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A variational characterization of contact metric structures

Tomasz Zawadzki[1]

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Abstract
On a manifold with a given nowhere vanishing vector field, we examine the squared L^2-norm of the integrability tensor of the orthogonal complement of the field, as a functional on the space of Riemannian metrics of fixed volume. We compute the first variation of this action and prove that its only critical points locally are metrics with integrable orthogonal complement of the field, or metrics of contact metric structures rescaled by a function. Moreover, in dimensions other than 5, that function is constant and the above characterization is global. We examine the second variation of the functional at the critical points and estimate it for some geometrically meaningful sets of variations.

Keywords Distribution · Variation · Contact structure · Riemannian metric

Mathematics Subject Classification 53C12 · 53C25

1 Introduction

Variations of functionals of Riemannian metric remain a source of many geometric problems and results. Classical examples of functionals, such as the Einstein–Hilbert action, are important not only because of their direct applications, but also due to interesting properties of their critical points, which can be viewed as natural choices for a Riemannian metric on a manifold [2]. Indeed, given a functional that depends on a well-understood geometric object on the manifold, one can argue that its critical points are best fitting to the particular geometric setting. An example of such setting is a manifold equipped with a distribution, i.e., a smooth field of tangent planes of constant dimension.

Distributions are encountered in various problems, e.g., as kernels of differential forms or tangent spaces of foliations, but their geometric nature is easily envisioned and of independent interest. The simplest example of a distribution is the one tangent to a given, nowhere vanishing vector field ξ on a manifold. The orthogonal complement of ξ, a codimension-one distribution that will be denoted by $\mathcal{D}$, is uniquely determined by a Riemannian metric on the manifold. Its various geometric properties depend on the choice of the metric and allow the

✉ Tomasz Zawadzki
tomasz.zawadzki@wmii.uni.lodz.pl

1 Faculty of Mathematics and Computer Science, University of Lodz, Banacha 22, 90-238 Lodz, Poland

formulation of several variational problems, e.g., about the energy or volume of the vector field ξ [5, 6], or the tension field of the submersion locally determined by ξ [1].

The aim of this paper is to examine a particularly simple functional of a Riemannian metric: the square of L^2-norm of the integrability tensor of $\mathcal{D}$. Aside from its own independent meaning, this action appears as one of the terms of other functionals, e.g., the total mixed scalar curvature of $\mathcal{D}$ [13], examined in [10], or the norm of the differential of the 1-form dual to ξ.

The fixed setting of a manifold equipped with a vector field influences the choice of metrics we should consider, therefore analogously as in [10], so-called $g^{\perp}$-variations will be used to investigate the functional. These are partial variations of Riemannian metric that preserve the length of the vector field ξ, which always remains unit. Thus, values of the action that we obtain will depend only on the position of $\mathcal{D}$ relative to ξ and the metric on $\mathcal{D}$. As our main interest is comparing metrics of various orthogonal complements and not merely rescaling one of them on a fixed distribution, we shall furthermore consider metrics of the same total volume of the manifold. This restriction can be easily generalized also to the case of a non-compact manifold of infinite volume, where we vary the metric only on its relatively compact subset, preserving its finite volume. Similar approach is usually taken in obtaining the Einstein equations from the Einstein–Hilbert action [2].

In this setting, we prove that the critical points of the squared L^2-norm of the integrability tensor of $\mathcal{D}$ can be classified to a large extent, as either metrics with integrable $\mathcal{D}$, or metrics arising from contact metric structures [3] by a simple rescaling. Moreover, on connected manifolds these two kinds of solutions can be combined together, but only in dimension 5. This result can be viewed as a variational characterization of contact metric structures, relating them to critical points of a rather simple functional of Riemannian metric—in particular, describing contact metric structures as defining "critically non-integrable" distributions in a rigorous sense. On the other hand, it may also help to solve the Euler–Lagrange equations for other distribution-related functionals [10], which contain similar terms.

We obtain general formula for the second variation of the action and prove that contact metric structures on 3- and 5-dimensional manifolds define critical points, where the second variation is nonnegative when restricted to variations that keep the orthogonal complement $\mathcal{D}$ of ξ fixed, i.e., variations of metric within a given almost-product structure [8]. This is no longer true in higher dimensions. We also consider a complementary case: variations initially vanishing on $\mathcal{D} \times \mathcal{D}$, and prove that for metrics of K-contact structures such second variation of the functional can be described by a rather simple formula. Moreover, it is nonnegative for certain variations of metric, related to the Riemannian foliation by flowlines of the Reeb field.

This paper has introduction and three sections: the first section contains necessary definitions, mostly following [10] in concepts as well as notation. In the second part, we characterize the critical points of the squared L^2-norm of the integrability tensor of $\mathcal{D}$, by stating and solving the Euler–Lagrange equations of this action. The third part of the paper describes the second variation of the considered functional at its critical points.

2 Preliminaries

Let (M, ξ) be a smooth, connected, oriented manifold of dimension $\dim M = m > 2$, with a nowhere vanishing vector field ξ. We admit non-compact M, so the Euler characteristic does not restrict the dimension m.

In what follows, we shall use some notation and terminology established in [10]. For a Riemannian metric g on (M,ξ), we denote by $\widetilde{\mathcal{D}}$ the one-dimensional distribution (i.e., subbundle of TM) spanned by ξ, and by $\mathcal{D}$ its g-orthogonal complement. Thus, $(M,\widetilde{\mathcal{D}},\mathcal{D},g)$ is an *almost product structure* [8]. We also denote by V the following subset of $TM\times TM$: $\mathrm{V}=(\mathcal{D}\times\widetilde{\mathcal{D}})\cup(\widetilde{\mathcal{D}}\times\mathcal{D})$. Let $\mathfrak{X}_M$ be the module over $C^\infty(M)$ of all vector fields on M, by $\mathfrak{X}_{\mathcal{D}}$ and $\mathfrak{X}_{\widetilde{\mathcal{D}}}$ we denote modules of sections of $\mathcal{D}$ and $\widetilde{\mathcal{D}}$, respectively. We shall consider only Riemannian metrics g that make ξ a unit vector field on M.

Let $x\in M$, restriction of $\mathcal{D}$ to x is a subspace of T_xM of dimension $p=m-1$, which will be denoted by $\mathcal{D}_x$. We denote orthogonal projections onto $\widetilde{\mathcal{D}}$ and $\mathcal{D}$ by $^\top$ and $^\perp$, respectively. We define for all $X,Y\in\mathfrak{X}_M$

$$g^\top(X,Y)=g(X^\top,Y^\top),\ \ g^\perp(X,Y)=g(X^\perp,Y^\perp).$$

In what follows, ∇ denotes the covariant derivative with respect to the Levi-Civita connection on (M,g) and $\{e_i\}, i\in\{1,\dots,p\}$ is a local orthonormal frame of $\mathcal{D}$. Let $\tilde{T},\tilde{h}:\mathfrak{X}_{\mathcal{D}}\times\mathfrak{X}_{\mathcal{D}}\to\mathfrak{X}_{\widetilde{\mathcal{D}}}$ be the integrability tensor and the second fundamental form of $\mathcal{D}$, respectively, given by formulas

$$\tilde{T}(X,Y)=(1/2)\,[X,\,Y]^\top,\quad \tilde{h}(X,Y)=(1/2)\,(\nabla_XY+\nabla_YX)^\top,\quad (X,Y\in\mathfrak{X}_{\mathcal{D}}).$$

Recall that $\tilde{H}=\sum_i\tilde{h}(e_i,e_i)$ and $H=\nabla_\xi\xi$ are mean curvature vector fields of $\mathcal{D}$ and $\widetilde{\mathcal{D}}$. Since ξ is a unit vector field, $H\in\mathfrak{X}_{\mathcal{D}}$ is the curvature of the integral curves of ξ. As $\widetilde{\mathcal{D}}$ is one-dimensional, the shape operator A_Z of $\widetilde{\mathcal{D}}$ with respect to $Z\in\mathcal{D}$ is given by:

$$A_Z\xi=g(H,Z)\xi.$$

To describe the extrinsic geometry of $\mathcal{D}$, we shall use tensor fields $\tilde{A}_\xi$ and $\tilde{T}^\sharp_\xi$ defined by the following formulas:

$$g(\tilde{A}_\xi X,Y)=g(\tilde{h}(X,Y),\xi),\quad g(\tilde{T}^\sharp_\xi X,Y)=g(\tilde{T}(X,Y),\xi),\quad (X,Y\in\mathfrak{X}_{\mathcal{D}}).$$

We note that $\tilde{T}^\sharp_\xi$ is antisymmetric, i.e., $g(\tilde{T}^\sharp_\xi X,Y)=-g(\tilde{T}^\sharp_\xi Y,X)$. We shall also use the symmetric $(0,2)$-tensor $\widetilde{\mathcal{T}}^\flat$, defined by the formula

$$\widetilde{\mathcal{T}}^\flat(X,Y)=g((\tilde{T}^\sharp_\xi)^2X,Y),\quad (X,Y\in\mathfrak{X}_{\mathcal{D}}).$$

For a $(1,2)$-tensor field P, we define a $(0,2)$-tensor field $\mathrm{div}P$ by

$$(\mathrm{div}P)(X,Y)=(\mathrm{div}^\top P)(X,Y)+(\mathrm{div}^\perp P)(X,Y)$$

where

$$(\mathrm{div}^\top P)(X,Y)=g((\nabla_\xi\,P)(X,Y),\xi)$$

and

$$(\mathrm{div}^\perp P)(X,Y)=\sum_{i=1}^p g((\nabla_{e_i}\,P)(X,Y),e_i)$$

for all $X,Y\in\mathfrak{X}_M$.

Let $Z\in\mathfrak{X}_M$. We define the following $(1,2)$-tensor fields:

$$\alpha(X,Y)=\frac{1}{2}\,(A_{X^\perp}(Y^\top)+A_{Y^\perp}(X^\top)),\ \tilde{\alpha}(X,Y)=\frac{1}{2}\big(\tilde{A}_{X^\top}(Y^\perp)+\tilde{A}_{Y^\top}(X^\perp)\big),$$
$$\theta(X,Y)=\frac{1}{2}\big(T^\sharp_{X^\perp}(Y^\top)+T^\sharp_{Y^\perp}(X^\top)\big),\ \tilde{\theta}(X,Y)=\frac{1}{2}\big(\tilde{T}^\sharp_{X^\top}(Y^\perp)+\tilde{T}^\sharp_{Y^\top}(X^\perp)\big),$$

$$\tilde{\delta}_Z(X,Y)=\frac{1}{2}\big(g(\nabla_{X^\top}Z,\,Y^\perp)+g(\nabla_{Y^\top}Z,\,X^\perp)\big),$$

for all $X, Y \in \mathfrak{X}_M$. Above, $T^\sharp$ is defined analogously as $\tilde{T}^\sharp$, and in our case vanishes, as $\widetilde{\mathcal{D}}$ is integrable.

For any (1, 2)-tensors P, Q at $x \in M$, we define a (0, 2)-tensor $\Lambda_{P,Q}$ by

$$\Lambda_{P,Q}(X,Y)=\sum_{\nu,\mu}\Big(g(X,P(E_\nu,E_\mu))g(Y,Q(E_\nu,E_\mu))\\ +g(X,Q(E_\nu,E_\mu))g(Y,P(E_\nu,E_\mu))\Big),$$

for all $X, Y \in T_xM$, where $\{E_\nu\}$ is an orthonormal basis of T_xM. We have

$$\Lambda_{P,Q}=\Lambda_{Q,P}\quad\text{and}\quad \Lambda_{P,Q_1+Q_2}=\Lambda_{P,Q_1}+\Lambda_{P,Q_2}$$

for all (1, 2)-tensors P, Q, Q_1, Q_2. Let $\langle\cdot,\cdot\rangle$ denote the inner product of tensors induced by g, i.e., for (0, 2)-tensors S, W and (1, 2)-tensors P, Q at x, we have

$$\langle S,W\rangle=\sum_{\mu,\nu}S(E_\mu,E_\nu)W(E_\mu,E_\nu),$$
$$\langle P,Q\rangle=\sum_{\mu,\nu}g(P(E_\mu,E_\nu),\,Q(E_\mu,E_\nu)),$$
$$\langle S,P\rangle=\sum_{\mu,\nu}S(E_\mu,E_\nu)P(E_\mu,E_\nu),$$

for any orthonormal basis $\{E_\mu\}$ of T_xM. We also use notation $\|P\| = \sqrt{\langle P, P\rangle}$. For a (1, 2)-tensor P and a vector $Z \in T_xM$, we define the (0, 2)-tensor $\langle P, Z\rangle$ by the formula

$$\langle P,Z\rangle(X,Y)=g(P(X,Y),Z)$$

for all $X, Y \in T_xM$. These pointwise definitions extend in the natural way to vector and tensor fields.

Let $\mathrm{Riem}(M)$ be the set of all Riemannian metrics on M, and let $\mathrm{Riem}(M,\xi)$ be the set of Riemannian metrics with respect to which ξ is a unit vector field. For a codimension-one distribution $\mathcal{D}$ on M, let $\mathrm{Riem}(M,\xi,\mathcal{D}) \subset \mathrm{Riem}(M,\xi)$ be the set of Riemannian metrics for which ξ is orthogonal to $\mathcal{D}$ and ξ is unit.

On the manifold (M, ξ), for a relatively compact open set $\Omega \subset M$, we consider the functional

$$J_\Omega : g \mapsto \int_\Omega \|\tilde{T}\|^2 \,\mathrm{vol}_g, \tag{1}$$

defined on $\mathrm{Riem}(M,\xi)$, where vol_g denotes the volume form of g. If M is compact, we shall consider $\Omega = M$. For a given codimension-one distribution $\mathcal{D}$ transverse to ξ, we denote by $J_{\Omega,\mathcal{D}}$ the functional defined by the formula (1), but considered only on the set $\mathrm{Riem}(M,\xi,\mathcal{D})$.

As in [10], a family of metrics $\{g_t \in \mathrm{Riem}(M,\xi) : |t| < \epsilon\}$ smoothly depending on the parameter t and such that $g_0 = g$ will be called a $g^\perp$-*variation* of the metric g. In other words, for $g^\perp$-variations the norm of ξ is preserved, but the orthogonal complement of ξ and the Riemannian metric on it may vary. A variation $\{g_t \in \mathrm{Riem}(M,\xi,\mathcal{D}) : |t| < \epsilon\}$, where $\mathcal{D}$ is the g_0-orthogonal complement of the distribution spanned by ξ, will be called *adapted* variation. Finally, we say that variation $\{g_t \in \mathrm{Riem}(M) : |t| < \epsilon\}$ of g is *volume-preserving* if for all $|t| < \epsilon$ we have $\int_\Omega \mathrm{vol}_{g_t} = \int_\Omega \mathrm{vol}_g$. We shall only consider volume-preserving variations and use the notation $\partial_t \equiv \frac{\partial}{\partial t}$ and $\partial^2_{tt} \equiv \frac{\partial^2}{\partial t^2}$ for differentiating with respect to the parameter of variation.

We say that a metric g is critical for the functional (1) with respect to $g^\perp$-variations if for every $g^\perp$-variation g_t of g we have

$$\partial_t J_\Omega(g_t)|_{t=0} = 0.$$

For a $g^\perp$-variation g_t of g, let $\mathcal{D}(t)$ be the g_t-orthogonal complement of the distribution $\widetilde{\mathcal{D}}$ spanned by ξ and let $\mathrm{V}(t) = (\mathcal{D}(t) \times \widetilde{\mathcal{D}}) \cup (\widetilde{\mathcal{D}} \times \mathcal{D}(t))$. Let $^\top$ and $^\perp$ denote the g_t-orthogonal projections onto $\widetilde{\mathcal{D}}$ and $\mathcal{D}(t)$, respectively. Let $B_t = \partial_t g_t$ and let $B_t^\sharp$ be a symmetric $(1,1)$-tensor field defined for all $x \in M$ by the formula: $g_t(B_t^\sharp X, Y) = B_t(X, Y)$ for all $X, Y \in T_x M$. We have the following lemma [10].

Lemma 1 *Let g_t be a $g^\perp$-variation of g with $B_t = \partial_t g_t$. Let $\{\xi, e_1, \dots e_p\}$ be a local g-orthonormal frame, and let $\{\xi, e_1(t), \dots e_p(t)\}$ be a t-dependent frame such that for all $i \in \{1, \dots, p\}$*

$$e_i(0) = e_i, \qquad \partial_t e_i(t) = -(1/2)\,(B_t^\sharp(e_i(t)))^\perp - (B_t^\sharp(e_i(t)))^\top. \tag{2}$$

Then for all t, $\{\xi, e_1(t), \dots, e_p(t)\}$ is a local g_t-orthonormal frame, i.e., $\{e_i(t)\}_{i=1}^p$ is a local g_t-orthonormal frame of $\mathcal{D}(t)$.

Similarly, we can describe evolution of $\widetilde{\mathcal{D}}$- and $\mathcal{D}(t)$-components of any vector $X \in TM$ [10].

Lemma 2 *Let g_t be a $g^\perp$-variation of g. Then for any t-dependent vector X_t on M, we have*

$$\partial_t(X_t^\top) = (\partial_t X_t)^\top + (B_t^\sharp(X_t^\perp))^\top, \qquad \partial_t(X_t^\perp) = (\partial_t X_t)^\perp - (B_t^\sharp(X_t^\perp))^\top.$$

To make equations easier to read, in further formulas we shall not explicitly indicate the dependence on t of all tensors, but we shall write g_t, ∇^t, $e_i(t)$ and B_t to emphasize that a formula holds for all values of the parameter t of the variation. Recall that for $t = 0$ we have $g_0 = g$, $e_i(0) = e_i$, $\nabla^0 = \nabla$; we shall also write B instead of B_0.

From the Koszul formula for the Levi-Civita connection ∇^t of g_t ($|t| < \epsilon$), it follows that [12]

$$2\, g_t(\partial_t(\nabla^t_X Y), Z) = (\nabla^t_X B_t)(Y, Z) + (\nabla^t_Y B_t)(X, Z) - (\nabla^t_Z B_t)(X, Y), \tag{3}$$

where X, Y, Z are vector fields on M and $(\nabla^t_Z B_t)$ is the first covariant derivative of a $(0,2)$-tensor B_t with respect to Z, given by

$$(\nabla^t_Z B_t)(Y, V) = Z(B_t(Y, V)) - B_t(\nabla^t_Z Y, V) - B_t(Y, \nabla^t_Z V)$$

for all $Y, V \in \mathfrak{X}_M$. We shall also use the formula for the variation of the volume form vol_g of a metric g. For a $g^\perp$-variation g_t, we have [12]

$$\partial_t\, \mathrm{vol}_{g_t} = \frac{1}{2}(\mathrm{Tr}\ B_t^\sharp)\ \mathrm{vol}_{g_t}, \tag{4}$$

where $\mathrm{Tr}\ B_t^\sharp = \sum_{i=1}^p B_t(e_i(t), e_i(t))$ for a g_t-orthonormal basis $e_i(t)$ of $\mathcal{D}(t)$; we have $\mathrm{Tr}\ B_t^\sharp = \langle g_t, B_t \rangle$.

Using (3) together with Lemmas 1 and 2, the following result was established in [10], for distribution $\widetilde{\mathcal{D}}$ of any dimension and the integrability tensor $\tilde{T}$ of its orthogonal complement.

Proposition 1 *Let g_t be a $g^\perp$-variation of g. Then,*

$$\partial_t \|\tilde{T}\|^2 = \partial_t \langle \tilde{T}, \tilde{T} \rangle = 2 \langle\, \tilde{\mathcal{T}}^\flat + \Lambda_{\tilde{\theta}, \theta-\alpha} - (\operatorname{div} \tilde{\theta})_{|\mathrm{V}(t)},\ B_t \,\rangle + 2 \operatorname{div} \langle \tilde{\theta},\ B_t \rangle, \tag{5}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product of tensor fields defined by g_t.

3 Critical points and contact metric structures

In this section, we determine critical points of the action (1) with respect to volume-preserving $g^\perp$-variations. From Proposition 1 and definitions in Sect. 2, we obtain the following.

Proposition 2 *A metric g is critical for the functional (1) with respect to volume-preserving $g^\perp$-variations if and only if the following Euler–Lagrange equations hold for some $\lambda \in \mathbb{R}$:*

$$2\tilde{\mathcal{T}}^\flat + \frac{1}{2}\|\tilde{T}\|^2 g^\perp = \lambda g^\perp, \tag{6}$$

$$-\Lambda_{\tilde{\theta},\alpha} - (\operatorname{div} \tilde{\theta})_{|\mathrm{V}} = 0. \tag{7}$$

Equations (6), (7) are equivalent to:

$$2g((\tilde{T}^\sharp_\xi)^2 X, Y) + \frac{1}{2}\|\tilde{T}\|^2 g(X, Y) = \lambda g(X, Y), \tag{8}$$

$$-\frac{1}{2}(\operatorname{div}^\perp \tilde{T}^\sharp_\xi)(X) - g(\tilde{T}^\sharp_\xi H, X) = 0, \tag{9}$$

respectively, for all $X, Y \in \mathfrak{X}_\mathcal{D}$.

Proof From Proposition 1 and (4), we obtain

$$\begin{aligned}
\partial_t J_\Omega(g_t) &= \int_\Omega (\partial_t \|\tilde{T}\|^2)\ \mathrm{vol}_{g_t} + \int_\Omega \|\tilde{T}\|^2\ (\partial_t\ \mathrm{vol}_{g_t}) \\
&= \int_\Omega (2\,\langle\, \tilde{\mathcal{T}}^\flat + \Lambda_{\tilde{\theta},\theta-\alpha} - (\operatorname{div}\tilde{\theta})_{|\mathrm{V}(t)},\ B_t\,\rangle + 2\operatorname{div}\langle \tilde{\theta},\ B_t\rangle \\
&\quad + \frac{1}{2}\|\tilde{T}\|^2\ \mathrm{Tr}\ B^\sharp_t)\ \mathrm{vol}_{g_t} \\
&= \int_\Omega (2\,\langle\, \tilde{\mathcal{T}}^\flat + \frac{1}{4}\|\tilde{T}\|^2 g_t + \Lambda_{\tilde{\theta},\theta-\alpha} - (\operatorname{div}\tilde{\theta})_{|\mathrm{V}(t)},\ B_t\,\rangle)\ \mathrm{vol}_{g_t}\,.
\end{aligned} \tag{10}$$

Separating terms depending on $B|_{\mathcal{D}\times\mathcal{D}}$ and $B|_V$, we obtain that $\partial_t J_\Omega(g_t)|_{t=0} = 0$ if and only if

$$\int_\Omega \langle\, 2\tilde{\mathcal{T}}^\flat + \frac{1}{2}\|\tilde{T}\|^2 g,\ B\,\rangle\ \mathrm{vol}_g = 0$$

and

$$\int_\Omega \langle \Lambda_{\tilde{\theta},\theta-\alpha} - (\operatorname{div}\tilde{\theta})_{|\mathrm{V}},\ B\,\rangle\ \mathrm{vol}_g = 0$$

for all $B = \partial_t g_t|_{t=0}$ defined by volume-preserving $g^\perp$-variations g_t. For such variations, we have

$$0 = \partial_t \int_\Omega\ \mathrm{vol}_{g_t}\,|_{t=0} = \frac{1}{2}\int_\Omega \langle g, B\rangle\ \mathrm{vol}_g$$

and hence, $\partial_t J_\Omega(g_t)|_{t=0} = 0$ for volume-preserving $g^\perp$-variations if and only if

$$\langle\, 2\tilde{\mathcal{T}}^\flat + \frac{1}{2}\|\tilde{T}\|^2 g,\ B\,\rangle = \lambda\langle g,\ B\,\rangle$$

for some $\lambda \in \mathbb{R}$ and

$$\langle\, \Lambda_{\tilde\theta,\theta-\alpha} - (\operatorname{div}\tilde\theta)_{|\mathrm{V}},\ B\,\rangle = 0$$

for all symmetric (0, 2)-tensor fields B on Ω. Since $\langle g, B\rangle = \langle g^\perp, B\rangle$ for $g^\perp$-variations, we obtain (6) and since $\theta = 0$ (as the one-dimensional distribution $\widetilde{\mathcal{D}}$ is integrable), we obtain (7). Using definitions from the previous section, we obtain (8) and (9). □

Proposition 3 *Let $g \in \mathrm{Riem}(M, \xi)$ be a metric satisfying (8). Then at all points where the integrability tensor $\tilde T$ of $\mathcal{D}$ does not vanish, (9) is equivalent to $H = 0$.*

Proof Taking trace to determine λ, we can write (8) in the following form:

$$(\tilde{T}^\sharp_\xi)^2 = -\frac{1}{p}\|\tilde{T}\|^2\ \mathrm{Id}, \tag{11}$$

where Id denotes the identity transformation on $\mathcal{D}$ and

$$\frac{p-4}{2p}\|\tilde{T}\|^2 = \lambda = \mathrm{const}. \tag{12}$$

Let $X \in \mathfrak{X}_M$. We have from the Koszul formula

$$\begin{aligned}
2(\operatorname{div}^\perp \tilde{T}^\sharp_\xi)(X) &= 2\sum_{i=1}^p g((\nabla_{e_i}\tilde{T}^\sharp_\xi)X, e_i)\\
&= 2\sum_{i=1}^p g(\nabla_{e_i}(\tilde{T}^\sharp_\xi X), e_i) + 2\sum_{i=1}^p g(\tilde{T}^\sharp_\xi(\nabla_{e_i}X), e_i)\\
&= 2\sum_{i=1}^p g(\nabla_{e_i}(\tilde{T}^\sharp_\xi X), e_i) - 2\sum_{i=1}^p g(\nabla_{e_i}X, \tilde{T}^\sharp_\xi e_i)\\
&= \sum_{i=1}^p \big(2g([e_i, \tilde{T}^\sharp_\xi X], e_i) + e_i(g(X, \tilde{T}^\sharp_\xi e_i)) - (\tilde{T}^\sharp_\xi e_i)(g(e_i, X))\\
&\quad + g([e_i, X], \tilde{T}^\sharp_\xi e_i) - g([e_i, \tilde{T}^\sharp_\xi e_i], X) - g([X, \tilde{T}^\sharp_\xi e_i], e_i)\big),
\end{aligned} \tag{13}$$

where e_i is any local orthonormal basis of $\mathcal{D}$. Let Φ be the 2-form defined by the formula $\Phi(X, Y) = g(X, \tilde{T}^\sharp_\xi Y)$ for all $X, Y \in \mathfrak{X}_M$ [3]. We shall compare (13) to the differential of Φ evaluated on particular vectors.

All the following formulas in the proof will be computed at a point $x \in M$, where $\tilde T \neq 0$. Using (11), we have

$$\begin{aligned}
3d\Phi(e_i, \frac{p}{\|\tilde T\|^2}\tilde{T}^\sharp_\xi X, \tilde{T}^\sharp_\xi e_i) &= e_i\left(\Phi\left(\frac{p}{\|\tilde T\|^2}\tilde{T}^\sharp_\xi X, \tilde{T}^\sharp_\xi e_i\right)\right) + \frac{p}{\|\tilde T\|^2}(\tilde{T}^\sharp_\xi X)(\Phi(\tilde{T}^\sharp_\xi e_i, e_i))\\
&\quad + (\tilde{T}^\sharp_\xi e_i)(\Phi(e_i, \frac{p}{\|\tilde T\|^2}\tilde{T}^\sharp_\xi X)) - \Phi([(e_i, \frac{p}{\|\tilde T\|^2}\tilde{T}^\sharp_\xi X], \tilde{T}^\sharp_\xi e_i)\\
&\quad - \Phi\left(\left[\frac{p}{\|\tilde T\|^2}\tilde{T}^\sharp_\xi X, \tilde{T}^\sharp_\xi e_i\right], e_i\right) - \Phi([\tilde{T}^\sharp_\xi e_i, e_i], \frac{p}{\|\tilde T\|^2}\tilde{T}^\sharp_\xi X)
\end{aligned}$$

$$
\begin{aligned}
&= e_i(g(X, \tilde{T}^\sharp_\xi e_i)) + \frac{1}{\|\tilde{T}\|^2} g(e_i, e_i) \cdot (\tilde{T}^\sharp_\xi X)(\|\tilde{T}\|^2) \\
&\quad -(\tilde{T}^\sharp_\xi e_i)(g(e_i, X)) + g([e_i, \tilde{T}^\sharp_\xi X], e_i) \\
&\quad +\|\tilde{T}\|^2 g(\tilde{T}^\sharp_\xi X, e_i) \cdot e_i\left(\frac{1}{\|\tilde{T}\|^2}\right) - \frac{p}{\|\tilde{T}\|^2} g([\tilde{T}^\sharp_\xi X, \tilde{T}^\sharp_\xi e_i], \tilde{T}^\sharp_\xi e_i) \\
&\quad +g(\tilde{T}^\sharp_\xi X, \tilde{T}^\sharp_\xi e_i) \cdot (\tilde{T}^\sharp_\xi e_i)\left(\frac{p}{\|\tilde{T}\|^2}\right) - g([e_i, \tilde{T}^\sharp_\xi e_i], X).
\end{aligned}
$$

It follows from (11) with $\tilde{T} \neq 0$ that distribution $\mathcal{D}$ is even-dimensional. From now on, we consider a local orthonormal frame of $\mathcal{D}$ consisting of the following vector fields:

$$
\{e_1, \ldots, e_{p/2},\ e_{p/2+1} = \frac{\sqrt{p}}{\|\tilde{T}\|} \tilde{T}^\sharp_\xi e_1,\ \ldots, e_p = \frac{\sqrt{p}}{\|\tilde{T}\|} \tilde{T}^\sharp_\xi e_{p/2}\}.
$$

Then, we have

$$
\begin{aligned}
&-\frac{p}{\|\tilde{T}\|^2} \sum_{i=1}^p g([\tilde{T}^\sharp_\xi X, \tilde{T}^\sharp_\xi e_i], \tilde{T}^\sharp_\xi e_i) \\
&\quad = -\frac{p}{\|\tilde{T}\|^2} \sum_{i=1}^p g([\tilde{T}^\sharp_\xi X, \tilde{T}^\sharp_\xi \frac{\sqrt{p}}{\|\tilde{T}\|} \tilde{T}^\sharp_\xi e_i], \tilde{T}^\sharp_\xi \frac{\sqrt{p}}{\|\tilde{T}\|} \tilde{T}^\sharp_\xi e_i) \\
&\quad = \sum_{i=1}^p g([e_i, \tilde{T}^\sharp_\xi X], e_i) - \frac{p}{\|\tilde{T}\|} (\tilde{T}^\sharp_\xi X)(\|\tilde{T}\|)
\end{aligned}
$$

and, similarly,

$$
\sum_{i=1}^p g(\tilde{T}^\sharp_\xi X, \tilde{T}^\sharp_\xi e_i) \cdot (\tilde{T}^\sharp_\xi e_i)\left(\frac{p}{\|\tilde{T}\|^2}\right) = -\frac{2}{\|\tilde{T}\|} (\tilde{T}^\sharp_\xi X)(\|\tilde{T}\|).
$$

We also have

$$
\|\tilde{T}\|^2 \sum_{i=1}^p g(\tilde{T}^\sharp_\xi X, e_i) \cdot e_i\left(\frac{1}{\|\tilde{T}\|^2}\right) = \frac{2p}{\|\tilde{T}\|} (\tilde{T}^\sharp_\xi X)(\|\tilde{T}\|)
$$

and

$$
\frac{1}{\|\tilde{T}\|^2} \sum_{i=1}^p g(e_i, e_i) \cdot (\tilde{T}^\sharp_\xi X)(\|\tilde{T}\|^2) = -\frac{2}{\|\tilde{T}\|} (\tilde{T}^\sharp_\xi X)(\|\tilde{T}\|).
$$

Hence, we obtain

$$
\begin{aligned}
3\sum_{i=1}^p \mathrm{d}\Phi(e_i, \frac{p}{\|\tilde{T}\|^2} \tilde{T}^\sharp_\xi X, \tilde{T}^\sharp_\xi e_i) &= \sum_{i=1}^p \big(2g([e_i, \tilde{T}^\sharp_\xi X], e_i) + e_i(g(X, \tilde{T}^\sharp_\xi e_i)) \\
&\quad -(\tilde{T}^\sharp_\xi e_i)(g(e_i, X)) - g([e_i, \tilde{T}^\sharp_\xi e_i], X)\big) \\
&\quad + \frac{p-4}{\|\tilde{T}\|} (\tilde{T}^\sharp_\xi X)(\|\tilde{T}\|). \qquad (14)
\end{aligned}
$$

For the choice of orthonormal basis as above, we also have

$$\begin{aligned}\sum_{i=1}^{p} g([e_i, X], \tilde{T}^{\sharp}_{\xi} e_i) &= \sum_{i=1}^{p} g\left(\left[\frac{\sqrt{p}}{\|\tilde{T}\|}\tilde{T}^{\sharp}_{\xi} e_i, X\right], \frac{\sqrt{p}}{\|\tilde{T}\|}(\tilde{T}^{\sharp}_{\xi})^2 e_i\right)\\ &= \sum_{i=1}^{p} g([X, \tilde{T}^{\sharp}_{\xi} e_i], e_i) + \|\tilde{T}\| \sum_{i=1}^{p} g(\tilde{T}^{\sharp}_{\xi} e_i, e_i) X\left(\frac{1}{\|\tilde{T}\|}\right)\\ &= \sum_{i=1}^{p} g([X, \tilde{T}^{\sharp}_{\xi} e_i], e_i)\end{aligned}$$

and using it in (13), we obtain

$$\begin{aligned}2(\operatorname{div}^{\perp} \tilde{T}^{\sharp}_{\xi})(X) = \sum_{i=1}^{p} \big(&2g([e_i, \tilde{T}^{\sharp}_{\xi} X], e_i) + e_i(g(X, \tilde{T}^{\sharp}_{\xi} e_i)) - (\tilde{T}^{\sharp}_{\xi} e_i)(g(e_i, X))\\ &- g([e_i, \tilde{T}^{\sharp}_{\xi} e_i], X)\big).\end{aligned}$$

Comparing the above with (14), we obtain

$$3\sum_{i=1}^{p} \mathrm{d}\Phi(e_i, \frac{p}{\|\tilde{T}\|^2}\tilde{T}^{\sharp}_{\xi} X, \tilde{T}^{\sharp}_{\xi} e_i) = 2(\operatorname{div}^{\perp} \tilde{T}^{\sharp}_{\xi})(X) + \frac{p-4}{\|\tilde{T}\|}(\tilde{T}^{\sharp}_{\xi} X)(\|\tilde{T}\|). \tag{15}$$

Let $\eta(X) = g(\xi, X)$. Then, we have for all $X, Y \in \mathfrak{X}_{\mathcal{D}}$:

$$\begin{aligned}\mathrm{d}\eta(X, Y) &= \frac{1}{2}\big(X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y])\big)\\ &= -\frac{1}{2} g(\xi, [X, Y]) = g(X, \tilde{T}^{\sharp}_{\xi} Y) = \Phi(X, Y)\end{aligned}$$

and

$$\begin{aligned}\mathrm{d}\eta(\xi, X) &= -\frac{1}{2} g(\xi, [\xi, X]) = -\frac{1}{2} g(\nabla_{\xi} X, \xi) + \frac{1}{2} g(\nabla_X \xi, \xi)\\ &= \frac{1}{2} g(X, \nabla_{\xi}\xi) + \frac{1}{4} X(g(\xi, \xi)) = \frac{1}{2} g(X, H).\end{aligned}$$

For all $X, Y, Z \in \mathfrak{X}_{\mathcal{D}}$, we have

$$\begin{aligned}0 &= 3\mathrm{d}^2\eta(X, Y, Z) = 3\mathrm{d}\Phi(X, Y, Z) - \mathrm{d}\eta([X, Y]^{\top}, Z) - \mathrm{d}\eta([Y, Z]^{\top}, X) - \mathrm{d}\eta([Z, X]^{\top}, Y)\\ &= 3\mathrm{d}\Phi(X, Y, Z) - \mathrm{d}\eta(\xi, Z) g([X, Y], \xi) - \mathrm{d}\eta(\xi, X) g([Y, Z], \xi) - \mathrm{d}\eta(\xi, Y) g([Z, X], \xi)\\ &= 3\mathrm{d}\Phi(X, Y, Z) - g(Z, H) g(\tilde{T}^{\sharp}_{\xi} X, Y) - g(X, H) g(\tilde{T}^{\sharp}_{\xi} Y, Z) - g(Y, H) g(\tilde{T}^{\sharp}_{\xi} Z, X).\end{aligned}$$

It follows that

$$\begin{aligned}3\sum_{i=1}^{p} \mathrm{d}\Phi(e_i, \frac{p}{\|\tilde{T}\|^2}\tilde{T}^{\sharp}_{\xi} X, \tilde{T}^{\sharp}_{\xi} e_i) &= \sum_{i=1}^{p} \big(g(\tilde{T}^{\sharp}_{\xi} e_i, H) g(\tilde{T}^{\sharp}_{\xi} e_i, \frac{p}{\|\tilde{T}\|^2}\tilde{T}^{\sharp}_{\xi} X)\\ &\quad + g(e_i, H) g(\frac{p}{\|\tilde{T}\|^2}(\tilde{T}^{\sharp}_{\xi})^2 X, \tilde{T}^{\sharp}_{\xi} e_i)\\ &\quad + g\left(\frac{p}{\|\tilde{T}\|^2}\tilde{T}^{\sharp}_{\xi} X, H\right) g((\tilde{T}^{\sharp}_{\xi})^2 e_i, e_i)\big)\\ &= \sum_{i=1}^{p} \big(g(H, \tilde{T}^{\sharp}_{\xi} e_i) g(e_i, X) - g(e_i, H) g(X, \tilde{T}^{\sharp}_{\xi} e_i)\big)\end{aligned}$$

$$-pg(\tilde{T}^\sharp_\xi X, H)$$
$$= (2-p)g(\tilde{T}^\sharp_\xi X, H).$$

Hence,

$$2(\operatorname{div}^\perp \tilde{T}^\sharp_\xi)(X) = (2-p)g(\tilde{T}^\sharp_\xi X, H) - \frac{p-4}{\|\tilde{T}\|}(\tilde{T}^\sharp_\xi X)(\|\tilde{T}\|) \tag{16}$$

and if (8) holds and $\tilde{T} \neq 0$, (9) takes the following form:

$$-(p+2)g(\tilde{T}^\sharp_\xi H, X) + \frac{p-4}{\|\tilde{T}\|}(\tilde{T}^\sharp_\xi X)(\|\tilde{T}\|) = 0. \tag{17}$$

For $p = 4$, the last term above vanishes and from (12) it follows that for $p \neq 4$ we have $(\tilde{T}^\sharp_\xi X)(\|\tilde{T}\|) = 0$. Hence, for all g satisfying (8), equation (9) at all points $x \in M$ where $\mathcal{D}$ is non-integrable is equivalent to $g(\tilde{T}^\sharp_\xi H, X) = 0$ for all $X \in \mathcal{D}_x$. Taking $X = \tilde{T}^\sharp_\xi Y$ and using (11), we obtain $g(H, Y) = 0$ for all $Y \in \mathcal{D}_x$ and hence $H = 0$.

We note that if $p \neq 4$ and (8) holds, it follows from (12) that the integrability tensor $\tilde{T}$ of $\mathcal{D}$ vanishes either everywhere, or nowhere on M. □

Recall that a manifold M^{2n+1} with a 1-form η such that for all $X \in \mathfrak{X}_M$

$$\iota_\xi \mathrm{d}\eta(X) \equiv \mathrm{d}\eta(\xi, X) = 0, \quad \eta(\xi) = 1,$$

is called a *contact manifold*, and ξ is called the *characteristic vector field* (or the *Reeb field*). A Riemannian metric g on a contact manifold (M^{2n+1}, η) is *associated* if there exists a $(1,1)$-tensor field ϕ such that for all $X, Y \in \mathfrak{X}_M$

$$\eta(X) = g(\xi, X), \quad \mathrm{d}\eta(X, Y) = g(X, \phi Y), \quad \phi^2 = -\mathrm{Id_M} + \eta \otimes \xi, \tag{18}$$

where $\mathrm{Id_M}$ denotes the identity transformation. The above (ϕ, ξ, η, g) is called a *contact metric structure* on M. Since for $X \in \mathfrak{X}_\mathcal{D}$, we have

$$\mathrm{d}\eta(\xi, X) = -\frac{1}{2}g(\xi, [\xi, X]) = -\frac{1}{2}g(\xi, \nabla_\xi X - \nabla_X \xi) = \frac{1}{2}g(H, X),$$

condition $\mathrm{d}\eta(\xi, X) = 0$ for all $X \in \mathfrak{X}_M$ is equivalent to $H = 0$, i.e., integral curves of the Reeb field are geodesics with respect to associated metric [3].

We note that $(18)_2$ implies that $\phi = \tilde{T}^\sharp_\xi$, and from $(18)_3$, it follows that g satisfies (8). Hence, contact metric structures are critical points of the action (1). Up to rescaling, they are in fact the only critical points of (1) with nowhere vanishing integrability tensor $\tilde{T}$ of the distribution $\mathcal{D}$, as we show below.

Proposition 4 *Let $g \in \mathrm{Riem}(M, \xi)$ be a metric satisfying (8) and (9). Then at all points where the integrability tensor $\tilde{T}$ of $\mathcal{D}$ does not vanish, we have $g = \bar{g}^\top + f\bar{g}^\perp$, where $\bar{g}$ is a metric of some contact metric structure on M and f is a smooth function on M. Moreover, if $p \neq 4$, f is constant on M.*

Proof We consider only the set of points of M, where $\tilde{T} \neq 0$. Let $\eta(X) = g(\xi, X)$ for all $X \in \mathfrak{X}_\mathcal{D}$. Then for all $X, Y \in \mathfrak{X}_\mathcal{D}$

$$\mathrm{d}\eta(X, Y) = g(X, \tilde{T}^\sharp_\xi Y)$$

and $(\tilde{T}^\sharp_\xi)^2 = -\frac{\|\tilde{T}\|^2}{p}\mathrm{Id}$. Let $\bar{g} = \frac{\|\tilde{T}\|}{\sqrt{p}}g^\perp + g^\top$. Define ϕ by formulas $\phi(X) = \frac{\sqrt{p}}{\|\tilde{T}\|}\tilde{T}^\sharp_\xi X$ for all $X \in \mathfrak{X}_\mathcal{D}$ and $\phi(\xi) = 0$. Then $\bar{g}(\xi, X) = g(\xi, X) = \eta(X)$ for all $X \in \mathfrak{X}_M$ and (18) holds.

Since g satisfies (9), by Proposition 3 we have $H = 0$ and from the Koszul formula it easily follows that also $\bar{\nabla}_\xi \xi = 0$, where $\bar{\nabla}$ is the Levi-Civita connection of $\bar{g}$. Hence, $\iota_\xi \mathrm{d}\eta = 0$.

It follows that $(\phi, \xi, \eta, \bar{g})$ is a contact metric structure. Finally, note that since for $p \neq 4$ by (12) we have $\|\tilde{T}\| = \text{const}$, the function $f = \frac{\|\tilde{T}\|}{\sqrt{p}}$ may be non-constant only if $p = 4$. □

In the following example, we construct a family of metrics critical for the action (1) for $p = 4$, with the integrability tensor $\tilde{T}$ of distribution $\mathcal{D}$ vanishing on non-empty, proper subset of M.

Example 1 Let x_1, x_2, y_1, y_2, z be coordinates on an open subset U of $\mathbb{R}^5$. Let $\eta = \frac{1}{2}(\mathrm{d}z - f_1(y_1)\mathrm{d}x_1 - f_2(y_2)\mathrm{d}x_2)$ be a 1-form, where $f_1(y_1)$ and $f_2(y_2)$ are smooth functions on U. Then, vector fields $X_i = \frac{\partial}{\partial x_i} + f_i(y_i)\frac{\partial}{\partial z}$ and $Y_i = \frac{\partial}{\partial y_i}$ for $i = 1, 2$ form a local frame of the distribution $\mathcal{D} = \ker \eta$. We also have

$$\mathrm{d}\eta = \frac{1}{4}\left(\frac{\partial f_1}{\partial y_1}\mathrm{d}x_1 \wedge \mathrm{d}y_1 + \frac{\partial f_2}{\partial y_2}\mathrm{d}x_2 \wedge \mathrm{d}y_2\right). \tag{19}$$

Let $\xi = 2\frac{\partial}{\partial z}$, then $\eta(\xi) = 1$. Define the metric g by equations: $g\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) = \frac{1}{4}$, $g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}) = 0$, $g\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial y_j}\right) = 0$, $g\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial x_i}\right) = -\frac{1}{4} f_i(y_i)$, $g\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right) = \delta_{ij}$, $g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = \delta_{ij}(1 + f_i(y_i)^2)$. Then, $\{X_1, X_2, Y_1, Y_2, \xi\}$ form a local g-orthonormal frame on U and $g(\xi, Z) = \eta(Z)$ for all vector fields on U. We also have

$$\mathrm{d}\eta(X_i, Y_j) = \frac{1}{4}\delta_{ij}\frac{\partial f_i}{\partial y_i} = g(X_i, \tilde{T}^\sharp_\xi Y_j), \tag{20}$$

and $\mathrm{d}\eta(X_i, X_j) = \mathrm{d}\eta(Y_i, Y_j) = 0$ for all $i, j \in \{1, 2\}$. It follows that $(\tilde{T}^\sharp_\xi)^2 = -\frac{\|\tilde{T}\|^2}{p}\mathrm{Id}$, and hence, g satisfies the first Euler–Lagrange equation (11). Since $\iota_\xi \mathrm{d}\eta = 0$, we also have $H = \nabla_\xi \xi = 0$, and by Proposition 3 also (9) holds. By (19) and (20), we have $\tilde{T} = 0$ at points where $\frac{\partial f_1}{\partial y_1} = \frac{\partial f_2}{\partial y_2} = 0$.

Remark 1 The above example admits various functions f_1, f_2 and can be used to construct an open set $V \subsetneq M$ on which $\tilde{T} = 0$. Metric g can be then modified on a smaller subset of V to obtain $H \neq 0$ at some point $x \in V$, e.g., by conformally rescaling by a function ψ with $(\nabla\psi)^\perp \neq 0$ at x. Thus, for dimension $p = 4$ there exist solutions of the Euler–Lagrange equations (8) and (9) with $H \neq 0$. Since this may occur only in this particular dimension and only on the set where the integrability tensor $\tilde{T}$ of $\mathcal{D}$ vanishes; in the next section, we shall only consider critical points of (1) with $H = 0$.

4 Second variation and extrema of the action

By (10), the first variation of J_Ω can be presented in the following form:

$$\partial_t J_\Omega(g_t) = \int_\Omega \langle \delta J_\Omega(g_t), B_t\rangle \, \mathrm{vol}_{g_t},$$

where $B_t = \partial_t g_t$ and

$$\delta J_\Omega(g_t) = 2\tilde{T}^\flat + \frac{1}{2}\|\tilde{T}\|^2 g_t^\perp + 2\Lambda_{\tilde{\theta},\theta-\alpha} - 2(\operatorname{div}\tilde{\theta})_{|\mathrm{V(t)}}.$$

Moreover, in the proof of Proposition 2 it was established that at a critical point g of J_Ω we have $\delta J_\Omega(g) = \lambda g^\perp$.

In this section, we compute and examine the second variation of J_Ω, i.e.,

$$\partial^2_{tt} J_\Omega(g_t)|_{t=0} = \partial_t \int_\Omega \langle \delta J_\Omega(g_t), B_t\rangle \ \mathrm{vol}_{g_t}|_{t=0}.$$

4.1 General formulas

First we obtain the variation formula for the vector field H that is implicitly present in $\delta J_\Omega(g_t)$. In what follows, we use a result from [10] to write $\partial_t H$ explicitly as a vector field, for arbitrary $g^\perp$-variation g_t.

Lemma 3 *Let g_t be a $g^\perp$-variation. Then,*

$$\partial_t H = (\mathrm{div}^\top(B_t^\sharp))^\perp - \langle \tilde\alpha - \tilde\theta, B_t\rangle^\perp - (B_t^\sharp H)^\top,$$

where $\mathrm{div}^\top(B_t^\sharp) = (\nabla_\xi B_t^\sharp)(\xi)$.

Proof For all $X \in \mathfrak{X}_M$, the following formula was obtained in [10,(21)]:

$$\begin{aligned} g_t(\partial_t H, X) &= \mathrm{div}((B_t^\sharp(X^\perp))^\top) - \langle \tilde\delta_{X^\perp}, B_t\rangle \\ &\quad + g(X, (B_t^\sharp \tilde H)^\perp - (B_t^\sharp H)^\perp - \langle \tilde\alpha - \tilde\theta, B_t\rangle^\perp - (B_t^\sharp H)^\top). \end{aligned} \tag{21}$$

We compute (without any assumptions on the covariant derivatives of the g_t-orthonormal frame $\{\xi, e_1(t), \dots, e_p(t)\}$)

$$\begin{aligned} \mathrm{div}((B_t^\sharp(X^\perp))^\top) - \langle \tilde\delta_{X^\perp}, B_t\rangle &= g_t(\nabla^t_\xi((B_t^\sharp(X^\perp))^\top), \xi) - g_t((B_t^\sharp(X^\perp))^\top, \tilde H) \\ &\quad - g_t(\nabla^t_\xi(X^\perp), e_i(t)) B_t(\xi, e_i(t)) \\ &= -g_t((B_t^\sharp(X^\perp))^\top, \tilde H) + \nabla^t_\xi g_t(B_t^\sharp\xi, X^\perp) \\ &\quad - g_t(B_t^\sharp(X^\perp), (\nabla^t_\xi \xi)^\top) - g_t(\nabla^t_\xi(X^\perp), B_t^\sharp \xi) \\ &= -g_t((B_t^\sharp(X^\perp))^\top, \tilde H) + g_t(X^\perp, \nabla^t_\xi(B_t^\sharp \xi)) \\ &\quad - g_t(X^\perp, B_t^\sharp(\nabla^t_\xi \xi)^\top) \\ &= -g_t(X^\perp, B_t^\sharp \tilde H) + g_t(X^\perp, (\nabla_\xi B_t^\sharp)\xi + B_t^\sharp(\nabla^t_\xi \xi)) \\ &\quad - g_t(X^\perp, B_t^\sharp(\nabla^t_\xi \xi)^\top) \\ &= -g_t(X^\perp, B_t^\sharp \tilde H) + g_t(X^\perp, (\nabla^t_\xi B_t^\sharp)\xi) \\ &\quad + g_t(X^\perp, B_t^\sharp(\nabla^t_\xi \xi)^\perp) \\ &= g_t(X, (B_t^\sharp H)^\perp - (B_t^\sharp \tilde H)^\perp + (\mathrm{div}^\top(B_t^\sharp))^\perp), \end{aligned}$$

and compare it with (21). □

To compute the second variation of J_Ω, we need few more technical lemmas. Recall that $\mathcal{D}(t)$ is the g_t-orthogonal complement of $\widetilde{D}$.

Definition 1 Let Q_t, $t \in (-\epsilon, \epsilon)$ be a one-parameter family of symmetric $(0, 2)$-tensors, such that $Q_t(\xi, \xi)$ is independent of t, let g_t be a $g^\perp$-variation and let $B_t = \partial_t g_t$. For all

$t \in (-\epsilon, \epsilon)$, we define the tensor $D_t Q_t$ by equations

$$(D_t Q_t)(\xi, \xi) = 0, \tag{22}$$

$$(D_t Q_t)(X, Y) = (\partial_t Q_t)(X, Y) - \frac{1}{2} g^{\perp}(Q_t^{\sharp} X, B_t^{\sharp} Y) - \frac{1}{2} g_t^{\perp}(Q_t^{\sharp} Y, B_t^{\sharp} X) - g_t^{\top}(Q_t^{\sharp} X, B_t^{\sharp} Y) - g_t^{\top}(Q_t^{\sharp} Y, B_t^{\sharp} X), \tag{23}$$

$$(D_t Q_t)(X, \xi) = (D_t Q_t)(\xi, X) = (\partial_t Q_t)(X, \xi) - \frac{1}{2} g_t^{\perp}(Q_t^{\sharp} \xi, B_t^{\sharp} X) - g_t^{\top}(Q_t^{\sharp} \xi, B_t^{\sharp} X), \tag{24}$$

for all $X, Y \in \mathcal{D}(t)$.

Lemma 4 *Let Q_t be as in Definition 1; then, for any $g^{\perp}$-variation g_t we have*

$$\partial_t \langle Q_t, B_t \rangle = \langle D_t Q_t, B_t \rangle + \langle Q_t, D_t B_t \rangle, \tag{25}$$

where $D_t B_t$ is defined by (22)-(24) with $Q_t = B_t$. Moreover, if $\{e_i(t)\}_{i=1}^{p}$ is an orthonormal frame of $\mathcal{D}(t)$ obtained as in Lemma 1, then

$$\partial_t Q_t(e_i(t), e_j(t)) = (D_t Q_t)(e_i(t), e_j(t)), \tag{26}$$

$$\partial_t Q_t(e_i(t), \xi) = (D_t Q_t)(e_i(t), \xi). \tag{27}$$

Proof We prove the last claim first. Let $\{\xi, e_1(t), \ldots, e_p(t)\}$ be an orthonormal frame, obtained as in Lemma 1. We have

$$\begin{aligned} \partial_t Q_t(e_i(t), e_j(t)) &= (\partial_t Q_t)(e_i, e_j) + Q_t(\partial_t e_i, e_j) + Q_t(e_i, \partial_t e_j) \\ &= (\partial_t Q_t)(e_i(t), e_j(t)) \\ &\quad - \frac{1}{2} \sum_{m=1}^{p} B_t(e_i(t), e_m(t)) Q_t(e_m(t), e_j(t)) \\ &\quad - \frac{1}{2} \sum_{m=1}^{p} B_t(e_j(t), e_m(t)) Q_t(e_m(t), e_i(t)) \\ &\quad - B_t(\xi, e_i(t)) Q_t(\xi, e_j(t)) - B_t(\xi, e_j(t)) Q_t(\xi, e_i(t)) \\ &= (\partial_t Q_t)(e_i(t), e_j(t)) - \frac{1}{2} g_t^{\perp}(B_t^{\sharp} e_i(t), Q_t^{\sharp} e_j(t)) \\ &\quad - \frac{1}{2} g_t^{\perp}(B_t^{\sharp} e_j(t), Q_t^{\sharp} e_i(t)) - g_t^{\top}(B_t^{\sharp} e_i(t), Q_t^{\sharp} e_j(t)) \\ &\quad - g_t^{\top}(B_t^{\sharp} e_j(t), Q_t^{\sharp} e_i(t)) \\ &= (D_t Q_t)(e_i(t), e_j(t)). \end{aligned}$$

On the other hand, using again Lemma 1, we obtain

$$\begin{aligned} \partial_t Q_t(e_i(t), \xi) &= (\partial_t Q_t)(e_i(t), \xi) + Q_t(\partial_t e_i(t), \xi) + Q_t(e_i(t), \partial_t \xi) \\ &= (\partial_t Q_t)(e_i(t), \xi) \\ &\quad - \frac{1}{2} \sum_{m=1}^{p} B_t(e_i(t), e_m(t)) Q_t(e_m(t), \xi) \\ &\quad - B_t(\xi, e_i(t)) Q_t(\xi, \xi) \\ &= (\partial_t Q_t)(e_i(t), \xi) - \frac{1}{2} g_t^{\perp}(B_t^{\sharp} e_i(t), Q_t^{\sharp} \xi) - g_t^{\top}(Q_t^{\sharp} \xi, B_t^{\sharp} e_i(t)) \end{aligned}$$

$$= (D_t Q_t)(e_i(t), \xi).$$

We have

$$\langle Q_t, B_t \rangle = \sum_{i,j=1}^{p} Q_t(e_i(t), e_j(t)) B_t(e_i(t), e_j(t)) \\ +2 \sum_{i=1}^{p} Q_t(\xi, e_i(t)) B_t(\xi, e_i(t)),$$

where $\{e_i(t)\}_{i=1}^p$ is any orthonormal frame of $\mathcal{D}(t)$. Hence,

$$\begin{aligned}\partial_t \langle Q_t, B_t \rangle = &\sum_{i,j=1}^{p} (\partial_t Q_t(e_i(t), e_j(t))) B_t(e_i(t), e_j(t)) \\ &+ \sum_{i,j=1}^{p} Q_t(e_i(t), e_j(t))(\partial_t B_t(e_i(t), e_j(t))) \\ &+2 \sum_{i=1}^{p} (\partial_t Q_t(\xi, e_i(t))) B_t(\xi, e_i(t)) \\ &+2 \sum_{i=1}^{p} Q_t(\xi, e_i(t))(\partial_t B_t(\xi, e_i(t))) \\ &+(\partial_t Q_t(\xi, \xi)) B_t(\xi, \xi) + Q_t(\xi, \xi)(\partial_t B_t(\xi, \xi)). \end{aligned} \tag{28}$$

The last two terms in (28) vanish by the assumption that $Q_t(\xi, \xi)$ does not depend on t, and the fact that g_t is a $g^\perp$-variation with $B_t(\xi, \xi) = 0$ for all t. Since B_t satisfies the same assumptions as Q_t, equations (26) and (27) hold also for $Q_t = B_t$.

As the product $\langle Q_t, B_t \rangle$ can be computed using any orthonormal frame—in particular, the one from Lemma 1—we can use (26) and (27) in (28), which completes the proof. □

The above lemma simplifies some further notation and shows that for $e_i(t)$ as in Lemma 1, derivatives $\partial_t Q(e_i(t), \xi)$ and $\partial_t Q(e_i(t), e_j(t))$ can be expressed as values of the tensor $D_t Q_t$ on vectors $e_i(t), e_j(t)$ and ξ. Later we shall use values of tensors $D_t Q_t$ on a special frame, e.g., satisfying assumptions $\nabla_X e_i(0) \in \widetilde{\mathcal{D}}$ for all $X \in T_x M$.

Lemma 5 *Let g_t be a volume-preserving $g^\perp$-variation and let $B_t = \partial_t g_t$. Then,*

$$\int_\Omega \langle g_t^\perp, D_t B_t \rangle \operatorname{vol}_{g_t} = -\frac{1}{2} \int_\Omega \langle g_t^\perp, B_t \rangle \langle g_t^\perp, B_t \rangle \operatorname{vol}_{g_t}. \tag{29}$$

Proof For volume-preserving variations, we have $\partial_t \int \operatorname{vol}_{g_t} = \frac{1}{2} \int \langle g_t, B_t \rangle \operatorname{vol}_{g_t} = 0$. Since g_t satisfies assumptions of Definition 1, from Lemma 4 and equation (4) it follows that

$$\begin{aligned} 0 &= \partial_t \int_\Omega \langle g_t, B_t \rangle \operatorname{vol}_{g_t} \\ &= \int_\Omega (\langle D_t g_t, B_t \rangle + \langle g_t, D_t B_t \rangle + \frac{1}{2} \langle g_t, B \rangle \langle g_t, B \rangle) \operatorname{vol}_{g_t} \\ &= \int_\Omega (\langle g_t, D_t B_t \rangle + \frac{1}{2} \langle g_t, B_t \rangle \langle g_t, B_t \rangle) \operatorname{vol}_{g_t}, \end{aligned}$$

where we used the fact that $D_t g_t = 0$, as for $X_t, Y_t \in \mathcal{D}(t)$ we have

$$(D_t g_t)(X_t, Y_t) = B_t(X_t, Y_t) - \frac{1}{2} g_t^{\perp}(X_t, B_t^{\sharp} Y_t) - \frac{1}{2} g_t^{\perp}(Y_t, B_t^{\sharp} X_t) = 0,$$

and similarly

$$(D_t g_t)(X_t, \xi) = B_t(X_t, \xi) - \frac{1}{2} g_t^{\perp}(\xi, B_t^{\sharp} X_t) - g_t(\xi, \xi) B_t(X_t, \xi) = 0.$$

Since $B_t(\xi, \xi) = 0$ for $g^{\perp}$-variations, we have $\langle g_t, B_t\rangle = \langle g_t^{\perp}, B_t\rangle$. Also, we have $(D_t B_t)(\xi, \xi) = 0$, and hence, $\langle g_t, D_t B_t\rangle = \langle g_t^{\perp}, D_t B_t\rangle$. □

Lemma 6 *Let g be a critical point of action (1) with respect to volume-preserving $g^{\perp}$-variations and let*

$$\partial_t J_{\Omega}(g_t) = \int_{\Omega} \langle \delta J_{\Omega}(g_t), B_t\rangle \ \mathrm{vol}_g .$$

Then,

$$\partial_{tt}^2 J_{\Omega}(g_t)|_{t=0} = \int_{\Omega} \langle D_t \delta J_{\Omega}(g_t)|_{t=0}, B\rangle \ \mathrm{vol}_g .$$

Proof Recall that at a critical point we have $\delta J_{\Omega}(g) = \lambda g^{\perp}$ where $\lambda \in \mathbb{R}$ is a constant. Also, $\delta J_{\Omega}(g_t)(\xi, \xi) = 0$, so we can use Lemma 4. Hence, using (29), we obtain

$$\begin{aligned}
\partial_{tt}^2 J_{\Omega}(g_t)|_{t=0} &= \partial_t \int_{\Omega} \langle \delta J_{\Omega}(g_t), B_t\rangle \ \mathrm{vol}_{g_t} |_{t=0} \\
&= \int_{\Omega} (\langle D_t \delta J_{\Omega}(g_t)|_{t=0}, B\rangle + \langle \delta J_{\Omega}(g), D_t B|_{t=0}\rangle \\
&\quad + \frac{1}{2} \langle \delta J_{\Omega}(g), B\rangle \langle g, B\rangle) \ \mathrm{vol}_g \\
&= \int_{\Omega} (\langle D_t \delta J_{\Omega}(g_t)|_{t=0}, B\rangle + \langle \lambda g^{\perp}, D_t B|_{t=0}\rangle + \frac{1}{2} \langle \lambda g^{\perp}, B\rangle \langle g, B\rangle) \ \mathrm{vol}_g \\
&= \int_{\Omega} \langle D_t \delta J_{\Omega}(g_t)|_{t=0}, B\rangle \ \mathrm{vol}_g .
\end{aligned}$$

□

Now we are ready to compute

$$\langle D_t \delta J_{\Omega}(g_t)|_{t=0}, B\rangle = \sum_{i,j} (\partial_t \delta J_{\Omega}(e_i(t), e_j(t)))|_{t=0} \cdot B(e_i(0), e_j(0))$$

for a g_t-orthonormal frame $\{e_i(t)\}$ of $\mathcal{D}(t)$ from Lemma 1. We shall consider only the case where $H = 0$ to make (already lengthy) computations somewhat easier—due to Proposition 3 and (12), it is in fact the general case for non-integrable distributions $\mathcal{D}$ of dimension $p \neq 4$ and distributions $\mathcal{D}$ of dimension $p = 4$ with nowhere vanishing integrability tensor $\tilde{T}$.

Proposition 5 *Let g be a critical point of the action (1) with respect to volume-preserving $g^{\perp}$-variations, such that $H = 0$. Then, for all volume-preserving $g^{\perp}$-variations g_t of g we have:*

$$\partial_{tt}^2 J_{\Omega}(g_t)|_{t=0} = \int_{\Omega} \delta^2 J_{\Omega} \ \mathrm{vol}_g ,$$

where for all $x \in M$

$$
\begin{aligned}
\delta^2 J_\Omega = & \sum_{i,j=1}^{p} B(e_i, e_j) B(e_i, \tilde{A}_\xi \tilde{T}_\xi^\sharp e_j) - \sum_{i,j=1}^{p} B(e_i, e_j) B(\tilde{T}_\xi^\sharp e_i, \tilde{T}_\xi^\sharp e_j) \\
& +7 \sum_{i=1}^{p} B(\xi, \tilde{A}_\xi e_i) B(\xi, \tilde{T}_\xi^\sharp e_i) + \frac{(7-p)\|\tilde{T}\|^2}{p} \sum_{i=1}^{p} B(\xi, e_i) B(\xi, e_i) \\
& -2 \sum_{i,j=1}^{p} B(\tilde{T}_\xi^\sharp e_i, e_j)(\nabla_{e_j} B)(e_i, \xi) + \sum_{i,j=1}^{p} B(\tilde{T}_\xi^\sharp e_i, e_j)(\nabla_{e_i} B)(e_j, \xi) \\
& -\frac{1}{p}\|\tilde{T}\|^2 (\operatorname{Tr}\ B^\sharp)^2 - 2 \sum_{i,j=1}^{p} B(\xi, e_i)(\nabla_{\tilde{T}_\xi^\sharp e_i} B)(e_j, e_j) \\
& -2 \sum_{i,j=1}^{p} B(e_j, \tilde{A}_\xi e_i)(\nabla_{e_i} B)(\xi, e_j) + 2 \sum_{i,j=1}^{p} B(e_i, \tilde{A}_\xi e_j)(\nabla_{e_i} B)(\xi, e_j) \\
& + \sum_{i,j=1}^{p} B(e_j, \tilde{A}_\xi e_i) B(e_j, \tilde{A}_\xi e_i) - \sum_{i,j=1}^{p} B(e_i, \tilde{A}_\xi e_j) B(\tilde{A}_\xi e_i, e_j) \\
& + \sum_{i,j=1}^{p} (\nabla_{e_i} B)(e_j, \xi)(\nabla_{e_i} B)(\xi, e_j) - \sum_{i,j=1}^{p} (\nabla_{e_j} B)(e_i, \xi)(\nabla_{e_i} B)(\xi, e_j) \\
& + \sum_{i,j=1}^{p} B(\xi, e_j) B(\xi, e_j) g(\tilde{T}_\xi^\sharp e_i, \tilde{A}_\xi e_i) \\
& - \sum_{i,j=1}^{p} B(\xi, e_j)(\nabla_{e_i} B)(\tilde{T}_\xi^\sharp e_i, e_j) - 4 \sum_{k=1}^{p} B(\xi, \tilde{T}_\xi^\sharp e_k)(\nabla_\xi B)(\xi, e_k)
\end{aligned}
\tag{30}
$$

for any orthonormal basis $\{e_i\}_{i=1}^p$ of $\mathcal{D}_x$.

Proof By (10), we have

$$
\begin{aligned}
\partial_t J_\Omega(g_t) &= \partial_t \int_\Omega \|\tilde{T}\|^2 \operatorname{vol}_{g_t} = \int_\Omega (\partial_t \|\tilde{T}\|^2 + \frac{1}{2}\|\tilde{T}\|^2 \langle g_t, B_t \rangle) \operatorname{vol}_{g_t} \\
&= \int_\Omega \langle \delta J_\Omega(g_t), B_t \rangle \operatorname{vol}_{g_t},
\end{aligned}
$$

where

$$
\delta J_\Omega(g_t) = 2\widetilde{\mathcal{T}}^\flat + 2\Lambda_{\tilde{\theta},\theta-\alpha} - 2 \operatorname{div} \tilde{\theta}_{|\mathrm{V}(t)} + \frac{1}{2}\|\tilde{T}\|^2 g_t .
$$

We have

$$
\begin{aligned}
\langle D_t \delta J_\Omega(g_t), B_t \rangle|_{t=0} &= \langle D_t (2\widetilde{\mathcal{T}}^\flat + 2\Lambda_{\tilde{\theta},\theta-\alpha} - 2 \operatorname{div} \tilde{\theta}_{|\mathrm{V}(t)} + \frac{1}{2}\|\tilde{T}\|^2 g_t)|_{t=0}, B \rangle \\
&= 2\langle D_t \widetilde{\mathcal{T}}^\flat |_{t=0}, B \rangle + 2\langle D_t \Lambda_{\tilde{\theta},\theta-\alpha}|_{t=0}, B \rangle \\
&\quad -2\langle D_t \operatorname{div} \tilde{\theta}_{|\mathrm{V}(t)} \,|_{t=0}, B \rangle + \frac{1}{2}\langle D_t (\|\tilde{T}\|^2 g_t)|_{t=0}, B \rangle .
\end{aligned}
$$

For a critical metric g we have, using (5), (6), (7) and (12):

$$\begin{aligned}\frac{1}{2}D_t(\|\tilde{T}\|^2 g_t)|_{t=0} &= \frac{1}{2}(\partial_t\|\tilde{T}\|^2|_{t=0})g\\ &= \frac{1}{2}\langle 2\widetilde{\mathcal{T}}^\flat + 2\Lambda_{\tilde{\theta},\theta-\alpha} - 2\operatorname{div}\tilde{\theta}_{|\mathrm{V}}, B\rangle g + \frac{1}{2}\cdot 2(\operatorname{div}\langle\tilde{\theta}_{|\mathrm{V}}, B\rangle)g\\ &= \frac{1}{2}\langle\lambda g^\perp - \frac{1}{2}\|\tilde{T}\|^2 g^\perp, B\rangle g + (\operatorname{div}\langle\tilde{\theta}_{|\mathrm{V}}, B\rangle)g\\ &= \frac{1}{2}\langle\frac{p-4}{2p}\|\tilde{T}\|^2 g^\perp - \frac{1}{2}\|\tilde{T}\|^2 g^\perp, B\rangle g + (\operatorname{div}\langle\tilde{\theta}_{|\mathrm{V}}, B\rangle)g\\ &= -\frac{1}{p}\langle\|\tilde{T}\|^2 g^\perp, B\rangle g + (\operatorname{div}\langle\tilde{\theta}_{|\mathrm{V}}, B\rangle)g.\end{aligned}$$

We also have

$$(\operatorname{div}\langle\tilde{\theta}_{|\mathrm{V}}, B\rangle)\langle g, B\rangle = \operatorname{div}((\mathrm{Tr}\ B^\sharp)\langle\tilde{\theta}_{|\mathrm{V}}, B\rangle) - \langle\tilde{\theta}_{|\mathrm{V}}, B\rangle(\mathrm{Tr}\ B^\sharp)$$

and hence

$$\begin{aligned}\frac{1}{2}\langle D_t(\|\tilde{T}\|^2 g_t)|_{t=0}, B\rangle = &-\frac{1}{p}\|\tilde{T}\|^2\langle g^\perp, B\rangle\langle g^\perp, B\rangle - \langle\tilde{\theta}_{|\mathrm{V}}, B\rangle(\mathrm{Tr}\ B^\sharp)\\ &+\operatorname{div}((\mathrm{Tr}\ B^\sharp)\langle\tilde{\theta}_{|\mathrm{V}}, B\rangle).\end{aligned}$$

We have

$$-\frac{1}{p}\|\tilde{T}\|^2\langle g^\perp, B\rangle\langle g^\perp, B\rangle = -\frac{1}{p}\|\tilde{T}\|^2(\mathrm{Tr}\ B^\sharp)^2.$$

In the following formulas of this proof, let $\{e_i\}_{i=1}^p$ be a local orthonormal frame of $\mathcal{D}$ at the point $x \in M$ at which the formula is considered, such that $\nabla_X e_i \in \widetilde{\mathcal{D}}$ for all $X \in T_x M$. Also, let $\{e_i(t)\}_{i=1}^p$ be the orthonormal frame obtained from $\{e_i\}_{i=1}^p$ as in Lemma 1. We have

$$\begin{aligned}-\langle\tilde{\theta}_{|\mathrm{V}}, B\rangle(\mathrm{Tr}\ B^\sharp) &= \sum_{i,j,k=1}^p B(\xi, e_i)g(\tilde{T}^\sharp_\xi e_i, e_k)\cdot e_k(B(e_j, e_j))\\ &= -\sum_{i,j,k=1}^p B(\xi, e_i)g(\tilde{T}^\sharp_\xi e_i, e_k)\cdot\big((\nabla_{e_k}B)(e_j, e_j)\\ &\quad -2\sum_{j,k=1}^p B(\nabla_{e_k}e_j, e_j)\big)\\ &= -\sum_{i,j=1}^p B(\xi, e_i)(\nabla_{\tilde{T}^\sharp_\xi e_i}B)(e_j, e_j)\\ &\quad -2\sum_{i,j,k=1}^p B(\xi, e_i)B(\xi, e_j)g(\tilde{T}^\sharp_\xi e_i, e_k)g((\tilde{A}_\xi + \tilde{T}^\sharp_\xi)e_k, e_j)\\ &= -\sum_{i,j=1}^p B(\xi, e_i)(\nabla_{\tilde{T}^\sharp_\xi e_i}B)(e_j, e_j)\\ &\quad -2\sum_{i,j=1}^p B(\xi, e_i)B(\xi, e_j)g(\tilde{T}^\sharp_\xi e_i, \tilde{A}_\xi e_j)\end{aligned}$$

$$+2\sum_{i,j=1}^{p} B(\xi,e_i)B(\xi,e_j)g(\tilde{T}^\sharp_\xi e_i,\tilde{T}^\sharp_\xi e_j).$$

We have

$$\begin{aligned}
-&\sum_{i,j=1}^{p} B(\xi,e_i)B(\xi,e_j)g(\tilde{T}^\sharp_\xi e_i,\tilde{A}_\xi e_j)\\
&=-\sum_{i,j,k=1}^{p} B(\xi,e_i)B(\xi,e_j)g(\tilde{T}^\sharp_\xi e_i,e_k)g(e_k,\tilde{A}_\xi e_j)\\
&=\sum_{k=1}^{p} B(\xi,\tilde{T}^\sharp_\xi e_k)B(\xi,\tilde{A}_\xi e_k)
\end{aligned}\tag{31}$$

and using (11), we obtain

$$\begin{aligned}
&\sum_{i,j=1}^{p} B(\xi,e_i)B(\xi,e_j)g(\tilde{T}^\sharp_\xi e_i,\tilde{T}^\sharp_\xi e_j)\\
&\quad=-\sum_{i,j=1}^{p} B(\xi,e_i)B(\xi,e_j)g(\tilde{T}^\sharp_\xi\tilde{T}^\sharp_\xi e_i,e_j)\\
&\quad=\frac{\|\tilde{T}\|^2}{p}\sum_{i=1}^{p} B(\xi,e_i)B(\xi,e_i).
\end{aligned}\tag{32}$$

Therefore,

$$\begin{aligned}
\frac{1}{2}\langle D_t(\|\tilde{T}\|^2 g_t)|_{t=0},B\rangle &= -\frac{1}{p}\|\tilde{T}\|^2(\operatorname{Tr} B^\sharp)^2-\sum_{i,j=1}^{p} B(\xi,e_i)(\nabla_{\tilde{T}^\sharp_\xi e_i}B)(e_j,e_j)\\
&\quad+\frac{2\|\tilde{T}\|^2}{p}\sum_{i=1}^{p} B(\xi,e_i)B(\xi,e_i)\\
&\quad+2\sum_{i=1}^{p} B(\xi,\tilde{T}^\sharp_\xi e_i)B(\xi,\tilde{A}_\xi e_i)\\
&\quad+\operatorname{div}((\operatorname{Tr} B^\sharp)\langle\tilde{\theta}_{|\mathrm{V}},B\rangle).
\end{aligned}\tag{33}$$

We have from (26)

$$\begin{aligned}
\langle D_t\widetilde{\mathcal{T}}^\flat|_{t=0},B\rangle &= \sum_{i,j=1}^{p} B(e_i,e_j)\partial_t g_t((\tilde{T}^\sharp_\xi)^2 e_i(t),e_j(t))|_{t=0}\\
&=-\sum_{i,j=1}^{p} B(e_i,e_j)\partial_t g_t(\tilde{T}^\sharp_\xi e_i(t),\tilde{T}^\sharp_\xi e_j(t))|_{t=0}\\
&=-\sum_{i,j,k=1}^{p} B(e_i,e_j)\partial_t(g_t(\tilde{T}^\sharp_\xi e_i(t),e_k(t))g_t(\tilde{T}^\sharp_\xi e_j(t),e_k(t)))|_{t=0}\\
&=-2\sum_{i,j,m=1}^{p} B(e_i,e_j)g(\tilde{T}(e_i,e_m),\xi)\partial_t g_t(\tilde{T}(e_j(t),e_m(t)),\xi)|_{t=0}
\end{aligned}$$

$$
\begin{aligned}
&= -2 \sum_{i,j,m=1}^{p} B(e_i, e_j) g(\tilde{T}(e_i, e_m), \xi) B(\tilde{T}(e_j, e_m), \xi) \\
&\quad -2 \sum_{i,j,m=1}^{p} B(e_i, e_j) g(\tilde{T}_\xi^\sharp e_i, e_m) g(\partial_t \tilde{T}(e_j(t), e_m(t)), \xi)|_{t=0} \\
&= -2 \sum_{i,j,m=1}^{p} B(e_i, e_j) g(\tilde{T}_\xi^\sharp e_i, e_m) g(\partial_t \tilde{T}(e_j(t), e_m(t)), \xi)|_{t=0}.
\end{aligned}
\tag{34}
$$

Using Lemma 2, $g((\partial_t \nabla)_{e_j} e_m, \xi) = g((\partial_t \nabla)_{e_m} e_j, \xi)$ and $H = 0$, we obtain

$$
\begin{aligned}
&2g(\partial_t \tilde{T}(e_j(t), e_m(t))|_{t=0}, \xi) \\
&\quad = g(\partial_t (\nabla_{e_j(t)} e_m(t) - \nabla_{e_m(t)} e_j(t))^\top |_{t=0}, \xi) \\
&\quad = g((\partial_t (\nabla_{e_j(t)} e_m(t) - \nabla_{e_m(t)} e_j(t))|_{t=0})^\top + (B^\sharp(\nabla_{e_j} e_m - \nabla_{e_m} e_j)^\perp)^\top, \xi) \\
&\quad = g((\partial_t (\nabla_{e_j(t)} e_m(t) - \nabla_{e_m(t)} e_j(t))|_{t=0}), \xi) \\
&\quad = -\frac{1}{2} \sum_{k=1}^{p} B(e_j, e_k) g(\nabla_{e_k} e_m, \xi) - B(e_j, \xi) g(\nabla_\xi e_m, \xi) \\
&\qquad -g\left(\nabla_{e_j} \left(\frac{1}{2} \sum_{k=1}^{p} B(e_m, e_k) e_k + B(e_m, \xi)\xi\right), \xi\right) + \frac{1}{2} \sum_{k=1}^{p} B(e_m, e_k) g(\nabla_{e_k} e_j, \xi) \\
&\qquad +B(e_m, \xi) g(\nabla_\xi e_j, \xi) + g\left(\nabla_{e_m} \left(\frac{1}{2} \sum_{k=1}^{p} B(e_j, e_k) e_k + B(e_j, \xi)\xi\right), \xi\right) \\
&\quad = -\frac{1}{2} \sum_{k=1}^{p} B(e_j, e_k) g(e_m, (\tilde{A}_\xi + \tilde{T}_\xi^\sharp) e_k) - \frac{1}{2} \sum_{k=1}^{p} B(e_m, e_k) g(\nabla_{e_j} e_k, \xi) \\
&\qquad -e_j(B(e_m, \xi)) + \frac{1}{2} \sum_{k=1}^{p} B(e_m, e_k) g(e_j, (\tilde{A}_\xi + \tilde{T}_\xi^\sharp) e_k) \\
&\qquad +\frac{1}{2} \sum_{k=1}^{p} B(e_j, e_k) g(\nabla_{e_m} e_k, \xi) + e_m(B(e_j, \xi)) \\
&\quad = -\sum_{k=1}^{p} B(e_j, e_k) g(e_k, \tilde{A}_\xi e_m) + \sum_{k=1}^{p} B(e_m, e_k) g(e_k, \tilde{A}_\xi e_j) \\
&\qquad -(\nabla_{e_j} B)(e_m, \xi) + (\nabla_{e_m} B)(e_j, \xi).
\end{aligned}
$$

Using the above in (34) and

$$
\begin{aligned}
&\sum_{i,j,k=1}^{p} B(e_i, \tilde{A}_\xi e_k) B(\tilde{T}_\xi^\sharp e_i, e_k) \\
&\quad = \sum_{i,j,k,m=1}^{p} B(e_i, e_j) g(\tilde{T}_\xi^\sharp e_i, e_m) B(e_m, e_k) g(e_k, \tilde{A}_\xi e_j)
\end{aligned}
$$

$$= - \sum_{j,k,m=1}^{p} B(\tilde{T}^{\sharp}_{\xi} e_m, e_j) B(e_m, \tilde{A}_{\xi} e_j) = 0, \tag{35}$$

we obtain

$$\begin{aligned} 2\langle D_t \widetilde{\mathcal{T}}^{\flat}|_{t=0}, B\rangle &= 2 \sum_{i,j=1}^{p} B(e_i, e_j) B(e_j, \tilde{A}_{\xi} \tilde{T}^{\sharp}_{\xi} e_i) \\ &\quad -2 \sum_{i,j=1}^{p} B(\tilde{T}^{\sharp}_{\xi} e_i, e_j)(\nabla_{e_j} B)(e_i, \xi) \\ &\quad +2 \sum_{i,j=1}^{p} B(\tilde{T}^{\sharp}_{\xi} e_i, e_j)(\nabla_{e_i} B)(e_j, \xi). \end{aligned} \tag{36}$$

Next we compute $\langle D_t \Lambda_{\tilde{\theta},\theta-\alpha}|_{t=0}, B\rangle$. Since $\dim \widetilde{\mathcal{D}} = 1$, we have

$$\begin{aligned} \langle D_t \Lambda_{\tilde{\theta},\theta-\alpha}|_{t=0}, B\rangle &= 2 \sum_{i=1}^{p} B(\xi, e_i) \partial_t (\Lambda_{\tilde{\theta},\theta-\alpha}(\xi, e_i))|_{t=0} \\ &= - \sum_{i,j=1}^{p} B(\xi, e_i) \partial_t (g(\tilde{T}^{\sharp}_{\xi} e_j, e_i) g(\xi, A_j \xi))|_{t=0} \\ &= - \sum_{i,j=1}^{p} B(\xi, e_i) \partial_t (g(\tilde{T}^{\sharp}_{\xi} e_j, e_i) g(H, e_j))|_{t=0}. \end{aligned}$$

We use Lemma 3 to compute $D_t \Lambda_{\tilde{\theta},\theta-\alpha}|_{t=0}$. Using $H = 0$, we obtain

$$\begin{aligned} \langle D_t \Lambda_{\tilde{\theta},\theta-\alpha}|_{t=0}, B\rangle &= - \sum_{i,j=1}^{p} B(\xi, e_i) g(\tilde{T}^{\sharp}_{\xi} e_j, e_i) g(\partial_t H|_{t=0}, e_j) \\ &= - \sum_{i,j=1}^{p} B(\xi, e_i) g(\tilde{T}^{\sharp}_{\xi} e_j, e_i) g((\operatorname{div}^{\top}(B^{\sharp})) - \langle \tilde{\alpha} - \tilde{\theta}, B\rangle, e_j) \\ &= \sum_{i=1}^{p} B(\xi, e_i) g(\tilde{T}^{\sharp}_{\xi} e_i, (\nabla_{\xi}(B^{\sharp}))\xi) \\ &\quad + \sum_{i,j,k=1}^{p} B(\xi, e_i) B(\xi, e_k) g(\tilde{T}^{\sharp}_{\xi} e_j, e_i) g(e_j, \tilde{A}_{\xi} e_k - \tilde{T}^{\sharp}_{\xi} e_k) \\ &= \sum_{i=1}^{p} B(\xi, e_i) g(\tilde{T}^{\sharp}_{\xi} e_i, (\nabla_{\xi}(B^{\sharp}))\xi) \\ &\quad - \sum_{i,j=1}^{p} B(\xi, e_i) B(\xi, e_j) g(\tilde{T}^{\sharp}_{\xi} e_i, \tilde{A}_{\xi} e_j - \tilde{T}^{\sharp}_{\xi} e_j) \\ &= \sum_{i=1}^{p} B(\xi, e_i) g(\tilde{T}^{\sharp}_{\xi} e_i, (\nabla_{\xi}(B^{\sharp}))\xi) \end{aligned}$$

$$- \sum_{i,j=1}^{p} B(\xi, e_i) B(\xi, e_j) g(\tilde{T}^\sharp_\xi e_i, \tilde{A}_\xi e_j)$$
$$+ \sum_{i,j=1}^{p} B(\xi, e_i) B(\xi, e_j) g(\tilde{T}^\sharp_\xi e_i, \tilde{T}^\sharp_\xi e_j),$$

from which using (31) and (32) in the last two terms above, we obtain

$$2\langle D_t \Lambda_{\tilde{\theta},\theta-\alpha}, B\rangle = -2\sum_{i=1}^{p} B(\xi, \tilde{T}^\sharp_\xi e_i)(\nabla_\xi B)(\xi, e_i) + 2\sum_{i=1}^{p} B(\xi, \tilde{T}^\sharp_\xi e_i) B(\xi, \tilde{A}_\xi e_i)$$
$$+ \frac{2\|\tilde{T}\|^2}{p} \sum_{i=1}^{p} B(\xi, e_i) B(\xi, e_i). \tag{37}$$

Next we compute $-\langle B, D_t((\operatorname{div}\theta)_{|\mathrm{V}(t)})|_{t=0}\rangle$. Since $g_t(\xi, \xi) = 1$ for all t, we have $\nabla^t_X \xi \in \mathcal{D}(t)_x$ for all $X \in T_x M$, and hence,

$$(\operatorname{div} \tilde{T}^\sharp)_\xi e_j(t) = \operatorname{div}(\tilde{T}^\sharp_\xi e_j(t)) - \sum_{k=1}^{p} g_t(\tilde{T}^\sharp_\xi(\nabla^t_{e_k(t)} e_j(t)), e_k(t)).$$

We have

$$-\langle B, D_t((\operatorname{div}\theta)_{|\mathrm{V}(t)})\,|_{t=0}\rangle = -\sum_{j=1}^{p} 2B(\xi, e_j)\partial_t((\operatorname{div}\theta)(\xi, e_j(t)))|_{t=0}$$
$$= -\sum_{j=1}^{p} B(\xi, e_j)(\partial_t(\operatorname{div} \tilde{T}^\sharp)_\xi e_j(t))|_{t=0}$$
$$= -\sum_{j,k=1}^{p} B(\xi, e_j)\partial_t(\operatorname{div}(\tilde{T}^\sharp_\xi e_j(t)) - g(\tilde{T}^\sharp_\xi(\nabla_{e_k(t)} e_j(t), e_k(t))))|_{t=0}$$
$$= -\sum_{j=1}^{p} B(\xi, e_j)\partial_t(\operatorname{div}(\tilde{T}^\sharp_\xi e_j(t)))|_{t=0}$$
$$+ \sum_{j,k=1}^{p} B(\xi, e_j)\partial_t(g(\tilde{T}^\sharp_\xi(\nabla_{e_k(t)} e_j(t), e_k(t))))|_{t=0}.$$

Using the formula $\partial_t(\operatorname{div} X) = \operatorname{div}(\partial_t X) + \frac{1}{2} X(\operatorname{Tr}\ B^\sharp_t)$, obtained in [10] (see the beginning of the proof of Theorem 1 there), and $\operatorname{div}(f X) = f \operatorname{div} X + X(f)$, for all $X \in \mathfrak{X}_M$ and $f \in C^\infty(M)$, we get

$$-\sum_{j=1}^{p} B(\xi, e_j)\partial_t(\operatorname{div}(\tilde{T}^\sharp_\xi e_j(t)))|_{t=0}$$
$$= -\sum_{j=1}^{p} B(\xi, e_j)(\operatorname{div}(\partial_t \tilde{T}^\sharp_\xi e_j(t))|_{t=0} + \frac{1}{2}(\tilde{T}^\sharp_\xi e_j)(\operatorname{Tr}\ B^\sharp))$$
$$= -\sum_{j=1}^{p} \operatorname{div}(B(\xi, e_j)\partial_t(\tilde{T}^\sharp_\xi e_j(t))|_{t=0}) + \sum_{j=1}^{p}(\partial_t \tilde{T}^\sharp_\xi e_j(t))|_{t=0}(B(\xi, e_j))$$

$$-\frac{1}{2}\sum_{j=1}^{p} B(\xi, e_j)\cdot(\tilde{T}^\sharp_\xi e_j)(\mathrm{Tr}\ B^\sharp).$$

Hence, we have

$$\begin{aligned}
-\langle B, D_t((\operatorname{div}\theta)_{|\mathrm{V}(t)})\,|_{t=0}\rangle &= -\operatorname{div}(\sum_{j=1}^{p} B(\xi, e_j)\partial_t \tilde{T}^\sharp_\xi e_j(t)|_{t=0})\\
&\quad+\sum_{j,k=1}^{p} B(\xi, e_j)\partial_t(g(\tilde{T}^\sharp_\xi(\nabla^t_{e_k(t)} e_j(t), e_k(t))))|_{t=0}\\
&\quad-\frac{1}{2}\sum_{j=1}^{p} B(\xi, e_j)(\tilde{T}^\sharp_\xi e_j)(\mathrm{Tr}\ B^\sharp)\\
&\quad+\sum_{j=1}^{p}(\partial_t \tilde{T}^\sharp_\xi e_j(t))|_{t=0}(B(\xi, e_j)).
\end{aligned}\tag{38}$$

Using the assumption that $\nabla_X e_j \in \widetilde{\mathcal{D}}$ for all $X \in T_x M$, we obtain

$$\begin{aligned}
&\sum_{k=1}^{p}\partial_t g(\tilde{T}^\sharp_\xi(\nabla^t_{e_k(t)} e_j(t), e_k(t)))|_{t=0}\\
&\quad=\sum_{k,m=1}^{p}(\partial_t g_t(\tilde{T}(e_m(t), e_k(t)), \xi))|_{t=0}\, g(\nabla_{e_k} e_j, e_m)\\
&\qquad+\sum_{k,m=1}^{p} g(\tilde{T}^\sharp_\xi e_m, e_k)\partial_t g(\nabla^t_{e_k(t)} e_j(t), e_m(t))|_{t=0}\\
&\quad=\sum_{k,m=1}^{p} g(\tilde{T}^\sharp_\xi e_m, e_k)\partial_t g(\nabla^t_{e_k(t)} e_j(t), e_m(t))|_{t=0}\\
&\quad=\sum_{k,m=1}^{p} g(\tilde{T}^\sharp_\xi e_m, e_k)\Big(B(\xi, e_m)g(\nabla_{e_k} e_j, \xi)-\frac{1}{2}g(\nabla_{e_k}(\sum_{i=1}^{p} B(e_j, e_i)e_i), e_m)\\
&\qquad-g(\nabla_{e_k}(B(\xi, e_j)\xi), e_m)-g(\nabla_{e_k} e_j, \xi)B(\xi, e_m)+g((\partial_t\nabla^t|_{t=0})_{e_k} e_j, e_m)\Big)\\
&\quad=\sum_{k,m=1}^{p} g(\tilde{T}^\sharp_\xi e_m, e_k)\Big(B(\xi, e_m)g(e_j, (\tilde{A}_\xi+\tilde{T}^\sharp_\xi)e_k)-\frac{1}{2}e_k(B(e_j, e_m))\\
&\qquad+B(\xi, e_j)g((\tilde{A}_\xi+\tilde{T}^\sharp_\xi)e_k, e_m)-B(\xi, e_m)g(e_j, (\tilde{A}_\xi+\tilde{T}^\sharp_\xi)e_k)\\
&\qquad+\frac{1}{2}(\nabla_{e_k} B)(e_j, e_m)+\frac{1}{2}(\nabla_{e_j} B)(e_k, e_m)-\frac{1}{2}(\nabla_{e_m} B)(e_k, e_j)\Big)
\end{aligned}$$

and eventually

$$\sum_{j,k=1}^{p} B(\xi, e_j)\partial_t g(\tilde{T}^\sharp_\xi(\nabla_{e_k(t)} e_j(t), e_k(t)))|_{t=0}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{k=1}^{p} B(\xi, \tilde{A}_\xi e_k) B(\xi, \tilde{T}^\sharp_\xi e_k) + \frac{1}{2}\sum_{k=1}^{p} B(\xi, \tilde{T}^\sharp_\xi e_k) B(\xi, \tilde{T}^\sharp_\xi e_k)\\
&+\frac{1}{2}\sum_{j,k=1}^{p} B(\xi, e_j) B(\xi, e_j) g(\tilde{T}^\sharp_\xi e_k, \tilde{A}_\xi e_k)\\
&-\frac{1}{2}\sum_{j,k=1}^{p} B(\xi, e_j) B(\xi, e_j) g(\tilde{T}^\sharp_\xi e_k, \tilde{T}^\sharp_\xi e_k)\\
&+\frac{1}{2}\sum_{j,k=1}^{p} B(\xi, e_j)(\nabla_{e_j} B)(\tilde{T}^\sharp_\xi e_k, e_k) - \frac{1}{2}\sum_{j,k=1}^{p} B(\xi, e_j)(\nabla_{e_k} B)(\tilde{T}^\sharp_\xi e_k, e_j).
\end{aligned}
$$

Using

$$
\begin{aligned}
\sum_{k=1}^{p} (\nabla_{e_j} B)(\tilde{T}^\sharp_\xi e_k, e_k) &= \sum_{k,m=1}^{p} (\nabla_{e_j} B)(e_m, e_k) g(\tilde{T}^\sharp_\xi e_k, e_m)\\
&= -\sum_{k,m=1}^{p} (\nabla_{e_j} B)(e_k, e_m) g(\tilde{T}^\sharp_\xi e_m, e_k)
\end{aligned}
$$

and (11) in

$$
\begin{aligned}
\sum_{k=1}^{p} B(\xi, \tilde{T}^\sharp_\xi e_k) B(\xi, \tilde{T}^\sharp_\xi e_k) &= \sum_{i,j,k=1}^{p} B(\xi, e_i) g(e_i, \tilde{T}^\sharp_\xi e_k) B(\xi, e_j) g(e_j, \tilde{T}^\sharp_\xi e_k)\\
&= \sum_{i,j=1}^{p} B(\xi, e_i) B(\xi, e_j) g(\tilde{T}^\sharp_\xi e_i, \tilde{T}^\sharp_\xi e_j)\\
&= \frac{\|\tilde{T}\|^2}{p} \sum_{i=1}^{p} B(\xi, e_i) B(\xi, e_i) \qquad (39)
\end{aligned}
$$

and

$$
\sum_{j,k=1}^{p} B(\xi, e_j) B(\xi, e_j) g(\tilde{T}^\sharp_\xi e_k, \tilde{T}^\sharp_\xi e_k) = \|\tilde{T}\|^2 \sum_{i=1}^{p} B(\xi, e_i) B(\xi, e_i), \qquad (40)
$$

we obtain

$$
\begin{aligned}
&\sum_{j,k=1}^{p} B(\xi, e_j) \partial_t g(\tilde{T}^\sharp_\xi (\nabla_{e_k(t)} e_j(t), e_k(t)))|_{t=0}\\
&= \frac{1}{2}\sum_{k=1}^{p} B(\xi, \tilde{A}_\xi e_k) B(\xi, \tilde{T}^\sharp_\xi e_k)\\
&+\frac{\|\tilde{T}\|^2(1-p)}{2p} \sum_{i=1}^{p} B(\xi, e_i) B(\xi, e_i)\\
&+\frac{1}{2}\sum_{j,k=1}^{p} B(\xi, e_j) B(\xi, e_j) g(\tilde{T}^\sharp_\xi e_k, \tilde{A}_\xi e_k)
\end{aligned}
$$

$$-\frac{1}{2}\sum_{j,k=1}^{p} B(\xi, e_j)(\nabla_{e_k} B)(\tilde{T}_\xi^\sharp e_k, e_j). \tag{41}$$

We have

$$\begin{aligned}
&-\frac{1}{2}\sum_{j=1}^{p} B(\xi, e_j)\cdot(\tilde{T}_\xi^\sharp e_j)(\mathrm{Tr}\ B^\sharp)\\
&= -\frac{1}{2}\sum_{j,k,m=1}^{p} B(\xi, e_j) g(\tilde{T}_\xi^\sharp e_j, e_m)\cdot e_m(B(e_k, e_k))\\
&= -\frac{1}{2}\sum_{j,k,m=1}^{p} B(\xi, e_j) g(\tilde{T}_\xi^\sharp e_j, e_m)\Big((\nabla_{e_m} B)(e_k, e_k) + 2B(\nabla_{e_m} e_k, e_k)\Big)\\
&= -\frac{1}{2}\sum_{j,k,m=1}^{p} B(\xi, e_j) g(\tilde{T}_\xi^\sharp e_j, e_m)\Big((\nabla_{e_m} B)(e_k, e_k)\\
&\quad +2g(e_k, (\tilde{A}_\xi + \tilde{T}_\xi^\sharp) e_m) B(\xi, e_k)\Big)\\
&= -\frac{1}{2}\sum_{j,k,m=1}^{p} B(\xi, e_j) g(\tilde{T}_\xi^\sharp e_j, e_m)(\nabla_{e_m} B)(e_k, e_k)\\
&\quad -\sum_{j,k,m=1}^{p} B(\xi, e_j) B(\xi, e_k) g(\tilde{T}_\xi^\sharp e_j, e_m) g(e_k, (\tilde{A}_\xi + \tilde{T}_\xi^\sharp) e_m).
\end{aligned}$$

Eventually, using (39), we obtain

$$\begin{aligned}
-\frac{1}{2}\sum_{j=1}^{p} B(\xi, e_j)(\tilde{T}_\xi^\sharp e_j)(\mathrm{Tr}\ B^\sharp) &= -\frac{1}{2}\sum_{j,k=1}^{p} B(\xi, e_j)(\nabla_{\tilde{T}_\xi^\sharp e_j} B)(e_k, e_k)\\
&\quad + \sum_{i=1}^{p} B(\xi, \tilde{T}_\xi^\sharp e_i) B(\xi, \tilde{A}_\xi e_i)\\
&\quad + \frac{\|\tilde{T}\|^2}{p}\sum_{i=1}^{p} B(\xi, e_i) B(\xi, e_i).
\end{aligned} \tag{42}$$

We have, since $\tilde{T}_\xi^\sharp e_j(t) \in \mathcal{D}(t)$ for all t,

$$\begin{aligned}
\partial_t \tilde{T}_\xi^\sharp e_j(t)|_{t=0} &= \partial_t \sum_{m=1}^{p} (g_t(\tilde{T}_\xi^\sharp e_j(t), e_m(t)) e_m(t))|_{t=0}\\
&= \partial_t \sum_{m=1}^{p} (g_t(\tilde{T}(e_j(t), e_m(t)), \xi) e_m(t))|_{t=0}\\
&= \sum_{m=1}^{p} B(\tilde{T}(e_j, e_m), \xi) e_m\\
&\quad + \sum_{m=1}^{p} g(\partial_t \tilde{T}(e_j(t), e_m(t))|_{t=0}, \xi) e_m
\end{aligned}$$

$$+\sum_{m=1}^{p} g(\tilde{T}(e_j, e_m), \xi)(\partial_t e_m(t))|_{t=0}.$$

For $g^{\perp}$-variations, we have $B(\tilde{T}(e_j, e_m), \xi)e_m = 0$. Also

$$\begin{aligned}
&\sum_{m=1}^{p} g(\tilde{T}(e_j, e_m), \xi)(\partial_t e_m(t))|_{t=0} \\
&= \sum_{m=1}^{p} g(\tilde{T}(e_j, e_m), \xi) \cdot \left(-\frac{1}{2} \sum_{k=1}^{p} B(e_m, e_k)e_k - B(e_m, \xi)\xi \right) \\
&= -\frac{1}{2} \sum_{k,m=1}^{p} B(e_m, e_k) g(\tilde{T}_{\xi}^{\sharp} e_j, e_m) e_k - \sum_{m=1}^{p} B(e_m, \xi) g(\tilde{T}_{\xi}^{\sharp} e_j, e_m)\xi.
\end{aligned}$$

Recall that for $H = 0$, we have

$$\begin{aligned}
g(\partial_t \tilde{T}(e_j(t), e_m(t))|_{t=0}, \xi) &= -\frac{1}{2} \sum_{k=1}^{p} B(e_j, e_k) g(e_k, \tilde{A}_{\xi} e_m) \\
&\quad + \frac{1}{2} \sum_{k=1}^{p} B(e_m, e_k) g(e_k, \tilde{A}_{\xi} e_j) \\
&\quad - \frac{1}{2} (\nabla_{e_j} B)(e_m, \xi) + \frac{1}{2} (\nabla_{e_m} B)(e_j, \xi).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sum_{j=1}^{p} (\partial_t \tilde{T}_{\xi}^{\sharp} e_j)(B(\xi, e_j)) \\
&= \sum_{j,m=1}^{p} \Bigg(-\frac{1}{2} \sum_{k=1}^{p} B(e_j, e_k) g(e_k, \tilde{A}_{\xi} e_m) \\
&\quad + \frac{1}{2} \sum_{k=1}^{p} B(e_m, e_k) g(e_k, \tilde{A}_{\xi} e_j) - \frac{1}{2} (\nabla_{e_j} B)(e_m, \xi) \\
&\quad + \frac{1}{2} (\nabla_{e_m} B)(e_j, \xi) \Bigg) \cdot e_m(B(\xi, e_j)) \\
&\quad - \frac{1}{2} \sum_{j,k,m=1}^{p} B(e_m, e_k) g(\tilde{T}_{\xi}^{\sharp} e_j, e_k) \cdot e_m(B(\xi, e_j)) \\
&\quad - \sum_{j,m=1}^{p} B(e_m, \xi) g(\tilde{T}_{\xi}^{\sharp} e_j, e_m) \cdot \xi(B(\xi, e_j)) \\
&= -\frac{1}{2} \sum_{j,k,m=1}^{p} B(e_j, e_k)(\nabla_{e_m} B)(\xi, e_j) g(e_k, \tilde{A}_{\xi} e_m) \\
&\quad + \frac{1}{2} \sum_{j,k,m=1}^{p} B(e_m, e_k)(\nabla_{e_m} B)(\xi, e_j) g(e_k, \tilde{A}_{\xi} e_j)
\end{aligned}$$

$$-\frac{1}{2}\sum_{j,k,m=1}^{p} B(e_m, e_k)g(\tilde{T}_\xi^\sharp e_j, e_k)(\nabla_{e_m} B)(\xi, e_j)$$
$$-\frac{1}{2}\sum_{i,j,m=1}^{p} B(e_i, e_j)(\nabla_{e_m} B)(e_j, \xi)g(e_i, (\tilde{A}_\xi + \tilde{T}_\xi^\sharp)e_m)$$
$$+\frac{1}{2}\sum_{i,j,m=1}^{p} B(e_i, e_j)(\nabla_{e_j} B)(e_m, \xi)g(e_i, (\tilde{A}_\xi + \tilde{T}_\xi^\sharp)e_m)$$
$$+\frac{1}{2}\sum_{i,j,k,m=1}^{p} B(e_j, e_k)B(e_i, e_j)g(e_k, \tilde{A}_\xi e_m)g(e_i, (\tilde{A}_\xi + \tilde{T}_\xi^\sharp)e_m)$$
$$-\frac{1}{2}\sum_{i,j,k,m=1}^{p} B(e_m, e_k)B(e_i, e_j)g(e_k, \tilde{A}_\xi e_j)g(e_i, (\tilde{A}_\xi + \tilde{T}_\xi^\sharp)e_m)$$
$$+\frac{1}{2}\sum_{i,j,k,m=1}^{p} B(e_m, e_k)B(e_i, e_j)g(\tilde{T}_\xi^\sharp e_j, e_k)g(e_i, (\tilde{A}_\xi + \tilde{T}_\xi^\sharp)e_m)$$
$$+\frac{1}{2}\sum_{j,m=1}^{p} (\nabla_{e_m} B)(e_j, \xi)(\nabla_{e_m} B)(\xi, e_j)$$
$$-\frac{1}{2}\sum_{j,m=1}^{p} (\nabla_{e_j} B)(e_m, \xi)(\nabla_{e_m} B)(\xi, e_j)$$
$$-\sum_{j,m=1}^{p} B(e_m, \xi)(\nabla_\xi B)(\xi, e_j)g(\tilde{T}_\xi^\sharp e_j, e_m).$$

Eventually, we obtain

$$\begin{aligned}
&\sum_{j=1}^{p}(\partial_t \tilde{T}_\xi^\sharp e_j)(B(\xi, e_j)) \\
&= \sum_{i,j=1}^{p}\Big(- B(e_j, \tilde{A}_\xi e_i)(\nabla_{e_i} B)(\xi, e_j) + B(e_i, \tilde{A}_\xi e_j)(\nabla_{e_i} B)(\xi, e_j) \\
&\quad -\frac{1}{2}B(\tilde{T}_\xi^\sharp e_i, e_j)(\nabla_{e_i} B)(e_j, \xi) + \frac{1}{2}B(e_j, \tilde{A}_\xi e_i)B(e_j, \tilde{A}_\xi e_i) \\
&\quad -\frac{1}{2}B(e_i, e_j)B(e_j, \tilde{A}_\xi \tilde{T}_\xi^\sharp e_i) - \frac{1}{2}B(\tilde{A}_\xi e_i, e_j)B(e_i, \tilde{A}_\xi e_j) \\
&\quad -\frac{1}{2}B(\tilde{T}_\xi^\sharp e_i, \tilde{T}_\xi^\sharp e_j)B(e_i, e_j) + \frac{1}{2}(\nabla_{e_i} B)(e_j, \xi)(\nabla_{e_i} B)(\xi, e_j) \\
&\quad -\frac{1}{2}(\nabla_{e_j} B)(e_i, \xi)(\nabla_{e_i} B)(\xi, e_j)\Big) - \sum_{j=1}^{p} B(\tilde{T}_\xi^\sharp e_j, \xi)(\nabla_\xi B)(\xi, e_j).
\end{aligned} \tag{43}$$

Using (41), (42) and (43) in (38), we obtain

$$-2\langle B, D_t((\operatorname{div}\theta)_{|\mathrm{V}(t)})\,|_{t=0}\rangle$$

$$
\begin{aligned}
&= -2\,\mathrm{div}(\sum_{j=1}^{p} B(\xi, e_j)\partial_t \tilde{T}^{\sharp}_{\xi} e_j(t)|_{t=0})\\
&\quad +3\sum_{i=1}^{p} B(\xi, \tilde{A}_{\xi} e_i)B(\xi, \tilde{T}^{\sharp}_{\xi} e_i) + \frac{\|\tilde{T}\|^2(3-p)}{p}\sum_{i=1}^{p} B(\xi, e_i)B(\xi, e_i)\\
&\quad + \sum_{j,k=1}^{p} B(\xi, e_j)B(\xi, e_j)g(\tilde{T}^{\sharp}_{\xi} e_k, \tilde{A}_{\xi} e_k)\\
&\quad - \sum_{j,k=1}^{p} B(\xi, e_j)(\nabla_{e_k} B)(\tilde{T}^{\sharp}_{\xi} e_k, e_j) - \sum_{j,k=1}^{p} B(\xi, e_j)(\nabla_{\tilde{T}^{\sharp}_{\xi} e_j} B)(e_k, e_k)\\
&\quad -2\sum_{i,j=1}^{p} B(e_j, \tilde{A}_{\xi} e_i)(\nabla_{e_i} B)(\xi, e_j) + 2\sum_{i,j=1}^{p} B(e_i, \tilde{A}_{\xi} e_j)(\nabla_{e_i} B)(\xi, e_j)\\
&\quad - \sum_{i,j=1}^{p} B(\tilde{T}^{\sharp}_{\xi} e_i, e_j)(\nabla_{e_i} B)(e_j, \xi) + \sum_{i,j=1}^{p} B(e_j, \tilde{A}_{\xi} e_i)B(e_j, \tilde{A}_{\xi} e_i)\\
&\quad + \sum_{i,j=1}^{p} B(e_j, \tilde{A}_{\xi} e_i)B(e_j, \tilde{T}^{\sharp}_{\xi} e_i) - \sum_{i,j=1}^{p} B(\tilde{A}_{\xi} e_i, e_j)B(e_i, \tilde{A}_{\xi} e_j)\\
&\quad - \sum_{i,j=1}^{p} B(\tilde{T}^{\sharp}_{\xi} e_i, \tilde{T}^{\sharp}_{\xi} e_j)B(e_i, e_j) + \sum_{i,j=1}^{p} (\nabla_{e_i} B)(e_j, \xi)(\nabla_{e_i} B)(\xi, e_j)\\
&\quad - \sum_{i,j=1}^{p} (\nabla_{e_j} B)(e_i, \xi)(\nabla_{e_i} B)(\xi, e_j) - 2\sum_{j=1}^{p} B(\tilde{T}^{\sharp}_{\xi} e_j, \xi)(\nabla_{\xi} B)(\xi, e_j)
\end{aligned}
\tag{44}
$$

We obtain (30) as the sum of (33), (36), (37) and (44), with removed divergences of vector fields with compact supports contained in Ω, such as $\mathrm{div}(B(\xi, e_j)\partial_t \tilde{T}^{\sharp}_{\xi} e_j|_{t=0})$—which can be written in a frame-independent form, albeit only after defining few new tensors—and $\mathrm{div}((\mathrm{Tr}\ B^{\sharp})\langle\tilde{\theta}_{|\mathrm{V}}, B\rangle)$, as these terms vanish after integration over Ω. It is easily seen that changing the frame $\{e_i\}_{i=1}^{p}$ by an orthogonal transformation leaves every term in (30) invariant, and thus, (30) is independent of the choice of an orthonormal basis of $\mathcal{D}$. □

Equation (30) is difficult to analyze in general form. As the existence of a tensor field B satisfying some assumptions on its covariant derivative may depend on a particular manifold, it is also difficult to construct generic variations with prescribed values of $\partial^2_{tt} J_{\Omega}(g_t)$. However, as we show below, (30) can be estimated in some special cases, also interesting from the geometric point of view.

4.2 Adapted variations

Recall that we denote by $\widetilde{\mathcal{D}}$ the one-dimensional distribution spanned by ξ and for a distribution $\mathcal{D}$ on M such that for all $x \in M$ we have $\widetilde{\mathcal{D}}_x \cap \mathcal{D}_x = \{0\}$, we denote by $\mathrm{Riem}(M, \xi, \mathcal{D})$ the space of all Riemannian metrics on M with respect to which $\widetilde{\mathcal{D}}$ and $\mathcal{D}$ are orthogonal.

Proposition 6 *Let g be a metric that is a critical point of the action (1), with respect to volume-preserving $g^{\perp}$-variations, such that $H = 0$. Let $\mathcal{D}$ denote the g-orthogonal complement of*

$\widetilde{\mathcal{D}}$. *If* $\mathcal{D}$ *is non-integrable, then for all adapted variations* $g_t \in \mathrm{Riem}(M,\xi,\mathcal{D})$ *of* g *such that* $\mathrm{Tr}\ B^\sharp = 0$*, we have* $\partial^2_{tt} J_{\Omega,\mathcal{D}}(g_t)|_{t=0} \ge 0$.

Proof Since we consider $J_{\Omega,\mathcal{D}} : \mathrm{Riem}(M,\xi,\mathcal{D}) \to \mathbb{R}$, let g_t be an adapted $g^\perp$-variation. Then, we have $g_t \in \mathrm{Riem}(M,\xi,\mathcal{D})$ for all t and

$$\delta J_{\Omega,\mathcal{D}}(g_t) = 2\widetilde{\mathcal{T}}^\flat + \frac{1}{2}\|\tilde{T}\|^2 g_t^\perp.$$

Suppose that $\mathcal{D}$ is non-integrable and g is a critical point of (1) with $H = 0$. We have by Lemma 6

$$\frac{1}{2}\partial^2_{tt} J_{\Omega,\mathcal{D}}(g_t)|_{t=0} = \int_\Omega \langle D_t(\widetilde{\mathcal{T}}^\flat + \frac{1}{4}\|\tilde{T}\|^2 g_t^\perp)|_{t=0}, B\rangle \ \mathrm{vol}_g|_{t=0}.$$

For an adapted $g^\perp$-variation g_t (i.e., with B restricted to $\mathcal{D}\times\mathcal{D}$), we obtain from (33) and (36)

$$\begin{aligned}
&\langle D_t(\widetilde{\mathcal{T}}^\flat + \frac{1}{4}\|\tilde{T}\|^2 g_t)|_{t=0}, B\rangle \\
&= \sum_{i,j=1}^p \Big(B(e_i,e_j)B(e_j,\tilde{A}_\xi\tilde{T}^\sharp_\xi e_i) - B(\tilde{T}^\sharp_\xi e_i,e_j)(\nabla_{e_j}B)(e_i,\xi) \\
&\quad + B(\tilde{T}^\sharp_\xi e_i,e_j)(\nabla_{e_i}B)(e_j,\xi)\Big) - \frac{1}{2p}\|\tilde{T}\|^2(\mathrm{Tr}\ B^\sharp)^2 \\
&= -\mathrm{Tr}\,(B^\sharp\tilde{T}^\sharp_\xi B^\sharp\tilde{T}^\sharp_\xi) + \frac{1}{p}\|\tilde{T}^\sharp_\xi\|^2\|B^\sharp\|^2 - \frac{1}{2p}\|\tilde{T}^\sharp_\xi\|^2(\mathrm{Tr}\ B^\sharp)^2.
\end{aligned}$$

Indeed, we have

$$\sum_{i,j=1}^p B(e_i,e_j)B(e_j,\tilde{A}_\xi\tilde{T}^\sharp_\xi e_i) = -\sum_{i,j=1}^p B(\tilde{T}^\sharp_\xi e_i,e_j)B(\tilde{A}_\xi e_i,e_j),$$

by (35) we have

$$\begin{aligned}
-\sum_{i,j=1}^p B(\tilde{T}^\sharp_\xi e_i,e_j)(\nabla_{e_j}B)(e_i,\xi) &= -\sum_{i,j=1}^p B(\tilde{T}^\sharp_\xi e_i,e_j)B(e_i,\tilde{A}_\xi e_j) \\
&\quad -\sum_{i,j=1}^p B(\tilde{T}^\sharp_\xi e_i,e_j)B(e_i,\tilde{T}^\sharp_\xi e_j) \\
&= -\sum_{i,j=1}^p B(\tilde{T}^\sharp_\xi e_i,e_j)B(e_i,\tilde{T}^\sharp_\xi e_j),
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i,j=1}^p B(\tilde{T}^\sharp_\xi e_i,e_j)(\nabla_{e_i}B)(e_j,\xi) &= \sum_{i,j=1}^p B(\tilde{T}^\sharp_\xi e_i,e_j)B(\tilde{A}_\xi e_i,e_j) \\
&\quad + \sum_{i,j=1}^p B(\tilde{T}^\sharp_\xi e_i,e_j)B(\tilde{T}^\sharp_\xi e_i,e_j).
\end{aligned}$$

It follows that

$$
\begin{aligned}
-\sum_{i,j=1}^{p} B(\tilde{T}^{\sharp}_{\xi}e_i, e_j)B(e_i, \tilde{T}^{\sharp}_{\xi}e_j) &= -\sum_{i=1}^{p} g(B^{\sharp}\tilde{T}^{\sharp}_{\xi}B^{\sharp}\tilde{T}^{\sharp}_{\xi}e_i, e_i)\\
&= -\operatorname{Tr}(B^{\sharp}\tilde{T}^{\sharp}_{\xi}B^{\sharp}\tilde{T}^{\sharp}_{\xi})
\end{aligned}
$$

and using (11) we obtain

$$
\begin{aligned}
\sum_{i,j=1}^{p} B(\tilde{T}^{\sharp}_{\xi}e_i, e_j)B(\tilde{T}^{\sharp}_{\xi}e_i, e_j) &= \sum_{i,j=1}^{p} -g(\tilde{T}^{\sharp}_{\xi}\tilde{T}^{\sharp}_{\xi}e_i, e_j)g(B^{\sharp}e_i, B^{\sharp}e_j)\\
&= \frac{1}{p}\|\tilde{T}^{\sharp}_{\xi}\|^2\|B^{\sharp}\|^2.
\end{aligned}
$$

Hence, for adapted $g^{\perp}$-variations,

$$
\begin{aligned}
\frac{1}{2}\partial^2_{tt}J_{\Omega,\mathcal{D}}(g_t)|_{t=0} = \int_{\Omega}\Big(&-\operatorname{Tr}(B^{\sharp}\tilde{T}^{\sharp}_{\xi}B^{\sharp}\tilde{T}^{\sharp}_{\xi}) + \frac{1}{p}\|\tilde{T}^{\sharp}_{\xi}\|^2\|B^{\sharp}\|^2\\
&-\frac{1}{2p}\|\tilde{T}^{\sharp}_{\xi}\|^2(\operatorname{Tr} B^{\sharp})^2\Big)\,\mathrm{vol}_g\,.
\end{aligned}
\tag{45}
$$

Since $(\tilde{T}^{\sharp}_{\xi})^2 = -(\|\tilde{T}^{\sharp}_{\xi}\|^2/p)\,\mathrm{Id}_{|\mathcal{D}}$ and $\tilde{T}^{\sharp}_{\xi}$ is antisymmetric, we can define an antisymmetric mapping U of $\mathcal{D}$, preserving norms of vectors from $\mathcal{D}$, by the formula $U = (\sqrt{p}/\|\tilde{T}^{\sharp}_{\xi}\|)\tilde{T}^{\sharp}_{\xi}$. Then

$$
-\operatorname{Tr}(B^{\sharp}\tilde{T}^{\sharp}_{\xi}B^{\sharp}\tilde{T}^{\sharp}_{\xi}) + \frac{1}{p}\|\tilde{T}^{\sharp}_{\xi}\|^2\|B^{\sharp}\|^2 = (\|\tilde{T}^{\sharp}_{\xi}\|^2/p)(-\operatorname{Tr}(B^{\sharp}UB^{\sharp}U) + \|B^{\sharp}\|^2).
$$

We estimate the above expression at a point $x \in M$. Let $(\cdot|\cdot)$ be the inner product on the space $\mathcal{L}(D_x)$ of linear operators on $\mathcal{D}_x$, defined for all $S_1, S_2 \in \mathcal{L}(D_x)$ by $(S_1|S_2) = \operatorname{Tr}(S_1S_2^T)$, with transpose $()^T$ defined by the Riemannian metric g on T_xM. For all $S \in \mathcal{L}(D_x)$, we define $\Phi(S) = (USU^T)^T$, Φ is an isometry of $\mathcal{L}(D_x)$ with respect to the product $(\cdot|\cdot)$, as we have

$$
\begin{aligned}
\|\Phi(S)\|^2 &= \operatorname{Tr}(\Phi(S)\,(\Phi(S))^T) = \operatorname{Tr}(USU^T(USU^T)^T)\\
&= \operatorname{Tr}(USU^TUS^TU^T) = \operatorname{Tr}(USS^TU^{-1}) = \operatorname{Tr}(SS^T) = \|S\|^2,
\end{aligned}
$$

where we used $U^T = U^{-1}$. We have

$$
-\operatorname{Tr}(B^{\sharp}UB^{\sharp}U) = (B^{\sharp}|(UB^{\sharp}U^T)^T) = (B^{\sharp}|\Phi(B^{\sharp})),
$$

from the Schwarz inequality it follows that

$$
|(B^{\sharp}|\Phi(B^{\sharp}))| \le \|B^{\sharp}\|\|\Phi(B^{\sharp})\| = \|B^{\sharp}\|^2
$$

and hence

$$
-\operatorname{Tr}(B^{\sharp}\tilde{T}^{\sharp}_{\xi}B^{\sharp}\tilde{T}^{\sharp}_{\xi}) + \frac{1}{p}\|\tilde{T}^{\sharp}_{\xi}\|^2\|B^{\sharp}\|^2 \ge 0
$$

for all symmetric $B^{\sharp} : T_xM \to T_xM$. □

Remark 2 We note that according to (4), $\operatorname{Tr} B^{\sharp} = 0$ holds for all variations g_t that preserve the volume form vol_g.

Example 2 For dim $M = 3$, and similarly for higher dimensions, we can explicitly construct B restricted to $\mathcal{D} \times \mathcal{D}$ for which $\partial^2_{tt} J_{\Omega,\mathcal{D}}(g_t)|_{t=0} = 0$.

Let W be an open set in M with local orthonormal frame $\{\xi, e_1, e_2\}$, such that the matrix of the map $\tilde{T}^\sharp_\xi$ in this frame has the following form

$$\mathfrak{T} = \frac{\|\tilde{T}\|}{\sqrt{p}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \tag{46}$$

Taking a (0, 2)-tensor field B such that the matrix representing $B^\sharp$ in considered frame at all $x \in W$ is

$$\mathfrak{B}(x) = b(x) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

where $b(x)$ is a smooth function with non-empty compact support in W, we obtain on the set W

$$-\operatorname{Tr}(B^\sharp \tilde{T}^\sharp_\xi B^\sharp \tilde{T}^\sharp_\xi) + \frac{1}{p}\|\tilde{T}^\sharp_\xi\|^2 \|B^\sharp\|^2 - \frac{1}{2p}\|\tilde{T}^\sharp_\xi\|^2 (\operatorname{Tr} B^\sharp)^2 = 0$$

and such tensor field B (extended by zero to $M \setminus W$) satisfies the constraint $\int_\Omega (\operatorname{Tr} B^\sharp) \operatorname{vol}_g = 0$, and hence can define a volume-preserving $g^\perp$-variation.

Proposition 7 *Let $p = 2$ and let g be a metric that is a critical point of the action (1), with respect to volume-preserving $g^\perp$-variations. Let $\mathcal{D}$ denote the g-orthogonal complement of $\widetilde{\mathcal{D}}$. If $\mathcal{D}$ is non-integrable, then for all adapted variations $g_t \in \mathrm{Riem}(M, \xi, \mathcal{D})$ of g we have $\partial^2_{tt} J_{\Omega,\mathcal{D}}(g_t)|_{t=0} \geq 0$ and if $\operatorname{Tr} B^\sharp \neq 0$, we have $\partial^2_{tt} J_{\Omega,\mathcal{D}}(g_t)|_{t=0} > 0$.*

Proof Here we do not need to assume $H = 0$, as for $p = 2$ and non-integrable $\mathcal{D}$ it follows from Proposition 3. Let $x \in M$ be a point where $\tilde{T} \neq 0$. Then, there exists an orthonormal frame $\{\xi, e_1, e_2\}$ at x with respect to which $\tilde{T}^\sharp_\xi$ and B are represented by matrices

$$\mathfrak{T}_2 = \frac{\|\tilde{T}^\sharp_\xi\|}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad \mathfrak{B}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b_{22} & b_{23} \\ 0 & b_{23} & b_{33} \end{pmatrix} \tag{47}$$

and for the integrand in (45), we obtain

$$\begin{aligned} &-\operatorname{Tr}(B^\sharp \tilde{T}^\sharp_\xi B^\sharp \tilde{T}^\sharp_\xi) + \frac{1}{2}\|\tilde{T}^\sharp_\xi\|^2 \|B^\sharp\|^2 - \frac{1}{4}\|\tilde{T}^\sharp_\xi\|^2 (\operatorname{Tr} B^\sharp)^2 \\ &= \frac{\|\tilde{T}^\sharp_\xi\|^2}{2}(-\operatorname{Tr}(\mathfrak{B}_2\mathfrak{T}_2\mathfrak{B}_2\mathfrak{T}_2) + \|\mathfrak{B}_2\|^2 - \frac{1}{2}(\operatorname{Tr}\mathfrak{B}_2)^2) \\ &= \frac{\|\tilde{T}^\sharp_\xi\|^2}{4}(b_{22}^2 + 2b_{22}b_{33} + b_{33}^2) \geq 0. \end{aligned}$$

□

Proposition 8 *Let $p = 4$ and let g be a metric that is a critical point of the action (1), with respect to volume-preserving $g^\perp$-variations, such that $H = 0$. Let $\mathcal{D}$ denote the g-orthogonal* *complement of $\widetilde{\mathcal{D}}$. If $\mathcal{D}$ is non-integrable, then for all adapted variations $g_t \in \mathrm{Riem}(M, \xi, \mathcal{D})$ of g we have $\partial_{tt}^2 J_{\Omega,\mathcal{D}}(g_t)|_{t=0} \geq 0$.*

Proof Let $x \in M$ be a point where $\tilde{T} \neq 0$. Then, there exists an orthonormal frame $\{\xi, e_1, \ldots, e_4\}$ at x, with respect to which matrices of $\tilde{T}^\sharp_\xi$ and B are, respectively,

$$\mathfrak{T}_4 = \frac{\|\tilde{T}^\sharp_\xi\|}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} \tag{48}$$

and a symmetric matrix $B_4 = (b_{ij})_{1\leq i,j\leq 5}$ with $b_{i1} = 0 = b_{1i}$ for all $1 \leq i \leq 5$. For the integrand in (45), we obtain

$$\begin{aligned} &-\mathrm{Tr}\,(B^\sharp \tilde{T}^\sharp_\xi B^\sharp \tilde{T}^\sharp_\xi) + \frac{1}{4}\|\tilde{T}^\sharp_\xi\|^2\|B^\sharp\|^2 - \frac{1}{8}\|\tilde{T}^\sharp_\xi\|^2(\mathrm{Tr}\ B^\sharp)^2\\ &= \frac{\|\tilde{T}^\sharp_\xi\|^2}{4}(-\mathrm{Tr}\,(\mathfrak{B}_4\mathfrak{T}_4\mathfrak{B}_4\mathfrak{T}_4) + \|\mathfrak{B}_4\|^2 - \frac{1}{2}(\mathrm{Tr}\ \mathfrak{B}_4)^2)\\ &= \frac{\|\tilde{T}^\sharp_\xi\|^2}{4}(2(b_{25}-b_{34})^2 + 2(b_{24}+b_{35})^2 + \frac{1}{2}(b_{44}+b_{55}-b_{22}-b_{33})^2) \geq 0. \end{aligned}$$

□

Proposition 9 *Let $p > 4$ and let g be a metric that is a critical point of the action (1), with respect to volume-preserving $g^\perp$-variations, such that $H = 0$. Let $\mathcal{D}$ denote the g-orthogonal complement of $\widetilde{\mathcal{D}}$. If $\mathcal{D}$ is non-integrable, then there exists an adapted variation $g_t \in \mathrm{Riem}(M, \xi, \mathcal{D})$ of g such that we have $\partial_{tt}^2 J_{\Omega,\mathcal{D}}(g_t)|_{t=0} < 0$.*

Proof Taking $B = f g^\perp$ for some function $f \in C^\infty(M)$, where $g^\perp(X, Y) = g(X^\perp, Y^\perp)$, we obtain in (45)

$$\begin{aligned} \frac{1}{2}\partial_{tt}^2 J_{\Omega,\mathcal{D}}(g_t)|_{t=0} &= \int_\Omega (-\mathrm{Tr}\,(B^\sharp \tilde{T}^\sharp_\xi B^\sharp \tilde{T}^\sharp_\xi) + \frac{1}{p}\|\tilde{T}^\sharp_\xi\|^2\|B^\sharp\|^2\\ &\qquad -\frac{1}{2p}\|\tilde{T}^\sharp_\xi\|^2(\mathrm{Tr}\ B^\sharp)^2)\ \mathrm{vol}_g\\ &= \int_\Omega f^2\frac{\|\tilde{T}^\sharp_\xi\|^2}{2}(4-p)\ \mathrm{vol}_g < 0. \end{aligned} \tag{49}$$

□

4.3 Transverse variations

In this part, we consider a complementary case to the adapted variations analyzed above: families of metrics g_t such that $B = \partial_t g_t|_{t=0}$ vanishes on $\mathcal{D} \times \mathcal{D}$.

Definition 2 A family of metrics $\{g_t \in \mathrm{Riem}(M, \xi) : |t| < \epsilon\}$ smoothly depending on the parameter t and such that $g_0 = g$ and $B(X, Y) = 0$ for all $X, Y \in \mathcal{D}$, where $B = \partial_t g_t|_{t=0}$, will be called a $g^\pitchfork$-*variation* of the metric g.

The following proposition gives a geometric interpretation of some $g^\pitchfork$-variations.

Proposition 10 *Let $\pi : (M, g) \to (N, g_N)$ be a Riemannian submersion with fibers being integral curves of ξ. Let g_t, $|t| < \epsilon$, be a $g^\perp$-variation of g. Then, $\pi : (M, g_t) \to (N, g_N)$ is a Riemannian submersion for all $|t| < \epsilon$ if and only if for all $X, Y \in \mathfrak{X}_M$ we have $\partial_t(g_t(X, Y) - g_t(X, \xi)g_t(Y, \xi)) = 0$ for all $|t| < \epsilon$.*

Proof Recall that $\pi : (M, g_t) \to (N, g_N)$ is a Riemannian submersion if and only if for all $x \in M$ for all $X, Y \in \mathcal{D}(t)_x$ we have $g_t(X, Y) = g_N(\pi_* X, \pi_* Y)$ [9]. Since $g_t(\xi, \xi) = 1$ for $g^\perp$-variations, this is equivalent to the following condition for all $X, Y \in \mathfrak{X}_M$

$$
\begin{aligned}
0 &= \partial_t g_N(\pi_* X, \pi_* Y) = \partial_t g_t(X^\perp, Y^\perp) \\
&= \partial_t g_t(X - g_t(X, \xi)\xi, Y - g_t(Y, \xi)\xi) \\
&= \partial_t(g_t(X, Y) - g_t(X, \xi)g_t(Y, \xi)).
\end{aligned} \tag{50}
$$

□

It follows that (50) is equivalent to

$$
B_t(X, Y) = g_t(X, \xi)B_t(\xi, Y) + g_t(Y, \xi)B_t(\xi, X), \tag{51}
$$

which shows that B_t is determined by g_t and $B_t^\sharp \xi$. Also, if $\pi : (M, g_t) \to (N, g_N)$ is a Riemannian submersion for all $|t| < \epsilon$, from (51) evaluated at $t = 0$ it follows that for all $X, Y \in \mathfrak{X}_\mathcal{D}$ we have

$$
B(X, Y) = g(X, \xi)B(\xi, Y) + g(Y, \xi)B(\xi, X) = 0.
$$

Let $\beta(X, Y) = (\nabla_X B)(Y, \xi)$ and $\beta^T(X, Y) = \beta(Y, X)$ for all $X, Y \in \mathfrak{X}_M$.

Proposition 11 *Let g be a metric that is a critical point of the action (1) with respect to volume-preserving $g^\perp$-variations and let g_t be a $g^\pitchfork$-variation of g. Then, at every point $x \in M$, where $H = 0$, (30) takes the following form:*

$$
\begin{aligned}
\frac{1}{2}\delta^2 J_\Omega &= -g([\tilde{A}_\xi, \tilde{T}_\xi^\sharp]B^\sharp\xi, B^\sharp\xi) + \frac{2\|\tilde{T}\|^2}{p} g(B^\sharp\xi, B^\sharp\xi) \\
&\quad + \frac{1}{2}(\langle\beta, \beta\rangle_{|\mathcal{D}\times\mathcal{D}} - \langle\beta^T, \beta\rangle_{|\mathcal{D}\times\mathcal{D}}) + 2g(\tilde{T}_\xi^\sharp B^\sharp\xi, \nabla_\xi B^\sharp\xi).
\end{aligned} \tag{52}
$$

Proof Let g_t be a $g^\pitchfork$-variation of g, and let $\{e_i\}_{i=1}^p$ be a local g-orthonormal frame of $\mathcal{D}$. Since $B(X, Y) = 0$ for all $X, Y \in \mathfrak{X}_\mathcal{D}$, using (40) and (39) we obtain at every point x, where $H = 0$,

$$
\begin{aligned}
\sum_{i,j=1}^p B(\xi, \tilde{T}_\xi^\sharp e_i)(\nabla_{e_i} B)(e_j, e_j) &= -2\sum_{i=1}^p B(\xi, \tilde{T}_\xi^\sharp e_i)B(\xi, \tilde{A}_\xi e_i) \\
&\quad - \frac{2\|\tilde{T}\|^2}{p} \sum_{i=1}^p B(\xi, e_i)B(\xi, e_i)
\end{aligned}
$$

and

$$
\begin{aligned}
-\frac{1}{2}\sum_{i,j=1}^p B(\xi, e_j)(\nabla_{e_i} B)(\tilde{T}_\xi^\sharp e_i, e_j) &= \frac{1}{2}\sum_{i=1}^p B(\xi, \tilde{A}_\xi e_i)B(\tilde{T}_\xi^\sharp e_i, \xi) \\
&\quad + \frac{1}{2}\sum_{i,j=1}^p B(\xi, e_j)B(\xi, e_j)g(\tilde{A}_\xi e_i, \tilde{T}_\xi^\sharp e_i)
\end{aligned}
$$

$$+\frac{(p+1)\|\tilde{T}\|^2}{2p}\sum_{i=1}^{p}B(\xi,e_i)B(\xi,e_i).$$

Hence, for B as above, we obtain from (30)

$$\begin{aligned}\frac{1}{2}\delta^2 J_\Omega &= 2\sum_{k=1}^{p}B(\xi,\tilde{A}_\xi e_k)B(\xi,\tilde{T}^\sharp_\xi e_k)+\frac{2\|\tilde{T}\|^2}{p}\sum_{i=1}^{p}B(\xi,e_i)B(\xi,e_i)\\ &+\frac{1}{2}\sum_{i,j=1}^{p}(\nabla_{e_i}B)(e_j,\xi)(\nabla_{e_i}B)(\xi,e_j)\\ &-\frac{1}{2}\sum_{i,j=1}^{p}(\nabla_{e_j}B)(e_i,\xi)(\nabla_{e_i}B)(\xi,e_j)\\ &+\sum_{i,j=1}^{p}B(\xi,e_j)B(\xi,e_j)g(\tilde{T}^\sharp_\xi e_i,\tilde{A}_\xi e_i)\\ &-2\sum_{k=1}^{p}B(\xi,\tilde{T}^\sharp_\xi e_k)(\nabla_\xi B)(\xi,e_k).\end{aligned}\tag{53}$$

Since $H=\nabla_\xi\xi=0$, we have

$$\begin{aligned}-\sum_{j=1}^{p}B(\xi,\tilde{T}^\sharp_\xi e_j)(\nabla_\xi B)(\xi,e_j) &= \sum_{j=1}^{p}g(\tilde{T}^\sharp_\xi B^\sharp\xi,e_j)g(\nabla_\xi B^\sharp\xi,e_j)\\ &= g(\tilde{T}^\sharp_\xi B^\sharp\xi,\nabla_\xi B^\sharp\xi).\end{aligned}$$

Let $[\tilde{A}_\xi,\tilde{T}^\sharp_\xi]=\tilde{A}_\xi\tilde{T}^\sharp_\xi-\tilde{T}^\sharp_\xi\tilde{A}_\xi$, we have

$$2\sum_{k=1}^{p}B(\xi,\tilde{A}_\xi e_k)B(\xi,\tilde{T}^\sharp_\xi e_k)=-2g(\tilde{A}_\xi B^\sharp\xi,\tilde{T}^\sharp_\xi B^\sharp\xi)=-g([\tilde{A}_\xi,\tilde{T}^\sharp_\xi]B^\sharp\xi,B^\sharp\xi)$$

and

$$\sum_{i,j=1}^{p}B(\xi,e_j)B(\xi,e_j)g(\tilde{T}^\sharp_\xi e_i,\tilde{A}_\xi e_i)=\frac{1}{2}g(B^\sharp\xi,B^\sharp\xi)\,\mathrm{Tr}\,([\tilde{A}_\xi,\tilde{T}^\sharp_\xi])=0.$$

Using the above together with the definition of β in (53), we obtain (52). □

Proposition 12 *Let g be a metric that is a critical point of the action (1) with respect to volume-preserving $g^\perp$-variations. Then, there exists a $g^\pitchfork$-variation g_t of g such that $\partial^2_{tt}J_\Omega(g_t)|_{t=0}>0$.*

Proof Suppose that $\mathcal{D}$ is non-integrable. From Proposition 3, we obtain that $H=0$ on some open subset $M_0\subset M$. In what follows, we shall restrict our considerations to M_0, effectively assuming that $\Omega=M_0$ in (1). We shall find a family of $g^\pitchfork$-variations g_t for which we have $\partial^2_{tt}J_\Omega(g_t)|_{t=0}>0$.

For β defined in Proposition 11, we clearly have $\langle\beta^T,\beta^T\rangle_{|\mathcal{D}\times\mathcal{D}}=\langle\beta,\beta\rangle_{|\mathcal{D}\times\mathcal{D}}$. Thus, from Schwarz inequality we obtain

$$|\langle\beta^T,\beta\rangle_{|\mathcal{D}\times\mathcal{D}}|\leq\|\beta_{|\mathcal{D}\times\mathcal{D}}\|\cdot\|\beta^T_{|\mathcal{D}\times\mathcal{D}}\|=\langle\beta,\beta\rangle_{|\mathcal{D}\times\mathcal{D}}$$

and hence

$$\langle \beta, \beta \rangle_{|\mathcal{D}\times\mathcal{D}} - \langle \beta^T, \beta \rangle_{|\mathcal{D}\times\mathcal{D}} \geq 0. \tag{54}$$

Using (54) in (52), we obtain the following estimate on M_0:

$$\begin{aligned}\frac{1}{2}\delta^2 J_\Omega \geq & -2\|\tilde{A}_\xi\|\|\tilde{T}^\sharp_\xi\| g(B^\sharp\xi, B^\sharp\xi) + \frac{2\|\tilde{T}\|^2}{p} g(B^\sharp\xi, B^\sharp\xi) \\ & +2g(\tilde{T}^\sharp_\xi B^\sharp\xi, \nabla_\xi B^\sharp\xi).\end{aligned} \tag{55}$$

Let $W \subset M_0$ be a compact set where $\tilde{T} \neq 0$ and let $(x_i)_{i=1}^m$ be coordinates on an open set $U \subset W$, such that $\xi = \frac{\partial}{\partial x_1}$. Let $L \subset U$ be a p-dimensional submanifold transverse to the integral curves of ξ and let $\zeta \in \mathfrak{X}_\mathcal{D}$ be a smooth unit vector field defined on L. Let $V \subset U$ be an open set such that there exists a smooth vector field Z_N on V that is the solution of the following equation:

$$\nabla_\xi Z_N = N \cdot \tilde{T}^\sharp_\xi (Z_N)^\perp \tag{56}$$

for some $N > 2p \cdot \max_W(\|\tilde{A}_\xi\|/\|\tilde{T}^\sharp_\xi\|)$, satisfying condition $Z_N|_{L\cap V} = \zeta$. We have $Z_N \in \mathcal{D}$, as

$$\xi(g(\xi, Z_N)) = g(\nabla_\xi Z_N, \xi) + g(\nabla_\xi \xi, Z_N) = N \cdot g(\tilde{T}^\sharp_\xi (Z_N)^\perp, \xi) = 0,$$

where we used $\nabla_\xi \xi = 0$ and $\tilde{T}^\sharp_\xi (Z_N)^\perp \in \mathcal{D}$. Also, $g(Z_N, Z_N) = 1$, as

$$\xi(g(Z_N, Z_N)) = 2g(\nabla_\xi Z_N, Z_N) = 2N \cdot g(\tilde{T}^\sharp_\xi (Z_N)^\perp, Z_N) = 0.$$

One can obtain such solution Z_N by taking at every $q \in L$ an orthonormal frame $\{e_i(q)\}$, $i = 1, \ldots, p$ in $\mathcal{D}$ and its parallel transport along flowlines of ξ. Then, we can write $Z_N = z_{N,i} e_i$ for smooth functions $z_{N,i}$ and solve (56) as a system of ODEs for $z_{N,i}$ with the initial condition $Z_N(q) = \zeta(q)$.

Let $f \geq 0$ be a smooth function on M, with non-empty compact support contained in V. Let S be a symmetric $(0, 2)$-tensor field on M such that on V we have $S(\xi, \xi) = 0$, $S(X, Y) = 0$ for $X, Y \in \mathfrak{X}_\mathcal{D}$ and $S(\xi, X) = g(Z_N, X)$ for all $X \in \mathfrak{X}_\mathcal{D}$. Then, for $B = f \cdot S$ we have

$$\begin{aligned}g(\tilde{T}^\sharp_\xi B^\sharp\xi, \nabla_\xi B^\sharp\xi) &= g(f\tilde{T}^\sharp_\xi Z_N, \nabla_\xi(fZ_N)) \\ &= f^2 g(\tilde{T}^\sharp_\xi Z_N, \nabla_\xi Z_N) + fg(\tilde{T}^\sharp_\xi Z_N, Z_N)\xi(f) \\ &= f^2 g(\tilde{T}^\sharp_\xi Z_N, \nabla_\xi Z_N)\end{aligned}$$

and hence from (55) and (56), we obtain

$$\begin{aligned}\frac{1}{2}\int_\Omega \delta^2 J_\Omega \ \mathrm{vol}_g \geq & \int_V f^2\big(-2\|\tilde{A}_\xi\|\|\tilde{T}^\sharp_\xi\| g(Z_N, Z_N) + \frac{2\|\tilde{T}\|^2}{p} g(Z_N, Z_N) \\ & +2g(\tilde{T}^\sharp_\xi Z_N, \nabla_\xi Z_N)\big) \ \mathrm{vol}_g \\ = & \int_V f^2\big(-2\|\tilde{A}_\xi\|\|\tilde{T}^\sharp_\xi\| + \frac{2\|\tilde{T}\|^2}{p} + 2Ng(\tilde{T}^\sharp_\xi Z_N, \tilde{T}^\sharp_\xi Z_N)\big) \ \mathrm{vol}_g \\ \geq & \int_V \frac{2f^2\|\tilde{T}\|^2}{p} \ \mathrm{vol}_g > 0.\end{aligned}$$

Thus, we obtain a symmetric, traceless (0, 2)-tensor field B on M that gives rise to a volume-preserving variation, e.g., $g_t = g + t \cdot B,\ t \in (-\epsilon, \epsilon)$ for small enough $\epsilon > 0$, such that $\partial^2_{tt} J_\Omega(g_t)|_{t=0} > 0$. □

We note that since its proof is local, Proposition 10 is valid also for Riemannian foliations, as they are locally defined by Riemannian submersions. Thus, using variation of metric constructed in the proof of Proposition 12 we can obtain the following.

Corollary 1 *Let g be the metric of a K-contact structure on (M, ξ) with the Reeb field ξ. Then there exists a volume-preserving $g^\perp$-variation $g_t, t \in (-\epsilon, \epsilon)$ of g such that the flowlines of ξ form a Riemannian foliation on every Riemannian manifold (M, g_t) and for every $0 \neq t \in (-\epsilon, \epsilon)$ we have $J_\Omega(g_t) > J_\Omega(g)$.*

Proof Let $B = f \cdot S$ be the tensor field obtained in the end of the proof of Proposition 12. Let $B_t(\xi, \xi) = 0$ and let $B_t(\xi, X) = B(\xi, X)$ for all $t \in \mathbb{R}$, $X \in \mathfrak{X}_M$. We use (51) to set $B_t(X, Y)$ for all $X, Y \in \mathfrak{X}_M$ as follows:

$$\begin{aligned} B_t(X, Y) = &\left(g(X, \xi) + \int_0^t B_s(X, \xi)\mathrm{d}s\right) B(\xi, Y) \\ &+ \left(g(Y, \xi) + \int_0^t B_s(Y, \xi)\mathrm{d}s\right) B(\xi, X) \\ &= g(X, \xi)B(\xi, X) + g(Y, \xi)B(Y, \xi) + 2tB(X, \xi)B(\xi, Y). \end{aligned}$$

Then, there exists $\epsilon > 0$ and variation g_t such that $g_0 = g$ and $\partial_t g_t = B_t$ for $t \in (-\epsilon, \epsilon)$. By (51), this variation preserves the Riemannian foliation by the flowlines of ξ, and since $B_0 = f \cdot S$, with f and S as in Proposition 12, we have $\partial^2_{tt} J_\Omega(g_t)|_{t=0} > 0$. Using (51) again, for a local orthonormal frame $e_i(t)$ obtained as in Lemma 1, we have

$$B_t(e_i(t), e_i(t)) = 2g_t(e_i(t), \xi)B_t(e_i(t), \xi) = 0,$$

as $g_t(e_i(t), \xi) = 0$. Together with $B_t(\xi, \xi) = 0$, it implies that B_t is traceless for all $t \in (-\epsilon, \epsilon)$, and thus, the variation g_t is volume-preserving. □

On a K-contact manifold, we can estimate (52) for a certain family of $g^\pitchfork$-variations.

Proposition 13 *Let (M, g) be a K-contact manifold, and let $\mathcal{F}$ be the Riemannian foliation with fibers being the integral curves of ξ. Then for all $g^\pitchfork$-variations g_t of g such that $B^\sharp\xi$ is a basic field (i.e., orthogonal to ξ and locally projectable to leaf space $M/\mathcal{F}$) we have $\partial^2_{tt} J_\Omega(g_t)|_{t=0} \geq 0$.*

Proof For K-contact manifolds, we have $\tilde{A}_\xi = 0$ and $(\tilde{T}^\sharp_\xi)^2 = -\mathrm{Id}$. Let $Z = B^\sharp\xi$. Projectability of Z is equivalent to condition $[Z, \xi] \in \widetilde{\mathcal{D}}$ [11]. Then, we have

$$g(\nabla_\xi Z, \tilde{T}^\sharp_\xi Z) = g([\xi, Z], \tilde{T}^\sharp_\xi Z) + g(\nabla_Z \xi, \tilde{T}^\sharp_\xi Z) = -g(\tilde{T}^\sharp_\xi Z, \tilde{T}^\sharp_\xi Z) = -g(Z, Z)$$

and (52) becomes

$$\frac{1}{2}\delta^2 J_\Omega = \frac{1}{2}(\langle \beta, \beta\rangle_{|\mathcal{D}\times\mathcal{D}} - \langle \beta^T, \beta\rangle_{|\mathcal{D}\times\mathcal{D}}) \geq 0. \tag{57}$$

□

Example 3 Let (M, g) be a compact regular K-contact manifold. We construct a $g^{\pitchfork}$-variation g_t of g such that $\partial^2_{tt} J_\Omega(g_t)|_{t=0} = 0$. Let $W \subset M$ be an open set such that $\pi : (W, g) \to (U, g_U)$ is a Riemannian submersion along ξ and $W = \pi^{-1}(U)$ is fibre bundle over U [4]. Let ψ be a closed 1-form on U with compact support. Let Z be the basic field on $\pi^{-1}(U)$ such that $g(Z, V) = \psi(\pi_* V) = g_U(\pi_* Z, \pi_* V)$ for all $V \in TM$. Let $B^\sharp \xi = Z$ and $B(X, Y) = 0$ for $X, Y \in \mathcal{D}$. Using $\nabla_{e_i}\xi \in \mathcal{D}$ for a local orthonormal frame $\{\xi, e_1, \dots, e_p\}$, we get

$$\begin{aligned}\frac{1}{2}(\langle\beta,\beta\rangle_{|\mathcal{D}\times\mathcal{D}} - \langle\beta^T,\beta\rangle_{|\mathcal{D}\times\mathcal{D}}) &= \frac{1}{2}\sum_{i,j=1}^{p}(\nabla_{e_i}B)(e_j,\xi)((\nabla_{e_i}B)(e_j,\xi) - (\nabla_{e_j}B)(e_i,\xi))\\ &= \frac{1}{2}\sum_{i,j=1}^{p}(e_i(B(e_j,\xi)) - B(\nabla_{e_i}e_j,\xi) - B(e_j,\nabla_{e_i}\xi))\\ &\quad\cdot(e_i(B(e_j,\xi)) - B(\nabla_{e_i}e_j,\xi) - B(e_j,\nabla_{e_i}\xi)\\ &\quad - e_j(B(e_i,\xi)) - B(\nabla_{e_j}e_i,\xi) - B(e_i,\nabla_{e_j}\xi))\\ &= \frac{1}{2}\sum_{i,j=1}^{p} g(\nabla_{e_i}B^\sharp\xi, e_j)(g(\nabla_{e_i}B^\sharp\xi, e_j) - g(\nabla_{e_j}B^\sharp\xi, e_i))\end{aligned}$$

and, similarly,

$$\begin{aligned}&\frac{1}{2}(\langle\beta,\beta\rangle_{|\mathcal{D}\times\mathcal{D}} - \langle\beta^T,\beta\rangle_{|\mathcal{D}\times\mathcal{D}})\\ &\quad = -\frac{1}{2}\sum_{i,j=1}^{p} g(\nabla_{e_j}B^\sharp\xi, e_i)(g(\nabla_{e_i}B^\sharp\xi, e_j) - g(\nabla_{e_j}B^\sharp\xi, e_i)).\end{aligned}$$

It follows that

$$\begin{aligned}&\frac{1}{2}(\langle\beta,\beta\rangle_{|\mathcal{D}\times\mathcal{D}} - \langle\beta^T,\beta\rangle_{|\mathcal{D}\times\mathcal{D}})\\ &\quad = \frac{1}{4}\sum_{i,j=1}^{p}(g(\nabla_{e_i}Z, e_j) - g(\nabla_{e_j}Z, e_i))^2.\end{aligned} \tag{58}$$

Then, as $g(\nabla_{e_i}Z, e_j) = g_U(\nabla^U_{\pi_* e_i}\pi_* Z, \pi_* e_j)$, where ∇^U is the Levi-Civita connection on (U, g_U) [9], and $\{\pi_* e_i, i = 1, \dots, p\}$ form a local orthonormal frame on U, we have

$$\begin{aligned}\frac{1}{2}(\langle\beta,\beta\rangle_{|\mathcal{D}\times\mathcal{D}} - \langle\beta^T,\beta\rangle_{|\mathcal{D}\times\mathcal{D}}) &= \frac{1}{4}\sum_{i,j=1}^{p}(g(\nabla^U_{\pi_* e_i}\pi_* Z, \pi_* e_j) - g(\nabla^U_{\pi_* e_j}\pi_* Z, \pi_* e_i))^2\\ &= \frac{1}{4}\sum_{i,j=1}^{p}((\pi_* e_i)(\psi(\pi_* e_j)) - (\pi_* e_j)(\psi(\pi_* e_i))\\ &\quad - \psi([\pi_* e_i, \pi_* e_j]))^2\\ &= \sum_{i,j}^{p}(\mathrm{d}\psi(\pi_* e_i, \pi_* e_j))^2 = \|\pi^*\mathrm{d}\psi\|^2\end{aligned}$$

and from (57) we obtain

$$\partial^2_{tt} J_\Omega(g_t)|_{t=0} = \int_{\pi^{-1}(U)} \|\pi^*\mathrm{d}\psi\|^2 \,\mathrm{vol}_g = 0.$$

The following is an alternative form of Proposition 11 for K-contact metrics.

Proposition 14 *Let g be a K-contact metric and let g_t be a $g^{\pitchfork}$-variation of g. Then,* (30) *takes the following form:*

$$\delta^2 J_\Omega = \langle \beta, \beta \rangle - \langle \beta^T, \beta \rangle - g(\nabla_\xi B^\sharp \xi, \nabla_\xi B^\sharp \xi), \tag{59}$$

where $\beta(X, Y) = (\nabla_X B)(Y, \xi)$ and $\beta^T(X, Y) = \beta(Y, X)$ for all $X, Y \in \mathfrak{X}_M$. Equivalently, we can write (59) *as*

$$\delta^2 J_\Omega = \|\mathrm{d}\omega\|^2 - \|\nabla_\xi \omega\|^2, \tag{60}$$

where ω is the 1-form dual to $B^\sharp \xi$.

Proof We have

$$\langle \beta, \beta \rangle = \langle \beta, \beta \rangle_{|\mathcal{D}\times\mathcal{D}} + \sum_{i=1}^p \beta(e_i, \xi)\beta(e_i, \xi) + \sum_{i=1}^p \beta(\xi, e_i)\beta(\xi, e_i)$$

and

$$\langle \beta, \beta^T \rangle = \langle \beta, \beta^T \rangle_{|\mathcal{D}\times\mathcal{D}} + 2\sum_{i=1}^p \beta(e_i, \xi)\beta(\xi, e_i).$$

From (11), we obtain

$$\begin{aligned}
\sum_{i=1}^p \beta(e_i, \xi)\beta(e_i, \xi) &= \sum_{i=1}^p (\nabla_{e_i} B)(\xi, \xi)(\nabla_{e_i} B)(\xi, \xi) \\
&= 4\sum_{i=1}^p B(\nabla_{e_i}\xi, \xi) B(\nabla_{e_i}\xi, \xi) \\
&= 4\sum_{i=1}^p g(\tilde{T}^\sharp_\xi e_i, B^\sharp \xi) g(\tilde{T}^\sharp_\xi e_i, B^\sharp \xi) \\
&= \frac{4\|\tilde{T}\|^2}{p} g(B^\sharp \xi, B^\sharp \xi).
\end{aligned}$$

We have

$$\begin{aligned}
\sum_{i=1}^p \beta(\xi, e_i)\beta(\xi, e_i) &= \sum_{i=1}^p (\nabla_\xi B)(e_i, \xi)(\nabla_\xi B)(e_i, \xi) \\
&= \sum_{i=1}^p (\xi(B(e_i, \xi)) - B(\nabla_\xi e_i, \xi))^2 \\
&= \sum_{i=1}^p g(e_i, \nabla_\xi B^\sharp \xi)^2 \\
&= g(\nabla_\xi B^\sharp \xi, \nabla_\xi B^\sharp \xi)
\end{aligned}$$

and

$$2\sum_{i=1}^p \beta(e_i, \xi)\beta(\xi, e_i) = 2\sum_{i=1}^p (\nabla_{e_i} B)(\xi, \xi)(\nabla_\xi B)(e_i, \xi)$$

$$= 4\sum_{i=1}^{p} g(\tilde{T}_{\xi}^{\sharp} e_i, B^{\sharp}\xi) g(e_i, \nabla_{\xi} B^{\sharp}\xi)$$
$$= -4g(\tilde{T}_{\xi}^{\sharp} B^{\sharp}\xi, \nabla_{\xi} B^{\sharp}\xi).$$

Using the above and $\tilde{A}_{\xi} = 0$ in (52) completes the proof of (59). Equation (60) can be obtained from (59) by a similar computation as (58). □

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