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Balanced picture fuzzy graph with application

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Abstract
Picture fuzzy graphs are an extension of intuitionistic fuzzy graphs. Balanced picture fuzzy graph is a special type of picture fuzzy graph (*PFG*). In this study, the definition and important properties of *PFG* like, average *PFG*, balanced *PFG*, size, order, density of a *PFG*, isomorphism, the direct product of two *PFG*, etc have been studied. The necessary and sufficient conditions for balanced picture fuzzy graphs have also been studied in this article. Beside this, we proposed an algorithm to test whether a *PFG* is balanced or not. The proof of correctness and an illustration of the proposed algorithm is presented in this article. Lastly, an application of balanced *PFG* to business alliance is presented.

Keywords Picture fuzzy graph · Balanced picture fuzzy graph · Average picture fuzzy graph · Density of a picture fuzzy graph

1 Introduction

A picture fuzzy set is a generalization of intuitionistic fuzzy set (Atanassov 1986). Picture fuzzy models give more precision, flexibility and compatibility to the system as compared to the intuitionistic fuzzy models. The concept of a picture fuzzy set was first introduced by Cuong and Kreinovich (2013). In addition to intuitionistic fuzzy sets, Coung appended new components which determine the degree of neutral membership. The intuitionistic fuzzy set gives the degree of membership and the degree of non-membership of an element, while the picture fuzzy set gives the degree of positive membership, the degree of neutral membership and the degree of negative membership of an element. These memberships

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are more or less independent from each other, the only requirement is that the sum of these three degrees is less than or equal to 1. Basically, picture fuzzy sets based models may be adequate in situations where we counter several opinions involving more answers of types: yes, no, abstain, refusal. If we take voting as an example , the human voters may be separated into four possible groups with distinct choices of opinions like vote for, abstain, vote against, refusal of the voting. Some properties of the picture fuzzy set and its operators have been studied in Cuong (2014), Dutta and Ganju (2017).

1.1 Review of literature

After the invention of fuzzy graph theory, it increases with its several extensions. In 2012 (Rashmanlou and Pal 2013) have defined balanced interval-valued fuzzy graphs and discuss some important properties, like product of two interval-valued fuzzy graphs, balanced and strictly balanced interval-valued fuzzy graphs, etc. (Samanta and Pal 2011) has introduced the definition of fuzzy threshold graph. Also, they have introduced some important definitions related to fuzzy threshold graphs like, fuzzy alternating four cycle, fuzzy ferrers digraph, threshold dimension of fuzzy graph. Fuzzy planar graphs have been introduced by Samanta et al. (2014),Samanta and Pal (2015). In this article they have discussed the concept of two different edges: effective edge and considerable edge. Also, in 2013, (Pal et al. 2013) have studied about fuzzy *k*-competition graphs. Interval-valued fuzzy threshold graphs are defined and studied several properties by (Pramanik et al. 2016a, b). They also have considered planarity in bipolar fuzzy graphs and they extended it to bipolar fuzzy planar graphs (Pramanik et al. 2018). In 2017, (Alvi et al. 2017) have studied an adaptive grayscale image de-noising technique by fuzzy inference system. Also, (Pramanik et al. 2016) have extended fuzzy planar graph to interval-valued fuzzy planar graph, interval-valued fuzzy graph. (Voskoglou and Pramanik 2020) have discussed and characterized several fuzzy graph theoretic structures and fuzzy hypergraphs. (Srivastav et al. 2020) have worked on Integration of Multiple Cache Server Scheme for User-Based Fuzzy Logic in Content Delivery Networks.

Sahoo and Pal (2015b) discussed the concept of intuitionistic fuzzy competition graphs. In 2012, (Akram and Davvaz 2012))defined strong intuitionistic fuzzy graphs. They also discuss intuitionistic fuzzy hypergraphs with applications (Akram and Dudek 2013). Also, in 2014, (Akram and Al-Shehrie, 2014) defined intuitionistic fuzzy cycles and intuitionistic fuzzy trees and intuitionistic fuzzy planar graphs (Al-Shehrie and Akram 2014). (Karunambigai et al. 2013) have defined density of intuitionistic fuzzy graph, Balanced intuitionistic fuzzy graph, direct product of intuitionistic fuzzy graph. (Sahoo and Pal 2015b) discussed the concept of intuitionistic fuzzy competition graphs. Also, (Sahoo and Pal 2015a, 2016) discussed intuitionistic fuzzy tolerance graphs with application, different types of products on intuitionistic fuzzy graphs. In 2019, (Zuo et al. 2019) introduced the concept of picture fuzzy set to graph theory and obtained *PFG*. In this article some types of picture fuzzy graphs such as strong *PFG*, regular *PFG*, complete *PFG*, and complement of *PFG* are introduced. Also, the isomorphism of *PFG*s, Cartesian product, composition, join, direct product, lexicographic and strong product on *PFG*s have been defined. (Ismayil and Bosley 2019) have introduced the domination in PFG and have defined the order and size of a *PFG*. Also, (Ismayil et al. 2019), have studied edge domination in *PFG*. (Akram and Habib 2019) have introduced q-rung picture fuzzy line graphs and developed a necessary condition for this graph. Recently, (Mohanta et al. 2020) have introduced the concept of a dombi operator to picture fuzzy graphs and obtained picture Dombi fuzzy graphs. In

2020, (Das and Ghorai 2020) have studied some properties of planer picture fuzzy graph like, strong (weak) edges, strong (weak) picture fuzzy planar graphs, strength of an edge, degree of planarity, picture fuzzy faces, strong (weak) picture fuzzy faces, etc. Many terminologies of fuzzy graphs and their variations and applications are mentioned in the recent published book written by (Pal et al. 2020).

1.2 Motivation

Picture fuzzy graph is an extension of intuitionistic fuzzy graphs. Intuitionistic fuzzy graphs have a lot of applications in the real world, because it has a capability to model several decision making problems in an uncertain environment. A number of generalizations of intuitionistic fuzzy graphs have been introduced to deal with the uncertainty of the complex real life problems. As uncertainties are well expressed using picture fuzzy sets, picture fuzzy graphs would be a prominent research direction for modeling the uncertain optimization problems. We see that one of the important concepts of neutrality degree is lacking in IFS theory. Concept of neutrality degree can be seen in situations when we face human opinions involving more answers of type: yes, abstain, no, refusal. For example, in a democratic election station, the council issues 600 voting papers for a candidate. The voting results are divided into four groups accompanied with the number of papers namely "vote for"(350), "abstain"(70), "vote against" (155) and "refusal of voting"(25). Group "abstain" means that the voting paper is a white paper rejecting both "agree" and "disagree" for the candidate but still takes the vote. Group "refusal of voting" is either invalid voting papers or bypassing the vote.

This situation can not be described by intuitionistic fuzzy graphs. Motivated from this point of view we consider balanced picture fuzzy graphs for this research study and have presented some definitions, properties, theorems, and algorithms to test whether a balanced picture fuzzy graph is balanced or not.

The remaining part of the paper is organized as follows. Some preliminaries are presented in Sect. 1. In Sect. 2, picture fuzzy graphs and some related terms are defined. In Sect. 3, balanced picture fuzzy graphs and its properties have been studied. Also an algorithm to check the balanced property of a *PFG* and illustration of the algorithm is also presented in this section. In Sect. 4 an application of balanced *PFG* is presented. Section 5 is for brief discussion and conclusion.

2 Preliminaries

Intuitionistic fuzzy graph is a generalization of fuzzy graphs. The definition of intuitionistic fuzzy graph is given below:

Definition 1 An intuitionistic fuzzy graph is of the form $G = (V, \sigma, \mu)$ where $\sigma = (\sigma_1, \sigma_2)$, $\mu = (\mu_1, \mu_2)$ and

(i) $V = \{p_1, p_2, \dots, p_n\}$ such that $\sigma_1 : V \to [0, 1]$ and $\sigma_2 : V \to [0, 1]$, denote the degree of membership and non-membership of the node $p_i \in V$ respectively and $0 \leq \sigma_1(p_i) + \sigma_2(p_i) \leq 1$ for every $p_i \in V$ $(i = 1, 2, \dots, n)$.

(ii) $\mu_1 : V \times V \to [0, 1]$ and $\mu_2 : V \times V \to [0, 1]$, where $\mu_1(p_i, p_j)$ and $\mu_2(p_i, p_j)$ denote the the degree of membership and non-membership value of the edge (p_i, p_j) respectively such that $\mu_1(p_i,p_j) \leq \min\{\sigma_1(p_i), \sigma_1(p_j)\}$ and $\mu_2(p_i,p_j) \leq \max\{\sigma_2(p_i), \sigma_2(p_j)\}$, $0 \leq \mu_1(p_i,p_j) + \mu_2(p_i,p_j) \leq 1$ for every $(p_i,p_j) \in V \times V$.

Picture fuzzy set is an extension of an intuitionistic fuzzy set. The definition of picture fuzzy set is presented below.

Definition 2 A picture fuzzy set A on a universe X is an object of the form $A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) : x \in X\}$, where $\mu_A(x) \in [0, 1]$ is called the degree of positive membership of x in A, $\eta_A(x) \in [0, 1]$ is called the degree of neutral membership of x in A, $\nu_A(x) \in [0, 1]$ is called the degree of negative membership of x in A and they satisfies $0 \leq \mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1$ for all $x \in X$.

Now $1 - \{\mu_A(x) + \eta_A(x) + \nu_A(x)\}$ is said to be the degree of refusal of x in A. Basically, picture fuzzy sets based models may be adequate in situations when we face human opinions involving more answers of types: yes, abstain, no , refusal. Voting can be a good example of such a situation as the human voters may be divided into four groups of those who: vote for, abstain, vote against, refusal of the voting.

3 Picture fuzzy graph

In this section, we introduce some basic definitions and properties, theorems related to picture fuzzy graphs.

Definition 3 A picture fuzzy graph (PFG) is of the type $G = (V, E, m_V, m_E)$, where $V = \{p_1, p_2, \dots, p_n\}$ be the set of nodes, m_V and m_E denotes the membership function of nodes and edges respectively, where $m_V = (\mu_V, \eta_V, \nu_V)$ and $m_E = (\mu_E, \eta_E, \nu_E)$ is such that,

(i) $\mu_V : V \to [0, 1]$, $\eta_V : V \to [0, 1]$ and $\nu_V : V \to [0, 1]$ are respectively positive membership, neutral membership and negative membership function and it satisfy the condition $0 \leq \mu_V(p_i) + \eta_V(p_i) + \nu_V(p_i) \leq 1$ for all $p_i \in V$, for $i = 1, 2, \dots, n$.

(ii) $\mu_E : V \times V \to [0, 1]$, $\eta_E : V \times V \to [0, 1]$ and $\nu_E : V \times V \to [0, 1]$, are respectively positive membership, neutral membership and negative membership function of edge (p_i,p_j), which satisfies the following condition

$\mu_E(p_i,p_j) \leq \min\{\mu_V(p_i), \mu_V(p_j)\}$, $\eta_E(p_i,p_j) \leq \min\{\eta_V(p_i), \eta_V(p_j)\}$, $\nu_E(p_i,p_j) \leq \max\{\nu_V(p_i), \nu_V(p_j)\}$ and $0 \leq \mu_E(p_i,p_j) + \eta_E(p_i,p_j) + \nu_E(p_i,p_j) \leq 1$ for every $(p_i,p_j) \in V \times V$, for $i = 1, 2, \dots, n$.

Here we notice that G is a picture fuzzy graph with node set V and E be the set of edges, $\mu_V(p_i), \eta_V(p_i), \nu_V(p_i)$ are respectively the positive, neutral, negative membership value of the node p_i, and $\mu_E(p_i,p_j), \eta_E(p_i,p_j), \nu_E(p_i,p_j)$ are respectively the positive, neutral, negative membership degree of the edge joining the nodes p_i, p_j.

A PFG is shown in Fig. 1. In this figure we see that the degree of positive membership, neutral membership and negative membership value of the node p_1 is 0.3, 0.2, 0.4 respectively. Also, the degree of positive membership, neutral membership and negative membership value of the edge joining the nodes p_1 and p_2 is 0.25, 0.1, 0.35 respectively. Same for the other nodes and edges.

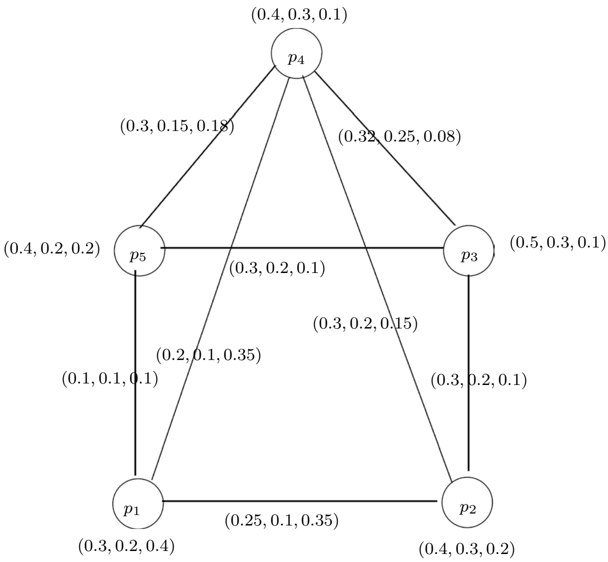


Fig. 1 A picture fuzzy graph

Throughout the paper we denote a *PFG* $G = (V, E, m_V, m_E)$, where $m_V = (\mu_V, \eta_V, \nu_V)$ and $m_E = (\mu_E, \eta_E, \nu_E)$.

Definition 4 A *PFG* $G = (V, E, m_V, m_E)$, where $m_V = (\mu_V, \eta_V, \nu_V)$ and $m_E = (\mu_E, \eta_E, \nu_E)$ is called complete picture fuzzy graph if $\mu_E(p_i, p_j) = \min\{\mu_V(p_i), \mu_V(p_j)\}$, $\eta_E(p_i, p_j) = \min\{\eta_V(p_i), \eta_V(p_j)\}$, $\nu_E(p_i, p_j) = \max\{\nu_V(p_i), \nu_V(p_j)\}$ for all $p_i, p_j \in V$.

Definition 5 $G = (V, E, m_V, m_E)$ be a *PFG* then G is said to be a single valued strong picture fuzzy graph if $\mu_E(p_i, p_j) = \min\{\mu_V(p_i), \mu_V(p_j)\}$,
$\eta_E(p_i, p_j) = \min\{\eta_V(p_i), \eta_V(p_j)\}$, $\nu_E(p_i, p_j) = \max\{\nu_V(p_i), \nu_V(p_j)\}$ for all $(p_i, p_j) \in E$.

Definition 6 A *PFG* $G = (V, E, m_V, m_E)$ where $m_V = (\mu_V, \eta_V, \nu_V)$ and $m_E = (\mu_E, \eta_E, \nu_E)$ is called single valued average picture fuzzy graph
if $\mu_E(p_i, p_j) = \frac{1}{2}\min\{\mu_V(p_i), \mu_V(p_j)\}$, $\eta_E(p_i, p_j) = \frac{1}{2}\min\{\eta_V(p_i), \eta_V(p_j)\}$, $\nu_E(p_i, p_j) = \frac{1}{2}\max\{\nu_V(p_i), \nu_V(p_j)\}$ for all $p_i, p_j \in V$.

Definition 7 The complement of a *PFG* $G = (V, E, m_V, m_E)$ is a *PFG* $\bar{G} = (\bar{V}, \bar{E}, \bar{m_V}, \bar{m_E})$, where $m_V = (\mu_V, \eta_V, \nu_V)$, $m_E = (\mu_E, \eta_E, \nu_E)$, $\bar{m_V} = (\bar{\mu_V}, \bar{\eta_V}, \bar{\nu_V})$ and $\bar{m_E} = (\bar{\mu_E}, \bar{\eta_E}, \bar{\nu_E})$

if (i) $\bar{V} = V$.
(ii) $\bar{\mu_V}(p_i) = \mu_V(p_i)$, $\bar{\eta_V}(p_i) = \eta_V(p_i)$, $\bar{\nu_V}(p_i) = \nu_V(p_i)$, for all $p_i \in V$.
(iii) $\bar{\mu_E}(p_i, p_j) = \min\{\mu_V(p_i), \mu_V(p_j)\} - \mu_E(p_i, p_j)$, $\bar{\eta_E}(p_i, p_j) = \min\{\eta_V(p_i), \eta_V(p_j)\} - \eta_E(p_i, p_j)$ and $\bar{\nu_E}(p_i, p_j) = \max\{\nu_V(p_i), \nu_V(p_j)\} - \nu_E(p_i, p_j)$, for all $(p_i, p_j) \in V \times V$.

Definition 8 Let $G = (V, E, m_V, m_E)$ be a *PFG* where $m_V = (\mu_V, \eta_V, \nu_V)$ and $m_E = (\mu_E, \eta_E, \nu_E)$. The order of G is denoted by $O(G)$ and it is defined by $O(G) = (O_\mu(G), O_\eta(G), O_\nu(G))$, where

$$O_\mu(G) = \frac{1}{n}\sum_{p_i \in V} \mu_V(p_i), O_\eta(G) = \frac{1}{n}\sum_{p_i \in V} \eta_V(p_i), O_\nu(G) = \frac{1}{n}\sum_{p_i \in V} \nu_V(p_i).$$

The value of $O_\mu(G)$, $O_\eta(G)$ and $O_\nu(G)$ lies between 0 and 1.

Definition 9 Let $G = (V, E, m_V, m_E)$ be a *PFG* where $m_V = (\mu_V, \eta_V, \nu_V)$ and $m_E = (\mu_E, \eta_E, \nu_E)$. Then the size of G is denoted by S_G and is defined by $S_G = (S_\mu(G), S_\eta(G), S_\nu(G))$, where

$$S_\mu(G) = \sum_{p_i \neq p_j} \mu_E(p_i, p_j), S_\eta(G) = \sum_{p_i \neq p_j} \eta_E(p_i, p_j), S_\nu(G) = \sum_{p_i \neq p_j} \nu_E(p_i, p_j).$$

4 Balanced picture fuzzy graph

In this section, we have studied the definition and properties of balanced picture fuzzy graphs. Also, we have studied the necessary and sufficient condition for a balanced picture fuzzy graph. An algorithm is also presented to test whether a *PFG* is balanced or not.

Definition 10 Let $G = (V, E, m_V, m_E)$ be a *PFG* where $m_V = (\mu_V, \eta_V, \nu_V)$ and $m_E = (\mu_E, \eta_E, \nu_E)$. Then the weight of G is denoted by $w(G)$ and is defined by

$$\begin{aligned}
&w(G) = (w_\mu(G), w_\eta(G), w_\nu(G)), \text{where} \\
&w_\mu(G) = \sum_{(p_i, p_j) \in E} \min\{\mu_V(p_i), \mu_V(p_j)\} \\
&w_\eta(G) = \sum_{(p_i, p_j) \in E} \min\{\eta_V(p_i), \eta_V(p_j)\} \\
&w_\nu(G) = \sum_{(p_i, p_j) \in E} \max\{\nu_V(p_i), \nu_V(p_j)\}
\end{aligned}$$

Now, one can defined the density of a *PFG* G based on weight and size of G.

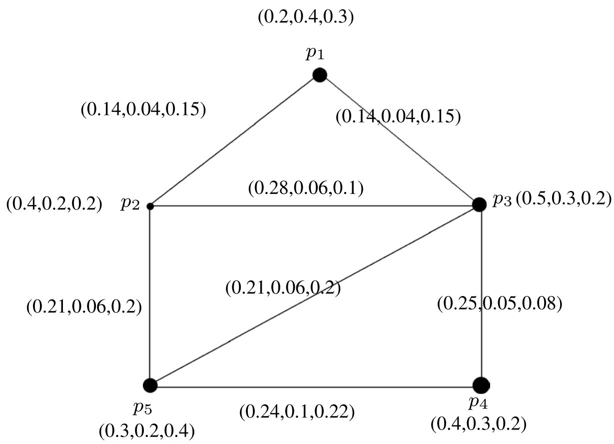
Definition 11 Let $G = (V, E, m_V, m_E)$ be a *PFG* where $m_V = (\mu_V, \eta_V, \nu_V)$ and $m_E = (\mu_E, \eta_E, \nu_E)$. Then the density of G is denoted by $\rho(G)$ and is defined by $\rho(G) = (\rho_\mu(G), \rho_\eta(G), \rho_\nu(G))$, where $\rho_\mu(G) = \frac{S_\mu(G)}{w_\mu(G)}$, $\rho_\eta(G) = \frac{S_\eta(G)}{w_\eta(G)}$, $\rho_\nu(G) = \frac{S_\nu(G)}{w_\nu(G)}$ for all $p_i, p_j \in V$. All the components $\rho_\mu(G)$, $\rho_\eta(G)$ and $\rho_\nu(G)$ lie between 0 and 1.

Definition 12 Let $G = (V, E, m_V, m_E)$ be a *PFG*. A subgraph S of G is called intense picture fuzzy subgraph if the node set of S is a subset of node set of G and $E(S) \subseteq E(G)$ and $\rho(S) \leq \rho(G)$.

Now $\rho(S) \leq \rho(G)$ holds if $\rho_\mu(S) \leq \rho_\mu(G)$, $\rho_\eta(S) \leq \rho_\eta(G)$, $\rho_\nu(S) \leq \rho_\nu(G)$.

Definition 13 Let $G = (V, E, m_V, m_E)$ be a *PFG*. A subgraph S of G is called feeble picture fuzzy subgraph if the node set of S is a subset of node set of G and $V(S) \subseteq V(G)$, $E(S) \subseteq E(G)$ and $\rho(S) > \rho(G)$, i.e.,$\rho_\mu(S) > \rho_\mu(G)$, $\rho_\eta(S) > \rho_\eta(G)$, $\rho_\nu(S) > \rho_\nu(G)$.

Fig. 2 A picture fuzzy graph



Example 1 Here we consider a graph in Fig. 2 and find out the intense picture fuzzy subgraph and feeble picture fuzzy subgraph of G. For this graph $\rho(G) = (0.7, 0.3, 0.5)$.

Let us consider a subgraph $H_1 = (V_1, E_1, m_{V_1}, m_{E_1})$ where $V_1 = \{p_3, p_4, p_5\}$, $E_1 = \{(p_3, p_5), (p_5, p_4), (p_3, p_4)\}$.

Then $\rho(H_1) = (0.7, 0.3, 0.5)$.

Let us consider a subgraph $H_2 = (V_2, E_2, m_{V_2}, m_{E_2})$ where $V_2 = \{p_4, p_5\}$, $E_2 = \{(p_4, p_5)\}$.

Then $\rho(H_2) = (0.8, 0.5, 0.55)$.

Again consider a subgraph $H_3 = (V_3, E_3, m_{V_3}, m_{E_3})$ where $V_3 = \{p_3, p_4\}$, $E_3 = \{(p_3, p_4)\}$.

Then $\rho(H_3) = (0.625, 0.167, 0.04)$. Also, we consider a subgraph $H_4 = (V_4, E_4, m_{V_4}, m_{E_4})$ where $V_4 = \{p_1, p_2, p_3\}$, $E_4 = \{(p_1, p_2), (p_2, p_3), (p_1, p_3), (p_3, p_4)\}$.

Then $\rho(H_4) = (0.675, 0.23, 0.48)$. Hence, the subgraphs H_1 and H_3, H_4 are intense picture fuzzy subgraphs of G and the subgraphs H_2 is a feeble picture fuzzy subgraph of G.

Theorem 1 *Let $G = (V, E, m_V, m_E)$ be a PFG. Then G is called self complementary if and only if G is an average picture fuzzy graph.*

Proof Since, $G = (V, E, m_V, m_E)$ be a *PFG*. Let $\bar{G} = (\bar{V}, \bar{E}, \bar{m_V}, \bar{m_E})$ be its complement, where $m_V = (\mu_V, \eta_V, \nu_V)$, $m_E = (\mu_E, \eta_E, \nu_E)$, $\bar{m_V} = (\bar{\mu_V}, \bar{\eta_V}, \bar{\nu_V})$ and $\bar{m_E} = (\bar{\mu_E}, \bar{\eta_E}, \bar{\nu_E})$. Since $\bar{G}$ is the complement of G, then

(*i*) $\bar{V} = V$.
(*ii*) $\bar{\mu_V}(p_i) = \mu_V(p_i)$, $\bar{\eta_V}(p_i) = \eta_V(p_i)$, $\bar{\nu_V}(p_i) = \nu_V(p_i)$, for all $p_i \in V$.
(*iii*) $\bar{\mu_E}(p_i, p_j) = \min\{\mu_V(p_i), \mu_V(p_j)\} - \mu_E(p_i, p_j)$, $\bar{\eta_E}(p_i, p_j) = \min\{\eta_V(p_i), \eta_p(p_j)\} - \eta_E(p_i, p_j)$ and $\bar{\nu_E}(p_i, p_j) = \max\{\nu_V(p_i), \nu_V(p_j)\} - \nu_E(p_i, p_j)$, for every $p_i, p_j \in V$.

Let G be a single valued average *PFG* then $\mu_E(p_i, p_j) = \frac{1}{2}\min\{\mu_V(p_i), \mu_V(p_j)\}$, $\eta_E(p_i, p_j) = \frac{1}{2}\min\{\eta_V(p_i), \eta_V(p_j)\}$, $\nu_E(p_i, p_j) = \frac{1}{2}\max\{\nu_V(p_i), \nu_V(p_j)\}$ for all $p_i, p_j \in V$. Now,

$$\begin{aligned}\bar{\mu}_E(p_i,p_j) =& \min\{\mu_V(p_i),\mu_V(p_j)\} - \frac{1}{2}\min\{\mu_V(p_i),\mu_V(p_j)\}\\ =&\frac{1}{2}\min\{\mu_V(p_i),\mu_V(p_j)\}\\ =&\mu_E(p_i,p_j) \text{ for all } p_i,p_j\in V.\end{aligned}$$

Similarly, $\bar{\eta}_E(p_i,p_j) = \eta_E(p_i,p_j)$ and $\bar{\nu}_E(p_i,p_j) = \nu_E(p_i,p_j)$. This shows that $\bar{G}$ is similar to G. Hence, G is a self complementary *PFG*.

Conversely, let G is a self complementary *PFG*. Then $\bar{\mu}_E(p_i,p_j) = \mu_E(p_i,p_j)$, $\bar{\eta}_E(p_i,p_j) = \eta_E(p_i,p_j)$, $\bar{\nu}_E(p_i,p_j) = \nu_E(p_i,p_j)$ for all $p_i,p_j\in V$. Now, $\mu_E(p_i,p_j) = \bar{\mu}_E(p_i,p_j)$ $= \min\{\mu_V(p_i),\mu_V(p_j)\} - \mu_E(p_i,p_j)$. So, $2\mu_E(p_i,p_j) = \min\{\mu_V(p_i),\mu_V(p_j)\}$. Therefore, $\mu_E(p_i,p_j) = \frac{1}{2}\min\{\mu_V(p_i),\mu_V(p_j)\}$. Similarly, $\eta_E(p_i,p_j) = \frac{1}{2}\min\{\eta_V(p_i),\eta_V(p_j)\}$, $\nu_E(p_i,p_j) = \frac{1}{2}\min\{\nu_V(p_i),\nu_V(p_j)\}$ for all $p_i,p_j\in V$. Hence, G is single valued average *PFG*. □

Theorem 2 *Let $G = (V, E, m_V, m_E)$ be an average PFG and $\rho(G)$ be the density of G then $\rho(G) = (0.5, 0.5, 0.5)$.*

Proof Since $\rho(G)$ is the density of the *PFG* G then $\rho(G) = (\rho_\mu(G), \rho_\eta(G), \rho_\nu(G))$

We know that,

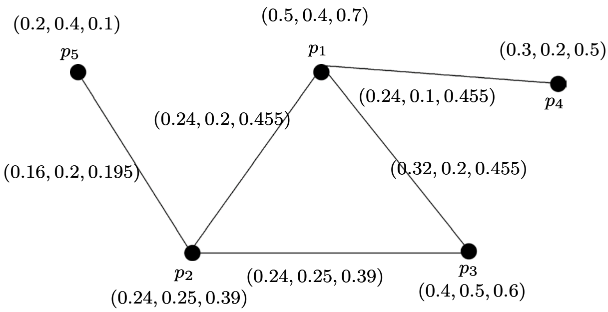
$$\begin{aligned}\rho_\mu(G) =& \frac{S_\mu(G)}{w_\mu(G)}\\ =&\frac{\sum_{p_i\neq p_j}\mu_E(p_i,p_j)}{\sum_{(p_i,p_j)\in E}\min\{\mu_V(p_i),\mu_V(p_j)\}}\\ =&\frac{\frac{1}{2}\sum_{p_i\neq p_j}\min\{\mu_V(p_i),\mu_V(p_j)\}}{\sum_{(p_i,p_j)\in E}\min\{\mu_V(p_i),\mu_V(p_j)\}}\\ =&0.5.\end{aligned}$$

Similarly, $\rho_\eta(G) = 0.5$ and $\rho_\nu(G) = 0.5$. Therefore, $\rho(G) = (0.5, 0.5, 0.5)$. □

Definition 14 A *PFG* $G = (V, E, m_V, m_E)$ is said to be balanced if all its subgraphs are intense in G, i.e., $\rho(S) \le \rho(G)$ for any subgraph S of G. $\rho(S) \le \rho(G)$ holds if $\rho_\mu(S) \le \rho_\mu(G)$, $\rho_\eta(S) \le \rho_\eta(G)$ and $\rho_\nu(S) \le \rho_\nu(G)$.

Example 2 We consider a *PFG* $G = (V, E, m_V, m_E)$ such that $V = \{p_1, p_2, p_3, p_4, p_5\}$, $E = \{(p_1, p_2), (p_1, p_3), (p_1, p_4), (p_2, p_3), (p_2, p_5)\}$ in Fig. 3 and then check whether this *PFG* is balanced or not. We know that the size of the graph G is $S_G = (S_\mu(G), S_\eta(G), S_\nu(G))$. For this graph, $S_\mu(G) = \sum_{p_i\neq p_j}\mu_E(p_i,p_j) = 1.2$, $S_\eta(G) = \sum_{p_i\neq p_j}\eta_E(p_i,p_j) = 0.95$, $S_\nu(G) = \sum_{p_i\neq p_j}\nu_E(p_i,p_j) = 1.95$. Again, the weight of G is $w(G) = (w_\mu(G), w_\eta(G), w_\nu(G))$. Then $w_\mu(G) = \sum\limits_{(p_i,p_j)\in E}\min\{\mu_V(p_i),\mu_V(p_j)\} = 1.5$.

Fig. 3 A *PFG*



$$w_\eta(G) = \sum_{(p_i,p_j)\in E} \min\{\eta_V(p_i), \eta_V(p_j)\} = 1.9.$$

$$w_\nu(G) = \sum_{(p_i,p_j)\in E} \min\{\nu_V(p_i), \nu_V(p_j)\} = 3.0.$$

Therefore, the density of G is $\rho(G) = (\rho_\mu(G), \rho_\eta(G), \rho_\nu(G))$, where $\rho_\mu(G) = \frac{S_\mu(G)}{w_\mu(G)} = 0.8$, $\rho_\eta(G) = \frac{S_\eta(G)}{w_\eta(G)} = 0.5$, $\rho_\nu(G) = \frac{S_\nu(G)}{w_\nu(G)} = 0.65$. So, $\rho(G) = (0.8, 0.5, 0.65)$. From Table 1, we see that the density of the subgroup S_i is (0.8, 0.5, 0.65) for $i = 1, 2, \dots, 19, 21, 23$ and the density of the subgroup S_j is (0, 0, 0) for $j = 20, 22, 24, 25, 26$. Here, all $\rho(S_r) \leq \rho(G)$ for all subgraph S_r of G which is shown in the following table.

Hence, G is balanced.

Definition 15 A *PFG* $G = (V, E, m_V, m_E)$ is said to be strictly balanced if $\rho(G) = \rho(S)$ for all subgraph S of G, i.e., $\rho(S) = \rho(G)$ holds if $\rho_\mu(S) = \rho_\mu(G)$, $\rho_\eta(S) = \rho_\eta(G)$ and $\rho_\nu(S) = \rho_\nu(G)$.

Theorem 3 *PFG* $G = (V, E, m_V, m_E)$ *is strictly balanced iff*

$\mu_E(p_i, p_j) = \lambda_1 \min\{\mu_V(p_i), \mu_V(p_j)\}$, $\eta_E(p_i, p_j) = \lambda_2 \min\{\eta_V(p_i), \eta_V(p_j)\}$, $\nu_E(p_i, p_j) = \lambda_3 \max\{\nu_V(p_i), \nu_V(p_j)\}$ *for all* $p_i, p_j \in V$, *where* $\rho(G) = (\lambda_1, \lambda_2, \lambda_3)$.

Proof Suppose that $G = (V, E, m_V, m_E)$ is strictly balanced *PFG* with n nodes and $V = \{p_1, p_2, \dots, p_n\}$. Then $\rho(G) = \rho(S)$ for all subgraph S of G. Also given that $\rho(G) = (\lambda_1, \lambda_2, \lambda_3)$. Since, G contains n nodes then G has $2^n - (n+1) = \Delta$ subgraph. Among the Δ subgraphs of G, ${}^n c_2 = \Delta_1$ subgraphs are the subgraphs, each containing 2 nodes.

In Δ_1 subgraph, let us consider any arbitrary subgraph, say S_r of G.

Let $S_r = (V_r, E_r, m_{V_r}, m_{E_r})$ where $V_r = \{p_{r_i}, p_{r_j}\}$.

Table 1 Density of all subgraph of the *PFG* in Fig. 3

Subgraph	Vertex set	Density
S_1	$\{p_1, p_2, p_3, p_4, p_5\}$	(0.8, 0.5, 0.65)
S_2	$\{p_1, p_2, p_3, p_4\}$	(0.8, 0.5, 0.65)
S_3	$\{p_1, p_2, p_3, p_5\}$	(0.8, 0.5, 0.65)
S_4	$\{p_1, p_2, p_4, p_5\}$	(0.8, 0.5, 0.65)
S_5	$\{p_1, p_3, p_4, p_5\}$	(0.8, 0.5, 0.65)
S_6	$\{p_2, p_3, p_4, p_5\}$	(0.8, 0.5, 0.65)
S_7	$\{p_1, p_2, p_3\}$	(0.8, 0.5, 0.65)
S_8	$\{p_1, p_2, p_4\}$	(0.8, 0.5, 0.65)
S_9	$\{p_1, p_2, p_5\}$	(0.8, 0.5, 0.65)
S_{10}	$\{p_1, p_3, p_4\}$	(0.8, 0.5, 0.65)
S_{11}	$\{p_1, p_3, p_5\}$	(0.8, 0.5, 0.65)
S_{12}	$\{p_1, p_4, p_5\}$	(0.8, 0.5, 0.65)
S_{13}	$\{p_2, p_3, p_4\}$	(0.8, 0.5, 0.65)
S_{14}	$\{p_2, p_3, p_5\}$	(0.8, 0.5, 0.65)
S_{15}	$\{p_2, p_4, p_5\}$	(0.8, 0.5, 0.65)
S_{16}	$\{p_3, p_4, p_5\}$	(0.8, 0.5, 0.65)
S_{17}	$\{p_1, p_2\}$	(0.8, 0.5, 0.65)
S_{18}	$\{p_1, p_3\}$	(0.8, 0.5, 0.65)
S_{19}	$\{p_1, p_4\}$	(0.8, 0.5, 0.65)
S_{20}	$\{p_1, p_5\}$	(0, 0, 0)
S_{21}	$\{p_2, p_3\}$	(0.8, 0.5, 0.65)
S_{22}	$\{p_2, p_4\}$	(0, 0, 0)
S_{23}	$\{p_2, p_5\}$	(0.8, 0.5, 0.65)
S_{24}	$\{p_3, p_4\}$	(0, 0, 0)
S_{25}	$\{p_3, p_5\}$	(0, 0, 0)
S_{26}	$\{p_4, p_5\}$	(0, 0, 0)

Now the density of S_r, $\rho(S_r) = (\rho_\mu(S_r), \rho_\eta(S_r), \rho_\nu(S_r))$.

Therefore, $\rho_\mu(S_r) = \frac{\mu_{E_r(p_{r_i}, p_{r_j})}}{\min\{\mu_{V_r}(p_{r_i}), \mu_{V_r}(p_{r_j})\}}$, $\rho_\eta(S_r) = \frac{\eta_{E_r(p_{r_i}, p_{r_j})}}{\min\{\eta_{V_r}(p_{r_i}), \eta_{V_r}(p_{r_j})\}}$,

$\rho_\nu(S_r) = \frac{\nu_{E_r(p_{r_i}, p_{r_j})}}{\max\{\nu_{V_r}(p_{r_i}), \nu_{V_r}(p_{r_j})\}}$, for $p_{r_i}, p_{r_j} \in V_r$. Since, S_r is an arbitrary subgraph G with two nodes and G is strictly balanced, so, $\rho(S_r) = \rho(G)$. This implies that $\rho_\mu(S_r) = \lambda_1$, $\rho_\eta(S_r) = \lambda_2$ and $\rho_\nu(S_r) = \lambda_3$.

That is $\mu_{E_r}(p_{r_i}, p_{r_j}) = \lambda_1 \min\{\mu_V(p_{r_i}), \mu_V(p_{r_j})\}$,

$\eta_{E_r}(p_{r_i}, p_{r_j}) = \lambda_2 \min\{\eta_V(p_{r_i}), \eta_V(p_{r_j})\}$, $\nu_{E_r}(p_{r_i}, p_{r_j}) = \lambda_3 \max\{\nu_V(p_{r_i}), \nu_V(p_{r_j})\}$ Since, S_r is arbitrary subgraph with two nodes of G, so the above relation is true for all $p_{r_i}, p_{r_j} \in V$.

Hence, $\mu_E(p_i, p_j) = \lambda_1 \min\{\mu_V(p_i), \mu_V(p_j)\}$, $\eta_E(p_i, p_j) = \lambda_2 \min\{\eta_V(p_i), \eta_V(p_j)\}$, $\nu_E(p_i, p_j) = \lambda_3 \max\{\nu_V(p_i), \nu_V(p_j)\}$, for all $p_i, p_j \in V$.

Conversely, let $G = (V, E, m_V, m_E)$ be a *PFG* where $V = \{p_1, p_2, \dots, p_n\}$. $\rho(G)$ be the density of G where, $\rho(G) = (\lambda_1, \lambda_2, \lambda_3)$. The membership function of all edges are satisfied the relations $\mu_E(p_i, p_j) = \lambda_1 \min\{\mu_V(p_i), \mu_V(p_j)\}$, $\eta_E(p_i, p_j) = \lambda_2 \min\{\eta_V(p_i), \eta_V(p_j)\}$, $\nu_E(p_i, p_j) = \lambda_3 \max\{\nu_V(p_i), \nu_V(p_j)\}$, for all $p_i, p_j \in V$. Now we have to prove that G is strictly balanced.

Let $S_t = (V_t, E_t, m_{V_t}, m_{E_t})$ be any subgraph of G, where $V_t = \{p_{t_1}, p_{t_2}, \dots, p_{t_m}\}$, $t_1, t_2, \dots, t_m \in \{1, 2, \dots, n\}$ and $t_i \neq t_j$ for all i, j.

Let $\rho(S_t) = (\rho_\mu(S_t), \rho_\eta(S_t), \rho_\nu(S_t))$ be the density of the subgraph S_t. Then,

$$\begin{aligned}\rho_\mu(S_t) =& \frac{\sum_{p_{t_i} \neq p_{t_j}} \mu_{E_t}(p_{t_i}, p_{t_j})}{\sum_{(p_{t_i}, p_{t_j}) \in E} \min\{\mu_{V_t}(p_{t_i}), \mu_{V_t}(p_{t_j})\}} \\ =& \frac{\lambda_1 \sum_{(p_{t_i}, p_{t_j}) \in E} \min\{\mu_{V_t}(p_{t_i}), \mu_{V_t}(p_{t_j})\}}{\sum_{(p_{t_i}, p_{t_j}) \in E} \min\{\mu_{V_t}(p_{t_i}), \mu_{V_t}(p_{t_j})\}} \\ =& \lambda_1\end{aligned}$$

Similarly, $\rho_\eta(S_t) = \lambda_2$ and $\rho_\nu(S_t) = \lambda_3$. Therefore, $\rho(S_t) = (\lambda_1, \lambda_2, \lambda_3)$. Since S_t is an arbitrary subgraph of *PFG* G, so, $\rho(S) = \rho(G)$ for all subgraph S of G. Hence, G is strictly balanced. □

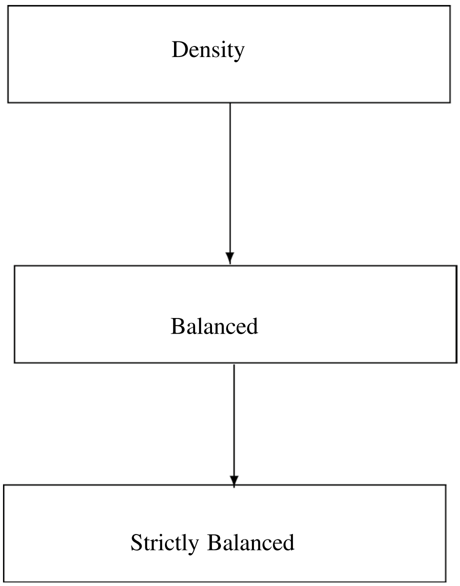
Corollary 1 *PFG* $G = (V, E, m_V, m_E)$ *is balanced iff*

$\mu_E(p_i, p_j) = \min\{\mu_V(p_i), \mu_V(p_j)\} \times \lambda_1$, $\eta_E(p_i, p_j) = \min\{\eta_V(p_i), \eta_V(p_j)\} \times \lambda_2$, $\nu_E(p_i, p_j) = \max\{\nu_V(p_i), \nu_V(p_j)\} \times \lambda_3$ *for all* $(p_i, p_j) \in E$, *where* $\rho(G) = (\lambda_1, \lambda_2, \lambda_3)$.

Corollary 2 *Let* $G = (V, E, m_V, m_E)$ *be a balanced PFG and S be any subgraph of G then* $\rho(S) = \rho(G)$ *or* $\rho(S) = (0, 0, 0)$.

Figure 4 shows the inclusion of density of a *PFG*, balanced *PFG* and strictly *PFG*.

Fig. 4 Workflow of definitions



4.1 An Algorithm

In this subsection we proposed an algorithm to test whether a *PFG* is balanced or not. A proof of correctness of the proposed algorithm is also given in this section.

Algorithm $BPFG$

Input: A PFG G.

Output: G is balanced or not balanced .

Step 1: Compute,

$$\lambda_1 = \frac{\sum\limits_{p_i \neq p_j} \mu_E(p_i, p_j)}{\sum\limits_{(p_i,p_j)\in E} \min\{\mu_V(p_i), \mu_V(p_j)\}}$$

$$\lambda_2 = \frac{\sum\limits_{p_i \neq p_j} \eta_E(p_i, p_j)}{\sum\limits_{(p_i,p_j)\in E} \min\{\eta_V(p_i), \eta_V(p_j)\}} \quad \text{and}$$

$$\lambda_3 = \frac{\sum\limits_{p_i \neq p_j} \nu_E(p_i, p_j)}{\sum\limits_{(p_i,p_j)\in E} \min\{\nu_V(p_i), \nu_V(p_j)\}}$$

where, λ_1, λ_2, λ_3 are respectively μ-density, η-density, ν-density of G, i.e., $\rho(G) = (\lambda_1, \lambda_2, \lambda_3)$.

Step 2: for $i = 1$ to n

for $j = 1$ to n $(i \neq j)$

if $(\mu_E(p_i, p_j), \eta_E(p_i, p_j), \nu_E(p_i, p_j)) = (0, 0, 0)$

or $(\mu_E(p_i, p_j), \eta_E(p_i, p_j), \nu_E(p_i, p_j))$

$= (\lambda_1 \min\{\mu_V(p_i), \mu_V(p_j)\}, \lambda_2 \min\{\eta_V(p_i), \eta_V(p_j)\}, \lambda_3 \min\{\nu_V(p_i), \nu_V(p_j)\})$

then G is balanced;

else

G is not balanced;

end $BPFG$.

The proof of correctness of the above algorithm is given by the following theorem.

Theorem 4 *Algorithm BPFG correctly tests whether a PFG is balanced or not.*

Proof Let $G = (V, E, m_V, m_E)$ be a *PFG* and let $\rho(G)$ be the density of the graph G. Then $\rho(G) = (\lambda_1, \lambda_2, \lambda_3)$. According to the definition of density function of G, we have

$$\begin{aligned}\lambda_1 &= \rho_\mu(G)\\ &= \frac{S_\mu(G)}{w_\mu(G)}\\ &= \frac{\sum_{p_i \neq p_j} \mu_E(p_i, p_j)}{\sum_{(p_i,p_j)\in E} \min\{\mu_V(p_i), \mu_V(p_j)\}}\\ \lambda_2 &= \rho_\eta(G)\\ &= \frac{S_\eta(G)}{w_\eta(G)}\\ &= \frac{\sum_{p_i \neq p_j} \eta_E(p_i, p_j)}{\sum_{(p_i,p_j)\in E} \min\{\eta_V(p_i), \eta_V(p_j)\}}\\ \lambda_3 &= \rho_\nu(G)\\ &= \frac{S_\nu(G)}{w_\nu(G)}\\ &= \frac{\sum_{p_i \neq p_j} \mu_E(p_i, p_j)}{\sum_{(p_i,p_j)\in E} \min\{\nu_V(p_i), \nu_V(p_j)\}}\end{aligned}$$

So, Algorithm *BPFG* correctly computes $\rho(G)$, the density of G.

Again from Theorem 3 and Corollary 1 of Theorem 3, we have a *PFG* $G = (V, E, m_V, m_E)$ is balanced iff $\mu_E(p_i, p_j) = \lambda_1 \min\{\mu_V(p_i), \mu_V(p_j)\}$, $\eta_E(p_i, p_j) = \lambda_2 \min\{\eta_V(p_i), \eta_V(p_j)\}$, $\nu_E(p_i, p_j) = \lambda_3 \max\{\nu_V(p_i), \nu_V(p_j)\}$, or $\mu_E(p_i, p_j) = 0$, $\eta_E(p_i, p_j) = 0$, $\nu_E(p_i, p_j) = 0$ for all $p_i, p_j \in V$.

That is G is balanced if $(\mu_E(p_i, p_j), \eta_E(p_i, p_j), \nu_E(p_i, p_j)) = (0, 0, 0)$ or $(\mu_E(p_i, p_j), \eta_E(p_i, p_j), \nu_E(p_i, p_j)) = (\lambda_1 \min\{\mu_V(p_i), \mu_V(p_j)\}, \lambda_2 \min\{\eta_V(p_i), \eta_V(p_j)\}, \lambda_3 \min\{\nu_V(p_i), \nu_V(p_j)\})$. Otherwise G is not balanced.

Hence, the above algorithm correctly check the balanced property of *PFG*. □

4.2 Illustration of Algorithm *BPFG*

The algorithm *BPFG* is illustrated for two different *PFG*s for considering all possible cases.

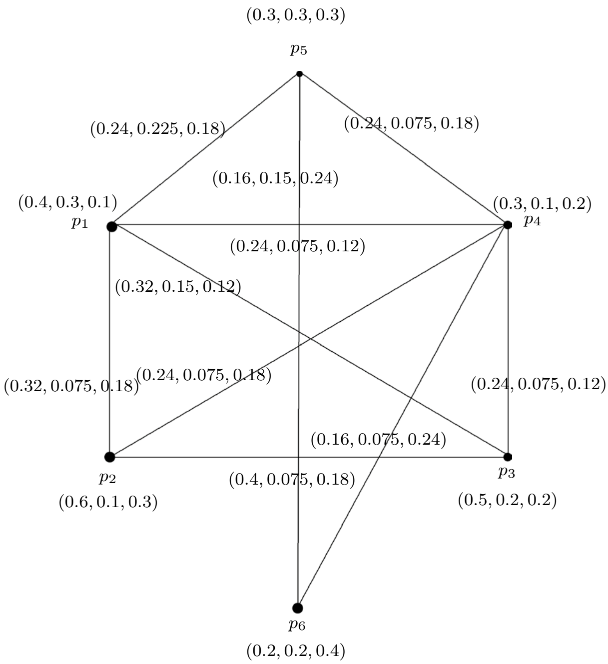
Illustration 1

Let us consider a *PFG* $G = (V, E, m_V, m_E)$ where $V = \{p_1, p_2, \dots, p_6\}$ and m_V, m_E are shown in Fig. 5.

For this graph, $\rho(G) = (0.8, 0.75, 0.6) = (\lambda_1, \lambda_2, \lambda_3)$. Therefore, $\lambda_1 = 0.8$, $\lambda_2 = 0.75$ and $\lambda_3 = 0.6$.

Iteration 1: For $i = 1$.

Fig. 5 A *PFG* for illustration 1



Let us consider a node p_1. Now compute, $(\mu_E(p_1, p_j), \eta_E(p_1, p_j), \nu_E(p_1, p_j))$ for $j = 2, 3, \ldots, 6$.

When $j = 2$.

$$\mu_E(p_1, p_2) = 0.32 = 0.8 \min\{0.4, 0.6\} = 0.8 \min\{\mu_V(p_1), \mu_V(p_2)\}.$$
$$\eta_E(p_1, p_2) = 0.075 = 0.75 \min\{0.3, 0.1\} = 0.7 \min\{\eta_V(p_1), \eta_V(p_2)\}.$$
$$\nu_E(p_1, p_2) = 0.18 = 0.6 \min\{0.1, 0.3\} = 0.6 \max\{\nu_V(p_1), \nu_V(p_2)\}.$$

Therefore, $(\mu_E(p_1, p_2), \eta_E(p_1, p_2), \nu_E(p_1, p_2)) = (\lambda_1 \min\{\mu_V(p_1), \mu_V(p_2)\}, \lambda_2 \min\{\eta_V(p_1), \eta_V(p_2)\}, \lambda_3 \min\{\nu_V(p_1), \nu_V(p_2)\})$

When $j = 3$.

$$\mu_E(p_1, p_3) = 0.32 = 0.8 \min\{0.4, 0.5\} = 0.8 \min\{\mu_V(p_1), \mu_V(p_3)\}.$$
$$\eta_E(p_1, p_3) = 0.15 = 0.75 \min\{0.3, 0.2\} = 0.7 \min\{\eta_V(p_1), \eta_V(p_3)\}.$$
$$\nu_E(p_1, p_3) = 0.12 = 0.6 \min\{0.1, 0.2\} = 0.6 \max\{\nu_V(p_1), \nu_V(p_3)\}.$$

Therefore, $(\mu_E(p_1, p_3), \eta_E(p_1, p_3), \nu_E(p_1, p_3)) = (\lambda_1 \min\{\mu_V(p_1), \mu_V(p_3)\}, \lambda_2 \min\{\eta_V(p_1), \eta_V(p_3)\}, \lambda_3 \min\{\nu_V(p_1), \nu_V(p_3)\})$

When $j = 4$.

$$\mu_E(p_1, p_4) = 0.24 = 0.8 \min\{0.4, 0.3\} = 0.8 \min\{\mu_V(p_1), \mu_V(p_4)\}.$$
$$\eta_E(p_1, p_4) = 0.075 = 0.75 \min\{0.3, 0.1\} = 0.7 \min\{\eta_V(p_1), \eta_V(p_4)\}.$$
$$\nu_E(p_1, p_3) = 0.12 = 0.6 \min\{0.1, 0.2\} = 0.6 \max\{\nu_V(p_1), \nu_V(p_4)\}.$$

Therefore, $(\mu_E(p_1, p_4), \eta_E(p_1, p_4), \nu_E(p_1, p_4)) = (\lambda_1 \min\{\mu_V(p_1), \mu_V(p_4)\}, \lambda_2 \min\{\eta_V(p_1), \eta_V(p_4)\}, \lambda_3 \min\{\nu_V(p_1), \nu_V(p_4)\})$

When $j = 5$.

$$\mu_E(p_1, p_5) = 0.24 = 0.8\min\{0.4, 0.3\} = 0.8\min\{\mu_V(p_1), \mu_V(p_5)\}.$$
$$\eta_E(p_1, p_5) = 0.225 = 0.75\min\{0.3, 0.3\} = 0.7\min\{\eta_V(p_1), \eta_V(p_5)\}.$$
$$\nu_E(p_1, p_5) = 0.18 = 0.6\min\{0.1, 0.3\} = 0.6\max\{\nu_V(p_1), \nu_V(p_5)\}.$$

Therefore, $(\mu_E(p_1, p_5), \eta_E(p_1, p_5), \nu_E(p_1, p_5)) = (\lambda_1 \min\{\mu_V(p_1), \mu_V(p_5)\}, \lambda_2 \min\{\eta_V(p_1), \eta_V(p_5)\}, \lambda_3 \min\{\nu_V(p_1), \nu_V(p_5)\})$

When $j = 6$.

$$\mu_E(p_1, p_6) = 0, \eta_E(p_1, p_6) = 0, \nu_E(p_1, p_6) = 0.$$

Therefore, $(\mu_E(p_1, p_6), \eta_E(p_1, p_6), \nu_E(p_1, p_6)) = (0, 0, 0)$.

So, we can write,

$$(\mu_E(p_1, p_j), \eta_E(p_1, p_j), \nu_E(p_1, p_j)) = \begin{cases} (\lambda_1 \min\{\mu_V(p_1), \mu_V(p_j)\}, \lambda_2 \min\{\eta_V(p_1), \eta_V(p_j)\}, \lambda_3 \min\{\nu_V(p_1), \nu_V(p_j)\}) & \text{for } j = 2, 3, 4, 5 \\ (0, 0, 0) & \text{for } j = 6 \end{cases}$$

Iteration 2: For $i = 2$.

Let us consider a node p_2. Now compute, $(\mu_E(p_2, p_j), \eta_E(p_2, p_j), \nu_E(p_2, p_j))$ for $j = 3, 4, 5, 6$.

When $j = 3$.

$$\mu_E(p_2, p_3) = 0.4 = 0.8\min\{0.6, 0.5\} = 0.8\min\{\mu_V(p_2), \mu_V(p_3)\}.$$
$$\eta_E(p_2, p_3) = 0.075 = 0.75\min\{0.1, 0.2\} = 0.7\min\{\eta_V(p_2), \eta_V(p_3)\}.$$
$$\nu_E(p_2, p_3) = 0.18 = 0.6\min\{0.3, 0.2\} = 0.6\max\{\nu_V(p_2), \nu_V(p_3)\}.$$

Therefore, $(\mu_E(p_2, p_3), \eta_E(p_2, p_3), \nu_E(p_2, p_3)) = (\lambda_1 \min\{\mu_V(p_2), \mu_V(p_3)\}, \lambda_2 \min\{\eta_V(p_2), \eta_V(p_3)\}, \lambda_3 \min\{\nu_V(p_2), \nu_V(p_3)\})$.

When $j = 4$.

$$\mu_E(p_2, p_4) = 0.24 = 0.8\min\{0.6, 0.3\} = 0.8\min\{\mu_V(p_2), \mu_V(p_4)\}.$$
$$\eta_E(p_2, p_4) = 0.075 = 0.75\min\{0.1, 0.1\} = 0.7\min\{\eta_V(p_2), \eta_V(p_4)\}.$$
$$\nu_E(p_2, p_3) = 0.18 = 0.6\min\{0.3, 0.2\} = 0.6\max\{\nu_V(p_2), \nu_V(p_4)\}.$$

Therefore, $(\mu_E(p_2, p_4), \eta_E(p_2, p_4), \nu_E(p_2, p_4)) = (\lambda_1 \min\{\mu_V(p_2), \mu_V(p_4)\}, \lambda_2 \min\{\eta_V(p_2), \eta_V(p_4)\}, \lambda_3 \min\{\nu_V(p_2), \nu_V(p_4)\})$.

When $j = 5$.

$$\mu_E(p_2, p_5) = 0, \eta_E(p_2, p_5) = 0, \nu_E(p_2, p_5) = 0.$$

Therefore, $(\mu_E(p_2, p_5), \eta_E(p_2, p_5), \nu_E(p_2, p_5)) = (0, 0, 0)$.

When $j = 6$.

$$\mu_E(p_2, p_6) = 0, \eta_E(p_2, p_6) = 0, \nu_E(p_2, p_6) = 0.$$

Therefore, $(\mu_E(p_2, p_6), \eta_E(p_2, p_6), \nu_E(p_2, p_6)) = (0, 0, 0)$.

Therefore, we can write,

$$(\mu_E(p_2,p_j),\eta_E(p_2,p_j),\nu_E(p_2,p_j)) = \begin{cases} (\lambda_1\min\{\mu_V(p_2),\mu_V(p_j)\},\lambda_2\min\{\eta_V(p_2),\eta_V(p_j)\},\lambda_3\min\{\nu_V(p_2),\nu_V(p_j)\}) & \text{for } j=1,3,4\\ (0,0,0) & \text{for } j=5,6\end{cases}$$

Similarly,

Iteration 3:

$$(\mu_E(p_3,p_j),\eta_E(p_3,p_j),\nu_E(p_3,p_j)) = \begin{cases} (\lambda_1\min\{\mu_V(p_3),\mu_V(p_j)\},\lambda_2\min\{\eta_V(p_3),\eta_V(p_j)\},\lambda_3\min\{\nu_V(p_3),\nu_V(p_j)\}) & \text{for } j=1,2,4\\ (0,0,0) & \text{for } j=5,6\end{cases}$$

Iteration 4:

$$(\mu_E(p_4,p_j),\eta_E(p_4,p_j),\nu_E(p_4,p_j)) = (\lambda_1\min\{\mu_V(p_4),\mu_V(p_j)\},\lambda_2\min\{\eta_V(p_4),\eta_V(p_j)\},\lambda_3\min\{\nu_V(p_4),\nu_V(p_j)\}) \text{ for } j=1,2,3,5,6$$

Iteration 5:

$$(\mu_E(p_5,p_j),\eta_E(p_5,p_j),\nu_E(p_5,p_j)) = \begin{cases} (\lambda_1\min\{\mu_V(p_5),\mu_V(p_j)\},\lambda_2\min\{\eta_V(p_5),\ \eta_V(p_j)\},\lambda_3\min\{\nu_V(p_5),\nu_V(p_j)\}) & \text{for} j=1,4,6\\ (0,0,0) & \text{for } j=2,3\end{cases}$$

Therefore, $(\mu_E(p_i,p_j),\eta_E(p_i,p_j),\nu_E(p_i,p_j)) = (\lambda_1\min\{\mu_V(p_i),\mu_V(p_j)\},\lambda_2\min\{\eta_V(p_i),\eta_V(p_j)\},\lambda_3\min\{\nu_V(p_i),\nu_V(p_j)\})$ or, $(\mu_E(p_i,p_j),\eta_E(p_i,p_j),\nu_E(p_i,p_j)) = (0,0,0)$ for all $i,j=1,2,\ldots,6$ and $i\neq j$.

Hence, G is a balanced *PFG*.

Illustration 2

Here we consider another *PFG* $G=(V,E,m_V,m_E)$ where $V=\{p_1,p_2,\ldots,p_5\}$ and m_V, m_E are shown in Fig. 6.

For this graph, $\rho(G)=(0.7,0.5,0.8)=(\lambda_1,\lambda_2,\lambda_3)$. Therefore, $\lambda_1=0.7$, $\lambda_2=0.5$ and $\lambda_3=0.8$. Now we check whether G is balanced or not using Algorithm *BBPFG*.

Iteration 1: For $i=1$.

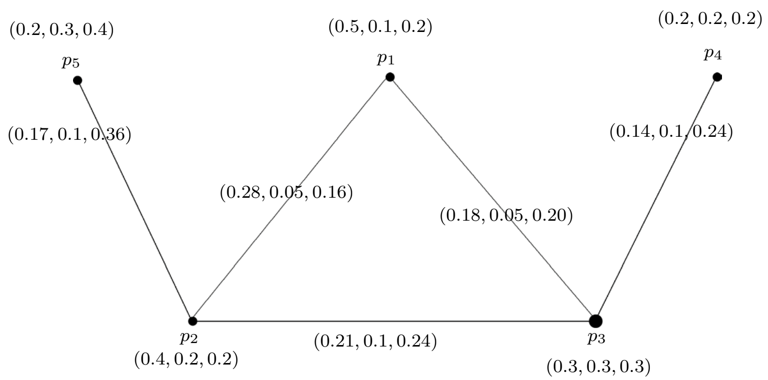


Fig. 6 A *PFG* for illustration 2

Let us consider a node p_1. Now compute, $(\mu_E(p_1,p_j),\eta_E(p_1,p_j),\nu_E(p_1,p_j))$ for $j=2,3,\ldots,5$.

When $j=2$.

$$\mu_E(p_1,p_2)=0.28=0.7\min\{0.5,0.4\}=0.7\min\{\mu_V(p_1),\mu_V(p_2)\}.$$
$$\eta_E(p_1,p_2)=0.05=0.5\min\{0.1,0.2\}=0.5\min\{\eta_V(p_1),\eta_V(p_2)\}.$$
$$\nu_E(p_1,p_2)=0.16=0.8\min\{0.2,0.2\}=0.8\max\{\nu_V(p_1),\nu_V(p_2)\}.$$

Therefore, $(\mu_E(p_1,p_2),\eta_E(p_1,p_2),\nu_E(p_1,p_2))=(\lambda_1\min\{\mu_V(p_1),\mu_V(p_2)\},\lambda_2\min\{\eta_V(p_1),\eta_V(p_2)\},\lambda_3\min\{\nu_V(p_1),\nu_V(p_2)\})$.

When $j=3$.

$$\mu_E(p_1,p_3)=0.18=0.7\min\{0.5,0.3\}=0.7\min\{\mu_V(p_1),\mu_V(p_3)\}.$$
$$\eta_E(p_1,p_3)=0.05=0.5\min\{0.1,0.3\}=0.5\min\{\eta_V(p_1),\eta_V(p_3)\}.$$
$$\nu_E(p_1,p_3)=0.20\neq0.8\min\{0.2,0.3\}=0.8\max\{\nu_V(p_1),\nu_V(p_3)\}.$$

Thus, $(\mu_E(p_1,p_3),\eta_E(p_1,p_3),\nu_E(p_1,p_3))\neq(\lambda_1\min\{\mu_V(p_1),\mu_V(p_3)\},\lambda_2\min\{\eta_V(p_1),\eta_V(p_3)\},\lambda_3\min\{\nu_V(p_1),\nu_V(p_3)\})$.

Hence, G is not a balanced PFG.

Theorem 5 *Let $G=(V,E,m_V,m_E)$ be a complete PFG then $\rho(G)=(1,1,1)$.*

Proof Let $G=(V,E,m_V,m_E)$ be a complete PFG, where $m_V=(\mu_V,\eta_V,\nu_V)$ and $m_E=(\mu_E,\eta_E,\nu_E)$, then $\mu_E(p_i,p_j)=\min\{\mu_V(p_i),\mu_V(p_j)\}$, $\eta_E(p_i,p_j)=\min\{\eta_V(p_i),\eta_V(p_j)\}$, $\nu_E(p_i,p_j)=\max\{\nu_V(p_i),\nu_V(p_j)\}$ for all $p_i,p_j\in V$. Since $\rho(G)$ is the density of G then $\rho(G)=(\rho_\mu(G),\rho_\eta(G),\rho_\nu(G))$

Now,

$$\begin{aligned}\rho_\mu(G)=&\frac{S_\mu(G)}{w_\mu(G)}\\ =&\frac{\sum_{p_i\neq p_j}\mu_E(p_i,p_j)}{\sum_{(p_i,p_j)\in E}\min\{\mu_V(p_i),\mu_V(p_j)\}}\\ =&\frac{\sum_{(p_i,p_j)\in E}\min\{\mu_V(p_i),\mu_V(p_j)\}}{\sum_{(p_i,p_j)\in E}\min\{\mu_V(p_i),\mu_V(p_j)\}}\\ =&1\end{aligned}$$

Similarly, $\rho_\eta(G)=1$ and $\rho_\nu(G)=1$. Therefore, $\rho(G)=(1,1,1)$. □

Theorem 6 *Any single valued complete picture fuzzy graph $G = (V, E, m_V, m_E)$ is strictly balanced.*

Proof Let $G = (V, E, m_V, m_E)$ be a complete *PFG*, where $m_V = (\mu_V, \eta_V, \nu_V)$ and $m_E = (\mu_E, \eta_E, \nu_E)$. Let $\rho(G)$ be the density of G. Since G is complete *PFG* then $\rho(G) = (1, 1, 1)$. So, $\rho_\mu(G) = 1$, $\rho_\eta(G) = 1$ and $\rho_\nu(G) = 1$. Again since G is complete *PFG* then, $\mu_E(p_i, p_j) = \min\{\mu_V(p_i), \mu_V(p_j)\}$, $\eta_E(p_i, p_j) = \min\{\eta_V(p_i), \eta_V(p_j)\}$, $\nu_E(p_i, p_j) = \max\{\nu_V(p_i), \nu_V(p_j)\}$ for all $p_i, p_j \in V$. Now the above relations can be written as $\mu_E(p_i, p_j) = \lambda_1 \min\{\mu_V(p_i), \mu_V(p_j)\}$, $\eta_E(p_i, p_j) = \lambda_2 \min\{\eta_V(p_i), \eta_V(p_j)\}$, $\mu_E(p_i, p_j) = \lambda_3 \min\{\nu_V(p_i), \nu_V(p_j)\}$, for all $p_i, p_j \in V$, where $\lambda_1 = \lambda_2 = \lambda_3 = 1$. Then by Theorem 3 we can say that G is strictly balanced. □

Observation 1 Every average *PFG* is strictly balanced.

Observation 2 Every strong *PFG* is balanced.

Theorem 7 *Let $G = (V, E, m_V, m_E)$ be a strictly balanced PFG then $\bar{G}$ is also strictly balanced and $\rho(G) + \rho(\bar{G}) = (1, 1, 1)$.*

Proof Since, $G = (V, E, m_V, m_E)$ is a strictly balanced *PFG*, therefore there exists three real numbers $\lambda_1, \lambda_2, \lambda_3$ in [0, 1] such that $\mu_E(p_i, p_j) = \lambda_1 \min\{\mu_V(p_i), \mu_V(p_j)\}$, $\eta_E(p_i, p_j) = \lambda_2 \min\{\eta_V(p_i), \eta_V(p_j)\}$, $\nu_E(p_i, p_j) = \lambda_3 \max\{\nu_V(p_i), \nu_V(p_j)\}$, for all $p_i, p_j \in V$ where $\rho(G) = (\lambda_1, \lambda_2, \lambda_3)$. Since, $\bar{G} = (\bar{V}, \bar{E}, \bar{m_V}, \bar{m_E})$ be the complement of G so $\bar{V} = V$ and $\bar{m_V} = (\bar{\mu_V}, \bar{\eta_V}, \bar{\nu_V})$, $\bar{m_E} = (\bar{\mu_E}, \bar{\eta_E}, \bar{\nu_E})$, $\bar{\mu_V}(p_i) = \mu_V(p_i)$, $\bar{\eta_V}(p_i) = \eta_V(p_i)$, $\bar{\nu_V}(p_i) = \nu_V(p_i)$, for all $p_i \in V$.

$\bar{\mu_E}(p_i, p_j) = \min\{\mu_V(p_i), \mu_V(p_j)\} - \mu_E(p_i, p_j)$, $\bar{\eta_E}(p_i, p_j) = \min\{\eta_V(p_i), \eta_p(p_j)\} - \eta_E(p_i, p_j)$ and $\bar{\nu_E}(p_i, p_j) = \max\{\nu_V(p_i), \nu_V(p_j)\} - \nu_E(p_i, p_j)$, for all $p_i, p_j \in V$.

Now we have $\bar{\mu_E}(p_i, p_j) = \min\{\mu_V(p_i), \mu_V(p_j)\} - \lambda_1 \min\{\mu_V(p_i), \mu_V(p_j)\} = (1 - \lambda_1) \min\{\mu_V(p_i), \mu_V(p_j)\}$, for all $p_i, p_j \in V$.

Similarly, $\bar{\eta_E}(p_i, p_j) = (1 - \lambda_1) \min\{\eta_V(p_i), \eta_V(p_j)\}$, for all $p_i, p_j \in V$ and $\bar{\eta_E}(p_i, p_j) = (1 - \lambda_1) \min\{\eta_V(p_i), \eta_V(p_j)\}$, for all $p_i, p_j \in V$. Therefore, for the graph $\bar{G}$ there exists three real numbers $1 - \lambda_1, 1 - \lambda_2, 1 - \lambda_3$ which lie on [0, 1] such that the above relation holds. Hence, by Theorem 3, $\bar{G}$ is strictly balanced with $\rho(\bar{G}) = (1 - \lambda_1, 1 - \lambda_2, 1 - \lambda_3)$. □

Example 3 A *PFG* G and its complement $\bar{G}$ are shown in Fig. 7. Here the graph G is strictly balanced and $\rho(G) = (0.7, 0.7, 0.7)$ and also $\bar{G}$ is strictly balanced and $\rho(\bar{G}) = (0.3, 0.3, 0.3)$. Therefore, $\rho(G) + \rho(\bar{G}) = (1, 1, 1)$.

Theorem 8 *Let $G = (V, E, m_V, m_E)$ be an average PFG then $\rho(\bar{G}) = (0.5, 0.5, 0.5)$.*

Proof Since, $G = (V, E, m_V, m_E)$ is an average *PFG* then from Theorem 3, $\rho(G) = (0.5, 0.5, 0.5)$. Since every *PFG* is strictly balanced then by Theorem 10, $\rho(G) + \rho(\bar{G}) = (1, 1, 1)$. That is $\rho(\bar{G}) = (0.5, 0.5, 0.5)$ □

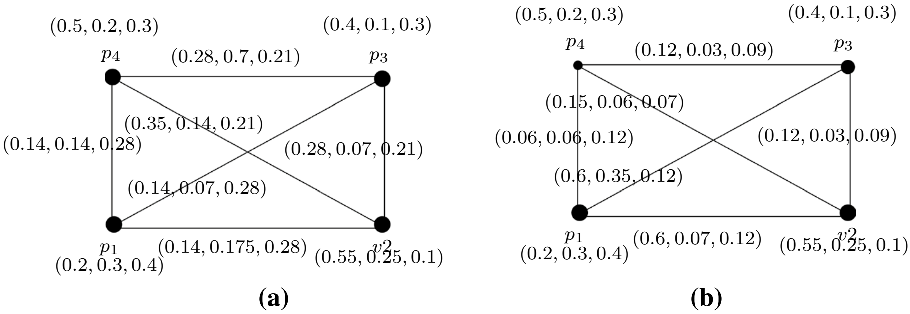


Fig. 7 **a** A *PFG G* and **b** its Complement $\bar{G}$

Definition 16 An isomorphism between two *PFG* $G_1 = (V_1, E_1, m_{V_1}, m_{E_1})$ and $G_2 = (V_2, E_2, m_{V_2}, m_{E_2})$ is a bijection mapping $f : V_1 \rightarrow V_2$ which satisfies the following condition:

(*i*) $\mu_{V_1}(p_i) = \mu_{V_2}(f(p_i))$, $\eta_{V_1}(p_i) = \eta_{V_2}(f(p_i))$, $\nu_{V_1}(p_i) = \nu_{V_2}(f(p_i))$ for all $p_i \in V$.
(*ii*) $\mu_{E_1}(p_i,p_j) = \mu_{E_2}(f(p_i),f(p_j))$, $\eta_{E_1}(p_i,p_j) = \eta_{E_2}(f(p_i),f(p_j))$, $\nu_{E_1}(p_i,p_j) = \nu_{E_2}(f(p_i),f(p_j))$ for all $(p_i,p_j) \in E_1$.

Theorem 9 *Let* $G_1 = (V_1, E_1, m_{V_1}, m_{E_1})$ *and* $G_2 = (V_2, E_2, m_{V_2}, m_{E_2})$ *be two isomorphic PFG. Then if* G_1 *is balanced then* G_2 *is balanced and vice versa.*

Proof An isomorphism between two *PFG* G_1 and G_2 is a mapping $f : V_1 \rightarrow V_2$ which satisfies the following conditions:

(*i*) $\mu_{V_1}(p_i) = \mu_{V_2}(f(p_i))$, $\eta_{V_1}(p_i) = \eta_{V_2}(f(p_i))$, $\nu_{V_1}(p_i) = \nu_{V_2}(f(p_i))$ for all $p_i \in V$.
(*ii*) $\mu_{E_1}(p_i,p_j) = \mu_{E_2}(f(p_i),f(p_j))$, $\eta_{E_1}(p_i,p_j) = \eta_{E_2}(f(p_i),f(p_j))$, $\nu_{E_1}(p_i,p_j) = \nu_{E_2}(f(p_i),f(p_j))$ for all $(p_i,p_j) \in E_1$. Then

$$\sum_{p_i \in V_1} \mu_{V_1}(p_i) = \sum_{q_i \in V_2} \mu_{V_2}(q_i)$$
$$\sum_{p_i \in V_1} \eta_{V_1}(p_i) = \sum_{q_i \in V_2} \eta_{V_2}(q_i)$$
$$\sum_{p_i \in V_1} \nu_{V_1}(p_i) = \sum_{q_i \in V_2} \nu_{V_2}(q_i)$$
$$\sum_{(p_i,p_j) \in E_1} \mu_{E_1}(p_i, p_j) = \sum_{(q_i,q_j) \in E_2} \mu_{E_2}(q_i, q_j)$$
$$\sum_{(p_i,p_j) \in E_1} \eta_{E_1}(p_i, p_j) = \sum_{(q_i,q_j) \in E_2} \eta_{E_2}(q_i, q_j)$$
$$\sum_{(p_i,p_j) \in E_1} \nu_{E_1}(p_i, p_j) = \sum_{(q_i,q_j) \in E_2} \nu_{E_2}(q_i, q_j)$$

Let H_1 be any arbitrary subgraph of a *PFG* G_1 and H_2 be that of G_2. Therefore H_1 and H_2 are also isomorphic.

Let $H_1 = (V_1', E_1', m_{V_1}', m_{E_1}')$, $H_2 = (V_2', E_2', m_{V_2}', m_{E_2}')$. Let G_1 is balanced. Therefore $\rho(H_1) \le \rho(G_1)$, that is $\rho_\mu(H_1) \le \rho_\mu(G_1)$, $\rho_\eta(H_1) \le \rho_\eta(G_1)$, $\rho_\nu(H_1) \le \rho_\nu(G_1)$.

Now

$$\rho_\mu(H_1) \le \rho_\mu(G_1)\&\text{gives}$$

$$\frac{\sum_{p_i,p_j \in V_1'} \mu_{E_1'}(p_i,p_j)}{\sum_{(p_i,p_j)\in E_1'} \min\{\mu_{V_1'}(p_i), \mu_{V_1'}(p_j)\}} \le \frac{\sum_{p_i,p_j \in V_1} \mu_{E_1}(p_i,p_j)}{\sum_{(p_i,p_j)\in E_1} \min\{\mu_{V_1}(p_i), \mu_{V_1}(p_j)\}}$$

$$\frac{\sum_{q_i,q_j r \in V_2'} \mu_{E_2'}(q_i,q_j)}{\sum_{(q_i,q_j)\in E_2'} \min\{\mu_{V_2'}(q_i), \mu_{V_2'}(q_j)\}} \le \frac{\sum_{q_i,q_j \in V_1} \mu_{E_2}(q_i,q_j)}{\sum_{(q_i,q_j)\in E_2} \min\{\mu_{V_1}(q_i), \mu_{V_1}(q_j)\}}$$

Therefore, $\rho_\mu(H_2) \le \rho_\mu(G_2)$. Similarly, $\rho_\eta(H_2) \le \rho_\eta(G_2)$ and $\rho_\nu(H_2) \le \rho_\nu(G_2)$. Since H_1 is arbitrary and H_2 be corresponding isomorphic subgraph of G_2, therefore H_2 is balanced. Similarly we can introduced a function $f_1 : V_2 \to V_1$ since G_1 and G_2 are isomorphic. We can proceed in similar way and prove that G_1 is balanced when G is balanced. □

Definition 17 $G_1 = (V_1, E_1, m_{V_1}, m_{E_1})$ and $G_2 = (V_2, E_2, m_{V_2}, m_{E_2})$ be two *PFG*s where,

(i) $V = V_1 \times V_2$ and $m_{V_i} = (\mu_{V_i}, \eta_{V_i}, \nu_{V_i})$, $m_{E_i} = (\mu_{E_i}, \eta_{E_i}, \nu_{E_i})$ for $i = 1, 2$.
(ii) $E = \{(p_i, q_i)(p_j, q_j) : (p_i, p_j) \in E_1, (q_i, q_j) \in E_2\}$.

Then the direct product of G_1 and G_2 is a *PFG* and is denoted by $G_1 \sqcap G_2 = (V, E, m_V, m_E)$ where,

$(\mu_{V_1} \sqcap \mu_{V_2})(p_i, q_i) = \min\{\mu_{V_1}(p_i), \mu_{V_2}(q_i)\}$, $(\eta_{V_1} \sqcap \eta_{V_2})(p_i, q_i) = \min\{\eta_{V_1}(p_i), \eta_{V_2}(q_i)\}$, $(\nu_{V_1} \sqcap \nu_{V_2})(p_i, q_i) = \min\{\nu_{V_1}(p_i), \nu_{V_2}(q_i)\}$ for all $(p_i, q_i) \in V_1 \times V_2$. and $(\mu_{E_1} \sqcap \mu_{E_2})$ $((p_i, q_i)(p_j, q_j)) = \min\{\mu_{E_1}(p_i, p_j), \mu_{E_2}(q_i, q_j)\}$, $(\eta_{E_1} \sqcap \eta_{E_2})((p_i, q_i)(p_j, q_j)) = \min\{\eta_{E_1}(p_i, p_j),$ $\eta_{E_2}(q_i, q_j)\}$, $(\nu_{E_1} \sqcap \nu_{E_2})((p_i, q_i)(p_j, q_j)) = \min\{\nu_{E_1}(p_i, p_j), \nu_{E_2}(q_i, q_j)\}$ for all $(p_i, p_j) \in E_1, (q_i, q_j) \in E_2$

Example 4 Here we consider two *PFG*s G_1 and G_2 (See Fig. 8) and their direct product $G_1 \sqcap G_2$ (See Fig. 9). The membership function of edges of $G_1 \sqcap G_2$ are shown in the Table 2. From this table we see that the positive, neutral, negative membership value of the edge joining the nodes p_2q_2 and p_1q_1 in the graph $G_1 \sqcap G_2$ are 0.28, 0.08, 0.3. Also from Table 2, we see that there is no edge between the nodes p_1q_2 and p_1q_1. Same for the other edges of the graph $G_1 \sqcap G_2$.

Theorem 10 $G_1 = (V_1, E_1, m_{V_1}, m_{E_1})$ *and* $G_2 = (V_2, E_2, m_{V_2}, m_{E_2})$ *be two complete PFGs. Then the direct product of* G_1 *and* G_2 *are strong PFG.*

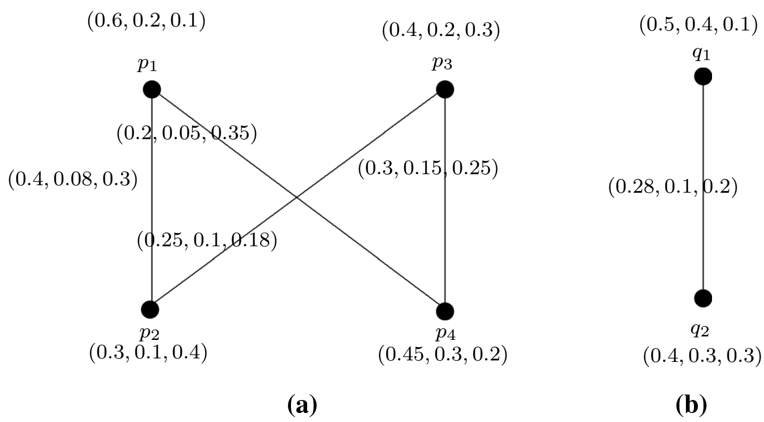


Fig. 8 **a** G_1 and **b** G_2

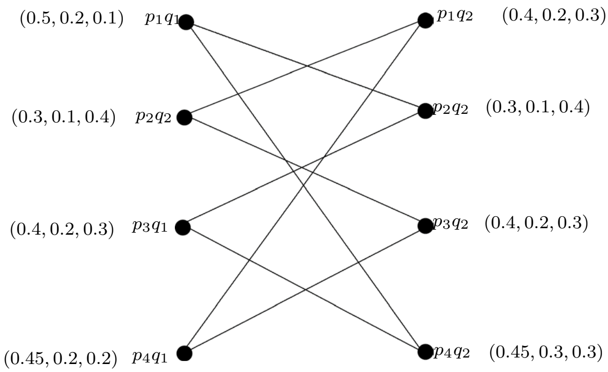


Fig. 9 The direct product $G_1 \sqcap G_2$

Proof Since, $G_1 = (V_1, E_1, m_{V_1}, m_{E_1})$ and $G_2 = (V_2, E_2, m_{V_2}, m_{E_2})$ be two PFGs be two complete PFG. Then $\mu_{E_1}(p_i, p_j) = \min\{\mu_{V_1}(p_i), \mu_{V_1}(p_j)\}$, $\eta_{E_1}(p_i, p_j) = \min\{\eta_{V_1}(p_i), \eta_{V_1}(p_j)\}$, $\nu_{E_1}(p_i, p_j) = \max\{\nu_{V_1}(p_i), \nu_{V_1}(p_j)\}$ for all $p_i, p_j \in V$.
and $\mu_{E_2}(p_i, p_j) = \min\{\mu_{V_2}(p_i), \mu_{V_2}(p_j)\}$, $\eta_{E_2}(p_i, p_j) = \min\{\eta_{V_2}(p_i), \eta_{V_2}(p_j)\}$, $\nu_{E_2}(p_i, p_j) = \max\{\nu_{V_2}(p_i), \nu_{V_2}(p_j)\}$ for all $p_i, p_j \in V$. Now, $G_1 \sqcap G_2$ be the direct product of G_1 and G_2, whose edge set $E = \{((p_i, q_i), (p_j, q_j)) : (p_i, p_j) \in E_1, (q_i, q_j) \in E_2\}$. Then

$$\begin{aligned}
(\mu_{E_1} \sqcap \mu_{E_2})((p_i, q_i)(p_j, q_j)) &= \min\{\mu_{E_1}(p_i, p_j), \mu_{E_2}(q_i, q_j)\} \\
&= \min\{\min\{\mu_{V_1}(p_i), \mu_{V_1}(p_j)\}, \min\{\mu_{V_2}(q_i), \mu_{V_2}(q_j)\}\} \\
&= \min\{\min\{\mu_{V_1}(p_i), \mu_{V_2}(q_i)\}, \min\{\mu_{V_1}(p_j), \mu_{V_2}(q_j)\}\} \\
&= \min\{\mu_{V_1} \sqcap \mu_{V_2}(p_i, q_i), \mu_{V_2} \sqcap \mu_{V_2}(p_j, q_j)\} \\
&\quad \text{for all}(p_i, p_j) \in E_1, (p_j, q_j) \in E_2. \\
&= \min\{\mu_{V_1} \sqcap \mu_{V_2}(p_i, q_i), \mu_{V_2} \sqcap \mu_{V_2}(p_j, q_j)\} \\
&\quad \text{for all}((p_i, q_i), (p_j, q_j)) \in E.
\end{aligned}$$

Similarly, $(\eta_{E_1} \sqcap \eta_{E_2})((p_i, q_i)(p_j, q_j)) = \min\{\eta_{V_1} \sqcap \eta_{V_2}(p_i, q_i), \eta_{V_2} \sqcap \eta_{V_2}(p_j, q_j)\}$

Table 2 Membership function of edges of $G_1 \sqcap G_2$

$G_1 \sqcap G_2$	p_1q_1	p_1q_2	p_2q_1	p_2q_2
p_1q_1	–	–	–	(0.28, 0.08, 0.3)
p_1q_2	–	–	(0.28, 0.08, 0.3)	–
p_2q_1	–	(0.28, 0.8, 0.3)	–	–
p_2q_2	(0.28, 0.08, 0.3)	–	–	–
p_3q_1	–	–	–	(0.2, 0.05, 0.35)
p_3q_2	–	–	(0.2, 0.05, 0.35)	–
p_4q_1	–	(0.25, 0.1, 0.2)	–	–
p_4q_2	(0.25, 0.1, 0.2)	–	–	–
$G_1 \sqcap G_2$	p_3q_1	p_3q_2	p_4q_1	p_4q_2
p_1q_1	–	–	–	(0.25, 0.1, 0.2)
p_1q_2	–	–	(0.25, 0.1, 0.2)	–
p_2q_1	–	(0.2, 0.05, 0.35)	–	–
p_2q_2	(0.2, 0.05, 0.35)	–	–	–
p_3q_1	–	–	–	(0.28, 0.1, 0.25)
p_3q_2	–	–	(0.28, 0.1, 0.25)	–
p_4q_1	–	(0.28, 0.1, 0.25)	–	–
p_4q_2	(0.28, 0.1, 0.25)	–	–	–

for all $((p_i, q_i), (p_j, q_j)) \in E$ and $(\nu_{E_1} \sqcap \nu_{E_2})((p_i, q_i)(p_j, q_j)) = \min\{\nu_{V_1} \sqcap \nu_{V_2}(p_i, q_i), \nu_{V_2} \sqcap \nu_{V_2}(p_j, q_j)\}$ for all $((p_i, q_i), (p_j, q_j)) \in E$. This shows that $G_1 \sqcap G_2$ is strong *PFG*. □

Theorem 11 *Let $G_1 = (V_1, E_1, m_{V_1}, m_{E_1})$ and $G_2 = (V_2, E_2, m_{V_2}, m_{E_2})$ are two balanced PFG and $\rho(G_1) = \rho(G_2)$, then $G_1 \sqcap G_2$ is balanced and $\rho(G_1) = \rho(G_2) = \rho(G_1 \sqcap G_2)$.*

Proof Let G_1 and G_2 be two balanced *PFG* and $\rho(G_1) = \rho(G_2) = (\lambda_1, \lambda_2, \lambda_3)$, where $\lambda_1, \lambda_2, \lambda_3$ are three real numbers belongs to [0,1]. Since, G_1 and G_2 be two balanced *PFG* then there exist $(p_i, p_j) \in E_1$ and $(q_i, q_j) \in E_2$ such that,

$$\mu_{E_1}(p_i, p_j) = \lambda_1 \min\{\mu_{V_1}(p_i), \mu_{V_1}(p_j)\}$$
$$\eta_{E_1}(p_i, p_j) = \lambda_2 \min\{\eta_{V_1}(p_i), \eta_{V_1}(p_j)\}$$
$$\nu_{E_1}(p_i, p_j) = \lambda_3 \max\{\nu_{V_1}(p_i), \nu_{V_1}(p_j)\}, \text{ for all } (p_i, p_j) \in E_1 \text{and}$$
$$\mu_{E_2}(q_i, q_j) = \lambda_1 \min\{\mu_{V_2}(q_i), \mu_{V_2}(q_j)\}$$
$$\eta_{E_2}(q_i, q_j) = \lambda_2 \min\{\eta_{V_2}(q_i), \eta_{V_2}(q_j)\}$$
$$\nu_{E_2}(q_i, q_j) = \lambda_3 \max\{\nu_{V_2}(q_i), \nu_{V_2}(q_j)\}, \text{ for all } (q_i, q_j) \in E_2$$

Now

Fig. 10 **a** G_1 and **b** G_2

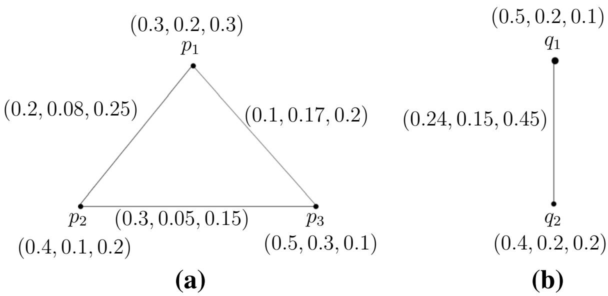
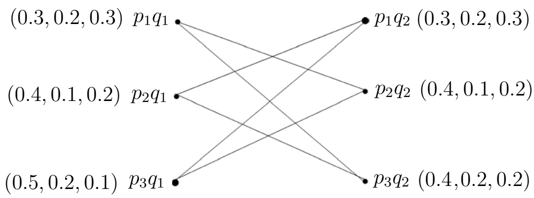


Fig. 11 $G_1 \sqcap G_2$



$$
\begin{aligned}
(\mu_{E_1} \sqcap \mu_{E_2})((p_i, q_i)(p_j, q_j)) &= \min\{\mu_{E_1}(p_i, p_j), \mu_{E_2}(q_i, q_j)\} \\
&= \min\{\lambda_1 \min\{\mu_{V_1}(p_i, \mu_{V_1}(p_j)\}, \\
&\quad \lambda_1 \min\{\mu_{V_2}(q_i), \mu_{V_2}(q_j)\}\} \\
&= \lambda_1 \min\{\min\{\mu_{V_1}(p_i), \mu_{V_2}(q_i)\}, \\
&\quad \min\{\mu_{V_1}(p_j), \mu_{V_2}(q_j)\}\} \\
&= \lambda_1 \min\{(\mu_{V_1} \sqcap \mu_{V_2})(p_i, q_i), (\mu_{V_1} \sqcap \mu_{V_2})(p_j, q_j)\} \\
&\quad \text{for all}(p_i, p_j) \in E_1, (p_j, q_j) \in E_2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&(\eta_{E_1} \sqcap \eta_{E_2})((p_i, q_i)(p_j, q_j)) = \lambda_1 \min\{(\eta_{V_1} \sqcap \eta_{V_2})(p_i, q_i), (\eta_{V_1} \sqcap \eta_{V_2})(p_j, q_j)\} \\
&(\nu_{E_1} \sqcap \nu_{E_2})((p_i, q_i)(p_j, q_j)) = \lambda_1 \min\{(\nu_{V_1} \sqcap \nu_{V_2})(p_i, q_i), (\nu_{V_1} \sqcap \nu_{V_2})(p_j, q_j)\} \\
&\text{for all}(p_i, p_j) \in E_1, (p_j, q_j) \in E_2.
\end{aligned}
$$

Hence, $G_1 \sqcap G_2$ is balanced and $\rho(G_1 \sqcap G_2) = (\lambda_1, \lambda_2, \lambda_3)$. □

Corollary 3 *Let G_1, G_2 be two PFG such that $\rho(G_1) = \rho(G_2)$ then the density of $G_1 \sqcap G_2$ may or may not be equal to density of G_i, for $i = 1, 2$.*

Example 5 Let us consider two *PFG*s G_1 and G_2 in Fig. 10 where G_1 is not balanced and their direct product $G_1 \sqcap G_2$ (See Fig. 11). The membership value of all edges of $G_1 \sqcap G_2$ are shown in the table below.

For this example, $\rho(G_1) = \rho(G_2) = (0.6, 0.75, 0.75)$ and $\rho(G_1 \sqcap G_2) = (0.54, 0.7, 0.75)$. So, for this example, $\rho(G_1) = \rho(G_2)$ but it is not equal to $\rho(G_1 \sqcap G_2)$, this holds since G_2 is not balanced.

5 Application of balanced picture fuzzy graph

In this section, a novel application of balanced *PFG* to business alliance in industry is proposed. The principal of the work is to find the business partner those may be allied under certain condition described below.

Here we consider eight companies, Coal India Limited (*CIL*), Tata Limited(*TL*), Hindustan limited(*HL*), Oil and Natural Gas corporation Limited(*ONGC*), Reliance Industries(*RI*), Life Insurance corporation (*LIC*), Infosys Limited(*IL*), Aditya Birla Group(*ABG*). Any company may engage in allied business with one or more companies. so we draw a *PFG* among the eight companies where, each companies are represented as node of the graph and alliance business between two companies are connected by an edge. For example, if Coal India Limited (*CIL*) alliance with Tata Limited(*TL*), then there is an edge between *CIL* and *TL*. If there is no alliance business between Coal India Limited (*CIL*) alliance with Tata Limited(*TL*) then there is no edge between *CIL* and *TL*. Now we consider the membership function of nodes and edges as following.

For Nodes:

1. The strength and operating style of each companies referred as a positive membership degree of the node.
2. The market placement of each companies referred as a neutral membership degree of the node.
3. Poor management system of each companies gives the negative membership degree of the node.

For Edges:

1. Alliance business between two companies are successfully increasing referred as a positive membership degree of each edges.
2. Alliance business between two companies are no growth referred as a neutral membership degree of each edges.
3. Alliance business between two companies are to be failure referred as a negative membership degree of each edges.

Also, the membership value of each node and edges are shown in Fig. 12 and the Table 3 respectively. From Fig. 12, we see that the membership value of the nodes *CIL* and *TL* are (0.5, 0.3, 0.2) and (0.7, 0.2, 0.1) respectively. Also, from Table 3, we have the membership value of the edge between the nodes *CIL* and *TL* is (0.4, 0.14, 0.8). Similarly, we can find the membership values of other nodes and edges from Fig. 12 and the Table 3.

Here, the business relationship rate (density) of the graph is (0.8, 0.7, 0.4). From the above graph, $S = \{CIL, TL, IS, ONGC, RI\}$ is largest subgraph in which the relationship rate are all equal for every pair of nodes. Hence, the subgraph $S = \{CIL, TL, IS, ONGC, RI\}$ is balanced. Therefore, these five companies namely Coal India Limited, Tata Limited, Infosys Limited, Oil and Natural Gas Corporation Limited,

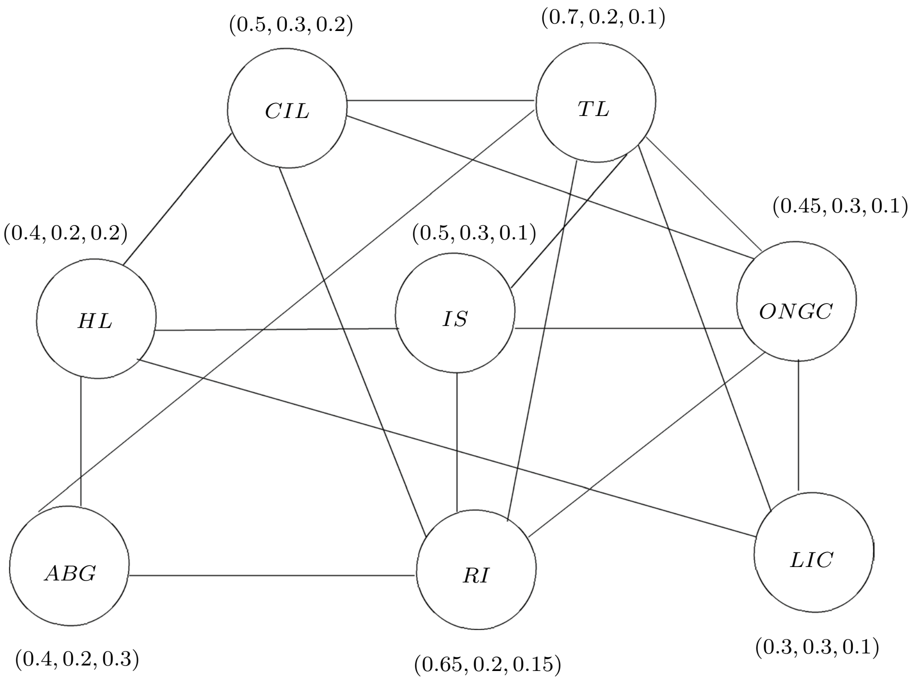


Fig. 12 A *PFG* corresponding to eight companies

Table 3 Membership values of edges

	HL	*IS*	*CIL*	*TL*
HL	–	(0.32, 0.14, 0.08)	(0.30, 0.14, 0.10)	–
IS	(0.32, 0.14, 0.08)	–	–	(0.4, 0.14, 0.04)
CIL	(0.30, 0.14, 0.10)	–	–	(0.4, 0.14, 0.08)
TL	–	(0.4, 0.14, 0.4)	(0.4, 0.14, 0.08)	–
ONGC	–	(0.36, 0.21, 0.04)	(0.36, 0.21, 0.08)	(0.36, 0.14, 0.04)
LIC	(0.24, 0.14, 0.08)	–	–	(0.24, 0.14, 0.08)
RI	–	(0.4, 0.14, 0.06)	(0.4, 0.14, 0.08)	(0.52, 0.14, 0.06)
ABG	(0.35, 0.1, 0.05)	–	–	(0.37, 0.18, 0.15)
	ONGC	*LIC*	*RI*	*ABG*
HL	–	(0.24, 0.14, 0.08)	–	(0.35, 0.1, 0.05)
IS	(0.36, 0.21, 0.04)	–	(0.4, 0.14, 0.06)	–
CIL	(0.36, 0.21, 0.08)	–	(0.4, 0.14, 0.08)	–
TL	(0.36, 0.14, 0.04)	(0.24, 0.14, 0.08)	(0.52, 0.14, 0.06)	(0.37, 0.18, 0.15)
ONGC	–	(0.21, 0.18, 0.07)	(0.36, 0.14, 0.06)	–
LIC	(0.21, 0.18, 0.07)	–	–	–
RI	(0.36, 0.14, 0.06)	–	–	(0.29, 0.17, 0.11)
ABG	–	–	(0.29, 0.17, 0.11)	–

Reliance Industries can be alliance properly. So our example helps to alliance a lot of companies with their strategies described above.

6 Discussion and conclusion

The density of a *PFG* is defined and showed that each component lie between 0 and 1. If we assign the neutral value to 0, then one can determine the density for *IFG*. If we assign neutral and negative membership values to 0, then we obtain the density for fuzzy graph. So, the density of *PFG* generalizes for *FG* and *IFG*. If the membership values of all nodes and edges are one then, we get the density for Crisp graph.

In this article, some new terminologies are defined. Some useful properties of *PFG* are studied. The definition and properties of *PFG* like, average *PFG*, balanced *PFG*, size, order, density of a *PFG* are given. The direct product of two *PFG*s is defined and presented some properties. Also, an algorithm is give to test whether a *PFG* is balanced or not. Beside this an application of balanced *PFG* to business alliance is presented. This paper will help to the new researchers to extend *PFG*. In future, we will study picture fuzzy threshold graph, picture fuzzy k-competition graph, picture fuzzy planer graph, etc.

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Declarations

Conflict of Interest There is no conflict of interest between the authors and the institute where the work has been carried out.

Ethical Approval The article does not contain any studies with human participants or animals performed by any of the authors.

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