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Instability of a family of examples of harmonic maps

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Abstract
The radial map $u(x) = \frac{x}{\|x\|}$ is a well-known example of a harmonic map from $\mathbb{R}^m - \{0\}$ into the spheres $\mathbb{S}^{m-1}$ with a point singularity at $x = 0$. In Nakauchi (Examples Counterexamples 3:100107, 2023), the author constructed recursively a family of harmonic maps $u^{(n)}$ into $\mathbb{S}^{m^n-1}$ with a point singularity at the origin ($n = 1, 2, \ldots$), such that $u^{(1)}$ is the above radial map. It is known that for $m \geq 3$, the radial map $u^{(1)}$ is not only *stable* as a harmonic map but also a *minimizer* of the energy of harmonic maps. In this paper, we show that for $n \geq 2$, $u^{(n)}$ may be *unstable* as a harmonic map. Indeed we prove that under the assumption $n > \dfrac{\sqrt{3}-1}{2}(m-1)$ ($m \geq 3$, $n \geq 2$), the map $u^{(n)}$ is *unstable* as a harmonic map. It is remarkable that they are unstable and our result gives many examples of *unstable* harmonic maps into the spheres with a point singularity at the origin.

Keywords Harmonic map · Stability · Instability · Radial map · Singularity

Mathematics Subject Classification Primary 58E20; Secondary 53C43

1 Introduction

The radial map u, defined by $u(x) = \frac{x}{\|x\|}$, is a well-known example of a harmonic map with a point singularity at $x = 0$ from the m-dimensional Euclidean space except the origin $\mathbb{R}^m - \{0\}$ into the $(m-1)$-dimensional sphere $\mathbb{S}^{m-1}$ in $\mathbb{R}^m$ (m is a positive integer). Several studies are given for this special example of harmonic maps ([1, 5, 6, 8], etc. See [2, 3] for harmonic maps.).

In [9], the author introduced a family of harmonic maps $u^{(n)}$ $(n = 1, 2, \ldots)$ from $\mathbb{R}^m - \{0\}$ into spheres of higher dimension, with a point singularity of a polynomial order of degree n at $x = 0$, such that $u^{(1)}$ is the above radial map:

Theorem A ([9]). *For any positive integers m, n with $m \geq n$, there exists a harmonic map*

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$$\begin{array}{cccc} u^{(n)} : & \mathbb{R}^m - \{0\} & \longrightarrow & \mathbb{S}^{m^n-1} \subset \mathbb{R}^{m^n} \\ & \Psi & & \Psi \\ & x = (x_1, \cdots, x_m) & \longmapsto & u^{(n)}(x) = \left(u^{(n)}_{i_1 \cdots i_n}(x)\right)_{1 \le i_1, \cdots, i_n \le m} \end{array}$$

such that

(1) $u^{(n)}$ is a smooth harmonic map, i.e., it satisfies the harmonic map equation

$$\triangle u^{(n)} + \|Du^{(n)}\|^2 u^{(n)} = 0 .$$

(2) Each component of $u^{(n)}(x)$ is a polynomial of $y_1, \ldots, y_m$ of degree n, where

$$y_i = \frac{x_i}{\|x\|} \quad (i = 1, \ldots, m).$$

More precisely the component $u^{(n)}_{i_1 \ldots i_n}(x)$ is a polynomial of $y_{i_1}, \ldots, y_{i_n}$ of degree n, Therefore, $u^{(n)}$ has a point singularity of the polynomial of degree n at $x = 0$.

(3) (the energy density)

$$\|Du^{(n)}\|^2 = \frac{n(n+m-2)}{\|x\|^2}$$

(4) (the initial map is the radial one)

$$u^{(1)}(x) = \frac{x}{\|x\|}$$

Theorem A gives a harmonic map with a point singularity of a polynomial of various general order, and recovers our previous paper [7] and Fujioka's paper [4].

For any fixed integer m, this family of examples is constructed recursively with respect to n ($\le m$) by the following defining equalities:

$$u^{(1)}_{i_1}(x) = \frac{x_{i_1}}{\|x\|} \tag{1.1}$$

$$u^{(n)}_{i_1 \ldots i_n}(x) = C_{m,n} \left(\frac{x_{i_n}}{\|x\|} u^{(n-1)}_{i_1 \ldots i_{n-1}}(x) - \frac{1}{n+m-3} \|x\| D_{i_n} u^{(n-1)}_{i_1 \ldots i_{n-1}}(x) \right) \quad (n \ge 2) \tag{1.2}$$

where D_i denotes the derivative with respect to x_i , i.e.,

$$D_i = \frac{\partial}{\partial x_i}$$

and

$$C_{m,n} = \sqrt{\frac{n+m-3}{2n+m-4}}. \tag{1.3}$$

It is known that for $m \ge 3$, the radial map $u^{(1)}$ is not only *stable* as a harmonic map but also a *minimizer* of the energy of harmonic maps ([6]). In this paper, we show that for $n \ge 2$, $u^{(n)}$ may be *unstable* as a harmonic map. Indeed we prove that for any integer $n \ge \frac{\sqrt{3}-1}{2}(m-1)$ $(m \ge 3, n \ge 2)$, the map $u^{(n)}$ is *unstable* as a harmonic map.

Main Theorem. Let $m \ge 3$ and $n \ge 2$. For $n \ge \frac{\sqrt{3}-1}{2}(m-1)$, the map $u^{(n)}$ is *unstable* as a harmonic map.

Main Theorem gives many examples of *unstable* harmonic maps into the spheres with a point singularity at the origin. For example, in the case of $m = 3$ and $n = 2$, Main Theorem implies that the map

$$\begin{array}{ccccc} u^{(2)} : & \mathbb{R}^3 - \{0\} & \longrightarrow & \mathbb{S}^8 \subset \mathbb{R}^9 \\ & \Downarrow & & \Downarrow \\ & x = (x_1, x_2, x_3) & \longmapsto & u^{(2)}(x) \end{array}$$

such that

$$u^{(2)}(x) = \sqrt{\frac{3}{2}} \left(\frac{x_1^2}{\|x\|^2} - \frac{1}{3}, \frac{x_1 x_2}{\|x\|^2}, \frac{x_1 x_3}{\|x\|^2}, \frac{x_2 x_1}{\|x\|^2}, \frac{x_2^2}{\|x\|^2} - \frac{1}{3}, \frac{x_2 x_3}{\|x\|^2}, \frac{x_3 x_1}{\|x\|^2}, \frac{x_3 x_2}{\|x\|^2}, \frac{x_3^2}{\|x\|^2} - \frac{1}{3} \right)$$

is an *unstable* harmonic map.

In Sect. 2, we recall basic concepts on stability. In Sect. 3, we give preliminary facts to prove our Main Theorem. We prove Main Theorem in Sect. 4.

2 Basic concepts on stability

In this section, we recall basic facts on harmonic maps, especially the stability of harmonic maps.

Let (M, g), (N, h) be Riemannian manifolds without boundary and let u be a smooth map from M into N. We know the L^2-energy

$$E(u) = \int_M \|du\|^2 \, dv_g$$

where

$$\begin{array}{lcl} du & : & \text{the differential map of } u \\ dv_g & : & \text{the volume form on } (M, g). \end{array}$$

We call it the *energy* or the energy functional. A map u is *harmonic* if it is stationary for the energy $E(\,)$, where u is *stationary* for the energy $E(\,)$ if the first variation of the energy $E(\,)$

$$(\delta E)(u)(X) = \frac{d}{dt} E(u_t) \Big|_{t=0}$$

vanishes for any variation u_t of u with compact support such that $u_0 = u$, and $X = dU\left(\frac{\partial}{\partial t}\right)\Big|_{t=0}$ is the variation vector field with $U(t, x) = u_t(x)$. In other words, it satisfies the Euler–Lagrange equation for the energy $E(\,)$, i.e., the *harmonic map equation*:

$$\sum_{i=1}^{m} \left(\nabla_{e_i} du\right)(e_i) = 0 \quad \left(\text{i.e., } \operatorname{tr}\left(\nabla du\right) = 0\right)$$

where e_i $(i = 1, \ldots, m)$ is a local orthonormal frame on M, and ∇ denotes the connection on the bundle $TM \otimes f^{-1}TN$. A harmonic map u is *unstable* (resp. *stable*) if the second variation

$$(\delta^2 E)(u)(X) = \frac{d^2}{dt^2} E(u_t) \Big|_{t=0}$$

is *negative* (resp. *nonnegative*) for some (resp. any) variation u_t with compact support.

In our situation such as $M = \mathbb{R}^m - \{0\}$ and $N = \mathbb{S}^n \subset \mathbb{R}^{n+1}$, we can write u as a map

$$x = (x_1, \ldots, x_m) \;\to\; u(x) = \big(u_1(x), \ldots, u_{n+1}(x)\big).$$

Take any function $\varphi \in C^\infty(\mathbb{R}^m - \{0\}, \mathbb{R}^{n+1})$ with compact support. Consider the variation u_t of u with the variation function φ:

$$u_t(x) \;=\; \frac{u(x) + t\varphi(x)}{\|u(x) + t\varphi(x)\|}.$$

We can see

$$\frac{\partial}{\partial t} u_t(x)\bigg|_{t=0} \;=\; \varphi(x) \;-\; \big(\varphi(x) \cdot u(x)\big)u(x),$$

where $\cdot$ denotes the inner product on $\mathbb{R}^{n+1}$. Then, we have the first variation

$$\begin{aligned}(\delta E)(u)(\varphi) &= \frac{d}{dt} E(u_t)\bigg|_{t=0} \\ &= \int_{\mathbb{R}^m - \{0\}} \Big(\langle Du,\; D\varphi\rangle \;-\; \|Du\|\, u \cdot \varphi\Big)\, dx \end{aligned} \tag{2.1}$$

for any variation function $\varphi \in C^\infty(\mathbb{R}^m - \{0\}, \mathbb{R}^{n+1})$ with compact support, where $Du = \left(\dfrac{\partial u_j}{\partial x_i}\right)_{\substack{1\le i\le m \\ 1\le j\le n+1}}$ and $dx = dx_1 \ldots dx_m$. Then, we know

u is a *harmonic map* $\iff$ $\triangle u \;+\; \|Du\|^2 u \;=\; 0$ (harmonic map equation).

We see the second variation

$$\begin{aligned}(\delta^2 E)(u)(\varphi) &= \frac{d^2}{dt^2} E(u_t)\bigg|_{t=0} \\ &= 2\int_{\mathbb{R}^m - \{0\}} \Big(\|D\varphi\|^2 \;-\; \|Du\|^2\|\varphi\|^2\Big)\, dx \end{aligned} \tag{2.2}$$

for any variation function $\varphi \in C^\infty(\mathbb{R}^m - \{0\}, \mathbb{R}^{n+1})$ with compact support satisfying the *orthogonality condition*

$$\varphi \cdot u \;=\; 0.$$

For a harmonic map u, we have the definition of instability:

u is *unstable* (resp. *stable*) $\overset{\text{def}}{\iff}$ the second variation $(\delta^2 E)(u)(\varphi) < 0$ (resp. $\geq\; 0$) for some (resp. any) variation function φ with compact support.

In the proof of Main Theorem, we give a special variation function φ with compact support, given by (4.6) later. For this variation function φ, we calculate the second variation $(\delta^2 E)(u)(\varphi)$ and prove that it is negative.

3 Preliminaries

In this section, we give preliminary facts for our proof of Main Theorem. We introduce the following two basic quantities. They play an important role in our proofs. See [7] and [9] for their details.

Two quantities

$$y = (y_i)_{1\le i\le m} \quad : \quad y_i = \frac{x_i}{\|x\|}$$

$$a = (a_{ij})_{1\le i,j\le m} \; : \; a_{ij} = \delta_{ij} - \frac{x_i x_j}{\|x\|^2} = \delta_{ij} - y_i y_j$$

These two quantities y_i and a_{ij} satisfy the following conditions:

Lemma 1 *(1)* $\sum_{i=1}^{m} y_i^2 = 1 \quad (i.e., \|y\| = 1)$

(2) $\sum_{i=1}^{m} a_{ii} = m - 1 \quad (i.e., tr\, a = m - 1)$

(3) $D_i y_j = \frac{1}{\|x\|} a_{ij} \quad \left(i.e., Dy = \frac{1}{\|x\|} a\right)$

We omit the proof of Lemma 1, because Lemma 1 follows from the definitions of y_i and a_{ij} with simple calculations.

In this paper, we use the following properties of $u^{(n)}_{i_1 \ldots i_n}$.

Lemma 2

$$\sum_{i_1, i_2=1}^{m} \delta_{i_1 i_2} u^{(n)}_{i_1 \ldots i_n} = 0 \tag{3.1}$$

for $n \ge 2$.

Proof We use the induction. We first prove (3.1) for $n = 2$. Equality (1.2) for $n = 2$ implies

$$u^{(2)}_{i_1 i_2} \overset{(1.2)}{=} C_{m,2}\left(y_{i_2} u^{(1)}_{i_1} - \frac{1}{m-1}\|x\| D_{i_2} u^{(1)}_{i_1}\right)$$
$$\overset{(1.1),\,\text{Lemma}1(3)}{=} C_{m,2}\left(y_{i_1} y_{i_2} - \frac{1}{m-1} a_{i_1 i_2}\right)$$

Then,

$$\sum_{i_1, i_2=1}^{m} \delta_{i_1 i_2} u^{(2)}_{i_1 i_2} = C_{m,2}\left(\|y\|^2 - \frac{1}{m-1}\sum_{i=1}^{m} a_{ii}\right)$$
$$\overset{\text{Lemma } 1(1),(2)}{=} 0$$

Thus, we have (3.1) for $n = 2$.

We assume that (3.1) holds for $n = k - 1$ $(k \ge 3)$, i.e.,

$$\sum_{i_1, i_2=1}^{m} \delta_{i_1 i_2} u^{(k-1)}_{i_1 \ldots i_{k-1}} = 0. \tag{3.2}$$

Then for $n = k$, we have from (1.2)

$$\begin{aligned}
&\sum_{i_1, i_2=1}^{m} \delta_{i_1 i_2} u^{(k)}_{i_1 \dots i_k} \\
&\overset{(1.2)}{=} C_{m,k} \sum_{i_1, i_2=1}^{m} \delta_{i_1 i_2} \left(y_{i_k} u^{(k-1)}_{i_1 \dots i_{k-1}} - \frac{1}{k+m-3} \|x\| D_{i_k} u^{(k-1)}_{i_1 \dots i_{k-1}} \right) \\
&= C_{m,k} \left\{ y_{i_k} \sum_{i_1, i_2=1}^{m} \delta_{i_1 i_2} u^{(k-1)}_{i_1 \dots i_{k-1}} - \frac{1}{k+m-3} \|x\| D_{i_k} \left(\sum_{i_1, i_2=1}^{m} \delta_{i_1 i_2} u^{(k-1)}_{i_1 \dots i_{k-1}} \right) \right\} \\
&\overset{(3.2)}{=} 0.
\end{aligned}$$

by the induction assumption (3.2). We have (3.1) for $n = k$. Thus, (3.1) holds for any $n \geq 2$. □

4 Instability of *u*

In this section, we prove the following result on the instability of u.

Main Theorem. Let $m \geq 3$ and $n \geq 2$. For $n \geq \dfrac{\sqrt{3}-1}{2}(m-1)$, the map $u^{(n)}$ is *unstable* as a harmonic map.

Let B_r denotes the open ball of radius r in $\mathbb{R}^m$ centered at the origin:

$$B_r = \left\{ x \in \mathbb{R}^m \mid \|x\| < r \right\}.$$

Take positive real numbers a, b, c and d satisfying $a < b < c < d$. Let $\eta_0(x)$ be a Lipschitz continuous function on $\mathbb{R}^m - \{0\}$ given by

$$\eta_0(x) = \begin{cases} 0 & \text{on } B_a - \{0\} \\ \dfrac{\|x\| - a}{(b-a)} & \text{on } B_b - B_a \\ 1 & \text{on } B_c - B_b \\ \dfrac{d - \|x\|}{d-c} & \text{on } B_d - B_c \\ 0 & \text{on } \mathbb{R}^m - B_d \end{cases} \tag{4.1}$$

Take a sufficiently small positive number ε which is determined later. Let $\eta(x)$ be a smooth cutoff function, approximating $\eta_0(x)$, satisfying the following four conditions:

$$\eta(x) \begin{cases} = 0 & \text{on } B_a - \{0\} \\ \in [0,\ 1] & \text{on } B_b - B_a \\ = 1 & \text{on } B_c - B_b \\ \in [0,\ 1] & \text{on } B_d - B_c \\ = 0 & \text{on } \mathbb{R}^m - B_d \end{cases} \tag{4.2}$$

$$|\eta(x) - \eta_0(x)| < \varepsilon \ \text{ for } \forall x \in \mathbb{R}^m - \{0\} \tag{4.3}$$

$$\|D\eta\| \leq \frac{1+\varepsilon}{(b-a)} \ \text{ on } B_b - B_a \tag{4.4}$$

$$\|D\eta\| \leq \frac{1+\varepsilon}{(d-c)} \ \text{ on } B_d - B_c. \tag{4.5}$$

Note that the support of $\eta(x)$ is compact since $\eta(x) = 0$ outside $B_d - B_a$.

For simplicity, we set $u := u^{(n)}$. Take the variation function φ with compact support, for $n \geq 2$, defined by

$$\varphi_{i_1 \dots i_n}(x) = \begin{cases} \eta(x)\, \delta_{i_1 i_2} & (n = 2) \\ \eta(x)\, \delta_{i_1 i_2} y_{i_3} \dots y_{i_n} & (n \geq 3) \end{cases} \tag{4.6}$$

for $x \in \mathbb{R}^m - \{0\}$. Note that the condition "$n \geq 2$" is necessary for this variation $\varphi = (\varphi_{i_1 \dots i_n})_{1 \leq i_1, \dots, i_n \leq m}$, since $\delta_{i_1 i_2}$ in the definition of $\varphi_{i_1 \dots i_n}$ requires *two* indices i_1 and i_2.

We can easily check the following three properties:

Lemma 3 *(1)* $\varphi \cdot u = 0$
(2) $\|\varphi\|^2 = m\eta^2$
(3) $\|D\varphi\|^2 = m\|D\eta\|^2$

Proof (1:) We have

$$\begin{aligned} \varphi \cdot u &= \sum_{i_1, \dots, i_n = 1}^{m} \varphi_{i_1 \dots i_n} u^{(n)}_{i_1 \dots i_n} \\ &\overset{(4.6)}{=} \begin{cases} \eta(x) \displaystyle\sum_{i_1, i_2 = 1}^{m} \delta_{i_1 i_2} u^{(2)}_{i_1 i_2} & (n = 2) \\ \eta(x) \displaystyle\sum_{i_3, \dots, i_n = 1}^{m} \left(\sum_{i_1, i_2 = 1}^{m} \delta_{i_1 i_2} u^{(n)}_{i_1 \dots i_n} \right) y_{i_3} \dots y_{i_n} & (n \geq 3) \end{cases} \\ &\overset{\text{Lemma 2}}{=} 0 \end{aligned}$$

(2:) We get

$$\begin{aligned}\|\varphi\|^2 &= \sum_{i_1,\ldots,i_n=1}^{m} \left(\varphi_{i_1,\ldots i_n}\right)^2 \\ &\overset{(4.6)}{=} \begin{cases} \eta^2 \sum_{i_1,i_2=1}^{m} \delta_{i_1 i_2}^2 & (n=2) \\ \eta^2 \sum_{i_1,i_2=1}^{m} \delta_{i_1 i_2}^2 \sum_{i_3=1}^{m} y_{i_3}^2 \cdots \sum_{i_n=1}^{m} y_{i_n}^2 & (n \geq 3) \end{cases} \\ &\overset{\text{Lemma 1(1)}}{=} m\eta^2\end{aligned}$$

(3:) We see

$$\begin{aligned}\|D\varphi\|^2 &\overset{(4.6)}{=} \begin{cases} \|D\eta\|^2 \sum_{i_1,i_2=1}^{m} \delta_{i_1 i_2}^2 & (n=2) \\ \|D\eta\|^2 \sum_{i_1,i_2=1}^{m} \delta_{i_1 i_2}^2 \sum_{i_3=1}^{m} y_{i_3}^2 \cdots \sum_{i_n=1}^{m} y_{i_n}^2 & (n \geq 3) \end{cases} \\ &\overset{\text{Lemma 1(1)}}{=} m\|D\eta\|^2\end{aligned}$$

Thus, we have Lemma 3. □

By Theorem A (3), we have

$$\|Du\|^2 = \frac{n(n+m-2)}{\|x\|^2}. \tag{4.7}$$

Take the variation

$$u_t(x) = \frac{u(x) + t\varphi(x)}{\|u(x) + t\varphi(x)\|}.$$

The support of this variation is compact, since it is contained in the closure of $B_d - B_a$. Then, we have the second variation

$$\begin{aligned}(\delta^2 E)(u)(\varphi) &= \frac{d^2}{dt^2} E(u_t)\Big|_{t=0} \\ &\overset{\substack{(2.2)\text{ with} \\ \text{Lemma 3 (1)}}}{=} 2\int_{\mathbb{R}^m - \{0\}} \left(\|D\varphi\|^2 - \|Du\|^2\|\varphi\|^2\right) dx_1 \ldots dx_m. \\ &\overset{\text{Lemma 3 (2),(3)}}{=} 2m \int_{\mathbb{R}^m - \{0\}} \left(\|D\eta\|^2 - \frac{n(n+m-2)}{\|x\|^2}\eta^2\right) dx_1 \ldots dx_m \\ &= 2m \int_{B_b - B_a} \left(\|D\eta\|^2 - \frac{n(n+m-2)}{\|x\|^2}\eta^2\right) dx_1 \ldots dx_m \\ &\quad + 2m \int_{B_c - B_b} \left(\|D\eta\|^2 - \frac{n(n+m-2)}{\|x\|^2}\eta^2\right) dx_1 \ldots dx_m \\ &\quad + 2m \int_{B_d - B_c} \left(\|D\eta\|^2 - \frac{n(n+m-2)}{\|x\|^2}\eta^2\right) dx_1 \ldots dx_m \\ &=: \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3. \end{aligned} \tag{4.8}$$

Since $\eta = 1$ on $B_c - B_b$, we have

$$\mathrm{I}_2 \overset{(4.2)}{=} -2m \int_{B_c - B_b} \frac{n(n+m-2)}{\|x\|^2} \, dx_1 \dots dx_m < 0. \tag{4.9}$$

To estimate I_1, we note

$$\eta(x)^2 \geq \left(\max\{\eta_0(x) - \varepsilon, 0\} \right)^2 \tag{4.10}$$

since $\eta(x) \geq \eta_0(x) - \varepsilon$ by (4.3). On $B_b - B_a$, we see

$$\eta_0(x) \geq \varepsilon \quad \Leftrightarrow \quad \frac{\|x\| - a}{b - a} \geq \varepsilon \quad \Leftrightarrow \quad \|x\| \geq a + \varepsilon(b - a)$$

and hence

$$\eta_0(x) \geq \varepsilon \text{ on } B_b - B_{(a+\varepsilon(b-a))} \tag{4.11}$$

$$\eta_0(x) \leq \varepsilon \text{ on } B_{(a+\varepsilon(b-a))} - B_a. \tag{4.12}$$

Thus from (4.10), (4.11) and (4.12), we have

$$\eta(x)^2 \geq (\eta_0(x) - \varepsilon)^2 \text{ on } B_b - B_{(a+\varepsilon(b-a))} \tag{4.13}$$

$$\eta(x)^2 \geq (\eta_0(x) - \varepsilon)^2 \text{ on } B_b - B_{(a+\varepsilon(b-a))} \tag{4.14}$$

Then, we have

$$\begin{aligned}
\mathrm{I}_1 &\overset{(4.4)}{\leq} 2m \int_{B_b - B_a} \frac{(1+\varepsilon)^2}{(b-a)^2} \, dx_1 \dots dx_m - 2m \int_{B_b - B_a} \frac{n(n+m-2)}{\|x\|^2} \eta^2 \, dx_1 \dots dx_m \\
&\overset{(4.13),(4.14)}{\leq} \frac{2m(1+\varepsilon)^2}{(b-a)^2} \int_{B_b - B_a} dx_1 \dots dx_m \\
&\quad - 2nm(n+m-2) \int_{B_b - B_{(a+\varepsilon(b-a))}} \frac{1}{\|x\|^2} (\eta_0 - \varepsilon)^2 \, dx_1 \dots dx_m \\
&\overset{(4.1)}{=} \frac{2m(1+\varepsilon)^2}{(b-a)^2} \int_{B_b - B_a} dx_1 \dots dx_m \\
&\quad - 2nm(n+m-2) \int_{B_b - B_{(a+\varepsilon(b-a))}} \frac{1}{\|x\|^2} \left(\frac{\|x\| - a}{b - a} - \varepsilon \right)^2 dx_1 \dots dx_m \\
&= \frac{2m(1+\varepsilon)^2}{(b-a)^2} \int_{B_b - B_a} dx_1 \dots dx_m \\
&\quad - \frac{2nm(n+m-2)}{(b-a)^2} \int_{B_b - B_{(a+\varepsilon(b-a))}} \frac{1}{\|x\|^2} \Big(\|x\| - \big(a + \varepsilon(b-a)\big) \Big)^2 dx_1 \dots dx_m
\end{aligned} \tag{4.15}$$

Using the polar coordinate in $\mathbb{R}^m - \{0\}$, we have

the right hand side of (4.15)

$$\begin{aligned}
= \quad & \frac{2m(1+\varepsilon)^2}{(b-a)^2} \mathrm{Vol}(\mathbb{S}^{m-1}) \int_a^b \rho^{m-1} d\rho \\
& - \frac{2nm(n+m-2)}{(b-a)^2} \mathrm{Vol}(\mathbb{S}^{m-1}) \\
& \times \int_{(a+\varepsilon(b-a))}^b \frac{1}{\rho^2} \Big(\rho - \big(a + \varepsilon(b-a)\big) \Big)^2 \rho^{m-1} d\rho
\end{aligned}$$

$$
\begin{aligned}
= \quad & \frac{2m}{(b-a)^2} \operatorname{Vol}(\mathbb{S}^{m-1}) \Bigg[(1+\varepsilon)^2 \int_a^b \rho^{m-1} d\rho \\
& - n(n+m-2) \Bigg\{ \int_{(a+\varepsilon(b-a))}^b \rho^{m-1} d\rho \\
& - 2\big(a+\varepsilon(b-a)\big) \int_{(a+\varepsilon(b-a))}^b \rho^{m-2} d\rho \\
& + \big(a+\varepsilon(b-a)\big)^2 \int_{(a+\varepsilon(b-a))}^b \rho^{m-3} d\rho \Bigg\} \Bigg] \\
= \quad & \frac{2m}{(b-a)^2} \operatorname{Vol}(\mathbb{S}^{m-1}) \Bigg[\frac{(1+\varepsilon)^2}{m} (b^m - a^m) \\
& - n(n+m-2) \Bigg\{ \frac{1}{m} \Big(b^m - \big(a+\varepsilon(b-a)\big)^m \Big) \\
& - \frac{2}{m-1} \big(a+\varepsilon(b-a)\big) \Big(b^{m-1} - \big(a+\varepsilon(b-a)\big)^{m-1} \Big) \\
& + \frac{1}{m-2} \big(a+\varepsilon(b-a)\big)^2 \Big(b^{m-2} - \big(a+\varepsilon(b-a)\big)^{m-2} \Big) \Bigg\} \Bigg] \\
= \quad & \frac{2m}{(b-a)^2} \operatorname{Vol}(\mathbb{S}^{m-1}) \Bigg[\frac{(1+\varepsilon)^2}{m} (b^m - a^m) \\
& - n(n+m-2) \Bigg\{ \frac{1}{m} b^2 - \frac{2}{m-1} b\big(a+\varepsilon(b-a)\big) \\
& + \frac{1}{m-2} \big(a+\varepsilon(b-a)\big)^2 \Bigg\} b^{m-2} \\
& + n(n+m-2) \left(\frac{1}{m} - \frac{2}{m-1} + \frac{1}{m-2} \right) \big(a+\varepsilon(b-a)\big)^m \Bigg] \\
\overset{\substack{\text{Lemma 4 (1) and (2),} \\ \text{which is given later,} \\ \text{for } A=b \text{ and } B=a+\varepsilon(b-a)}}{=} \quad & \frac{2m}{(b-a)^2} \operatorname{Vol}(\mathbb{S}^{m-1}) \Bigg[\frac{(1+\varepsilon)^2}{m} (b^m - a^m) \\
& - \frac{n(n+m-2)}{m(m-1)(m-2)} \Big\{ (1-\varepsilon)^2 m(m-1)(b-a)^2 \\
& - 2(1-\varepsilon) m b (b-a) + 2b^2 \Big\} b^{m-2} \\
& + \frac{2n(n+m-2)}{m(m-1)(m-2)} \big(a+\varepsilon(b-a)\big)^m \Bigg]. \qquad (4.16)
\end{aligned}
$$

In the last equality, we use Lemma 4 (1) and (2), given later, for $A = b$ and $B = a + \varepsilon(b - a)$. Thus from (4.15) and (4.16), we have

$$
\begin{aligned}
\mathrm{I}_1 \le & \frac{2m}{(b-a)^2} \operatorname{Vol}(\mathbb{S}^{m-1}) \Bigg[\frac{(1+\varepsilon)^2}{m} (b^m - a^m) \\
& - \frac{n(n+m-2)}{m(m-1)(m-2)} \Big\{ (1-\varepsilon)^2 m(m-1)(b-a)^2 - 2(1-\varepsilon) m b (b-a) + 2b^2 \Big\} b^{m-2} \\
& + \frac{2n(n+m-2)}{m(m-1)(m-2)} \big(a+\varepsilon(b-a)\big)^m \Bigg]. \qquad (4.17)
\end{aligned}
$$

Take a large positive numbers α and we set $b = (\alpha + 1)a$. We know $(\alpha + 1)^m = \alpha^m + O(\alpha^{m-1})$, where $O(\ )$ denotes Landau's symbol, i.e., $O(\alpha^\ell)$ is a term satisfying $\dfrac{O(\alpha^\ell)}{\alpha^\ell}$ is bounded as $\alpha \to \infty$. Then, (4.17) implies

$$
\begin{aligned}
\mathrm{I}_1 \leq 2m \operatorname{Vol}(\mathbb{S}^{m-1}) \Bigg[& \frac{(1+\varepsilon)^2}{m} \frac{(\alpha+1)^m - 1}{\alpha^2} a^{m-2} \\
& - \frac{n(n+m-2)}{m(m-1)(m-2)} \Bigg\{ (1-\varepsilon)^2 m(m-1) \\
& \quad - 2(1-\varepsilon)m \frac{\alpha+1}{\alpha} + \frac{2(\alpha+1)^2}{\alpha^2} \Bigg\} (\alpha+1)^{m-2} a^{m-2} \\
& \quad + \frac{2n(n+m-2)}{m(m-1)(m-2)} \frac{\left(1+\varepsilon\alpha\right)^m}{\alpha^2} a^{m-2} \Bigg] \\
= 2m \operatorname{Vol}(\mathbb{S}^{m-1}) a^{m-2} \Bigg[& \frac{(1+\varepsilon)^2}{m} \alpha^{m-2} \\
& - \frac{n(n+m-2)}{m(m-1)(m-2)} \Big\{ (1-\varepsilon)^2 m(m-1) - 2(1-\varepsilon)m + 2 \Big\} \alpha^{m-2} \\
& \quad + \frac{2n(n+m-2)}{m(m-1)(m-2)} \frac{\left(1+\varepsilon\alpha\right)^m}{\alpha^2} + O\left(\alpha^{m-3}\right) \Bigg].
\end{aligned}
\tag{4.18}
$$

Take a sufficiently small positive number ε such that

$$
\varepsilon < \frac{1}{\alpha}. \tag{4.19}
$$

Then, $\varepsilon\alpha^{m-2} = O\left(\alpha^{m-3}\right)$ and $\varepsilon^2\alpha^{m-2} = O\left(\alpha^{m-4}\right)$, we see

$$
\frac{(1+\varepsilon)^2}{m} \alpha^{m-2} = \frac{1}{m} \alpha^{m-2} + O\left(\alpha^{m-3}\right),
$$

$$
\begin{aligned}
& \Big\{ (1-\varepsilon)^2 m(m-1) - 2(1-\varepsilon)m + 2 \Big\} \alpha^{m-2} \\
&= \Big\{ m(m-1) - 2m + 2 \Big\} \alpha^{m-2} + O\left(\alpha^{m-3}\right) \\
&= (m-1)(m-2)\, \alpha^{m-2} + O\left(\alpha^{m-3}\right)
\end{aligned}
$$

and

$$
\frac{2n(n+m-2)}{m(m-1)(m-2)} \frac{\left(1+\varepsilon\alpha\right)^m}{\alpha^2} = O\left(\alpha^{-2}\right).
$$

Then for sufficiently large α, we have

$$
\begin{aligned}
\mathrm{I}_1 &\leq 2m \operatorname{Vol}(\mathbb{S}^{m-1}) a^{m-2} \left\{ \left(\frac{1}{m} - \frac{n(n+m-2)}{m} \right) \alpha^{m-2} + O\left(\alpha^{m-3}\right) \right\} \\
&= -2 \operatorname{Vol}(\mathbb{S}^{m-1}) a^{m-2} \left\{ \left((n^2-1) + (m-2)n \right) \alpha^{m-2} + O\left(\alpha^{m-3}\right) \right\} \\
&< 0,
\end{aligned}
\tag{4.20}
$$

since the assumptions $m \geq 3$ and $n \geq 2$ imply $(n^2 - 1) + (m-2)n \geq 5 > 0$.

Similarly on $B_d - B_c$, we see

$$\eta_0(x) \geq \varepsilon \quad \Leftrightarrow \quad \frac{d - \|x\|}{d - c} \geq \varepsilon \quad \Leftrightarrow \quad \|x\| \leq d - \varepsilon(d-c)$$

and hence

$$\eta(x)^2 \geq (\eta_0(x) - \varepsilon)^2 \quad \text{on } B_{(d-\varepsilon(d-c))} - B_c \tag{4.21}$$

$$\eta(x)^2 \geq 0 \quad \text{on } B_d - B_{(d-\varepsilon(d-c))}. \tag{4.22}$$

and we have

$$\begin{aligned}
\mathrm{I}_3 \overset{\substack{(4.5),(4.21)\\ \text{and }(4.22)}}{\leq}\ & 2m \int_{B_d - B_c} \frac{(1+\varepsilon)^2}{(d-c)^2}\, dx_1 \ldots dx_m \\
& - 2m \int_{B_{(d-\varepsilon(d-c))} - B_c} \frac{n(n+m-2)}{\|x\|^2} (\eta_0 - \varepsilon)^2 dx_1 \ldots dx_m \\
\overset{(4.1)}{=}\ & \frac{2m(1+\varepsilon)^2}{(d-c)^2} \int_{B_d - B_c} dx_1 \ldots dx_m \\
& - 2nm(n+m-2) \int_{B_{(d-\varepsilon(d-c))} - B_c} \frac{1}{\|x\|^2} \left(\frac{d - \|x\|}{d-c} - \varepsilon \right)^2 dx_1 \ldots dx_m \\
=\ & \frac{2m(1+\varepsilon)^2}{(d-c)^2} \int_{B_d - B_c} dx_1 \ldots dx_m \\
& - \frac{2nm(n+m-2)}{(d-c)^2} \\
& \times \int_{B_{(d-\varepsilon(d-c))} - B_c} \frac{1}{\|x\|^2} \Big(\big(d - \varepsilon(d-c)\big) - \|x\| \Big)^2 dx_1 \ldots dx_m \\
=\ & \frac{2m(1+\varepsilon)^2}{(d-c)^2} \operatorname{Vol}(\mathbb{S}^{m-1}) \int_c^d \rho^{m-1} d\rho \\
& - \frac{2nm(n+m-2)}{(d-c)^2} \operatorname{Vol}(\mathbb{S}^{m-1}) \\
& \times \int_c^{(d-\varepsilon(d-c))} \frac{1}{\rho^2} \Big(\big(d - \varepsilon(d-c)\big) - \rho \Big)^2 \rho^{m-1} d\rho \\
=\ & \frac{2m}{(d-c)^2} \operatorname{Vol}(\mathbb{S}^{m-1}) \Bigg[\frac{(1+\varepsilon)^2}{m} (d^m - c^m) \\
& - n(n+m-2) \Bigg\{ \frac{1}{m} \Big(\big(d - \varepsilon(d-c)\big)^m - c^m \Big) \\
& - \frac{2}{m-1} \Big(\big(d - \varepsilon(d-c)\big)^{m-1} - c^{m-1} \Big) \big(d - \varepsilon(d-c)\big) \\
& + \frac{1}{m-2} \Big(\big(d - \varepsilon(d-c)\big)^{m-2} - c^{m-2} \Big) \big(d - \varepsilon(d-c)\big)^2 \Bigg\} \Bigg] \\
=\ & \frac{2m}{(d-c)^2} \operatorname{Vol}(\mathbb{S}^{m-1}) \Bigg[\frac{(1+\varepsilon)^2}{m} (d^m - c^m) \\
& - n(n+m-2) \left(\frac{1}{m} - \frac{2}{m-1} + \frac{1}{m-2} \right) \big(d - \varepsilon(d-c)\big)^m \\
& + n(n+m-2) \Bigg\{ \frac{1}{m} c^2 - \frac{2}{m-1} c \big(d - \varepsilon(d-c)\big) \\
& + \frac{1}{m-2} \big(d - \varepsilon(d-c)\big)^2 \Bigg\} c^{m-2} \Bigg]
\end{aligned}$$

$$\overset{\substack{\text{Lemma4 (1) and (2),}\\ \text{which is given later,}\\ \text{for } A=c \text{ and } B=d-\varepsilon(d-c)}}{=} \frac{2m}{(d-c)^2} \operatorname{Vol}(\mathbb{S}^{m-1}) \Bigg[\frac{(1+\varepsilon)^2}{m}(d^m - c^m) - \frac{2n(n+m-2)}{m(m-1)(m-2)} \big(d - \varepsilon(d-c)\big)^m + \frac{n(n+m-2)}{m(m-1)(m-2)} \Big\{ (1-\varepsilon)^2 m(m-1)(d-c)^2 + 2(1-\varepsilon)mc(d-c) + 2c^2 \Big\} c^{m-2} \Bigg] \tag{4.23}$$

Take a large positive number β and let ε be sufficiently small satisfying $\varepsilon < \dfrac{1}{\beta}$. We set $d = (\beta+1)c$. Then, (4.23) implies

$$\begin{aligned} \mathrm{I}_3 &\le 2m \operatorname{Vol}(\mathbb{S}^{m-1})\, c^{m-2} \left\{ \frac{1}{m}\beta^{m-2} - \frac{2n(n+m-2)}{m(m-1)(m-2)} \beta^{m-2} + O\big(\beta^{m-3}\big) \right\} \\ &= -\frac{2}{(m-1)(m-2)} \operatorname{Vol}(\mathbb{S}^{m-1})\, c^{m-2} \\ &\quad \times \left\{ \Big(2n^2 + 2(m-2)n - (m-1)(m-2) \Big) \beta^{m-2} + O\big(\beta^{m-3}\big) \right\}. \end{aligned}$$

Then using Lemma 5 mentioned later, we have

$$\mathrm{I}_3 < 0 \quad \text{for} \quad n \ge \frac{\sqrt{3}-1}{2}(m-1) \tag{4.24}$$

for sufficiently large number β. Thus by (4.9), (4.20) and (4.24), we conclude

$$(\delta^2 E)(u)(\varphi) < 0$$

and we finish the proof of our Main Theorem. □

We give here the following two lemmas which are used in the proof of Main Theorem. Lemma 4 is easy to prove and then we omit the proof. We give a proof of Lemma 5 only.

Lemma 4 *(1)* $\dfrac{1}{m} - \dfrac{2}{m-1} + \dfrac{1}{m-2} = \dfrac{2}{m(m-1)(m-2)}$

(2) $\dfrac{1}{m}A^2 - \dfrac{2}{m-1}AB + \dfrac{1}{m-2}B^2 = \dfrac{1}{m(m-1)(m-2)}\Big\{ m(m-1)(A-B)^2 - 2mA(A-B) + 2A^2 \Big\}$

Lemma 5 *If* $x \ge \dfrac{\sqrt{3}-1}{2}(m-1)$, *then we have*

$$2x^2 + 2(m-2)x - (m-1)(m-2) > 0.$$

Proof of Lemma 5 Let $\omega = \dfrac{\sqrt{3}-1}{2}$ and we note

$$2\omega^2 + 2\omega - 1 = 0. \tag{4.25}$$

Let

$$f(x) = 2x^2 + 2(m-2)x - (m-1)(m-2)$$

and then we have

$$f'(x) = 4x + 2(m-2) = 2(2x+m-2) > 0$$

for any $x > 0$. Therefore, $f(x)$ is monotone increase on $\{x > 0\}$ and we have

$$f(x) \geq f\left(\frac{\sqrt{3}-1}{2}(m-1)\right)$$

for any $x \geq \frac{\sqrt{3}-1}{2}(m-1)$. The right hand side of this inequality is:

$$\begin{aligned} f\left(\frac{\sqrt{3}-1}{2}(m-1)\right) &= f\big((m-1)\omega\big) \\ &= 2(m-1)^2\omega^2 + 2(m-1)(m-2)\omega - (m-1)(m-2) \\ &= (m-1)(m-2)\big(2\omega^2 + 2\omega - 1\big) + 2(m-1)\omega^2 \\ &\overset{(4.25)}{=} 2(m-1)\omega^2 > 0. \end{aligned}$$

Thus, we have $f(x) > 0$. □

At the end of this paper, we give two remarks on Main Theorem.

Remark 1 Though the map $u^{(n)}$ in Main Theorem has a singularity at $x = 0$, it is a *weakly harmonic map from* $\mathbb{R}^m$ $(m \geq 3)$, where

u is a *weakly harmonic map from* $\mathbb{R}^m$ $\overset{\text{def}}{\Longleftrightarrow}$ $u \in \mathrm{L}^{1,2}_{\mathrm{loc}}\big(\mathbb{R}^m, \mathbb{S}^{m^n-1}\big)$ and
$$\int_{\mathbb{R}^m} \Big(\langle Du, D\varphi\rangle - \|Du\|\, u \cdot \varphi\Big)\, dx = 0$$
for any $\varphi \in \mathrm{C}^\infty(\mathbb{R}^m, \mathbb{R}^{n+1})$ with compact support (a weak solution of the harmonic map equation).

Here, $\mathrm{L}^{1,2}_{\mathrm{loc}}\big(\mathbb{R}^m, \mathbb{S}^{m^n-1}\big)$ denotes the Sobolev space of $\mathbb{S}^{m^n-1}$-valued functions u on $\mathbb{R}^m$ such that both u and the weak derivative Du are in L^2 on any compact subset K of $\mathbb{R}^m$. The fact that $u^{(n)}$ is a *weakly harmonic map from* $\mathbb{R}^m$ $(m \geq 3)$ follows from the finiteness of the local energy near $x = 0$, i.e.,

$$\int_{B_r} \|Du^{(n)}\|^2\, dx = n(n+m-2)\mathrm{Vol}\,(\mathbb{S}^{m-1}) \int_0^r \rho^{m-3} d\rho < \infty \quad (r > 0)$$

for any $m \geq 3$, by the condition (3) in Theorem A. Then, Main Theorem implies that $u^{(n)}$ is an *unstable weakly harmonic map from* $\mathbb{R}^m$. Furthermore rescaling radially, we can obtain an *unstable weakly harmonic map* $\tilde{u}^{(n)}$ *from* B_1. Indeed, we take a large radius $R > 0$ satisfying that the support of the variation function φ in our proof is contained in B_R, and then we define

$$\tilde{u}^{(n)} \;\; B_1 \to \mathbb{S}^{m^n-1} \;\; \text{s.t. } \tilde{u}^{(n)}(x) = u^{(n)}(Rx)$$

which is an *unstable weakly harmonic map*.

Remark 2 As we have seen, our proof of Main Theorem needs only the quadratic inequality $2n^2 + 2(m-2)n - (m-1)(m-2) \geq 0$ with respect to n in Lemma 5, and therefore, we may assume the weaker condition

$$\begin{aligned} n &\geq \frac{-(m-2) + \sqrt{(m-2)^2 + 2(m-1)(m-2)}}{2} \\ &= \frac{(m-1)(m-2)}{\sqrt{(m-2)(3m-4)} + m - 2} \end{aligned}$$

in place of the assumption $n \geq \frac{\sqrt{3}-1}{2}(m-1)$.

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