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Two-step homogeneous geodesics in pseudo-Riemannian manifolds

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Abstract

Given a homogeneous pseudo-Riemannian space $(G/H, \langle\ ,\ \rangle)$, a geodesic $\gamma : I \to G/H$ is said to be two-step homogeneous if it admits a parametrization $t = \phi(s)$ (s affine parameter) and vectors X, Y in the Lie algebra $\mathfrak{g}$, such that $\gamma(t) = \exp(tX)\exp(tY) \cdot o$, for all $t \in \phi(I)$. As such, two-step homogeneous geodesics are a natural generalization of homogeneous geodesics (i.e., geodesics which are orbits of a one-parameter group of isometries). We obtain characterizations of two-step homogeneous geodesics, both for reductive homogeneous spaces and in the general case, and undertake the study of two-step g.o. spaces, that is, homogeneous pseudo-Riemannian manifolds all of whose geodesics are two-step homogeneous. We also completely determine the left-invariant metrics $\langle\ ,\ \rangle$ on the unimodular Lie group $SL(2,\mathbb{R})$ such that $\big(SL(2,\mathbb{R}), \langle\ ,\ \rangle\big)$ is a two-step g.o. space.

Keywords Homogeneous space · Pseudo-Riemannian manifold · Homogeneous geodesic · Geodesic orbit space · Two-step homogeneous geodesic · Two-step geodesic orbit space · Generalized geodesic lemma · Lorentzian Lie group

Mathematics Subject Classification Primary 53C22 · Secondary 53C30 · 53C50

1 Introduction

Let (M, g) be a homogeneous pseudo-Riemannian manifold. Denoted by $G \subset I_0(M, g)$, a connected Lie group of isometries acting transitively on M determines a corresponding realization of the manifold, given by the pseudo-Riemannian homogeneous space $(G/H, g)$. Here, H denotes the isotropy group at a point $o \in M$, chosen the origin.

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A geodesic $\gamma : I \to G/H$ through o is called *homogeneous* if it is the (reparametrization of an) orbit of a one-parameter subgroup. For homogeneous pseudo-Riemannian manifolds, homogeneous geodesics and related topics have been studied extensively in past years. In particular, a pseudo-Riemannian homogeneous space $(G/H, g)$ is called *g.o. space* if every geodesic in G/H is homogeneous. The terminology was introduced by Kowalski and Vanhecke in [13] for a Riemannian space. For comprehensive reviews and various results on the subject, we refer to [2, 5–7, 9, 15] and references therein.

In the work [3], the first and third author considered a generalization of homogeneous geodesics, namely geodesics of the form

$$\gamma(t) = \exp(tX)\exp(tY)\cdot o, \quad X, Y \in \mathfrak{g}, \tag{1.1}$$

which were called *two-step homogeneous geodesics*. Geodesics of the form (1.1) had previously appeared in semisimple Lie groups G equipped with a metric induced by a Cartan involution of the Lie algebra $\mathfrak{g}$ of G. More specifically, in [19] it was shown that if B is the Killing form and θ is an involution of $\mathfrak{g}$ then the geodesics through $e \in G$, with respect to the metric $\langle X, Y\rangle = -B(X, \theta Y)$, have the form $\gamma(t) = \exp t(-\theta(Z))\exp t(Z + \theta(Z))$, $Z = \dot\gamma(0)$. The above result was generalized for Riemannian homogeneous spaces $(G/H, \langle\ ,\ \rangle)$ in [10]; there it was proven that if the tangent space at o decomposes into the orthogonal sum of two spaces $\mathfrak{m}_1, \mathfrak{m}_2$ such that $\langle [X, Y]_{\mathfrak{m}_2}, Z\rangle + c\langle X, [Z, Y]\rangle = 0$ for $X, Y \in \mathfrak{m}_1$, $Z \in \mathfrak{m}_2$, and under certain algebraic conditions for $\mathfrak{m}_1, \mathfrak{m}_2$, then the geodesics have the form $\exp t(X_1 + cX_2)\exp t(1-c)X_2 \cdot o$. One of those algebraic conditions requires that

$$[\mathfrak{m}_1, \mathfrak{m}_2] \subseteq \mathfrak{m}_1. \tag{1.2}$$

In [3], the first and the third author proved that condition (1.2) is sufficient for a Riemannian homogeneous space to admit two-step homogeneous geodesics. In particular, if $(G/H, \langle\ ,\ \rangle)$ is a compact homogeneous Riemannian space and $\mathfrak{m}_1, \mathfrak{m}_2 \subset T_o(G/H)$ are eigenspaces of the metric endomorphism satisfying (1.2), then any geodesic tangent to $\mathfrak{m}_1 \oplus \mathfrak{m}_2$ is two-step homogeneous ([3], Theorem 2.3). A Riemannian homogeneous space G/H such that any geodesic of G/H passing through the origin is two-step homogeneous is called a *two-step g.o. space*.

As Remark 1.3 shows, form (1.1) is invariant by left translations. The same invariance holds for any curve of the form

$$\gamma(t) = \exp(tX_1)\dots\exp(tX_n)\cdot o, \quad X_1, \dots, X_n \in \mathfrak{g}. \tag{1.3}$$

It is then natural to investigate the cases where geodesics in homogeneous spaces have the general form (1.3).

The aim of the present paper is to initiate a systematic study of two-step homogeneous geodesics and two-step g.o. spaces in the pseudo-Riemannian setting. As it will turn out, the theory is not a direct generalization of the Riemannian case. We start with the following.

Definition 1.1 Let $(G/H, \langle\ ,\ \rangle)$ be a homogeneous pseudo-Riemannian space and consider a point $o \in G/H$. A geodesic $\gamma : I \to G/H$ through o, with an affine parameter s, is called *two-step homogeneous* if there exists a parametrization $t = \phi(s)$ of γ and vectors X, Y in the Lie algebra $\mathfrak{g}$ of G, such that

$$\gamma(t) = \exp(tX)\exp(tY) \cdot o \quad \text{for all } t \in \phi(I),$$

where $\cdot$ denotes the action of G on G/H.

Obviously, setting $X = 0$ or $Y = 0$, a two-step homogeneous geodesic reduces to a homogeneous geodesic. A *g.o. space* ("geodesic orbit space") is a coset representation $(M = G/H, \langle\ ,\ \rangle)$ of a homogeneous pseudo-Riemannian manifold M, so that all geodesics are homogeneous. We extend the concept of g.o. space to the following:

Definition 1.2 A *two-step geodesic orbit space* (or *two-step g.o. space*) is a pseudo-Riemannian homogeneous space $(G/H, \langle\ ,\ \rangle)$ such that every geodesic through a point $o \in G/H$ is two-step homogeneous.

Remark 1.3 Similarly to the case of g.o. spaces, Definition 1.2 is independent of the choice of the point $o \in G/H$. Indeed, if the curve $\gamma : I \to G/H$ with $\gamma(t) = \exp(tX)\exp(tY) \cdot o$ is a geodesic through o and $o' = g \cdot o$ is another point in G/H, then $\tau_g \circ \gamma$ is a geodesic through o', where $\tau_g : G/H \to G/H$ denotes the left translation by g in G/H. Moreover, it satisfies

$$\begin{aligned}(\tau_g \circ \gamma)(t) &= g\exp(tX)\exp(tY) \cdot o = \left(g\exp(tX)g^{-1}\right)\left(g\exp(tY)g^{-1}\right)(g \cdot o)\\ &= \exp(\mathrm{Ad}(g)tX)\exp(\mathrm{Ad}(g)tY)(g \cdot o) = \exp(\widetilde{X})\exp(\widetilde{Y}) \cdot o',\end{aligned}$$

where $\widetilde{X} = \mathrm{Ad}(g)tX$ and $\widetilde{Y} = \mathrm{Ad}(g)tY$. Therefore, $\tau_g \circ \gamma$ is also a two-step homogeneous geodesic.

It is clear that both the notions of g.o. and two-step g.o. spaces are properties of the specific coset representation of the homogeneous pseudo-Riemannian manifold. For this reason, a pseudo-Riemannian homogeneous manifold (M, g) is said to be a *g.o. manifold* (respectively, a *two-step g.o. manifold*) if it admits a coset representation given by a g.o. space (respectively, by a two-step g.o. space). Clearly, not all the representations of a g.o. manifold need to be g.o. spaces, and not all the representations of a two-step g.o. manifold are necessarily two-step g.o..

The paper is organized as follows. In Sect. 2, we provide the appropriate background for homogeneous spaces and reparametrization of geodesics in pseudo-Riemannian homogeneous spaces. The main results of Sect. 3 will provide some criteria to determine whether a geodesic is two-step homogeneous, both for general (not necessarily reductive) homogeneous pseudo-Riemannian spaces, and in the special case of reductive homogeneous pseudo-Riemannian spaces. This leads to an algebraic characterization of two-step homogeneous geodesics, which generalizes the well-known algebraic characterization of homogeneous geodesics for reductive homogeneous spaces (known as "Geodesic Lemma", cf. [11]). Further characterizations of two-step homogeneous geodesics are given in Sect. 4, with particular regard to the case of left-invariant metrics on Lie groups. In Sect. 5 we turn our attention to two-step g.o. spaces and illustrate some ways to construct such examples. Finally, in Sect. 6 we provide some explicit examples of homogeneous pseudo-Riemannian spaces which are two-step g.o. but not g.o. spaces. In particular, we completely determine the left-invariant (Lorentzian and Riemannian) metrics $\langle , \rangle$ on the unimodular Lie group $SL(2, \mathbb{R})$ such that $\left(SL(2, \mathbb{R}), \langle\ ,\ \rangle\right)$ is a two-step g.o. space.

2 Preliminaries

2.1 Invariant metrics and killing vector fields in homogeneous spaces

Consider a homogeneous pseudo-Riemannian manifold $(M = G/H, \langle\ ,\ \rangle)$. Let $\pi : G \to G/H$ denote the projection and $o = \pi(e)$ be the origin of G/H. For $g \in G$, let $\tau_g : G/H \to G/H$ be the *left translation* by g, i.e., $\tau_g(g'H) = (gg')H$. For $g \in G$, denote by $L_g, R_g : G \to G$ the left and the right translations by g and let $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ denote the adjoint representation of G. Recall also the relation $\pi \circ L_g = \tau_g \circ \pi$. For $X \in \mathfrak{g}$ let X^L (resp. X^R) be the left-invariant (resp. right-invariant) vector field in G induced by X. In other words, $X^L_g := (L_g)_*(X)$ and $X^R_g := (R_g)_*(X)$.

A metric $\langle\ ,\rangle$ on G/H is called *G-invariant* if the left translations are isometries of $(G/H, \langle\ ,\rangle)$. The G-invariant metrics on G/H are in one-to-one correspondence with $\mathrm{Ad}(H)$-invariant scalar products in $T_o(G/H)$. Let $\mathfrak{g}$, $\mathfrak{h}$ be the Lie algebras of G, H, respectively. The space G/H is called *reductive* if there exists a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \tag{2.1}$$

such that $\mathrm{Ad}(H)\mathfrak{m} \subseteq \mathfrak{m}$. The decomposition (2.1) is also called reductive. When G/H is reductive, we naturally identify $\mathfrak{m}$ with the tangent space $T_o(G/H) = \pi_*(\mathfrak{g})$, where $\pi_* : \mathfrak{g} \to T_o(G/H)$ is the differential of the projection at e.

For any $W \in \mathfrak{g}$, the correspondence $W^* : G/H \to T(G/H)$, with

$$W^*_{aH} = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \exp(tW)aH = (\pi \circ R_a)_* W, \quad aH \in G/H, \tag{2.2}$$

is a well-defined vector field in G/H which is a Killing vector field for all G-invariant metrics in G/H. Moreover, since π is a submersion, the tangent space of G/H at each point aH is spanned by the vectors W^*_{aH}, $W \in \mathfrak{g}$.

2.2 Reparametrizations of geodesics in homogeneous spaces

Let $\big(M = G/H, \langle\ ,\rangle\big)$ be a homogeneous pseudo-Riemannian manifold with the Levi-Civita connection ∇. A curve $\gamma : J \to M$ is called *a geodesic up to a reparametrization* if its tangent vector field $\dot\gamma$ is parallel along γ, that is

$$\nabla_{\dot\gamma}\dot\gamma = k\dot\gamma, \tag{2.3}$$

where k is a real function of the affine parameter t of γ (see, for example, [8, p. 14]). It is always possible to find a new parameter s for which $k = 0$ along γ, so that the geodesic equation reduces to $\nabla_{\dot\gamma}\dot\gamma = 0$.

Given a curve $\gamma : J \to G/H$, a vector $W \in \mathfrak{g}$ and a real function k, we introduce the function $\mathcal{G}^W_k : J \to \mathbb{R}$ defined by

$$\mathcal{G}^W_k(t) = \langle \nabla_{\dot\gamma}\dot\gamma - k\dot\gamma, W^* \rangle_{\gamma(t)},$$

where $\dot\gamma$ denotes a local extension of the vector field $\dot\gamma(t)$ along γ and W^* is the vector field defined by Eq. (2.2). We have the following.

Proposition 2.1 *Let* $(G/H, \langle\ ,\ \rangle)$ *be a homogeneous pseudo-Riemannian space and let* $\gamma : J \to G/H$ *be a curve. Then,* γ *is a geodesic up to reparametrization if and only if there exists a function* $k : J \to \mathbb{R}$ *such that* $\mathcal{G}_k^W(t) = 0$ *for any* $W \in \mathfrak{g}$ *and for any* $t \in J$.

Proof Using the nondegeneracy of $\langle\ ,\ \rangle$, we have that Eq. (2.3) holds, if and only if there exists a function $k : J \to \mathbb{R}$ such that

$$\langle \nabla_{\dot\gamma}\dot\gamma - k\dot\gamma, V\rangle_{\gamma(t)} = 0, \tag{2.4}$$

for any vector field V in G/H and for any $t \in J$. Since the tangent space of G/H at each point aH is spanned by the vectors W^*_{aH}, $W \in \mathfrak{g}$, we have $T_{\gamma(t)}(G/H) = \operatorname{span}\{(W_1^*)_{\gamma(t)}, \dots, (W_n^*)_{\gamma(t)}\}$ for some $W_i \in \mathfrak{g}$. Write $V_{\gamma(t)} = \sum_{i=1}^n c_i (W_i)^*_{\gamma(t)}$, $c_i \in \mathbb{R}$. Then, $\langle \nabla_{\dot\gamma}\dot\gamma - k\dot\gamma, V\rangle_{\gamma(t)} = \sum_{i=1}^n c_i \langle \nabla_{\dot\gamma}\dot\gamma - k\dot\gamma, (W_i)^*\rangle_{\gamma(t)}$. Hence, if $\langle \nabla_{\dot\gamma}\dot\gamma - k\dot\gamma, W^*\rangle_{\gamma(t)} = 0$ for any $W \in \mathfrak{g}$ then $\langle \nabla_{\dot\gamma}\dot\gamma - k\dot\gamma, V\rangle_{\gamma(t)} = 0$ for any vector field V in G/H. Conversely, if $\langle \nabla_{\dot\gamma}\dot\gamma - k\dot\gamma, V\rangle_{\gamma(t)} = 0$ for any vector field V in G/H, then $\langle \nabla_{\dot\gamma}\dot\gamma - k\dot\gamma, W^*\rangle_{\gamma(t)} = 0$ for any $W \in \mathfrak{g}$. Therefore, it suffices to replace V in (2.4) with any vector field W^*, $W \in \mathfrak{g}$.

We conclude that γ is a geodesic in G/H if and only if there exists a function $k : J \to \mathbb{R}$ such that

$$\mathcal{G}_k^W(t) = \langle \nabla_{\dot\gamma}\dot\gamma - k\dot\gamma, W^*\rangle_{\gamma(t)} = 0,$$

for any $W \in \mathfrak{g}$ and for any $t \in J$. □

3 The generalized geodesic lemma

We start with the following general characterization of two-step homogeneous geodesics.

Theorem 3.1 *Let* $(G/H, \langle\ ,\ \rangle)$ *be a homogeneous pseudo-Riemannian space with the natural projection* $\pi : G \to G/H$, *and let* $o = \pi(e)$ *be the origin in* G/H. *Let* $\gamma : J \to G/H$ *be the curve*

$$\gamma(t) = \pi\big(\exp(tX)\exp(tY)\big), \quad X, Y \in \mathfrak{g}. \tag{3.1}$$

Moreover, let $T : J \to \operatorname{Aut}(\mathfrak{g})$ *be the map*

$$T(t) = \operatorname{Ad}(\exp(-tY)) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\operatorname{ad}^n(-Y). \tag{3.2}$$

Then, γ *is a geodesic up to reparametrization (i.e., a two-step homogeneous geodesic) if and only if there exists a function* $k : J \to \mathbb{R}$, *such that the function* $\mathcal{G}_k^W(t)$ *defined by the formula*

$$\begin{aligned}\mathcal{G}_k^W(t) =& \langle \pi_*(T(t)X + Y), \pi_*([W, T(t)X + Y])\rangle_o + \langle \pi_*(W), \pi_*([T(t)X, Y])\rangle_o \\ &- k(t)\langle \pi_*(W), \pi_*(T(t)X + Y)\rangle_o = 0,\end{aligned} \tag{3.3}$$

for any $W \in \mathfrak{g}$ *and for any* $t \in J$.

To prove Theorem 3.1 we need the following lemma.

Lemma 3.2 *Let $X, Y \in \mathfrak{g}$, let γ be the curve described by (3.1) and let $\alpha : J \to G$ be the curve defined by*

$$\alpha(t) = \exp(tX)\exp(tY).$$

Then, the velocity of γ is given by

$$\dot{\gamma}(t) = (\pi_*)_{\alpha(t)}\left(\left(X^R + Y^L\right)_{\alpha(t)}\right) = ((\tau_{\alpha(t)} \circ \pi)_*)_e(T(t)X + Y). \tag{3.4}$$

Proof We have that $\gamma = \pi \circ \alpha$. Therefore,

$$\begin{aligned}
\dot{\gamma}(t) =& \pi_*(\dot{\alpha}(t)) = \pi_*\left(\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \alpha(t+s)\right) = \pi_*\left(\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \exp(t+s)X\exp(t+s)Y\right)\\
=& \pi_*\left(\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \exp(t+s)X\exp tY + \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \exp tX\exp(t+s)Y\right)\\
=& \pi_*\left(\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \exp sX\exp tX\exp tY + \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \exp tX\exp tY\exp sY\right)\\
=& \pi_*\left(\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \exp sX\alpha(t) + \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \alpha(t)\exp sY\right) = \pi_*\left((R_{\alpha(t)})_*(X) + (L_{\alpha(t)})_*Y\right)\\
=& (\pi_*)_{\alpha(t)}\left(\left(X^R + Y^L\right)_{\alpha(t)}\right),
\end{aligned}$$

which proves the first equality of (3.4). Moreover, it equals

$$\begin{aligned}
(\pi_*)_{\alpha(t)}\left(\left(X^R + Y^L\right)_{\alpha(t)}\right) =& ((\pi \circ L_{\alpha(t)} \circ L_{\alpha(t)^{-1}})_*)_{\alpha(t)}((R_{\alpha(t)})_*(X) + (L_{\alpha(t)})_*Y)\\
=& ((\pi \circ L_{\alpha(t)})_*)_{\alpha(t)}(\mathrm{Ad}(\alpha(t)^{-1})X + Y)\\
=& ((\tau_{\alpha(t)} \circ \pi)_*)_e(\mathrm{Ad}(\alpha(t)^{-1})X + Y)\\
=& ((\tau_{\alpha(t)} \circ \pi)_*)_e(\mathrm{Ad}(\exp(-tY)\exp(-tX))X + Y)\\
=& ((\tau_{\alpha(t)} \circ \pi)_*)_e(\mathrm{Ad}(\exp(-tY))\mathrm{Ad}(\exp(-tX))X + Y)\\
=& ((\tau_{\alpha(t)} \circ \pi)_*)_e(\mathrm{Ad}(\exp(-tY))X + Y)\\
=& ((\tau_{\alpha(t)} \circ \pi)_*)_e(T(t)X + Y),
\end{aligned}$$

which proves the second equality of (3.4). □

We now proceed to the proof of Theorem 3.1.

Proof of Theorem 3.1 Using Koszul formula, we have that

$$\begin{aligned}
\mathcal{G}_k^W(t) =& \langle \nabla_{\dot{\gamma}}\dot{\gamma} - k\dot{\gamma}, W^* \rangle_{\gamma(t)}\\
=& \left(\dot{\gamma}\langle W^*, \dot{\gamma}\rangle + \langle \dot{\gamma}, [W^*, \dot{\gamma}]\rangle - \frac{1}{2}W^*\langle \dot{\gamma}, \dot{\gamma}\rangle - k\langle W^*, \dot{\gamma}\rangle\right)_{\gamma(t)}.
\end{aligned} \tag{3.5}$$

Moreover, using the compatibility of the Levi-Civita connection ∇ with the metric along with its torsion-free property (see [16]), we have the following:

$$W^*\langle\dot\gamma,\dot\gamma\rangle = 2\langle\nabla_{W^*}\dot\gamma,\dot\gamma\rangle \tag{3.6}$$

$$\nabla_{W^*}\dot\gamma - \nabla_{\dot\gamma}W^* = [W^*,\dot\gamma]. \tag{3.7}$$

Furthermore, since W is a Killing vector field we have that

$$\langle\nabla_{\dot\gamma}W^*,\dot\gamma\rangle = 0 \tag{3.8}$$

(see [16]). By taking into account Eqs. (3.6)–(3.8), we see that $\langle\dot\gamma,[W^*,\dot\gamma]\rangle - \frac{1}{2}W^*\langle\dot\gamma,\dot\gamma\rangle = 0$ and thus Eq. (3.5) is equivalent to

$$\mathcal{G}_k^W(t) = (\dot\gamma\langle W^*,\dot\gamma\rangle)_{\gamma(t)} - k(t)\langle W^*,\dot\gamma\rangle_{\gamma(t)}. \tag{3.9}$$

We will describe explicitly each term of the right hand side of Eq. (3.9). Using Eq. (2.2), the G-invariance of the metric as well as Lemma 3.2, the first term of the right-hand side of Eq. (3.9) becomes

$$\begin{aligned}
(\dot\gamma\langle W^*,\dot\gamma\rangle)_{\gamma(t)} &= \dot\gamma_{\gamma(t)}\langle W^*,\dot\gamma\rangle_{\gamma(t)} = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\langle W^*_{\gamma(t+s)},\dot\gamma_{\gamma(t+s)}\rangle_{\gamma(t+s)}\\
&= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\Big\langle(\pi\circ R_{\alpha(t+s)})_*W,(\tau_{\alpha(t+s)}\circ\pi)_*(T(t+s)X+Y)\Big\rangle_{\gamma_{(t+s)}}\\
&= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\Big\langle(\pi\circ L_{\alpha(t+s)}\circ L_{\alpha^{-1}(t+s)}\circ R_{\alpha(t+s)})_*W,(\tau_{\alpha(t+s)}\circ\pi)_*(T(t+s)X+Y)\Big\rangle_{\gamma_{(t+s)}}\\
&= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\Big\langle(\tau_{\alpha(t+s)}\circ\pi)_*\big(\mathrm{Ad}(\alpha^{-1}(t+s))W\big),(\tau_{\alpha(t+s)}\circ\pi)_*(T(t+s)X+Y)\Big\rangle_{\gamma_{(t+s)}}\\
&= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\Big\langle\pi_*(\mathrm{Ad}(\alpha^{-1}(t+s))W),\pi_*(T(s)T(t)X+Y)\Big\rangle_o,
\end{aligned} \tag{3.10}$$

By Lemma 3.2, we obtain $\dot\alpha(t) = (L_{\alpha(t)})_*(T(t)X+Y)$ [one can see this for example by assuming that γ is a curve in G by setting $\pi_* := \mathrm{id}$ in Eq. (3.4)]. So by differentiating the relation $\alpha^{-1}(t)\alpha(t) = e$ we have $(R_{\alpha(t)})_*\dot\alpha^{-1}(t) = -(L_{\alpha^{-1}(t)})_*\dot\alpha(t) = -(T(t)X+Y)$ (here by $\dot\alpha^{-1}(t)$ we denote the quantity $\frac{\mathrm{d}}{\mathrm{d}t}\alpha^{-1}(t)$). Along with the fact that $\mathrm{Ad}: G\to\mathrm{Aut}(\mathfrak{g})$ is a homomorphism and by setting

$$\widetilde{W} := \mathrm{Ad}(\alpha^{-1}(t))W,$$

we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\mathrm{Ad}(\alpha^{-1}(t+s))W &= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\mathrm{Ad}(\alpha^{-1}(t+s)\alpha(t))\mathrm{Ad}(\alpha^{-1}(t))W\\
&=(\mathrm{Ad}_*)_e\left(\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\alpha^{-1}(t+s)\alpha(t)\right)\mathrm{Ad}(\alpha^{-1}(t))W\\
&=(\mathrm{Ad}_*)_e\big((R_{\alpha(t)})_*\dot\alpha^{-1}(t)\big)\widetilde{W}\\
&=(\mathrm{Ad}_*)_e\big(-(L_{\alpha^{-1}(t)})_*\dot\alpha(t)\big)\widetilde{W}\\
&=(\mathrm{Ad}_*)_e(-(T(t)X+Y))\widetilde{W} = [\widetilde{W},T(t)X+Y].
\end{aligned} \tag{3.11}$$

Moreover, by taking into account Eq. (3.2) we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \pi_*(T(s)T(t)X + Y) = \pi_*\left(\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \mathrm{Ad}(\exp(-sY))T(t)X\right) = [T(t)X, Y]. \quad (3.12)$$

Using Eqs. (3.11) and (3.12), Eq. (3.10) implies that the first term of the right-hand side Eq. (3.9) becomes

$$(\dot\gamma\langle W^*, \dot\gamma\rangle)_{\gamma(t)} = \langle \pi_*([\widetilde{W}, T(t)X + Y]), \pi_*(T(t)X + Y)\rangle_o + \langle \pi_*(\widetilde{W}), \pi_*([T(t)X, Y])\rangle_o. \quad (3.13)$$

Finally, the second term at the right-hand side of Eq. (3.9) becomes

$$\begin{aligned} -k(t)\langle W^*_{\gamma(t)}, \dot\gamma(t)\rangle_{\gamma(t)} &= - k(t)\langle (\tau_{\alpha(t)}\circ\pi)_*(\mathrm{Ad}(\alpha^{-1}(t))W), (\tau_{\alpha(t)}\circ\pi)_*(T(t)X + Y)\rangle_{\gamma(t)} \\ &= - k(t)\langle \pi_*(\widetilde{W}), \pi_*(T(t)X + Y)\rangle_o. \end{aligned} \quad (3.14)$$

We substitute (3.13) and (3.14) into (3.9) to obtain that γ is a geodesic if and only if there exists a function $k : J \to \mathbb{R}$ such that

$$\begin{aligned} \mathcal{G}_k^{\widetilde{W}}(t) =& \langle \pi_*(T(t)X + Y), \pi_*([\widetilde{W}, T(t)X + Y])\rangle_o + \langle \pi_*(\widetilde{W}), \pi_*([T(t)X, Y])\rangle_o \\ & - k(t)\langle \pi_*(\widetilde{W}), \pi_*(T(t)X + Y)\rangle_o = 0, \end{aligned}$$

for any $t \in J$ and for any $\widetilde{W} = \mathrm{Ad}(\alpha^{-1}(t))W$, $W \in \mathfrak{g}$. But $\mathrm{Ad}(\alpha^{-1}(t))$ is an automorphism of $\mathfrak{g}$ and thus we may substitute "for any $\widetilde{W} \in \mathfrak{g}$" with "for any $W \in \mathfrak{g}$", which concludes the proof of the theorem. □

For the rest of this paper, we will use the notation $\langle\ ,\ \rangle$ to denote both the metric on G/H and the corresponding inner product on $T_o(G/H)$. For the reductive case, Theorem 3.1 is simplified in the following way.

Corollary 3.3 (Generalized Geodesic Lemma) *Let $(G/H, \langle\ ,\ \rangle)$ be a pseudo-Riemannian reductive homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Then, the curve γ in Theorem 3.1 is a geodesic up to reparametrization if and only if there exists a function $k : J \to \mathbb{R}$, such that*

$$\begin{aligned} \mathcal{G}_k^{W}(t) =& \langle (T(t)X + Y)_{\mathfrak{m}}, [W, T(t)X + Y]_{\mathfrak{m}}\rangle + \langle W, [T(t)X, Y]_{\mathfrak{m}}\rangle \\ & - k(t)\langle W, (T(t)X + Y)_{\mathfrak{m}}\rangle = 0, \end{aligned} \quad (3.15)$$

for any $W \in \mathfrak{m}$ and for any $t \in J$.

Proof From the reductive decomposition, the tangent space $T_o(G/H)$ is naturally identified with $\mathfrak{m}$ via the differential π_*. In particular, for $W \in \mathfrak{g}$ the vector $\pi_*(W)$ is identified with $W_{\mathfrak{m}} \in \mathfrak{m}$. Under the above identification, Eq. (3.3) is equivalent to

$$\begin{aligned} \mathcal{G}_k^{W}(t) =& \langle (T(t)X + Y)_{\mathfrak{m}}, [W, T(t)X + Y]_{\mathfrak{m}}\rangle + \langle W_{\mathfrak{m}}, [T(t)X, Y]_{\mathfrak{m}}\rangle \\ & - k(t)\langle W_{\mathfrak{m}}, (T(t)X + Y)_{\mathfrak{m}}\rangle = 0, \end{aligned} \quad (3.16)$$

for any $W \in \mathfrak{g}$ and for any $t \in J$. Moreover, using the $\mathrm{ad}(\mathfrak{h})$-invariance of $\langle\ ,\ \rangle$, for any $Z \in \mathfrak{g}$ and $a \in \mathfrak{h}$ we obtain that

$$\begin{aligned}\langle Z_{\mathfrak{m}}, [a, Z]_{\mathfrak{m}} \rangle =& \langle Z_{\mathfrak{m}}, [a, Z_{\mathfrak{h}} + Z_{\mathfrak{m}}]_{\mathfrak{m}} \rangle = \langle Z_{\mathfrak{m}}, [a, Z_{\mathfrak{m}}]_{\mathfrak{m}} \rangle = \langle Z_{\mathfrak{m}}, [a, Z_{\mathfrak{m}}] \rangle \\ =& - \langle [a, Z_{\mathfrak{m}}], Z_{\mathfrak{m}} \rangle,\end{aligned}$$

and thus $\langle Z_{\mathfrak{m}}, [a, Z]_{\mathfrak{m}} \rangle = 0$. Using the above relation, it follows that

$$\begin{aligned}\langle Z_{\mathfrak{m}}, [W, Z]_{\mathfrak{m}} \rangle =& \langle Z_{\mathfrak{m}}, [W_{\mathfrak{h}} + W_{\mathfrak{m}}, Z]_{\mathfrak{m}} \rangle = \langle Z_{\mathfrak{m}}, [W_{\mathfrak{h}}, Z]_{\mathfrak{m}} \rangle + \langle Z_{\mathfrak{m}}, [W_{\mathfrak{m}}, Z]_{\mathfrak{m}} \rangle \\ =& \langle Z_{\mathfrak{m}}, [W_{\mathfrak{m}}, Z]_{\mathfrak{m}} \rangle, \quad \text{for all } W \in \mathfrak{g}.\end{aligned}$$

For $Z = T(t)X + Y$, the above equation yields

$$\langle (T(t)X + Y)_{\mathfrak{m}}, [W, T(t)X + Y]_{\mathfrak{m}} \rangle = \langle (T(t)X + Y)_{\mathfrak{m}}, [W_{\mathfrak{m}}, T(t)X + Y]_{\mathfrak{m}} \rangle.$$

Substituting the above into the first term of Eq. (3.16), we obtain

$$\begin{aligned}\mathcal{G}_k^W(t) =& \langle (T(t)X + Y)_{\mathfrak{m}}, [W_{\mathfrak{m}}, T(t)X + Y]_{\mathfrak{m}} \rangle + \langle W_{\mathfrak{m}}, [T(t)X, Y]_{\mathfrak{m}} \rangle \\ &- k(t)\langle W_{\mathfrak{m}}, (T(t)X + Y)_{\mathfrak{m}} \rangle = 0,\end{aligned}$$

for any $t \in J$ and $W_{\mathfrak{m}} \in \mathfrak{m}$, $W \in \mathfrak{g}$. Hence, we may assume without any loss of generality that $W \in \mathfrak{m}$, and thus the above equation is equivalent to

$$\begin{aligned}\mathcal{G}_k^W(t) =& \langle (T(t)X + Y)_{\mathfrak{m}}, [W, T(t)X + Y]_{\mathfrak{m}} \rangle + \langle W, [T(t)X, Y]_{\mathfrak{m}} \rangle \\ &- k(t)\langle W, (T(t)X + Y)_{\mathfrak{m}} \rangle = 0,\end{aligned}$$

for any $W \in \mathfrak{m}$ and $t \in J$. □

Remark 3.4 By setting $X = 0$, Eq. (3.15) reduces to

$$\langle Y_{\mathfrak{m}}, [W, Y]_{\mathfrak{m}} \rangle = k(t)\langle W, Y_{\mathfrak{m}} \rangle, \quad \text{for all } W \in \mathfrak{m},\ t \in J.$$

The above equation implies that $k(t)$ is independent of t and so, $k(t) = k$ is a constant. Hence, for $X = 0$, Corollary 3.3 implies that the curve γ with $\gamma(t) = \exp(tY) \cdot o$ is a geodesic up to some parameter if and only if there exists a constant k such that

$$\langle Y_{\mathfrak{m}}, [W, Y]_{\mathfrak{m}} \rangle = k\langle W, Y_{\mathfrak{m}} \rangle \quad \text{for all } W \in \mathfrak{m}.$$

This is exactly the Geodesic Lemma proved in [11]. For this reason, we called Lemma 3.3 “Generalized Geodesic Lemma”.

4 Two-step homogeneous geodesics in pseudo-Riemannian spaces

We shall now obtain various characterizations of two-step homogeneous geodesics in pseudo-Riemannian homogeneous spaces. In particular, we describe such geodesics in pseudo-Riemannian Lie groups. We start with the following.

Proposition 4.1 *Let $(G/H, \langle\ ,\ \rangle)$ be a pseudo-Riemannian space with the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, and let $X, Y \in \mathfrak{g}$ with $Y = a + \widetilde{Y}$, $a \in \mathfrak{h}, \widetilde{Y} \in \mathfrak{m}$. The following properties are equivalent.*

(1) The curve $\gamma : J \to G/H$ with $\gamma(t) = \exp(tX)\exp(tY) \cdot o$ is a geodesic up to reparametrization.

(2) There exists a function $k : J \to \mathbb{R}$ such that

$$\langle (T(t)X + Y)_{\mathfrak{m}}, [W, T(t)X + Y]_{\mathfrak{m}} \rangle + \langle W, [T(t)X, Y]_{\mathfrak{m}} \rangle = k(t)\langle W, (T(t)X + Y)_{\mathfrak{m}} \rangle,$$

for any $W \in \mathfrak{m}$, $t \in J$.

(3) There exists a function $k : J \to \mathbb{R}$ such that

$$\langle (T(t)X)_{\mathfrak{m}} + \widetilde{Y}, [W, T(t)X + \widetilde{Y}]_{\mathfrak{m}} \rangle + \langle W, [T(t)X, \widetilde{Y}]_{\mathfrak{m}} \rangle + \langle W, [a, \widetilde{Y}] \rangle = k(t)\langle W, (T(t)X)_{\mathfrak{m}} + \widetilde{Y} \rangle,$$

for any $W \in \mathfrak{m}$, $t \in J$. Moreover, assume that the following property is satisfied: (P) There exists an Ad-invariant inner product B in $\mathfrak{g}$ such that the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is B-orthogonal and let $\Lambda : \mathfrak{m} \to \mathfrak{m}$ be the symmetric, nondegenerate, $\mathrm{Ad}(H)$-equivariant operator determined by the metric $\langle\ ,\ \rangle$, i.e.,

$$\langle Z, W \rangle = B(\Lambda(Z), W), \quad Z, W \in \mathfrak{m}.$$

Then, any of the above properties (1)-(3) are equivalent to the following.

(4) There exists a function $k : J \to \mathbb{R}$ such that

$$[T(t)X + Y, \Lambda\big((T(t)X + Y)_{\mathfrak{m}}\big)] + \Lambda\big([T(t)X, Y]_{\mathfrak{m}}\big) = k(t)\Lambda\big((T(t)X + Y)_{\mathfrak{m}}\big),$$

for any $t \in J$.

Proof The equivalence of (1) and (2) follows from Theorem 3.1. Setting $Y = a + \widetilde{Y}$, Eq. (3.15) becomes

$$\begin{aligned}\langle (T(t)X)_{\mathfrak{m}} + \widetilde{Y}, [W, T(t)X + \widetilde{Y}]_{\mathfrak{m}} \rangle &+ \langle (T(t)X)_{\mathfrak{m}} + \widetilde{Y}, [W, a]_{\mathfrak{m}} \rangle + \langle W, [T(t)X, a]_{\mathfrak{m}} \rangle \\ &+ \langle W, [T(t)X, \widetilde{Y}]_{\mathfrak{m}} \rangle = k(t)\langle W, (T(t)X)_{\mathfrak{m}} + \widetilde{Y} \rangle,\end{aligned}$$

where $W \in \mathfrak{m}$, $t \in J$. Moreover, using the $\mathrm{Ad}(H)$-invariance of $\langle\ ,\ \rangle$, the second and third terms of the left-hand side of the above equation add to

$$\begin{aligned}\langle (T(t)X)_{\mathfrak{m}} + \widetilde{Y}, [W, a]_{\mathfrak{m}} \rangle + \langle W, [T(t)X, a]_{\mathfrak{m}} \rangle &= \langle (T(t)X)_{\mathfrak{m}} + \widetilde{Y}, [W, a] \rangle + \langle W, [T(t)X, a] \rangle \\ &= \langle (T(t)X)_{\mathfrak{m}} + \widetilde{Y}, [W, a] \rangle - \langle (T(t)X)_{\mathfrak{m}}, [W, a] \rangle \\ &= \langle \widetilde{Y}, [W, a] \rangle = \langle W, [a, \widetilde{Y}] \rangle,\end{aligned}$$

which implies the equivalence of (2) and (3). Finally, we will prove the equivalence of (2) and (4) under the additional assumption (P). Eq. (3.15) is equivalent to

$$\begin{aligned}&B\big(\Lambda\big((T(t)X + Y)_{\mathfrak{m}}\big), [W, T(t)X + Y]_{\mathfrak{m}}\big) + B\big(W, \Lambda\big([T(t)X, Y]_{\mathfrak{m}}\big)\big) \\ &\quad = k(t)B\big(W, \Lambda\big((T(t)X + Y)_{\mathfrak{m}}\big)\big), \\ &\quad \text{or } B\big(\Lambda\big((T(t)X + Y)_{\mathfrak{m}}\big), [W, T(t)X + Y]\big) + B\big(W, \Lambda\big([T(t)X, Y]_{\mathfrak{m}}\big)\big) \\ &\quad = k(t)B\big(W, \Lambda\big((T(t)X + Y)_{\mathfrak{m}}\big)\big),\end{aligned}$$

for any $W \in \mathfrak{m}$, $t \in J$. By the Ad-invariance and the bilinearity of B, the above is in turn equivalent to

$$B\Big([T(t)X+Y,\Lambda\big((T(t)X+Y)_{\mathfrak{m}}\big)]) + \Lambda\big([T(t)X,Y]_{\mathfrak{m}}\big) - k(t)\Lambda\big((T(t)X+Y)_{\mathfrak{m}}\big), W\Big) = 0,$$

for any $W \in \mathfrak{m}$, $t \in J$. Taking into account the B-orthogonality of the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, the above equation is equivalent to

$$[T(t)X+Y,\Lambda\big((T(t)X+Y)_{\mathfrak{m}}\big)] + \Lambda\big([T(t)X,Y]_{\mathfrak{m}}\big) - k(t)\Lambda\big((T(t)X+Y)_{\mathfrak{m}}\big) \in \mathfrak{h}. \quad (4.1)$$

It suffices to show that the left-hand side of Eq. (4.1) is also an element of $\mathfrak{m}$. Indeed, this will imply that the right-hand side of Eq. (4.1) is zero, which will yield the equivalence of (2) and (4). Since $\Lambda(\mathfrak{m}) \subset \mathfrak{m}$, it suffices to show that

$$[T(t)X+Y,\Lambda\big((T(t)X+Y)_{\mathfrak{m}}\big)] \in \mathfrak{m}. \quad (4.2)$$

Indeed, using the symmetry of Λ, the B-orthogonality of $\mathfrak{h}$ and $\mathfrak{m}$, the Ad-invariance of B, the fact that $[\mathfrak{h},\mathfrak{m}] \subset \mathfrak{m}$ as well as the ad($\mathfrak{h}$)-equivariance of Λ, for any $Z \in \mathfrak{g}$ and $b \in \mathfrak{h}$ we have

$$\begin{aligned} B([Z,\Lambda(Z_{\mathfrak{m}})],b) =& B(\Lambda(Z_{\mathfrak{m}}),[b,Z]) = B(\Lambda(Z_{\mathfrak{m}}),[b,Z]_{\mathfrak{m}}) \\ =& B(\Lambda(Z_{\mathfrak{m}}),[b,Z_{\mathfrak{m}}]) = B(Z_{\mathfrak{m}},\Lambda([b,Z_{\mathfrak{m}}])) \\ =& B(Z_{\mathfrak{m}},[b,\Lambda(Z_{\mathfrak{m}})]) = B(Z,[b,\Lambda(Z_{\mathfrak{m}})]) \\ =& -B([Z,\Lambda(Z_{\mathfrak{m}})],b), \end{aligned}$$

which implies that $B([Z,\Lambda(Z_{\mathfrak{m}})],b) = 0$. Therefore,

$$[Z,\Lambda(Z_{\mathfrak{m}})] \in \mathfrak{m}, \quad \text{for any} \quad Z \in \mathfrak{g},$$

which verifies Eq. (4.2) and this concludes the proof of the Proposition. □

By setting $X = 0$ in Proposition 4.1 we obtain the following conditions for homogeneous geodesics (see also [1]).

Corollary 4.2 *Let $(G/H,\langle\ ,\ \rangle)$ be a pseudo-Riemannian space with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, and let $X, Y \in \mathfrak{g}$ with $Y = a + \widetilde{Y}$, $a \in \mathfrak{h}$, $\widetilde{Y} \in \mathfrak{m}$. The following properties are equivalent.*

(1) The curve $\gamma : J \to G/H$ with $\gamma(t) = \exp t(a + \widetilde{Y}) \cdot o$ is a geodesic up to reparametrization.
(2) There exists a constant k, such that $\langle \widetilde{Y}, [W,Y]_{\mathfrak{m}}\rangle = k\langle W, \widetilde{Y}\rangle$, for any $W \in \mathfrak{m}$.
(3) There exists a constant k, such that $\langle \widetilde{Y}, [W,\widetilde{Y}]_{\mathfrak{m}}\rangle + \langle W, [a,\widetilde{Y}]\rangle = k\langle W, \widetilde{Y}\rangle$, for any $W \in \mathfrak{m}$. Moreover, if Property (P) is satisfied, then any of the above properties (1)-(3) are equivalent to the following.
(4) There exists a constant k, such that $[a + \widetilde{Y}, \Lambda(\widetilde{Y})] = k\Lambda(\widetilde{Y})$.

The function $T(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\mathrm{ad}^n(-Y)$ is in general hard to compute. The following proposition simplifies the conditions (2) and (4) of Proposition 4.1 when $H = \{e\}$ and $\mathrm{ad}(Y)$ is skew-symmetric.

Proposition 4.3 *Let $(G,\langle\ ,\ \rangle)$ be a pseudo-Riemannian Lie group, where $\langle\ ,\ \rangle$ is a left-invariant metric. Assume that for some $Y \in \mathfrak{g}$ the endomorphism* $\mathrm{ad}(Y)$ *is skew-symmetric* *with respect to* $\langle\ ,\ \rangle$. *Let k be a real function and* $\gamma(t) = \exp(tX)\exp(tY)$. *Then,* $\mathcal{G}_k^W(t) = 0$ *for all t if and only if* $\mathcal{G}_k^W(0) = 0$. *In particular, the following properties are equivalent.*

(1) The curve $\gamma : J \to G$ with $\gamma(t) = \exp(tX)\exp(tY)$, $X, Y \in \mathfrak{g}$, is a geodesic up to reparametrization.

(2) There exists a constant function $k : J \to \mathbb{R}$ such that

$$\mathcal{G}_k^W(0) = \langle X + Y, [W, X] - kW\rangle = 0,$$

for any $W \in \mathfrak{g}$. *Moreover, assume that there exists an* Ad *-invariant inner product on* $\mathfrak{g}$. *Then, properties* (1) *and* (2) *are equivalent to the following.* (3) *There exists a constant k such that*

$$[X + Y, \Lambda(X + Y))] + \Lambda([X, Y]) = k\Lambda(X + Y). \tag{4.3}$$

Proof We have that $\mathfrak{m} = \mathfrak{g}$. Moreover, since $\mathrm{Ad}(g) \in \mathrm{Aut}(\mathfrak{g})$, $g \in G$, we may replace W by $T(t)W$ in Eq. (3.15). Finally, by the definition of $T(t)$, we have that $T(t)Y = Y$. Taking into account Eqs. (4.3), (3.15) can be rewritten as

$$\begin{aligned}\mathcal{G}_k^W(t) = &\langle T(t)(X + Y), T(t)[W, X + Y]\rangle \\ &+ \langle T(t)W, T(t)[X, Y]\rangle - k(t)\langle T(t)W, T(t)(X + Y)\rangle = 0,\end{aligned} \tag{4.4}$$

for $W \in \mathfrak{g}$ and $t \in J$.

Using the skew-symmetry of $\mathrm{ad}(Y)$ with respect to $\langle\ ,\ \rangle$, for any $Z, Z' \in \mathfrak{g}$ we obtain

$$\begin{aligned}\langle T(t)Z, T(t)Z'\rangle &= \sum_{n=0}^{\infty} \frac{t^n}{n!}\langle \mathrm{ad}^n(-Y)Z, T(t)Z'\rangle = \sum_{n=0}^{\infty} \frac{t^n}{n!}\langle Z, \mathrm{ad}^n(Y)T(t)Z'\rangle \\ &= \left\langle Z, \sum_{n=0}^{\infty} \frac{t^n}{n!}\mathrm{ad}^n(Y)T(t)Z' \right\rangle = \langle Z, T(-t)T(t)Z'\rangle \\ &= \langle Z, T(t)^{-1}T(t)Z'\rangle = \langle Z, Z'\rangle.\end{aligned}$$

Using the above equality, Eq. (4.4) is equivalent to

$$\mathcal{G}_k^W(0) = \langle X + Y, [W, X + Y]\rangle + \langle W, [X, Y]\rangle - k(t)\langle W, X + Y\rangle = 0, \tag{4.5}$$

for any $W \in \mathfrak{g}$ and $t \in J$. We deduce that $k(t) = k(0)$ is a constant, and the equivalence between equations $\mathcal{G}_k^W(t) = 0$ for all t and $\mathcal{G}_k^W(0) = 0$ follows. Moreover, writing $[X, Y] = [X + Y, Y]$ in the second term of the right-hand side of Eq. (4.5) and then using the skew-symmetry of $\mathrm{ad}(Y)$ with respect to $\langle\ ,\ \rangle$, Eq. (4.5) becomes

$$\mathcal{G}_k^W(0) = \langle X + Y, [W, X + Y]\rangle - \langle [W, Y], X + Y\rangle - k\langle W, X + Y\rangle = \langle X + Y, [W, X] - kW\rangle = 0.$$

Therefore, (1) and (2) are equivalent. The equivalence of (2) and (3) is obtained by the equivalence of (2) and (4) in Proposition 4.1, setting $t = 0$. □

5 Two-step g.o. spaces

In the present section, we characterize two-step g.o. spaces and obtain large classes of such spaces. We start with the following characterization.

Proposition 5.1 *A pseudo-Riemannian space $(G/H, \langle\ ,\ \rangle)$ with origin $o = \pi(e) \in G/H$ is a two-step g.o. space if and only if for any $V \in T_o(G/H)$ there exist $X, Y \in \mathfrak{g}$ such that*

(1) $\pi_*(X + Y) = V$ and
(2) there exists a function $k : J \to \mathbb{R}$ such that the curve $\gamma : J \to G/H$, defined by $\gamma(t) = \exp(tX)\exp(tY) \cdot o$, satisfies the equation

$$\mathcal{G}_k^W(t) = \langle \nabla_{\dot\gamma}\dot\gamma - k\dot\gamma, W^* \rangle_{\gamma(t)} = 0,$$

for all $W \in \mathfrak{g}$ and $t \in J$. In particular, if $(G/H, \langle\ ,\ \rangle)$ is reductive and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a reductive decomposition of its Lie algebra, then the space is two-step g.o. if and only if for any $V \in \mathfrak{m}$ there exist $X, Y \in \mathfrak{g}$ such that

(1)′ $(X + Y)_{\mathfrak{m}} = V$ and
(2)′ one of the equivalent conditions (1)-(4) of Proposition 4.1 is satisfied, where Condition (4) is satisfied if Property (P) holds.

Proof Assume that $(G/H, \langle\ ,\ \rangle)$ is a two-step g.o. space and consider $V \in T_o(G/H)$. Then, there exists a unique geodesic $\gamma : I \to G/H$ with $\gamma(0) = o$ and $\dot\gamma(0) = V$. Let s be the (affine) parameter of the geodesic γ. By assumption γ is two-step homogeneous. Therefore, there exists a parameter $t = \phi(s)$, $\phi : I \to J$, as well as vectors $X, Y \in \mathfrak{g}$, such that

$$\gamma(t) = \gamma(\phi(s)) = \pi\big(\exp(tX)\exp(tY)\big), \quad t \in J. \tag{5.1}$$

Without loss of generality, we may assume that $\phi(0) = 0$. By virtue of Theorem 3.1, there exists a function $k : J \to \mathbb{R}$ such that the curve γ satisfies the condition $\mathcal{G}_k^W(t) = 0$ for any $W \in \mathfrak{g}$, $t \in J$. On the other hand, by differentiating Eq. (5.1) at $s = 0$, we obtain

$$\dot\gamma(\phi(0))\phi'(0) = \phi'(0)\pi_*(X + Y),$$

which is equivalent to $\pi_*(X + Y) = V$.

Conversely, assume that for any $V \in T_o(G/H)$, there exist $X, Y \in \mathfrak{g}$ such that conditions (1) and (2) are satisfied. We will show that the unique geodesic through the arbitrary V is two-step homogeneous. By Theorem 3.1 the curve $\gamma : J \to \mathbb{R}$ with $\gamma(t) = \pi\big(\exp(tX)\exp(tY)\big)$ is a geodesic with respect to some parameter $s = \phi(t)$, $\phi : J \to I$.

Again, we can use an affine transformation so that $\phi(0) = 0$ and $\phi'(0) = 1$. Then, the geodesic $\widetilde{\gamma} = \gamma \circ \phi$ passes through o, it is two-step homogeneous and $\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\widetilde{\gamma}(t) = \dot\gamma(\phi(0))\phi'(0) = V$, which concludes the proof. □

Large classes of pseudo-Riemannian two-step g.o. spaces can be obtained by applying the following proposition.

Proposition 5.2 *Let G be a compact Lie group and let G/H be a homogeneous space with origin o. Consider the homogeneous fibration $K/H \to G/H \to G/K$, where K is a closed subgroup of G such that $H \subset K \subset G$. We assume that B is an* Ad *-invariant inner product in $\mathfrak{g}$, and we endow G/H with a 1-parameter family of pseudo-Riemannian metrics $\langle\ ,\ \rangle_\lambda$, $\lambda \in \mathbb{R}^*$, constructed as deformations*

$$\langle\ ,\ \rangle_\lambda = B|_{T(G/K)\times T(G/K)} + \lambda B|_{T(K/H)\times T(K/H)}$$

of B along the fiber K/H. Then, $(G/H, \langle\ ,\ \rangle_\lambda)$ *is a two-step g.o. space.*

Proof Let $\mathfrak{k}$ and $\mathfrak{h}$ be the Lie algebras of K and H respectively. We consider the B-orthogonal reductive decompositions $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}_1$ and $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}_2$. Then, we have the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$ with $\mathfrak{m}_1 = T_{eK}(G/K)$, $\mathfrak{m}_2 = T_o(K/H)$, and $\mathfrak{m}_1 \oplus \mathfrak{m}_2 = \mathfrak{m} = T_o(G/H)$. Moreover, we have

$$[\mathfrak{m}_1, \mathfrak{m}_2] \subseteq \mathfrak{m}_1. \tag{5.2}$$

The deformation metric is induced by the scalar product $\langle\ ,\ \rangle = B|_{\mathfrak{m}_1\times\mathfrak{m}_1} + \lambda B|_{\mathfrak{m}_2\times\mathfrak{m}_2}$, $\lambda \in \mathbb{R}^*$.

Let $V = X_1 + X_2 \in \mathfrak{m}$, $X_i \in \mathfrak{m}_i$. Then, the metric endomorphism $\Lambda : \mathfrak{m} \to \mathfrak{m}$ corresponding to the deformation metric has the form $\Lambda(V) = X_1 + \lambda X_2$. Consider the vectors $X = X_1 + \lambda X_2$ and $Y = (1-\lambda)X_2$. Then, $X + Y = X_1 + X_2 = V$. Moreover, by virtue of relation (5.2), we have

$$T(t)X_1 = \sum_{n=0}^{\infty} \frac{t^n}{n!}\mathrm{ad}^n((\lambda-1)X_2)X_1 \in \sum_{n=0}^{\infty} \mathrm{ad}^n(\mathfrak{m}_2)\mathfrak{m}_1 \subseteq \mathfrak{m}_1.$$

Also, we have $T(t)X_2 = \sum_{n=0}^{\infty}\frac{t^n}{n!}\mathrm{ad}^n((\lambda-1)X_2)X_2 = X_2$. Therefore, $T(t)X + Y = T(t)(X+Y)$. Taking into account the above facts and setting $k(t) = 0$, the left-hand side of the equation in part (4) of Proposition 4.1 becomes

$$\begin{aligned}
&[T(t)X + Y, \Lambda\big((T(t)X+Y)_{\mathfrak{m}}\big)] + \Lambda\big([T(t)X, Y]_{\mathfrak{m}}\big)\\
&\quad = [T(t)(X+Y), \Lambda\big((T(t)(X+Y))_{\mathfrak{m}}\big)] + \Lambda\big([T(t)X, Y]_{\mathfrak{m}}\big)\\
&\quad = [T(t)(X_1+X_2), \Lambda\big((T(t)(X_1+X_2))_{\mathfrak{m}}\big)] + (1-\lambda)\Lambda\big([T(t)(X_1+\lambda X_2), X_2]_{\mathfrak{m}}\big)\\
&\quad = [T(t)X_1 + X_2, \Lambda\big(T(t)X_1 + X_2\big)] + (1-\lambda)\Lambda\big([T(t)X_1 + \lambda X_2), X_2]_{\mathfrak{m}}\big)\\
&\quad = [T(t)X_1, \Lambda(T(t)X_1)] + [T(t)X_1, \Lambda(X_2)] + [X_2, \Lambda(T(t)X_1)]\\
&\qquad + [X_2, \Lambda(X_2)] + (1-\lambda)\Lambda\big([T(t)X_1, X_2]_{\mathfrak{m}}\big)\\
&\quad = [T(t)X_1, T(t)X_1] + [T(t)X_1, \lambda X_2] + [X_2, T(t)X_1] + [X_2, \lambda X_2]\\
&\qquad + (1-\lambda)\Lambda\big([T(t)X_1, X_2]_{\mathfrak{m}_1}\big)\\
&\quad = (\lambda-1)[T(t)X_1, X_2] + (1-\lambda)[T(t)X_1, X_2]_{\mathfrak{m}_1}\\
&\quad = (\lambda-1)[T(t)X_1, X_2] + (1-\lambda)[T(t)X_1, X_2] = 0.
\end{aligned}$$

The result then follows from Propositions 5.1 and 4.1. □

6 Lorentzian two-step g.o. Lie groups

In the present section, we discuss a concrete example of a two-step g.o. space, namely a three-dimensional unimodular Lorentzian Lie group. We recall that a three-dimensional Lorentzian Lie group is a three-dimensional Lie group G endowed with a Lorentzian left-invariant metric $\langle\ ,\ \rangle$. We also recall the following classification result.

Theorem 6.1 [17] *A three-dimensional simply connected unimodular Lorentzian Lie group G admits a pseudo-orthonormal frame field* $\{e_1, e_2, e_3\}$, *with* e_3 *time-like, such that the Lie algebra* $\mathfrak{g}$ of *G is one of the following:*

$$\mathfrak{g}_1 : \begin{aligned} [e_1, e_2] &= \alpha e_1 - \beta e_3, \\ [e_1, e_3] &= -\alpha e_1 - \beta e_2, \\ [e_2, e_3] &= \beta e_1 + \alpha e_2 + \alpha e_3 \qquad \alpha \neq 0. \end{aligned}$$

If $\beta \neq 0$, *then* $G = \widetilde{SL(2,\mathbb{R})}$, *while for* $\beta = 0$ *it is* $G = E(1,1)$, *the group of rigid motions of the Minkowski two-space.*

$$\mathfrak{g}_2 : \begin{aligned} [e_1, e_2] &= -\gamma e_2 - \beta e_3, \\ [e_1, e_3] &= -\beta e_2 + \gamma e_3, \qquad \gamma \neq 0, \\ [e_2, e_3] &= \alpha e_1. \end{aligned}$$

In this case, $G = \widetilde{SL(2,\mathbb{R})}$ *if* $\alpha \neq 0$, *while* $G = E(1,1)$ *if* $\alpha = 0$.

$$\mathfrak{g}_3 : \quad [e_1, e_2] = -\gamma e_3, \quad [e_1, e_3] = -\beta e_2, \quad [e_2, e_3] = \alpha e_1.$$

Table 1 (where $\widetilde{E}(2)$ and H_3, *respectively, denote the universal covering of the group of rigid motions in the Euclidean two-space and the Heisenberg group) lists all the Lie groups G which admit a Lie algebra* $\mathfrak{g}_3$, *according to the different possibilities for the signs of* α, β *and* γ:

$$\mathfrak{g}_4 : \begin{aligned} [e_1, e_2] &= -e_2 + (2\varepsilon - \beta) e_3, \qquad \varepsilon = \pm 1, \\ [e_1, e_3] &= -\beta e_2 + e_3, \\ [e_2, e_3] &= \alpha e_1. \end{aligned}$$

Table 2 *describes all Lie groups G admitting a Lie algebra* $\mathfrak{g}_4$:

Table 1 3D Lorentzian Lie groups with Lie algebra $\mathfrak{g}_3$

Lie group	α	β	γ
$\widetilde{SL(2,\mathbb{R})}$	+	+	+
$\widetilde{SL(2,\mathbb{R})}$	+	−	−
$SU(2)$	+	+	−
$\widetilde{E}(2)$	+	+	0
$\widetilde{E}(2)$	+	0	−
$E(1,1)$	+	−	0
$E(1,1)$	+	0	+
H_3	+	0	0
H_3	0	0	−
$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$	0	0	0

We observe that it would be possible to unify the two cases described in Table 2. However, in order to unify them, the second column should list conditions for εa instead of a, which would make the whole table less readable.

We see from the above classification that the Lie algebra $\mathfrak{sl}_2$ of $SL(2,\mathbb{R})$ is the one occurring most frequently in the classification of three-dimensional unimodular Lorentzian Lie algebras. Observe that $SL(2,\mathbb{R})$ has been an important source of examples for very different topics (see, for example, [4, 18]).

The main result of this section is the following.

Theorem 6.2 *The Lorentzian Lie group* $(SL(2,\mathbb{R}),\langle\ ,\ \rangle)$ *is a two-step g.o. space if and only if one of the following properties holds for its Lie algebra* $\mathfrak{sl}_2$:

(a) $\mathfrak{sl}_2=\mathfrak{g}_3$ with either $\alpha=\beta$ or $\alpha=\gamma$ or $\beta=\gamma$.
(b) $\mathfrak{sl}_2=\mathfrak{g}_4$ with $\alpha=\beta-\varepsilon$.

To prove our main result, we need the following lemma.

Lemma 6.3 *Let* $(G,\langle\ ,\ \rangle)$ *be a Lie group such that* $\mathfrak{g}=\mathrm{span}\{e_1,\dots,e_n\}$. If $(G,\langle\ ,\ \rangle)$ *is a two-step g.o. space, then for any* $V\in\mathfrak{g}$ *there exist* $X,Y\in\mathfrak{g}$, *such that*

(i) $X+Y=V$ and
(ii) there exists a function k such that $\mathcal{G}_k^{e_i}(0)=0$ for all indices $i=1,\dots,n$, where $\mathcal{G}_k^W(t)$ is the function

$$\mathcal{G}_k^W(t)=\langle T(t)X+Y,[W,T(t)X+Y]\rangle+\langle W,[T(t)X,Y]\rangle-k(t)\langle W,T(t)X+Y\rangle.$$

Conversely, if for any $V\in\mathfrak{g}$ there exist $X,Y\in\mathfrak{g}$ satisfying conditions (i), (ii) and
(iii) $\mathrm{ad}(Y)$ is skew-symmetric with respect to $\langle\ ,\ \rangle$, then $(G,\langle\ ,\ \rangle)$ is a two-step g.o. space.

Proof If $(G,\langle\ ,\ \rangle)$ is a two-step g.o. space, then conditions (i) and (ii) follow immediately by Proposition 5.1. To prove the converse, we observe that the quantity $\mathcal{G}_k^W(t)$ is linear with respect to W. Since $\mathfrak{g}=\mathrm{span}\{e_1,\dots e_n\}$, condition (ii) implies that $\mathcal{G}_k^W(0)=0$ for any $W\in\mathfrak{g}$. By taking into account condition (iii) and Proposition 4.3, we obtain that $\mathcal{G}_k^W(t)=0$, for all $W\in\mathfrak{g}$ and for all $t\in J$. Also, by taking into account condition (i) and Proposition 5.1 we conclude that $(G,\langle\ ,\ \rangle)$ is a two-step g.o. space.

Proof of Theorem 6.2 We assume that there exists a left-invariant Lorentzian two-step g.o. metric $\langle\ ,\ \rangle$ in $SL(2,\mathbb{R})$. We will use the classification of three-dimensional Lorentzian Lie

Table 2 3D Lorentzian Lie groups with Lie algebra $\mathfrak{g}_4$

Lie group $(\varepsilon=1)$	α	β	Lie group $(\varepsilon=-1)$	α	β
$\widetilde{SL(2,\mathbb{R})}$	$\neq 0$	$\neq 1$	$\widetilde{SL(2,\mathbb{R})}$	$\neq 0$	$\neq -1$
$E(1,1)$	0	$\neq 1$	$E(1,1)$	0	$\neq -1$
$E(1,1)$	<0	1	$E(1,1)$	>0	-1
$\widetilde{E}(2)$	>0	1	$\widetilde{E}(2)$	<0	-1
H_3	0	1	H_3	0	-1

algebras mentioned in Theorem 6.1, and we will examine each of the four cases in which the Lie algebra $\mathfrak{sl}_2$ occurs. We recall that for $X, Y \in \mathfrak{g}$ we have

$$\mathcal{G}_k^W(0) = \langle X + Y, [W, X + Y]\rangle + \langle W, [X, Y]\rangle - k\langle W, X + Y\rangle.$$

Case 1 $\mathfrak{sl}_2 = \mathfrak{g}_1$ with $\beta \neq 0$. Let $V = 2\alpha e_1 + \beta e_2 - \beta e_3 \in \mathfrak{g}_1$.

By Lemma 6.3, there exist $X, Y \in \mathfrak{g}$ such that $X + Y = V$ and $\mathcal{G}_k^{e_i}(0) = 0$, $i = 1, 2, 3$. We set

$$X = x_1 e_1 + x_2 e_2 + x_3 e_3,$$

so that

$$Y = (2\alpha - x_1)e_1 + (\beta - x_2)e_2 + (-\beta - x_3)e_3.$$

Taking into account the Lie bracket relations for the algebra $\mathfrak{g}_1$ as well as the expressions of X, Y, then system $\mathcal{G}_k^{e_i}(0) = 0$, $i = 1, 2, 3$, is equivalent to

$$\begin{aligned}
&2\alpha\beta x_1 - \left(2\alpha^2 + \beta^2\right)x_2 + \left(2\alpha^2 - \beta^2\right)x_3 = 2k\alpha\\
&\beta^2 x_1 - \alpha\beta x_2 + \alpha\beta x_3 = k\beta + 2\alpha\left(2\alpha^2 + \beta^2\right)\\
&\beta^2 x_1 - \alpha\beta x_2 + \alpha\beta x_3 = k\beta - 2\alpha\left(2\alpha^2 + \beta^2\right).
\end{aligned}$$

The last two equations imply that $\alpha = 0$, which is a contradiction. Therefore, $\mathfrak{sl}_2 \neq \mathfrak{g}_1$.

Case 2 $\mathfrak{sl}_2 = \mathfrak{g}_2$ with $\alpha \neq 0$. Let $V = v_1 e_1 + v_2 e_2 + v_3 e_3 \in \mathfrak{g}_2$.

By Lemma 6.3, there exist $X, Y \in \mathfrak{g}$, such that $X + Y = V$ and $\mathcal{G}_k^{e_i}(0) = 0$, $i = 1, 2, 3$. We consider again $X = x_1 e_1 + x_2 e_2 + x_3 e_3$, so that

$$Y = (v_1 - x_1)e_1 + (v_2 - x_2)e_2 + (v_3 - x_3)e_3.$$

Taking into account the Lie bracket relations for the Lie algebra $\mathfrak{g}_2$ as well as the expressions of X, Y we deduce that the system $\mathcal{G}_k^{e_i}(0) = 0$, $i = 1, 2, 3$, is equivalent to

$$\begin{aligned}
&\alpha v_3 x_2 - \alpha v_2 x_3 = k v_1 + \gamma(v_2^2 + v_3^2)\\
&-(\gamma v_2 + \beta v_3)x_1 + \gamma v_1 x_2 + \beta v_1 x_3 = k v_2 + (\beta - \alpha)v_1 v_3 - \gamma v_1 v_2\\
&(\beta v_2 - \gamma v_3)x_1 - \beta v_1 x_2 + \gamma v_1 x_3 = -k v_3 + (\alpha - \beta)v_1 v_2 - \gamma v_1 v_3
\end{aligned}$$

with unknowns x_1, x_2, x_3 and parameters v_1, v_2, v_3, k. The determinant D of the above system is zero. The determinant D_{x_1} is given by $D_{x_1} = Ak + B$, where $A = v_1\left((\beta^2 + \gamma^2)v_1^2 - 2\alpha\gamma v_2 v_3 - \alpha\beta v_3^2 + \alpha\beta v_2^2\right)$ and $B = v_1^2(v_2^2 + v_3^2)\gamma((\alpha - \beta)^2 + \gamma^2)$. Here, D_{x_1} is the determinant obtained by replacing first column of the above system, by the column of constant terms.

If $v_1, v_2, v_3 \neq 0$ then $B \neq 0$, because $\gamma \neq 0$. Moreover, as $\alpha\gamma \neq 0$, there exist non zero real numbers v_1, v_2, v_3, such that $A = 0$. Indeed, it suffices to choose

$$v_1 = \sqrt{\frac{2\varepsilon\alpha\gamma}{\beta^2 + \gamma^2}}, \quad v_2 = \varepsilon, \quad v_3 = 1.$$

Here, $\varepsilon = 1$ if $\alpha\gamma > 0$ and $\varepsilon = -1$ if $\alpha\gamma < 0$. With the above choices for v_1, v_2, v_3, we obtain that $D_{x_1} \neq 0$. Since $D = 0$, this implies that there exist no solutions x_i for the system $\mathcal{G}_k^{e_i}(0) = 0$, $i = 1, 2, 3$, which is a contradiction. Therefore, $\mathfrak{sl}_2 \neq \mathfrak{g}_2$

Case 3 $\mathfrak{sl}_2 = \mathfrak{g}_3$ with $\alpha, \beta, \gamma > 0$ or $\alpha > 0$ and $\beta, \gamma < 0$. We set $V = v_1 e_1 + v_2 e_2 + v_3 e_3 \in \mathfrak{g}_3$.

Assume that $(SL(2,\mathbb{R}), \langle\ ,\ \rangle)$ is two-step g.o. and α, β, γ are all distinct. By Lemma 6.3, there exist $X, Y \in \mathfrak{g}_3$ such that $X + Y = V$ and $\mathcal{G}_k^{e_i}(0) = 0$, $i = 1, 2, 3$. We set $X = x_1 e_1 + x_2 e_2 + x_3 e_3$, so that

$$Y = (v_1 - x_1)e_1 + (v_2 - x_2)e_2 + (v_3 - x_3)e_3.$$

Taking into account the Lie bracket relations for the algebra $\mathfrak{g}_3$ as well as the expressions of X, Y, the system $\mathcal{G}_k^{e_i}(0) = 0$, $i = 1, 2, 3$ is equivalent to

$$\alpha v_3 x_2 - \alpha v_2 x_3 = (\beta - \gamma) v_2 v_3 + k v_1 \tag{6.1}$$

$$-\beta v_3 x_1 + \beta v_1 x_3 = (\gamma - \alpha) v_1 v_3 - k v_2 \tag{6.2}$$

$$\gamma v_2 x_1 - \gamma v_1 x_2 = (\alpha - \beta) v_1 v_2 - k v_3, \tag{6.3}$$

with unknowns x_1, x_2, x_3 and parameters v_1, v_2, v_3, k.

The determinant D of the above system is zero. The determinant D_{x_1} is equal to

$$D_{x_1} = Ak + B,$$

where $A = v_1(\beta\gamma v_1^2 - \alpha\beta v_3^2 + \alpha\gamma v_2^2)$ and $B = v_1^2 v_2 v_3 (\alpha - \gamma)(\alpha - \beta)(\beta - \gamma)$. If $\alpha, \beta, \gamma > 0$, we set

$$v_1 = v_2 = 1, \quad v_3 = \sqrt{\frac{\gamma(\alpha + \beta)}{\alpha\beta}},\text{''}$$

whereas if $\alpha > 0$ and $\beta, \gamma < 0$ we set

$$v_1 = v_3 = 1, \quad v_2 = \sqrt{\frac{\beta(\gamma - \alpha)}{-\alpha\gamma}}.$$

With the above choices for v_1, v_2, v_3 and since α, β, γ are all distinct, we obtain that $A = 0$ and $B \neq 0$. Therefore, $D_{x_1} \neq 0$. Since $D = 0$, this implies that there exist no solutions x_i for the system $\mathcal{G}_k^{e_i}(0) = 0$, $i = 1, 2, 3$, which is a contradiction. Hence, at least two of the structure constants α, β, γ are equal.

Conversely, assume that at least two of the structure constants α, β, γ coincide, so that we have one of the cases below:

If $\alpha = \beta$, we set $x_1 = v_1$, $x_2 = v_2$, $x_3 = \frac{\gamma}{\alpha} v_3$, so that

$$X = v_1 e_1 + v_2 e_2 + \frac{\gamma}{\alpha} v_3 e_3 \quad \text{and} \quad Y = \left(1 - \frac{\gamma}{\alpha}\right) v_3 e_3.$$

If $\alpha = \gamma$, we set $x_1 = v_1$, $x_3 = v_3$, $x_2 = \frac{\beta}{\alpha} v_2$, so that

$$X = v_1 e_1 + \frac{\beta}{\alpha} v_2 e_2 + v_3 e_3 \quad \text{and} \quad Y = \left(1 - \frac{\beta}{\alpha}\right) v_2 e_2.$$

Finally, if $\beta = \gamma$, we set $x_2 = v_2$, $x_3 = v_3$, $x_1 = \frac{\alpha}{\beta} v_1$, so that

$$X = \frac{\alpha}{\beta} v_1 e_1 + v_2 e_2 + v_3 e_3 \quad \text{and} \quad Y = \left(1 - \frac{\alpha}{\beta}\right) v_1 e_1.$$

In any of the above cases, the following conditions are satisfied:

(i) $X + Y = V$.
(ii) Equations (6.1)–(6.3) are satisfied for $k = 0$; therefore, $\mathcal{G}_0^{e_i}(0) = 0, i = 1, 2, 3$.
(iii) The endomorphism $\mathrm{ad}(Y)$ is skew symmetric with respect to $\langle , \rangle$.

By virtue of Lemma 6.3, we then conclude that $(SL(2,\mathbb{R}), \langle\ ,\ \rangle)$ is a two-step g.o. space.

□

Case 4 $\mathfrak{sl}_2 = \mathfrak{g}_4$ with $\alpha \neq 0$ and $\beta \neq \varepsilon$.

Assume that $(SL(2,\mathbb{R}), \langle\ ,\ \rangle)$ is two-step g.o. and $\alpha \neq \beta - \varepsilon$. By virtue of Lemma 6.3, there exist $X, Y \in \mathfrak{g}_4$ and a function k, such that $X + Y = V$ and $\mathcal{G}_k^{e_i}(0) = 0, i = 1, 2, 3$. We set $X = x_1 e_1 + x_2 e_2 + x_3 e_3$, so that

$$Y = (v_1 - x_1)e_1 + (v_2 - x_2)e_2 + (v_3 - x_3)e_3.$$

Taking into account the Lie bracket relations for the algebra $\mathfrak{g}_4$ as well as the expressions of X, Y, the system $\mathcal{G}_k^{e_i}(0) = 0, i = 1, 2, 3$ is equivalent to

$$\alpha v_3 x_2 - \alpha v_2 x_3 = k v_1 + (v_2 + \varepsilon v_3^2) \tag{6.4}$$

$$(-v_2 - \beta v_3)x_1 + v_1 x_2 + \beta v_1 x_3 = k v_2 - v_1 v_2 - v_1 v_2(\alpha + 2\varepsilon - \beta) \tag{6.5}$$

$$((\beta - 2\varepsilon)v_2 - v_3)x_1 - (\beta - 2\varepsilon)v_1 x_2 + v_1 x_3 = -k v_3 + (\alpha - \beta)v_1 v_2 - v_1 v_3, \tag{6.6}$$

with unknowns x_1, x_2, x_3 and parameters v_1, v_2, v_3, k. The determinant D of the above system is zero. The determinant D_{x_1} is given by

$$D_{x_1} = Ak + B,$$

where $A = k v_1 \left(v_1^2(\beta - \varepsilon)^2 - 2\alpha v_2 v_3 + \alpha\beta(v_2^2 - v_3^2) + 2\varepsilon v_2\right)$ and $B = v_1^2(v_2 + \varepsilon v_2)^2(\alpha - \beta + \varepsilon)^2$.

If $\alpha < 0$ and $\beta \leq, 0$ we set

$$v_2 = -\varepsilon, \quad v_3 = 2\varepsilon, \quad v_1 = \sqrt{\frac{-4\alpha + 3\alpha\beta + 2}{(\beta - \varepsilon)^2}}.$$

If $\alpha < 0$ and $\beta > 0$, we set

$$v_2 = -2\varepsilon, \quad v_3 = \varepsilon, \quad v_1 = \sqrt{\frac{-4\alpha - 3\alpha\beta + 4}{(\beta - \varepsilon)^2}}.$$

If $\alpha > 0$ and $\beta \leq 0$, we set

$$v_2 = -2\varepsilon, \quad v_3 = -\varepsilon, \quad v_1 = \sqrt{\frac{4\alpha - 3\alpha\beta + 4}{(\beta - \varepsilon)^2}}.$$

If $\alpha > 0$ and $\beta > 0$, we set

$$v_2 = -\varepsilon, \quad v_3 = -2\varepsilon, \quad v_1 = \sqrt{\frac{4\alpha + 3\alpha\beta + 2}{(\beta - \varepsilon)^2}}.$$

With the above choices for v_1, v_2, v_3, we obtain that $A = 0$ and $B \neq 0$. Therefore, $D_{x_1} \neq 0$. Since $D = 0$, this implies that there exist no solutions x_i for the system $\mathcal{G}_k^{e_i}(0) = 0$, $i = 1, 2, 3$, which is a contradiction. Hence, $\alpha = \beta - \varepsilon$. Conversely, assume that $\alpha = \beta - \varepsilon$. We set

$$x_1 = v_1, \quad x_2 = \frac{\beta v_2 + v_3}{\beta - \varepsilon}, \quad x_3 = \frac{-v_2 + (\beta - \varepsilon)v_3}{\beta - \varepsilon},$$

so that

$$X = v_1 e_1 + \frac{\beta v_2 + v_3}{\beta - \varepsilon} e_2 - \frac{-v_2 + (\beta - \varepsilon)v_3}{\beta - \varepsilon} e_3, \quad Y = -\frac{\varepsilon v_2 + v_3}{\beta - \varepsilon} e_2 + \frac{v_2 + \varepsilon v_3}{\beta - \varepsilon} e_3.$$

Then, the following conditions are satisfied:

(i) $X + Y = V$.
(ii) Equations (6.4)–(6.6) are satisfied for $k = 0$; therefore $\mathcal{G}_0^{e_i}(0) = 0$, $i = 1, 2, 3$.
(iii) The endomorphism $\mathrm{ad}(Y)$ is skew symmetric with respect to $\langle\ ,\ \rangle$. By virtue of Lemma 6.3, we obtain that $(SL(2,\mathbb{R}), \langle\ ,\ \rangle)$ is a two-step g.o. space, which concludes Case **4.** □

Remark 6.4 Observe that for the trivial coset realization $(SL(2,\mathbb{R}), \langle, \rangle)$ not every geodesic is homogeneous ([5]). Hence, the geodesics obtained in Theorem 6.2 are examples of proper two-step g.o. spaces. On the other hand, with respect to the full isometry group, those examples are g.o. (in fact, they are naturally reductive, cf. [5]).

For the Riemannian case, a similar argument proves the following.

Theorem 6.5 *The Riemannian Lie group* $(SL(2,\mathbb{R}), \langle\ ,\ \rangle)$ *is a two-step g.o. space if and only its Lie algebra* $\mathfrak{sl}_2$ *admits an orthonormal basis* $\{e_1, e_2, e_3\}$, *such that*

$$[e_1, e_2] = \gamma e_3, \qquad [e_2, e_3] = \alpha e_1, \qquad [e_3, e_1] = \beta e_2,$$

with either $\alpha = \beta$ *or* $\alpha = \gamma$ *or* $\beta = \gamma$.

Remark 6.6 By exactly the same argument used in the proof of the "only if "part of Theorem 6.2, the result remains true if we start from any three-dimensional unimodular Lorentzian Lie algebra (as described in Theorem 6.1). Hence, we have the following.

Theorem 6.7 *A unimodular Lorentzian Lie group* $(G, \langle\ ,\ \rangle)$ *is a two-step g.o. space, if one of the following properties holds for its Lie algebra* $\mathfrak{g}$:

(a) either $\mathfrak{g} = \mathfrak{g}_3$ and at least two among α, β and γ coincide, or
(b) $\mathfrak{g} = \mathfrak{g}_4$ with $\alpha = \beta - \varepsilon$.

In particular, $G = Sl(2, \mathbb{R})$.

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