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Core reduction for singular Riemannian foliations and applications to positive curvature

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Abstract
We expand upon the notion of a pre-section for a singular Riemannian foliation $(M, \mathcal{F})$, i.e. a proper submanifold $N \subset M$ retaining all the transverse geometry of the foliation. This generalization of a polar foliation provides a similar reduction, allowing one to recognize certain geometric or topological properties of $(M, \mathcal{F})$ and the leaf space $M/\mathcal{F}$. In particular, we show that if a foliated manifold M has positive sectional curvature and contains a nontrivial pre-section, then the leaf space $M/\mathcal{F}$ has nonempty boundary. We recover as corollaries the known result for the special case of polar foliations as well as the well-known analogue for isometric group actions.

Keywords Singular Riemannian foliation · Positive sectional curvature · Alexandrov spaces

Mathematics Subject Classification 53C12 · 53C23

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Introduction

All known examples of positively curved Riemannian manifolds have, in some sense or another, 'large' symmetry. This observation led to the initiation of the Grove Symmetry Program in 1991, birthing several systematic approaches to explore the link between the isometries of positively curved manifolds and their topology. Many techniques used rely not so much on the particular subgroup of the isometry group considered, but on how the orbits of those isometries decompose the given manifold. Such orbit decompositions are special cases of what are more generally known as *singular Riemannian foliations*. Galaz-Garcia and Radeschi used this more general framework to study positively curved manifolds carrying such foliations whose regular leaves are tori [6], generalizing some known results for torus actions as well as pointing out some interesting differences. Mendes and Radeschi [15], Corro [5], and Moreno [18] have continued this approach, establishing singular Riemannian foliations as a possible notion of symmetry for positively curved manifolds.

When the leaves of the foliation are closed, the so-called *leaf space* of the foliation is equipped with a natural metric which inherits lower curvature bounds in the comparison sense (see Burago, Burago, and Ivanov [3]). This allows one to study such quotients using Alexandrov geometry. In particular, the notions of spaces of directions and boundary are easy to describe in terms of the foliation and can hence be employed to study singular Riemannian foliations and the manifolds which admit them.

For positively curved leaf spaces, the presence of boundary already places topological restrictions on both the leaf space and the manifold (see Moreno [17, Theorem 4.2.3]). Moreover, placing some simple additional hypotheses on the topology of the boundary can yield strong topological implications on not just the leaf space and manifold, but also the leaves of the foliation, for example as in [17, Theorem 4.3.1]. Taking a step back, one is inclined to ask: what are some sufficient conditions that guarantee the presence of nonempty boundary?

In [22], Wilking showed that for positively curved manifolds, an isometric action with non-trivial principal isotropy group will have orbit space with nonempty boundary. This does not immediately generalize to leaf spaces of singular Riemannian foliations, where there is no group action. However, non-trivial principal isotropy guarantees a non-trivial *core reduction* as defined for group actions by Grove and Searle in [9].

Recall that the core ${}_cM \subset M$ of an isometric group action (M, G) together with its *core group* ${}_cG$, form a 'reduction' of the group action (see Sect. 2.1 for definitions). In particular, ${}_cG < G$ and the orbit spaces M/G and ${}_cM/{}_cG$ are isometric. Observe for a group action on M by G whose principal isotropy group is trivial, the core is the whole manifold. In this case though, there may exist $G' > G$ which acts orbit equivalently on M with M/G isometric to M/G', though containing non-trivial principal isotropy group. The group G' then has a core different from M, and thus a reduction. By the work of Straume in [21], we have examples of such phenomena when $G \subset \mathrm{O}(n)$ is acting with cohomogeneity 2 or 3 on $\mathbb{R}^n$. In general it is not clear when such a group exists. Recall that an isometric group action is a *polar action* when there exists a connected, complete totally geodesic embedded submanifold Σ, called a *section*, with a finite action by a so-called *polar group* W such that M/G and Σ/W are isometric. Despite the geometric similarities, a section of a polar action is not necessarily a *core* of that action. A simple example is the $\mathbf{S}^1$ action on $\mathbf{S}^2$ by rotation about a fixed axis. This is a polar action, whose section is a great circle and whose polar group is $\mathbb{Z}_2$. On the other hand, the core is all of $\mathbf{S}^2$. This action is given by a representation $\rho\colon \mathrm{SO}(2) \to \mathrm{O}(3) = \mathrm{Isom}(\mathbb{R}^3)$ and can be extended in the sense above to a new representation $\rho\colon \mathrm{O}(2) \to \mathrm{O}(3) = \mathrm{Isom}(\mathbb{R}^3)$ with the same orbits. The core of *this* $\mathrm{O}(2)$

action is indeed the polar section of the action of SO(2). Observe that the action of $SO(2)$ on $\mathbf{S}^2$ satisfies the hypothesis of the work in [21]. Nonetheless given an effective Lie group action G on a manifold M, it is difficult in general to find a lower dimensional Lie group $\bar{G}$ acting effectively on a smaller space N, such that $M/G = N/\bar{G}$. When we consider group actions on finite dimensional vector spaces given by effective representations, the Gorodski and Lytchak point to irreducible representations as a starting point for tackling this problem [7][Question 1.4].

Gorodski et al. [8] introduced the notion of *copolarity* to define a *k-section* of an isometric group action, providing a broader framework to discuss such reductions. In this language, a section of a polar action is at one extreme: it is a 0-section, while in general the core of a group action (M, G) is some k-section, where k is the difference between the dimension of the core and the dimension of the orbit space M/G. The core of the SO(2) action on $\mathbf{S}^2$ above is the entire original manifold, which has dimension 1 greater than the orbit space, hence is an example of a 1-section. In [14], Magata referred to k-sections of group actions as *fat sections* and proved that they are a form of reduction in the sense above (see [14][Theorem 3.1]).

Clearly, these reductions rely on the presence of a group action. *Polar foliations* generalize polar actions to the setting of singular Riemannian foliations by using the geometric properties of sections of polar actions to define a section of a foliation. This is more than simply an expanding of language, as there are polar foliations whose leaves are not the orbits of an isometric group action. For example, most FKM-type foliations by isoparametric hypersurfaces are not induced by a group action (see Radeschi [20]). This sort of reduction is extreme in the foliation setting as well. An expansion of this notion, similar to what was done in [8], was proposed in the thesis of Magata [13], where he referred to them as *pre-sections*, though it was not developed beyond a definition. Roughly, a pre-section is a totally geodesic connected submanifold which intersects all leaves of the foliation, and is transversal to most leaves (see Definition 2.1 for a precise definition). Any foliation M has at least one pre-section, namely M itself. When we refer to a non-trivial presection, we mean that we have a proper submanifold of positive dimension.

Motivated by the guarantee of boundary in the presence of a non-trivial core and the more general framework of pre-sections to which cores belong, we prove the following:

Theorem A *Let $\mathcal{F}$ be a singular Riemannian foliation with closed leaves on a positively curved manifold M. If $(M, \mathcal{F})$ has a non-trivial pre-section, then $\partial(M/\mathcal{F}) \neq \emptyset$.*

To see the need for the positive curvature assumption, consider the singular Riemannian foliation of $\mathbb{R}^2$ by horizontal lines. The vertical axis is then an example of a pre-section (a polar section, even) and the quotient is $\mathbb{R}$. The real issue here is that the codimension of each leaf in the "ambient" foliation is the same as one of the corresponding subleaves in the 'smaller' foliation, which is impossible in positive curvature (see Lemma 3.2 below).

Moreover, not every leaf space of a singular Riemannian foliation on a Riemannian manifold with positive sectional curvature has non empty boundary. The Hopf fibration is an example of a Riemannian foliation on the round $\mathbf{S}^3$ whose leaf space has empty boundary. Taking the foliated spherical join of two copies of $\mathbf{S}^3$ with the Hopf fibration yields a singular Riemannian foliation on the round $\mathbf{S}^7$ by the 2-torus, whose leaf space is $\mathbf{S}^5$. In both cases due to Theorem A if follows that there does not exist a non-trivial pre-section.

Since sections of polar sections of polar foliations are a special type of pre-section, we have:

Corollary B *If $(M, \mathcal{F})$ is a closed, polar singular Riemannian foliation that is not a single leaf or a foliation by points on a positively curved manifold, then $M/\mathcal{F}$ has nonempty boundary.*

The induced linear action on the round sphere coming from any irreducible representation is an example of a group action with no non-trivial pre-section, e.g. the Hopf fibration on $\mathbf{S}^3$. For actions with non-trivial pre-section, called k-sections, we have the following special case regarding orbit spaces of isometric group actions:

Corollary C *Let G be a compact lie group acting isometrically and not transitively on a positively curved Riemannian manifold M. If M contains a non-trivial k-section, then M/G has nonempty boundary.*

Remark 1 Corollary B is not actually new. In [12][Theorem 1.6], Lytchak arrives at the same result without the positive curvature condition. With the methods employed here, these curvature conditions are guaranteed once one restricts to the normal spheres of a given leaf (see Lemma 2.11). Moreover, we avoid the issue of exceptional leaves by restricting to the spheres normal to the stratum (see Lemma 2.10). In any case, we arrive at this result for entirely different reasons, so include it as a consequence of the Main Theorem.

Remark 2 Though it is not explicitly written in the literature, it is worth mentioning that one may also reach Corollary C through different means as well. Namely, [14][Theorem 4.2] gives that "fat sections" of isometric actions induce "fat sections" of the isotropy representations. If one supposes that such a "fat section" is non-trivial (i.e. not the entire manifold), then the isotropy representation is not reduced and by the contrapositive of Proposition 5.2 in [7], we arrive at nonempty boundary. Again, since our methods are quite different here, we include this result as a proper corollary.

1 Preliminaries

In this section, we provide a brief rundown of relevant concepts about singular Riemannian foliations, including a local description of such foliations and the Alexandrov geometry of their associated quotient spaces.

1.1 Singular Riemannian foliations

We begin with the definition of a singular Riemannian foliation on a fixed smooth Riemannian manifold (M, g).

Definition 1.1 Given a Riemannian manifold M, a *singular Riemannian foliation*, which we denote by $(M, \mathcal{F})$, is a partition of M by a collection $\mathcal{F} = \{L_p \mid p \in M\}$ of connected, complete, immersed submanifolds L_p, called *leaves*, which may not be of the same dimension, such that the following conditions hold:

(i) Every geodesic meeting one leaf perpendicularly, stays perpendicular to all the leaves it meets.
(ii) There exists a family of vectors fields on M which at any point $p \in M$, span the tangent space to the leaf through p.

If the partition $(M, \mathcal{F})$ satisfies the first condition, then we say that $(M, \mathcal{F})$ is a *transnormal system*. If it satisfies the second condition, we say that $(M, \mathcal{F})$ is a *smooth singular foliation*. When all the leaves have the same dimension, we say that the foliation is a *regular Riemannian foliation* or just a *Riemannian foliation*. We will deal mainly with *closed* singular Riemannian foliations—those in which every leaf is closed (compact without boundary). The interested reader can consult Alexandrino, Briquet, and Töben [2], or [5] and [18] for a more detailed discussion of singular Riemannian foliations.

A standard example of a singular Riemannian foliation is the orbit decomposition of a Riemannian manifold under some connected group action by isometries. Such foliations are called *homogeneous*, in reference to their leaves being homogeneous manifolds. Although the geometry of singular Riemannian foliations closely resembles that of orbit decompositions, there are important examples of singular Riemannian foliations which do not come from isometric group actions. The fibers of a Riemannian submersion also provide examples of singular Riemannian foliations, and a well known *inhomogeneous* such foliation is given by the $\mathbf{S}^7$ fibers of the Hopf map $\mathbf{S}^{15} \to \mathbf{S}^8$ (see [20] for more examples).

It is often important to distinguish the leaves of a foliation by their dimension, as the local picture of the foliation differs depending on this dimension. For a connected manifold M, the *dimension* of a foliation $\mathcal{F}$, denoted by $\dim \mathcal{F}$, is the maximal dimension of the leaves of $\mathcal{F}$. The *codimension* of a foliation is,

$$\operatorname{codim}(M, \mathcal{F}) = \dim M - \dim \mathcal{F}.$$

Leaves of maximal dimension are called *regular leaves* and the remaining leaves are called *singular leaves*. Since $\mathcal{F}$ gives a partition of M, for each point $p \in M$ there is a unique leaf, which we denote by L_p, that contains p. We say that L_p is the *leaf through* p.

Definition 1.2 For an integer $0 \leqslant k < \dim(M)$ we define the *(coarse) stratum of dimension* k as

$$\Sigma_k = \{p \in M \mid \dim(L_p) = k\}.$$

For $k = \dim(\mathcal{F})$ the stratum Σ_k is known as the regular part of $\mathcal{F}$ and denoted by M_{reg}.

The quotient space $M/\mathcal{F}$ obtained from the partition of M, is known as the *leaf space* and the quotient map $\pi : M \to M/\mathcal{F}$ is the *leaf projection map*. The topology of M yields a topology on $M/\mathcal{F}$, namely the quotient topology. With respect to this topology the quotient map is continuous. We discuss the geometry of this quotient space in Sect. 1.3.

Given a singular Riemannian foliation $(M, \mathcal{F})$, we denote by $O(M, \mathcal{F})$ the group of isometries of M which respect the foliation, i.e. map leaves to leaves, and by $O(\mathcal{F})$ the group of isometries which leaves any leaf of the foliation invariant. Observe that $O(M, \mathcal{F})/O(\mathcal{F})$ is the group of bijections of the leaf space $M/\mathcal{F}$ which lift to isometries of M (see [15]).

We end this section with a characterization of singular foliations using the language of distributions, as presented by Lavau [11].

On a smooth manifold M, a *(generalized) distribution* $\mathcal{D}$ is the assignment to each point $p \in M$, of a subspace $\mathcal{D}(p)$ of the tangent space T_pM .

A distribution $\mathcal{D}$ is smooth at a point p if any tangent vector $X(p) \in \mathcal{D}(p)$ can be locally extended to a smooth vector field X on some open set $U \subset M$ such that $X(q) \in \mathcal{D}(q)$ for every $q \in U$.

A distribution $\mathcal{D}$ is *generated by a family of (possibly locally defined) vector fields* F if the following holds

$$\mathcal{D}(p) = \operatorname{span}\{X(p) \mid X \in F\},$$

for every $p \in M$.

Condition (ii) of Definition 1.1 says that the leaves of a Singular Riemannian foliation are the integral manifolds of the distribution generated by a family of smooth vector fields on M

Any $X \in F$ defines a flow $t \mapsto \phi_t^X$. For every $t \in \mathbb{R}$, the map ϕ_t^X is a (local) diffeomorphism of M, with inverse ϕ_{-t}^X. We say that a distribution $\mathcal{D}$ is F-*invariant* if for any $X \in F$ we have:

$$\left(\phi_t^X\right)_* (\mathcal{D}(y)) \subset \mathcal{D}\left(\phi_t^X(y)\right),$$

for every y in the domain of X and $t \in \mathbb{R}$.

Theorem 1.3 (Steffan-Sussmann, Theorem 4 in [11]) *Let M be a smooth manifold and let $\mathcal{D}$ be a smooth distribution. Then $\mathcal{D}$ is integrable if and only if it is generated by a family F of smooth vector fields, and is invariant with respect to F.*

1.2 Infinitesimal foliations

Let $(M, \mathcal{F})$ be a closed manifold with a closed singular Riemannian foliation. In this section, we present definitions and results related to the infinitesimal, or local, structure of a foliation near a fixed point in the manifold.

For fixed $p \in M$ and $\varepsilon > 0$ sufficiently small, let $\mathbf{S}_p^{\perp}(\varepsilon)$ denote the unit sphere of radius ε in $\nu_p(M, L_p) \subset T_pM$ with respect to the inner product g_p.

Definition 1.4 The *infinitesimal foliation* $\mathcal{F}_p$ on $\mathbf{S}_p^{\perp}(\varepsilon)$ is given by taking the connected components of the preimages under the exponential map at p of the intersection between the leaves of $\mathcal{F}$ and $\exp_p(\mathbf{S}_p^{\perp}(\varepsilon))$.

It was shown by Molino in [16][Proposition 6.5] that this partition is a singular Riemannian foliation when we consider the round metric on $\mathbf{S}_p^{\perp}(\varepsilon)$. Moreover by [16][Proposition 6.2], this foliation does not depend on the radius ε chosen. Thus from now on we only consider the unit sphere $\mathbf{S}_p^{\perp} = \mathbf{S}_p^{\perp}(1)$, equipped with $\mathcal{F}_p$. Traditionally, an *infinitesimal foliation* $(V, \mathcal{F})$ refers to a singular Riemannian foliation of a Euclidean space V containing the origin as a leaf. Since any such foliation is the cone of the foliation on the unit sphere in V, we use this term to refer to both foliations.

To each loop in $\pi_1(L_p, p)$, one can construct a foliated isometry of $(\mathbf{S}_p^{\perp}, \mathcal{F}_p)$ which leaves invariant the leaves of $\mathcal{F}_p$. In fact, there is a group morphism $\rho_p \colon \pi_1(L_p, p) \to O(\mathbf{S}_p^{\perp}, \mathcal{F}_p)/O(\mathcal{F}_p)$ (see [15][Sect. 3.2] or [5][Sect. 2.5]).

Definition 1.5 Denote by Γ_p the image of $\pi_1(L_p, p)$ under the morphism $\rho_p \colon \pi_1(L_p, p) \to O(\mathbf{S}_p^{\perp}, \mathcal{F}_p)/O(\mathcal{F}_p)$. The group Γ_p is known as the *leaf holonomy group of* L_p.

In particular, the group Γ_p acts effectively by isometries on the leaf space $\mathbf{S}_p^{\perp}/\mathcal{F}_p$. Leaves of maximal dimension with Γ_p equal to the trivial group are called *principal leaves* (see [5]).

For a closed singular Riemannian foliation, the infinitesimal foliation at a point $p \in M$ and the holonomy group Γ_p are sufficient to determine how a tubular neighborhood of the leaf L_p looks up to foliated diffeomorphism.

Theorem 1.6 *(Slice Theorem in [15]) Let $(M, \mathcal{F})$ be a singular Riemannian foliation, and let L be a closed leaf with infinitesimal foliation $(\nu_p(M, L), \mathcal{F}_p)$ at a point $p \in L$. Let $P \to L$ be the G-principal covering associated to $G = \ker\{\rho_p \colon \pi(L, p) \to \Gamma_p\}$. Then for a small enough $\varepsilon > 0$, the ε-tube U around L is foliated diffeomorphic to $(P \times_{\Gamma_p} \nu_p^{\varepsilon}(M, L), \mathcal{F}_p), P \times_{\Gamma_p} \mathcal{F}_p)$.*

The following lemma is used in the proof of Theorem 2.9 and we include it for the sake of completeness.

Lemma 1.7 (Lemma 4.1 in [14]) *Let N be a totally geodesic submanifold of the Riemannian manifold M and let $\gamma\colon I \to N$ be a geodesic. Then every Jacobi field J along γ splits uniquely into Jacobi fields Y and Z along γ such that Y is a Jacobi field in N and Z is perpendicular to N. Furthermore, every derivative of Z is perpendicular to N.*

1.3 Alexandrov geometry of leaf spaces

We briefly mention some concepts from Alexandrov geometry that we will later need.

Definition 1.8 A locally compact, locally complete inner metric space is an *Alexandrov space* (X, d) if it satisfies local lower curvature bounds as in Topogonov's Theorem (see Burago, Gromov, and Perel'man [4]). In the case that the lower curvature bound is k, we write either $curv(X) \geq k$ or $X \in Alex(k)$.

The *dimension* of an Alexandrov space X is equal to the Hausdorff dimension of X. In particular for $(M, \mathcal{F})$ a closed singular Riemannian foliation on a complete connected manifold, the dimension of $M/\mathcal{F}$ equals the codimension of $\mathcal{F}$.

Similar to how honest lower curvature bounds are inherited by the target of a Riemannian submersion, lower curvature bounds in the sense of Toponogov are inherited by the targets of the metric space analogue of such maps, which we define below.

Definition 1.9 A map $f\colon (X, d_X) \to (Y, d_Y)$ between metric spaces is called a *submetry* if for any $x \in X$, and any $r > 0$ the following holds

$$f(B_r(x)) = B_r(f(x)).$$

We say that the submetry f is a *discrete submetry* if for all points $y \in f(X)$ the fibers $f^{-1}(y)$ are discrete subspaces of X.

Remark 3 Given a closed singular Riemannian foliation $(M, \mathcal{F})$, the leaf projection map $\pi : M \to M/\mathcal{F}$ is an example of a submetry. Thus, if $(M, \mathcal{F})$ is a singular Riemannian foliation with closed leaves and $sec(M) \geq k$, then the leaf space $M/\mathcal{F}$ is an Alexandrov space with $curv(M/\mathcal{F}) \geq k$ with respect to the metric induced by the Hausdorff distance between the leaves in M. The dimension of $M/\mathcal{F}$ is the codimension of $\mathcal{F}$.

Without tangent spaces, Alexandrov spaces do not have the usual 'tangent sphere' as manifolds do. Instead, one describes an analogous concept, the so-called *space of directions*, using the metric and comparisons to a model space as follows.

Consider $X \in Alex(k)$. Given two curves $c_1\colon [0, 1] \to X$ and $c_2\colon [0, 1] \to X$ with $c_1(0) = c_2(0) = x \in X$, we define the *angle between c_1 and c_2* as

$$\angle(c_1, c_2) := \lim_{s,t \to 0} \tilde{\angle}(c_1(s), x, c_2(t)).$$

where $\tilde{\angle}(c_1(s), x, c_2(t))$ is the angle in the comparison triangle in the appropriate model space of constant curvature k.

A curve $c\colon [0, 1] \to X$ is a *geodesic* if the length of c equals the distance $d(c(0), c(1))$. Two geodesics $c_1\colon [0, 1] \to X$ and $c_2\colon [0, 1] \to X$ emanating from a common fixed point $x \in X$ are said to be *equivalent* if the angle between them is zero. The set $\widetilde{\Sigma}_x$ of these equivalence classes becomes a metric space by declaring the distance between two classes to be the angle formed between any two representatives of each class. The *space of directions* $\Sigma_x(X)$ at x of X is the metric completion of the space $\widetilde{\Sigma}_x$. The following is a well-known collection of results which will be crucial for our "inductive" proof of Theorem A.

Theorem 1.10 (Corollaries 7.10, 7.11 in Burago, Gromov and Perelman [4]) *Let X be an Alexandrov space of dimension n. Then for any $x \in X$, the space of directions $\Sigma_x(X)$ is a compact Alexandrov space with curvature at least* 1*, and of dimension* $(n-1)$.

For leaf spaces of singular Riemannian foliations, we have:

Proposition 1.11 (see p. 4 in [18]) *The space of directions of the Alexandrov space $M/\mathcal{F}$ at p^*, consists of geodesic directions and is isometric to $(\mathbf{S}^{\perp}_p/\mathcal{F}_p)/\Gamma_p$.*

It is worth mentioning here that for a principal leaf $L_p \subset M$, the infinitesimal foliation $(\mathbf{S}^{\perp}_p, \mathcal{F}_p)$ is a foliation by points and the leaf holonomy Γ_p is trivial. Thus for $p \in M$ contained in a principal leaf, the space of directions at $p^* \in M/\mathcal{F}$ is $\Sigma_{p^*} \cong \mathbf{S}^{\perp}_p$.
For Alexandrov spaces, the *boundary* is defined inductively from the spaces of the directions:

Definition 1.12 Let X be an Alexandrov space. The *boundary* of X, denoted $\partial(X)$ is defined inductively as

$$\partial X := \{x \in X \mid \partial \Sigma_x(X) \neq \emptyset\}.$$

Here we use the fact that spaces of directions are compact positively curved Alexandrov spaces with $\dim(\Sigma_x(X)) = \dim(X) - 1$, and the only such 1-dimensional Alexandrov spaces are circles or closed intervals, both with diameter $\leq \pi$.

For leaf spaces then, the boundary will consist of all points $p^* \in M/\mathcal{F}$ such that

$$\partial\left((\mathbf{S}^{\perp}_p/\mathcal{F}_p)/\Gamma_p\right) \neq \emptyset.$$

All strata of $\mathcal{F}$ whose closure contains the leaf L_p appear as strata of $(\mathbf{S}^{\perp}_p, \mathcal{F}_p)$ (see [17]). In particular, nearby leaves of the same dimension as L_p appear as 0-dimensional leaves in $\mathcal{F}_p$. If $\mathbf{S}^{\perp}_p$ is an n-dimensional sphere, we have the following splitting given by Radeschi in [19]:

$$(\mathbf{S}^n, \mathcal{F}_p) \cong (\mathbf{S}^k, \mathcal{F}_0) * (\mathbf{S}^{n-k-1}, \mathcal{F}_1),$$

where $\mathcal{F}_0$ is a foliation by points and $\mathcal{F}_1$ is a foliation containing no point leaves. We refer to $(\mathbf{S}^{n-k-1}, \mathcal{F}_1)$ as the *infinitesimal foliation normal to the stratum of* L_p, and refer to the quotient $\mathbf{S}^{n-k-1}/\mathcal{F}_1$ as the *space of directions normal to the stratum of* L_p. We focus on this space of normal directions in our proof of Theorem A.

From the description of singular Riemannian foliations on round spheres in Radeschi [19], and the discussion in [17][pp. 25–28], we have the following lemma, which allows us to recognize regular leaves of a foliation at the infinitesimal level.

Lemma 1.13 *Consider $(M, \mathcal{F})$ a closed singular Riemannian foliation. Fix $p \in M$ and consider $v' \in \mathbf{S}^{\perp}_p$. Assume that for $v = \lambda v'$, the point $\exp_p(v)$ is contained in the tubular neighborhood determined by the Slice Theorem. Then $\mathcal{L}_{v'}$ is a regular leaf of $(\mathbf{S}^{\perp}_p, \mathcal{F}_p)$ if and only if for $q = \exp_p(v)$, the leaf L_q is a regular leaf. Moreover, if q is in a regular leaf, then for $t \in (0, 1]$, the leaf $L_{\gamma(t)}$ is a regular leaf of $\mathcal{F}$.*

2 Pre-sections and local pre-sections

In this section we present the definition of a pre-section for a closed singular Riemannian foliation $(M, \mathcal{F})$. We show that the intersections of the leaves of $\mathcal{F}$ with a pre-section $N \subset M$

form a singular Riemannian foliation $(N, \mathcal{F}')$ in Lemma 2.6. We also show that for a fixed point $p \in N \subset M$, the infinitesimal foliation at p with respect to $(N, \mathcal{F}')$, forms a pre-section of the infinitesimal foliation with respect to $(M, \mathcal{F})$ in Theorem 2.9. This allows us to locally inductively "reduce" the foliation in the presence of a non-trivial pre-section $N \subset M$, which is key for the proof of Theorem A. We finish the section with Lemma 2.11, which allows us to further restrict our attention to foliations of spheres.

2.1 Pre-sections

Definition 2.1 Let $(M, \mathcal{F})$ be a closed singular Riemannian foliation on a complete manifold. A connected embedded submanifold $N \subset M$ is a *pre-section of* $(M, \mathcal{F})$ if the following are satisfied:

(A) N is complete, totally geodesic.
(B) N intersects every leaf of $\mathcal{F}$.
(C) For every point p in $N \cap M_{\text{reg}}$ we have $\nu_p(M, L_p) \subset T_pN$.

We say that a pre-section $N \subset M$ is *non-trivial* if N is a proper submanifold and is not a single point.

Note that condition (C) implies that a pre-section intersects the regular leaves transversally. The main theorem of this paper was motivated by the development of a *core* for an isometric group action in [9]. We show that what they called a core is indeed a special case of the more general notion of a pre-section developed in this paper.
Let M be a smooth Riemannian manifold with a smooth effective action via isometries by a compact Lie group G. Recall that for fixed $p \in M$, the *orbit through* p is the subset $G(p) = \{g \cdot p \in M \mid g \in G\}$, and the *isotropy subgroup at* p is the subgroup $G_p = \{g \in G \mid g \cdot p = p\}$. An orbit $G(p)$ is a *principal* orbit when the isotropy group G_p acts trivially on $\nu_p(M, G(p))$ (see Alexandrino and Bettiol [1][Exercise 3.77]). Denote by M_0 the subset consisting of all principal orbits in M. Given p_1 and p_2 in M_0, their isotropy groups are conjugate to each other in G. Fix an principal isotropy group H, and consider the action of H on M_0. The *core of the group action*, ${}_cM$ is the closure of M_0^H, the fixed point set of the action of H on M_0. The *core group*, ${}_cG$, is $N(H)/H$, where $N(H)$ is the normalizer of H in G. We now show that the core satisfies all three conditions of Definition 2.1.

Example 1 (Cores are Pre-sections) Let ${}_cM$ be a core of an isometric group action as above. By [9][Proposition 1.2] ${}_cM$ consists of the connected components F in M^H, the set of points in M fixed by H, such that $F \cap M_0 \neq \emptyset$. Recall that each of these connected components is a totally geodesic submanifold, and that M_0 is an open set. Thus, cM is a totally geodesic submanifold. By [9][Proposition 1.4] the inclusion of ${}_cM \subset M$ induces and isometry between ${}_cM/{}_cG$ and M/G. Thus we conclude that each G-orbit intersects ${}_cM$. We now prove that ${}_cM$ satisfies 2.1 C. Consider $p \in {}_cM \cap M_{\text{reg}}$. By [1][Exercise 3.86] an exceptional orbit has the same dimension as a principal orbit, but more connected components. Moreover for any q close enough to p, contained in a principal orbit, we have a principal G_p/G_q-covering of $G(q)$ by $G(p)$. Since condition C is a local condition, we may assume that p lives in a principal orbit. Since ${}_cM \subset M^H$ we conclude that $H \subset G_p$. This implies that $H = G_p$, since p is in a principal orbit. By [1][Exercise 3.77] H acts trivially on the normal space $\nu_p(M, G(p))$. This implies that the slice through p is contained in $M_0^H \subset {}_cM$. Thus we conclude that $\nu_p(M, G(p)) \subset T_p({}_cM)$. The interested reader can consult [9] for more properties of the core.

We start by proving the following lemma for regular leaves of $\mathcal{F}$.

Lemma 2.2 *Let $(M, \mathcal{F})$ be a closed singular foliation on a complete manifold, and N a pre-section. Consider $p \in N$ such that L_p is a regular leaf of $\mathcal{F}$. Assume $q \in N$ is a closest point in $L_q \cap N$ to p. Then q is a closest point to p in L_q and $q \in \exp_p(\nu_p(M, L_p))$. Note that this implies that the geodesic from p to q in M is contained in N.*

Proof Assume there exists $q' \in L_q$ different from q which is a closest point to p in L_q. From this it follows that $q' = \exp_p(v)$ for some $v \in \nu_p(M, L_p)$. Since p is contained in a regular leaf, the condition (C) implies that $\nu_p(M, L_p) \subset T_pN$. Thus the minimizing geodesic in M joining p to q' given by $\gamma(t) = \exp_p(tv)$, is a geodesic in N. But this implies that $q' \in L_q \cap N$, and the distance from q to p is larger or equal than the distance from q' to p. The distance cannot be strictly larger, since this would contradict the fact that q is a closest point to p in $L_q \cap N$. Thus the distance from q to p in M realizes the distance from L_q to p. Therefore, there exists $v_0 \in \nu_p(M, L_p)$ with $q = \exp_p(v_0)$. □

Lemma 2.3 *(Lemma 2.5 in [14]) Let $(M, \mathcal{F})$ be a closed singular Riemannian foliation on a complete manifold. Let $p \in M$ be contained in a regular leaf L_p. Then $\exp_p(\nu_p(M, L_p)$ intersects each leaf of $\mathcal{F}$.*

Proof Take $L \in \mathcal{F}$ to be an arbitrary leaf. Then there exists a geodesic $\gamma^*\colon I \to M/\mathcal{F}$ joining L^* to p^* by Hopf-Rinow [3][Remark 2.5.29]. Since around a small ball of p^* in $M/\mathcal{F}$ the projection map $\pi: M \to M/\mathcal{F}$ is a submersion, there exists a unique geodesic $\gamma\colon I \to M$ starting at p, with $\gamma'(0) \in \nu_p(M, L_p)$, lifting γ^*. Thus the conclusion follows. □

Lemma 2.4 *(Lemma 5.2 in [8], Lemma 2.7 in [14]) Let $(M, \mathcal{F})$ be a closed singular Riemannian foliation on a complete manifold, and N be a pre-section of $\mathcal{F}$. Then for any $p \in N$, there exists $v \in T_pN \cap \nu_p(M, L_p)$ such that for $q = \exp_p(v)$, the leaf L_q is regular. In particular the leaf $\mathcal{L}_v \in \mathcal{F}_p$ is a regular leaf.*

Proof Let L be a regular leaf of $\mathcal{F}$, and denote by L' a connected component of $L \cap N$. There exists a minimizing geodesic $c\colon [0, \ell] \to N$ from p to L'. Thus we have by construction that $c'(\ell) \in \nu_{c(\ell)}(N, L')$. By definition, N and L are transversal, which implies $T_{c(\ell)}L' = T_{c(\ell)}N \cap T_{c(\ell)}L$. Therefore by writing $T_{c(\ell)}M = T_{c(\ell)}L \oplus \nu_{c(\ell)}(M, L)$, and using the fact that $\nu_{c(\ell)}(M, L) \subset T_{c(\ell)}N$ we get that $T_{c(\ell)}N = T_{c(\ell)}L' \oplus \nu_{c(\ell)}(M, L)$. Thus we conclude that $c'(\ell) \in \nu_{c(\ell)}(M, L)$. Since $\mathcal{F}$ is a singular Riemannian foliation and c is also a geodesic of M, we get that $c'(0) \in T_pN \cap \nu_p(M, L_p)$. Taking $v = c'(0)$, we get that $q = \exp_p(v)$ lies in a regular leaf of $\mathcal{F}$, and thus v lies in a regular leaf of $\mathcal{F}_p$. □

Lemma 2.5 *Let $(M, \mathcal{F})$ be a closed singular Riemannian foliation, and let $N \subset M$ be a pre-section of $\mathcal{F}$. Then on N the distribution $\mathcal{D}'(p) = T_pL \cap T_pN$ is integrable and induces a smooth foliation $\mathcal{F}'$ on N. Over an open and dense set the leaves of $\mathcal{F}'$ are connected components of the intersections of N with the leaves of $\mathcal{F}$.*

Proof Observe that $M_{\mathrm{reg}} \cap N$ is an open and dense subset of N. Moreover for $p \in M_{\mathrm{reg}} \cap N$, by condition 2.1 C the intersection $L_p \cap N$ is a smooth manifold, and we have $T_p(L_p \cap N) = T_pL_p \cap T_pN$. This implies that over $M_{\mathrm{reg}} \cap N$ the distribution $\mathcal{D}'(p) = T_pL_p \cap T_pN$ is integrable.

We now show that $\mathcal{D}'$ is integrable over N. Consider $p \in N \setminus (M_{\mathrm{reg}} \cap N)$, and take a sequence $\{p_n\} \subset M_{\mathrm{reg}} \cap N$, converging to p in N. Take $Y, Z \in \mathcal{D}'(p)$. By Theorem 1.3, we have to show that for t sufficiently small it holds

$$\left(\phi_t^Y\right)_*(Z(p)) \in \mathcal{D}'\left(\phi_t^Y(p)\right).$$

Observe that we have by continuity

$$\lim_{n\to\infty}\left(\phi_t^Y\right)_*(Z(p_n)) = \left(\phi_t^Y\right)_*(Z(p)),$$

and

$$\lim_{n\to\infty}\mathcal{D}'\left(\phi_t^Y(p_n)\right) = \mathcal{D}'\left(\phi_t^Y(p)\right).$$

Since for each n we have $\left(\phi_t^Y\right)_*(Z(p_n)) \in \mathcal{D}'\left(\phi_t^Y(p_n)\right)$, we conclude that

$$\left(\phi_t^Y\right)_*(Z(p)) \in \mathcal{D}'\left(\phi_t^Y(p)\right).$$

Therefore $\mathcal{D}'$ induces a smooth foliation $\mathcal{F}'$ on N, such that for $p \in M_{\text{reg}} \cap N$ the leaf of $\mathcal{F}'$ containing p is the connected component of $L_p \cap N$ containing p. We denote this leaf of $\mathcal{F}'$ by L'_p. □

Remark 4 Observe that for $p \in N$ fixed, and a curve $\alpha\colon I \to L'_p$ we have that $\alpha'(t) \in \mathcal{D}'(\alpha(t)) = T_{\alpha(t)}L_{\alpha(t)} \cap T_{\alpha(t)}N_{\alpha(t)}$. This implies that the curve α is a curve contained in $L_p \cap N$. Thus we conclude that $L'_p \subset L_p \cap N$.

Lemma 2.6 *Let $(M, \mathcal{F})$ be a closed singular Riemannian foliation on a complete manifold. Let $N \subset M$ be a pre-section of $(M, \mathcal{F})$. Then the partition $(N, \mathcal{F}')$ is a singular Riemannian foliation with respect to the induced metric of M on N.*

Proof By (2.5), $\mathcal{F}'$ is a partition of N by submanifolds. To prove that $\mathcal{F}'$ is a singular Riemannian foliation we show that conditions (i) and (ii) of Definition 1.1 hold with respect to the induced Riemannian metric on N‘ coming from M.

Condition (i) is given by Lemma 2.5.

We prove now condition (ii): that $\mathcal{F}'$ gives a transnormal system, i.e. for any geodesic $\gamma\colon I \to N$ with $\gamma'(0) \perp L_{\gamma(0)} \cap N$, we have $\gamma'(t) \perp L_{\gamma(t)} \cap N$ for all $t \in I$. Note that because N is totally geodesic, such a γ is also a geodesic of M.

We first show that for $p \in M_{\text{reg}} \cap N$, if γ emanates orthogonally to $L'_p = L_p \cap N$, then its intersections with the elements of the partition $\mathcal{F}'$ are orthogonal.

Let $p \in N$ be contained in a regular leaf of $\mathcal{F}$. Since $\nu_p(M, L_p) \subset T_pN$, it follows that N intersects L_p transversally. Thus, conclude that

$$T_pL'_p \oplus \nu_p(N, L'_p) = T_pN = T_pL'_p \oplus \nu_p(M, L_p).$$

From this it follows that $\nu_p(N, L'_p) = \nu_p(M, L_p)$. Thus any geodesic $\gamma\colon I \to N$ starting at p with γ perpendicular to L'_p is perpendicular to L_p. Thus γ is perpendicular to every leaf of $\mathcal{F}$ it intersects. Since the distribution $\mathcal{D}'(\gamma(t))$ is the tangent space to the leaves $L_{\gamma(t)}$, we have that for any $Y \in \mathcal{D}'(\gamma(t))$ that $g(Y(\gamma(t)), \gamma'(t)) = 0$. Thus we conclude that γ is perpendicular to every leaf of $\mathcal{F}'$ it intersects.

Now let p be an arbitrary point in N, and $\gamma\colon I \to N$ a geodesic starting at p with $\gamma'(0) \in \nu_p(N, L'_p)$. Take q a point on γ close enough to p. By Lemma 2.4, there exists $v \in T_qN \cap \nu_q(M, L_q)$ such that for $t > 0$ the points $\exp_q(tv)$ are contained in regular leaves of $\mathcal{F}$. From this it follows that taking $q_i = \exp_q((1/i)v)$ we have a sequence $\{q_i\}_{i\in\mathbb{N}} \subset M_{\text{reg}} \cap N$ converging to q in N. From each of these points, there is some geodesic γ_i in N emanating orthogonally from L'_{q_i} at q_i minimizing the distance between L'_{q_i} and L'_p, hence meeting both orthogonally. These γ_i converge to the geodesic $-\gamma(t) = \gamma(1-t)$, from which it follows that γ meets L'_q orthogonally, by continuity of the metric. This completes the proof of transnormality. □

Theorem 2.7 *Let $(M, \mathcal{F})$ be a closed singular Riemannian foliation on a complete manifold. Let $N \subset M$ be a pre-section. Then the inclusion $i\colon N \hookrightarrow M$ induces a discrete submetry $i^*\colon N/\mathcal{F}' \to M/\mathcal{F}$, given by $i^*(L'_p) = L_p$*

Proof Recall that the distance between L'_p and L'_q in $N/\mathcal{F}'$ is given by $d_N(L'_p, L'_q) = \inf\{d_N(x', y') \mid x' \in L'_p,\ y' \in L'_q\}$. On the other hand the distance between L_p and L_q in $M/\mathcal{F}$, is equal to $\inf\{d_M(x, y) \mid x \in L_p,\ y \in L_q\}$. From the fact that $L'_p \subset L_p \cap N$ and $L'_q \subset L_q \cap N$ we see that in general for $p, q \in N$, we have $d_M(L_p, L_q) \leqslant d_N(L'_p, L'_q)$. Thus, a ball of radius r centered at L'_p in $N/\mathcal{F}'$ gets mapped into the ball of radius r centered at L_p in $M/\mathcal{F}$.

To prove that $i^*\colon N/\mathcal{F}' \to M/\mathcal{F}$ is a submetry, we have to prove that for any $r > 0$ and any leaf L_p of $\mathcal{F}$, if L_q is such that $d_M(L_q, L_p) < r$, then $d_N(L'_q, L'_p) < r$. That is, the map from the ball of radius r around L'_p in $N/\mathcal{F}'$ to the ball of radius r around L_p in $M/\mathcal{F}$ is onto. It is enough to prove this for sufficiently small radius.

We will now prove that for $p \in M_{\text{reg}} \cap N$, the map $i^*\colon N/\mathcal{F}' \to M/\mathcal{F}$ is a local isometry for a sufficiently small ball around L_p. Fix $r > 0$ with r smaller than the injectivity radius of M at p. Take L_q such that $d_M(L_p, L_q) < r$, and let γ be the minimizing geodesic between L_p and L_q starting at p; i.e. $\ell(\gamma) = d_M(L_p, L_q)$ and $\gamma(0) = p$. Then γ is perpendicular to L_p at $p \in M$. Since N is a pre-section we have $\nu_p(M, L_p) \subset T_pN$, and since N is totally geodesic, we conclude that γ is a geodesic in N. Let $q' \in L'_q$ be a closest point to p in L'_q. By Lemma 2.2 we have that q' is the closest point to p in L_q. Thus we have

$$d_N(L'_q, L'_p) = d_N(q', p) = d_M(q', p) = d_M(L_q, L_p) < r.$$

Now consider the case when $p \in N$ does not belong to a regular leaf of $\mathcal{F}$, and take r smaller that the injectivity radius of M at p. Fix L_q of $\mathcal{F}$ such that $d_M(L_p, L_q) < r$. Let $\gamma\colon [0, 1] \to M$ be the minimizing geodesic of M joining L_p to L_q starting at p. Since M_{reg} is dense, for any $n > 2$ there exists $p_n \in M_{\text{reg}}$ such that for the middle point $\gamma(1/2)$ we have

$$d_M(p_n, \gamma(1/2)) < \frac{r}{n}.$$

By applying the triangle inequality and the fact that $d_M(\gamma(1/2), q) \leqslant r/2$ and $d_M(p, \gamma(1/2)) \leqslant r/2$ we conclude that:

$$d_M(p_n, q) \leqslant d_M(p_n, \gamma(1/2)) + d_M(\gamma(1/2), q) \leqslant r/n + r/2 < r;$$
$$d_M(p, p_n) \leqslant d_M(p, \gamma(1/2)) + d_M(\gamma(1/2), p_n) \leqslant r/n + r/2 < r.$$

Thus we get,

$$\begin{aligned} d_N(L'_p, L'_q) &\leqslant d_N(L'_p, L'_{p_n}) + d_N(L'_{p_n}, L'_q) \\ &= d_M(L_p, L_{p_n}) + d_M(L_{p_n}, L_q) \leqslant \frac{2r}{n} + r. \end{aligned}$$

Here we use that $p_n \in M_{\text{reg}}$, so by Lemma 2.2, we have that $d_N(L'_p, L'_{p_n}) = d_M(L_p, L_{p_n})$ and $d_N(L'_{p_n}, L'_q) = d_M(L_{p_n}, L_q)$. Now taking the limit as n goes to infinity, we conclude that

$$d_N(L'_p, L'_q) \leqslant r.$$

□

2.2 Reductions & restrictions

Lemma 2.8 (Lemma 5.10 in [8]) *For any $q \in N$, we have the following orthogonal decomposition*

$$T_q N = T_q L'_q \oplus \left(T_q N \cap \nu_q(M, L_q)\right).$$

Proof We only have to prove that $\nu_q(N, L'_q) = T_q N \cap \nu_q(M, L_q)$. First consider $w \in T_q N \cap \nu_q(M, L_q) \subset \nu_q(M, L_q)$. Since $T_q L'_q = T_q N \cap T_q L_q \subset T_q L_q$, then for any $v \in T_q L'_q$ it holds $g_q(v, w) = 0$. That is, $T_q N \cap \nu_q(M, L_q) \subset \nu_q(N, L'_q)$.

Now we prove the other inclusion. Consider $v \in \nu_q(N, L'_q)$ such that $p = \exp_q(v) \in M_{\text{reg}} \cap N$. Observe that the geodesic $\gamma(t) = \exp_q(tv)$ is by construction perpendicular to all leaves of $\mathcal{F}'$ it intersects, since $(N, \mathcal{F}')$ is a singular Riemannian foliation by Lemma 2.6. Since p is contained in a regular leaf, then $\gamma'(1)$ is actually orthogonal to L_p: the point $q = \gamma(0)$ is the closest point in $L_q \cap N$ to p, so by Lemma 2.2, it follows that q is the closest point to p in L_q. This implies that γ is orthogonal to L_p, and thus orthogonal to L_q. Therefore $v \in \nu_q(M, L_q) \cap T_q N$.

Now there exists $w \in \nu_q(N, L'_q)$ such that $\exp_q(w) \in M_{\text{reg}} \cap N$. Moreover since $M_{\text{reg}} \cap N$ is open, and $\exp_q(\nu_q(N, L'_q))$ is the slice at q in N, we conclude that there exists an open set $U \subset \nu_q(N, L'_q)$ containing w, such that for any $v \in U$, we have $\exp_q(v) \in M_{\text{reg}} \cap N$. Since U contains a basis $\{v_i\}$ of $\nu_q(N, L'_q)$, and for each index i we have by the previous paragraph that $v_i \in \nu_q(M, L_q) \cap T_q N$. This implies that $\nu_q(N, L'_q) = \nu_q(M, L_q) \cap T_q N$. □

Theorem 2.9 *(Infinitesimal Reduction) Let $(M, \mathcal{F})$ be a closed singular Riemannian foliation on a complete manifold containing a pre-section $N \subset M$. For any $q \in N$, the space $V_q = \nu_q(M, L_q) \cap T_q N$ is a pre-section for the infinitesimal foliation $(\nu_q(M, L_q), \mathcal{F}_q)$.*

Proof Observe that V_q is a linear subspace of $\nu_q(M, L_q)$, so it is totally geodesic and complete. Hence, it satisfies condition (A) of Definition 2.1.

We now prove that V_q satisfies condition (C) of Definition 2.1. Fix $v \in V_q$ such that $\mathcal{L}_v$ is a regular leaf of the infinitesimal foliation $\mathcal{F}_q$, and observe that, as in [14], the property (C) is equivalent to $\nu_v(\nu_q(M, L_q), V_q) \subset T_v \mathcal{L}_v$. Since the infinitesimal foliation is invariant under homotheties, we may assume that v is small enough, so that $p = \exp_q(v)$ is contained in a tubular neighborhood given by the Slice Theorem in [15].

Recall that $(\nu_q(M, L_q), \mathcal{F}_q)$ is a singular Riemannian foliation with respect to the Euclidean metric (see [19][Sect. 1.2]); thus we fix this metric on $\nu_q(M, L_q)$. We observe that since $\nu_q(M, L_q)$ is a linear space, we can identify $T_v \nu_q(M, L_q)$ with $\nu_q(M, L_q)$. Since V_q is a linear subspace of $\nu_q(M, L_q) \subset T_q M$, under this identification we identify $T_v V_q$ with V_q. Moreover, for the Euclidean metric we have the following splittings: $T_v \nu_q(M, L_q) = T_v V_q \oplus \nu_v(\nu_q(M, L_q), V_q)$ and $\nu_q(M, L_q) = V_q \oplus V_q^{\perp}$. Thus, we can identify $\nu_v(\nu_q(M, L_q), V_q)$ with $V_q^{\perp}$.

By Lemma 2.8 we have that $T_q N = T_q L'_q \oplus V_q$, and by construction $T_q L'_q \subset T_q L_q$. This implies that $V_q^{\perp} \subset \nu_q(M, N)$.

We consider $w \in \nu_v(\nu_q(M, L_q), V_q)$ arbitrary. Let $w' \in \nu_q(M, L_q)$ be the vector corresponding to w under the identification of $T_v(\nu_q(M, L_q))$ with $\nu_q(M, L_q)$. Then by the previous paragraphs we have, $w' \in V_q^{\perp} \subset \nu_q(M, N)$. Observe that since $v \in V_q \subset T_q N$ and N is totally geodesic, the geodesic $\alpha\colon I \to M$ given by $\alpha(s) = \exp_q(sv)$ is really a geodesic in N. Let $J(s)$ be the Jacobi field along $\alpha(s)$ determined by $J(0) = 0$, and $J'(0) = w'$. By Lemma 1.7 we have that for all $s \in I$, $J(s) \in \nu_{\alpha(s)}(M, N)$. For $p = \alpha(1)$, we have by a result of Lang [10][Chapter IX, Thm 3.1]:

$$D_v(\exp_q)(w) = J(1) \in \nu_p(M, N).$$

Since $v \in V_q$ is such that $\mathcal{L}_v$ is a regular leaf of $\mathcal{F}_q$, we have by Lemma 1.13 that L_p is a regular leaf of $\mathcal{F}$. By property (C), we have that $\nu_p(M, N) \subset T_p L_p$. Thus $D_v(\exp_q)(w) \in T_p L_p$. Moreover the exponential map of M at q induces a local diffeomorphism $\exp_q : \mathcal{L}_v \to L_p$. By considering the derivative at v, we have an isomorphism $D_v(\exp_q)\colon T_v\mathcal{L}_v \to T_p L_p$. Thus we conclude that $w \in T_v\mathcal{L}_v$ as desired.

It remains to prove condition (B) of Definition 2.1. Observe that by Lemma 2.4 there exists $v \in (\nu_q(M, L_q), \mathcal{F}_q)$ contained in a regular leaf . And by Lemma 2.3 we have that the image under the exponential map, $\exp_v$, of the normal space $\nu_v(\nu_q(M, L_q), \mathcal{L}_v))$ to $\mathcal{L}_v$ intersects every leaf of $\mathcal{F}_q$. Recall that we are considering a fixed Euclidean metric on $\nu_q(M, L_q)$, so in particular $\exp_v$ is a global diffeomorphism. Since V_q satisfies condition (C), i.e. $\nu_v(\nu_q(M, L_q), \mathcal{L}_v)) = T_v V_q$, we conclude that V_q intersects all leaves of $\mathcal{F}_q$. □

Remark 5 Since V_q is a subspace of $\nu_q(M, L_q)$, we have $\dim(V_q) \leqslant \dim(\nu_q(M, L_q))$. This obvious statement provides a comparison of the relative codimensions of leaves in a pre-section with that of the associated “ambient” leaves. Namely, for a given leaf $L_q \cap N$ of the pre-section foliation $\mathcal{F}'$, we have that $\mathrm{codim}(N, L_q \cap N) \leq \mathrm{codim}(M, L_q)$. Observe that for any $q \in M_{\mathrm{reg}} \cap N$, we have $V_q = \nu_q(M, L_q)$ since by definition $\nu_q(M, L_q) \subset T_q N$. Thus we have $\dim(V_q) = \dim(\nu_q(M, L_q))$.

The technique employed to prove the Main Theorem will involve “chasing the codimension drop” and inductively reducing via pre-sections. For our argument, we will need to focus on the infinitesimal foliation normal to such a leaf’s *stratum* - the component of leaves of the same dimension containing that leaf. In the infinitesimal foliation, nearby leaves of the same dimension appear as point leaves and we have the splitting

$$(V, \mathcal{F}) = (V_0 \times V_0^{\perp}, \{pts\} \times \mathcal{F}_{>0})$$

where V_0 is the linear subspace of V foliated by points, which corresponds to the tangent space to the stratum of the central leaf. The normal space to this stratum, $V_0^{\perp}$, is the orthogonal complement of V_0 with respect to the Euclidean metric, and $\mathcal{F}_{>0}$ is a foliation whose only point leaf is the origin. If $(W, \mathcal{F}')$ is a pre-section of $(V, \mathcal{F})$, then since W intersects all leaves of $\mathcal{F}$, we must have that $V_0 \subset W$. With this, we have the splitting

$$(W, \mathcal{F}') = \left(V_0 \times (V_0^{\perp} \cap W), \{pts\} \times (\mathcal{F}_{>0})'\right)$$

where $(\mathcal{F}_{>0})'$ is the partition of $V_0^{\perp} \cap W$ by its intersection with the leaves $\mathcal{F}$.

Lemma 2.10 *With the notation above, if $(W, \mathcal{F}')$ is a pre-section of an infinitesimal foliation $(V, \mathcal{F})$, then $\left((V_0^{\perp} \cap W), (\mathcal{F}_{>0})'\right)$ is a pre-section of $(V_0^{\perp}, \mathcal{F}_{>0})$.*

Proof Since $V_0^{\perp} \cap W$ is a linear subspace of $V_0^{\perp}$, it is totally geodesic and we have condition A. Moreover, a leaf of $\mathcal{F}$ is of the form $\{a\} \times L$, where $a \in V_0$ is a point leaf and $L \in \mathcal{F}_{>0}$. Since $(W, \mathcal{F}')$ is a pre-section of $(V, \mathcal{F})$, we have

$$W \cap (\{a\} \times L) \neq \emptyset$$

$$\implies \Big(V_0 \times (V_0^\perp \cap W)\Big) \cap (\{a\} \times L) \neq \emptyset$$
$$\implies (V_0 \cap \{a\}) \times \Big((V_0^\perp \cap W) \cap L\Big) \neq \emptyset$$
$$\implies \{a\} \times ((V_0^\perp \cap W) \cap L) \neq \emptyset$$

Hence $V_0^\perp \cap W$ intersects every leaf of $\mathcal{F}_{>0}$ and we have condition B. Now let $L \in \mathcal{F}_{>0}$ be a regular leaf through $p \in L \cap (V_0^\perp \cap W)$. Again, since $(W, \mathcal{F}')$ is a pre-section of $(V, \mathcal{F})$, we have $\nu_{(a,p)}(V, \{a\} \times L) \subset T_{(a,p)}(W)$. Given the splittings above, we have

$$\nu_{(a,p)}\Big(V_0 \times V_0^\perp, \{a\} \times L\Big) \subset T_{(a,p)}\Big(V_0 \times (V_0^\perp \cap W)\Big)$$
$$\implies \nu_a(V_0, a) \times \nu_p(V_0^\perp, L) \subset T_a(V_0) \times T_p(V_0^\perp \cap W)$$

and since $\nu_a(V_0, a) = T_a(V_0)$, it follows that $\nu_p(V_0^\perp, L) \subset T_p(V_0^\perp \cap W)$, so condition (C) is satisfied. □

Since the leaves of an infinitesimal foliation are contained in distance spheres centered at the origin, we get the following lemma from the previous one.

Lemma 2.11 *Let $(V, \mathcal{F})$ be an infinitesimal foliation and $(\mathbf{S}_V, \mathcal{F}|_{\mathbf{S}_V})$ denote the singular Riemannian foliation given by its restriction to the round unit sphere in V. If $(W, \mathcal{F}')$ is a pre-section of $(V, \mathcal{F})$, then $(\mathbf{S}_W, \mathcal{F}'|_{\mathbf{S}_W})$ is a pre-section of $(\mathbf{S}_V, \mathcal{F}|_{\mathbf{S}_V})$.*

Proof Since W is totally geodesic, and V is a Euclidean space, we conclude that W is a linear subspace of V. Thus the unit sphere $\mathbf{S}_W$ is a totally geodesic submanifold of $\mathbf{S}_V$, so we need only show that properties (B) and (C) are satisfied by $(\mathbf{S}_W, \mathcal{F}'|_{\mathbf{S}_W})$. Since V is an infinitesimal foliation, the leaves of $\mathcal{F}$ are contained in distance spheres about the origin, and since $(W, \mathcal{F}')$ is a pre-section of $(V, \mathcal{F})$, it follows that each distance sphere about the origin in W intersects every leaf of the distance sphere of the same radius about the origin in V. Thus, property (B) is satisfied.
For property (C), first note that because leaves of $(V, \mathcal{F})$ are contained in distance spheres about the origin, a regular leaf of $(\mathbf{S}_V, \mathcal{F}|_{\mathbf{S}_V})$ is exactly a regular leaf of $(V, \mathcal{F})$ (i.e. $L_p \cap \mathbf{S}_V = L_p$). So let L_p be such a leaf with $p \in W$. We wish to show that $\nu_p(\mathbf{S}_V, L_p \cap \mathbf{S}_V) \subset T_p(\mathbf{S}_W)$. Now

$$\begin{aligned}\nu_p(\mathbf{S}_V, L_p \cap \mathbf{S}_V) &= \nu_p(\mathbf{S}_V, L_p)\\ &= \nu_p(V, L_p) \cap T_p(\mathbf{S}_V)\\ &\subset T_p(W) \cap T_p(\mathbf{S}_V)\\ &= T_p(W \cap \mathbf{S}_V)\\ &= T_p(\mathbf{S}_W)\end{aligned}$$

where the third line uses that $\nu_p(V, L_p) \subset T_p(W)$ since $(W, \mathcal{F}')$ is a pre-section of $(V, \mathcal{F})$. The fourth line follows from the fact that W and $\mathbf{S}_V$ intersect transversally in V. □

3 An application in positive curvature

We conclude this note with the proof of A. The terminology and theorem below from Wilking [23] is central to our proof.

Let $(M, \mathcal{F})$ be a singular Riemannian foliation. A piecewise smooth curve c is called *horizontal* with respect to the foliation $\mathcal{F}$, if $c'(t)$ is in the normal space $\nu_{c(t)}(L_{c(t)})$ of the leaf $L_{c(t)}$ at $c(t)$. The *dual foliation* $(M, \mathcal{F}^{\#})$ of $\mathcal{F}$ is given by defining for a point $p \in M$ the leaf as

$$L_p^{\#} = \{q \in M \mid \text{there is a piecewise smooth horizontal curve from } p \text{ to } q\}.$$

Theorem 3.1 (Theorem 1 in [23]) *Suppose that M is a complete positively curved manifold with a singular Riemannian foliation $\mathcal{F}$. Then the dual foliation has only one leaf, M.*

From this result, we can prove the following Lemma, on which our proof of Theorem A hinges. The Lemma says that in positive curvature, the presence of a non-trivial pre-section $N \subset M$ guarantees the existence of the leaf whose "relative codimension" drops.

Lemma 3.2 *Let $(M, \mathcal{F})$ be a closed singular Riemannian foliation on a complete manifold with positive sectional curvature. Let $N \subset M$ be a non-trivial pre-section of $\mathcal{F}$, and for $p \in N$ set $V_p = T_pN \cap \nu_p(M, L_p)$. Then there exists $q \in N$ such that* $\dim(V_q) < \dim(\nu_q(M, L_q))$. *That is,* $\operatorname{codim}(N, L_q \cap N) < \operatorname{codim}(M, L_q)$.

Proof Assume that $\dim(V_q) = \dim(\nu_q(M, L_q))$ for all $q \in N$. Since $V_q = \nu_q(M, L_q) \cap T_qN$, it follows that $\nu_q(M, L_q) \subset T_qN$. Because N is totally geodesic, it follows that all horizontal geodesics from q belong to N. Thus, the dual leaf $L_q^{\#}$ (see [23]) is contained in N. Since M is positively curved, it follows from Theorem 3.1 in [23], that the dual leaf of $\mathcal{F}$ through p is equal to M. This implies that $N = M$, which is a contradiction. Thus $\dim(V_q) < \dim(\nu_q(M, L_q))$ for some $q \in N$. □

With Theorem 2.9 and Lemma 3.2, we have the necessary ingredients to prove the main theorem:

Proof of Theorem A Let $(N, \mathcal{F}')$ be a non-trivial pre-section of $(M, \mathcal{F})$. We are assuming M is positively curved, so by Lemma 3.2, there exists a point $q \in N$ (necessarily belonging to a singular leaf of $\mathcal{F}$) such that $\operatorname{codim}(N, L_q \cap N) < \operatorname{codim}(M, L_q)$. From Theorem 2.9, we have that $V_q = \nu_q(M, L_q) \cap T_qN$ is a pre-section for the infinitesimal foliation $(\nu_q(M, L_p), \mathcal{F}_q)$. By using Lemma 2.10, we will focus on the foliation normal to the stratum of L_q and its pre-section. By Lemma 2.11, we restrict this (normal to the stratum of L_q) infinitesimal foliation and its pre-section to their respective unit spheres and refer to them as $(M_1, \mathcal{F}_1)$ and $(N_1, \mathcal{F}_1')$. Observe that N_1 is a proper submanifold of M_1. In particular, we have $\dim(M_1) < \dim(M)$ and M_1 is positively curved (it is a round sphere) and $\mathcal{F}_1$ contains no point leaves. With this, we point out that N_1 is a trivial pre-section of $(M_1, \mathcal{F}_1)$ only when N_1 is a point.

For as long as the hypothesis of Lemma 3.2 are satisfied, N_1 is not a point, so we can repeat this process at a point $q_2 \in N_1$ where the relative codimension drops as in Lemma 3.2 to form $(M_2, \mathcal{F}_2)$ with pre-section $(N_2, \mathcal{F}_2')$. Moreover by construction we have $\dim(N_2) < \dim(M_2)$, and

$$\dim(M_2) < \dim(M_1) < \dim(M).$$

Next we observe that when N_i is a trivial pre-section of $(M_i, \mathcal{F}_i)$, since $\dim(N_i) < \dim(M_i)$ by construction, then N_i is a point. From the requirement that N_i intersects all leaves of $\mathcal{F}_i$ we conclude that $\mathcal{F}_i$ consists of only one leaf, i.e. $\mathcal{F}_i = \{M_i\}$. Second we point that since M_i is a sphere, M_i fails to have positive curvature only when $\dim(M_i) = 1$. So the process of applying Lemma 3.2 ends only when $(M_i, \mathcal{F}_i)$ is a single leaf foliation, or when M_i is $\mathbf{S}^1$.

Now we point out that if we encounter $M_i = \mathbf{S}^1$, then since $\mathcal{F}_i$ cannot contain point leaves, it must be that this is a single leaf foliation. Thus, this process necessarily ends with a single leaf foliation $(M_i, \mathcal{F}_i)$. In this case, let $L_{q_i} \in \mathcal{F}_{i-1}$ be the chosen leaf of M_{i-1} whose relative codimension dropped. The fact that $(M_i, \mathcal{F}_i)$ is a single leaf foliation means precisely that the space of directions normal to the stratum of $L_{q_i} \subset M_{i-1}$ is a single point. This implies that this stratum forms a boundary face (see [18][p. 5]) of the Alexandrov leaf space $M_{i-1}/\mathcal{F}_{i-1}$. By the inductive definition of boundary for Alexandrov spaces, this implies that $\partial(M_{i-2}/\mathcal{F}_{i-2}) \neq \emptyset$, which implies that $\partial(M_{i-3}/\mathcal{F}_{i-3}) \neq \emptyset$, and ultimately, that $\partial(M/F) \neq \emptyset$

□

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