 2, (0.50, 0.05, 0.50)\rangle$	$\langle 3, (0.79, 0.015, 0.1)\rangle$	$\langle 3, (0.65, 0.01, 0.35)\rangle$	$\langle 3, (0.70, 0.01, 0.39)\rangle$
$(p_4, t, 1)$	$\langle 3, (0.70, 0.01, 0.35)\rangle$	$\langle 2, (0.55, 0.10, 0.59)\rangle$	$\langle 2, (0.45, 0.10, 0.45)\rangle$	$\langle 3, (0.75, 0.01, 0.39)\rangle$
$(p_2, r, 0)$	$\langle 3, (0.65, 0.01, 0.35)\rangle$	$\langle 4, (0.90, 0.01, 0.20)\rangle$	$\langle 2, (0.45, 0.10, 0.55)\rangle$	$\langle 2, (0.50, 0.10, 0.50)\rangle$
$(p_3, t, 0)$	$\langle 2, (0.50, 0.10, 0.50)\rangle$	$\langle 3, (0.70, 0.01, 0.39)\rangle$	$\langle 3, (0.65, 0.01, 0.25)\rangle$	$\langle 3, (0.70, 0.01, 0.30)\rangle$

Table 19 Restricted intersection of SF5SESs $(\gamma, Y, 5)$ and $(\delta, Z, 5)$ in Example 8

$(H_R, A, 5)$	$\mathfrak{u}_1$	$\mathfrak{u}_2$	$\mathfrak{u}_3$	$\mathfrak{u}_4$
$(p_1, r, 1)$	$\langle 2, (0.45, 0.09, 0.50)\rangle$	$\langle 1, (0.33, 0.11, 0.75)\rangle$	$\langle 3, (0.70, 0.01, 0.39)\rangle$	$\langle 2, (0.50, 0.10, 0.50)\rangle$
$(p_3, s, 1)$	$\langle 2, (0.50, 0.05, 0.55)\rangle$	$\langle 3, (0.75, 0.015, 0.2)\rangle$	$\langle 3, (0.60, 0.01, 0.39)\rangle$	$\langle 3, (0.70, 0.01, 0.39)\rangle$
$(p_4, t, 1)$	$\langle 3, (0.60, 0.01, 0.39)\rangle$	$\langle 2, (0.55, 0.10, 0.59)\rangle$	$\langle 2, (0.40, 0.10, 0.55)\rangle$	$\langle 3, (0.70, 0.01, 0.39)\rangle$
$(p_3, t, 0)$	$\langle 2, (0.50, 0.10, 0.50)\rangle$	$\langle 3, (0.60, 0.01, 0.39)\rangle$	$\langle 3, (0.65, 0.01, 0.30)\rangle$	$\langle 3, (0.60, 0.01, 0.39)\rangle$

Table 20 Restricted union of SF5SESs $(\gamma, Y, 5)$ and $(\delta, Z, 5)$ in Example 8

$(J_R, A, 5)$	$\mathfrak{u}_1$	$\mathfrak{u}_2$	$\mathfrak{u}_3$	$\mathfrak{u}_4$
$(p_1, r, 1)$	$\langle 2, (0.55, 0.09, 0.40)\rangle$	$\langle 1, (0.35, 0.11, 0.70)\rangle$	$\langle 3, (0.75, 0.01, 0.30)\rangle$	$\langle 2, (0.55, 0.10, 0.50)\rangle$
$(p_3, s, 1)$	$\langle 2, (0.55, 0.05, 0.50)\rangle$	$\langle 3, (0.79, 0.015, 0.1)\rangle$	$\langle 3, (0.65, 0.01, 0.35)\rangle$	$\langle 3, (0.70, 0.01, 0.39)\rangle$
$(p_4, t, 1)$	$\langle 3, (0.70, 0.01, 0.35)\rangle$	$\langle 2, (0.59, 0.10, 0.55)\rangle$	$\langle 2, (0.45, 0.10, 0.45)\rangle$	$\langle 3, (0.75, 0.01, 0.35)\rangle$
$(p_3, t, 0)$	$\langle 2, (0.50, 0.10, 0.50)\rangle$	$\langle 3, (0.70, 0.01, 0.39)\rangle$	$\langle 3, (0.70, 0.01, 0.25)\rangle$	$\langle 3, (0.70, 0.01, 0.30)\rangle$

Table 21 Extended intersection

$(H_{\mathcal{E}}, A, 5)$	$\mathfrak{u}_1$	$\mathfrak{u}_2$	$\mathfrak{u}_3$	$\mathfrak{u}_4$
$(p_1, r, 1)$	$\langle 2, (0.45, 0.09, 0.50)\rangle$	$\langle 1, (0.33, 0.11, 0.75)\rangle$	$\langle 3, (0.70, 0.01, 0.39)\rangle$	$\langle 2, (0.50, 0.10, 0.50)\rangle$
$(p_2, r, 1)$	$\langle 3, (0.70, 0.01, 0.30)\rangle$	$\langle 4, (0.90, 0.01, 0.10)\rangle$	$\langle 2, (0.55, 0.10, 0.45)\rangle$	$\langle 2, (0.50, 0.10, 0.50)\rangle$
$(p_2, s, 1)$	$\langle 3, (0.65, 0.01, 0.35)\rangle$	$\langle 4, (0.90, 0.00, 0.10)\rangle$	$\langle 2, (0.45, 0.09, 0.55)\rangle$	$\langle 2, (0.55, 0.01, 0.40)\rangle$
$(p_3, s, 1)$	$\langle 2, (0.50, 0.05, 0.55)\rangle$	$\langle 3, (0.75, 0.015, 0.2)\rangle$	$\langle 3, (0.60, 0.01, 0.39)\rangle$	$\langle 3, (0.70, 0.01, 0.39)\rangle$
$(p_4, t, 1)$	$\langle 3, (0.60, 0.01, 0.39)\rangle$	$\langle 2, (0.55, 0.10, 0.59)\rangle$	$\langle 2, (0.40, 0.10, 0.55)\rangle$	$\langle 3, (0.70, 0.01, 0.39)\rangle$
$(p_1, s, 0)$	$\langle 2, (0.50, 0.10, 0.55)\rangle$	$\langle 1, (0.00, 0.12, 0.75)\rangle$	$\langle 3, (0.60, 0.01, 0.35)\rangle$	$\langle 2, (0.55, 0.10, 0.50)\rangle$
$(p_2, r, 0)$	$\langle 3, (0.65, 0.01, 0.35)\rangle$	$\langle 4, (0.90, 0.01, 0.20)\rangle$	$\langle 2, (0.45, 0.10, 0.55)\rangle$	$\langle 2, (0.50, 0.10, 0.50)\rangle$
$(p_3, t, 0)$	$\langle 2, (0.50, 0.10, 0.50)\rangle$	$\langle 3, (0.60, 0.01, 0.39)\rangle$	$\langle 3, (0.65, 0.01, 0.30)\rangle$	$\langle 3, (0.60, 0.01, 0.39)\rangle$

$$H_{\mathcal{E}}(x) = \begin{cases} \gamma(x), & \textit{if}\, x \in X - Y, \\ \delta(x), & \textit{if}\, x \in Y - X, \\ \gamma(x) \cap \delta(x), & \textit{if}\, x \in X \cap Y, \end{cases}$$

such that $\forall\, x \in X \cap Y$

$$\gamma(x) \cap \delta(x) = \left\{ \langle (\mathfrak{u}, \max(g_x^1, g_x^2)), \kappa(\mathfrak{u}, g_x^1) \cup_S \kappa(\mathfrak{u}, g_x^2) \rangle \,:\, (\mathfrak{u}, g_x^1) \in \gamma(x),\, (\mathfrak{u}, g_x^2) \in \delta(x) | \mathfrak{u} \in \mathcal{U} \right\}$$

with $\kappa(\mathfrak{u}, g_x^1) \cap_S \kappa(\mathfrak{u}, g_x^2)$ as the spherical fuzzy intersection of $\kappa(\mathfrak{u}, g_x^1)$ and $\kappa(\mathfrak{u}, g_x^2)$. This relation is denoted by $(\gamma, X, N)\bar{\cap}_{\mathcal{E}}(\delta, Y, N)$.

Example 9 Reconsider Example 8 and two SF5SESs $(\gamma, Y, 5)$ and $(\delta, Z, 5)$ in Tables 17 and 18, respectively. Then, by Definition 20, the extended intersection $(H_{\mathcal{E}}, A, 5)$ of the SF5SESs $(\gamma, Y, 5)$ and $(\delta, Z, 5)$ is given in Table 21.

Definition 21 The extended union of two SFNSESs (γ, X, N) and (δ, Y, N) over the universe $\mathcal{U}$ is the SFNSES $(J_{\mathcal{E}}, A, N)$ over $\mathcal{U}$, where $A = X \cup Y$ and $\forall\, x \in A$,

$$J_{\mathcal{E}}(x) = \begin{cases} \gamma(x), & \textit{if}\, x \in X - Y, \\ \delta(x), & \textit{if}\, x \in Y - X, \\ \gamma(x) \cup \delta(x), & \textit{if}\, x \in X \cap Y, \end{cases}$$

such that $\forall\, x \in X \cap Y$

$$\gamma(x) \cup \delta(x) = \left\{ \langle (\mathfrak{u}, \max(g_x^1, g_x^2)), \kappa(\mathfrak{u}, g_x^1) \cup_S \kappa(\mathfrak{u}, g_x^2) \rangle \,:\, (\mathfrak{u}, g_x^1) \in \gamma(x),\, (\mathfrak{u}, g_x^2) \in \delta(x) | \mathfrak{u} \in \mathcal{U} \right\},$$

with $\kappa(\mathfrak{u}, g_x^1) \cup_S \kappa(\mathfrak{u}, g_x^2)$ as the spherical fuzzy union of $\kappa(\mathfrak{u}, g_x^1)$ and $\kappa(\mathfrak{u}, g_x^2)$. This relation is denoted by $(\gamma, X, N)\bar{\cup}_{\mathcal{E}}(\delta, Y, N)$.

Example 10 Reconsider Example 8 and two SF5SESs $(\gamma, Y, 5)$ and $(\delta, Z, 5)$ in Tables 17 and 18, respectively. Then, by Definition 21, the extended union $(J_{\mathcal{E}}, A, 5)$ of SF5SESs $(\gamma, Y, 5)$ and $(\delta, Z, 5)$ is given in Table 22.

Table 22 Extended union

$(J_{\mathcal{E}}, A, 5)$	$\mathfrak{u}_1$	$\mathfrak{u}_2$	$\mathfrak{u}_3$	$\mathfrak{u}_4$
$(p_1, r, 1)$	$\langle 2, (0.55, 0.09, 0.40)\rangle$	$\langle 1, (0.35, 0.11, 0.70)\rangle$	$\langle 3, (0.75, 0.01, 0.30)\rangle$	$\langle 2, (0.55, 0.10, 0.50)\rangle$
$(p_2, r, 1)$	$\langle 3, (0.70, 0.01, 0.30)\rangle$	$\langle 4, (0.90, 0.01, 0.10)\rangle$	$\langle 2, (0.55, 0.10, 0.45)\rangle$	$\langle 2, (0.50, 0.10, 0.50)\rangle$
$(p_2, s, 1)$	$\langle 3, (0.65, 0.01, 0.35)\rangle$	$\langle 4, (0.90, 0.00, 0.10)\rangle$	$\langle 2, (0.45, 0.09, 0.55)\rangle$	$\langle 2, (0.55, 0.01, 0.40)\rangle$
$(p_3, s, 1)$	$\langle 2, (0.55, 0.05, 0.50)\rangle$	$\langle 3, (0.79, 0.015, 0.1)\rangle$	$\langle 3, (0.65, 0.01, 0.35)\rangle$	$\langle 3, (0.70, 0.01, 0.39)\rangle$
$(p_4, t, 1)$	$\langle 3, (0.70, 0.01, 0.35)\rangle$	$\langle 2, (0.59, 0.10, 0.55)\rangle$	$\langle 2, (0.45, 0.10, 0.45)\rangle$	$\langle 3, (0.75, 0.01, 0.35)\rangle$
$(p_1, s, 0)$	$\langle 2, (0.50, 0.10, 0.55)\rangle$	$\langle 1, (0.00, 0.12, 0.75)\rangle$	$\langle 3, (0.60, 0.01, 0.35)\rangle$	$\langle 2, (0.55, 0.10, 0.50)\rangle$
$(p_2, r, 0)$	$\langle 3, (0.65, 0.01, 0.35)\rangle$	$\langle 4, (0.90, 0.01, 0.20)\rangle$	$\langle 2, (0.45, 0.10, 0.55)\rangle$	$\langle 2, (0.50, 0.10, 0.50)\rangle$
$(p_3, t, 0)$	$\langle 2, (0.50, 0.10, 0.50)\rangle$	$\langle 3, (0.70, 0.01, 0.39)\rangle$	$\langle 3, (0.70, 0.01, 0.25)\rangle$	$\langle 3, (0.70, 0.01, 0.30)\rangle$

Proposition 3 *If (γ, X, N), (δ, Y, N) and (ψ, Z, N) are three SFNSESs over a common universe $\mathcal{U}$, then*

1. $(\gamma, X, N)\,\bar{\cup}_{\mathcal{E}}\,(\delta, Y, N) = (\delta, Y, N)\,\bar{\cup}_{\mathcal{E}}\,(\gamma, X, N)$.
2. $(\gamma, X, N)\,\bar{\cap}_{\mathcal{E}}\,(\delta, Y, N) = (\delta, Y, N)\,\bar{\cap}_{\mathcal{E}}\,(\gamma, X, N)$.
3. $(\gamma, X, N)\,\bar{\cup}_{\mathcal{E}}\,((\delta, Y, N)\,\bar{\cup}_{\mathcal{E}}\,(\psi, Z, N)) = ((\gamma, X, N)\,\bar{\cup}_{\mathcal{E}}\,(\delta, Y, N))\,\bar{\cup}_{\mathcal{E}}\,(\psi, Z, N)$.
4. $(\gamma, X, N)\,\bar{\cap}_{\mathcal{E}}\,((\delta, Y, N)\,\bar{\cap}_{\mathcal{E}}\,(\psi, Z, N)) = ((\gamma, X, N)\,\bar{\cap}_{\mathcal{E}}\,(\delta, Y, N))\,\bar{\cap}_{\mathcal{E}}\,(\psi, Z, N)$.

Proof

1. Let $(\zeta, X \cup Y, N) = (\gamma, X, N)\,\bar{\cup}_{\mathcal{E}}\,(\delta, Y, N)$, then from Definition 21, $\forall\, x \in X \cup Y$, we have
$$\zeta(x) = \begin{cases} \gamma(x), & \textit{if}\ x \in X - Y, \\ \delta(x), & \textit{if}\ x \in Y - X, \\ \gamma(x) \cup \delta(x), & \textit{if}\ x \in X \cap Y. \end{cases}$$
   Considering only non-trivial case when $x \in X \cap Y$, we have
$$\begin{aligned} \zeta(x) &= \gamma(x) \cup \delta(x) = \langle \big(\mathfrak{u}, \max(g_x^1, g_x^2)\big), \kappa(\mathfrak{u}, g_x^1) \cup_S \kappa(\mathfrak{u}, g_x^2)\rangle, \\ &= \langle \big(\mathfrak{u}, \max(g_x^2, g_x^1)\big), \kappa(\mathfrak{u}, g_x^2) \cup_S \kappa(\mathfrak{u}, g_x^1)\rangle, \\ &= \delta(x) \cup \gamma(x). \end{aligned}$$
   Here $\kappa(\mathfrak{u}, g_x^1) \cup_S \kappa(\mathfrak{u}, g_x^2) = \kappa(\mathfrak{u}, g_x^2) \cup_S \kappa(\mathfrak{u}, g_x^1)$. Hence it proves that
$$(\gamma, X, N)\,\bar{\cup}_{\mathcal{E}}\,(\delta, Y, N) = (\delta, Y, N)\,\bar{\cup}_{\mathcal{E}}\,(\gamma, X, N).$$

The remaining parts can be proved similarly. □

Proposition 4 *Let (γ, X, N) and (δ, X, N) be two SFNSESs over $\mathcal{U}$, we have*

1. $(\gamma, X, N)\,\bar{\cup}_{\mathcal{E}}\,(\delta, X, N) = (\gamma, X, N)\,\bar{\cup}_{R}\,(\delta, X, N)$.
2. $(\gamma, X, N)\,\bar{\cap}_{\mathcal{E}}\,(\delta, X, N) = (\gamma, X, N)\,\bar{\cap}_{R}\,(\delta, X, N)$.

Table 23 SF5SES $(\gamma, Y, 5)$ in Example 11

$(\gamma, Y, 5)$	$\mathfrak{u}_1$	$\mathfrak{u}_2$	$\mathfrak{u}_3$	$\mathfrak{u}_4$
$(p_1, s, 1)$	$\langle 2, (0.59, 0.10, 0.41)\rangle$	$\langle 1, (0.20, 0.12, 0.60)\rangle$	$\langle 3, (0.7, 0.015, 0.25)\rangle$	$\langle 2, (0.45, 0.08, 0.55)\rangle$
$(p_2, r, 1)$	$\langle 3, (0.70, 0.01, 0.30)\rangle$	$\langle 4, (0.90, 0.01, 0.10)\rangle$	$\langle 2, (0.55, 0.10, 0.45)\rangle$	$\langle 2, (0.50, 0.10, 0.50)\rangle$
$(p_4, t, 0)$	$\langle 3, (0.70, 0.01, 0.30)\rangle$	$\langle 2, (0.45, 0.10, 0.55)\rangle$	$\langle 2, (0.55, 0.10, 0.45)\rangle$	$\langle 3, (0.60, 0.01, 0.25)\rangle$

Table 24 SF5SES $(\delta, Z, 5)$ in Example 11

$(\delta, Z, 5)$	$\mathfrak{u}_1$	$\mathfrak{u}_2$	$\mathfrak{u}_3$	$\mathfrak{u}_4$
$(p_1, s, 1)$	$\langle 2, (0.50, 0.08, 0.55)\rangle$	$\langle 1, (0.30, 0.10, 0.79)\rangle$	$\langle 3, (0.60, 0.01, 0.35)\rangle$	$\langle 2, (0.50, 0.08, 0.55)\rangle$
$(p_3, t, 1)$	$\langle 2, (0.50, 0.10, 0.50)\rangle$	$\langle 3, (0.75, 0.01, 0.39)\rangle$	$\langle 3, (0.75, 0.01, 0.39)\rangle$	$\langle 3, (0.60, 0.01, 0.35)\rangle$

3. $(\gamma, X, N)\, \bar{\cup}_{\mathcal{E}}\, (\gamma, X, N) = (\gamma, X, N)$.
4. $(\gamma, X, N)\, \bar{\cap}_{\mathcal{E}}\, (\gamma, X, N) = (\gamma, X, N)$.

Proof

1. For all $x \in X \cup X = X \cap X$, we have
$$(\gamma\, \bar{\cup}_{\mathcal{E}}\, \delta)(x) = \langle \big(\mathfrak{u}, \max(g_x^1, g_x^2)\big), \kappa(\mathfrak{u}, g_x^1) \cup_S \kappa(\mathfrak{u}, g_x^2)\rangle,$$
$$= (\gamma\, \bar{\cup}_R\, \delta)(x).$$
Hence, it is proved that
$$(\gamma, X, N)\, \bar{\cup}_{\mathcal{E}}\, (\delta, X, N) = (\gamma, X, N)\, \bar{\cup}_R\, (\delta, X, N).$$

The remaining parts can be proved similarly. □

Definition 22 Let (γ, X, N) and (δ, Y, N) be two SFNSESs over $\mathcal{U}$. Then the 'AND' operation between them denoted by $(\gamma, X, N) \barwedge (\delta, Y, N)$, is defined as

$$(\gamma, X, N) \barwedge (\delta, Y, N) = (\rho, X \times Y, N),$$

where $\rho(\alpha, \beta) = \gamma(\alpha) \cap \delta(\beta)$, $\forall\, (\alpha, \beta) \in X \times Y$.

Example 11 Reconsider Example 1. Assume $Y = \{(p_1, s, 1), (p_2, r, 1), (p_4, t, 0)\}$ and $Z = \{(p_1, s, 1), (p_3, t, 1)\}$, and the SF5SESs $(\gamma, Y, 5)$ and $(\delta, Z, 5)$ are defined on them as shown in Tables 23 and 24, respectively.

Using Definition 22, the AND operation $(\rho, Y \times Z, 5)$ between these SF5SESs is shown in Table 25.

Table 25 AND operation between $(\gamma, Y, 5)$ and $(\delta, Z, 5)$ in Example 11

$(\rho, Y \times Z, 5)$	$\mathfrak{u}_1$	$\mathfrak{u}_2$	$\mathfrak{u}_3$	$\mathfrak{u}_4$
$\big((p_1, s, 1), (p_1, s, 1)\big)$	$\langle 2, (0.50, 0.08, 0.55)\rangle$	$\langle 1, (0.20, 0.10, 0.79)\rangle$	$\langle 3, (0.60, 0.01, 0.35)\rangle$	$\langle 2, (0.45, 0.08, 0.55)\rangle$
$\big((p_1, s, 1), (p_3, t, 1)\big)$	$\langle 2, (0.50, 0.10, 0.50)\rangle$	$\langle 1, (0.20, 0.01, 0.60)\rangle$	$\langle 3, (0.70, 0.01, 0.39)\rangle$	$\langle 2, (0.45, 0.01, 0.55)\rangle$
$\big((p_2, r, 1), (p_1, s, 1)\big)$	$\langle 2, (0.50, 0.01, 0.55)\rangle$	$\langle 1, (0.30, 0.01, 0.79)\rangle$	$\langle 2, (0.55, 0.01, 0.45)\rangle$	$\langle 2, (0.50, 0.08, 0.55)\rangle$
$\big((p_2, r, 1), (p_3, t, 1)\big)$	$\langle 2, (0.50, 0.01, 0.50)\rangle$	$\langle 3, (0.75, 0.01, 0.39)\rangle$	$\langle 2, (0.55, 0.01, 0.45)\rangle$	$\langle 2, (0.50, 0.01, 0.50)\rangle$
$\big((p_4, t, 0), (p_1, s, 1)\big)$	$\langle 2, (0.50, 0.01, 0.55)\rangle$	$\langle 1, (0.30, 0.10, 0.79)\rangle$	$\langle 2, (0.55, 0.01, 0.45)\rangle$	$\langle 2, (0.50, 0.01, 0.55)\rangle$
$\big((p_4, t, 0), (p_3, t, 1)\big)$	$\langle 2, (0.50, 0.01, 0.50)\rangle$	$\langle 2, (0.45, 0.01, 0.55)\rangle$	$\langle 2, (0.55, 0.01, 0.45)\rangle$	$\langle 3, (0.60, 0.01, 0.35)\rangle$

Table 26 OR operation between $(\gamma, Y, 5)$ and $(\delta, Z, 5)$ in Example 12

$(\varrho, Y \times Z, 5)$	$\mathfrak{u}_1$	$\mathfrak{u}_2$	$\mathfrak{u}_3$	$\mathfrak{u}_4$
$\big((p_1, s, 1), (p_1, s, 1)\big)$	$\langle 2, (0.59, 0.08, 0.41)\rangle$	$\langle 1, (0.30, 0.10, 0.60)\rangle$	$\langle 3, (0.70, 0.01, 0.25)\rangle$	$\langle 2, (0.50, 0.08, 0.55)\rangle$
$\big((p_1, s, 1), (p_3, t, 1)\big)$	$\langle 2, (0.59, 0.10, 0.41)\rangle$	$\langle 3, (0.75, 0.01, 0.39)\rangle$	$\langle 3, (0.75, 0.01, 0.39)\rangle$	$\langle 3, (0.60, 0.01, 0.35)\rangle$
$\big((p_2, r, 1), (p_1, s, 1)\big)$	$\langle 3, (0.70, 0.01, 0.30)\rangle$	$\langle 4, (0.90, 0.01, 0.10)\rangle$	$\langle 3, (0.60, 0.01, 0.35)\rangle$	$\langle 2, (0.50, 0.08, 0.50)\rangle$
$\big((p_2, r, 1), (p_3, t, 1)\big)$	$\langle 3, (0.70, 0.01, 0.30)\rangle$	$\langle 4, (0.90, 0.01, 0.10)\rangle$	$\langle 3, (0.75, 0.01, 0.39)\rangle$	$\langle 3, (0.60, 0.01, 0.35)\rangle$
$\big((p_4, t, 0), (p_1, s, 1)\big)$	$\langle 3, (0.70, 0.01, 0.30)\rangle$	$\langle 2, (0.45, 0.10, 0.55)\rangle$	$\langle 3, (0.60, 0.01, 0.35)\rangle$	$\langle 3, (0.60, 0.01, 0.25)\rangle$
$\big((p_4, t, 0), (p_3, t, 1)\big)$	$\langle 3, (0.70, 0.01, 0.30)\rangle$	$\langle 3, (0.75, 0.01, 0.39)\rangle$	$\langle 3, (0.75, 0.01, 0.39)\rangle$	$\langle 3, (0.60, 0.01, 0.25)\rangle$

Definition 23 Let (γ, X, N) and (δ, Y, N) be two SFNSESs over $\mathcal{U}$. Then the 'OR' operation between them denoted by $(\gamma, X, N) \veebar (\delta, Y, N)$, is defined as

$$(\gamma, X, N) \veebar (\delta, Y, N) = (\varrho, X \times Y, N),$$

where $\varrho(\alpha, \beta) = \gamma(\alpha) \cup \delta(\beta)$, $\forall\, (\alpha, \beta) \in X \times Y$.

Example 12 Reconsider Example 11 and two SF5SESs $(\gamma, Y, 5)$ and $(\delta, Z, 5)$ in Tables 23 and 24, respectively. Then by Definition 23, the OR operation $(\varrho, Y \times Z, 5)$ between them is shown in Table 26.

Proposition 5 *If* $(\gamma, X, N), (\delta, Y, N)$ *and* (ψ, Z, N) *are three SFNSESs over the universe* $\mathcal{U}$, *then*

1. $(\gamma, X, N) \veebar ((\delta, Y, N) \veebar (\psi, Z, N)) = ((\gamma, X, N) \veebar (\delta, Y, N)) \veebar (\psi, Z, N)$.
2. $(\gamma, X, N) \barwedge ((\delta, Y, N) \barwedge (\psi, Z, N)) = ((\gamma, X, N) \barwedge (\delta, Y, N)) \barwedge (\psi, Z, N)$.
3. $(\gamma, X, N) \veebar ((\delta, Y, N) \barwedge (\psi, Z, N)) = ((\gamma, X, N) \veebar (\delta, Y, N)) \barwedge ((\gamma, X, N) \veebar (\psi, Z, N))$.
4. $(\gamma, X, N) \barwedge ((\delta, Y, N) \veebar (\psi, Z, N)) = ((\gamma, X, N) \barwedge (\delta, Y, N)) \veebar ((\gamma, X, N) \barwedge (\psi, Z, N))$.

Proof

1. Let $\delta(\beta) \veebar \psi(\lambda) = \delta(\beta) \cup \psi(\lambda)$, where for all $(\beta, \lambda) \in Y \times Z$,

$$\delta(\beta) \cup \psi(\lambda) = \langle \big(\mathfrak{u}, \max(g_\beta, g_\lambda)\big), \kappa(\mathfrak{u}, g_\beta) \cup_S \kappa(\mathfrak{u}, g_\lambda)\rangle.$$

Then,

$$\begin{aligned}
\gamma(\alpha) \veebar \big(\delta(\beta) \veebar \psi(\lambda)\big) &= \gamma(\alpha) \cup \big(\delta(\beta) \cup \psi(\lambda)\big), \quad \forall\, (\alpha, (\beta, \lambda)) \in X \times Y \times Z \\
&= \big(\gamma(\alpha) \cup \delta(\beta)\big) \cup \psi(\lambda), \quad \forall\, ((\alpha, \beta), \lambda) \in X \times Y \times Z \\
&\quad (\because \text{By Proposition 2}) \\
&= \big(\gamma(\alpha) \veebar \delta(\beta)\big) \veebar \psi(\lambda).
\end{aligned}$$

Hence it is proved that

$$(\gamma, X, N) \veebar ((\delta, Y, N) \veebar (\psi, Z, N)) = ((\gamma, X, N) \veebar (\delta, Y, N)) \veebar (\psi, Z, N).$$

2. Let $\delta(\beta) \barwedge \psi(\lambda) = \delta(\beta) \cap \psi(\lambda)$, where for all $(\beta, \lambda) \in Y \times Z$,

$$\delta(\beta) \cap \psi(\lambda) = \langle \big(\mathfrak{u}, \min(g_\beta, g_\lambda)\big), \kappa(\mathfrak{u}, g_\beta) \cap_S \kappa(\mathfrak{u}, g_\lambda)\rangle.$$

Then,

$$\begin{aligned}
\gamma(\alpha) \barwedge \big(\delta(\beta) \barwedge \psi(\lambda)\big) &= \gamma(\alpha) \cap \big(\delta(\beta) \cap \psi(\lambda)\big), \quad \forall\, (\alpha, (\beta, \lambda)) \in X \times Y \times Z \\
&= \big(\gamma(\alpha) \cap \delta(\beta)\big) \cap \psi(\lambda), \quad \forall\, ((\alpha, \beta), \lambda) \in X \times Y \times Z \\
&\quad (\because \text{By Proposition 2}) \\
&= \big(\gamma(\alpha) \barwedge \delta(\beta)\big) \barwedge \psi(\lambda).
\end{aligned}$$

Hence it is proved that

$$(\gamma, X, N) \barwedge ((\delta, Y, N) \barwedge (\psi, Z, N)) = ((\gamma, X, N) \barwedge (\delta, Y, N)) \barwedge (\psi, Z, N).$$

3. Let $\pi(\beta, \lambda) = \delta(\beta) \barwedge \psi(\lambda)$, then

$$\begin{aligned}
\gamma(\alpha) \veebar (\delta(\beta) \barwedge \psi(\lambda)) &= \gamma(\alpha) \veebar \pi(\beta, \lambda), \\
&= \gamma(\alpha) \cup_S \pi(\beta, \lambda), \\
&= \langle \big(\mathfrak{u}, \max(g_\alpha, g_{(\beta,\lambda)})\big), \kappa(\mathfrak{u}, g_\alpha) \cup_S \kappa(\mathfrak{u}, g_{(\beta,\lambda)})\rangle, \\
&= \langle \big(\mathfrak{u}, \max\{g_\alpha, \min(g_\beta, g_\lambda)\}\big), \kappa(\mathfrak{u}, g_\alpha) \cup_S \{\kappa(\mathfrak{u}, g_\beta) \cap_S \kappa(\mathfrak{u}, g_\lambda)\}\rangle, \\
&= \langle \big(\mathfrak{u}, \min\{\max(g_\alpha, g_\beta), \max(g_\alpha, g_\lambda)\}\big), \\
&\quad \{(\kappa(\mathfrak{u}, g_\alpha) \cup_S \kappa(\mathfrak{u}, g_\beta)) \cap_S (\kappa(\mathfrak{u}, g_\alpha) \cup_S \kappa(\mathfrak{u}, g_\lambda))\}\rangle, \\
&= \big(\gamma(\alpha) \cup \delta(\beta)\big) \cap \big(\gamma(\alpha) \cup \psi(\lambda)\big), \\
&= \big(\gamma(\alpha) \veebar \delta(\beta)\big) \barwedge \big(\gamma(\alpha) \veebar \psi(\lambda)\big).
\end{aligned}$$

Hence proved that

$$(\gamma, X, N) \veebar \big((\delta, Y, N) \barwedge (\psi, Z, N)\big) = \big((\gamma, X, N) \veebar (\delta, Y, N)\big) \barwedge \big((\gamma, X, N) \veebar (\psi, Z, N)\big).$$

4. Similar to proof of part 3. □

Proposition 6 *Let (γ, X, N), (δ, Y, N) and (ψ, Z, N) be any three SFNSESs over a common universe $\mathcal{U}$. Then*

1. $(\gamma, X, N)\bar{\cap}_R((\delta, Y, N)\,\bar{\cup}_{\mathcal{E}}\,(\psi, Z, N)) = ((\gamma, X, N)\bar{\cap}_R(\delta, Y, N))\,\bar{\cup}_{\mathcal{E}}\,((\gamma, X, N)\bar{\cap}_R(\psi, Z, N))$.
2. $(\gamma, X, N)\bar{\cup}_R((\delta, Y, N)\,\bar{\cap}_{\mathcal{E}}\,(\psi, Z, N)) = ((\gamma, X, N)\bar{\cup}_R(\delta, Y, N))\,\bar{\cap}_{\mathcal{E}}\,((\gamma, X, N)\bar{\cup}_R(\psi, Z, N))$.
3. $(\gamma, X, N)\,\bar{\cap}_{\mathcal{E}}\,((\delta, Y, N)\bar{\cup}_R(\psi, Z, N)) = ((\gamma, X, N)\,\bar{\cap}_{\mathcal{E}}\,(\delta, Y, N))\bar{\cup}_R((\gamma, X, N)\,\bar{\cap}_{\mathcal{E}}\,(\psi, Z, N))$.
4. $(\gamma, X, N)\,\bar{\cup}_{\mathcal{E}}\,((\delta, Y, N)\bar{\cap}_R(\psi, Z, N)) = ((\gamma, X, N)\,\bar{\cup}_{\mathcal{E}}\,(\delta, Y, N))\bar{\cap}_R((\gamma, X, N)\,\bar{\cup}_{\mathcal{E}}\,(\psi, Z, N))$.
5. $(\gamma, X, N)\bar{\cap}_R((\delta, Y, N)\bar{\cup}_R(\psi, Z, N)) = ((\gamma, X, N)\bar{\cap}_R(\delta, Y, N))\bar{\cup}_R((\gamma, X, N)\bar{\cap}_R(\psi, Z, N))$.
6. $(\gamma, X, N)\bar{\cup}_R((\delta, Y, N)\bar{\cap}_R(\psi, Z, N)) = ((\gamma, X, N)\bar{\cup}_R(\delta, Y, N))\bar{\cap}_R((\gamma, X, N)\bar{\cup}_R(\psi, Z, N))$.

Proof

1. Suppose that $x \in X \cap (Y \cup Z)$. Then, there are three possibilities:

   (a) If $x \in X \cap (Y - Z)$, then

   $$\gamma(x)\,\bar{\cap}_R\,(\delta\,\bar{\cup}_{\mathcal{E}}\,\psi)(x) = \gamma(x)\,\bar{\cap}_R\,\delta(x) = (\gamma\,\bar{\cap}_R\,\delta)(x),$$

   and

   $$(\gamma\,\bar{\cap}_R\,\delta)(x)\,\bar{\cup}_{\mathcal{E}}\,(\gamma\,\bar{\cap}_R\,\psi)(x) = (\gamma\,\bar{\cap}_R\,\delta)(x)\,\bar{\cup}_{\mathcal{E}}\,\emptyset = (\gamma\,\bar{\cap}_R\,\delta)(x).$$

   (b) If $x \in X \cap (Z - Y)$, then

   $$\gamma(x)\,\bar{\cap}_R\,(\delta\,\bar{\cup}_{\mathcal{E}}\,\psi)(x) = \gamma(x)\,\bar{\cap}_R\,\psi(x) = (\gamma\,\bar{\cap}_R\,\psi)(x),$$

   and

   $$(\gamma\,\bar{\cap}_R\,\delta)(x)\,\bar{\cup}_{\mathcal{E}}\,(\gamma\,\bar{\cap}_R\,\psi)(x) = \emptyset\,\bar{\cup}_{\mathcal{E}}\,(\gamma\,\bar{\cap}_R\,\delta)(x) = (\gamma\,\bar{\cap}_R\,\psi)(x).$$

   (c) If $x \in X \cap (Y \cap Z)$, then for $(\mathfrak{u}, g_x^1) \in \gamma(x)$, $(\mathfrak{u}, g_x^2) \in \delta(x)$ and $(\mathfrak{u}, g_x^3) \in \psi(x)$, we have

   $$\begin{aligned}\gamma(x)\bar{\cap}_R(\delta\,\bar{\cup}_{\mathcal{E}}\,\psi)(x) &= \gamma(x)\,\bar{\cap}_R\,(\delta\,\bar{\cup}_R\,\psi)(x),\\ &= \langle(\mathfrak{u}, g_x^1), \kappa(\mathfrak{u}, g_x^1)\rangle\,\bar{\cap}_R\\ &\quad \langle\big(\mathfrak{u}, \max(g_x^2, g_x^3)\big), \kappa(\mathfrak{u}, g_x^2) \cup_S \kappa(\mathfrak{u}, g_x^3)\rangle,\\ &= \langle\big(\mathfrak{u}, \min\{g_x^1, \max(g_x^2, g_x^3)\}\big),\\ &\quad \kappa(\mathfrak{u}, g_x^1) \cap_S \{\kappa(\mathfrak{u}, g_x^2) \cup_S \kappa(\mathfrak{u}, g_x^3)\}\rangle,\end{aligned}$$

   and

   $$\begin{aligned}(\gamma\,\bar{\cap}_R\,\delta)(x)\,\bar{\cup}_{\mathcal{E}}\,(\gamma\,\bar{\cap}_R\,\psi)(x) &= (\gamma\,\bar{\cap}_R\,\delta)(x)\,\bar{\cup}_R\,(\gamma\,\bar{\cap}_R\,\psi)(x),\\ &= \langle\big(\mathfrak{u}, \min(g_x^1, g_x^2)\big), \kappa(\mathfrak{u}, g_x^1) \cap_S \kappa(\mathfrak{u}, g_x^2)\rangle\,\bar{\cup}_R\\ &\quad \langle\big(\mathfrak{u}, \min(g_x^2, g_x^3)\big), \kappa(\mathfrak{u}, g_x^2) \cap_S \kappa(\mathfrak{u}, g_x^3)\rangle,\\ &= \langle\big(\mathfrak{u}, \max\{min(g_x^1, g_x^2), \min(g_x^1, g_x^3)\}\big),\\ &\quad \{(\kappa(\mathfrak{u}, g_x^1) \cap_S \kappa(\mathfrak{u}, g_x^2)) \cup_S (\kappa(\mathfrak{u}, g_x^2) \cap_S \kappa(\mathfrak{u}, g_x^3))\}\rangle,\\ &= \langle\big(\mathfrak{u}, \min\{g_x^1, \max(g_x^2, g_x^3)\}\big),\\ &\quad \kappa(\mathfrak{u}, g_x^1) \cap_S \{\kappa(\mathfrak{u}, g_x^2) \cup_S \kappa(\mathfrak{u}, g_x^3)\}\rangle.\end{aligned}$$

   In all three cases, we see that

$$\gamma(x)\,\bar{\cap}_R\,(\delta\,\bar{\cup}_{\mathcal{E}}\,\psi)(x) = (\gamma\,\bar{\cap}_R\,\delta)(x)\,\bar{\cup}_{\mathcal{E}}\,(\gamma\,\bar{\cap}_R\,\psi)(x).$$

Hence it is proved that

$$(\gamma, X, N)\bar{\cap}_R\big((\delta, Y, N)\,\bar{\cup}_{\mathcal{E}}\,(\psi, Z, N)\big) = \big((\gamma, X, N)\bar{\cap}_R(\delta, Y, N)\big)\,\bar{\cup}_{\mathcal{E}}\,\big((\gamma, X, N)\bar{\cap}_R(\psi, Z, N)\big).$$

The remaining parts can be proved similarly. □

4 Application of SFNSESs in MAGDM problems

This section provides a real-life problem-solving method based on the proposed SFNSES model. Before getting on to the application, a few new notions need to be defined as follows.

Definition 24 For a SFNSES (γ, X, N) over the universe $\mathcal{U}$, an associated SFSES $\text{þ} = (\gamma, X)$ is defined as

$$\text{Þ} = \big\{\langle w, (\mu_{\gamma(x)}, \tau_{\gamma(x)}, \nu_{\gamma(x)})\rangle |\ \mathfrak{u} \in \mathcal{U},\ x \in X\big\}.$$

Here $\mu_{\gamma(x)}$, $\tau_{\gamma(x)}$ and $\nu_{\gamma(x)}$ are the positive, neutral and negative memberships, respectively, corresponding to the parameterized opinions $x \in X$ (with respect to the grading criteria defined for the corresponding SFNSES).

Definition 25 Consider $\text{þ} = (\gamma, X)$ be an associated SFSES over $\mathcal{U}$. Let $\lambda : X \to [0, 1]^3$ be a threshold function, such that $\lambda(x) = (\mathfrak{p}(x), \mathfrak{q}(x), \mathfrak{r}(x))$, $\forall\, x \in X$. Then, the level SES of þ with respect to λ will be a crisp SES denoted by $\mathcal{L}(\text{þ}; \lambda)$ defined as

$$\mathcal{L}(\text{Þ}; \lambda) = \{\mathfrak{u} \in \mathcal{U} | \mu_{\gamma(x)} \geq \mathfrak{p}(x), \tau_{\gamma(x)} \leq \mathfrak{q}(x), \nu_{\gamma(x)} \leq \mathfrak{r}(x)\},\quad \forall\, x \in X.$$

Definition 26 The level-agree score ξ_j of an object $\mathfrak{u} \in \mathcal{U}$ is defined as

$$\xi_j = \sum_i l_{ij}$$

where l_{ij} is the ij-th entry of a level agree SES table.

Definition 27 The level-disagree score η_j of an object $\mathfrak{u} \in \mathcal{U}$ is defined as

$$\eta_j = n(P \times E) - \sum_i d_{ij}$$

where d_{ij} is the ij-th entry of a level disagree SES table, and $n(P \times E)$ is the number of pairs in Cartesian product $P \times E$.

Now we present an algorithm, which will be used to solve the group decision-making problems under SFNSESs.

For the above Algorithm 1, a flowchart diagram is displayed in Fig. 3.

Algorithm Algorithm for solving MAGDM problems in SFNSES environment

1. **Input:**
   (a) The universe $\mathcal{U}$ of n objects,
   (b) The set $G = \{0, 1, 2, ..., N-1\}$ of grades with $N \in \{2, 3, \ldots, \}$,
   (c) The set P of parameters,
   (d) The set E of experts,
   (e) For $X \subseteq R$, where $R = P \times E \times \mathcal{O}$, insert the SFNSES (γ, X, N) according to the opinions of different experts.
2. Input the threshold function λ (any one of the mid, top-bottom-bottom, bottom-bottom-bottom, or med-threshold functions) for the associated SFSES Þ.
3. Calculate the level SES $\mathcal{L}(\text{Þ}; \lambda)$ of Þ in a tabular form.
4. Find the level agree and level disagree SES tables with entries l_{ij} and d_{ij}, respectively.
5. Put the level-agree score $\xi_j = \sum_i l_{ij}$ as the last row in the level-agree SES table.
6. Put the level-disagree score $\eta_j = n(P \times E) - \sum_i d_{ij}$ as the last row in the level-disagree SES table.
7. Calculate the final score $\mathfrak{s}_j = \xi_j - \eta_j$.
8. Find k for which $\mathfrak{s}_k = \max(\mathfrak{s}_j)$.

Output: Step 8 declares $\mathfrak{u}_k$ to be the best alternative for selection. For multiple values of k, any one of the corresponding alternatives can be selected as the best option.

We now present two applications of our proposed model and Algorithm 1 in solving different real-life-based uncertain decision-making problems.

Example 13 (*Prediction of winning candidate using survey before a local election*)

Elections, whether local or general, play an important role in determining the future of a community, a town, or a state. Either direct elections (by the people, for the people) or indirect elections (by the legislatives) at state and local levels prove to be very important in setting the directions for the future developments, solutions of existing and upcoming issues, and keeping the state sovereignty.

Although both direct and indirect elections have pros and cons in their manners, direct elections are much more encouraged as compared to indirect elections in most cases. One of the most important reasons is the direct decision of the people of such respective town or city or state under election, about which person or political party best meets their requirements. Unlike the indirect elections, where instead of the people's choice, a few legislatives assign representatives to different locations, the direct elections make it better aligned with the democratic principles. Despite their disadvantages as creating logistical issues and polarizing the political systems, direct elections represent citizens equally, encourage voter turnouts and provide better democratic choices.

For either of the direct and indirect elections, various tools respective to the two types can be efficient in predicting the winners. Considering only the direct elections, various techniques like social media analysis, including the Twitter trends, critics, Facebook polls, physical and online surveys (Liu et al. 2021b; Chin and Wang 2021), and a few more, help in predicting the elections significantly.

Among these techniques, surveys are among the most accurate and inexpensive methods in forecasting election outcomes. If surveys are conducted from the appropriate samples with suitable conditions at the proper time, they prove to be consistent with the actual poll results. For example, a survey taken from the voters before a week or a month can give more accurate results as compared to a survey taken from people of any age (including under 18 or voting age) or conducted several months before the actual election. In this example, we will model a similar situation, where experts will predict the election using the survey ratings provided by the people.

Consider a direct election is to be done for electing the local representative of a town containing 8000 voters. A survey is conducted from the maximum of voters, three weeks

Input

- The universe $\mathcal{U}$ of n objects,
- The set $G = \{0, 1, 2, ..., N-1\}$ of grades with $N \in \{2, 3, \ldots, \}$,
- The set P of parameters,
- The set E of experts,
- For $X \subseteq R$, where $R = P \times E \times \mathcal{O}$, insert the SFNSES (γ, X, N) according to the opinions of different experts,
- The threshold function λ (any one of the mid, top-bottom-bottom, bottom-bottom-bottom, or med-threshold functions) for the associated SFSES Þ.

Level SES

Calculate the level SES $\mathcal{L}(Þ; \lambda)$ of Þ in a tabular form.

Level SESs

Find the level agree and level disagree SES tables with entries l_{ij} and d_{ij}, respectively.

Agree score

Put the level-agree score $\xi_j = \sum_i l_{ij}$ as the last row in the level-agree SES table.

Disagree score

Put the level-disagree score $\eta_j = n(P \times E) - \sum_i d_{ij}$ as the last row in the level-disagree SES table.

Final score

Calculate the final score $\mathfrak{s}_j = \xi_j - \eta_j$.

Optimal object

Find k for which $\mathfrak{s}_k = \max(\mathfrak{s}_j)$.

Output

The last step declares $\mathfrak{u}_k$ to be the best alternative for selection. For multiple values of k, any one of the corresponding alternatives can be selected as the best option.

Fig. 3 Flowchart

Table 27 Survey outcomes

$\mathcal{U}/P$	p_1	p_2	p_3	p_4	p_5
$\mathfrak{u}_1$	⋆⋆⋆	⋆⋆⋆	⋆	⋆⋆	⋆⋆⋆⋆
$\mathfrak{u}_2$	⋆⋆	⋆⋆⋆	⋆⋆⋆	⋆⋆⋆	⋆⋆
$\mathfrak{u}_3$	⋆⋆	⋆⋆	⋆	⋆	⋆
$\mathfrak{u}_4$	⋆⋆⋆	⋆⋆	⋆⋆	⋆⋆	⋆

before the actual election. In this survey, the selected voters are asked to give ratings from 0 to 4 stars to the election candidates. These star ratings are interpreted as follows:

‘0 star’ for ‘Worst’;
‘1 star’ for ‘Bad’;
‘2 stars’ for ‘Average’;
‘3 stars’ for ‘Better’;
‘4 stars’ for ‘Best’.

Survey takes ratings from voters corresponding to the key-parameters $P = \{p_1, p_2, p_3, p_4, p_5\}$, declaring how much a candidate will meet their requirements. These parameters $p_i\,(i = 1, 2, \ldots, 5)$ are defined as:

- p_1—*Personality* How good is the candidate's behavior towards the area's problems? How much is his personal and political history supportive for him or her getting elected?
- p_2—*Provision of facilities* How much will the candidate succeed in providing the basic facilities as sanitation, health, transport, electricity, food and other utilities to the people in a good manner?
- p_3—*Discipline in the area* How good will the candidate manage in keeping law enforcement and peace in the area, and assure equal rights for the community?
- p_4—*Development* How good is his aptitude towards the development of the area, whether structural or financial?
- p_5—*Local understanding* How much is the candidate familiar with the local issues? Will he or she be able to keep in touch with the residents while making necessary decisions?

Consider there are four candidates as in the set $\mathcal{U} = \{\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3, \mathfrak{u}_4\}$ standing for the position of local representative in the election. The voter's ratings for the candidates corresponding to the above parameters are given in Table 27. These star ratings can be related to the natural numbers as discussed in Example 1.

Assume there are two experts comprising the set $E = \{r, s\}$. These experts are assigned the task by the research organization (that conducted survey) to predict the election's outcome. The experts calculate the spherical fuzzy memberships from the survey outcomes using the same criteria as in Example 1. The resulting SF5SES is represented in Table 28.

The experts choose the mid-level decision method for calculating the predictions. Thus for the associated SFSES $\text{þ} = (\gamma, X)$, we get the following:

Table 28 SF5SES $(\gamma, X, 5)$ from survey outcomes

$(\gamma, X, 5)$	$\mathfrak{u}_1$	$\mathfrak{u}_2$	$\mathfrak{u}_3$	$\mathfrak{u}_4$
$(p_1, r, 1)$	$\langle 3,(0.78,0.01,0.39)\rangle$	$\langle 2,(0.55,0.08,0.42)\rangle$	$\langle 2,(0.45,0.10,0.55)\rangle$	$\langle 3,(0.75,0.01,0.35)\rangle$
$(p_1, s, 1)$	$\langle 3,(0.75,0.01,0.35)\rangle$	$\langle 2,(0.59,0.10,0.45)\rangle$	$\langle 2,(0.50,0.10,0.50)\rangle$	$\langle 3,(0.65,0.01,0.39)\rangle$
$(p_2, r, 1)$	$\langle 3,(0.75,0.01,0.30)\rangle$	$\langle 3,(0.75,0.01,0.30)\rangle$	$\langle 2,(0.50,0.10,0.59)\rangle$	$\langle 2,(0.59,0.08,0.50)\rangle$
$(p_2, s, 1)$	$\langle 3,(0.79,0.01,0.22)\rangle$	$\langle 3,(0.75,0.01,0.33)\rangle$	$\langle 2,(0.45,0.08,0.50)\rangle$	$\langle 2,(0.55,0.10,0.40)\rangle$
$(p_3, r, 1)$	$\langle 1,(0.35,0.12,0.65)\rangle$	$\langle 3,(0.73,0.01,0.32)\rangle$	$\langle 1,(0.30,0.10,0.75)\rangle$	$\langle 2,(0.50,0.08,0.55)\rangle$
$(p_3, s, 1)$	$\langle 1,(0.39,0.10,0.75)\rangle$	$\langle 3,(0.78,0.01,0.30)\rangle$	$\langle 1,(0.30,0.11,0.79)\rangle$	$\langle 2,(0.49,0.10,0.59)\rangle$
$(p_4, r, 1)$	$\langle 2,(0.59,0.05,0.46)\rangle$	$\langle 3,(0.70,0.01,0.35)\rangle$	$\langle 1,(0.25,0.12,0.75)\rangle$	$\langle 2,(0.55,0.10,0.55)\rangle$
$(p_4, s, 1)$	$\langle 2,(0.55,0.07,0.45)\rangle$	$\langle 3,(0.75,0.01,0.30)\rangle$	$\langle 1,(0.32,0.08,0.75)\rangle$	$\langle 2,(0.45,0.10,0.59)\rangle$
$(p_5, r, 1)$	$\langle 4,(0.95,0.01,0.09)\rangle$	$\langle 2,(0.55,0.05,0.40)\rangle$	$\langle 1,(0.38,0.12,0.60)\rangle$	$\langle 1,(0.35,0.10,0.65)\rangle$
$(p_5, s, 1)$	$\langle 4,(0.90,0.01,0.12)\rangle$	$\langle 2,(0.45,0.10,0.56)\rangle$	$\langle 1,(0.36,0.10,0.70)\rangle$	$\langle 1,(0.39,0.10,0.76)\rangle$
$(p_1, r, 0)$	$\langle 3,(0.75,0.01,0.25)\rangle$	$\langle 2,(0.40,0.10,0.50)\rangle$	$\langle 2,(0.59,0.10,0.45)\rangle$	$\langle 3,(0.65,0.01,0.30)\rangle$
$(p_1, s, 0)$	$\langle 3,(0.65,0.01,0.25)\rangle$	$\langle 2,(0.58,0.10,0.55)\rangle$	$\langle 2,(0.55,0.10,0.45)\rangle$	$\langle 3,(0.76,0.01,0.30)\rangle$
$(p_2, r, 0)$	$\langle 3,(0.60,0.01,0.30)\rangle$	$\langle 3,(0.65,0.01,0.39)\rangle$	$\langle 2,(0.59,0.10,0.40)\rangle$	$\langle 2,(0.45,0.08,0.56)\rangle$
$(p_2, s, 0)$	$\langle 3,(0.79,0.01,0.39)\rangle$	$\langle 3,(0.65,0.01,0.35)\rangle$	$\langle 2,(0.55,0.08,0.40)\rangle$	$\langle 2,(0.40,0.10,0.55)\rangle$
$(p_3, r, 0)$	$\langle 1,(0.25,0.10,0.75)\rangle$	$\langle 3,(0.62,0.01,0.30)\rangle$	$\langle 1,(0.39,0.11,0.60)\rangle$	$\langle 2,(0.59,0.09,0.45)\rangle$
$(p_3, s, 0)$	$\langle 1,(0.20,0.10,0.79)\rangle$	$\langle 3,(0.65,0.01,0.30)\rangle$	$\langle 1,(0.35,0.10,0.65)\rangle$	$\langle 2,(0.55,0.10,0.45)\rangle$
$(p_4, r, 0)$	$\langle 2,(0.50,0.10,0.41)\rangle$	$\langle 3,(0.60,0.01,0.39)\rangle$	$\langle 1,(0.39,0.12,0.69)\rangle$	$\langle 2,(0.40,0.10,0.50)\rangle$
$(p_4, s, 0)$	$\langle 2,(0.45,0.10,0.50)\rangle$	$\langle 3,(0.65,0.01,0.39)\rangle$	$\langle 1,(0.39,0.10,0.60)\rangle$	$\langle 2,(0.55,0.10,0.40)\rangle$
$(p_5, r, 0)$	$\langle 4,(0.80,0.01,0.19)\rangle$	$\langle 2,(0.45,0.10,0.55)\rangle$	$\langle 1,(0.30,0.10,0.75)\rangle$	$\langle 1,(0.29,0.10,0.73)\rangle$
$(p_5, s, 0)$	$\langle 4,(0.85,0.01,0.02)\rangle$	$\langle 2,(0.56,0.07,0.43)\rangle$	$\langle 1,(0.27,0.10,0.61)\rangle$	$\langle 1,(0.26,0.10,0.65)\rangle$

$$
mid_{\text{Þ}} = \left\{ \begin{array}{l}
\langle (p_1,r,1),(0.63,0.05,0.43)\rangle,\ \langle (p_1,s,1),(0.62,0.06,0.42)\rangle,\\
\langle (p_2,r,1),(0.65,0.05,0.42)\rangle,\ \langle (p_2,s,1),(0.64,0.05,0.36)\rangle,\\
\langle (p_3,r,1),(0.47,0.08,0.57)\rangle,\ \langle (p_3,s,1),(0.49,0.08,0.61)\rangle,\\
\langle (p_4,r,1),(0.52,0.08,0.53)\rangle,\ \langle (p_4,s,1),(0.52,0.07,0.50)\rangle,\\
\langle (p_5,r,1),(0.56,0.07,0.44)\rangle,\ \langle (p_5,s,1),(0.53,0.08,0.54)\rangle,\\
\langle (p_1,r,0),(0.60,0.06,0.38)\rangle,\ \langle (p_1,s,0),(0.64,0.06,0.39)\rangle,\\
\langle (p_2,r,0),(0.57,0.05,0.41)\rangle,\ \langle (p_2,s,0),(0.60,0.05,0.42)\rangle,\\
\langle (p_3,r,0),(0.46,0.08,0.53)\rangle,\ \langle (p_3,s,0),(0.44,0.08,0.55)\rangle,\\
\langle (p_4,r,0),(0.47,0.08,0.50)\rangle,\ \langle (p_4,s,0),(0.51,0.08,0.47)\rangle,\\
\langle (p_5,r,0),(0.46,0.08,0.56)\rangle,\ \langle (p_5,s,0),(0.49,0.07,0.43)\rangle.
\end{array} \right\}
$$

From the above mid-level thresholds calculated, the mid-level soft expert set $\mathcal{L}(\text{Þ}; mid_{\text{Þ}})$ of þ is obtained as in Table 29. The mid-level agree SES and mid-level disagree SES corresponding to Table 29 are given in Tables 30 and 31, respectively.

At last, using the level-agree scores ξ_j and level-disagree scores η_j from Tables 30 and 31, respectively, the final scores $\mathfrak{s}_j$ in Table 32 are enough to predict the winner of the upcoming local election.

From Table 32, it is predicted that $\mathfrak{u}_1$ with the maximum final score (i.e., $\mathfrak{s}_1 = \max(\mathfrak{s}_j) = 4$) will be the new local representative of the town.

Example 14 (*Ranking credibility of the smartphones using customer feedback*) It is the 21st century, where almost everything is getting more and more technology-driven each day. With the ever-increasing demand and ever-revolutionizing ideas in this domain, more and more gadgets are coming into the market, dominating their predecessor technologies.

Table 29 Mid-Level SES $\mathcal{L}(\text{Þ}; mid_{\text{Þ}})$ of þ

$\mathcal{L}(\text{Þ}; mid_{\text{Þ}})$	$\mathfrak{u}_1$	$\mathfrak{u}_2$	$\mathfrak{u}_3$	$\mathfrak{u}_4$
$(p_1, r, 1)$	1	0	0	1
$(p_1, s, 1)$	1	0	0	1
$(p_2, r, 1)$	1	1	0	0
$(p_2, s, 1)$	1	1	0	0
$(p_3, r, 1)$	0	1	0	1
$(p_3, s, 1)$	0	1	0	0
$(p_4, r, 1)$	1	1	0	0
$(p_4, s, 1)$	1	1	0	0
$(p_5, r, 1)$	1	0	0	0
$(p_5, s, 1)$	1	0	0	0
$(p_1, r, 0)$	1	0	0	1
$(p_1, s, 0)$	1	0	0	1
$(p_2, r, 0)$	1	1	0	0
$(p_2, s, 0)$	1	1	0	0
$(p_3, r, 0)$	0	1	0	0
$(p_3, s, 0)$	0	1	0	0
$(p_4, r, 0)$	0	1	0	0
$(p_4, s, 0)$	0	1	0	0
$(p_5, r, 0)$	1	0	0	0
$(p_5, s, 0)$	1	1	0	0

Table 30 Mid-level agree SES $\mathcal{L}(\text{Þ}; mid_{\text{Þ}})_1$ of þ

$\mathcal{L}(\text{Þ}; mid_{\text{Þ}})_1$	$\mathfrak{u}_1$	$\mathfrak{u}_2$	$\mathfrak{u}_3$	$\mathfrak{u}_4$
(p_1, r)	1	0	0	1
(p_1, s)	1	0	0	1
(p_2, r)	1	1	0	0
(p_2, s)	1	1	0	0
(p_3, r)	0	1	0	1
(p_3, s)	0	1	0	0
(p_4, r)	1	1	0	0
(p_4, s)	1	1	0	0
(p_5, r)	1	0	0	0
(p_5, s)	1	0	0	0
$\xi_j = \sum_i l_{ij}$	8	6	0	3

Particularly when it comes to telecommunication and connectivity, smartphones prove to be one of the most engaging devices. The long-way journey from traditional telephones to the latest smartphones has made many revolutionary changes in global communication. Now, instead of relying on your computer for sending emails, taking a camera along with you for photography, and carrying a compass for keeping yourself in the right direction, you only need a smartphone with all these apps and features integrated with it. It makes smartphones more valuable as they take responsibility for many of our daily life tasks.

Table 31 Mid-level disagree SES $\mathcal{L}(\text{Þ}; mid_{\text{Þ}})_0$ of þ

$\mathcal{L}(\text{Þ}; mid_{\text{Þ}})_0$	$\mathfrak{u}_1$	$\mathfrak{u}_2$	$\mathfrak{u}_3$	$\mathfrak{u}_4$
(p_1, r)	1	0	0	1
(p_1, s)	1	0	0	1
(p_2, r)	1	1	0	0
(p_2, s)	1	1	0	0
(p_3, r)	0	1	0	0
(p_3, s)	0	1	0	0
(p_4, r)	0	1	0	0
(p_4, s)	0	1	0	0
(p_5, r)	1	0	0	0
(p_5, s)	1	1	0	0
$\eta_j = 10 - \sum_i d_{ij}$	4	3	10	8

Table 32 Final scores

$\mathcal{U}$	Agree score ξ_j	Disagree score η_j	Final score $\mathfrak{s}_j = \xi_j - \eta_j$
$\mathfrak{u}_1$	8	4	4
$\mathfrak{u}_2$	6	3	3
$\mathfrak{u}_3$	0	10	−10
$\mathfrak{u}_4$	3	8	−5

Today, there are numerous smartphone manufacturers making hundreds of thousands of smartphones each day. We have a variety of smartphones competing to dominate the smartphone market with their varying specifications, unique features, and elegant designs. According to the recent statistics, there had been 1.38 billion unit smartphones sales globally in 2020, comprising 1.32 billion Android smartphones. The global smartphone revenue has reached an amount of massive 409 billion US dollars. Considering such a revenue, smartphone giants like Apple, Samsung, HTC, Oppo, etc., are competing in the market by launching new smartphone models with features like foldable screens, high-resolution cameras, large memory, and super speed. However, sometimes they fail in doing good business and keeping their stance. This year, LG[3] decided to quit its smartphone business after being in continuous loss, despite its highly innovative ideas in the field. One of the main reasons is that most consumers considered many of those innovations of no use. That is, they failed somehow to keep up with their customers' actual requirements. One example is Nokia,[4] which fell from being the World's best mobile phone seller to completely losing it till 2013. The most important reason behind their failure was that they did not adopt the change. Instead of opting for the leading OSs like Android and IOS, they kept up with their featured phones, which led them to sell themselves to Microsoft.

[3] https://nypost.com/2021/04/05/lg-to-stop-making-smartphones-after-years-of-losses/.

[4] https://medium.com/multiplier-magazine/why-did-nokia-fail-81110d981787.

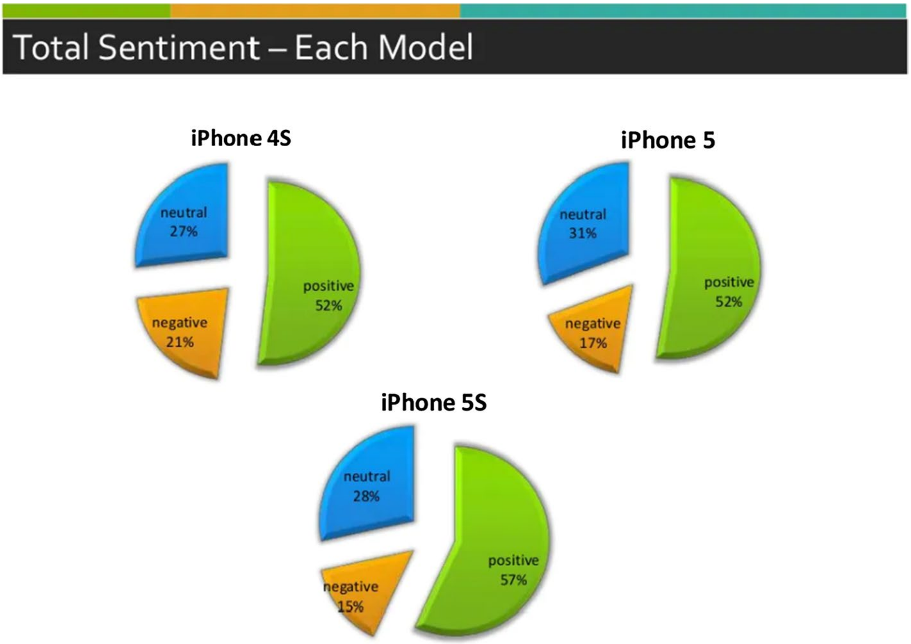

Fig. 4 Sentiment analysis representing sentiments on different Iphone models

The above examples indicate that customer satisfaction is one of the keys to building a business's success, which is achievable by keeping track of the customers' requirements and opinions. Customer feedbacks, consumer/user reviews, and product satisfaction ratings prove to be quite handy in dealing with this task. Another effective tool is sentiment analysis, which offers a deep insight into the users' sentiments. Numerous websites, online and conventional promoting companies utilize sentiment analysis through reviews, social media, including Twitter posts, Facebook patterns, and more, to keep track of their customer's prerequisites and interests, as well as an insight into their brand and item validity. Figure 4 represents the users' sentiment analysis summary[5] for iPhone 4S, iPhone 5, and iPhone 5S models.

Requesting for direct costumer feedbacks within the app, or product as smartphone, or as an email to the customer, can also be an important way of knowing exactly, what needs to be known. Different companies conduct customer feedback in the form of reviews, in-app surveys, and ratings. Figure 5 shows an example of an in-app customer feedback[6] survey.

Consider a smartphone manufacturing company launches several new smartphone models in the market. To gain an insight into its products' credibilities and the user requirements, the company decides to conduct a costumer survey within the smartphones interfaces, three months after the launch. A prompt appears in the smartphone screen asking for users to share their experiences about their new smartphone. In this way, the company asks

5 https://www.slideshare.net/joellecool/it651-project-report/14.

6 https://monkeylearn.com/customer-feedback/.

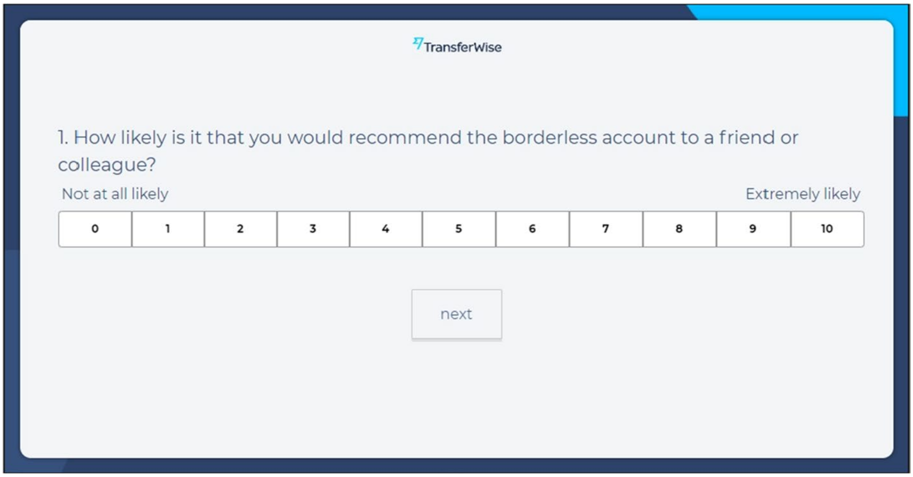

Fig. 5 An example of in-app survey

its customers directly for the questions, that the company oughts to be the most important feedbacks. In this customer feedback survey, each question asks for a rating from 0 to 4 stars. These star ratings are interpreted as follows:

'0 star' for 'extremely dissatisfactory';
'1 star' for 'dissatisfactory';
'2 stars' for 'average';
'3 stars' for 'satisfactory';
'4 stars' for 'extremely satisfactory'.

The short customer feedback survey asks for customer satisfaction ratings considering the parameters $P = \{p_1, p_2, p_3, p_4\}$ in order to understand, which smartphones require modifications, and which ones are going well. The parameters p_i $(i = 1, 2, 3, 4)$ are defined as:

- p_1—*Technology* Including camera, display colours, processing speed, refresh rate, and overall usage experience.
- p_2—*Design* Including the screen ratio, colour, dimensions, and design.
- p_3—*Material* Including battery, body, screen-protection etc.
- p_4—*Price* Suitability of the price with the service provided.

Consider there are 8 smartphones models comprising the set $\mathcal{U} = \{u_1, u_2, \ldots, u_8\}$ launched in the market. The customer feedback summary ratings are provided in Table 33. These ratings can be related to the numbers as discussed in Example 1.

Table 33 User ratings

$\mathcal{U}/P$	p_1	p_2	p_3	p_4
$\mathfrak{u}_1$	⋆⋆⋆	⋆⋆	⋆⋆⋆	⋆⋆⋆
$\mathfrak{u}_2$	⋆⋆⋆⋆	⋆⋆⋆	⋆⋆⋆	⋆⋆
$\mathfrak{u}_3$	⋆⋆⋆	⋆⋆	⋆⋆⋆	⋆
$\mathfrak{u}_4$	⋆⋆	⋆⋆⋆	⋆⋆	⋆⋆⋆
$\mathfrak{u}_5$	⋆⋆⋆	⋆⋆	⋆⋆	⋆⋆
$\mathfrak{u}_6$	⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆	⋆⋆⋆
$\mathfrak{u}_7$	⋆⋆	⋆⋆	⋆	⋆⋆
$\mathfrak{u}_8$	⋆⋆⋆⋆	⋆⋆	⋆⋆⋆	⋆⋆⋆⋆

Assume two experts as in the set $E = \{r, s\}$ are assigned the task to process the ratings. The experts calculate the spherical fuzzy memberships from customer survey using the same standard as discussed in Example 1. A SF5SES is obtained as represented in Table 34.

Using the med-level decision method, the med-level thresholds for the associated SFSES $\mathfrak{p} = (\gamma, X)$ are calculated, and we get the following results:

$$med_{\mathfrak{p}} = \left\{ \begin{array}{l} \langle(p_1, r, 1), (0.73, 0.01, 0.28)\rangle, \langle(p_1, s, 1), (0.75, 0.01, 0.33)\rangle, \\ \langle(p_2, r, 1), (0.55, 0.08, 0.48)\rangle, \langle(p_2, s, 1), (0.55, 0.05, 0.45)\rangle, \\ \langle(p_3, r, 1), (0.74, 0.01, 0.36)\rangle, \langle(p_3, s, 1), (0.75, 0.01, 0.35)\rangle, \\ \langle(p_4, r, 1), (0.62, 0.05, 0.39)\rangle, \langle(p_4, s, 1), (0.63, 0.05, 0.38)\rangle, \\ \langle(p_1, r, 0), (0.63, 0.01, 0.35)\rangle, \langle(p_1, s, 0), (0.67, 0.01, 0.35)\rangle, \\ \langle(p_2, r, 0), (0.55, 0.09, 0.45)\rangle, \langle(p_2, s, 0), (0.53, 0.04, 0.45)\rangle, \\ \langle(p_3, r, 0), (0.60, 0.01, 0.37)\rangle, \langle(p_3, s, 0), (0.63, 0.01, 0.36)\rangle, \\ \langle(p_4, r, 0), (0.53, 0.04, 0.45)\rangle, \langle(p_4, s, 0), (0.60, 0.06, 0.38)\rangle. \end{array} \right\}$$

From the above med-level thresholds calculated, the med-level SES $\mathcal{L}(\mathfrak{p}; med_{\mathfrak{p}})$ of $\mathfrak{p}$ is obtained as in Table 35. The med-level agree SES and med-level disagree SES corresponding to Table 35 are given in Tables 36 and 37, respectively.

Finally, using the level-agree scores ξ_j and level-disagree scores η_j from Tables 36 and 37, respectively, the final scores $\mathfrak{s}_j$ in Table 38 rank the credibility of the smartphones.

From Table 38, company gets the ranking as $u_6 > u_8 > u_2 > u_1 > u_4 > u_3 > u_5 > u_7$. Thus, the company decides to make a little bit improvements in the smartphones u_1, u_2, u_3, u_4 addressing to the respective ratings. Moreover, the company stops further production of u_5 and u_7, since they failed to meet the customer requirements. Here u_6 and u_8 came out to be the most successful and appreciated models of all the eight newly launched smartphones.

Table 34 SF5SES $(\gamma, X, 5)$ from user ratings

$(\gamma, X, 5)$	$\mathfrak{u}_1$	$\mathfrak{u}_2$	$\mathfrak{u}_3$	$\mathfrak{u}_4$
$(p_1, r, 1)$	$\langle 3, (0.70, 0.01, 0.20)\rangle$	$\langle 4, (1.00, 0.00, 0.00)\rangle$	$\langle 3, (0.75, 0.01, 0.25)\rangle$	$\langle 2, (0.50, 0.08, 0.45)\rangle$
$(p_1, s, 1)$	$\langle 3, (0.75, 0.00, 0.30)\rangle$	$\langle 4, (0.90, 0.01, 0.15)\rangle$	$\langle 3, (0.78, 0.01, 0.20)\rangle$	$\langle 2, (0.55, 0.10, 0.40)\rangle$
$(p_2, r, 1)$	$\langle 2, (0.55, 0.08, 0.45)\rangle$	$\langle 3, (0.70, 0.01, 0.35)\rangle$	$\langle 2, (0.55, 0.08, 0.50)\rangle$	$\langle 3, (0.75, 0.01, 0.30)\rangle$
$(p_2, s, 1)$	$\langle 2, (0.45, 0.08, 0.55)\rangle$	$\langle 3, (0.75, 0.01, 0.20)\rangle$	$\langle 2, (0.44, 0.10, 0.45)\rangle$	$\langle 3, (0.75, 0.01, 0.30)\rangle$
$(p_3, r, 1)$	$\langle 3, (0.78, 0.01, 0.35)\rangle$	$\langle 3, (0.73, 0.00, 0.37)\rangle$	$\langle 3, (0.78, 0.01, 0.30)\rangle$	$\langle 2, (0.55, 0.10, 0.40)\rangle$
$(p_3, s, 1)$	$\langle 3, (0.77, 0.01, 0.33)\rangle$	$\langle 3, (0.78, 0.01, 0.35)\rangle$	$\langle 3, (0.75, 0.01, 0.25)\rangle$	$\langle 2, (0.58, 0.05, 0.45)\rangle$
$(p_4, r, 1)$	$\langle 3, (0.65, 0.01, 0.38)\rangle$	$\langle 2, (0.58, 0.08, 0.45)\rangle$	$\langle 1, (0.35, 0.11, 0.60)\rangle$	$\langle 3, (0.78, 0.01, 0.30)\rangle$
$(p_4, s, 1)$	$\langle 3, (0.70, 0.01, 0.35)\rangle$	$\langle 2, (0.55, 0.10, 0.40)\rangle$	$\langle 1, (0.30, 0.12, 0.75)\rangle$	$\langle 3, (0.78, 0.01, 0.32)\rangle$
$(p_1, r, 0)$	$\langle 3, (0.60, 0.01, 0.35)\rangle$	$\langle 4, (0.80, 0.01, 0.19)\rangle$	$\langle 3, (0.65, 0.01, 0.35)\rangle$	$\langle 2, (0.55, 0.08, 0.40)\rangle$
$(p_1, s, 0)$	$\langle 3, (0.65, 0.00, 0.35)\rangle$	$\langle 4, (0.85, 0.01, 0.18)\rangle$	$\langle 3, (0.68, 0.01, 0.38)\rangle$	$\langle 2, (0.45, 0.10, 0.55)\rangle$
$(p_2, r, 0)$	$\langle 2, (0.55, 0.10, 0.45)\rangle$	$\langle 3, (0.78, 0.01, 0.25)\rangle$	$\langle 2, (0.50, 0.08, 0.45)\rangle$	$\langle 3, (0.65, 0.01, 0.30)\rangle$
$(p_2, s, 0)$	$\langle 2, (0.50, 0.08, 0.44)\rangle$	$\langle 3, (0.65, 0.01, 0.35)\rangle$	$\langle 2, (0.55, 0.10, 0.41)\rangle$	$\langle 3, (0.65, 0.01, 0.38)\rangle$
$(p_3, r, 0)$	$\langle 3, (0.60, 0.01, 0.35)\rangle$	$\langle 3, (0.65, 0.01, 0.35)\rangle$	$\langle 3, (0.60, 0.01, 0.25)\rangle$	$\langle 2, (0.40, 0.10, 0.55)\rangle$
$(p_3, s, 0)$	$\langle 3, (0.63, 0.01, 0.37)\rangle$	$\langle 3, (0.62, 0.01, 0.35)\rangle$	$\langle 3, (0.65, 0.01, 0.35)\rangle$	$\langle 2, (0.43, 0.06, 0.55)\rangle$
$(p_4, r, 0)$	$\langle 3, (0.75, 0.01, 0.25)\rangle$	$\langle 2, (0.45, 0.10, 0.55)\rangle$	$\langle 1, (0.25, 0.11, 0.75)\rangle$	$\langle 3, (0.60, 0.01, 0.35)\rangle$
$(p_4, s, 0)$	$\langle 3, (0.70, 0.01, 0.35)\rangle$	$\langle 2, (0.45, 0.10, 0.59)\rangle$	$\langle 1, (0.35, 0.12, 0.70)\rangle$	$\langle 3, (0.65, 0.01, 0.30)\rangle$
$(\gamma, X, 5)$	$\mathfrak{u}_5$	$\mathfrak{u}_6$	$\mathfrak{u}_7$	$\mathfrak{u}_8$
$(p_1, r, 1)$	$\langle 3, (0.70, 0.01, 0.35)\rangle$	$\langle 3, (0.79, 0.01, 0.30)\rangle$	$\langle 2, (0.55, 0.10, 0.42)\rangle$	$\langle 4, (0.94, 0.01, 0.15)\rangle$
$(p_1, s, 1)$	$\langle 3, (0.75, 0.01, 0.35)\rangle$	$\langle 3, (0.75, 0.01, 0.35)\rangle$	$\langle 2, (0.50, 0.10, 0.50)\rangle$	$\langle 4, (0.90, 0.01, 0.18)\rangle$
$(p_2, r, 1)$	$\langle 2, (0.45, 0.08, 0.55)\rangle$	$\langle 4, (0.92, 0.01, 0.10)\rangle$	$\langle 2, (0.55, 0.10, 0.50)\rangle$	$\langle 2, (0.50, 0.09, 0.55)\rangle$
$(p_2, s, 1)$	$\langle 2, (0.55, 0.10, 0.45)\rangle$	$\langle 4, (0.89, 0.01, 0.15)\rangle$	$\langle 2, (0.53, 0.01, 0.45)\rangle$	$\langle 2, (0.55, 0.10, 0.48)\rangle$
$(p_3, r, 1)$	$\langle 2, (0.52, 0.07, 0.48)\rangle$	$\langle 3, (0.78, 0.01, 0.25)\rangle$	$\langle 1, (0.36, 0.11, 0.72)\rangle$	$\langle 3, (0.74, 0.01, 0.34)\rangle$
$(p_3, s, 1)$	$\langle 2, (0.54, 0.10, 0.45)\rangle$	$\langle 3, (0.75, 0.01, 0.35)\rangle$	$\langle 1, (0.34, 0.10, 0.75)\rangle$	$\langle 3, (0.75, 0.01, 0.30)\rangle$
$(p_4, r, 1)$	$\langle 2, (0.56, 0.10, 0.40)\rangle$	$\langle 3, (0.70, 0.01, 0.20)\rangle$	$\langle 2, (0.52, 0.10, 0.43)\rangle$	$\langle 4, (0.95, 0.01, 0.19)\rangle$
$(p_4, s, 1)$	$\langle 2, (0.52, 0.10, 0.44)\rangle$	$\langle 3, (0.77, 0.01, 0.35)\rangle$	$\langle 2, (0.49, 0.08, 0.45)\rangle$	$\langle 4, (0.90, 0.01, 0.10)\rangle$
$(p_1, r, 0)$	$\langle 3, (0.75, 0.01, 0.30)\rangle$	$\langle 3, (0.61, 0.01, 0.37)\rangle$	$\langle 2, (0.45, 0.10, 0.52)\rangle$	$\langle 4, (0.82, 0.01, 0.19)\rangle$
$(p_1, s, 0)$	$\langle 3, (0.65, 0.01, 0.30)\rangle$	$\langle 3, (0.70, 0.01, 0.35)\rangle$	$\langle 2, (0.45, 0.10, 0.55)\rangle$	$\langle 4, (0.85, 0.01, 0.10)\rangle$
$(p_2, r, 0)$	$\langle 2, (0.55, 0.10, 0.44)\rangle$	$\langle 4, (0.86, 0.01, 0.11)\rangle$	$\langle 2, (0.48, 0.10, 0.50)\rangle$	$\langle 2, (0.55, 0.10, 0.50)\rangle$
$(p_2, s, 0)$	$\langle 2, (0.45, 0.08, 0.55)\rangle$	$\langle 4, (0.90, 0.01, 0.15)\rangle$	$\langle 2, (0.47, 0.01, 0.54)\rangle$	$\langle 2, (0.45, 0.07, 0.48)\rangle$
$(p_3, r, 0)$	$\langle 2, (0.46, 0.10, 0.47)\rangle$	$\langle 3, (0.78, 0.01, 0.23)\rangle$	$\langle 1, (0.25, 0.10, 0.69)\rangle$	$\langle 3, (0.63, 0.01, 0.39)\rangle$
$(p_3, s, 0)$	$\langle 2, (0.45, 0.09, 0.45)\rangle$	$\langle 3, (0.76, 0.01, 0.30)\rangle$	$\langle 1, (0.28, 0.11, 0.75)\rangle$	$\langle 3, (0.70, 0.01, 0.33)\rangle$
$(p_4, r, 0)$	$\langle 2, (0.43, 0.06, 0.55)\rangle$	$\langle 3, (0.70, 0.01, 0.30)\rangle$	$\langle 2, (0.46, 0.10, 0.55)\rangle$	$\langle 4, (0.81, 0.01, 0.10)\rangle$
$(p_4, s, 0)$	$\langle 2, (0.45, 0.10, 0.50)\rangle$	$\langle 3, (0.67, 0.01, 0.33)\rangle$	$\langle 2, (0.55, 0.10, 0.40)\rangle$	$\langle 4, (0.85, 0.01, 0.14)\rangle$

5 Comparative analysis and discussion

In this section, we provide the advantages and limitations of our model and compare the proposed model with the already existing ones, including SFSESs and NSSs.

Table 35 Med-level SES $\mathcal{L}(Þ; med_Þ)$ of þ

$\mathcal{L}(Þ; med_Þ)$	$\mathfrak{u}_1$	$\mathfrak{u}_2$	$\mathfrak{u}_3$	$\mathfrak{u}_4$	$\mathfrak{u}_5$	$\mathfrak{u}_6$	$\mathfrak{u}_7$	$\mathfrak{u}_8$
$(p_1, r, 1)$	0	1	1	0	0	0	0	1
$(p_1, s, 1)$	1	1	1	0	0	0	0	1
$(p_2, r, 1)$	1	1	0	1	0	1	0	0
$(p_2, s, 1)$	0	1	0	1	0	1	0	0
$(p_3, r, 1)$	1	0	1	0	0	1	0	1
$(p_3, s, 1)$	1	1	1	0	0	1	0	1
$(p_4, r, 1)$	1	0	0	1	0	1	0	1
$(p_4, s, 1)$	1	0	0	1	0	1	0	1
$(p_1, r, 0)$	0	1	1	0	1	0	0	1
$(p_1, s, 0)$	0	1	0	0	0	1	0	1
$(p_2, r, 0)$	0	1	0	1	0	1	0	0
$(p_2, s, 0)$	0	1	0	1	0	1	0	0
$(p_3, r, 0)$	1	1	1	0	0	1	0	0
$(p_3, s, 0)$	0	0	1	0	0	1	0	1
$(p_4, r, 0)$	1	0	0	1	0	1	0	1
$(p_4, s, 0)$	1	0	0	1	0	1	0	1

Table 36 Med-level agree SES $\mathcal{L}(Þ; med_Þ)_1$ of þ

$\mathcal{L}(Þ; med_Þ)_1$	$\mathfrak{u}_1$	$\mathfrak{u}_2$	$\mathfrak{u}_3$	$\mathfrak{u}_4$	$\mathfrak{u}_5$	$\mathfrak{u}_6$	$\mathfrak{u}_7$	$\mathfrak{u}_8$
(p_1, r)	0	1	1	0	0	0	0	1
(p_1, s)	1	1	1	0	0	0	0	1
(p_2, r)	1	1	0	1	0	1	0	0
(p_2, s)	0	1	0	1	0	1	0	0
(p_3, r)	1	0	1	0	0	1	0	1
(p_3, s)	1	1	1	0	0	1	0	1
(p_4, r)	1	0	0	1	0	1	0	1
(p_4, s)	1	0	0	1	0	1	0	1
$\xi_j = \sum_i l_{ij}$	6	5	4	4	0	6	0	6

Table 37 Med-level disagree SES $\mathcal{L}(Þ; med_Þ)_0$ of þ

$\mathcal{L}(Þ; med_Þ)_0$	$\mathfrak{u}_1$	$\mathfrak{u}_2$	$\mathfrak{u}_3$	$\mathfrak{u}_4$	$\mathfrak{u}_5$	$\mathfrak{u}_6$	$\mathfrak{u}_7$	$\mathfrak{u}_8$
(p_1, r)	0	1	1	0	1	0	0	1
(p_1, s)	0	1	0	0	0	1	0	1
(p_2, r)	0	1	0	1	0	1	0	0
(p_2, s)	0	1	0	1	0	1	0	0
(p_3, r)	1	1	1	0	0	1	0	0
(p_3, s)	0	0	1	0	0	1	0	1
(p_4, r)	1	0	0	1	0	1	0	1
(p_4, s)	1	0	0	1	0	1	0	1
$\eta_j = 8 - \sum_i d_{ij}$	5	3	5	4	7	1	8	3

Table 38 Final scores

$\mathcal{U}$	Agree score ξ_j	Disagree score η_j	Final score $\mathfrak{s}_j = \xi_j - \eta_j$
$\mathfrak{u}_1$	6	5	1
$\mathfrak{u}_2$	5	3	2
$\mathfrak{u}_3$	4	5	−1
$\mathfrak{u}_4$	4	4	0
$\mathfrak{u}_5$	0	7	−7
$\mathfrak{u}_6$	6	1	5
$\mathfrak{u}_7$	0	8	−8
$\mathfrak{u}_8$	6	3	3

Table 39 Comparison of final scores obtained by proposed and some existing models for the Example 13

$\mathcal{U}$	NSSs (Fatimah et al. 2018)	SFSESs (Perveen et al. 2020)	Proposed SFNSESs
$\mathfrak{u}_1$	13	4	(13, 4)
$\mathfrak{u}_2$	13	3	(13, 3)
$\mathfrak{u}_3$	7	−10	(7, −10)
$\mathfrak{u}_4$	10	−5	(10, −5)

Table 40 Comparison of ranking orders for the Example 13

Models	Ranking order
NSSs (Fatimah et al. 2018)	$\mathfrak{u}_1 = \mathfrak{u}_2 > \mathfrak{u}_4 > \mathfrak{u}_3$
SFSESs (Perveen et al. 2020)	$\mathfrak{u}_1 > \mathfrak{u}_2 > \mathfrak{u}_4 > \mathfrak{u}_3$
Proposed SFNSESs	$\mathfrak{u}_1 > \mathfrak{u}_2 > \mathfrak{u}_4 > \mathfrak{u}_3$

1. *Advantages*

   Different uncertain situations require different methodologies to deal with them. In the last few decades, researchers have developed a number of uncertain models to deal with different uncertain and vague scenarios. The need for to develop more such models and their hybridization is ever increasing due to the ever-increasing problems and challenges in dealing with them. Among these models, SFSESs have proved their efficiency in dealing with uncertain spherical fuzzy information under the opinions of multiple experts. But the limitation is that the model can not deal with multinary information and deals only with data under the binary category ($N = 2$). Nowadays, in many problems, the uncertain information is based on grades and ratings, requiring methods capable of dealing with multinary information. The NSSs work well under these scenarios but are inefficient in dealing with spherical fuzzy information. This emerges the need for a new model that fills the gaps in both the above models. In this paper, we propose a novel hybrid model named SFNSESs, combining the properties of both the above models. The newly developed model can deal with spherical fuzzy data under the opinion of multiple experts based on multinary information. Thus using the model, the user can interpret the multinary category information with spherical fuzzy information under the

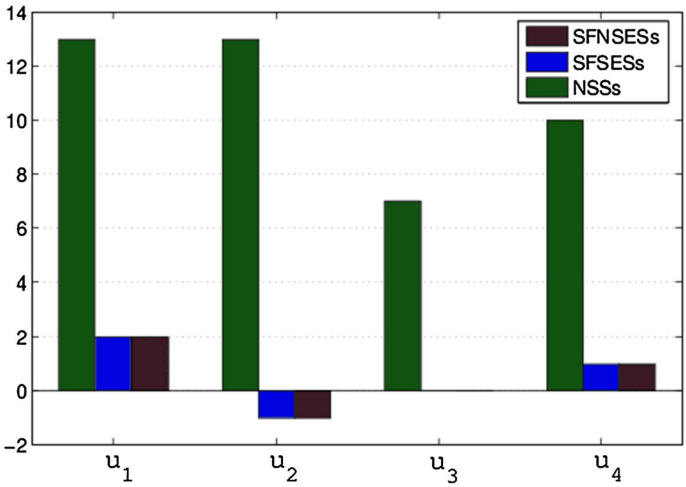

Fig. 6 Comparison between proposed SFNSESs and existing NSSs (Fatimah et al. 2018) and SFSESs (Perveen et al. 2020) on Example 13

Table 41 Comparison of final scores obtained by proposed model and some existing methods for the Example 14

$\mathcal{U}$	NSSs (Fatimah et al. 2018)	SFSESs (Perveen et al. 2020)	Proposed SFNSESs
$\mathfrak{u}_1$	11	1	(11, 1)
$\mathfrak{u}_2$	12	2	(12, 2)
$\mathfrak{u}_3$	9	− 1	(9, −1)
$\mathfrak{u}_4$	10	0	(10, 0)
$\mathfrak{u}_5$	9	− 7	(9, −7)
$\mathfrak{u}_6$	13	5	(13, 5)
$\mathfrak{u}_7$	7	− 8	(7, −8)
$\mathfrak{u}_8$	13	3	(13, 3)

Table 42 Comparison of ranking orders for the Example 13

Models	Ranking order
NSSs (Fatimah et al. 2018)	$\mathfrak{u}_6 = \mathfrak{u}_8 > \mathfrak{u}_2 > \mathfrak{u}_1 > \mathfrak{u}_4 > \mathfrak{u}_3 = \mathfrak{u}_5 > \mathfrak{u}_7$
SFSESs (Perveen et al. 2020)	$\mathfrak{u}_6 > \mathfrak{u}_8 > \mathfrak{u}_2 > \mathfrak{u}_1 > \mathfrak{u}_4 > \mathfrak{u}_3 > \mathfrak{u}_5 > \mathfrak{u}_7$
Proposed SFNSESs	$\mathfrak{u}_6 > \mathfrak{u}_8 > \mathfrak{u}_2 > \mathfrak{u}_1 > \mathfrak{u}_4 > \mathfrak{u}_3 > \mathfrak{u}_5 > \mathfrak{u}_7$

opinion of more than one expert, hence dealing with the uncertain data more efficiently as compared to the models already developed till now.

2. *Comparison*

   When dealing with decision-making problems, both SFSESs and NSSs prove their strength in their respective domains under their different structures. SFSESs are very powerful in dealing with the uncertain fuzzy information as compared to the previous

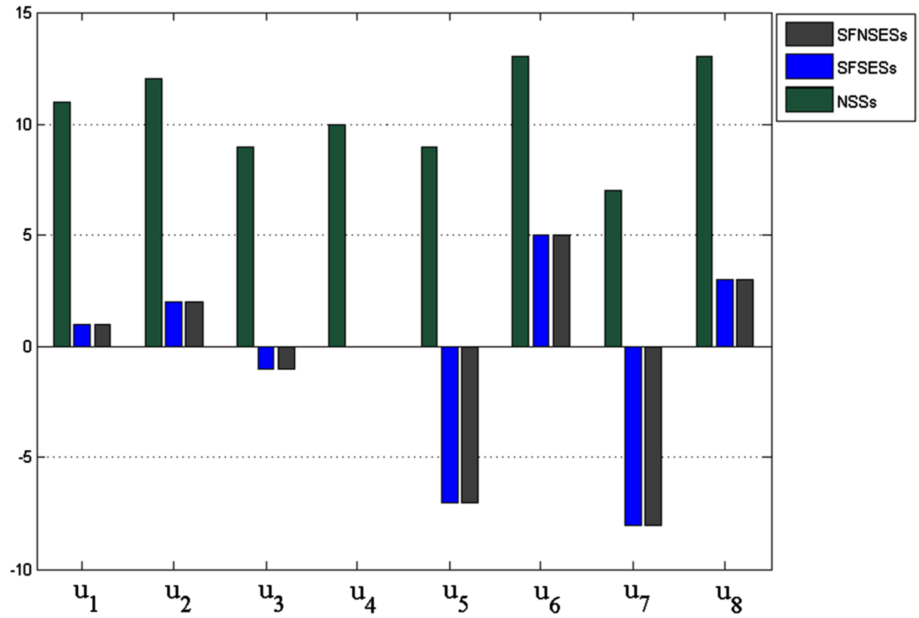

Fig. 7 Comparison between proposed SFNSESs and existing NSSs (Fatimah et al. 2018) and SFSESs (Perveen et al. 2020) on Example 14

models like fuzzy SESs, intuitionistic fuzzy SESs, PyFSESs, etc. Similarly, NSSs can deal with multinary information efficiently, as in the case of ratings or grades. But both these models have limitations too. SFSESs can not deal with multinary information and therefore are restricted to only binary data. Similarly, NSSs fail to interpret the information in the spherical fuzzy form, thus not efficient enough when dealing with uncertain fuzzy data. The newly proposed model called SFNSESs not only combines the properties of the above two models but is also free of the restrictions and limitations as discussed above. To show the diversity and accuracy of our model, Tables 39 and 40 compare the results of the application (Example 13, Sect. 4) under NSSs, SFSESs, and proposed SFNSESs. From Table 39 and Fig. 6, it can be clearly seen that NSSs fail to identify the difference between the objects with the same ratings (that is, objects $\mathfrak{u}_1$ and $\mathfrak{u}_2$ have similar grades), whereas SFNSESs are capable of providing a distinction between these equally rated alternatives. Similarly, Tables 41 and 42 compare the results of the application (Example 14, Sect. 4) under NSSs, SFSESs, and proposed SFNSESs. From Table 42 and Fig. 7, it can be clearly observed that NSS model fails to identify the difference between the objects with the same ratings (that is, objects $\mathfrak{u}_6$ and $\mathfrak{u}_8$ have similar grades), whereas SFNSES model is capable of providing a distinction between these alternatives. In addition, Table 43 shows the diversity and applicability of the proposed model compared to some of the already existing models.

3. *Limitations*

The developed model has some limitations, firstly its complicated structure having grades and spherical fuzzy information estimated by multiple experts. Due to complex calculations, it can be hard to handle the data having several alternatives or parameters and its execution during the decision-making process. Thus, despite the high applicability of the model, this model may sometimes make it more difficult to handle situations

Table 43 Comparing applicability with some existing models

Models	Set of experts (E)	Applicability	Domain
Soft sets (Molodtsov 1999)	$\lvert E\rvert = 1$	Least	Binary data in discrete form
SFSSs (Perveen et al. 2019)	$\lvert E\rvert = 1$	Higher than soft sets	Spherical fuzzy data in soft environment
NSSs (Fatimah et al. 2018)	$\lvert E\rvert = 1$	Higher than soft sets	Multinary data in discrete form
SESs (Alkhazaleh and Salleh 2011)	$\lvert E\rvert \geq 1$	Higher than soft sets	Binary data in discrete form
NSESs (Ali and Akram 2020)	$\lvert E\rvert \geq 1$	Higher than SESs and NSSs	Multinary data in discrete form
SFSESs (Perveen et al. 2020)	$\lvert E\rvert \geq 1$	Higher than SFSSs and SESs	Spherical fuzzy data in soft expert environment
Proposed SFNSESs	$\lvert E\rvert \geq 1$	Highest	Spherical fuzzy data in *N*-soft expert environment

considering somewhat easier scenarios. Software like MATLAB may implement the algorithm and handle lengthy calculations, thus easing this difficulty. Secondly, the choice of thresholds for the decision-making algorithm during the decision-making may slightly affect the outcomes. Thirdly, with the addition or removal of any new parameters (or objects) in the MAGDM scenario, there may occur a variation in the optimal (or sub-optimal) objects' ranking order. It occurs as a consequence of the independent behavior of parameters and objects. Adding to it, the proposed model is bounded to counter limited three-dimensional ambiguous information, for instance, in a scenario considering 0.8, 0.5, and 0.6 as positive, negative, and neutral membership values, respectively, the developed model fails to deal such situations, since $0.8^2 + 0.5^2 + 0.6^2 = 0.64 + 0.25 + 0.36 = 1.25 \nless 1$.

6 Conclusions and future directions

As a powerful mathematical model for dealing with uncertainties, the SFSESs (Perveen et al. 2020) have proved their effectiveness in dealing with problems concerning uncertainties and vague information under multiple experts' opinions. The emerging NSS model (Fatimah et al. 2018) is a strong tool when dealing with uncertainties in multinary data. The model has proved to effectively deal with various daily life problems, as illustrated in Fatimah et al. (2018). Despite their effectiveness, these models have some limitations. SFSESs fail to deal with multinary data, while NSSs cannot handle spherical fuzzy information. This paper proposes the SFNSES model, which is more applicable and accurate in several group decision-making situations. Some operations on the model, including intersections, unions, complements, AND and OR operations, are investigated with their basic properties. Our proposed idea of SFNSESs is illustrated by Example 1, in which a novel for the title 'best novel of the year' has to be selected based on the reader's ratings. Two detailed applications of the model along with the algorithm are also provided in Examples 13 and 14, in which the winner of an upcoming local election and ranking credibility of the smartphones using customer feedback are respectively predicted on the basis of survey ratings and experts' opinions in SFNSES environment. Finally, a comparison analysis is provided between the proposed model and existing decision-making tools, including NSSs and SFSESs. Some potential applications of the proposed model include market research, artificial intelligence-based product analytics, election predictions, sentiment analytics involving public sentiments, business analytics, and more. Despite the high applicability, the limitations of the model also exist due to its complicated structure, lengthy algorithm, and a massive number of alternatives and parameters. The mathematical software can help to overcome these limitations.

In the future, we are expanding our research work to (1) Complex spherical fuzzy *N*-soft expert sets, (2) *m*-Polar spherical fuzzy *N*-soft expert sets, and (3) Spherical fuzzy *N*-soft expert graphs.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest regarding the publication of this article.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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