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CORRECTION

Correction to: Hypercohomologies of truncated twisted holomorphic de Rham complexes

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Abstract
In the original article [1], Theorem 1.2 (Künneth theorem) is incorrect.

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In the proof of [1, Theorem 1.2] (see [1, p. 531]), we said that

"Fix two integers s and t. Consider the double complexes

$$K^{\bullet,\bullet} = \bigoplus_{\substack{u+w=s\\ v+w=t}} ss(S^{\bullet,\bullet}(X,\mathcal{L},u,v)\otimes_{\mathbb{C}} S^{\bullet,\bullet}(Y,\mathcal{H},w,w)),$$
$$L^{\bullet,\bullet} = S^{\bullet,\bullet}(X\times Y,\mathcal{L}\boxtimes\mathcal{H},s,t)$$

and a morphism $f = pr_1^*(\bullet)\wedge pr_2^*(\bullet) : K^{\bullet,\bullet}\to L^{\bullet,\bullet}$."

In general cases, f is *not* a morphism.

Example 0.1 Assume that $X=\{pt\}$ is a single-point set and Y is an n-dimensional complex manifold. Let pr_1, pr_2 be projections from $\{pt\}\times Y$ onto $\{pt\}$, Y, respectively. Consider the double complexes

$$K^{\bullet,\bullet} = \bigoplus_{\substack{u+w=0\\ v+w=n}} ss(S^{\bullet,\bullet}(\{pt\},\mathbb{C},u,v)\otimes_{\mathbb{C}} S^{\bullet,\bullet}(Y,\mathbb{C},w,w)),$$
$$L^{\bullet,\bullet} = S^{\bullet,\bullet}(\{pt\}\times Y,\mathbb{C},0,n)$$

and a *map* $f = pr_1^*(\bullet)\wedge pr_2^*(\bullet) : K^{\bullet,\bullet}\to L^{\bullet,\bullet}$.

In such case, we have

$$L^{a,b} = A^{a,b}(\{pt\}\times Y),$$
$${}_L D_1^{a,b} = \partial,\quad {}_L D_2^{a,b} = \bar{\partial}.$$

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and

$$
\begin{aligned}
K^{a,b} &= \bigoplus_{\substack{u+w=0\\ v+w=n}} \bigoplus_{\substack{p+q=a\\ r+s=b}} S^{p,r}(\{pt\}, \mathbb{C}, u, v) \otimes_{\mathbb{C}} S^{q,s}(Y, \mathbb{C}, w, w) \\
&= \bigoplus_{\substack{u+w=0\\ v+w=n}} S^{0,0}(\{pt\}, \mathbb{C}, u, v) \otimes_{\mathbb{C}} S^{a,b}(Y, \mathbb{C}, w, w) \\
&= S^{0,0}(\{pt\}, \mathbb{C}, -a, n-a) \otimes_{\mathbb{C}} S^{a,b}(Y, \mathbb{C}, a, a) \\
&= \mathcal{A}^{0,0}(\{pt\}) \otimes_{\mathbb{C}} \mathcal{A}^{a,b}(Y), \\
{}_K D_1^{a,b} &= \sum_{\substack{u+w=0\\ v+w=n}} \sum_{\substack{p+q=a\\ r+s=b}} d_{\{pt\}1}^{p,r} \otimes 1_{S^{q,s}(Y,\mathbb{C},w,w)} \\
&\quad + \sum_{\substack{u+w=0\\ v+w=n}} \sum_{\substack{p+q=a\\ r+s=b}} (-1)^{p+r} 1_{S^{p,r}(\{pt\},\mathbb{C},u,v)} \otimes d^{q,s}_{S^{\bullet,\bullet}(Y,\mathbb{C},w,w)1} \\
&= \sum_{\substack{u+w=0\\ v+w=n}} 1_{S^{0,0}(\{pt\},\mathbb{C},u,v)} \otimes d^{a,b}_{S^{\bullet,\bullet}(Y,\mathbb{C},w,w)1} \\
&= 0 \quad \left(\text{since } d^{a,b}_{S^{\bullet,\bullet}(Y,\mathbb{C},w,w)1} = 0\right), \\
{}_K D_2^{a,b} &= \sum_{\substack{u+w=0\\ v+w=n}} \sum_{\substack{p+q=a\\ r+s=b}} d_{\{pt\}2}^{p,r} \otimes 1_{S^{q,s}(Y,\mathbb{C},w,w)} \\
&\quad + \sum_{\substack{u+w=0\\ v+w=n}} \sum_{\substack{p+q=a\\ r+s=b}} (-1)^{p+r} 1_{S^{p,r}(\{pt\},\mathbb{C},u,v)} \otimes d^{q,s}_{S^{\bullet,\bullet}(Y,\mathbb{C},w,w)2} \\
&= \sum_{\substack{u+w=0\\ v+w=n}} 1_{S^{0,0}(\{pt\},\mathbb{C},u,v)} \otimes d^{a,b}_{S^{\bullet,\bullet}(Y,\mathbb{C},w,w)2} \\
&= \sum_{w=0}^{n} 1_{S^{0,0}(\{pt\},\mathbb{C},-w,n-w)} \otimes d^{a,b}_{S^{\bullet,\bullet}(Y,\mathbb{C},w,w)2} \\
&= 1_{\mathcal{A}^{0,0}(\{pt\})} \otimes \left(\sum_{w=0}^{n} d^{a,b}_{S^{\bullet,\bullet}(Y,\mathbb{C},w,w)2}\right) \\
&= 1 \otimes \bar{\partial} \quad \left(\text{since } \sum_{w=0}^{n} d^{a,b}_{S^{\bullet,\bullet}(Y,\mathbb{C},w,w)2} = d^{a,b}_{\bigoplus_{w=0}^{n} S^{\bullet,\bullet}(Y,\mathbb{C},w,w)2} = d^{a,b}_{\mathcal{A}^{\bullet,\bullet}(Y)2} = \bar{\partial}\right).
\end{aligned}
$$

We easily check that $f \circ_K D_1 = 0 \neq_L D_1 \circ f$ and $f \circ_K D_2 =_L D_2 \circ f$. So f is *not* a morphism of double complexes.

Example 0.1 also gives a counterexample to [1, Theorem 1.2]. Let $X = \{pt\}$ be a single-point set, Y an n-dimensional complex manifold, $\mathcal{L} = \mathcal{O}_X$, $\mathcal{H} = \mathcal{O}_Y$ and $s = 0$, $t = n$. Then

$$\bigoplus_{\substack{a+b=c\\u+w=0\\v+w=n}} \mathbb{H}^a\left(X, \Omega_X^{[u,v]}(\mathcal{L})\right) \otimes_{\mathbb{C}} \mathbb{H}^b\left(Y, \Omega_Y^{[w,w]}(\mathcal{H})\right)$$
$$= \bigoplus_{\substack{a+b=c\\0\le w\le n}} \mathbb{H}^a\left(\{pt\}, \Omega_{\{pt\}}^{[-w,n-w]}\right) \otimes_{\mathbb{C}} H^{b-w}\left(Y, \Omega_Y^w\right)$$
$$= \bigoplus_{0\le w\le n} \mathbb{H}^0\left(\{pt\}, \Omega_{\{pt\}}^{[0,0]}\right) \otimes_{\mathbb{C}} H^{c-w}\left(Y, \Omega_Y^w\right)$$
$$= \bigoplus_{0\le w\le n} H^0\left(\{pt\}, \mathcal{O}_{\{pt\}}\right) \otimes_{\mathbb{C}} H^{c-w}\left(Y, \Omega_Y^w\right)$$
$$= \bigoplus_{p+q=c} H^q\left(Y, \Omega_Y^p\right)$$

and

$$\mathbb{H}^c\left(X \times Y, \Omega_{X\times Y}^{[0,n]}(\mathcal{L} \boxtimes \mathcal{H})\right) = \mathbb{H}^c\left(\{pt\} \times Y, \Omega_{\{pt\}\times Y}^{[0,n]}\right) = \mathbb{H}^c\left(Y, \Omega_Y^{[0,n]}\right) = H_{dR}^c(Y, \mathbb{C})$$

However, $\bigoplus_{p+q=c} H^q(Y, \Omega_Y^p) \neq H_{dR}^c(Y, \mathbb{C})$ for a general complex manifold Y. So [1, Theorem 1.2] *doesn't* hold.

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Reference

1. Meng, L.: Hypercohomologies of truncated twisted holomorphic de Rham complexes. Ann. Glob. Anal. Geom. **57**(4), 519–535 (2020). https://doi.org/10.1007/s10455-020-09711-y