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Reverse comparison theorems with upper integral Ricci curvature condition

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Abstract
We prove some reverse Laplacian comparison and relative volume comparison results under the situation where one has an integral bound for the part of the Ricci curvature which lies above a prescribed continuous function of the distance parameter. These extend parts of results of Ding (Chin Ann Math Ser B 15(1):35–42, 1994) and Kura (Proc Jpn Acad Ser A Math Sci 78(1):7–9, 2002) from pointwise Ricci curvature to integral Ricci curvature.

Keywords Laplacian comparison · Volume comparison · Integral Ricci curvature · Reverse inequalities

Mathematics Subject Classification 53C20 · 53B20

1 Introduction

In Riemannian geometry, there have been many classical comparison theorems on various corresponding geometric quantities of an n-dimensional Riemannian manifold M and the n-dimensional space form $\mathbb{M}^n_K$ of constant sectional curvature K under the condition $\mathrm{Ric}_M \geq (n-1)K$, such as Laplacian comparison, Bishop–Gromov's relative volume comparison, Myers' diameter comparison, the first eigenvalue comparison, the fundamental group and the first Betti number control. The readers can refer to some nice books [2,8] and survey articles [12,15] and the references therein.

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An approach of generalizing the classical comparison results is replacing the pointwise lower Ricci bound condition by the integral Ricci curvature condition, and a lot of results and their applications have been obtained (cf. [1,5,9–11]).

Let M be an n-dimensional Riemannian manifold. We denote by $\| f \|_{q,\Omega}$ the usual q-norm on a domain $\Omega \subset M$, namely,

$$\| f \|_{q,\Omega} = \left(\int_\Omega |f|^q \right)^{1/q}.$$

For each $x \in M$, let $\underline{\mathrm{Ric}}(x)$ (or $\overline{\mathrm{Ric}}(x)$) be the smallest (or biggest) eigenvalue of the Ricci tensor at x. Denote

$$\mathrm{Ric}_-^K(x) = \max\{0, (n-1)K - \underline{\mathrm{Ric}}(x)\},$$
$$\mathrm{Ric}_+^K(x) = \max\{0, \overline{\mathrm{Ric}}(x) - (n-1)K\}$$

for a real number K. Let $B(x, R) \subset M$ denote the geodesic ball of radius R centered at $x \in M$. Then $\| \mathrm{Ric}_-^K \|_{q,B(x,R)}$ (or $\| \mathrm{Ric}_+^K \|_{q,B(x,R)}$) measures the amount of Ricci curvature lying below (or above) the given bound $(n-1)K$ in $B(x, R)$. It is easy to see that $\| \mathrm{Ric}_-^K \|_{q,B(x,R)}$(or $\| \mathrm{Ric}_+^K \|_{q,B(x,R)}$) $= 0$ if and only if $\mathrm{Ric}^M \geq$ (or $\leq$) $(n-1)K$ in $B(x, R)$.

In general, Laplacian comparison theorem is a foundation of other comparison results such as volume comparison, heat kernel comparison, eigenvalue comparison and so on. Unlike Hessian comparison, there exists no reverse Laplacian comparison without any extra assumptions except that the Ricci curvature has an upper bound. However, if an upper bound of sectional curvature is additionally given, we may have a reverse Laplacian comparison, which is sometimes better than the one obtained directly from the Hessian comparison. For example, Q. Ding proved the following reverse Laplacian comparison theorem between two *Cartan–Hadamard manifolds* (i.e., complete simply-connected Riemannian manifolds of non-positive sectional curvature).

Theorem 1.1 ([4, Theorem 2.1]). *Let M and $\widetilde{M}$ be two n-dimensional Cartan–Hadamard manifolds. Let*

$$\gamma : [0, l] \to M \text{ and } \tilde{\gamma} : [0, l] \to \widetilde{M}$$

be unit-speed geodesics starting from $x = \gamma(0)$ and $\tilde{x} = \tilde{\gamma}(0)$, respectively. Let r, Δ and Ric *be the distance function from x, the Laplacian and the Ricci curvature of M, respectively. We add ˜ to denote the corresponding quantities on $\widetilde{M}$.*

If

$$\mathrm{Ric}(\gamma', \gamma')(t) \leq \frac{1}{n-1} \widetilde{\mathrm{Ric}}(\tilde{\gamma}', \tilde{\gamma}')(t), \quad \forall t \in [0, l],$$

then

$$\Delta r(\gamma(t)) \geq \frac{1}{n-1} \widetilde{\Delta} \tilde{r}(\tilde{\gamma}(t)), \quad \forall t \in (0, l].$$

This reverse Laplacian comparison can be generalized (cf. [13,14] for Finsler manifolds cases). In order to state the next results, we introduce the following settings and notations used throughout this article.

Settings: Let $k, k_1 : [0, \infty) \to \mathbb{R}$ be continuous functions satisfying $k_1(t) \leq (n-1)k(t)$. Let f and f_1 be, respectively, the solutions of

$$\begin{cases} f'' + k(t) f = 0, \ f(t) > 0 & \text{for } 0 < t < l, \\ f(0) = 0, \ f'(0) = 1; \end{cases}$$

and

$$\begin{cases} f_1'' + \left[k_1(t) - (n-2)k(t)\right] f_1 = 0 & \text{for } 0 < t < l, \\ f_1(0) = 0,\ f_1'(0) = 1. \end{cases}$$

Notations:

$$\bar{\omega}(t) = f^{n-2}(t) f_1(t), \quad V(r) = \int_0^r \int_{\mathbb{S}^{n-1}} \bar{\omega}(s) \mathrm{d}\theta_{n-1} \mathrm{d}s. \tag{1.1}$$

Based on the above settings, Kura [6] proved the following theorem.

Theorem 1.2 ([6, Theorem 1]). *Let M be an n-dimensional complete Riemannian manifold. Let $\gamma : [0, l) \to M$ $(0 < l \leq \infty)$ be a unit-speed minimal geodesic with $\gamma(0) = x$. Assume that $f'(t) \geq 0$ on $t \in (0, l)$ and the sectional curvature* sec *of M satisfies*

$$\sec(\gamma'(t), X) \leq k(t), \ \forall t \in (0, l), \forall X \in T_{\gamma(t)} M, X \perp \gamma'(t).$$

If $\mathrm{Ric}(\gamma', \gamma')(t) \leq k_1(t)$ *on $t \in (0, l)$, then we have*

$$\Delta r(\gamma(t)) \geq (n-2)\frac{f'(t)}{f(t)} + \frac{f_1'(t)}{f_1(t)}, \quad \forall t \in (0, l).$$

Remark 1.3 In fact, Theorem 1.2 implies Theorem 1.1. Indeed, if we take $k(t) \equiv 0$, $k_1(t) = \frac{1}{n-1}\widetilde{\mathrm{Ric}}(\tilde{\gamma}', \tilde{\gamma}')(t)$, then we obtain

$$\Delta r(\gamma(t)) \geq (n-2)\frac{1}{t} + \frac{f_1'(t)}{f_1(t)} > \frac{f_1'(t)}{f_1(t)} \geq \frac{1}{n-1}\widetilde{\Delta}\tilde{r}(\tilde{\gamma}(t)).$$

In this paper, we extend the above two theorems to the upper integral Ricci curvature condition. We prove the following theorems.

Theorem 1.4 *Settings and notations as in Theorem 1.1. Denote*

$$\psi(t) = \max\left\{0, \frac{1}{n-1}\widetilde{\Delta}\tilde{r}(\tilde{\gamma}(t)) - \Delta r(\gamma(t))\right\},$$
$$\rho(t) = \max\left\{0, \mathrm{Ric}(\gamma', \gamma')(t) - \frac{1}{n-1}\widetilde{\mathrm{Ric}}(\tilde{\gamma}', \tilde{\gamma}')(t)\right\}.$$

Then for $q \geq 1$, we have

$$\int_0^a \psi^{2q} \omega \, \mathrm{d}t \leq \int_0^a \rho^q \omega \, \mathrm{d}t, \quad \forall a \in [0, l]. \tag{1.2}$$

Here ω is given in (2.5).

Theorem 1.5 *Settings and notations as in Theorem 1.2. Denote*

$$\bar{\psi}(t) = \max\left\{0, (n-2)\frac{f'(t)}{f(t)} + \frac{f_1'(t)}{f_1(t)} - \Delta r(\gamma(t))\right\},$$
$$\bar{\rho}(t) = \max\left\{0, \mathrm{Ric}(\gamma', \gamma')(t) - k_1(t)\right\}.$$

Then for $q \geq \max\{1, \frac{n+1}{4}\}$, we have

$$\int_0^a \bar{\psi}^{2q} \omega \, \mathrm{d}t \leq \int_0^a \bar{\rho}^q \omega \, \mathrm{d}t, \quad \forall a \in [0, l). \tag{1.3}$$

Here ω is given in (2.5).

Remark 1.6 When $\rho(t) \equiv 0$ (or $\bar{\rho}(t) \equiv 0$), Theorem 1.4 (or Theorem 1.5) implies Theorem 1.1 (or Theorem 1.2). But unlike in the pointwise case, Theorem 1.4 doesn't follow directly from Theorem 1.5 because of the difference of the restriction on q. Actually, the left side of (1.2) is not bigger than the left side of (1.3) from Remark 1.3 so the restriction on q can be relaxed, which is reasonable.

As an application of Theorem 1.5, we can prove the following reverse relative volume comparison theorem.

Theorem 1.7 *Settings and notations as in Theorem 1.2 and (1.1). Given $q > n/2$, then there is a constant $C = C(n, q, R)$ such that*

$$\left(\frac{\operatorname{vol} B(x,r)}{V(r)}\right)^{\frac{1}{2q}} - \left(\frac{\operatorname{vol} B(x,R)}{V(R)}\right)^{\frac{1}{2q}} \leq C(n,q,R)\|\bar{\rho}\|_{q,B(x,R)}^{\frac{1}{2}} \tag{1.4}$$

for $r < R < \operatorname{inj}_x$, where inj_x denotes the injectivity radius at x.

In particular, we have

$$\operatorname{vol} B(x,R) \geq \left(1 - C(n,q,R)\|\bar{\rho}\|_{q,B(x,R)}^{\frac{1}{2}}\right)^{2q} V(R). \tag{1.5}$$

Remark 1.8 When $\bar{\rho} \equiv 0$, Theorem 1.7 recovers Theorem 2 in [6].

We point out that the constant $C = C(n, q, R)$ in Theorem 1.7 depends on $k(t)$ and $k_1(t)$ as well, but we omit them in the expression. We would also like to mention that, the integral curvature conditions in Theorems 1.5 and 1.7 can be viewed as so-called integral *radial* (Ricci or sectional) curvatures condition in Mao's recent work [7], where lots of comparison results were obtained.

The paper is organized as follows. In Sect. 2, we give some preliminaries, including basic facts on relations between the Hessian/Laplacian of the distance function and the ordinary differential equations, notations for quantities of space forms and an algebra inequality. In Sect. 3, we prove the Laplacian comparison Theorems 1.4 and 1.5 . In Sect. 4, we prove the relative volume comparison Theorem 1.7.

2 Preliminaries

2.1 Second order ODE and Riccati ODE

We briefly recall some well-known facts on the relations between the Hessian/Laplacian of the distance function starting from a fixed point and the Jacobi fields along the geodesic, which can be easily found in some textbooks or survey articles, e.g., [8].

Let $\{e_i(t)\}_{i=1}^n$ be a parallel orthonormal frame along the unit speed geodesic $\gamma(t)$ such that $e_n(t) = \gamma'(t)$. Let $J_i(t)(1 \leq i \leq n-1)$ be the Jacobi field along $\gamma(t)$ such that $J_i(0) = 0$, $J_i'(t) = e_i(0)$. Denote

$$\begin{pmatrix} J_1 \\ \vdots \\ J_{n-1} \end{pmatrix}(t) = A(t) \begin{pmatrix} e_1 \\ \vdots \\ e_{n-1} \end{pmatrix}(t),$$

then $A(t)$ satisfies the second order ODE

$$\begin{cases} A''(t) + A(t)K(t) = 0; \\ A(0) = 0,\ A'(0) = I_{n-1}, \end{cases} \tag{2.1}$$

where $K(t) = (K_{ij}(t))$ and $K_{ij}(t) = \langle R(e_i, \gamma')\gamma', e_j\rangle(t)(1 \le i, j \le n-1)$.

When $A(t)$ is invertible, denote $U(t) = A^{-1}(t)A'(t), u(t) = \mathrm{tr}U(t)$, then by a direct calculation, we have (cf. [4])

$$\left(\mathrm{Hess}\, r(e_i, e_j)(t)\right)_{1\le i,j\le n-1} = U(t), \quad u(t) = \Delta r(\gamma(t)). \tag{2.2}$$

On the other hand, from (2.1), $U(t)$ and $u(t)$ satisfy the Riccati ODE (cf. [11,12]):

$$\begin{aligned} &U'(t) + U^2(t) + K(t) = 0; \\ &u'(t) + \mathrm{tr}U^2(t) + \mathrm{Ric}(\gamma'(t), \gamma'(t)) = 0. \end{aligned} \tag{2.3}$$

Similarly, for $\widetilde{M}$ we have the Riccati ODE

$$\tilde{u}'(t) + \mathrm{tr}\tilde{U}^2(t) + \widetilde{\mathrm{Ric}}(\tilde{\gamma}'(t), \tilde{\gamma}'(t)) = 0, \tag{2.4}$$

where $\tilde{u} = \mathrm{tr}\tilde{U} = \widetilde{\Delta}\tilde{r}(\tilde{\gamma}(t))$.

2.2 Volume element of space forms

For a fixed point $x \in M$, let $r(y) = d(x, y)$ be the distance function starting from y. Under the geodesic polar coordinate (t, θ) around y, the volume element $\mathrm{d\,vol}$ of M has the following expression:

$$\mathrm{d\,vol} = \omega\, \mathrm{d}t \wedge \mathrm{d}\theta_{n-1}, \tag{2.5}$$

where $\mathrm{d}\theta_{n-1}$ (sometimes we use $\mathrm{d}\theta$ for simplicity) represents the standard volume element on the unit sphere $\mathbb{S}^{n-1}$.

Let $\mathbb{M}^n_K$ denote the n-dimensional (complete, simply-connected) space form of constant curvature K. Then the metric can be written as $g_K = \mathrm{d}t^2 + \mathrm{sn}^2_K(t) g_{\mathbb{S}^{n-1}}$, and the volume element on $\mathbb{M}^n_K$ is given by $\mathrm{d\,vol}_K = \omega_K \mathrm{d}t \wedge \mathrm{d}\theta_{n-1}$ (when $K > 0$, generally $t < \pi/\sqrt{K}$ is required). Here by abuse of notation, we have (cf. [11,12])

$$\omega_K(t) = \omega_K(t, \cdot) = \mathrm{sn}^{n-1}_K(t), \quad \omega'_K = u_K \omega_K,$$

$$u_K(t) = u_K(t, \cdot) = (n-1)\frac{\mathrm{sn}'_K(t)}{\mathrm{sn}_K(t)} = (n-1)\,\mathrm{ctn}_K(t),$$

$$\mathrm{sn}_K(t) = \begin{cases} \frac{1}{\sqrt{K}}\sin(\sqrt{K}t), & \text{for } K > 0; \\ t, & \text{for } K = 0; \\ \frac{1}{\sqrt{-K}}\sinh(\sqrt{-K}t), & \text{for } K < 0. \end{cases} \quad \mathrm{cn}_K(t) = \begin{cases} \cos(\sqrt{K}t), & \text{for } K > 0; \\ 1, & \text{for } K = 0; \\ \cosh(\sqrt{-K}t), & \text{for } K < 0. \end{cases}$$

$$\mathrm{ctn}_K(t) = \frac{\mathrm{cn}_K(t)}{\mathrm{sn}_K(t)} = \begin{cases} \sqrt{K}\cot(\sqrt{K}t), & \text{for } K > 0; \\ 1/t, & \text{for } K = 0; \\ \sqrt{-K}\coth(\sqrt{-K}t), & \text{for } K < 0. \end{cases}$$

According to Sect. 2.1, we have $\Delta^K r_K(t) = u_K(t)$, where $r_K(t)$ means the distance function from any point $x = \gamma(0) \in \mathbb{M}^n_K$ along a unit-speed geodesic $\gamma(t)$ and Δ^K is the Laplacian on $\mathbb{M}^n_K$.

2.3 An algebra lemma

Now we recall the following algebra lemma which will be used later.

Lemma 2.1 ([6, Lemma1]). *Given $a \geq 0, b \geq ma^2$, consider the set*

$$D = \Big\{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid a \leq x_1 \leq \cdots \leq x_m, \sum_{i=1}^{m} x_j^2 \geq b\Big\}$$

and define a function $\Phi : D \to \mathbb{R}$ by $\Phi(x_1, \cdots, x_n) = \sum_{i=1}^{m} x_i$. Then

$$\min \Phi(D) \geq (m-1)a + \{b - (m-1)a^2\}^{1/2}.$$

Remark 2.2 In fact, we just need to consider $\min \Phi(D_b)$ on the compact set

$$D_b = \Big\{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid a \leq x_1 \leq \cdots \leq x_m, \sum_{i=1}^{m} x_j^2 = b\Big\}.$$

So Φ attains its minimum on D_b at either the interior critical points or the boundary point. Intuitively, this lemma says that the minimum of Φ on D_b is attained at the boundary point $x_1 = \cdots = x_{m-1} = a, x_m = \sqrt{b - (m-1)a^2}$.

3 Laplacian comparison

Proof of Theorem 1.4 Since M is a Cartan–Hadamard manifold, by the Hessian comparison theorem, we know that $U(t)$ is well-defined and positive definite for $t \in (0, l]$ (cf. [4] and Theorem 6.4.3 in [8]). So are for $\widetilde{M}$ and $\tilde{U}$.

Since $\mathrm{tr}U^2 \leq (\mathrm{tr}U)^2$ ($U(t)$ is positive definite) and $\mathrm{tr}\tilde{U}^2 \geq \frac{1}{n-1}(\mathrm{tr}\tilde{U})^2$ (Cauchy–Schwarz inequality), from (2.3) and (2.4) we have

$$u'(t) + u^2(t) + \mathrm{Ric}(\gamma'(t), \gamma'(t)) \geq 0,$$
$$\Big(\frac{\tilde{u}}{n-1}\Big)'(t) + \Big(\frac{\tilde{u}}{n-1}\Big)^2(t) + \frac{1}{n-1}\widetilde{\mathrm{Ric}}(\tilde{\gamma}'(t), \tilde{\gamma}'(t)) \leq 0.$$

Hence,

$$\Big(\frac{\tilde{u}}{n-1} - u\Big)'(t) + \Big(\frac{\tilde{u}}{n-1}\Big)^2(t) - u^2(t) \leq \mathrm{Ric}(\gamma'(t), \gamma'(t)) - \frac{1}{n-1}\widetilde{\mathrm{Ric}}(\tilde{\gamma}'(t), \tilde{\gamma}'(t)).$$

By the definitions of ψ, ρ, we have

$$\psi'(t) + \psi^2(t) + 2u(t)\psi(t) \leq \rho(t). \tag{3.1}$$

Inspired by the approach in [11], multiplying by $\psi^{2q-2}\omega$ both sides of (3.1) and then integrating from 0 to a, we obtain

$$\int_0^a \psi'\psi^{2q-2}\omega\,\mathrm{d}t + \int_0^a \psi^{2q}\omega\,\mathrm{d}t + 2\int_0^a u\psi^{2q-1}\omega\,\mathrm{d}t \leq \int_0^a \rho\psi^{2q-2}\omega\,\mathrm{d}t. \tag{3.2}$$

Integration by parts yields

$$\begin{aligned}\int_0^a \psi'\psi^{2q-2}\omega\,\mathrm{d}t &= \frac{1}{2q-1}\psi^{2q-1}\omega\Big|_0^a - \int_0^a \frac{1}{2q-1}\psi^{2q-1}\omega'\,\mathrm{d}t \\ &\geq -\int_0^a \frac{1}{2q-1}\psi^{2q-1}u\omega\,\mathrm{d}t,\end{aligned}$$

where we used the relation $\omega' = u\omega$, $\psi^{2q-1}\big|_{t=a} \geq 0$, $\psi^{2q-1}\big|_{t=0} = 0$. Inserting this into (3.2) we obtain

$$\int_0^a \psi^{2q}\omega\,\mathrm{d}t + \Big(2 - \frac{1}{2q-1}\Big)\int_0^a u\psi^{2q-1}\omega\,\mathrm{d}t \leq \int_0^a \rho\psi^{2q-2}\omega\,\mathrm{d}t. \tag{3.3}$$

When $q \geq 1$, by the Hölder inequality, we derive that

$$\int_0^a \psi^{2q}\omega\,\mathrm{d}t \leq \int_0^a \rho\psi^{2q-2}\omega\,\mathrm{d}t \leq \Big(\int_0^a \rho^q\omega\,\mathrm{d}t\Big)^{1/q}\Big(\int_0^a \psi^{2q}\omega\,\mathrm{d}t\Big)^{1-1/q}.$$

Therefore,

$$\int_0^a \psi^{2q}\omega\,\mathrm{d}t \leq \int_0^a \rho^q\omega\,\mathrm{d}t.$$

□

Corollary 3.1 *Let M be an n-dimensional Cartan–Hadamard manifold. Given $K < 0$, for $q \geq 1$, we have*

$$\|\tilde{\psi}\|_{2q,B(x,R)} \leq \Big(\|\operatorname{Ric}_+^{\frac{K}{n-1}}\|_{q,B(x,R)}\Big)^{1/2}, \tag{3.4}$$

where

$$\tilde{\psi} = \max\Big\{0, \operatorname{ctn}_K(r) - \Delta r\Big\},$$

$r(\cdot) = d(x,\cdot)$ is the distance function from $x \in M$, and Δ is the Laplacian on M.

Proof We use exponential polar coordinate around $x \in M$. Suppose the coordinate of $y \in M$ is (t,θ), then $r(y) = d(x,y) = t$, and (cf. [4])

$$\Delta = \frac{\partial^2}{\partial t^2} + u(t,\theta)\frac{\partial}{\partial t}.$$

By taking $\tilde{M} = \mathbb{M}_K^n$ in Theorem 1.4 and noting that the Laplacian on $\mathbb{M}_K^n$ (cf. Sect. 2.2), we have

$$\begin{aligned}\int_{B(x,R)} \tilde{\psi}^{2q}\,d\,\mathrm{vol} &= \int_{\mathbb{S}^{n-1}}\int_0^R \tilde{\psi}^{2q}\omega\,\mathrm{d}t\,\mathrm{d}\theta \\ &\leq \int_{\mathbb{S}^{n-1}}\int_0^R \big(\operatorname{Ric} - K\big)_+^q\omega\,\mathrm{d}t\,\mathrm{d}\theta \\ &= \int_{\mathbb{S}^{n-1}}\int_0^R (\operatorname{Ric}_+^{\frac{K}{n-1}})^q\omega\,\mathrm{d}t\,\mathrm{d}\theta = \int_{B(x,R)} (\operatorname{Ric}_+^{\frac{K}{n-1}})^q\,d\,\mathrm{vol},\end{aligned}$$

which implies (3.4). □

Proof of Theorem 1.5 The main process of the proof is almost the same as the proof of Theorem 1.4 but some extra steps are needed.

Firstly from Sturm's comparison theorem we have $f_1(t) > 0$ on $(0,l)$. So we can denote $F(t) = f'(t)/f(t)$, $F_1(t) = f_1'(t)/f_1(t)$ for convenience, and it is easily checked that F and F_1 satisfy the following Riccati equations, respectively:

$$F' + F^2 + k(t) = 0, \tag{3.5}$$

$$F_1' + F_1^2 + \big[k_1(t) - (n-2)k(t)\big] = 0. \tag{3.6}$$

By the Hessian comparison, we know that each eigenvalue of $U(t)$ is not less than $F(t)$, so we can apply Lemma 2.1 to the eigenvalues of $U(t)$ by taking $m = n-1, a = F(t) \geq 0, b = \mathrm{tr}U^2(t)$ and then obtain

$$\left[u(t) - (n-2)F(t)\right]^2 \geq \mathrm{tr}U^2(t) - (n-2)F^2(t).$$

So from (2.3) and (3.5) we have

$$v'(t) + v^2 + \mathrm{Ric}(\gamma'(t), \gamma'(t)) - (n-2)k(t) \geq 0,$$

where $v(t) = u(t) - (n-2)F(t)$. Combining this with (3.6) and noting the definitions of $\bar{\psi}$ and $\bar{\rho}$, we have

$$\bar{\psi}'(t) + \bar{\psi}^2(t) + 2v(t)\bar{\psi}(t) \leq \bar{\rho}(t). \tag{3.7}$$

Multiplying by $\psi^{2q-2}\omega$ both sides of (3.7) and then integrating from 0 to a, we obtain

$$\int_0^a \bar{\psi}'\bar{\psi}^{2q-2}\omega\,\mathrm{d}t + \int_0^a \bar{\psi}^{2q}\omega\,\mathrm{d}t + 2\int_0^a v\bar{\psi}^{2q-1}\omega\,\mathrm{d}t \leq \int_0^a \bar{\rho}\bar{\psi}^{2q-2}\omega\,\mathrm{d}t.$$

By using the analogous arguments to (3.3) (we also have $\bar{\psi}^{2q-1}\big|_{t=0} = 0$ from the initial value condition), we obtain

$$\int_0^a \bar{\psi}^{2q}\omega\,\mathrm{d}t + 2\int_0^a v\bar{\psi}^{2q-1}\omega\,\mathrm{d}t - \frac{1}{2q-1}\int_0^a u\bar{\psi}^{2q-1}\omega\,\mathrm{d}t \leq \int_0^a \bar{\rho}\bar{\psi}^{2q-2}\omega\,\mathrm{d}t. \tag{3.8}$$

In order to replace v with u in the second term of the left side of (3.8), we use $v = u - (n-2)F \geq u - \frac{n-2}{n-1}u = \frac{1}{n-1}u$ by the Hessian comparison and then we obtain

$$\int_0^a \bar{\psi}^{2q}\omega\,\mathrm{d}t + \left(\frac{2}{n-1} - \frac{1}{2q-1}\right)\int_0^a u\bar{\psi}^{2q-1}\omega\,\mathrm{d}t \leq \int_0^a \bar{\rho}\bar{\psi}^{2q-2}\omega\,\mathrm{d}t.$$

Now by using the same method of dealing with (3.3), we derive

$$\int_0^a \bar{\psi}^{2q}\omega\,\mathrm{d}t \leq \int_0^a \bar{\rho}^q\omega\,\mathrm{d}t$$

provided $q \geq \max\{1, \frac{n+1}{4}\}$. □

4 Relative volume comparison

In this section, we prove Theorem 1.7. Inspired by the proof of [3, Lemma 2.1], we firstly prove the following

Lemma 4.1 *Settings and notations as in Theorem 1.7, we have*

$$\frac{d}{dr}\frac{\mathrm{vol}\,B(x,r)}{V(r)} \geq -C_1(n,r)\left(\frac{\mathrm{vol}\,B(x,r)}{V(r)}\right)^{1-\frac{1}{2q}}\|\bar{\psi}\|_{2q,B(x,r)}\Big(V(r)\Big)^{-\frac{1}{2q}}. \tag{4.1}$$

Proof Denote $\bar{u}(t) = (n-2)\frac{f'(t)}{f(t)} + \frac{f_1'(t)}{f_1(t)}$ and recall (2.2), (1.1) and Theorem 1.5 for the definitions and properties of $u, \bar{\omega}$ and $\bar{\psi}$, then we have

$$-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\omega(t,\theta)}{\bar{\omega}(t)} \leq (\bar{u} - u)\frac{\omega}{\bar{\omega}} \leq \bar{\psi}\frac{\omega}{\bar{\omega}},$$

which implies that

$$-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\int_{\mathbb{S}^{n-1}}\omega(t,\theta)\,\mathrm{d}\theta}{\int_{\mathbb{S}^{n-1}}\bar{\omega}(t)\,\mathrm{d}\theta}=\frac{1}{\mathrm{vol}(\mathbb{S}^{n-1})}\int_{\mathbb{S}^{n-1}}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\omega}{\bar{\omega}}\,\mathrm{d}\theta\leq\frac{1}{\mathrm{vol}(\mathbb{S}^{n-1})}\int_{\mathbb{S}^{n-1}}\bar{\psi}\frac{\omega}{\bar{\omega}}\,\mathrm{d}\theta.$$

Thus for $t<r$, we have

$$\frac{\int_{\mathbb{S}^{n-1}}\omega(t,\theta)\,\mathrm{d}\theta}{\int_{\mathbb{S}^{n-1}}\bar{\omega}(t)\,\mathrm{d}\theta}-\frac{\int_{\mathbb{S}^{n-1}}\omega(r,\theta)\,\mathrm{d}\theta}{\int_{\mathbb{S}^{n-1}}\bar{\omega}(r)\,\mathrm{d}\theta}\leq\frac{1}{\mathrm{vol}(\mathbb{S}^{n-1})}\int_t^r\int_{\mathbb{S}^{n-1}}\bar{\psi}(s)\frac{\omega(s,\theta)}{\bar{\omega}(s)}\,\mathrm{d}\theta\,\mathrm{d}s,$$

which derives

$$\begin{aligned}Q(t)&:=\Big(\int_{\mathbb{S}^{n-1}}\omega(t,\theta)\,\mathrm{d}\theta\Big)\Big(\int_{\mathbb{S}^{n-1}}\bar{\omega}(r)\,\mathrm{d}\theta\Big)-\Big(\int_{\mathbb{S}^{n-1}}\omega(r,\theta)\,\mathrm{d}\theta\Big)\Big(\int_{\mathbb{S}^{n-1}}\bar{\omega}(t)\,\mathrm{d}\theta\Big)\\&\leq\Big(\int_t^r\frac{1}{\mathrm{vol}(\mathbb{S}^{n-1})\bar{\omega}(s)}\big(\int_{\mathbb{S}^{n-1}}\bar{\psi}(s)\omega(s,\theta)\,\mathrm{d}\theta\big)\,\mathrm{d}s\Big)\Big(\int_{\mathbb{S}^{n-1}}\bar{\omega}(t)\,\mathrm{d}\theta\Big)\Big(\int_{\mathbb{S}^{n-1}}\bar{\omega}(r)\,\mathrm{d}\theta\Big).\end{aligned}$$

By integrating this with respect to t from 0 to r, we have

$$\begin{aligned}\int_0^r Q(t)\,\mathrm{d}t&\leq\Big(\int_{\mathbb{S}^{n-1}}\bar{\omega}(r)\,\mathrm{d}\theta\Big)\int_0^r\Big[\Big(\int_{\mathbb{S}^{n-1}}\bar{\omega}(t)\,\mathrm{d}\theta\Big)\Big(\int_t^r\frac{1}{\mathrm{vol}(\mathbb{S}^{n-1})\bar{\omega}(s)}\big(\int_{\mathbb{S}^{n-1}}\bar{\psi}(s)\omega(s,\theta)\,\mathrm{d}\theta\big)\,\mathrm{d}s\Big)\Big]\,\mathrm{d}t\\&=V'(r)\int_0^r\Big[\int_0^s\Big(\frac{V'(t)}{V'(s)}\big(\int_{\mathbb{S}^{n-1}}\bar{\psi}(s)\omega(s,\theta)\,\mathrm{d}\theta\big)\,\mathrm{d}t\Big)\Big]\,\mathrm{d}s\\&=V'(r)\int_0^r\Big[\frac{V(s)}{V'(s)}\big(\int_{\mathbb{S}^{n-1}}\bar{\psi}(s)\omega(s,\theta)\,\mathrm{d}\theta\big)\Big]\,\mathrm{d}s\\&\leq V'(r)\left(\max_{s\in[0,r]}\frac{V(s)}{V'(s)}\right)\int_0^r\big(\int_{\mathbb{S}^{n-1}}\bar{\psi}(s)\omega(s,\theta)\,\mathrm{d}\theta\big)\,\mathrm{d}s\\&\leq V'(r)\left(\max_{s\in[0,r]}\frac{V(s)}{V'(s)}\right)\cdot\big(\mathrm{vol}\,B(x,r)\big)^{1-\frac{1}{2q}}\cdot\|\bar{\psi}\|_{2q,B(x,r)},\end{aligned}$$

where we used the Fubini's theorem in the first equality and the Hölder inequality in the last inequality. Now we derive that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\frac{\mathrm{vol}\,B(x,r)}{V(r)}&=\frac{\mathrm{d}}{\mathrm{d}r}\frac{\int_0^r\int_{\mathbb{S}^{n-1}}\omega(t,\theta)\,\mathrm{d}\theta\,\mathrm{d}t}{\int_0^r\int_{\mathbb{S}^{n-1}}\bar{\omega}(t)\,\mathrm{d}\theta\,\mathrm{d}t}=\frac{-\int_0^r Q(t)\,\mathrm{d}t}{(V(r))^2}\\&\geq-\frac{V'(r)\left(\max_{s\in[0,r]}\frac{V(s)}{V'(s)}\right)\cdot\big(\mathrm{vol}\,B(x,r)\big)^{1-\frac{1}{2q}}\cdot\|\bar{\psi}\|_{2q,B(x,r)}}{(V(r))^2}\\&\geq-C_1(n,r)\cdot\Big(\frac{\mathrm{vol}\,B(x,r)}{V(r)}\Big)^{1-\frac{1}{2q}}\cdot\|\bar{\psi}\|_{2q,B(x,r)}(V(r))^{-\frac{1}{2q}},\end{aligned}$$

where

$$C_1(n,r)=\frac{V'(r)}{V(r)}\left(\max_{s\in[0,r]}\frac{V(s)}{V'(s)}\right).$$

We remark that $C_1(n,r)\to1$ as $r\to0$. □

Proof of Theorem 1.7 Combining (4.1) with Theorem 1.5, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}\frac{\mathrm{vol}\,B(x,r)}{V(r)}\geq-C_1(n,r)\Big(\frac{\mathrm{vol}\,B(x,r)}{V(r)}\Big)^{1-\frac{1}{2q}}\|\bar{\rho}\|_{q,B(x,r)}^{\frac{1}{2}}\big(V(r)\big)^{-\frac{1}{2q}},$$

then by separation of variables we obtain

$$\Big(\frac{\mathrm{vol}\,B(x,R)}{V(R)}\Big)^{\frac{1}{2q}}-\Big(\frac{\mathrm{vol}\,B(x,r)}{V(r)}\Big)^{\frac{1}{2q}}\geq-\frac{1}{2q}\int_r^R C_1(n,s)\|\bar{\rho}\|_{q,B(x,s)}^{\frac{1}{2}}\big(V(s)\big)^{-\frac{1}{2q}}\,\mathrm{d}s$$

$$\geq -C(n,q,R)\|\bar{\rho}\|_{q,B(x,R)}^{\frac{1}{2}},$$

that is (1.4). Here $C(n,q,R) = \frac{1}{2q}\int_0^R C_1(n,s)\big(V(s)\big)^{-\frac{1}{2q}}\,\mathrm{d}s$ and we remark that the integral indeed converges when $q > n/2$ since the integrand $\approx s^n$ as $s \to 0$.

By letting $r \to 0$ in (1.4) and noticing that $\frac{\operatorname{vol} B(x,r)}{V(r)} \to 1$, we obtain (1.5). □

Remark 4.2 When $k(t)$ and $k_1(t)$ are both constant, one can show that $\max_{s\in[0,r]} \frac{V(s)}{V'(s)} = \frac{V(r)}{V'(r)}$ and then $C_1(n,r) \equiv 1$, so $C(n,q,R)$ is increasing with respect to R (cf. [3]).

Remark 4.3 It is interesting that for the Laplacian comparison Theorem 1.4 and Theorem 1.5, $q \geq 1$ and $q \geq \max\{1, \frac{n+1}{4}\}$ are required, respectively, while $q > \frac{n}{2}$ is required when considering Ric_{-}^{K} (cf. [11]). It is reasonable since the assumptions on curvatures in our theorems are stronger. But for relative volume comparison Theorem 1.7, the range of q is the same as in the Ric_{-}^{K} case because we need the convergence of the integral $\int_0^R C_1(n,s)\big(V(s)\big)^{-\frac{1}{2q}}\,\mathrm{d}s$.

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