Annals of Global Analysis and Geometry (2024) 65:13
https://doi.org/10.1007/s10455-023-09943-8

# Some remarks on almost Hermitian functionals

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Received: 1 October 2023 / Accepted: 30 December 2023 / Published online: 31 January 2024


**Abstract**
We study critical points of natural functionals on various spaces of almost Hermitian structures on a compact manifold $M^{2n}$. We present a general framework, introducing the notion of gradient of an almost Hermitian functional. As a consequence of the diffeomorphism invariance, we show that a Schur's type theorem still holds for general almost Hermitian functionals, generalizing a known fact for Riemannian functionals. We present two concrete examples, the Gauduchon's functional and a close relative of it. These functionals have been studied previously, but not in the most general setup as we do here, and we make some new observations about their critical points.

**Keywords** Almost Hermitian functionals · Critical almost Hermitian structures

## 1 Introduction and general setup

### 1.1 Introduction

Let $M^{2n}$ be a compact even-dimensional oriented smooth manifold admitting almost complex structures. Then, $M^{2n}$ admits (many) almost Hermitian structures, that is, triples $(g, J, \omega)$ of a Riemannian metric $g$, an almost complex structure $J$, and a non-degenerate 2-form $\omega$, related by the compatibility relation

$$\omega(X, Y) = g(JX, Y), \ \forall X, Y \in TM. \tag{1}$$

Note that the compatibility relation (1) determines any one of the elements of the triple $(g, J, \omega)$ in terms of other two.

Let $\mathcal{AH}$ denote the space of all almost Hermitian structures on $M^{2n}$. This has a natural structure of an infinite dimensional manifold endowed even with interesting Riemannian metrics as described for example in [19].

Dedicated to Professor David E. Blair.

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We call $\mathcal{L} : \mathcal{AH} \to \mathbb{R}$ an *almost Hermitian functional*, a map of the type

$$\mathcal{L}(g, J, \omega) = \int_M \ell(g, J, \omega)\, \mu_g,$$

where $\ell(g, J, \omega)$ is some scalar function that depends on the almost Hermitian structure $(g, J, \omega)$ and which behaves naturally with respect diffeomorphisms:

$$\ell(\varphi(g, J, \omega)) = \varphi^*\big(\ell(g, J, \omega)\big)\ .$$

Here $\varphi(g, J, \omega) = (\varphi^* g, \varphi_*^{-1} J \varphi_*, \varphi^* \omega)$. Since the volume form

$$\mu_g = \sqrt{\det(g_{ij})}\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge ... \wedge \mathrm{d}x^{2n} = \frac{\omega^n}{n!}$$

also behaves naturally with respect to diffeomorphisms, $\mu_{\varphi^* g} = \varphi^*(\mu_g)$, the above condition on $\ell$ implies that any almost Hermitian function is, by definition, invariant under diffeomorphisms:

$$\mathcal{L}(\varphi(g, J, \omega)) = \mathcal{L}(g, J, \omega),\ \ \forall \varphi \in \mathcal{D}iff(M)\ . \tag{2}$$

In all interesting examples (and certainly in the examples we present), almost Hermitian functionals will also re-scale nicely with respect to the action of homotheties, $\lambda(g, J, \omega) = (\lambda g, J, \lambda\omega)$, that is,

$$\mathcal{L}(\lambda(g, J, \omega)) = \lambda^k \mathcal{L}(g, J, \omega),$$

for some power $k \in \mathbb{R}$ and any scalar $\lambda > 0$.

Various almost Hermitian functionals, although not given this name, have been considered in many previous works. One goal of this note is to present a general setup for critical points of almost Hermitian functionals on natural subspaces of $\mathcal{AH}$. As such, some of the results of our note are not new, but a recast in somewhat more general terms of known facts (see, for example, Propositions 1.5, 1.6, 3.4, Theorem 3.5). However, as a byproduct of our setup, we do make some new observations, which we hope to be of interest and open up roads for further investigations.

### 1.2 Riemannian functionals

When restricted to $\mathcal{AH}$ (and subspaces of it), *Riemannian functionals* are particular examples of almost Hermitian functionals. As there is a vast literature on Riemannian functionals (for example, see Chapter 4 of Besse [6]), we only briefly review here some basic facts and terminology relevant to us later. A *Riemannian functional*, $\mathcal{F}$, is a map, invariant under diffeomorphisms,

$$\mathcal{F} : \mathcal{M} \to \mathbb{R}, \quad \mathcal{F}(g) = \int_M f(g)\, \mu_g,$$

where $f(g)$ is some scalar quantity depending on the metric. Here $\mathcal{M}$ denotes the space of all Riemannian metrics on $M$. There is extensive work on the geometry of the infinite-dimensional manifold $\mathcal{M}$ (e.g., see [6, 14, 15]).

Given a Riemannian metric $g$, a variation $(g_t)_t$ of Riemannian metrics has the form

$$g_t = g + th + o\left(t^2\right),$$

where $h$ is a symmetric tensor on $M$, and, conversely, given any symmetric tensor $h$, one can find a variation $g_t$ with $\dot{g} := \frac{\mathrm{d}}{\mathrm{d}t}(g_t)|_{t=0} = h$. Thus,

$$T_g\mathcal{M} = \Gamma\left(S^2M\right) ,$$

where $S^2M$ denotes the bundle of symmetric tensors and $\Gamma(V)$ denotes here and everywhere below the $C^\infty$-sections of a vector bundle $V$. We assume that $\mathcal{F}$ is differentiable and admits a gradient, that is, there is a symmetric tensor $(\operatorname{grad}\mathcal{F})_g$ (that depends on $g$), so that for any variation $g_t$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(g_t)|_{t=0} = \int_M \left((\operatorname{grad}\mathcal{F})_g, h\right)_g \mu_g .$$

The diffeomorphism invariance of $\mathcal{F}$

$$\mathcal{F}\left(\varphi^* g\right) = \mathcal{F}(g),\ \ \forall\varphi \in \mathcal{D}iff(M)$$

yields the fact that the gradient of $\mathcal{F}$ is divergence free, that is,

$$\delta^g(\operatorname{grad}\mathcal{F}_g) = 0. \tag{3}$$

For Riemannian functionals which rescale with homotheties but are not invariant under them, i.e., $\mathcal{F}(\lambda g) = \lambda^k\mathcal{F}(g)$ with $k \neq 0$, critical metrics $g$ of $\mathcal{F}$ on the whole space $\mathcal{M}$ will always have critical value $\mathcal{F}(g) = 0$. For this reason, critical points are most often considered for $\mathcal{F}$ restricted to smaller subspaces of $\mathcal{M}$, for example, to $\mathcal{M}_1$, the space of all of Riemannian metrics of total volume 1 (alternatively, the functional can be normalized by a suitable power of the total volume), or even to the space $\mathcal{M}^\mu$ of all Riemannian metrics with a fixed volume form $\mu$ (assumed to yield total volume 1, so that $\mathcal{M}^\mu \subset \mathcal{M}_1$ ). Note that since

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu_{g_t})|_{t=0} &= \frac{1}{2}(h, g)_g = \frac{1}{2}\operatorname{tr}_g h \ , \\
T_g\mathcal{M}^\mu &= \left\{h \in \Gamma\left(S^2(M)\right) \mid \operatorname{tr}_g h = 0\right\} , \\
T_g\mathcal{M}_1 &= \left\{h \in \Gamma\left(S^2(M)\right) \mid \int_M \operatorname{tr}_g h\, \mu_g = 0\right\} .
\end{aligned} \tag{4}$$

One immediately obtains the following well-known fact:

**Proposition 1.1** *A metric $g$ is a critical metric for the Riemannian functional $\mathcal{F}$ restricted to $\mathcal{M}^\mu$ if and only if* $(\operatorname{grad}\mathcal{F})_g = c\, g$ *for a constant $c$ if and only if $g$ is a critical metric for the Riemannian functional $\mathcal{F}$ on $\mathcal{M}_1$.*

The second equivalence follows straight from the description of $T_g\mathcal{M}_1$. Using the description of the tangent space $T_g\mathcal{M}^\mu$ above, the critical condition at a metric $g$ for $\mathcal{F}$ restricted to $\mathcal{M}^\mu$, would seem to be the weaker $(\operatorname{grad}\mathcal{F})_g = \lambda\, g$ for a function (possibly, non-constant) $\lambda$. However, the divergence free condition on the gradient (3) implies that $\lambda$ must be a constant. This fact can be thought of as a Schur's type theorem for each Riemannian functional $\mathcal{F}$, a consequence of the diffeomorphisms invariance. If $\mathcal{F}$ is the Hilbert functional

$$\mathcal{H}(g) = \int_M s_g\ \mu_g ,$$

then this phenomenon is exactly the Schur's theorem for (pseudo-)Riemannian metrics (usually seen as a consequence of Bianchi's differential identity): if the trace-free part of the Ricci tensor vanishes, $Ric_0 = 0$, then $Ric = cg$, with $c = \frac{s}{\dim M}$ a constant, i.e., the metric is Einstein. Indeed, as is well known (see, e.g., [6]), the gradient of the Hilbert functional is

$$(\mathrm{grad}\,\mathcal{H})_g = \frac{1}{2}s_g\, g - Ric_g\ . \tag{5}$$

We will investigate below whether this Schur's type theorem remains valid in case of general almost Hermitian functionals (see part (b) of Theorem 1.4).

### 1.3 General setup for almost Hermitian functionals

Let us now come back to general almost Hermitian functionals on the space of all almost Hermitian metrics $\mathcal{AH}$ and natural subspaces of it. Fix an almost Hermitian structure $(g, J, \omega)$ on $M^{2n}$. A general variation in the space $\mathcal{AH}$ has the form $(g_t, J_t, \omega_t)$, with

$$g_t = g + th + o\left(t^2\right), \quad J_t = J + tK + o\left(t^2\right), \quad \omega_t = \omega + t\alpha + o\left(t^2\right)\ , \tag{6}$$

where $h$ is a symmetric tensor, $\alpha$ is a 2-form, and $K$ is a certain endomorphism of $TM$. It is important to note that the three elements $h, K, \alpha$ are *not* independent of each other. In fact, the variation of $J_t^2 = -id$, yields

$$JK + KJ = 0\ , \tag{7}$$

and the variation of compatibility relation (1) for $(g_t, J_t, \omega_t)$ implies

$$\alpha(X, Y) = h(JX, Y) + g(KX, Y)\ , \forall X, Y \in TM\ . \tag{8}$$

Relations (8) and (7) imply that the $J$-invariant parts of $h$ and $\alpha$ must satisfy

$$h'(JX, Y) = \alpha'(X, Y)\ , \forall X, Y \in TM,$$

which can be rewritten as

$$h' \circ J = \alpha'\ . \tag{9}$$

Further, relations (8) and (7) imply that the $J$-anti-invariant parts of $h$ and $\alpha$ completely determine the endomorphism $K$.

$$g(KX, Y) = \alpha''(X, Y) - h''(JX, Y)\ . \tag{10}$$

It can be easily checked that the endomorphism $K$ determined by (10) satisfies relation (7).

Let us make at this point an interlude on notation. In the relations above and throughout the paper, we will denote with superscript $'$ (resp. $''$) the $J$-invariant part (resp. the $J$-anti-invariant part) of a real symmetric tensor or of a two form. Thus, if $\Phi$ is a 2-form (or a symmetric tensor),

$$\Phi'(\cdot,\cdot) = \frac{1}{2}\big(\Phi(\cdot,\cdot) + \Phi(J\cdot, J\cdot)\big)\, ,\ \Phi''(\cdot,\cdot) = \frac{1}{2}\big(\Phi(\cdot,\cdot) - \Phi(J\cdot, J\cdot)\big)\ .$$

This corresponds to the $J$-induced decomposition of the bundle of (real) two forms, or the bundle of symmetric tensors:

$$\Lambda^2 M = \Lambda^{1,1}_{\mathbb{R}} M \oplus [\![\Lambda^{0,2} M]\!], \quad S^2 M = S^{1,1}_{\mathbb{R}} M \oplus [\![S^{0,2} M]\!].$$

As the notation already suggests, these are closely related to the familiar type decomposition induced by $J$ on complex bundles of forms and tensors. Here and henceforth $[\![\ \cdot\ ]\!]$ denotes the real vector bundle underlying a given complex bundle. The real bundle $[\![\Lambda^{0,2}M]\!]$ (resp. $[\![S^{0,2}M]\!]$) inherits a canonical complex structure, still denoted by $J$, which is given by

$$(J\Phi)(X,Y) := -\Phi(JX,Y),\ \forall\Phi\in[\![\Lambda^{0,2}M]\!],$$

so that $([\![\Lambda^{0,2}M]\!], J)$ becomes isomorphic to the complex bundle $\Lambda^{0,2}M$. We adopt a similar definition for the action of $J$ on $[\![S^{0,2}M]\!]$. Notice that, using the metric $g$, $[\![S^{0,2}M]\!]$ can be also viewed as the bundle of symmetric, $J$-anti-commuting endomorphisms of $TM$. Note also that via a composition with $J$ as in (9), the bundles $S^{1,1}_{\mathbb{R}}M$ and $\Lambda^{1,1}_{\mathbb{R}}M$ are isomorphic.

Coming back, as consequence of relations (9) and (10), the tangent space to $\mathcal{AH}$ at $(g,J,\omega)$ is described by pairs $(h,\alpha)$, where $h$ is a symmetric tensor and $\alpha$ is a 2-form, whose $J$-invariant parts are identified via relation (9),

$$T_{(g,J,\omega)}\mathcal{AH} = \{(h,\alpha)\,|\,h\in\Gamma\left(S^2(M)\right)\ ,\ \ \alpha\in\Gamma(\Lambda^2M)\ ,\ \ h'\circ J=\alpha'\}.$$

In fact, the above arguments only show the inclusion "⊆" in the claimed description of the tangent space above. For the other inclusion, one shows that a suitable exponential map will yield from any pair $(h,\alpha)$ as above a family $(g_t, J_t, \omega_t)$ of almost Hermitian structures with $(\dot g, \dot J, \dot\omega) = (h, K, \alpha)$, where $K$ is determined by $(h,\alpha)$ via (10). This is the case for all other tangent spaces of subspaces of $\mathcal{AH}$ that we consider below (see [19] for details).

Next we consider various natural subspaces of $\mathcal{AH}$. Let $\mathcal{AH}_1$ be the subspace of most Hermitian structures of total volume 1, and let $\mathcal{AH}^\mu$ the subspace of almost Hermitian structures with fixed volume $\mu$ (as before, $\mu$ is taken to give a total volume 1, so that $\mathcal{AH}^\mu\subset\mathcal{AH}_1$). Then,

$$T_g\mathcal{AH}^\mu = \left\{(h,\alpha)\,|\,h'\circ J=\alpha',\ (h,g)_g = 2(\alpha,\omega)_g = 0\right\}\ ,$$
$$T_g\mathcal{AH}_1 = \left\{(h,\alpha)\,|\,h'\circ J=\alpha',\ \int_M \mathrm{tr}_g h\,\mu_g = 0\right\}\ .$$

Also very important are the subspaces of $\mathcal{AH}$ when one of the elements of the triple $(g,J,\omega)$ are fixed. Denote by $\mathcal{AH}^\omega$, $\mathcal{AH}^J$, $\mathcal{AH}^g$, respectively, the subspaces when $\omega$, $J$ and $g$, respectively, are fixed. Variational problems in each of these subspaces have been considered before. For $\mathcal{AH}^\omega$ (assuming that $\omega$ is also closed, thus symplectic), see, for example, [7–10, 26, 31] and others. For $\mathcal{AH}^J$, especially assuming that the fixed $J$ is integrable, see, for example, [11, 17, 18] and many others. For $\mathcal{AH}^g$, see, for example, [32]. The tangent spaces for each of these subspaces have been determined in previous works (but see [19] for all of them). In the case of $\mathcal{AH}^\omega$, for instance, notice that all variations in this space must have $\alpha$ identically 0. From equation (9), we also deduce that the $J$-invariant part of $h$ must be identically 0; thus, we obtain as in [7] that the tangent space of $\mathcal{AH}^\omega$ at a point $(g,J,\omega)$ consists of the space of symmetric, $J$-anti-invariant tensors:

$$T_{(g,J,\omega))}\mathcal{AH}^\omega = \{(h,0)|h'=0\}\cong\Gamma([\![S^{0,2}M]\!])\ . \tag{11}$$

For $\mathcal{AH}^J$, as $J$ is fixed, all variations in this space must have $K$ identically 0. Therefore, in this case relation (10) implies that the $J$-anti-invariant parts of both $h$ and $\alpha$ must vanish (note that $h''(JX,Y)$ is symmetric in $X,Y$). As relation (9) must still be satisfied, we get that the tangent space of $\mathcal{AH}^J$ is expressed either by the space of $J$-invariant symmetric tensors, or the space of $J$-invariant 2-forms:

$$T_{(g,J,\omega)}\mathcal{AH}^J = \{(h,\alpha)|h''=0,\ \alpha''=0,\ h'\circ J=\alpha'\}\cong\Gamma(S^{1,1}_{\mathbb{R}}M)\cong\Gamma(\Lambda^{1,1}_{\mathbb{R}}M)\ . \tag{12}$$

Finally, all variations in the space $\mathcal{AH}^g$ must have $h$ identically 0; thus, the $J$-invariant part of $\alpha$ must also vanish, so the tangent space of $\mathcal{AH}^g$ is identified with the space of $J$-anti-invariant

2-forms:

$$T_{(g,J,\omega)}\mathcal{AH}^g = \{(0,\alpha)|\alpha' = 0\} \cong \Gamma([\![\Lambda^{0,2}M]\!]) \ . \tag{13}$$

Let now $\mathcal{L} : \mathcal{AH} \to \mathbb{R}$ be an almost Hermitian functional, which we will assume to be differentiable in the sense that the first variation in $\mathcal{L}$ in $\mathcal{AH}$ is given by a pair $(T, \Psi)$, where $T \in \Gamma(S^2M)$ is a symmetric tensor and $\Psi \in \Gamma(\Lambda^2 M)$ is a 2-form on $M$ via

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathcal{L}(g_t, J_t, \omega_t)\Big)|_{t=0} = \int_M \Big((T,h)_g + 2(\Psi,\alpha)_g\Big)\, \mu_g, \tag{14}$$

for any path with $\frac{\mathrm{d}}{\mathrm{d}t}(g_t, J_t, \omega_t)|_{t=0} = (h, K, \alpha)$ satisfying relations (7), (8), (9), (10) as above.

***Remark 1.2*** It is tempting to think of the pair $(T, \Psi)$ in (14) as the "gradient" of the almost Hermitian functional $\mathcal{L}$, but note that some precaution should be taken. Given that relation (9) holds between $h$ and $\alpha$, note that $(T, \Psi)$ are **not** uniquely determined. Rather, more appropriately, the triple $(T'', T' \circ J + \Psi', \Psi'')$, or $(T'', T' - \Psi' \circ J, \Psi'')$ should be considered the gradient of $\mathcal{L}$, as these **are** uniquely determined. For this reason, (14) is better written in one of the following equivalent forms

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathcal{L}(g_t, J_t, \omega_t)\Big)|_{t=0} = \int_M \Big((T'',h)_g + (T' - \Psi' \circ J, h)_g + 2(\Psi'',\alpha)_g\Big)\, \mu_g \ , \tag{15}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathcal{L}(g_t, J_t, \omega_t)\Big)|_{t=0} = \int_M \Big((T'',h)_g + 2(T' \circ J + \Psi', \alpha)_g + 2(\Psi'',\alpha)_g\Big)\, \mu_g \ . \tag{16}$$

We will prove in next section the following theorem regarding the first variation of the invariance under diffeomorphisms for an almost Hermitian functional:

**Theorem 1.3** *Suppose that $\mathcal{L}$ is an almost Hermitian functional whose gradient is given by the triple $(T'', T' - \Psi' \circ J, \Psi'')$ as in (15). Then, for any almost Hermitian structure $(g, J, \omega)$, we have*

$$\delta T'' + \delta(T' - \Psi' \circ J) + \big(\Psi'', \nabla.\omega\big)_g - \delta(J\Psi'') = 0 \ . \tag{17}$$

Note that if $\mathcal{L}$ is a Riemannian functional, then $\Psi = 0$ and $T$ is the gradient, so relation (17) is immediately seen to be equivalent to $\delta T = 0$ as we noted in the previous subsection.

From Theorem 1.3 and the previous description of the tangent spaces of $\mathcal{AH}$, $\mathcal{AH}_1$, and $\mathcal{AH}^\mu$ one obtains the following result (also partially proved in [24] for some specific functionals on $\mathcal{AH}$ and $\mathcal{AH}_1$). The less obvious part is the first equivalence of part (b). This is the Schur's type theorem which still holds for almost Hermitian functionals (compare with Proposition 1.1).

**Theorem 1.4** *Suppose that $\mathcal{L}$ is an almost Hermitian functional whose gradient is given by the triple $(T'', T' - \Psi' \circ J, \Psi'')$ as in (15). Then,*

(a) *$(g, J, \omega)$ is a critical point for $\mathcal{L}$ on $\mathcal{AH}$ if and only if*

$$T'' = 0, \quad \Psi'' = 0, \quad T' - \Psi' \circ J = 0 \ ; \tag{18}$$

(b) *$(g, J, \omega)$ is a critical point for $\mathcal{L}$ on $\mathcal{AH}^\mu$ if and only if*

$$T'' = 0, \quad \Psi'' = 0, \quad T' - \Psi' \circ J = cg \ , \textit{ for some constant } c, \tag{19}$$

*if and only if $(g, J, \omega)$ is a critical point for $\mathcal{L}$ on $\mathcal{AH}_1$.*

***Proof*** Part (a) is immediate from the description of the tangent space for $\mathcal{AH}$. For part (b), the fact that the critical point condition for $\mathcal{L}$ on $\mathcal{AH}_1$ consists exactly of the requirements in (19) follows again from the description of the tangent space for $\mathcal{AH}_1$ (and this was also shown in [24] for the functionals considered there). From the description of the tangent space for $\mathcal{AH}^\mu$, it would seem that we get the apparently weaker critical condition: $(g, J, \omega)$ is a critical point for $\mathcal{L}$ on $\mathcal{AH}^\mu$ if and only if

$$T'' = 0, \quad \Psi'' = 0, \quad T' - \Psi' \circ J = \lambda g \ , \ \text{for some function } \lambda. \tag{20}$$

However, at a critical point we use (17) from Theorem 1.3 together with (20), to conclude

$$0 = \delta(T' - \Psi' \circ J) = \delta(\lambda g) = -d\lambda \ .$$

Thus, $\lambda$ must be a constant. □

We also have the following general result, particular cases of which have been proved before by various authors (see for example [7, 10, 18, 24, 25, 32]).

**Proposition 1.5** *Suppose that $\mathcal{L}$ is an almost Hermitian functional whose first variation is given by* (15) *(or, equivalently, by* (16)*). Then*

(a) *$(g, J, \omega)$ is a critical point for $\mathcal{L}$ on $\mathcal{AH}^\omega$ if and only if $T'' = 0$.*
(b) *$(g, J, \omega)$ is a critical point for $\mathcal{L}$ on $\mathcal{AH}^J$ if and only if $T' - \Psi' \circ J = 0$.*

*Part (b) has the following subcases:*

(b1) *$(g, J, \omega)$ is a critical point for $\mathcal{L}$ on $\mathcal{AH}_1^J$ if and only if $T' - \Psi' \circ J = cg$, for some constant $c$.*
(b2) *$(g, J, \omega)$ is a critical point for $\mathcal{L}$ on $\mathcal{AH}^{\mu,J}$ if and only if $T' - \Psi' \circ J = \lambda g$, for some function (possibly non-constant) $\lambda$.*
(b3) *$(g, J, \omega)$ is a critical point for $\mathcal{L}$ on $\mathcal{AH}_1^{[g],J}$ if and only if $\operatorname{tr}_g(T' - \Psi' \circ J) = c$, for some constant $c$. Here $\mathcal{AH}_1^{[g],J}$ denotes the space of almost Hermitian structures of total volume 1, having a fixed almost complex structure $J$, and whose metric is in a fixed conformal class $[g]$.*

(c) *$(g, J, \omega)$ is a critical point for $\mathcal{L}$ on $\mathcal{AH}^g$ if and only if $\Psi'' = 0$.*

***Proof*** The proof for parts (a), (b), (c) follows directly from the description of the tangent spaces of those subspaces of $\mathcal{AH}$. We will only give some brief details for (b3) and (b1) and leave the rest to the reader. It is easy to see that the tangent space to $\mathcal{AH}_1^{[g],J}$ is given by

$$T_{(g,J,\omega)}\mathcal{AH}_1^{[g],J} = \{f(g, \omega) | \int_M f \, \mu_g = 0\} \ . \tag{21}$$

Assuming $(g, J, \omega)$ is a critical point for $\mathcal{L}$ on $\mathcal{AH}_1^{[g],J}$, for a variation with $(h, \alpha) = f(g, \omega)$ as above, Eq. (15) yields:

$$0 = \int_M f\lambda \, \mu_g \ , \ \text{ where here we denote } \lambda = \operatorname{tr}_g(T' - \Psi' \circ J) \ .$$

As the above relation holds for any function $f$ with zero integral, it follows that $\lambda = \operatorname{tr}_g(T' - \Psi' \circ J)$ must be a constant. (Take $f = \lambda - \int_M \lambda \, \mu_g$, in the above equality.)

For (b1), now assume that $(g, J, \omega)$ is a critical point for $\mathcal{L}$ on $\mathcal{AH}_1^J$. As $\mathcal{AH}_1^{[g],J}$ is a subspace of $\mathcal{AH}_1^J$, we already know that $\mathrm{tr}_g(T' - \Psi' \circ J)$ must be a constant. We show next that the trace-free part of $(T' - \Psi' \circ J)$ must vanish. The tangent space to $\mathcal{AH}_1^J$ is given by

$$T_{(g,J,\omega)}\mathcal{AH}_1^J = \{(h, \alpha) | h'' = 0,\ \alpha'' = 0,\ h' \circ J = \alpha',\ \int_M \mathrm{tr}_g h\, \mu_g = 0\} .$$

Note that we could take a variation so that $h = (T' - \Psi' \circ J)_0$, where the subscript denotes the trace-free part of the symmetric tensor, and just take $\alpha = h \circ J$. This chosen pair $(h, \alpha)$ is in the tangent space $T_{(g,J,\omega)}\mathcal{AH}_1^J$, and used in equation (15) together with the critical point condition for $(g, J, \omega)$, it implies that the $L^2$-norm of $(T' - \Psi' \circ J)_0$ is zero, hence $(T' - \Psi' \circ J)_0 = 0$. Thus, we proved that at a critical point $T' - \Psi' \circ J = cg$, for some constant $c$. The converse is immediate. □

In regard to this result, note that the conditions (b1) and (b2) might be genuinely different. As $T''$ and $\Psi''$ are not necessarily zero in this case, we cannot apply the general Schur's type theorem we proved in Theorem 1.4 (b). We added part (b3) to make the connection with the variational problems studied recently in [5]. Surely, there are other interesting subspaces of $\mathcal{AH}$ worth considering, but for now we limit ourselves to the above.

Another point worth noting is that the difficulty in applying Proposition 1.5 to various concrete almost Hermitian functionals is computing the pair $(T, \Psi)$ giving the first variation of the functional on $\mathcal{AH}$, or, rather more precisely, the triple $(T'', T' - \Psi' \circ J, \Psi'')$. We will consider in subsequent sections a couple of concrete examples of almost Hermitian functionals for which we can effectively compute their gradients on $\mathcal{AH}$.

Finally, note that for any Riemannian functional $\mathcal{F}$, $\Psi$ is identically 0, and $T$ is just the gradient of the Riemannian functional grad $\mathcal{F}$, so in Proposition 1.5, the $J$-invariant and the $J$-anti-invariant components of the gradient will determine the critical metric conditions (of course, part (c) is trivial for any Riemannian functional). As an initial example, we apply Proposition 1.5 for the Hilbert functional $\mathcal{H}$, whose gradient is well known (see (5)). We immediately obtain the following critical point conditions, which could all be considered as weakening of the Einstein condition.

**Proposition 1.6** *Let $(M^{2n}, g, J, \omega)$ be a compact almost Hermitian manifold. Then:*

(a) *$(g, J, \omega)$ is a critical point for the Hilbert functional $\mathcal{H}$ restricted to $\mathcal{AH}^\omega$ if and only if the Ricci tensor is $J$-invariant (i.e., $Ric'' = 0$).*

(b) *$(g, J, \omega)$ is a critical point for the Hilbert functional $\mathcal{H}$ restricted to $\mathcal{AH}^J$ if and only if the Ricci tensor is $J$-anti-invariant (i.e., $Ric' = 0$).*

(b1) *$(g, J, \omega)$ is a critical point for the Hilbert functional $\mathcal{H}$ restricted to $\mathcal{AH}_1^J$ if and only if $Ric' = cg$, for some constant $c$.*

(b2) *$(g, J, \omega)$ is a critical point for the Hilbert functional $\mathcal{H}$ restricted to $\mathcal{AH}^{\mu,J}$ if and only if $Ric' = \lambda g$, for some function (possibly non-constant) $\lambda$.*

(b3) *$(g, J, \omega)$ is a critical point for the Hilbert functional $\mathcal{H}$ restricted to $\mathcal{AH}_1^{[g],J}$ if and only if the scalar curvature $s_g$ is a constant.*

Some further remarks on the proposition above. Regarding part (a), of course Kähler structures automatically have $J$-invariant Ricci tensor, but note that nearly Kähler structures also do. In the case $\omega$ is also assumed to be closed, there is quite a bit of work regarding almost Kähler structures having $J$-invariant Ricci tensor (for example, see [1, 2, 12, 13, 16, 22]). Most of these works stem from a question of Blair and Ianus [10], where part (a) of the above proposition was first proved. The question of Blair and Ianus, in turn, is related to a famous, still open conjecture of Goldberg [20] regarding compact Einstein almost Kähler manifolds (see [30] for the most important partial result on the Golberg conjecture). Recently, there is an even higher interest investigating Hermitian manifolds (that is with $J$ assumed integrable) having $J$-invariant Ricci tensor and constant scalar curvature (for example, see [3] and the references therein) due to their connections to Riemannian Einstein–Maxwell metrics as introduced by LeBrun [23] and extremal Kähler metrics.

Not much is known in general about critical almost Hermitian structures for the Hilbert functional as in parts (b), (b1), (b2). However, on complex surfaces, Apostolov and Muskarov [4] obtained some very nice results regarding Hermitian metrics satisfying conditions (b1), (b2). They showed that there exist compact, non-Einstein examples of such metrics, and under some additional assumptions they gave classification results.

To finish with these remarks on the Hilbert functional $\mathcal{H}$, note that restricted to $\mathcal{AH}^{\mu}$, or $\mathcal{AH}_1$, its critical points are only the almost Hermitian structures with an Einstein metric.

## 2 More preliminaries and Proof of Theorem 1.3

Let $(M^{2n}, g, J, \omega)$ be an almost Hermitian manifold. We denote by $\nabla$ the Levi-Civita connection and by $N$ the Nijenhuis tensor of $J$. Consider the Lefschetz operator on forms $L_\omega = \omega\wedge\cdot$ and its adjoint $\Lambda_\omega := L_\omega^*$. The *Lee form* (or *the torsion 1-form*) $\theta \in \Lambda^1 M$ of the almost Hermitian structure $(g, J, \omega)$ is defined by

$$\theta = \Lambda_\omega(\mathrm{d}\omega) = J\delta\omega. \tag{22}$$

Alternatively, $\theta$ is identified with $d\left(\omega^{n-1}\right)$ via the isomorphism $L_\omega^{n-1} : \Lambda^1 M \to \Lambda^{2n-1} M$ by

$$d\left(\omega^{n-1}\right) = \theta \wedge \omega^{n-1}. \tag{23}$$

The following proposition is well known (e.g., see [27]).

**Proposition 2.1** *Let $(M^{2n}, g, J, \omega)$ be an almost Hermitian manifold. The covariant derivative $\nabla\omega$ is given in terms of $d\omega$ and the Nijenhuis tensor by*

$$\nabla_X\omega = (i_X \mathrm{d}\omega)'' + \frac{1}{2} N_{JX} , \tag{24}$$

*where $N_{JX}(Y, Z) := (JX, N(Y, Z))_g$, for all $X, Y, Z \in TM$.*
*In dimension $2n = 4$, as $\mathrm{d}\omega = \theta \wedge \omega$, the above relation takes the form*

$$\nabla_X\omega = (X^\flat \wedge J\theta)'' + \frac{1}{2} N_{JX} . \tag{25}$$

In Proposition 2.3, we establish some formulas that we need later for the $J$-invariant and $J$-anti-invariant components of $\mathrm{d}J\theta$ and $\mathrm{d}\theta$ on an arbitrary almost Hermitian manifold. These may also be known, but, for completeness, we present their proofs. We first start with a lemma valid for an arbitrary 1-form.

**Lemma 2.2** *Let $(M^{2n}, g, J, \omega)$ be an almost Hermitian structure with Lee 1-form $\theta$. For any 1-form $\tau$ on $M$, we have*

$$(\mathrm{d}J\tau)(X,Y) = -(i_{\tau^\sharp}\mathrm{d}\omega)(X,Y) + (\nabla_{\tau^\sharp}\omega)(X,Y) - \Big((\nabla_X\tau)(JY) - (\nabla_Y\tau)(JX)\Big) \tag{26}$$

$$(\mathrm{d}J\tau)'' = \frac{1}{2}N_{J\tau^\sharp} + J(\mathrm{d}\tau)'' \tag{27}$$

$$(\mathrm{d}J\tau)' = -(i_{\tau^\sharp}\mathrm{d}\omega)' + 2((\nabla\tau)^{sym})' \circ J \tag{28}$$

$$(\mathrm{d}J\tau, \omega)_g = -\big(\theta, \tau\big)_g - \delta\tau\ . \tag{29}$$

***Proof*** Computing the differential of the 1-form $J\tau$, we get

$$\begin{aligned}(\mathrm{d}J\tau)(X,Y) &= (\nabla_X J\tau)(Y) - (\nabla_Y J\tau)(X)\\ &= [(\nabla_X J\tau) + J(\nabla_X\tau)]\,(Y) - [(\nabla_Y J\tau) + J(\nabla_Y\tau)]\,(X)\\ &= -\tau((\nabla_X J)Y - (\nabla_Y J)X) - (\nabla_X\tau)(JY) + (\nabla_Y\tau)(JX)\\ &= (\nabla_X\omega)(\tau^\sharp, Y) + (\nabla_Y\omega)(X, \tau^\sharp) - \Big((\nabla_X\tau)(JY) - (\nabla_Y\tau)(JX)\Big)\\ &= (\mathrm{d}\omega)(X, \tau^\sharp, Y) - (\nabla_{\tau^\sharp}\omega)(Y, X) - \Big((\nabla_X\tau)(JY) - (\nabla_Y\tau)(JX)\Big),\end{aligned}$$

which can be immediately seen to be equivalent to the formula (26) claimed in the lemma. The formulas (27) and (28) follow by taking the $J$-anti-invariant, respectively, $J$-invariant components of the (26), also using relation (24) from Proposition 2.1. Finally, formula (29) follows by taking the inner product of (28) with $\omega$. We are also using that $i_X\ \cdot$ and $X^\flat \wedge\ \cdot$ are point-wise adjoints for inner product on forms, as are $\Lambda_\omega$ and $L_\omega$. That is, the following computation holds:

$$-\big(i_{\tau^\sharp}\mathrm{d}\omega, \omega\big)_g = -\big(\mathrm{d}\omega, \tau\wedge\omega\big)_g = -\big(\mathrm{d}\omega, L_\omega\tau\big)_g = -\big(\Lambda_\omega\mathrm{d}\omega, \tau\big)_g = -\big(\theta, \tau\big)_g\ .$$

□

Specializing the previous Lemma with $\tau = \theta = J\delta\omega$, and $\tau = J\theta = -\delta\omega$, we get the following:

**Proposition 2.3** *Let $(M^{2n}, g, J, \omega)$ be an almost Hermitian structure with Lee 1-form $\theta$. Then,*

$$-\,(\mathrm{d}\delta\omega)'' = (\mathrm{d}J\theta)'' = \frac{1}{2}N_{J\theta^\sharp} + J(\mathrm{d}\theta)'' \tag{30}$$

$$-(\mathrm{d}\delta\omega)' = (\mathrm{d}J\theta)' = -(i_{\theta^\sharp}\mathrm{d}\omega)' + 2((\nabla\theta)^{\mathrm{sym}})' \circ J \tag{31}$$

$$(\mathrm{d}\theta)' = (i_{J\theta^\sharp}\mathrm{d}\omega)' - 2((\nabla J\theta)^{\mathrm{sym}})' \circ J \tag{32}$$

$$\big(\mathrm{d}\delta\omega, \omega\big)_g = -\big(\mathrm{d}J\theta, \omega\big)_g = |\theta|_g^2 + \delta\theta \tag{33}$$

$$\big(\mathrm{d}\theta, \omega\big)_g = 0\ . \tag{34}$$

We end this section by proving Theorem 1.3 stated in the previous section.

***Proof of Theorem 1.3*** Assume $(g_t, J_t, \omega_t) = \phi_t^*(g, J, \omega)$, where $\phi_t$ is a family of diffeomorphisms. Then, $h = L_X g = 2\delta^*(i_X g)$ and $\alpha = L_X\omega = d(i_X\omega) + i_X(d\omega)$, where $X$ is the vector field induced by $\phi_t$ and $L_X$ denotes the Lie derivative in its direction. Then, using the diffeomorphism invariance of $\mathcal{L}$ and its 1st variation formula (15), we have:

$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}\Big(\phi_t^*(g, J, \omega)\Big) \\
&= \int_M \big(T - \Psi' \circ J \ ,\ 2\delta^*(i_X g)\big)_g + 2\big(\Psi'', \mathrm{d}(i_X\omega) + i_X(\mathrm{d}\omega)\big)_g \ \mu_g \\
&= 2\int_M \big(\delta T'' + \delta(T' - \Psi' \circ J), X^\flat\big)_g - \big(J\delta\Psi'', X^\flat\big)_g + \big(\delta(X^\flat \wedge \Psi''), \omega\big)_g \ \mu_g \ ,
\end{aligned}
$$

where we also used that $\delta^*$ and $\delta$ are adjoint operators (w.r.t $L^2$-inner product) on symmetric tensors, that $d$ and $\delta$ are adjoint operators on forms and that $i_X$ and $X^\flat\wedge$ are point-wise adjoint on forms, that is,

$$
\big(i_X\beta, \alpha\big)_g = \big(\beta, X^\flat \wedge \alpha\big)_g \ , \ \text{for any} \beta \in \Lambda^{k+1}M, \alpha \in \Lambda^k M \ .
$$

Then, using repeatedly that $(\Psi'', \omega)_g = 0$, a computation in coordinates[1] for the last term yields:

$$
\begin{aligned}
\big(\delta(X^\flat \wedge \Psi''), \omega\big)_g &= \frac{1}{2}\delta(X^\flat \wedge \Psi'')_{ab}\omega_{ab} \\
&= -\frac{1}{2}\Big[X_t^\flat(\nabla_t\Psi''_{ab})\omega_{ab} + 2(\nabla_t X_a^\flat)\Psi''_{bt}\omega_{ab} + 2X_a^\flat(\nabla_t\Psi''_{bt})\omega_{ab}\Big] \\
&= \frac{1}{2}X_t^\flat\Psi''_{ab}(\nabla_t\omega_{ab}) + (\nabla_t X_a^\flat)(J\Psi'')_{at} - X_a^\flat(\nabla_t\Psi''_{bt})\omega_{ab} \\
&= \big(\Psi'', \nabla_X\omega\big)_g + \nabla_t(X_a^\flat(J\Psi'')_{at}) - X_a^\flat(\nabla_t(J\Psi''))_{at} + \big(J\delta\Psi'', X^\flat\big)_g \\
&= \big(\Psi'', \nabla_X\omega\big)_g - \delta\big(i_X(J\Psi'')\big) - \big(\delta(J\Psi''), X^\flat\big)_g + \big(J\delta\Psi'', X^\flat\big)_g \ .
\end{aligned}
$$

Integrating this relation, using Stokes' theorem and plugging in the result in the formula above, we get

$$
0 = \int_M \big(\delta T'' + \delta(T' - \Psi' \circ J) + \big(\Psi'', \nabla.\omega\big)_g - \delta(J\Psi'') \ ,\ X^\flat\big)_g\mu_g \ .
$$

Since the above relation is true for any vector field $X$, we obtain relation (17) claimed in the statement. ◻

## 3 The Gauduchon functional

In this section, we consider the functional

$$
\mathcal{L}_G(g, J, \omega) = \int_M |\theta|_g^2 \ \mu_g \ , \tag{35}
$$

where $\theta = J\delta\omega$ is the Lee 1-form of the almost Hermitian structure $(g, J, \omega)$ (or the torsion 1-form in the terminology of [18]). This functional was first considered by Gauduchon [18] on the space $\mathcal{AH}_1^J$ when an integrable almost complex structure $J$ is fixed. Here we will not make any assumption on the existence of an integrable almost complex structure, and we consider critical points of the functional $\mathcal{L}_G$ on various subspaces of the space $\mathcal{AH}$. For the first variation of the functional in the space $\mathcal{AH}$, we have:

[1] We use Einstein's convention, that is, repeated index sums.

**Proposition 3.1** *The functional $\mathcal{L}_G$ satisfies a first variation as in (16) with*

$$T'' = (J\theta \otimes J\theta)'' = -(\theta \otimes \theta)'', \quad \Psi'' = -(\mathrm{d}J\theta)'', \text{ and}$$
$$2(T' \circ J + \Psi') = 2(\mathrm{d}J\theta)' + \theta \wedge J\theta + (|\theta|^2 + 2\delta\theta)\omega \,.$$

***Proof*** We follow computational ideas from [18], with some adjustments as we work on a larger space. First of all, the functional is better expressed in terms of the wedge product as

$$\mathcal{L}_G(g, J, \omega) = \frac{1}{(n-1)!}\int_M \theta \wedge J\theta \wedge \omega^{n-1}. \tag{36}$$

For brevity of the notation, when computing the first variation we will denote with a dot quantities that are differentiated with respect to $t$. Thus, we will denote

$$\dot{\mathcal{L}_G} = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathcal{L}_G(g_t, J_t, \omega_t)\Big)|_{t=0}\ , \quad \dot\theta = \frac{\mathrm{d}}{\mathrm{d}t}(\theta_t)|_{t=0}\ , \text{ etc,}$$

where $(g_t, J_t, \omega_t)$ is a variation (6) in the space $\mathcal{AH}$. We have

$$\dot{\mathcal{L}_G} = \frac{1}{(n-1)!}\Big[\int_M \dot\theta \wedge J\theta \wedge \omega^{n-1} + \int_M \theta \wedge J\dot\theta \wedge \omega^{n-1} + \int_M \theta \wedge \dot J\theta \wedge \omega^{n-1} + \int_M \theta \wedge J\theta \wedge (\dot{\omega^{n-1}})\Big]. \tag{37}$$

Since $\dot\theta \wedge J\theta \wedge \omega^{n-1} = \theta \wedge J\dot\theta \wedge \omega^{n-1}$ are point-wise the same, the previous formula takes the form

$$\dot{\mathcal{L}_G} = \frac{1}{(n-1)!}\Big[2\int_M \dot\theta \wedge J\theta \wedge \omega^{n-1} + \int_M \theta \wedge \dot J\theta \wedge \omega^{n-1} + \int_M \theta \wedge J\theta \wedge (\dot{\omega^{n-1}})\Big] \tag{38}$$

Next, note that the variation at $t = 0$ of $\mathrm{d}\omega_t^{n-1} = \theta_t \wedge \omega_t^{n-1}$ implies

$$\mathrm{d}(\dot{\omega^{n-1}}) = \dot\theta \wedge \omega^{n-1} + \theta \wedge (\dot{\omega^{n-1}})\ .$$

This, combined with

$$\mathrm{d}(J\theta \wedge (\dot{\omega^{n-1}})) = \mathrm{d}(J\theta) \wedge (\dot{\omega^{n-1}}) - J\theta \wedge \mathrm{d}(\dot{\omega^{n-1}}),$$

and Stokes' theorem lead to the following computation

$$\begin{aligned}
&\int_M \Big(\dot\theta \wedge J\theta \wedge \omega^{n-1} + \theta \wedge J\theta \wedge (\dot{\omega^{n-1}})\Big) \\
&= -\int_M J\theta \wedge (\dot\theta \wedge \omega^{n-1} + \theta \wedge (\dot{\omega^{n-1}})) = -\int_M J\theta \wedge \mathrm{d}(\dot{\omega^{n-1}}) \\
&= \int_M \mathrm{d}\big(J\theta \wedge (\dot{\omega^{n-1}})\big) - \mathrm{d}(J\theta) \wedge (\dot{\omega^{n-1}}) = -\int_M \mathrm{d}(J\theta) \wedge (\dot{\omega^{n-1}}).
\end{aligned}$$

Rearranging terms, this gives

$$\int_M \dot\theta \wedge J\theta \wedge \omega^{n-1} = -\int_M (\theta \wedge J\theta + \mathrm{d}J\theta) \wedge (\dot{\omega^{n-1}}). \tag{39}$$

Thus, by using (39) back into (38), we get

$$\dot{\mathcal{L}_G} = \frac{1}{(n-1)!}\Big[\int_M (-\theta \wedge J\theta - 2\mathrm{d}J\theta) \wedge (\dot{\omega^{n-1}}) + \int_M \theta \wedge \dot J\theta \wedge \omega^{n-1}\Big]. \tag{40}$$

For the second integral in (40), first note that

$$(\dot{J}\theta)(X) = \frac{\mathrm{d}}{\mathrm{d}t}(J_t\theta)(X) = -\frac{\mathrm{d}}{\mathrm{d}t}(\theta)(J_t X) = -\theta(KX)\ .$$

Then, as $\omega^{n-1} = (n-1)!\star\omega$, where $\star$ denotes the Hodge operator of the metric $g$,

$$\begin{aligned}\frac{1}{(n-1)!}\theta\wedge\dot{J}\theta\wedge\omega^{n-1} &= \theta\wedge\dot{J}\theta\wedge(\star\omega) = \big(\theta\wedge\dot{J}\theta,\omega\big)_g\,\mu_g\\ &= \omega(\theta^\sharp,(\dot{J}\theta)^\sharp)\,\mu_g = g(J\theta^\sharp,(\dot{J}\theta)^\sharp)\,\mu_g = (\dot{J}\theta)(J\theta^\sharp)\,\mu_g\\ &= -\theta(KJ\theta^\sharp)\,\mu_g = \theta(JK\theta^\sharp)\,\mu_g = -g(K\theta^\sharp,J\theta^\sharp)\,\mu_g\\ &= -(\alpha''(\theta^\sharp,J\theta^\sharp) - h''(J\theta^\sharp,J\theta^\sharp))\,\mu_g = \big(h'',(J\theta\otimes J\theta)''\big)_g\,\mu_g\ ,\end{aligned}$$

where in the first equality of the last line we used relation (10), and for the second equality we used that $\alpha''$ is $J$-anti-invariant. Thus, we get

$$\frac{1}{(n-1)!}\int_M \theta\wedge\dot{J}\theta\wedge\omega^{n-1} = \int_M \big(h'',(J\theta\otimes J\theta)''\big)_g\,\mu_g\ . \tag{41}$$

For the first integral in (40), we follow directly Gauduchon's computation in [18], but we present the details for completion. Using

$$(\dot{\omega^{n-1}}) = \frac{\mathrm{d}}{\mathrm{d}t}(\underbrace{\omega_t\wedge\omega_t\wedge\ldots\wedge\omega_t}_{(n-1)-times}) = (n-1)\,\alpha\wedge\omega^{n-2}\ ,$$

we get

$$\frac{1}{(n-1)!}\int_M(-\theta\wedge J\theta - 2\mathrm{d}J\theta)\wedge(\dot{\omega^{n-1}}) = \frac{-1}{(n-2)!}\int_M(2\mathrm{d}J\theta+\theta\wedge J\theta)\wedge\alpha\wedge\omega^{n-2} \tag{42}$$

We next use the remark of Gauduchon (see... [18]) that for any 2-form $\phi$, we have

$$\frac{1}{(n-2)!}\phi\wedge\omega^{n-2} = \star_g\big((tr\,\phi)\,\omega - \phi' + \phi''\big)\ ,$$

where $\star_g$ denotes the Hodge-star operator of $g$. Using this in (42), the first integral on the right side of (40) is equal to

$$-\int_M \alpha\wedge\star_g\big((tr\,\phi)\,\omega - \phi' + \phi''\big) = \int_M\Big(\alpha,\big((-tr\,\phi)\,\omega + \phi' - \phi''\big)\Big)_g\,\mu_g\ ,$$

where $\phi = 2(dJ\theta) + \theta\wedge J\theta$. Note that

$$\phi'' = 2(dJ\theta)''\ ,\quad \phi' = 2(dJ\theta)' + \theta\wedge J\theta\ ,\quad tr\,\phi = (\phi,\omega)_g = -(|\theta|^2 + 2\delta\theta)\ ,$$

where for the last equality we used formula (33) from Proposition 2.3.

Putting all together back in (40), we get

$$\begin{aligned}\dot{\mathcal{L}_G} &= \int_M\big(h'',(J\theta\otimes J\theta)''\big)_g\,\mu_g + \int_M\big(\alpha'',-2(dJ\theta)''\big)_g\,\mu_g\\ &\quad + \int_M\big(\alpha',2(dJ\theta)'+\theta\wedge J\theta+(|\theta|^2+2\delta\theta)\omega\big)_g\,\mu_g\end{aligned} \tag{43}$$

From (43), we see that the gradient of the functional $\mathcal{L}_G$ is given as stated. □

We continue with some consequences of Propositions 3.1 and 1.5.

**Corollary 3.2** *Restricted to the space $\mathcal{AH}^{\omega}$, the Gauduchon's functional $\mathcal{L}_G$ is either identically zero, when $\omega$ satisfies $\mathrm{d}(\omega^{n-1}) = 0$ (so when all $\omega$-compatible almost Hermitian structures are semi-Kähler structures), or, otherwise, $\mathcal{L}_G$ has no critical points on $\mathcal{AH}^{\omega}$.*

***Proof*** From Propositions 3.1 and 1.5, the critical point condition for $\mathcal{L}_G$ on $\mathcal{AH}^{\omega}$ is

$$T'' = -(\theta \otimes \theta)'' = 0.$$

This is immediately seen to imply that $\theta$ vanishes identically which, in turn, is equivalent to $d(\omega^{n-1}) = 0$, or $\delta\omega = 0$. □

In dimension 4, as is well known (and also apparent from the above) the semi-Kähler condition $\delta\omega = 0$ is equivalent to $\omega$ being closed. Thus, on a compact 4-manifold with a fixed non-degenerate 2-form $\omega$, if $\omega$ is not closed, the Gauduchon functional $\mathcal{L}_G$ has no critical points on $\mathcal{AH}^{\omega}$.

The next corollary gives the critical point condition of the Gauduchon's functional on $\mathcal{AH}^{g}$.

**Corollary 3.3** *An almost Hermitian structure $(g, J, \omega)$ is a critical point of the Gauduchon's functional $\mathcal{L}_G$ on the space $\mathcal{AH}^{g}$ if and only if $(\mathrm{d}J\theta)'' = 0$. By relation (30) in Proposition 2.3, this condition is also equivalent to*

$$\frac{1}{2} N_{J\theta^{\sharp}} + J(\mathrm{d}\theta)'' = 0.$$

Of course locally conformal Kähler structures are critical points, as $N^J = 0$ and $d\theta = 0$ in this case, but it would be desirable to know if there are other interesting critical points. For example, are there locally conformal almost Kähler structures ($d\theta = 0$) with $N_{J\theta^{\sharp}} = 0$? We leave for further work the search of such examples.

The final observations of this section are regarding the Gauduchon's functional $\mathcal{L}_G$ restricted on subspaces of $\mathcal{AH}^{J}$, for a fixed almost complex structure $J$. The following result is also an immediate consequence of Propositions 3.1 and 1.5, but most of the statements are already implicitly contained in results of Gauduchon in [18]. For example, for part (d), see the proposition on page 516 of [18].

**Proposition 3.4** *Let $M^{2n}$ be a compact manifold with a fixed almost complex structure $J$. Then,*

(a) *$(g, J, \omega)$ is a critical point for $\mathcal{L}_G$ on $\mathcal{AH}^{J}$ if and only if $\theta = 0$, that is, the structure is semi-Kähler.*
(b) *$(g, J, \omega)$ is a critical point for $\mathcal{L}_G$ on $\mathcal{AH}^{\mu,J}$ if and only if*

$$2(\mathrm{d}J\theta)' + \theta \wedge J\theta = -\frac{1}{n}(|\theta|^2 + 2\delta\theta)\,\omega\,. \tag{44}$$

(c) *$(g, J, \omega)$ is a critical point for $\mathcal{L}_G$ on $\mathcal{AH}^{J}_1$ if and only if*

$$2(\mathrm{d}J\theta)' + \theta \wedge J\theta = -\frac{1}{n}(|\theta|^2 + 2\delta\theta)\,\omega, \quad |\theta|^2 + 2\delta\theta = c, \;\; c \in \mathbb{R}, \;\; c \geq 0. \tag{45}$$

(d) *$(g, J, \omega)$ is a critical point for $\mathcal{L}_G$ on $\mathcal{AH}^{[g],J}_1$ if and only if*

$$|\theta|^2 + 2\delta\theta = c, \;\; c \in \mathbb{R}, \;\; c \geq 0. \tag{46}$$

***Proof*** Part (a) can be seen from Propositions 3.1 and 1.5, but it also follows immediately in a different way. Because of the behavior of $\mathcal{L}_G$ under homotheties, any critical point on $\mathcal{AH}^J$ must have critical value 0; thus, $\theta = 0$.

Part (b) follows from part (b2) of Proposition 1.5 and the computations of the gradient in Proposition 3.1. Indeed, the critical point condition in this case is equivalent with $2(T' \circ J + \Psi') = \lambda\, \omega$, for some function $\lambda$, and using Proposition 3.1 this becomes

$$2(\mathrm{d}J\theta)' + \theta \wedge J\theta + (|\theta|^2 + 2\delta\theta)\, \omega = \lambda\, \omega\ .$$

But taking the pointwise inner product of this with $\omega$ and using

$$\big(2(\mathrm{d}J\theta)' + \theta \wedge J\theta, \omega\big)_g = -(|\theta|^2 + 2\delta\theta)\ ,$$

which follows from (33), we get that

$$\lambda = \frac{n-1}{n}(|\theta|^2 + 2\delta\theta)\ .$$

Plugging this back in the formula above, we get relation (44).

Part (c) also follows, as the additional condition for a critical point on $\mathcal{AH}_1^J$ compared to $\mathcal{AH}^{\mu,J}$ is the requirement that $\lambda$ be a constant. Thus, $|\theta|^2 + 2\delta\theta = c$, for a constant $c$, but, by integration, $c = \int_M |\theta|^2\, \mu_g \geq 0$.

Part (d) is immediate from part (b3) of Proposition 1.5. □

As observed in Remark V.2 in [18], note that on a manifold with Euler class nonzero, an almost Hermitian structure $(g, J, \omega)$ with $\delta\theta = 0$ (i.e., with a Gauduchon metric) cannot be critical for $\mathcal{L}_G$ on $\mathcal{AH}_1^{[g],J}$ unless $\theta = 0$.

We end this section with the observation that Théorèm III.4 of Gauduchon [18] fully extends to the non-integrable case for compact 4-dimensional almost complex manifolds. This does not seem to be well known (e.g., see the presentation in [5]), although a close investigation of the proof in [18] reveals that it does not use the integrability of the almost complex structure. For completeness, we present the proof with our set up, but using the same idea as in [18].

**Theorem 3.5** *Let $(M^4, J)$ be a compact 4-dimensional almost Hermitian manifold. The only critical points of the Gauduchon functional $\mathcal{L}_G$ on the space $\mathcal{AH}_1^J$ are $J$-compatible almost Kähler structures.*

***Proof*** In all dimensions the critical point condition of $\mathcal{L}_G$ on $\mathcal{AH}_1^J$ is given as in part (c) of Proposition 1.5. Adding $2(\mathrm{d}J\theta)''$ to both sides of the first relation in (45), we have (still in all dimensions):

$$2(\mathrm{d}J\theta) = -\theta \wedge J\theta - \frac{1}{n}(|\theta|^2 + 2\delta\theta)\, \omega + 2(\mathrm{d}J\theta)''\ . \tag{47}$$

Finally, specializing to dimension $2n = 4$ and using $|\theta|^2 + 2\delta\theta = c$, we get from (47)

$$0 = 4\int_M (\mathrm{d}J\theta) \wedge (\mathrm{d}J\theta) = \int_M \left(2c^2 + 2c|\theta|^2 + |(\mathrm{d}J\theta)''|^2\right) \mu_g\ .$$

Since $c = \int_M |\theta|^2\, \mu_g \geq 0$, the above equality can hold if and only if $c = 0$, i.e., if and only if $\theta = 0$. □

## 4 The $L^2$-norm of $d\omega$

In this section, we consider the functional

$$\mathcal{A}(g, J, \omega) = \int_M |\mathrm{d}\omega|_g^2 \, \mu_g \,, \tag{48}$$

defined on $\mathcal{AH}$ and subspaces of this space. It is obvious that almost Kähler structures are absolute minima for the functional $\mathcal{A}$. We are interested if there are any other potentially interesting critical almost Hermitian structures for this functional. Note also that in dimension 4, as $\mathrm{d}\omega = \theta \wedge \omega$, we have $|\mathrm{d}\omega|_g^2 = |\theta|_g^2$. Thus, in this dimension, the functional $\mathcal{A}$ is identical with the Gauduchon functional $\mathcal{L}_G$.

We are able to compute the first variation of $\mathcal{A}$ in all dimensions, but in dimensions higher than 4 we don't have an optimal description of the components of $T$ and $\Psi$, so we don't have a general geometric characterization of critical points.

**Proposition 4.1** *If $(g_t, J_t, \omega_t) = (g, J, \omega) + t(h, K, \alpha) + o\left(t^2\right)$ is a variation in almost Hermitian structures on a compact manifold $M^{2n}$, then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathcal{A}(g_t, J_t, \omega_t)\Big)|_{t=0} = \int_M \big((T, h)_g + 2(\Psi, \alpha)_g\big) \, \mu_g \,, \tag{49}$$

*where*

$$T(X, Y) = -\big((i_X \mathrm{d}\omega), (i_Y \mathrm{d}\omega)\big)_g + \frac{1}{2}|\mathrm{d}\omega|_g^2 \, g(X, Y) \,, \quad \Psi = \delta \mathrm{d}\omega \,. \tag{50}$$

*In dimension 4, the components of $T$ and $\Psi$ are as follows:*

$$T' = (\theta \otimes \theta)' - \frac{1}{2}|\theta|^2 g \,, \quad T'' = (J\theta \otimes J\theta)'' = -(\theta \otimes \theta)'' \,; \tag{51}$$

$$\Psi' = (dJ\theta)' + (|\theta|^2 + \delta\theta)\, \omega \,, \quad \Psi'' = -(dJ\theta)'' \,. \tag{52}$$

***Proof*** By a direct computation in a coordinate system

$$\begin{aligned}
\dot{\mathcal{A}} &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_M \frac{1}{6}(\mathrm{d}\omega(t))_{ijk}(\mathrm{d}\omega(t))_{abc} g(t)^{ia} g(t)^{jb} g(t)^{kc} \, \mu_{g(t)}\Big)|_{t=0} \\
&= \int_M \Big(2(\mathrm{d}\alpha, \mathrm{d}\omega)_g - \frac{3}{6} h^{ia} (\mathrm{d}\omega)_{ijk} (\mathrm{d}\omega)_{abc} g^{jb} g^{kc} + \frac{1}{2}|\mathrm{d}\omega|^2 (h, g)_g\Big) \, \mu_g \\
&= \int_M \Big(-\big(h, (i.\mathrm{d}\omega, i.\mathrm{d}\omega)_g\big)_g + \frac{1}{2}|\mathrm{d}\omega|^2 (h, g)_g + 2(\alpha, \delta \mathrm{d}\omega)_g\Big) \, \mu_g \,.
\end{aligned}$$

The claimed expressions for $T$ and $\Psi$ follow.

In dimension 4, we use $d\omega = \theta \wedge \omega$. Thus,

$$i_X d\omega = i_X(\theta \wedge \omega) = \theta(X)\omega - \theta \wedge i_X\omega = \theta(X)\omega - \theta \wedge JX^\flat \,,$$

from which

$$\begin{aligned}
\big(i_X \mathrm{d}\omega, i_Y \mathrm{d}\omega\big)_g = &\ \big(\theta(X)\omega - \theta \wedge JX^\flat, \theta(Y)\omega - \theta \wedge JY^\sharp\big)_g \\
&= |\theta|^2 g(X, Y) - (J\theta)(X)(J\theta)(Y) \,.
\end{aligned}$$

Using this in (50), we get

$$T = J\theta \otimes J\theta - \frac{1}{2}|\theta|^2 g \,,$$

and,thus, the claimed components $T'$ and $T''$ in (51).

For the components of $\Psi$ in dimension 4, one may again proceed by direct computations using $\mathrm{d}\omega = \theta \wedge \omega$, but there is a faster way. Note that in dimension 4, we have the well-known self-dual, anti-self-dual decomposition of the bundle of 2-forms induced by the Hodge operator: $\Lambda^2 M = \Lambda^+ M \oplus \Lambda^- M$. Using the superscripts $\pm$ to denote the self-dual (resp. anti-self-dual) part of a 2-form and the relations $\delta = -\star\mathrm{d}\star$, $\mathrm{d} = \star\delta\star$, one quickly observes that

$$(\delta\mathrm{d}\omega)^+ = (\mathrm{d}\delta\omega)^+ \ , \quad (\delta\mathrm{d}\omega)^- = -(\mathrm{d}\delta\omega)^- .$$

On the other hand, the self-dual, anti-self-dual decomposition of $\Lambda^2 M$ is related to the $U(2)$-decomposition of the same bundle by

$$\Lambda^+ M = \mathbb{R}\,\omega \oplus [\![\Lambda^{0,2} M]\!] \ , \quad \Lambda^- = \Lambda^{1,1}_{0,\mathbb{R}} M \ ,$$

where $\Lambda^{1,1}_{0,\mathbb{R}} M$ denotes the bundle of $J$-invariant, primitive (trace-free) 2-forms. Thus, we get

$$\Psi = (\delta\mathrm{d}\omega)^+ + (\delta\mathrm{d}\omega)^- = (\mathrm{d}\delta\omega)^+ - (\mathrm{d}\delta\omega)^- = (\mathrm{d}J\theta)^- - (\mathrm{d}J\theta)^+,$$

while the self-dual, anti-self-dual components of $\mathrm{d}J\theta$ are related to its $J$-invariant, $J$-anti-invariant components by

$$\begin{aligned}(\mathrm{d}J\theta)^+ &= (\mathrm{d}J\theta)'' + \frac{1}{2}\big(\mathrm{d}J\theta, \omega\big)_g\, \omega = (\mathrm{d}J\theta)'' - \frac{1}{2}(|\theta|_g^2 + \delta\theta)\,\omega \ ,\\ (\mathrm{d}J\theta)^- &= (\mathrm{d}J\theta)' - \frac{1}{2}\big(\mathrm{d}J\theta, \omega\big)_g\, \omega = (\mathrm{d}J\theta)' + \frac{1}{2}(|\theta|_g^2 + \delta\theta)\,\omega \ .\end{aligned}$$

The claimed components for $\Psi$ in dimension 4 follow. □

Note that in dimension 4, the components we get for $T$ and $\Psi$ in Proposition 4.1 match the ones we obtained in Proposition 3.1. As functionals $\mathcal{A}$ and $\mathcal{L}_G$ coincide in dimension 4, from Corollary 3.2, it follows that in this dimension the functional $\mathcal{A}$ has no other critical points on $\mathcal{AH}^\omega$ except almost Kähler metrics. In contrast to $\mathcal{L}_G$, however, for higher dimensions the functional $\mathcal{A}$ does have other critical points on $\mathcal{AH}^\omega$ and even on the larger space $\mathcal{AH}_1$.

**Theorem 4.2** *On a compact manifold $M^{2n}$, $2n \geq 6$, a nearly Kähler structure $(g, J, \omega)$ is a critical point for the functional $\mathcal{A}$ restricted to $\mathcal{AH}^\omega$. Even stronger, in dimension $2n = 6$, a nearly Kähler structure $(g, J, \omega)$ is a critical point for the functional $\mathcal{A}$ on the space $\mathcal{AH}_1$.*

***Proof*** For a nearly Kähler structure $(g, J, \omega)$, we have, by definition,

$$(\nabla_X J)Y + (\nabla_Y J)X = 0 \ , \tag{53}$$

so, as a consequence,

$$(\mathrm{d}\omega)(X, Y, Z) = 3g((\nabla_X J)Y, Z) = 3(\nabla_X \omega)(Y, Z) \ . \tag{54}$$

Thus, the tensor $T$ is given in this case by

$$T(X, Y) = -9\big(\nabla_X\omega, \nabla_Y\omega\big)_g + \frac{1}{2}|\mathrm{d}\omega|^2 g \ . \tag{55}$$

It is well known that a nearly Kähler manifold is automatically quasi-Kähler, that is, the following identity holds

$$(\nabla_{JX} J)JY + (\nabla_X J)Y = 0 \ , \tag{56}$$

and then, via (55) and (56), it is clear that $T'' = 0$.

For the stronger statement in dimension 6, note that in this dimension Gray [21] showed that any 6-dimensional nearly Kähler structure $(g, J, \omega)$ must have constant type, that is,

$$|(\nabla_X J)(Y)|^2 = \alpha\{|X|^2|Y|^2 - g(X, Y)^2 - g(JX, Y)^2\}$$

holds for some constant $\alpha \geq 0$, for all vector fields $X, Y$. As a consequence of this, as is remarked in [28], we have

$$\begin{aligned} &g((\nabla_U J)X, (\nabla_V J)Y) \\ &\quad = \alpha\{g(U, V)g(X, Y) - g(U, Y)g(X, V) - \omega(U, V)\omega(X, Y) + \omega(U, Y)\omega(X, V)\}. \end{aligned}$$

From this, it can be easily seen that $T$ must be a multiple of $g$. In fact, it actually follows that $T = 0$

Regarding $\Psi$, note that on any nearly Kähler manifold $\delta\omega = 0$, thus

$$\Psi = \delta d\omega = \Delta\omega, \text{ where } \Delta \text{ denotes the Hodge–deRham Laplacian.}$$

But as once again observed in [28], Proposition 2.8, Gray's result about the constant type of 6-dimensional nearly Kähler manifolds implies that $\omega$ is an eigenform of the Laplacian. More precisely, $\Delta\omega = 12\alpha\omega$. It is thus clear that we have $T'' = 0$, $\Psi'' = 0$, $T' \circ J + \Psi' = 12\alpha\omega$; hence, the conclusion follows by Theorem 1.4, (b). □

Note that our result above is optimal, as nearly Kähler structures of dimension higher than 6 are, in general, not critical points for the functional $\mathcal{A}$ on the space $\mathcal{AH}_1$. To see this, let $M = M_1 \times M_2$, where $M_1$ is a compact 6-dimensional manifold endowed with a strict nearly Kähler structure $(g_1, J_1, \omega_1)$ and $M_2$ is any compact manifold endowed with a Kähler structure $(g_2, J_2, \omega_2)$. Then, $M$ obviously has the product nearly Kähler structure $(g = g_1 + g_2, J = J_1 + J_2, \omega = \omega_1 + \omega_2)$. The gradient $(T, \Psi)$ for the functional $\mathcal{A}$ at $(g, J, \omega)$ will be the sum $T = T_1 + T_2$, $\Psi = \Psi_1 + \Psi_2$. But as the structure $(g_2, J_2, \omega_2)$ is assumed to be Kähler, the pair $(T_2, \Psi_2)$ identically vanishes. From the computation in the proof of the theorem, we thus get in this case

$$T' \circ J + \Psi' = \Psi_1' = 12\alpha_1\omega_1.$$

But this cannot be a multiple of $\omega = \omega_1 + \omega_2$.

There is a local classification of nearly Kähler manifolds obtained by [29], but we have not attempted computations for $T' \circ J + \Psi'$ on all possible irreducible factors in the classification. But as we see in the paragraph above, the critical point condition on the space $\mathcal{AH}_1$ does not remain satisfied for products. Note, however, that if $(M_1, g_1, J_1, \omega_1)$ is a 6-dimensional strict nearly Kähler structure, then $M = M_1 \times M_1$ with the product structure is still a critical point for the functional $\mathcal{A}$ on the space $\mathcal{AH}_1$. So the functional $\mathcal{A}$ does have critical points with non-zero critical values on $\mathcal{AH}_1$ even in dimensions higher than 6.

It would also be of interest to determine what kind of critical points are the 6-dimensional nearly Kähler structures for the functional $\mathcal{A}$. This requires a computation of the second variation and we leave this for future work.

Finally, considering the functional $\mathcal{A}$ on the space $\mathcal{AH}_1^{[g],J}$, one recovers results from Sect. 4 of [5].

**Proposition 4.3** (Proposition 18, [5]) *On a compact manifold $M^{2n}$ a structure $(g, J, \omega)$ is critical for the functional $\mathcal{A}$ restricted to $\mathcal{AH}_1^{[g],J}$ if and only if*

$$(n-1)|\mathrm{d}\omega|^2 + 2\delta\theta = k\,, \tag{57}$$

*where $k$ is a constant, $k \geq 0$.*

***Proof*** It follows straight from Proposition 4.1 and part (b3) of Proposition 1.5. The fact that the constant $k$ is non-negative follows from integrating (57). □

The following is a slight improvement in Corollary 19 of [5].

**Corollary 4.4** *On a compact manifold $M^{2n}$ a Gauduchon almost Hermitian structure $(g, J, \omega)$ is critical for the functional $\mathcal{A}$ restricted to $\mathcal{AH}_1^{[g],J}$ if and only if $|d\omega|^2 = k$, where $k$ is a constant. Moreover, if the Euler class of the bundle $\Lambda^3 M$ is not zero (or, in dimension 4, if $\chi(M) \neq 0$, where $\chi(M)$ is the Euler class of $M$), then $k = 0$, so the structure must be almost Kähler.*

**Acknowledgements** The second author (CS) has been supported by The Scientific and Research Council of Turkey (TUBITAK) to conduct this research under the TUBITAK-2219-International Postdoctoral Research Fellowship Program for Turkish Citizens with the project number 1059B192000164. Both authors would like to thank Giovanni Russo and Weiyi Zhang for useful comments.

**Author Contributions** We jointly did the work and wrote the paper.

**Funding** The second author, Cem Sayar, has been supported by The Scientific and Research Council of Turkey (TUBITAK) to conduct this research under the TUBITAK-2219-International Postdoctoral Research Fellowship Program for Turkish Citizens with the project number 1059B192000164.

**Availability of data and materials** Not applicable.

## Declarations

**Conflict of interest** The authors declare no competing interest.

**Ethical Approval** Not applicable.

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