Annals of Global Analysis and Geometry (2024) 65:15
https://doi.org/10.1007/s10455-024-09946-z

Some regularity of submetries

Alexander Lytchak[1]

Received: 2 October 2023 / Accepted: 23 January 2024 / Published online: 21 February 2024

Abstract
We discuss regularity statements for equidistant decompositions of Riemannian manifolds and for the corresponding quotient spaces. We show that any stratum of the quotient space has curvature locally bounded from both sides.

Keywords Alexandrov space · Submetry · Orbit space · Singular Riemannian foliation · Equidistant decomposition

Mathematics Subject Classification 53C20 · 53C21 · 53C23

1 Introduction

A *submetry* is a map $P : X \to Y$ between metric spaces which sends balls in X onto balls of the same radius in Y. Submetries as metric generalization of Riemannian submersions have been introduced by Berestovskii [4]. Berestovksii and Guijarro verified that a submetry between smooth complete Riemannian manifolds always is a $\mathcal{C}^{1,1}$ Riemannian submersion, but it does not need to be $\mathcal{C}^2$ [6]. Another example of a submetry is provided by a distance function $P : \mathbb{R}^n \to \mathbb{R}$ to a convex nowhere dense subset $C \subset \mathbb{R}^n$.

Submetries $P : X \to Y$ with given *total space* X are in one-to-one correspondence with equidistant decompositions of X. The correspondence assigns to P the decomposition of X into fibers of P [16, Section 2.2]. Seen this way, submetries generalize quotient maps for isometric group actions and decompositions of a complete smooth Riemannian manifold into leaves of a singular Riemannian foliation with closed leaves [1, 22, 29].

Recent appearances of submetries in many unrelated settings, [5, 10, 11, 14, 14, 20, 27, 30, 31, 34, 36, 37], make investigations of the properties of submetries a natural task, especially if the total space is a Riemannian manifold. A systematic study of submetries $P : M \to Y$ with total space a sufficiently smooth Riemannian manifold has been initiated in [16]. The present paper continues the investigations of [16] and improves some regularity statements provided there.

The author was partially supported by the DFG grants Project-ID 281071066 - TRR 191 and SPP 2026.

✉ Alexander Lytchak
alexander.lytchak@kit.edu

1 Institute of Algebra and Geometry, KIT Englerstr. 2, 76131 Karlsruhe, Germany

If the total space X is a connected (sufficiently smooth) complete Riemannian manifold M, the following structural results on the base space Y of a submetry $P : M \to Y$ have been derived in [16].

The quotient space Y locally has curvature bounded from below, [16, Proposition 3.1], see also [7, Section 4.6]. There is a canonical stratification $Y = \cup_{l=0}^{m} Y^l$, where m is the dimension of Y and Y^l consists of all points $y \in Y$ such that the tangent space $T_y Y$ has $\mathbb{R}^l$ as a direct factor, [16, Theorem 1.6]. The subset Y^l is locally convex in Y, for any l, and it is an l-dimensional manifold. The maximal-dimensional stratum Y^m, the set of *regular points* of Y, is open, dense and convex in Y.

For any point $y \in Y$, there exists some $r > 0$, such that exponential map $\exp_y$ is a well-defined homeomorphism $\exp_y : B_r(0) \to B_r(y)$ between the open r-ball $B_r(0)$ in the tangent cone $T_y Y$ around the origin 0 and the open r-ball in Y around y, [16, Theorem 1.3]. This *injectivity radius* is locally bounded from below on each stratum Y^l, but it goes to 0, when points on Y^l converge to a lower-dimensional stratum.

Our first result improves the regularity of the exponential map:

Theorem 1.1 *Let Y be a base of a submetry $P : M \to Y$ of a Riemannian manifold with locally bounded curvature. Then, for any $y \in Y$, there exist $r_0, C > 0$, such that for all $r < r_0$ the exponential map $\exp_y : B_r(0) \to B_r(y)$ is $(1 + Cr^2)$-biLipschitz.*

Here and below, we use the notion of a *Riemannian manifold with locally bounded curvature* to describe a manifold without boundary with a continuous Riemannian metric, which has curvature bounded locally from above and below in the sense of Alexandrov see [9, 15]. Any sufficiently smooth Riemannian manifold is in this class and any $\mathcal{C}^{1,1}$-submanifold of a Riemannian manifold with locally bounded curvature has locally bounded curvature, [15, Proposition 1.7].

Theorem 1.1 can be informally understood as the existence of a pointwise both-sided curvature bound at any point $y \in Y$. Indeed, for a smooth Riemannian manifold $M = Y$, the optimal number C in the statement of Theorem 1.1 is equivalent (up to a factor) to the optimal bound on the norm of the sectional curvatures at y.

In Theorem 1.1, the constant $r_0(y)$ *always* goes to 0 and $C(y)$ *usually* goes to infinity, when y converges to a lower stratum, [16, Proposition 8.9], [22, Theorem 1.1]. But both constants can be chosen *locally uniformly* on any stratum, Proposition 6.1 below. This has the following consequence, which answers [16, Question 1.12]:

Corollary 1.2 *Let M be a Riemannian manifold with locally bounded curvature and let $P : M \to Y$ be a submetry. Then, any stratum Y^l of Y is a Riemannian manifold with locally bounded curvature.*

For smooth Riemannian manifolds M, the result will be strengthened in the continuation [23]. If M is analytic, the analyticity of the maximal stratum Y^m has been verified in [18].

In general, fibers of a submetry $P : M \to Y$ can be arbitrary subsets of positive reach in M (this is a common generalization of convex subsets and $\mathcal{C}^{1,1}$ submanifolds [12, 25, 35]). However, most fibers are $\mathcal{C}^{1,1}$-submanifolds and any fiber L of P contains a $\mathcal{C}^{1,1}$-submanifold, open and dense in L. A by-product of the proof of Theorem 1.1 is the following result saying that for any submetry $P : M \to Y$, the projections from nearby P-fibers onto any manifold P-fiber is *almost* a submetry. We formulate it as a global result for compact fibers and refer to Theorem 5.2 for a more general local version.

Proposition 1.3 *Let $P : M \to Y$ be a submetry, where M has locally bounded curvature. Let L be a fiber of P which is a compact manifold. Then there exist constants $C, r_0 > 0$ such that* *for all fibers L' of P at distance $r < r_0$ from L, the closest point projection $\Pi^L : L' \to L$ is $(1 + Cr)$-Lipschitz and locally $(1 + Cr)$-open.*

Recall that a map $f : X \to Y$ between metric spaces is *locally C-open*, (other terms used are *Lipschitz open* and *co-Lipschitz*) if for any $z \in X$ there exists $r_0 > 0$, such that, for any $r < r_0$ and any $x \in B_{r_0}(z)$,

$$B_r(f(x)) \subset f(B_{Cr}(x)).$$

A submetry $P : M \to Y$ is called *transnormal* if all fibers of P are $\mathcal{C}^{1,1}$-submanifolds. Thus, for transnormal submetries with compact fibers the conclusion of Proposition 1.3 is true for all fibers. Moreover, for transnormal submetries, the constants C, r_0 appearing in Proposition 1.3 and in Theorem 1.1 depend only on the following data: A bound on the curvature and the injectivity radius of M, a bound on the injectivity radius of Y at $y = P(L)$ and a lower volume bound of Y around y, see Corollary 7.4 below. This seems to be useful for applications to the theory of Laplacian algebras developed by Ricardo Mendes and Marco Radeschi, [19].

Theorem 1.1 implies that the local decomposition of the base space Y in strata around a point y corresponds to the decomposition in strata of the tangent space at y, see Corollary 6.3 below. This has the following consequence for *transnormal submetries*:

Corollary 1.4 *Let $P : M \to Y$ be a transnormal submetry, where M has locally bounded curvature. Let $\gamma : I \to M$ be a horizontal geodesic. Then, up to discretely many values $t_i \in I$, the connected component of the fiber of P through $\gamma(t)$ has the same dimension $k = k(\gamma)$ and $P(\gamma(t))$ is contained in the stratum Y^l, with $l = l(\gamma)$.*

As a related consequence of Proposition 1.3, we prove that all holonomy maps between fibers of transnormal submetries are Lipschitz open, see Proposition 7.2 below.

We mention that all results stated here and below do not require completeness of M and are valid for *local submetries*, see Sect. 3.1.

2 Preliminaries: manifolds with bounded curvature

2.1 Notation

By d, we denote the distance in metric spaces. For a subset A of a metric space X, we denote by $d_A : X \to \mathbb{R}$ the distance function to A. A *geodesic* will denote an isometric (i.e., globally distance preserving) embedding of an interval. A local geodesic $\gamma : I \to X$ is a curve whose restrictions to small sub-intervals are geodesics.

2.2 Curvature bounds and bounds of geometry

We assume some familiarity with spaces with curvature bounded in the sense of Alexandrov. We refer the reader to [2, 9].

By a *manifold with locally bounded curvature M*, we mean a length metric space homeomorphic to a manifold without boundary, such that any point $x \in M$ has a convex neighborhood in M, which is a $\mathrm{CAT}(\kappa)$ space and an Alexandrov space of curvature bounded from below by $-\kappa$, for some $\kappa \in \mathbb{R}$. We allow the manifold M to be non-complete and the value κ to be not globally bounded on M.

The *distance coordinates* define a $\mathcal{C}^{1,1}$-atlas on any such manifold M and the Riemannian metric is Lipschitz continuous in these coordinates, [9]. Any $\mathcal{C}^{1,1}$-submanifold $N \subset M$ also has locally bounded curvature in its intrinsic metric [15, Proposition 1.7].

In any manifold M with locally bounded curvature, there is a notion of parallel translation along any Lipschitz curve [9, Section 13].

Let x be a point in a manifold M with locally bounded curvature and let $\rho > 0$ be given. We say that the *the geometry is bounded by ρ at x* if the following conditions hold true:

The ball $B = \bar{B}_{\frac{10}{\rho}}(x)$ is compact, convex and uniquely geodesic and the curvature in B is bounded from below and above by $\pm\frac{\rho^2}{100}$.

If the geometry of M at x is bounded by ρ, then the metric space $\lambda \cdot M$ rescaled by $\lambda > 0$ has at x geometry bounded by $\frac{\rho}{\lambda}$.

Let the geometry of M at x be bounded by ρ and consider the ball $B = B_{\frac{1}{\rho}}(x)$. Consider some distance coordinates on B and the Lipschitz continuous metric tensor g defining the metric of B in these coordinates. Then, there exist a sequence of smooth, uniquely geodesic metrics g_n on B such that the sectional curvatures of g_n are bounded in norm by $\frac{\rho^2}{100}$, with the following properties [9, Section 15]: The metric tensors g_n converge to g in $\mathcal{C}^{0,1}$ and the parallel transport for g_n uniformly converges to the parallel transport for g.

This approximation result allows us to prove metric statements in the smooth case first and then to obtain the general case by a limiting procedure. Mostly, a more direct but technical explanation is available without using the approximation theorem. The main additional tool available in the smooth situation are Jacobi-fields, which only have almost everywhere analogs in the general case. Readers not acquainted with the theory of [9] may always assume the total manifold M to be smooth. We prefer to stick to the more general setting of manifolds with locally bounded curvature, since this setting seems to be appropriate for the study of submetries, see [16].

2.3 Comparison of tangent vectors at different points

Let M be a Riemannian manifold with locally bounded curvature. Let $O \subset M$ be open, uniquely geodesic and convex. Given $x, z \in O$ and vectors $v \in T_x O$, $w \in T_z O$, we define $|v - w|$ to be the distance in $T_x O$ between v and the parallel transport w' of w to $T_x O$ along the geodesic zx.

This "quasi-distance" is symmetric but satisfies the triangle inequality only up to a defect depending on the geometry of O, see (2.2) below.

For linear subspaces $W_x \subset T_x O$ and $W_z \subset T_z O$ with $\dim(W_x) = \dim(W_z)$, we denote by $|W_x - W_z|$ the symmetric "quasi-distance":

$$|W_z - W_x| := \sup\{d(W_x, w')\}.$$

Here, the distance $d(W_x, w')$ to the subspace W_x is measured in $T_x O$, and the supremum is taken over all parallel translates $w' \in W_x$ of unit vectors $w \in W_z$ along the geodesic zx.

2.4 Almost flat domains

We fix $\varepsilon = 10^{-4}$ for the rest of the paper.

We say that M is *almost flat* at $x \in M$ if the geometry of M at x is bounded by ε.

If M has geometry bounded by ρ at x then, for any $\lambda \geq \frac{\rho}{\varepsilon}$, the rescaled manifold $\lambda \cdot M$ is almost flat at x.

Let M be almost flat at x and consider the open ball $O = B_{10}(x)$. Thus, O is convex and uniquely geodesic and the curvature in O is bounded from both sides by $\pm\frac{\varepsilon^2}{100} = \pm 10^{-10}$.

We refer to [8, Section 6] for the estimates stated below.

For any ball $B_r(x) \subset O$, the exponential map $\exp_x : B_r(0) \to B_r(x)$ is $(1+\varepsilon^2 \cdot r^2)$-biLipschitz on $B_r(0) \subset T_x O$, [8, Proposition 6.4].

For a triangle $\Delta = xyz \subset O$ with two sides of length l_1, l_2, the holonomy along Δ of any $v \in T_x O$ satisfies [8, Section 6.2.1]

$$||Hol^{\Delta}(v) - v|| \leq \varepsilon^2 \cdot l_1 \cdot l_2 \cdot ||v||. \tag{2.1}$$

For such Δ and arbitrary $v_x \in T_x O$, $v_y \in T_y O$, $v_z \in T_z O$, set $a := \min\{||v_x||, ||v_y||, ||v_z||\}$. Then the holonomy bound (2.1) implies a triangle inequality with a defect:

$$|v_x - v_z| \leq |v_x - v_y| + |v_y - v_z| + \varepsilon^2 \cdot l_1 \cdot l_2 \cdot a. \tag{2.2}$$

The next result is a direct consequence of [8, Proposition 6.6].

Lemma 2.1 *Assume $B_4(x) \subset O$ and $u, w \in T_x O$ with $||u||, ||w|| < 2$. Set $z = \exp_x(w)$ and $p = \exp_x(u)$ and $q = \exp_x(w+u)$. Let $\tilde{u}$ be the parallel translation of u to z along xz and $q' := \exp_z(\tilde{u})$. Then*

$$d(q, q') \leq \varepsilon^2 \cdot (||u|| + ||w||) \cdot ||u|| \cdot ||w||. \tag{2.3}$$

Therefore,

$$\angle qpq' \leq 2 \cdot \varepsilon^2 \cdot ||u|| \cdot (||u|| + ||w||). \tag{2.4}$$

And

$$|w - \exp_p^{-1}(q')| \leq 2 \cdot \varepsilon^2 \cdot (||u|| + ||w||) \cdot ||u|| \cdot ||w||. \tag{2.5}$$

The estimate (2.3) implies:

Corollary 2.2 *Let $w, u \in T_x O$ be given with $||w|| = 1$. Consider the curve $\eta(t) := \exp_x(u + tw)$. Then, for all sufficiently small t, the starting direction w_j of the geodesic connecting $\eta(0)$ and $\eta(t)$ satisfies*

$$|w_j - w| \leq 2 \cdot \varepsilon^2 \cdot ||u||^2.$$

We will need the following (definitely not optimal) lemma:

Lemma 2.3 *Let $\eta, \gamma : [0,1] \to O$ be geodesics. Set $x_t := \eta(t)$, $z_t := \gamma(t)$ and $a_t = d(x_t, z_t)$. Finally, set $v_t := \exp_{x_t}^{-1}(z_t) \in T_{x_t} M$.*

If $2 > 2a_0 > a_1$ then, for any $1 \geq t > 0$,

$$|v_t - v_0| \leq 5 \cdot a_0 \cdot t. \tag{2.6}$$

Proof Let $h \in T_{x_0} O$ be the starting direction of γ. Let $\tilde{h}$ be the parallel translation of h to z_0. Set $\tilde{z}_t = \exp_{z_0}(t \cdot \tilde{h})$ and $\tilde{v}_t = \exp_{x_t}^{-1}(\tilde{z}_t)$. From (2.4), we deduce

$$d(x_1, \tilde{z}_1) \leq a_0 + 2\varepsilon^2(1 + a_0) \cdot a_0 < 2a_0$$

Thus, $d(z_1, \tilde{z}_1) < 4a_0$. The biLipschitz property of $\exp_{z_0}$ implies

$$d(\tilde{z}_t, z_t) \leq (4 + \varepsilon) \cdot a_0 \cdot t. \tag{2.7}$$

Applying (2.4) again, we deduce

$$|\tilde{v}_t - v_0| \leq 2\varepsilon^2 \cdot (a_0 + t) \cdot t \cdot a_0 < 4\varepsilon^2 \cdot a_0 \cdot t.$$

Together with (2.7) this implies $|v_t - v_0| < 5a_0 \cdot t$. □

2.5 Subsets of positive reach

Let M be a Riemannian manifold with locally bounded curvature. A locally closed subset $L \subset M$ has *positive reach* in M if the closest point projection Π^L is uniquely defined on a neighborhood U of L in M. In this case, Π^L is locally Lipschitz on U and the distance function d_L is $\mathcal{C}^{1,1}$ on $U \setminus L$ [15].

A subset L of positive reach in M is a topological manifold if and only if L is a $\mathcal{C}^{1,1}$ submanifold of M, [25, Proposition 1.4]. On the other hand, any set L of positive reach contains a subset L' open and dense in L, which is a $\mathcal{C}^{1,1}$-submanifold, possibly with components of different dimensions [35, Theorem 7.5].

The following result is essentially contained in [25, Theorem 1.6, Theorem 1.2]. It formalizes the following observation: For a $\mathcal{C}^{1,1}$ submanifold a lower bound on the reach is equivalent to an upper bound of the second fundamental form, as well as to a $\mathcal{C}^{1,1}$-bound of the submanifold. The proof consists just of a few citations.

Lemma 2.4 *There exists $c > 0$ with the following properties.*

Let the geometry of M at p be bounded by ρ. Let L be a closed subset containing p, such that the closest point projection $\Pi = \Pi^L$ onto L is uniquely defined in $B_{\frac{10}{\rho}}(p)$.

Then, for $r \leq \frac{3}{\rho}$, the Lipschitz constant of Π on $B_r(p)$ is at most

$$\mathrm{Lip}(\Pi^L : B_r(p) \to L) \leq 1 + c \cdot \rho \cdot r. \tag{2.8}$$

If, in addition, L is a $\mathcal{C}^{1,1}$-submanifold then, for all $q \in L \cap B_r(p)$,

$$|T_pL - T_qL| \leq c \cdot \rho \cdot r. \tag{2.9}$$

Proof Upon rescaling, it suffices to prove the statement just for $\rho = 1$.

The distance function d_L to L is semiconvex on $B_{10}(p)$, see [15, Proposition 1.1, Theorem 1.8]. Moreover, as shown in [17], see also the proof of [15, Proposition 1.1, Proposition 1.3], the semiconvexity constant depends only on the curvature bound and the reach. Thus, there is a universal constant C such that d_L is C-semiconvex on $B_9(0)$.

The gradient flow $(x,t) \to \Phi(x,t)$ of $-d_L$ retracts $B_4(p)$ along the shortest geodesics to L. The C-semiconcavity of $-d_L$ implies that the map $x \to \Phi(x,t)$ is $e^{C \cdot t}$-Lipschitz continuous [33, Lemma 2.1.4].

We find a constant $c = c(C)$ with $e^{Ct} \leq 1 + c \cdot t$, for all $|t| \leq 4$. Since $\Phi(x,r) = \Pi^L(x)$, for $d(L,x) \leq r$, we obtain (2.8), for all $0 < r \leq 4$.

We turn now to (2.9). The existence of some constant $c = c(L,M)$ satisfying (2.9) is equivalent to the property that L is a $\mathcal{C}^{1,1}$-submanifold. The claim is that c can be chosen independently of L and M

We fix a sufficiently small (but universally chosen) $1 >> \delta > 0$ to be determined later. It is sufficient to prove (2.9) for all $r < \delta$.

We fix $B := B_{\frac{1}{10}}(p)$ and some distance coordinates $\Psi : B \to U \subset \mathbb{R}^n$. The distance coordinates are uniform $\mathcal{C}^{1,1}$ in the following sense, [9, Theorem 13.2], [15, Section 3]. For some constant $C_1 = C_1(C)$:

- The map Ψ is C_1-biLipschitz.
- For any v_1, v_2 in the tangent bundle TB

$$\frac{1}{C_1} \cdot |v_1 - v_2| \leq ||d\Phi(v_1) - d\Phi(v_2)|| \leq C_1 \cdot |v_1 - v_2|.$$

- The function $d_L \circ \Psi^{-1} : U \to \mathbb{R}$ is C_1-semiconvex.

The first two properties imply that it suffices to prove the estimate (2.9) on $L' := \Psi(L) \subset U$ instead of $L \subset M$. The third property implies by [3, Lemma] that the closest point projection onto $L' \subset U$ is uniquely defined on $B_{\delta'}(p') \subset U$ with $p' = \Psi(p)$, once $\delta' = \delta'(C_1)$ is small enough.

Then for some $\delta_1 = \delta_1(\delta')$, the intersection K of L' with the closed ball $B' := \bar{B}_{\delta_1}(p')$ is a compact set of reach $\geq \delta_1$ in the Euclidean space $\mathbb{R}^n$, [35, Lemma 3.4]. Thus, it is a CAT(κ) space with respect to its intrinsic metric, for some $\kappa = \kappa(\delta_1)$ [24].

Since L' is a manifold, local geodesics in K starting in p' are extendable as local geodesics until the relative boundary of K in L' [21, Theorem 1.5]. Moreover, these local geodesics are minimizing in K on intervals of length $\delta = \delta(\kappa)$, due to the CAT(κ) property. Now a uniform Lipschitz estimate for the map $x \to T_x L'$ in $L' \cap B_\delta(p')$ is a direct consequence of [28, Proposition 2.4]. □

3 Basics on submetries

3.1 (Local) submetries

Recall that $P : X \to Y$ is a *submetry* if for any $x \in X$ and any $r > 0$ the equality $P(B_r(x)) = B_r(P(x))$ holds.

Remark 3.1 In [4], submetries have been introduced using closed balls and not the open balls as here and in [16]. For our considerations here, this distinction does not matter, cf. [16, Remark 2.2].

The map P is called a *local submetry* if for any $x \in X$ there exists some $s > 0$ such that the condition $P(B_r(z)) = B_r(P(z))$ holds true for any $z \in B_s(x)$ and any $r < s$. We call X the *total space* and Y the *base* of the local submetry P.

P is a local submetry if and only if it is locally 1-Lipschitz and locally 1-open. A restriction of a (local) submetry $P : X \to Y$ to an open subset $O \subset X$ is a local submetry $P : O \to Y$. A local submetry $P : X \to Y$ is a global submetry, if X and Y are length spaces and X is proper [16, Corollary 2.9].

Let $P : X \to Y$ be a local submetry and let X be a length space. Replacing Y by $P(X)$, we may assume that the local submetries are surjective. Replacing the metric on Y by the induced length metric, P remains a local submetry [16, Corollary 2.10]. Thus, we may assume without loss of generality that the base space Y is a length space.

For a local submetry $P : X \to Y$, a rectifiable curve $\gamma : I \to X$ is *horizontal* (with respect to P) if $\ell(\gamma) = \ell(P \circ \gamma)$.

3.2 Structure of the base

From now on let M denote a manifold with locally bounded curvature. Let $P : M \to Y$ be a surjective local submetry. Let $y \in Y$ be arbitrary.

There exists some $r = r(y) > 0$, such that any geodesic $\gamma : [0, t] \to Y$ starting in y can be extended to a geodesic $\gamma : [0, r] \to Y$ up to the distance sphere $\partial B_r(y)$ [16, Theorem 1.3]. In this case, we will say that the *injectivity radius at y is at least r*. Under the above assumptions, any point $y' \in B_r(y)$ is connected to y by a unique geodesic.

Set $m = \dim(Y)$. Then, Y admits a canonical decomposition $Y = \cup_{l=0}^{m} Y^l$ into *strata* Y^l. Here, Y^l is the set of all points $y \in Y$, for which the tangent space $T_y Y$ splits off $\mathbb{R}^l$ but not $\mathbb{R}^{l+1}$ as a direct factor. Y^l is an l-dimensional manifold with a canonical $\mathcal{C}^{1,1}$-atlas, which is locally convex in Y, [16, Theorem 1.6]. The metric on Y^l is given by a Lipschitz continuous Riemannian metric; the tangent space $T_y Y^l$ is the maximal Euclidean factor of $T_y Y$ [16, Theorem 11.1].

For any point $y \in Y^l$, there exists some $r_0 = r_0(y) > 0$ with the following properties [16, Lemma 10.1, Theorem 11.1]: The open ball $B_{2r_0}(y)$ does not contain points in $\cup_{i=0}^{l-1} Y^i$ and, for any $y' \in B_{r_0}(y) \cap Y^l$, the injectivity radius at y' is at least r_0.

3.3 Fibers

Let $P : M \to Y$ be a surjective local submetry. Let $y \in Y^l \subset Y$. Then the fiber $L = P^{-1}(y)$ and the preimage $S = P^{-1}(Y^l)$ are subsets of positive reach in M [16, Theorems 1.1, 1.7].

Neither L nor S have to be manifolds. However, for every $y \in Y \backslash \partial Y$, the fiber $L = P^{-1}(y)$ is a $\mathcal{C}^{1,1}$-submanifold of M [16, Theorem 1.8]. In particular, this applies to all $y \in Y^m$ with $m = \dim(Y)$.

3.4 Infinitesimal structure

Let $P : M \to Y$ be a local submetry, let $x \in M$ be arbitrary, $y = P(x)$ and denote by L the fiber $P^{-1}(y)$.

There exists a differential $D_x P : T_x M \to T_y Y$, which is itself a submetry. The tangent space $T_x L$ is the preimage $D_x P^{-1}(0)$ and it is a convex cone in $T_x M$ [16, Proposition 3.3, Corollary 3.4]. We call $T_x L$ the *vertical space* at x and denote it by V^x.

The *horizontal space* H^x is the dual cone of $T_x L$ in $T_x M$. The cone H_x consists of all $h \in T_x M$ such that $||h|| = |D_x P(h)|$, where $|\cdot|$ on the right side denotes the distance to the origin of $T_y Y$.

A Lipschitz curve $\gamma : I \to M$ is horizontal if and only if the vector $\gamma'(t)$ is horizontal, for almost all $t \in I$.

4 *P*-almost flatness

4.1 A single point

Let $P : M \to Y$ be a local submetry, where M has locally bounded curvature. Fix $y \in Y$ and consider $L = P^{-1}(y)$. Let $x \in L$ be such that a neighborhood of x in L is a $\mathcal{C}^{1,1}$-submanifold.

For any sequence $z_j \to x$ in M, any Gromov–Hausdorff limit of (any subsequence of) the vertical spaces V^{z_j} contains V^x, [16, Corollary 8.4]. Thus, we find some $r_1 > 0$ such that $B_{r_1} \cap L$ is a $\mathcal{C}^{1,1}$-submanifold and such that the following holds true: For any $z \in B_{r_1}(x)$ and any unit vector $v \in V^x$, there exists some $v' \in V^z$ with

$$|v - v'| \leq \varepsilon. \tag{4.1}$$

We call r_1 as above the *vertical semicontinuity radius* of P at x.

Lemma 4.1 *There exists $\lambda > 0$ with the following property.*

Let $r_1 > 0$ be given. Let M be a manifold with geometry bounded at x by $\frac{1}{r_1}$. Let $P : M \to Y$ be a local submetry, $y = P(x)$ and $L = P^{-1}(y)$. Let the vertical semicontinuity radius of P at x and the injectivity radius of Y at y be at least r_1.

Then, upon rescaling M and Y by the constant $\frac{\lambda}{r_1}$, we have

- *M is almost flat at x.*
- *If $r \leq 10$, the closed ball $\bar{B}_r(y)$ is strictly convex in Y.*
- *If $r \leq 10$, the projection $\Pi^L : B_r(x) \to L$ has Lipschitz constant:*

$$\mathrm{Lip}(\Pi^L : B_r(x) \to L) \leq 1 + \varepsilon \cdot r. \tag{4.2}$$

- *For all x_1, x_2 in the $\mathcal{C}^{1,1}$-manifold $B_{10}(x) \cap L$, we have*

$$|V^{x_1} - V^{x_2}| \leq \varepsilon \cdot d(x_1, x_2). \tag{4.3}$$

- *For any $z \in B_{10}(x)$ and any unit $v \in V^x$, there is $v' \in V^z$ with*

$$|v - v'| \leq \varepsilon. \tag{4.4}$$

Proof We may assume that the constant c appearing in Lemma 2.4 is at least 1. We set $\lambda := \frac{10 \cdot c}{\varepsilon}$ and rescale M and Y with $\frac{\lambda}{r_1}$.

Upon this rescaling, the geometry of M is bounded at x by $\frac{1}{\lambda} < \varepsilon$. Hence, the rescaled M is almost flat at x.

The injectivity radius of the rescaled Y at y is at least λ. Therefore, the closest point projection onto L (in the rescaled M) is uniquely defined in $B_\lambda(x)$. Applying Lemma 2.4, we deduce (4.2) and (4.3).

The last point (4.4) follows from the definition of the vertical semicontinuity radius and $\lambda > 10$.

Finally, the statement that for some r_0 and all $r < r_0$ the balls $\bar{B}_r(y)$ are strictly convex is exactly [16, Theorem 9.2]. Moreover, the proof actually shows that in the present situation, one can take $r_0 = 10$. □

For a local submetry $P : M \to Y$, we say that x is a P-*almost flat point* if the conclusions of Lemma 4.1 hold true without rescaling. Due to Lemma 4.1, for any local submetry $P : M \to Y$ and any point $x \in M$, such that a neighborhood of x in the fiber $L := P^{-1}(P(x))$ is a manifold, M becomes P-almost flat at x upon some rescaling.

4.2 Stability along strata

The bound r_1 appearing in Lemma 4.1 can be chosen locally uniformly along strata:

Lemma 4.2 *Let $P : M \to Y$ be a local submetry. Let L be a fiber $P^{-1}(y)$ and let $x \in L$ be a point, such that a neighborhood of x in L is a manifold. Let Y^l be the stratum through y and $S = P^{-1}(Y)$. Then upon rescaling by some $\mu = \mu(M, Y, P, x, y) > 0$ the following holds:*

Any $z \in B_{10}(x) \cap S$ is a P-almost flat point.

Proof The curvature and injectivity radii of M are bounded in a fixed ball around x. The injectivity radius of Y is uniformly bounded from below in a neighborhood of y in Y^l [16, Theorem 11.1].

Applying Lemma 4.1, it remains to obtain a uniform lower bound on the vertical semicontinuity radii in a neighborhood of x in S.

For some choice of a neighborhood U of x in M, the restriction $P : U \cap S \to Y^l$ is a fiber bundle [16, Proposition 11.3]. Thus, $U \cap S$ and $L' := U \cap P^{-1}(y')$ for any $y' \in Y^l$ are topological manifolds. Since S and L' are subsets of positive reach [16, Theorem 1.1, Theorem 1.7], both subsets $U \cap S$ and $U \cap L'$ are $\mathcal{C}^{1,1}$-submanifolds of M.

The submanifold $S \cap U$ in its intrinsic metric is a manifold with locally bounded curvature [15, Proposition 1.7]. The restriction $P : S \cap U \to Y^l$ is a local submetry with all fibers being regular. Thus, this restriction is a $\mathcal{C}^{1,1}$-Riemannian submersion [16, Theorem 1.2].

In particular the distribution $z \to V^z$ is continuous on $U \cap S$. Thus, the semicontinuity of vertical spaces in M around x implies the following. For a sufficiently small $2\delta > 0$, any point $x_1 \in B_{2\delta}(x) \cap S$, any $z_1 \in B_{2\delta}(x)$ and any unit vector $v \in V^{x_1}$ there exists some $v' \in V^{z_1}$ such that (4.1) holds true.

Thus, δ is the required uniform bound on the vertical semicontinuity radii in a neighborhood of x in S. This finishes the proof. $\square$

5 Projection onto a fiber

We aim to strengthen (4.2), (4.3), (4.4).

Lemma 5.1 *Let $P : M \to Y$ be a local submetry, let L be a fiber of P. Assume $x_0 \in L$ is a P-almost flat point. Then, $\Pi^L : L' \cap B_2(x_0) \to L$ is locally 2-open, for any fiber L' of P.*

Proof Consider any $z' \in L' \cap B_2(x_0)$. Set $r = d(L, z') < 2$. Consider some $\delta < \varepsilon = 10^{-4}$, so that $B_{10\delta r}(z') \subset B_2(x_0)$.

Consider an arbitrary $q' \in B_{\delta r}(z') \cap L'$. Set $q = \Pi^L(q')$. Consider any $p \in L$ with $t := d(p, q) < \delta r$. It is sufficient to find $p' \in L'$ with $\Pi^L(p') = p$ and $d(p', q') \leq 2t$.

Consider the ball $B = B_3(0)$ in the horizontal space H^p and the subset $K = \exp_p(B_3(0)) \subset M$. Note that $\Pi^L(K) = p$. Consider the distance function $f = d_K : B_2(x_0) \to \mathbb{R}$. We are looking for a point $p' \in L'$ with $f(p') = 0$ and $d(p', q') \leq 2t$.

By the open map theorem [26, Lemma 4.1], it suffices to prove

$$f(q') \leq \frac{5}{4} \cdot t \tag{5.1}$$

and that the absolute gradient of the restriction $-f : L' \to \mathbb{R}$ satisfies

$$|\nabla_{q''}(-f)| \geq \frac{4}{5} \tag{5.2}$$

at every point $q'' \in L' \cap B_{2t}(q')$.

Consider $h = \exp_q^{-1}(q') \in H^q$. By (4.3), we find some $\tilde{h} \in H^p$ with

$$|\tilde{h} - h| \leq \varepsilon \cdot t \cdot r.$$

Then, $\exp_p(\tilde{h}) \in K$ and from Lemma 2.1 we deduce

$$f(q') \leq d(q', \exp_p(\tilde{h})) \leq t + 4\varepsilon^2 \cdot t \cdot r + 2 \cdot \varepsilon \cdot t \cdot r < \frac{5}{4}t.$$

This proves (5.1).

In order to prove (5.2), we fix a point $q'' \in L' \cap B_{2t}(q')$. Consider a point $\hat{p} \in K$ with

$$f(q'') = d_K(q'') = d(q'', \hat{p}).$$

Denote by $w \in T_{q''}M$ the starting direction of the geodesic $q''\hat{p}$.

If there exists a vertical unit vector $v \in V^{q''} = T_{q''}(L')$ which encloses an angle less than $\arccos(\frac{4}{5})$ with w, then the first variation formula would imply (5.2).

Assume on the contrary, that such a vertical vector $v \in V^{q''}$ does not exist. Then, there exists a unit horizontal vector $u \in H^{q''}$ which encloses with w an angle at most

$$\angle(u, w) \leq \frac{\pi}{2} - \arccos(\frac{4}{5}) < 1.$$

Since x_0 is a P-almost flat point, we apply (4.4) and (4.3) and find a unit horizontal vector $u' \in H^p$ with $|u' - u| \leq 2\varepsilon$. Then, the angle between the parallel translate $\hat{w}$ of w to $\hat{p}$ and $\hat{u}$ of u' to $\hat{p}$ is at most

$$\angle(\hat{u}, \hat{w}) \leq 1 + 3\varepsilon.$$

Note, that $\hat{w}$ is just the starting direction at $\hat{p}$ of the geodesic $\hat{p}q''$.

Consider the vector $\hat{h} = \exp_p^{-1}(\hat{p}) \in H^p$ and the curve $\eta(t) := \exp_p(\hat{h} + t \cdot u')$ contained in K and starting at $\hat{p}$. By Corollary 2.2, the curve η encloses with vector $\hat{u}$ an angle less than ε.

Thus, the angle between η and $\hat{w}$ at $\hat{p}$ is less than $1 + 4\varepsilon < \frac{\pi}{2}$. Now the first formula of variation implies that $d(q'', \eta(t)) < d(q'', \hat{p})$ for all sufficiently small t.

This contradicts the choice of $\hat{p}$. The contradiction finishes the proof of (5.2) and of the Lemma. □

Using a combination of Lemmas 5.1 and 2.3, we now provide:

Theorem 5.2 *$P : M \to Y$ a local submetry. Let $x_0 \in M$ be such that a neighborhood of x_0 in $L = P^{-1}(P(x_0))$ is a $\mathcal{C}^{1,1}$-submanifold. Then, there exist $r_0 > 0, C > 0$ with the following properties, for any $0 < r < r_0$ and any $z \in B_r(x_0)$.*

(1) For any $v \in V^{x_0}$, there exists $v' \in V^z$ with $|v - v'| \leq C \cdot r \cdot ||v||$.
(2) For any $h \in H^z$, there exists $h' \in H^{x_0}$ with $|h - h'| \leq C \cdot r \cdot ||h||$.
(3) For $L^z = P^{-1}(P(z))$, the closest point projection $\Pi^L : L^z \cap B_r(x_0) \to L$ is $(1 + Cr)$-Lipschitz and locally $(1 + Cr)$-open.

The numbers r_0, C depend only on a bound of the geometry of M at x_0, a bound on the injectivity radius of Y at $P(x_0)$ and the vertical semicontinuity radius of P at x_0.

Proof After rescaling, we may assume that x_0 is a P-almost flat point. Due to Lemma 4.1, the rescaling constant depends only on a bound of the geometry of M at x_0, a bound on the injectivity radius of Y at $P(x_0)$ and the vertical semicontinuity radius of P at x_0.

Thus, it suffices to find universal constants $C, r_0 > 0$ satisfying (1), (2), (3), under the assumption that x_0 is a P-almost flat point.

We are going to prove (3) first. By the definition of P-almost flat points, the projection Π^L is $(1 + \varepsilon \cdot r)$-Lipschitz on the whole ball $B_r(x_0)$, for any $r < 10$. Thus, also the restriction of Π^L to $L' \cap B_r(x_0)$ has the same Lipschitz constant. It suffices to improve the openness constant of Π^L on L' provided by Lemma 5.1.

Set $r_0 := \frac{1}{5}$. Let $r \leq r_0$ and $z \in B_r(x_0)$ be arbitrary. Set $x = \Pi^L(z)$ and let p be a point on L with $a_0 := d(x, p) < \varepsilon \cdot r$.

We are going to find a point $q \in L^z$ satisfying $\Pi^L(q) = p$ and

$$d(q, z) \leq (1 + 5 \cdot r) \cdot a_0. \tag{5.3}$$

Extend the geodesic xz to a point $\tilde{z}$ with $d(x, \tilde{z}) = 1$. Lemma 5.1 provides a point $\tilde{q}$ on the fiber $\tilde{L}$ through $\tilde{z}$ such that $d(\tilde{q}, \tilde{z}) \leq 2a_0$ and $\Pi^L(\tilde{q}) = p$. The geodesics $\gamma := x\tilde{z}$ and $\eta := p\tilde{q}$ are horizontal and $P \circ \gamma = P \circ \eta$.

Consider the point q on the η with $d(p, q) = d(x, z)$. Then $P(q) = P(z)$, hence $q \in L^z$. From Lemma 2.3, we deduce (5.3), finishing the proof of (3).

In order to prove (1), we fix $r \leq \frac{1}{5}$ and $z \in B_r(x_0)$. Set again $x = \Pi^L(z)$. Due to (2.9), we have $|V^x - V^{x_0}| \leq 2\varepsilon r$.

Consider an arbitrary unit vector $v \in V^{x_0}$. We find a unit vector $\hat{v} \in V^x$ with $|\hat{v}-v| \leq 3\varepsilon r$.

For any sufficiently small $\delta > 0$ consider a point $p_\delta \in L$ with $d(x, p_\delta) = \delta$, such that the geodesic xp_δ starts in a direction $v_\delta \in T_x M$ with $|v_\delta - \hat{v}| \leq \varepsilon r$.

Extend as above the geodesic xz until a point $\tilde{z}$ with $d(x, \tilde{z}) = 1$. Due to Lemma 5.1, we find a point $\tilde{q}_\delta$ in the fiber $L^{\tilde{z}}$ of $\tilde{z}$ with $d(\tilde{q}, \tilde{z}) \leq 2\delta$. Let q_δ be the point on the geodesic $p_\delta\tilde{q}_\delta$ with $d(p_\delta, q_\delta) = d(x, z)$.

Then, $q_\delta \in L^z$. From Lemma 2.3, we deduce that the starting direction v_δ of the geodesic zq_δ satisfies

$$|v_\delta - \tilde{v}| \leq 5 \cdot r.$$

The directions v_δ subconverge to a vertical direction $v' \in V^z$ such that $|v' - v| \leq 6r$. This proves (1).

By duality of horizontal and vertical cones, (1) implies (2). □

We now easily deduce:

Proof of Proposition 1.3 We cover the compact manifold fiber L by finitely many balls as provided by Theorem 5.2. Choosing a tubular neighborhood $B_{r_0}(L)$ of L contained in the union of these balls, we obtain the conclusion directly from Theorem 5.2(3). □

6 Exponential map in the base

6.1 Exponential map in the base

The following result is a localization of Theorem 1.1.

Proposition 6.1 *There exists some $\mu > 1$ with the following properties. Let $P : M \to Y$, $x \in M$, $y = P(x)$ and r_1 be as in Lemma 4.1. Then for $r_0 := \frac{r_1}{\mu}$ and $C := \mu \cdot r_1^2$ and all $r < r_0$, the exponential map $\exp_y : B_r(0) \to B_r(y)$ is $(1 + Cr^2)$-biLipschitz on the ball $B_r(0) \subset T_yY$.*

Proof The statement is invariant under rescalings. Upon a rescaling, we may assume that x is a P-almost flat point. Let C, r_0 be as provided by Theorem 5.2. Upon a further rescaling, depending only on C and r_0, we may assume that the constants satisfy $r_0 = 1$ and $C = \frac{1}{10}$. It suffices to prove that, for any $r < \frac{1}{10}$, the map $\exp_y : B_r(0) \to B_r(y)$ is $(1+2\cdot r^2)$-biLipschitz on the ball $B_r(0) \subset T_yY$. We fix $r < \frac{1}{10}$.

The ball $\bar{B}_r(y) \subset Y$ inherits the lower curvature bound $-\frac{1}{100}\varepsilon^2$ from the ball $\bar{B}_r(x)$, [16, Proposition 3.1]. By Toponogov's theorem, the exponential map $\exp_y$ is $(1+\varepsilon^2 r^2)$-Lipschitz on $B_r(0) \subset T_yY$. It remains to bound the Lipschitz constant of $\exp_y^{-1}$ on $B_r(y)$.

Since M is almost flat at x, the exponential map $\exp_x : B_r(0) \to B_r(x)$ is $(1 + \varepsilon^2 r^2)$-biLipschitz on the ball $B_r(0) \subset T_x M$, for any $r \leq 1$.

We consider the ball $Q = B_r(0) \subset H^x$ in the horizontal space at x and its exponential image $Z = \exp_x(Q) \subset M$. For all $h \in Q \subset H^x$ we have, [16, Proposition 7.3]:

$$\exp_y \circ D_x P(h) = P \circ \exp_x(h).$$

Thus, the map $P : Z \to Y$ can be written as

$$P = \exp_y \circ D_x P \circ \exp_x^{-1}.$$

The map $\exp_x^{-1} : Z \to Q$ is locally $(1+\varepsilon^2 r^2)$-biLipschitz and the map $D_x P : H^x \to T_y Y$ is a local submetry. If we knew that $P : Z \to Y$ is locally $(1 + r^2)$-open, we would infer that $\exp_y$ is locally $(1+2 \cdot r^2)$-open. Since $\exp_y$ is a homeomorphism and $(1+\varepsilon^2 \cdot r^2)$-Lipschitz, this would prove that $\exp_y$ is locally $(1 + 2 \cdot r^2)$-biLipschitz.

It remains to prove that $P : Z \to Y$ is locally $(1 + r^2)$-open.

Thus, consider any $h \in Q$ and $z = \exp_x(h) \in Z$. Set $t_0 := \varepsilon \cdot (r - |h|)$. Let $z_1 = \exp_x(h_1) \in Z$ with $d(z, z_1) < t_0$ be given. Set $y_1 = P(z_1)$ and let $y_2 \in Y$ be such that $t := d(y_1, y_2) < t_0$. We need to find a point $z_2 \in Z \cap P^{-1}(y_2)$, such that $d(z_1, z_2) \le (1+r^2) \cdot t$.

We set $L' := P^{-1}(y_2)$ and denote by f the distance function $f := d_{L'}$. We are looking for $z_2 \in Z$ with $f(z_2) = 0$ and $d(z_1, z_2) \le (1 + r^2) \cdot t$.

Since $P : M \to Y$ is a local submetry, we have

$$f(z_1) = d(L', z_1) = d(y_1, y_2) = t.$$

Due to the open map theorem [26, Lemma 4.1], it suffices to prove that the absolute gradient of $-f$ on Z at every point $p \in Z \setminus L'$ is at least $1 - \frac{1}{2}r^2$.

We fix a point $p \in Z \setminus L'$ and a shortest geodesic from p to L'. This geodesic is horizontal, since L' is a fiber of P. Let $u \in H^p$ be the starting direction of this geodesic.

Due to Theorem 5.2 and the rescaling chosen above, we find some unit vector $u' \in H^x$ with $|u - u'| \le \frac{1}{5} \cdot r$. Consider $w := \exp_x^{-1}(p)$ and the curve $\eta : [0, t_0] \to Z$ starting at p:

$$\eta(s) := \exp_x(w + s \cdot u') \subset Z.$$

Due to Corollary 2.2, any starting direction $\tilde{u}$ of η at p satisfies

$$|\tilde{u} - u| \le \frac{1}{4} \cdot r.$$

Therefore, by the first formula of variation, we deduce that $-f$ grows at p at least with velocity

$$\cos\left(\frac{1}{4}r\right) \ge 1 - \frac{1}{2}r^2.$$

This provides the right estimate for the absolute gradient of the function $-f : Z \to \mathbb{R}$ at p and finishes the proof. □

6.2 Applications

As a consequence of Proposition 6.1, we derive a local generalization of Corollary 1.2:

Corollary 6.2 *Let $P : M \to Y$ be a surjective local submetry. Then, any stratum Y^l of Y is a manifold with locally bounded curvature.*

Proof Let $y_0 \in Y^l$ be arbitrary. Consider a point $x_0 \in L = P^{-1}(y_0)$ such that a neighborhood of y_0 in L is a $\mathcal{C}^{1,1}$-submanifold.

Due to Proposition 6.1 and Lemma 4.2, we find some $r_0 > 0$ and $C > 0$ such that the following holds true. For any $y \in B_{r_0}(y_0) \cap Y^l$ and any $r < r_0$, the exponential map $\exp_y : B_r(0) \to B_r(y)$ is $(1 + C \cdot r^2)$-biLipschitz from the ball in the tangent space $B_r(0) \subset T_y Y$.

Upon rescaling, we may assume that M is almost flat at x_0, that $r_0 = 10$ and $C = \varepsilon$. Due to Lemmas 4.1, 4.2, the ball $\bar{B}_{10}(y_0) \cap Y^l$ is compact and convex in Y^l. It inherits the lower curvature bound $-\varepsilon^2$ from $B := B_{10}(x_0)$. It remains to prove that the convex subset $Z := \bar{B}_1(y_0) \cap Y^l$ is CAT(1).

Consider 3 points y, p, q in this Z. Choose $\bar{y}, \bar{p}, \bar{q}$ in the round sphere $\mathbb{S}^l$ of dimension l, such that $d(y, p) = d(\bar{y}, \bar{p})$, $d(y, q) = d(\bar{y}, \bar{q})$ and $\angle pyq = \angle \bar{p}\bar{y}\bar{q}$. We need to prove $d(p, q) \geq d(\bar{p}, \bar{q})$.

Identify the tangent spaces $T_y Y^l$ and $T_{\bar{y}} \mathbb{S}^l$ through an isometry I, which sends the starting directions of yp and yq to the starting directions of $\bar{y}\bar{p}$ and $\bar{y}\bar{q}$, respectively. It suffices to prove that the map

$$f := \exp_{\bar{y}} \circ I \circ \exp_y^{-1} : B_1(y) \to B_1(\bar{y})$$

on the ball $B_1(y) \subset Y^l$ is 1-Lipschitz.

Due to Proposition 6.1, the map f is biLipschitz. By construction, f sends geodesics starting at y to geodesics starting at $\bar{y}$. Hence, f sends spheres around y onto spheres around $\bar{y}$ of the same radius.

The restriction of $\exp_{\bar{y}}$ to the concentric sphere $\partial B_s(0)$ in $T_{\bar{y}} \mathbb{S}^n$ is $(1 - \frac{1}{10} s^2)$-Lipschitz, if we equip this sphere with its intrinsic metric. Thus, the restriction $f : \partial B_s(y) \to \partial B_s(\bar{y})$ is 1-Lipschitz, if both spheres are equipped with their intrinsic metrics.

The biLipschitz map f is differentiable almost everywhere with linear differential, by Rademacher's theorem. By above, at any point z at which f is differentiable, the differential $D_z f$ is 1-Lipschitz. We claim that this is enough to conclude that f is 1-Lipschitz.

Indeed, the ball $B_1(y_0)$ can be considered as a Euclidean subset $O \subset \mathbb{R}^l$ with a Lipschitz continuous Riemannian metric. For any vector v in $\mathbb{R}^l$, Fubini's theorem implies that for *almost every* segment γ in O in direction of v, the length of γ in Y is not less than the length of $f \circ \gamma$ in $\mathbb{S}^n$. On segments parallel to v in $O \subset Y$, the length functional is *continuous* with respect to uniform convergence. On the other hand, the length of the images $f \circ \gamma$ is (as always) lower semi-continuous. Thus, by a limiting procedure, the length of $f \circ \gamma$ is not larger than the length of γ for *every* segment γ in the direction of v. Therefore, the map f is 1-Lipschitz and $B_1(y_0)$ has curvature at most 1. □

Another consequence of Proposition 6.1 is the following:

Corollary 6.3 *Let $P : M \to Y$ be a surjective local submetry as above. Let $y \in Y$ be an arbitrary point, let r be smaller than the injectivity radius of y and let $v \in T_y Y$ be a vector with $|v| < r$. Then, the tangent cones $T_v(T_y Y)$ and $T_{\exp_y(v)} Y$ are isometric.*

In particular, if $\exp_y(v)$ is contained in the l-dimensional stratum Y^l then v is contained in the l-dimensional stratum $(T_y Y)^l$.

Proof Consider the geodesic $\gamma_v : [0, r) \to Y$ in the direction of v parametrized by arc length. For $t \in (0, r)$, the tangent spaces at $\gamma_v(t)$ do not depend on t, [32]. Moreover, the tangent space $T_v(T_y Y)$ in the Euclidean cone $T_y Y$ is isometric $T_{s \cdot v}(T_y Y)$, for all $s > 0$.

Due to Proposition 6.1, for small $s > 0$, a neighborhood of $(s \cdot v)$ in $T_y Y$ is $(1 + C s^2)$-biLipschitz to a neighborhood of $\exp_y(s \cdot v)$ in Y, for some C independent of s. Rescaling, letting s go to 0 and using that the tangent cones at $s \cdot v$, respectively, at $\exp_y(s \cdot v)$ do not depend on s, we deduce the claim. □

7 Transnormal submetries

7.1 Horizontal geodesics

Recall that a local submetry $P : M \to Y$ is transnormal if all fibers of P are $\mathcal{C}^{1,1}$-submanifold of M. A local submetry $P : M \to Y$ is transnormal if and only if any local geodesic $\gamma : I \to M$ is horizontal once $\gamma'(t)$ is horizontal for some $t \in I$, [16, Proposition 12.5]. In this case, for any horizontal local geodesic $\gamma : I \to M$, the projection $\bar{\gamma} := P \circ \gamma : I \to Y$ is a discrete concatenation of geodesics in Y, [16, Corollary 7.2].

The following Lemma is stated as [16, Proposition 12.7] for global submetries, but the proof remains unchanged in the local case:

Lemma 7.1 *Let $P : M \to Y$ be a transnormal local submetry. Let $\gamma_1, \gamma_2 : I \to M$ be horizontal local geodesics. Set $\bar{\gamma}_i := P \circ \gamma_i$. Assume that, for some $t \in I$, we have $\bar{\gamma}_1(t) = \bar{\gamma}_2(t)$ and $\bar{\gamma}_1'(t) = \bar{\gamma}_2'(t)$. Then, $\bar{\gamma}_1$ and $\bar{\gamma}_2$ coincide on I.*

We can now provide

Proof of Corollary 1.4 The statement is local. We may assume that I is a compact interval $[a, b]$. The projection $\bar{\gamma} : [a, b] \to Y$ is a finite concatenation of geodesics $\bar{\gamma} : [s_i, s_{i+1}] \to Y$, for $a = s_0 < ... < s_k = b$.

For all $t \in (s_i, s_{i+1})$, the tangent spaces $T_{\bar{\gamma}(t)}Y$ are pairwise isometric [32, Theorem 1.1]. By Corollary 6.3, they are also isometric to $T_{\bar{\gamma}^+(s_i)}(T_{\bar{\gamma}(s_i)}Y)$ and to $T_{\bar{\gamma}^-(s_{i+1})}(T_{\bar{\gamma}(s_{i+1})}Y)$. Here and below $\bar{\gamma}^{\pm}(s)$ denotes the outgoing and the incoming direction of $\bar{\gamma}$ in $T_{\bar{\gamma}(s)}Y$.

On the other hand, for any $t \in (s_i, s_{i+1})$, the incoming and the outgoing directions $\bar{\gamma}^{\pm(t)}$ are contained in the line factor of $T_{\bar{\gamma}(t)}Y$. Hence, $T_{\bar{\gamma}'(t)}(T_{\bar{\gamma}(t)}Y)$ is isometric to $T_{\bar{\gamma}(t)}Y$, for any such t.

It only remains to prove that for any $s = s_1, \ldots, s_{k-1}$, the two tangent cones $T_{\bar{\gamma}^{\pm}(s)}(T_{\bar{\gamma}(s)}Y)$ are isometric two each other. In order to prove this, it suffices to find, for any such s, an isometry $I : T_{\bar{\gamma}(s)}Y \to T_{\bar{\gamma}(s)}Y$ which sends $\bar{\gamma}^+$ to $\bar{\gamma}^-$.

In order to find such I, we consider $x := \gamma(s) \in M$ and the differential $D_x P : T_x M \to T_y Y$. The restriction of $D_x P$ to the unit sphere K in the horizontal space H^x is a transnormal submetry $D_x P : K \to \Sigma_y Y$, onto the space of directions at y [16, Proposition 12.5].

The incoming and the outgoing directions $\gamma^{\pm}(t) \in K$ satisfy $\gamma^+(t) = -\gamma^-(t)$ and $D_x P(\gamma^{\pm}(t)) = \bar{\gamma}^{\pm}(t)$.

Due to [16, Proposition 12.7], the decomposition of K into the fibers of the submetry $D_x P$ is equivariant under the multiplication of K with -1. Thus, $-Id : K \to K$ induces an isometry $\overline{-Id} : \Sigma_y Y \to \Sigma_y Y$. The cone over this isometry $\overline{-Id}$ is the required isometry $I : T_y Y \to T_y Y$, which satisfies $I(\bar{\gamma}^+) = \bar{\gamma}^-$.

This proves the claim and implies that the spaces $T_{\bar{\gamma}'(t)}(T_{\bar{\gamma}(t)})Y$ are pairwise isometric.

Let now l denote the dimension of the maximal Euclidean factor of the pairwise isometric spaces $T_{\bar{\gamma}'(t)}(T_{\bar{\gamma}(t)}Y)$. Then, for all $t \neq s_0, s_1, \ldots, s_k$ as above, the iterated tangent cone $T_{\bar{\gamma}'(t)}(T_{\bar{\gamma}(t)}Y)$ is isometric to $T_{\bar{\gamma}(t)}Y$. By definition, $\gamma(t)$ is contained in Y^l, for all such t. □

7.2 Holonomy map along a horizontal geodesic

Let $P : M \to Y$ be a transnormal local submetry. Let $\gamma : [a, b] \to M$ be a horizontal local geodesic with projection $\bar{\gamma} = P \circ \gamma$. Due to Corollary 1.4, there exist some $1 \leq l \leq m$, such that $\bar{\gamma}(t) \in Y^l$, for all but finitely many times $t \in [a, b]$. For $t \in [a, b]$, consider the fiber

$$L^t := L^{\gamma(t)} = P^{-1}(P(\gamma(t))).$$

Set $S = P^{-1}(Y^l)$. Let $t \in [a, b]$ be such that $\bar{\gamma}(t) \in Y^l$. Set $x = \gamma(t) \in L^t \subset S$. Then, S is a $\mathcal{C}^{1,1}$-submanifold of M and the restriction $P : S \to Y^l$ is a $\mathcal{C}^{1,1}$ Riemannian submersion. Therefore, the normal vector $\gamma'(t) \in T_x S$ to L^t extends to a unique locally Lipschitz continuous normal field $z \to \nu_z \in T_z S$ along L^t, such that

$$D_z P(\nu_z) = D_x P(\nu_x) = D_x P(\gamma'(t)).$$

Denote by Q^t the set of all $z \in L^t$ such that the geodesic $\gamma^z : [a, b] \to M$ with $(\gamma^z)'(t) = \nu_z$ is defined. Then, Q^t is an open subset of L^t and, if M is complete, $Q^t = L^t$. Due to Lemma 7.1, for all $z \in Q^t$

$$P \circ \gamma^z = P \circ \gamma = \bar{\gamma}.$$

For all $s \in [a, b]$, we obtain a map $Hol^{\gamma}_{t,s} : Q^t \to L^s$, *the holonomy along* γ, given as

$$Hol^{\gamma}_{t,s}(z) := \gamma^z(s).$$

Since ν and the exponential map on M are locally Lipschitz, the map $Hol^{\gamma}_{t,s}$ is locally Lipschitz.

If $\gamma(s) \in Y^l$, then $Hol^{\gamma}_{t,s}(Q^t) = Q^s$ and $Hol^{\gamma}_{t,s}$ and $Hol^{\gamma}_{s,t}$ are inverse to each other. Thus, $Hol^{\gamma}_{t,s}$ is locally biLipschitz in this case.

Let now $r \in [a, b]$ be arbitrary. Find some $s \in [a, b]$ such that $\bar{\gamma}(s) \in Y^l$ and $|s - r|$ is smaller than the injectivity radius at $\bar{\gamma}(r)$. Then,

$$Hol^{\gamma}_{s,r} \circ Hol^{\gamma}_{t,s} = Hol^{\gamma}_{t,r}.$$

The map $Hol^{\gamma}_{t,s}$ is locally biLipschitz, as we have seen above. And the map $Hol^{\gamma}_{s,r} : Q^s \to L^r$ is the closest point projection to L^r. Once s has been chosen close enough to r, we can apply Theorem 5.2 and deduce that the map $Hol^{\gamma}_{s,r} : Q^s \to L^r$ is locally Lipschitz open.

Altogether we have verified the following

Proposition 7.2 *In the notation above, the holonomy map along* γ $Hol^{\gamma}_{t,r} : Q^t \to L^r$ *is locally Lipschitz continuous and locally Lipschitz open. If* $\bar{\gamma}(r) \in Y^l$*, then* $Hol^{\gamma}_{t,r}$ *is locally biLipschitz.*

7.3 A uniform bound in terms of the volume

We finally prove:

Lemma 7.3 *For any* $n, k, \rho, \nu > 0$*, there exists* $r = r(n, k, \rho, \nu) > 0$ *with the following property. Let* M *be a manifold with locally bounded curvature and let* $P : M \to Y$ *be a transnormal local submetry. Let* $n = \dim(M)$ *and* $k = \dim(Y)$*. Let the geometry of* M *at* x *be bounded by* $\frac{1}{\rho}$ *and let the injectivity radius of* Y *at* $y = P(x)$ *be at least* ρ*. Let, finally, the volume of the ball* $B_\rho(y)$ *be at least* $\mathcal{H}^k(B_\rho(y)) \geq \nu \cdot \rho^k$*. Then, the vertical semicontinuity radius of* P *at* x *is at least* r*.*

Proof of Lemma 7.3 Assume the contrary and let $P_j : (M_j, x_j) \to (Y_j, y_j)$ be a contradicting sequence. Thus, the vertical semicontinuity radii r_j of P_j at x_j converge to 0. Hence, there exist $z_j \in M_j$ with $s_j = d(z_j, x_j) \to 0$ and unit vectors $v_j \in V^{x_j}$ such that $d(v_j, V^{z_j}) > \varepsilon$.

Hence, there exists a unit horizontal vector $h_j \in H^{z_j}$ such that

$$\angle(h_j, \tilde{v}_j) < \frac{\pi}{2} - \varepsilon,$$

where $\tilde{v}_j$ is the parallel translation of v_j to z_j.

Upon rescaling we may assume that $\rho = \varepsilon$. In particular, M_j are almost flat at x_j. Choosing a subsequence we may assume that $\bar{B}_{10}(x_j)$ converge to a space M^∞ which is a closed ball of radius 10 around the limit point x of the sequence x_j. Moreover, we may assume that the balls $\bar{B}_{10}(y_j)$ converge to an Alexandrov space Z and that the restrictions of P_j converge to a 1-Lipschitz map $P : M^\infty \to Z$. The open ball $B = B_{10}(x)$ is again a manifold with locally bounded curvature and the restriction of P to B is a local submetry onto the ball $B_{10}(y) \subset Z$.

Consider the geodesic $\gamma_j : [0, 1] \to M_j$ starting in z_j in the direction of h_j. Then γ_j is a horizontal curve, by the definition of transnormality. The images $P_j(\gamma_j)$ are quasi-geodesics in Y_j, [16, Proposition 3.2]. Since the convergence $\bar{B}_{10}(y_j) \to Z$ is non-collapsed by the volume assumption, the limit of the curves $P_j(\gamma_j)$ is a quasigeodesic in Z [33, Section 5.1(6)]. Therefore, the limit geodesic $\gamma_\infty : [0, 1] \to M^\infty$ starting in x is horizontal. Thus, its starting direction $h \in T_x M^\infty$ is horizontal.

On the other hand, the fibers $L_j := P_j^{-1}(y_j)$ converge to $L = P_j^{-1}(y)$, [16, Lemma 2.4]. Since the manifolds L_j are uniformly $\mathcal{C}^{1,1}$ in distance coordinates by (2.9), the tangent space $T_{x_j} L_j$ converge to $T_x L$. Thus, any limit vector v of v_j is contained in $T_x L$. By assumptions on h_j and v_j, the angle between the vertical vector v and the horizontal vector h is at most $\frac{\pi}{2} - \varepsilon$. This contradiction finishes the proof. $\square$

As a direct consequence, we obtain:

Corollary 7.4 *For transnormal submetries $P : M \to Y$, the constants C, r_0 appearing in Theorems 1.1, 5.2, Propositions 6.1 and 1.3 depend only on a bound of the geometry of M around L, the injectivity radius of Y at $y := P(L)$ and a lower volume bound of a ball in Y around y.*

Acknowledgements The author is grateful to Ricardo Mendes and Marco Radeschi for their interest and helpful comments and to the anonymous referee for careful reading and helpful criticism.

Author Contributions A.L: has written the manuscript.

Funding Open Access funding enabled and organized by Projekt DEAL.

Declaration

Conflicts of interest The authors declare no conflict of interests.

References

1. Alexandrino, M., Caramello, F.: Leaf closures of Riemannian foliations: a survey on topological and geometric aspects of Killing foliations. Expo. Math. **40**(2), 177–230 (2022)
2. Alexander, S., Kapovitch, V., Petrunin, A.: Alexandrov geometry. Preprint arXiv:1903.08539, (2019)
3. Bangert, V.: Sets with positive reach. Arch. Math. (Basel) **38**(1), 54–57 (1982)
4. Berestovskii, V.N.: Submetries of space-forms of non-negative curvature. Sib. Math. J. **28**(4), 552–562 (1987)
5. Berestovskii, V.: Geodesics and curvatures of special sub-Riemannian metrics on Lie groups. Siberian. Math. J. **59**, 31–42 (2018)
6. Berestovskii, V.N., Guijarro, L.: A metric characterization of Riemannian submersions. Ann. Global Anal. Geom. **18**(6), 577–588 (2000)
7. Burago, Y., Gromov, M., Perelman, G.: A.D. Aleksandrov spaces with curvatures bounded below. Uspekhi Mat. Nauk **47**(284), 3–51 (1992)
8. Buser, P., Karcher, H.: Gromov's almost flat manifolds. Astérisque, vol. 81. Société Mathématique de France, Paris (1981)
9. Berestovskij, V. N., Nikolaev, I. G.: Multidimensional generalized Riemannian spaces. In: Geometry, IV, vol. 70 of Encyclopaedia Math. Sci., pp. 165–243, 245–250. Springer, Berlin (1993)
10. Berestovskiĭ, V.N., Nikonorov, Y.G.: Finite homogeneous metric spaces. Sibirsk. Mat. Zh. **60**(5), 973–995 (2019)
11. Chen, X., Grove, K.: Rigidity theorems for submetries in positive curvature. Adv. Math. **289**, 784–796 (2016)
12. Federer, H.: Curvature measures. Trans. Am. Math. Soc. **93**, 418–491 (1959)
13. Graf, M., Grant, J., Kunzinger, M., Steinbauer, R.: The Hawking-Penrose singularity theorem for $C^{1,1}$-Lorentzian metrics. Commun. Math. Phys. **360**(3), 1009–1042 (2018)
14. Guijarro, L., Wilhelm, F.: Focal radius, rigidity, and lower curvature bounds. Proc. Lond. Math. Soc. **116**(6), 1519–1552 (2018)
15. Kapovitch, V., Lytchak, A.: Remarks on manifolds with two-sided curvature bounds. Anal. Geom. Metr. Spaces **9**(1), 53–64 (2021)
16. Kapovitch, V., Lytchak, A.: The structure of submetries. Geom. Topol. **26**(6), 2649–2711 (2022)
17. Kleinjohann, N.: Nächste Punkte in der Riemannschen Geometrie. Math. Z. **176**(3), 327–344 (1981)
18. Lempert, L.: On riemannian submersions. Acta Math. Hungar. **158**, 363–372 (2019)
19. Lin, S., Mendes, R., Radeschi, M.: A Weyl law for singular Riemannian foliations with applications to invariant theory. arXiv:2312.06533 (2023)
20. Lebedeva, N., Petrunin, A., Zolotov, V.: Bipolar comparison. Geom. Funct. Anal. **29**(1), 258–282 (2019)
21. Lytchak, A., Schroeder, V.: Affine functions on CAT(κ)-spaces. Math. Z. **255**(2), 231–244 (2007)
22. Lytchak, A., Thorbergsson, G.: Curvature explosion in quotients and applications. J. Differ. Geom. **85**(1), 117–139 (2010)
23. Lytchak, A., Wilking, B.: The smoothness of submetries. In preparation (2024)
24. Lytchak, A.: On the geometry of subsets of positive reach. Manuscripta Math. **115**(2), 199–205 (2004)
25. Lytchak, A.: Almost convex subsets. Geom. Dedicata. **115**, 201–218 (2005)
26. Lytchak, A.: Open map theorem for metric spaces. Algebra i Analiz **17**(3), 139–159 (2005)
27. Lytchak, A.: Rigidity of spherical buildings and joins. Geom. Funct. Anal. **15**(3), 720–752 (2005)
28. Lytchak, A.: A note on subsets of positive reach, to appear. Manuscripta Math. (2023)
29. Molino, P.: Riemannian foliations. Progress in Mathematics, vol. 73. Birkhäuser Boston Inc, Boston, MA (1988)
30. Mendes, R., Radeschi, M.: Laplacian algebras, manifold submetries and the inverse invariant theory problem. Geom. Funct. Anal. **30**, 536–573 (2020)
31. Mendes, R., Radeschi, M.: Singular Riemannian foliations and their quadratic basic polynomials. Trasnf. Groups **25**, 251–277 (2020)
32. Petrunin, A.: Parallel transportation for Alexandrov space with curvature bounded below. Geom. Funct. Anal. **8**(1), 123–148 (1998)
33. Petrunin, A.: Semiconcave functions in Alexandrov's geometry. In: Surveys in Differential Geometry. Vol. XI, volume 11 of Surv. Differ. Geom., pp. 137–201. Int. Press, Somerville, MA (2007)
34. Radeschi, M.: Clifford algebras and new singular Riemannian foliations in spheres. Geom. Funct. Anal. **24**(5), 1660–1682 (2014)
35. Rataj, J., Zajíček, L.: On the structure of sets with positive reach. Math. Nachr. **290**(11–12), 1806–1829 (2017)
36. Stadler, S.: CAT(0) Spaces of higher rank i. Geom. Funct. Anal. (2022). https://doi.org/10.1007/s00039-024-00661-2

37. Stadler, S.: CAT(0) Spaces of higher rank ii. Invent. Math. **235**, 709–743 (2022)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.