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Pizzetti formula on the Grassmannian of 2-planes

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Abstract
This paper is devoted to the role played by the Higgs algebra H_3 in the generalisation of classical harmonic analysis from the sphere S^{m-1} to the (oriented) Grassmann manifold $\mathrm{Gr}_o(m, 2)$ of 2-planes. This algebra is identified as the dual partner (in the sense of Howe duality) of the orthogonal group $\mathrm{SO}(m)$ acting on functions on the Grassmannian. This is then used to obtain a Pizzetti formula for integration over this manifold. The resulting formulas are finally compared to formulas obtained earlier for the Pizzetti integration over Stiefel manifolds, using an argument involving symmetry reduction.

1 Introduction

The theory of spherical harmonics on $\mathbb{R}^m$ is a beautiful piece of mathematics, with many applications in, for instance, representation theory, physics and even engineering. It has grown out of, and is centred around, the notion of the Laplace operator on $\mathbb{R}^m$, given by

$$\Delta_x = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_m^2} = \sum_{j=1}^{m} \frac{\partial^2}{\partial x_j^2}.$$

Null solutions for Δ_x in the polynomial ring $\mathcal{P}(\mathbb{R}^m, \mathbb{C}) := \mathbb{C}[x_1, \dots, x_m]$ are commonly referred to as harmonic polynomials on $\mathbb{R}^m$. Their restrictions to the sphere $S^{m-1} \subset \mathbb{R}^m$ are the so-called spherical harmonics (indexed by a positive integer $k \in \mathbb{Z}^+$, which then refers to the degree of homogeneity of the harmonic polynomial it uniquely extends to). These functions realise the eigenspaces of the Laplace–Beltrami operator on the homogeneous space S^{m-1}, which is closely connected to the Casimir operator of order 2 for the Lie algebra $\mathfrak{so}(m)$. From a purely algebraic point of view, the operator Δ_x arises if one wants to understand the behaviour of the space $\mathcal{P}(\mathbb{R}^m, \mathbb{C})$ as a representation for the (special) orthogonal group, under the regular action

$$\mathrm{H} : \mathrm{SO}(m) \to \mathrm{Aut}\big(\mathcal{P}(\mathbb{R}^m, \mathbb{C})\big)$$

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with $P(\underline{x}) \mapsto \mathrm{H}(g)[P(\underline{x})] := P(g^{-1}\underline{x})$, where $\underline{x} = (x_1, \dots, x_m)^T$ and with g an arbitrary element of the group $\mathrm{SO}(m)$. It is well known that under this action, one has that

$$\mathcal{P}(\mathbb{R}^m, \mathbb{C}) = \bigoplus_{k=0}^{\infty} \mathcal{P}_k(\mathbb{R}^m, \mathbb{C}) = \bigoplus_{k=0}^{\infty} \bigoplus_{j=0}^{\lfloor \frac{k}{2} \rfloor} r^{2j} \mathcal{H}_{k-2j}(\mathbb{R}^m, \mathbb{C}), \tag{1}$$

where $\mathcal{H}_k = \mathcal{P}_k \cap \ker \Delta_x$ and $r^2 = x_1^2 + \cdots + x_m^2$ denotes the squared norm of the vector $\underline{x} \in \mathbb{R}^m$. Here, the spaces $\mathcal{H}_k$ of k-homogeneous harmonic polynomials define an irreducible module for $\mathrm{SO}(m)$ with the highest weight $(k, 0, \dots, 0)$. A crucial observation which can be made here is that the Lie algebra spanned by Δ_x and r^2, seen as a subalgebra of the Weyl algebra $\mathcal{W}(\mathbb{R}^m, \mathbb{C})$ acting on the space $\mathcal{P}(\mathbb{R}^m, \mathbb{C})$, is given by

$$\mathfrak{sl}(2) = \mathrm{Alg}(X, Y, H) \cong \mathrm{Alg}\left(\frac{1}{2}r^2, -\frac{1}{2}\Delta_x, \mathbb{E}_x + \frac{m}{2}\right), \tag{2}$$

with $\mathbb{E}_x = \sum_j x_j \partial_{x_j}$ the so-called Euler operator (acting as a constant k on homogeneous polynomials of degree k). This had led to the celebrated Howe duality theorem, which in this particular case allows to turn formula (1) into a decomposition which is multiplicity-free (see, for instance, [11, 16]). There are several ways in which the theory of spherical harmonics can be generalised, but the topic of this paper is based on the observation that the function space $\mathcal{P}(\mathbb{R}^m, \mathbb{C})$ can be seen as the homogeneous coordinate ring for the projective space $\mathbb{P}^{m-1}$ of lines through the origin in $\mathbb{R}^m$. Since this is merely the simplest example of a flag manifold, the extension to other Grassmann varieties is an obvious generalisation. In the present paper, we will therefore consider the (oriented) Grassmannian of 2-planes in $\mathbb{R}^m$: this (projective) variety also has a homogeneous coordinate ring, which defines a module for a suitable action of $\mathrm{GL}(m)$ that can thus be decomposed into irreducible representations for the (special) orthogonal group $\mathrm{SO}(m)$. Howe and Lee observed that the summands in this decomposition can be defined in terms of a differential operator which then generalises the role of Δ_x. (This operator is sometimes referred to as the Cayley–Laplace operator.) In [17], this was done for spaces of k-planes in $\mathbb{C}^m$, using the general language of representation theory. In this paper, we will consider the case $k = 2$, as we will focus on certain issues related to this Cayley–Laplace operator Δ_G (see further in this paper for its explicit definition). In particular, we will explicitly describe the Howe duality underlying the decomposition of the aforementioned coordinate ring into irreducibles for $\mathrm{SO}(m)$. This is then used to obtain a generalisation of formula (1) for the operator Δ_G in which the Higgs algebra H_3 (a polynomial deformation of the Lie algebra $\mathfrak{su}(2)$, cfr. infra) arises as a dual partner. As an application, we will then derive a Pizzetti formula for the integral over the Grassmannian $\mathrm{Gr}_o(m, 2)$.

2 Howe duality on the Grassmannian

In order to define the manifold on which we will define a class of functions, we first introduce the Stiefel manifold $\mathrm{St}(m, 2)$: it can either be defined as the collection of matrices $M_{xu} = (\underline{x}, \underline{u})$ in $\mathbb{R}^{m \times 2}$ for which $M_{xu}^T M_{xu} = I$, or as the homogeneous space

$$\mathrm{SO}(m)/\mathrm{SO}(m-2) = G/H \ni [gH] \mapsto g\begin{pmatrix} I \\ 0 \end{pmatrix} \in \mathrm{St}(m, 2), \tag{3}$$

with $I = \mathrm{Id}_2$ the identity matrix in $\mathbb{R}^{2\times 2}$ and $0 = 0_{m-2,2}$ the zero matrix in $\mathbb{R}^{m-2,2}$. The former allows us to restrict functions on $\mathbb{R}^{m\times 2}$ to $\mathrm{St}(m, 2)$ by putting $|\underline{x}|^2 = |\underline{u}|^2 = 1$ and $\langle \underline{x}, \underline{u}\rangle = 0$. (This will be useful later.) The oriented Grassmannian (which is defined as the set of oriented 2-planes in $\mathbb{R}^m$) can be defined as a homogeneous space

$$\mathrm{Gr}_o(m,2) = \frac{\mathrm{SO}(m)}{\mathrm{SO}(m-2)\times \mathrm{SO}(2)},$$

or as a quotient space of $\mathrm{St}(m,2)$. Indeed, for $A, B \in \mathrm{St}(m,2)$, we say that $A \sim B$ if and only if there exists a matrix $g_2 \in \mathrm{SO}(2)$ such that $B = Ag_2$ (in $\mathbb{R}^{m\times 2}$). The equivalence class of $A \in \mathrm{St}(m,2)$, which can be seen as the orbit of A under this action of $\mathrm{SO}(2)$ from the right, is denoted by $[A]$, and we then define a quotient map

$$\pi\,:\,\mathrm{St}(m,2) \to \mathrm{Gr}_o(m,2)\,:\,A \mapsto [A]. \tag{4}$$

It is easily seen that this is a (right) principal $\mathrm{SO}(2)$-bundle, with natural action $\mathrm{St}(m,2)\times\mathrm{SO}(2) \longrightarrow \mathrm{St}(m,2)\,:\,(A,g_2)\mapsto Ag_2$, in such a way that $\pi(A) = \pi(Ag_2) = [A]$. The group $G = \mathrm{SO}(m)$ also acts on $\mathrm{Gr}_o(m,2)$ from the left, by means of $(g,[A]) \mapsto [gA]$ for all $g \in \mathrm{SO}(m)$ and $[A] \in \mathrm{Gr}_o(m,2)$. This action is transitive, and the isotropy group of this left G-action at the equivalence class of the base point $(I\ 0)^t \in \mathbb{R}^{m\times 2}$ for $\mathrm{St}(m,2)$ used in (3) is seen to be the subgroup

$$\begin{pmatrix} \mathrm{SO}(2) & 0 \\ 0 & \mathrm{SO}(m-2) \end{pmatrix} \subset \mathrm{SO}(m).$$

Indeed, one has that

$$\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} I \\ 0 \end{pmatrix}\right] = \left[\begin{pmatrix} A \\ C \end{pmatrix}\right] = \left[\begin{pmatrix} I \\ 0 \end{pmatrix}\right]$$

if $C = 0$ and $A \in \mathrm{SO}(2)$. This then implies that $B = 0$ and $D \in \mathrm{SO}(m-2)$.

Remark 1 It is worth pointing out here that the right action of $\mathrm{SO}(2)$ used to construct $\mathrm{Gr}_o(m,2)$ as a quotient space is independent from the left action of $\mathrm{SO}(m)$. Indeed, the right action on an arbitrary $[A]$ in $\mathrm{Gr}_o(m,2)$ has no effect, whereas the left action of $\mathrm{SO}(2) \subset \mathrm{SO}(m)$ under the embedding as an upper left slot moves $[A]$ to a different point (unless A is the base point of the Stiefel manifold).

As is known from, for example, [8, 23], the homogeneous coordinate ring of the oriented real Grassmannian $\mathrm{Gr}_o(m,2)$ can be identified with a polynomial algebra:

$$\mathcal{R}(\mathrm{Gr}_o(m,2)) = \mathcal{P}(\mathbb{R}^{m\times 2},\mathbb{R})^{\mathrm{SL}(2)}, \tag{5}$$

whereby the action of $h \in \mathrm{SL}(2)$ is defined as the (right) regular action on polynomials in $(\underline{x},\underline{u}) \in \mathbb{R}^{m\times 2}$, by means of $\mathrm{H}(g)[P(\underline{x},\underline{u})] := P((\underline{x},\underline{u})g)$. The invariance under $\mathrm{SL}(2)$, which is what the upper index notation refers to in (5), can also be expressed in terms of the corresponding derived action dH. This action is defined in terms of the Lie algebra

$$\mathfrak{sl}(2) = \mathrm{Alg}(X,Y,H) \cong \mathrm{Alg}\big(\langle \underline{x},\underline{\partial}_u\rangle, \langle \underline{u},\underline{\partial}_x\rangle, \mathbb{E}_x - \mathbb{E}_u\big), \tag{6}$$

whereby the inner product notation stands for

$$X = \langle \underline{x}, \underline{\partial}_u \rangle = \sum_{j=1}^{m} x_j \frac{\partial}{\partial u_j} \quad \text{and} \quad Y = \langle \underline{u}, \underline{\partial}_x \rangle = \sum_{j=1}^{m} u_j \frac{\partial}{\partial x_j}. \tag{7}$$

Throughout this paper, these non-constant coefficient differential operators will often be referred to as the 'skew Euler operators' X and Y. Note that whereas the regular (derived) action of SL(2) is a right (matrix) action, X and Y are acting from the left as differential operators.

Remark 2 Note that in contrast to Howe and Lee [17], we restrict ourselves to real Grassmannians in this paper, so we, for instance, have that $\mathrm{Gr}_o(m,2) = G/K$ with $G = \mathrm{SO}(m)$ and $K = \mathrm{SO}(m-2) \times \mathrm{SO}(2)$. That being said though, one can easily extend the action of $\mathrm{SO}(m)$ on the homogeneous coordinate ring $\mathcal{R}(\mathrm{Gr}_o(m,2))$, or the associated polynomial algebra (5), to its complexification $\mathcal{R} \otimes \mathbb{C}$. This is the reason why we will always consider $\mathbb{C}$-valued polynomials in this paper.

We can thus work with the following model for $\mathcal{R}(\mathrm{Gr}_o(m,2)) \otimes \mathbb{C}$:

$$\mathcal{P}(\mathbb{R}^{m\times 2}, \mathbb{C})^{\mathrm{SL}(2)} := \{P(\underline{x}, \underline{u}) \in \mathcal{P}(\mathbb{R}^{m\times 2}, \mathbb{C}) : \langle \underline{x}, \underline{\partial}_u \rangle P = \langle \underline{u}, \underline{\partial}_x \rangle P = 0\}.$$

Note that the trivial action of the Cartan element $H = \mathbb{E}_x - \mathbb{E}_u \in \mathfrak{sl}(2)$ in the realisation (6) from above implies that the degree of homogeneity in $\underline{x}$ and $\underline{u}$ has to be equal ($\mathbb{E}_x P = \mathbb{E}_u P$). This leads to the following natural grading with respect to the degree in $(\underline{x}, \underline{u})$:

$$\mathcal{P}(\mathbb{R}^{m\times 2}, \mathbb{C})^{\mathrm{SL}(2)} = \bigoplus_{k=0}^{\infty} \mathcal{P}_{k,k}(\mathbb{R}^{m\times 2}, \mathbb{C})^{\mathrm{SL}(2)}.$$

In [17], the abstract decomposition for these graded subspaces as a module for the (left) regular action of the orthogonal group $\mathrm{SO}(m)$ was obtained, and these summands were given meaning as solution spaces for the operator Δ_G mentioned in the introduction, hereby thus generalising the notion of classical harmonics. This operator, which was dubbed the Cayley–Laplace operator in for instance [22], is given by

$$\Delta_G := \Delta_x \Delta_u - \langle \underline{\partial}_x, \underline{\partial}_u \rangle^2 = \sum_{a<b} \left(\partial_{x_a} \partial_{u_b} - \partial_{x_b} \partial_{u_a} \right)^2. \tag{8}$$

Note that the operator Δ_G arises naturally if one takes into account that the space $\mathbb{V}_2 := \mathrm{span}_{\mathbb{C}}(\Delta_x, \langle \underline{\partial}_x, \underline{\partial}_u \rangle, \Delta_u)$ defines a model for an irreducible representation of dimension 3 for $\mathfrak{sl}(2)$ as realised in terms of the skew Euler operators. The action is hereby defined as the commutator, and Δ_u serves as the highest weight vector, since it is easily verified that $[\langle \underline{x}, \underline{\partial}_u \rangle, \Delta_u] = 0$. The operator Δ_G then arises as the $\mathfrak{sl}(2)$-invariant inside the tensor product $\mathbb{V}_2 \otimes \mathbb{V}_2$ of this space with itself. Using the notations from [7], as these lie closer to the spirit of the present paper, one has the following analogue of the classical decomposition (1) for a fixed space $\mathcal{P}_k(\mathbb{R}^m, \mathbb{C})$ from the introduction (see [7, 17] for a proof):

Theorem 1 *The space of k-homogeneous polynomials in the skew variables* $X_{ab} := x_a u_b - x_b u_a$ *(with* $1 \le a < b \le m$*) decomposes as*

$$\mathcal{P}_{k,k}(\mathbb{R}^{m\times 2},\mathbb{C})^{\mathrm{SL}(2)} = \bigoplus_{a=0}^{\lfloor \frac{k}{2} \rfloor} |\underline{x} \wedge \underline{u}|^{2a} \mathcal{H}^G_{k-2a}(\mathbb{R}^{m\times 2},\mathbb{C}), \tag{9}$$

with $\mathcal{H}^G_k(\mathbb{R}^{m\times 2},\mathbb{C})$ *defined as* $\mathcal{P}_{k,k}(\mathbb{R}^{m\times 2},\mathbb{C})^{\mathrm{SL}(2)} \cap \ker \Delta_G$*, and where we have used the shorthand notation* $|\underline{x} \wedge \underline{u}|^2 = |\underline{x}|^2|\underline{u}|^2 - \langle \underline{x}, \underline{u}\rangle^2$.

Note that the skew variables X_{ab} hereby play the role of natural 'variables' on $\mathrm{Gr}_o(m,2)$, but one has to bear in mind that they satisfy Plücker relations of the form $X_{ab}X_{cd} + X_{ad}X_{bc} = X_{ac}X_{bd}$. In sharp contrast to the spaces of k-homogeneous harmonic polynomials, which are irreducible under the action of $\mathrm{SO}(m)$, these spaces $\mathcal{H}^G_k(\mathbb{R}^{m\times 2},\mathbb{C})$ decompose further under the regular H-action of $\mathrm{SO}(m)$. This was, for example, proved in [17], using an argument based on the celebrated Schur-Weyl duality. In this paper, we will give an alternative explanation which describes this decomposition in full detail. By this, we mean that also the embedding maps (realising the irreducible components) are explicitly described. This involves the spaces $\mathcal{H}_{k,\ell}(\mathbb{R}^{m\times 2},\mathbb{C})$ of so-called simplicial harmonics (for $k \geq \ell \in \mathbb{Z}^+$). Using the shorthand notation $\ker(D_1,\dots,D_p) := \ker(D_1) \cap \cdots \cap \ker(D_p)$, we recall that these spaces are defined as

$$\mathcal{H}_{k,\ell}(\mathbb{R}^{m\times 2},\mathbb{C}) := \mathcal{P}_{k,\ell}(\mathbb{R}^{m\times 2},\mathbb{C}) \cap \ker\left(\Delta_x, \Delta_u, \langle \underline{\partial}_x, \underline{\partial}_u\rangle, \langle \underline{x}, \underline{\partial}_u\rangle\right). \tag{10}$$

They realise the irreducible representation with highest weight $(k,\ell,0,\dots,0)$ for $\mathrm{SO}(m)$ inside the space of polynomials in two vector variables $(\underline{x},\underline{u}) \in \mathbb{R}^{m\times 2}$ (see, e.g. [3, 10]).

Theorem 2 *The space of k-homogeneous solutions for the operator* Δ_G *decomposes as*

$$\mathcal{H}^G_k(\mathbb{R}^{m\times 2},\mathbb{C}) = \bigoplus_{j=0}^{\lfloor \frac{k}{2} \rfloor} \pi_0\left(|\underline{u}|^{2j}\mathcal{H}_{k,k-2j}(\mathbb{R}^{m\times 2},\mathbb{C})\right). \tag{11}$$

The meaning of the operator π_0 *is described in the proof below. Note also that this space can be indexed by a single parameter* $k \in \mathbb{Z}^+$*, as the degrees of homogeneity in x and u are always equal.*

Proof we first note that each nonzero polynomial $H_{k,\ell}(\underline{x},\underline{u}) \in \mathcal{H}_{k,\ell}(\mathbb{R}^{m\times 2},\mathbb{C})$ generates a model for the finite irreducible representation $\mathbb{V}_{k-\ell}$ of dimension $d = k-\ell+1$ for $\mathfrak{sl}(2)$, realised in terms of the skew Euler operators X and Y. In explicit form, and using the notation $\mathbb{C}v$ to denote the one-dimensional space spanned by a vector v, one then has that

$$\mathbb{V}_{k-\ell} \cong \bigoplus_{j=0}^{k-\ell} \langle \underline{u}, \underline{\partial}_x\rangle^j \mathbb{C}H_{k,\ell}(\underline{x},\underline{u}). \tag{12}$$

Similarly, also the $\mathrm{SO}(m)$-invariant polynomial $|\underline{x}|^{2j}$ generates a model for the representation $\mathbb{V}_{2j}$ of dimension $d = 2j+1$ (containing $|\underline{u}|^{2j} \in \ker(Y)$ as the lowest weight element). This means that for $\ell = k-2j$, the space $\mathcal{H}_{k,k-2j}(\mathbb{R}^{m\times 2},\mathbb{C})$ can in a unique way be embedded in the coordinate ring $\mathcal{R}(\mathrm{Gr}_o(m,2))$. This was already found by Howe and Lee in [17] in an abstract version, but to obtain the explicit embedding maps we first of all note that

$$|\underline{u}|^{2j}H_{k,k-2j}(\underline{x},\underline{u}) \in \mathbb{V}_{2j} \otimes \mathbb{V}_{2j},$$

where the trivial representation $\mathbb{V}_0$ sits inside this tensor product $\mathbb{V}_{2j} \otimes \mathbb{V}_{2j}$ (easy application of the Clebsch–Gordan rule). At the same time, we also have that $|\underline{u}|^{2j}\mathcal{H}_{k,k-2j}(\mathbb{R}^{m\times 2}, \mathbb{C}) \subset \mathcal{P}_{k,k}(\mathbb{R}^{m\times 2}, \mathbb{C})$, which means that the projection π_0 on the trivial component $\mathbb{V}_0$ realises our embedding:

$$\pi_0\left(|\underline{u}|^{2j}H_{k,k-2j}(\underline{x},\underline{u})\right) \in \mathcal{P}_{k,k}(\mathbb{R}^{m\times 2}, \mathbb{C})^{\mathrm{SL}(2)}.$$

To see that this projection is non-trivial, it suffices to observe that any nonzero polynomial $H_{k,k-2j}(\underline{x},\underline{u}) \in \mathcal{H}_{k,k-2j}(\mathbb{R}^{m\times 2}, \mathbb{C})$ serves as a highest weight vector w^+ for $\mathbb{V}_{2j}$, whereas the invariant polynomial $|\underline{u}|^{2j}$ serves as a lowest weight vector w^-. The tensor product $w^+ \otimes w^-$ then clearly has total weight zero and can be projected onto the trivial summand. Indeed, it suffices to determine constants $c_p \in \mathbb{R}$ such that

$$w^+ \otimes w^- + \sum_{p=1}^{2j} c_p Y^p[w^+] \otimes X^p[w^-] \in \ker(X).$$

Using standard commutation relations in the Lie algebra $\mathfrak{sl}(2)$, this reduces to a simple recursive system (choosing the operator Y instead of X leads to the same system of equations fixing these constants). Finally, in order to prove that this summand belongs to $\ker(\Delta_G)$, we note that it suffices to verify that Δ_G acts trivially on the polynomials $|\underline{u}|^{2j}H_{k,k-2j}(\underline{x},\underline{u})$. As Δ_G is invariant under the action of $\mathfrak{sl}(2)$, this immediately implies that this operator acts as a multiple of the identity on all the irreducible components inside $\mathbb{V}_{2j} \otimes \mathbb{V}_{2j}$, so, in particular, on the trivial summand constructed above. A simple direct calculation shows that $\Delta_G|\underline{u}|^{2j}H_{k,k-2j}(\underline{x},\underline{u}) = 0$. □

In [7], this projection on $\mathbb{V}_0 \subset \mathbb{V}_{2j} \otimes \mathbb{V}_{2j}$ was obtained in a different way, using the power of the extremal projection operator for the Lie algebra $\mathfrak{sl}(2)$, see, e.g. [25]. This led to the $\mathrm{SO}(m)$-invariant mapping R_j, given by

$$R_j := \sum_{i=0}^{2j} \frac{(-1)^i(2j-i)!}{(2j+1)!i!}\left(\langle \underline{x}, \underline{\partial}_u\rangle^i |\underline{u}|^{2j}\right)\langle \underline{u}, \underline{\partial}_x\rangle^i. \tag{13}$$

This mapping also arises in another way, which we would like to describe here as it sheds light on how functions on the Grassmannian are related to functions on the Stiefel manifold. Recall that functions on the latter can be obtained through restriction of functions $f(\underline{x},\underline{u})$ on $\mathbb{R}^{m\times 2}$: it suffices to put $|\underline{x}|^2 = |\underline{u}|^2 = 1$ and $\langle \underline{x},\underline{u}\rangle = 0$. In particular, one may start from the space $\mathcal{P}(\mathbb{R}^{m\times 2}, \mathbb{C})$ of polynomials in two vector variables, which decomposes as

$$\mathcal{P}(\mathbb{R}^{m\times 2}, \mathbb{C}) \cong \bigoplus_{k\geq \ell} \mathcal{I} \otimes d_{k,\ell}\mathbb{V}_{k-\ell},$$

with $\mathcal{I} := \mathrm{Alg}(|\underline{x}|^2, |\underline{u}|^2, \langle \underline{x},\underline{u}\rangle)$ the algebra of polynomial invariants in two vector variables, where the modules $\mathbb{V}_{k-\ell}$ are realised as in (12), and where $d_{k,\ell}$ denotes the dimension of the space $\mathcal{H}_{k,\ell}(\mathbb{R}^{m\times 2}, \mathbb{C})$ as there is actually one copy for each linearly independent polynomial $H_{k,\ell}(\underline{x},\underline{u})$. We refer to the work of Howe [16] for more details. (Note that he adopts the convention that $\mathbb{V}_{k,\ell}$ is realised by the full space $\mathcal{H}_{k,\ell}$, so that there is no need to work with the integers $d_{k,\ell}$.) This decomposition shows that the restriction to $\mathrm{St}(m, 2)$ makes the tensor product with $\mathcal{I}$ obsolete, which means that the space of polynomial functions on the Stiefel manifold can actually be identified with the space

$$\bigoplus_{k\geq\ell} d_{k,\ell}\mathbb{V}_{k-\ell} = \mathcal{P}(\mathbb{R}^{m\times 2},\mathbb{C}) \cap \ker\left(\Delta_x, \Delta_u, \langle \underline{\partial}_x, \underline{\partial}_u\rangle\right). \tag{14}$$

This space then provides a model for $L^2(\mathrm{St}(m,2))$, see [20]. It is crucial to point out that the operator $\langle \underline{u}, \underline{\partial}_x\rangle$ does *not* appear here, in sharp contrast to the earlier definition for the spaces $\mathcal{H}_{k,\ell}(\mathbb{R}^{m\times 2},\mathbb{C})$ in (10). We then have the following result:

Theorem 3 *The irreducible components in* $L^2(\mathrm{St}(m,2))$ *which exhibit an additional invariance with respect to the regular action of* $\mathrm{SO}(2)$ *correspond to the summands in the space* $\mathcal{P}(\mathbb{R}^{m\times 2},\mathbb{C})$ *which are invariant with respect to the regular* $\mathrm{SL}(2)$*-action, when putting* $|\underline{x}\wedge\underline{u}|^2 = 1$.

Proof In order to locate the SO(2)-invariant polynomials inside the modules $\mathbb{V}_{k-\ell}$, we will make use of the fact that this rotation group sits inside SL(2) as the exponential map acting on the element $(X-Y) \in \mathfrak{sl}(2)$. We therefore choose an new basis for the Lie algebra $\mathfrak{sl}(2) = \mathrm{Alg}(X,Y,H)$ from (6), where we will use the subscript 'N' for 'new':

$$2X_N := X + Y + iH \quad \text{and} \quad 2Y_N := X + Y - iH,$$

with $H_N := [X_N, Y_N] = i(X-Y)$. Projection on the zero weight space for the Cartan element H_N then gives the desired invariance with respect to SO(2). This immediately tells us that for $k-\ell$ odd, no such element can be found as the projection is trivial in that case. If $k-\ell$ is even, which means that $\ell = k-2j$, it is readily verified that this element is given by

$$\varphi_j[H_{k,k-2j}(\underline{x},\underline{u})] := \sum_{i=0}^{j} \frac{\Gamma\left(j+\frac{1}{2}-i\right)}{2^{2i} i!\,\Gamma\left(j-\frac{1}{2}\right)} \langle \underline{u}, \underline{\partial}_x\rangle^{2i}[H_{k,k-2j}(\underline{x},\underline{u})].$$

We hereby introduced the notation φ_j for this projection, so we get

$$\mathrm{Res}\Big|_{\mathrm{St}(m,2)} \circ \varphi_j : \mathcal{H}_{k,k-2j}(\mathbb{R}^{m\times 2},\mathbb{C}) \to L^2(\mathrm{St}(m,2))^{\mathrm{SO}(2)}.$$

To show that this gives a smooth function on $\mathrm{Gr}_o(m,2)$, we will show that the same mapping (up to a constant) is obtained by restricting elements in $\pi_0\left(|\underline{u}|^{2j}\mathcal{H}_{k,k-2j}(\mathbb{R}^{m\times 2},\mathbb{C})\right)$ to $\mathrm{Gr}_o(m,2)$, by putting $|\underline{x}\wedge\underline{u}|^2 = 1$. It is now easily seen that this amounts to putting $|\underline{x}|^2 = |\underline{u}|^2 = 1$ and $\langle \underline{x},\underline{u}\rangle = 0$, which means that expression (13) will reduce to a single term (with $i = 2j$). This follows from the fact that

$$\langle \underline{x}, \underline{\partial}_u\rangle^i |\underline{u}|^{2j} = \sum_{t=0}^{\lfloor \frac{i}{2}\rfloor} c_t(i,j)|\underline{u}|^{2j-2i+2t}|\underline{x}|^{2t}\langle \underline{x},\underline{u}\rangle^{i-2t},$$

where the constant $c_t(i,j)$ is expressed in terms of rising factorials as

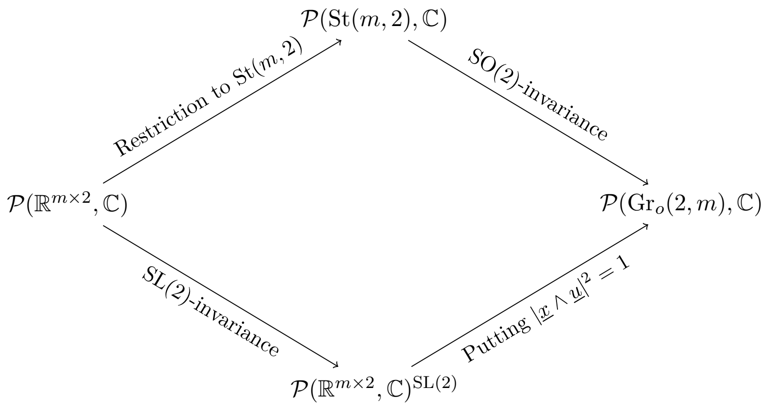
$$c_t(i,j) = \frac{(-1)^{i-t}2^{i-2t}(-i)^{(2t)}(-j)^{(i-t)}}{t!},$$

and plugging this into the definition for R_j leads to

$$\left(R_j H_{k,k-2j}\right)\Big|_{\mathrm{St}(m,2)} = \frac{2^{2j} j!\Gamma\left(j-\frac{1}{2}\right)}{(2j+1)!\Gamma\left(\frac{1}{2}\right)}\varphi_j\left[H_{k,k-2j}\right] = \frac{\varphi_j\left[H_{k,k-2j}\right]}{2j-1},$$

which thus proves the theorem. □

Remark 3 This result is somewhat remarkable, as it is a priori not clear that picking up invariant components with respect to the subgroup SO(2) gives the same as considering invariants for the action of the larger group SL(2). However, the restriction $|\underline{x} \wedge \underline{u}|^2 = 1$ does ensure that this is indeed equivalent.



The reader may wonder where the condition $|\underline{x} \wedge \underline{u}|^2 = 1$ comes from. (It is the equivalent of putting $|\underline{x}|^2 = 1$ when restricting a function on $\mathbb{R}^m$ to the sphere S^{m-1}.) This has to do with the fact that this invariant polynomial, both for the action of SO(m) and SL(2), serves as the adjoint of the operator Δ_G. This can be understood with respect to the Fischer inner product on $\mathcal{P}(\mathbb{R}^{m\times 2},\mathbb{C})$, defined by means of

$$[P(\underline{x},\underline{u}),Q(\underline{x},\underline{u})]_F := P^c(\partial_{\underline{x}},\partial_{\underline{u}})Q(\underline{x},\underline{u})\Big|_{\underline{x}=\underline{u}=\underline{0}}.$$

Here, $P^c(\partial_{\underline{x}},\partial_{\underline{u}})$ stands for the complex conjugate of the given polynomial $P(\underline{x},\underline{u})$ in which all the variables are replaced by the corresponding partial derivatives. Just like in the classical case (harmonic analysis on $\mathbb{R}^m$), we then arrive at a full decomposition of the space of polynomials involving the operator Δ_G and its adjoint $|\underline{x} \wedge \underline{u}|^2$. In the following theorem, we give a multiplicity-free version of this decomposition (which already appeared in, for instance, [7, 17]). This thus involves a dual symmetry algebra, which will be identified as the so-called Higgs algebra:

Definition 1 The Higgs algebra H_3 is defined as a polynomial deformation of the Lie algebra $\mathfrak{su}(2)$, generated by three elements D and $A_\pm$ satisfying the relations $[D,A_\pm] = \pm 4A_\pm$ and $[A_+,A_-] = -D^3 + \alpha_1 D + \alpha_2$, with α_1,α_2 constants (or, more generally, central elements).

This algebra first appeared in [14], as the algebra realised by the conserved quantities of the Coulomb problem and harmonic oscillator on the sphere $S^2 \subset \mathbb{R}^3$. Since then, it has reappeared under several different guises in the mathematical literature, for instance, in connection with the Hahn and Racah algebra (see for instance [5, 9]). In [24], this algebra was studied in connection with the quantum algebra $SU_q(2)$, as the Higgs relations can be seen as a second-order approximation (in the deformation parameter ν) for the relations

$$[J_0, J_\pm] = \pm J_\pm \qquad \text{and} \qquad [J_+, J_-] = \frac{\sinh(2\nu J_0)}{\sinh(\nu)}.$$

We now claim that this algebra also appears in the framework of harmonic analysis on the Grassmannian $\mathrm{Gr}_o(m, 2)$:

Theorem 4 *The operators $A_+ := |\underline{x} \wedge \underline{u}|^2$, $A_- := \Delta_G$ and $D := \mathbb{E}_x + \mathbb{E}_u + m$, considered as differential operators acting on functions $f(\underline{x}, \underline{u})$ in the kernel of the skew Euler operators, generate a copy of the Higgs algebra H_3. As a matter of fact, one has that*

$$\begin{aligned} [2\mathbb{E} + m, |\underline{x} \wedge \underline{u}|^2] &= +4|\underline{x} \wedge \underline{u}|^2 \\ [2\mathbb{E} + m, \Delta_G] &= -4\Delta_G \\ [|\underline{x} \wedge \underline{u}|^2, \Delta_G] &= -(2\mathbb{E} + m)^3 + \alpha_1(2\mathbb{E} + m), \end{aligned}$$

where the operator $\alpha_1 := (m^2 - 6m + 6) - 2C_2(H)$, with $C_2(H)$ the second-order Casimir operator for the regular action of the Lie algebra $\mathfrak{so}(m)$ on functions $f(\underline{x}, \underline{u})$ on $\mathbb{R}^{m\times 2}$. Although the operator $C_2(H)$ is not a numerical constant, it commutes with D and $A_\pm$ and can hence be seen as a central element.

Remark 4 Note that, as pointed out earlier, we can replace $\mathbb{E}_x + \mathbb{E}_u$ by $2\mathbb{E}$ because we are considering functions on which $X, Y \in \mathfrak{sl}(2)$ act trivially. This implies that $(\mathbb{E}_x - \mathbb{E}_u)f(\underline{x}, \underline{u}) = 0$, so $\deg(\underline{x}) = \deg(\underline{u})$. Alternatively, one can use the notation $\mathbb{E}_T$ here, where 'T' refers to the total degree. One then has that $2\mathbb{E} = \mathbb{E}_T$.

Proof First of all, we note that a tedious albeit straightforward calculation (see, for instance, [19]) gives that

$$[\Delta_G, |\underline{x} \wedge \underline{u}|^2] = 2(2\mathbb{E} + m)\big(3(2\mathbb{E} + m - 1) + T_{x,u}\big), \tag{15}$$

where the operator $T_{x,u} = |\underline{x}|^2\Delta_x + |\underline{u}|^2\Delta_u + 2\langle \underline{x}, \underline{u}\rangle\langle \underline{\partial}_x, \underline{\partial}_u\rangle$ is the operator which corresponds to the trivial component $\mathbb{V}_0 \subset \mathbb{V}_2 \otimes \mathbb{V}_2$, with $\mathbb{V}_2$ realised as the $\mathfrak{sl}(2)$-module generated by the highest weights $|\underline{x}|^2$ and Δ_u, respectively. It then suffices to note that the Casimir operator $C_2(H)$ for the regular representation of $\mathfrak{so}(m)$ on functions in two variables $(\underline{x}, \underline{u}) \in \mathbb{R}^{m\times 2}$ can (in full generality) be written as (see, for instance, [3])

$$\begin{aligned} C_2(H) &= \sum_{a<b} (L^x_{ab} + L^u_{ab})^2 \\ &= \Delta^x_{LB} + \Delta^u_{LB} + 2\big(\langle \underline{x}, \underline{u}\rangle\langle \underline{\partial}_x, \underline{\partial}_u\rangle - \langle \underline{u}, \underline{\partial}_x\rangle\langle \underline{x}, \underline{\partial}_u\rangle + \mathbb{E}_u\big), \end{aligned}$$

with, e.g. $\Delta^x_{LB} = |\underline{x}|^2\Delta_x - \mathbb{E}_x(\mathbb{E}_x + m - 2)$ the Laplace–Beltrami operator (or, equivalently, the Casimir operator $C_2(H)$ for the regular action of $\mathfrak{so}(m)$ in one vector variable $\underline{x} \in \mathbb{R}^m$). When acting on functions $f(\underline{x}, \underline{u})$ in the kernel of X and $Y \in \mathfrak{sl}(2)$, we thus find that $T_{x,u} = C_2(H) + 2\mathbb{E}(\mathbb{E} + m - 3)$, after which the result easily follows from relation (15). □

In view of the fact that a Casimir element can be rescaled and shifted by a numerical constant, the central element α_1 appearing in the theorem above is itself a Casimir operator of second order for the regular representation of $\mathfrak{so}(m)$ on functions $f(\underline{x}, \underline{u})$ in two vector variables. Its eigenvalues on the spaces $\mathcal{H}_{k,\ell}(\mathbb{R}^{m\times 2}, \mathbb{C})$ easily follows from the fact that

$$-2C_2(H)[\mathcal{H}_{k,\ell}] = 2\big(k(k+m-2) + \ell(\ell+m-4)\big)\mathcal{H}_{k,\ell},$$

see, for instance, [3]. Completing the squares above and adding $(m^2 - 6m + 6)$, one finds that $\alpha_1[\mathcal{H}_{k,\ell}] = a_{k,\ell}\mathcal{H}_{k,\ell}$ with

$$a_{k,\ell} = 2\left(\left(k + \frac{m}{2}\right)\left(k + \frac{m}{2} - 2\right) + \left(\ell + \frac{m}{2} - 1\right)\left(\ell + \frac{m}{2} - 3\right)\right). \tag{16}$$

Especially, the case where $\ell = k - 2j$ will be useful for us, see below.

In what follows, we will sometimes need the following operator identity on functions in the kernel of the skew Euler operators (which can be seen as an identity in the universal enveloping algebra of the Higgs algebra H_3 as realised above):

Proposition 1 *For all positive integers $a \in \mathbb{Z}^+$, one has that*

$$[A_-, A_+^a] = aA_+^{a-1}P_3(D), \tag{17}$$

where $P_3(D) = D^3 + c_2D^2 + c_1D + c_0$ is the cubic polynomial defined by

$$\begin{aligned} c_2 &= 6(a-1) \\ c_1 &= 2C_2(H) - m(m-6) + 2(2a+1)(4a-5) - 12(a-1) \\ c_0 &= 2(a-1)(2C_2(H) - m(m-6) + 2(2a+1)(2a-3)). \end{aligned}$$

Proof This can be proved by induction on $a \in \mathbb{Z}^+$, hereby using the previous theorem for the basic case $a = 1$. □

In the classical case (harmonic analysis in one variable in $\mathbb{R}^m$), a crucial role is played by the algebra $\mathfrak{sl}(2)$, appearing as the Howe dual partner (2) generated by the invariants. In particular, one has that the decomposition (1) becomes multiplicity-free via the introduction of certain Verma modules for $\mathfrak{sl}(2)$. In the present situation, a similar conclusion can be made if one defines a suitable class of infinite-dimensional irreducible representations for the Higgs algebra H_3. More information on unitary ladder representations for H_3 can, for instance, be found in [24]. For our purpose, we will first define them ad hoc here and compare our modules with the ones appearing in [24] afterwards.

Theorem 5 *Each space $\pi_0\big(|\underline{u}|^{2j}\mathcal{H}_{k,k-2j}(\mathbb{R}^{m\times 2}, \mathbb{C})\big)$ appearing in $\mathcal{H}_k^G$ (see theorem 2) generates an irreducible module $\mathbb{V}_{k,j}^\infty$ for the Higgs algebra H_3 from above. This module is lowest weight, is infinite-dimensional and can be completely decomposed into weight spaces on which the operator $D = 2\mathbb{E} + m$ acts diagonally. The eigenvalues for this D-action are given by the integers $(4a + 2k + m)$, with $a \in \mathbb{Z}^+$.*

Proof It is obvious that for each nonzero polynomial $H_{k,k-2j}(\underline{x},\underline{u})$ in the space $\mathcal{H}_{k,k-2k}(\mathbb{R}^{m\times 2},\mathbb{C})$ one has that $v_{k,j} := \pi_0\big(|\underline{u}|^{2j}H_{k,k-2j}(\underline{x},\underline{u})\big)$ serves as the lowest weight as it is killed by the operator Δ_G, and that the module generated by this vector can be defined as

$$\mathbb{V}_{k,j}^{\infty} := \bigoplus_{a=0}^{\infty} |\underline{x}\wedge\underline{u}|^{2a}\mathbb{C}v_{k,j} \subset \mathcal{P}(\mathbb{R}^{m\times 2},\mathbb{C})^{\mathrm{SL}(2)}.$$

To prove that this module is indeed irreducible, we note that for any element $v \in \mathbb{V}_{k,j}^{\infty}$, there exists a unique integer $n \in \mathbb{Z}^+$ such that $\Delta_G^n v = \lambda v_{k,j}$, with $\lambda \in \mathbb{R}_0$ a non-trivial constant. (As a matter of fact, this n will be the maximal index a appearing in the expression for v as a linear combination.) Indeed, introducing the shorthand notation $v_{k,j}(a)$ for the weight vector $|\underline{x}\wedge\underline{u}|^{2a}v_{k,j}$, relation (17) leads to

$$\begin{aligned}\Delta_G v_{k,j}(a) &= [\Delta_G, |\underline{x}\wedge\underline{u}|^{2a}]v_{k,j}\\ &= 2a|\underline{x}\wedge\underline{u}|^{2a-2}\big(D+2a-2\big)\big((2a+1)(D+2a-3)+T_{x,u}\big)v_{k,j}.\end{aligned}$$

As the operator $C_2(H)$ acts on $v_{k,j}$ by means of

$$C_2(H)v_{k,j} = -2k(k+m-3)v_{k,j} + 2j\big(2(k-j)+m-4\big)v_{k,j},$$

see, for instance, [3], we find that $T_{x,u}v_{k,j} = 2j\big(2(k-j)+m-4\big)v_{k,j}$. Plugging this into the expression for $\Delta_G v_{k,j}(a)$, we can easily see that for all $a \neq 0$, one has that $\Delta_G v_{k,j}(a) = \lambda_a v_{k,j}(a-1)$ with $\lambda_a \neq 0$ (recall that $m \geq 4$). An n-fold application of the operator Δ_G then indeed leads to an expression of the form $\Delta_G^n v_{k,j}(n) = \lambda v_{k,j}$, with $\lambda \neq 0$. Once the lowest weight space has been reached, the complete module $\mathbb{V}_{k,j}^{\infty}$ can be reconstructed. □

Bringing everything together, we thus arrive at the following conclusion:

Theorem 6 *The polynomial space* $\mathcal{P}(\mathbb{R}^{m\times 2},\mathbb{C})^{\mathrm{SL}(2)}$ *decomposes under the joint action of the product* $\mathrm{SO}(m)\times H_3$ *by means of*

$$\mathcal{P}(\mathbb{R}^{m\times 2},\mathbb{C})^{\mathrm{SL}(2)} \cong \bigoplus_{k=0}^{\infty}\bigoplus_{j=0}^{\lfloor \frac{k}{2}\rfloor} \mathcal{H}_{k,k-2j}(\mathbb{R}^{m\times 2},\mathbb{C}) \otimes \mathbb{V}_{k,j}^{\infty},$$

with $\mathbb{V}_{k,j}^{\infty}$ *the infinite-dimensional* H_3*-module defined above.*

To conclude this section, we will compare our modules $\mathbb{V}_{k,j}^{\infty}$ with the ones obtained by Zhedanov in [24]. To do so, we first identify our generators from theorem 4 with the operators (N_+, N_-, N_0) used in that paper:

$$N_+ := \frac{1}{4}|\underline{x}\wedge\underline{u}|^2 \qquad N_- := \frac{1}{4}\Delta_G \qquad N_0 := \frac{1}{2}\mathbb{E} + \frac{m}{4}.$$

The defining relation $[N_+, N_-] = 2N_0(a + 2hN_0^2)$ is then satisfied in our model for the central elements $h = -1$ and $8a = \alpha_1$. In [24], the author then proceeds by defining unitary ladder representations of the Higgs algebra in terms of an orthonormal basis $\{w_p\}$, with p a real discrete variable with unit step:

$$N_0 w_p = p w_p \qquad N_+ w_p = \bar{A}_{p+1} w_{p+1} \qquad N_- w_p = A_p w_{p-1}.$$

As the Casimir operator for the algebra H_3 is defined as

$$C_h := N_+ N_- + a(N_0 - 1)N_0 + hN_0^2(N_0 - 1)^2, \tag{18}$$

one can define the number C_μ as the eigenvalue of the Casimir operator C_h for the Higgs algebra on an irreducible module $\mathbb{V}_\mu = \text{span}(w_p)$. One then finds that

$$|A_p|^2 = C_\mu - ap(p-1) - hp^2(p-1)^2. \tag{19}$$

For our representations $\mathbb{V}_\mu = \mathbb{V}_{k,j}^\infty$ from theorem 5, we note that A_p is real and that the eigenvalues of the operator N_0 on the basis elements w_p are given by

$$N_0 w_p := \left(\frac{1}{2}\mathbb{E} + \frac{m}{4}\right)\left(c_{a,k,j}|\underline{x} \wedge \underline{u}|^{2a} v_{k,j}\right) = \left(\frac{k}{2} + \frac{m}{4} + a\right)\left(c_{a,k,j}|\underline{x} \wedge \underline{u}|^{2a} v_{k,j}\right),$$

with $c_{a,k,j}$ a suitable constant to ensure that $\{w_p\}$ is an orthonormal basis. If we now introduce the shorthand notation $K_m := \frac{k}{2} + \frac{m}{4}$, so that $p = K_m + a$ (with $a \in \mathbb{Z}^+$), one can easily verify that the eigenvalue of C_h on the module $\mathbb{V}_\mu = \mathbb{V}_{k,j}^\infty$ is given by

$$C_\mu = \frac{\alpha_1}{8} K_m(K_m - 1) - K_m^2(K_m - 1)^2.$$

Plugging this into expression (19) for A_p^2, using (16) to replace the central Casimir element α_1 by the number

$$a_{k,k-2j} = 8\Big(K_m(K_m - 1) + \Big(K_m - j - \frac{1}{2}\Big)\Big(K_m - j - \frac{3}{2}\Big)\Big),$$

an easy calculation shows that

$$A_p^2 = (f(K+a) - f(K))\Big(f(K+a) - f\Big(K - j - \frac{1}{2}\Big)\Big),$$

where $f(x) := x(x-1)$. Taking into account that $m \geq 4$ in our analysis, together with the fact that $k \geq 2j$, it is clear that $A_p > 0$ for $a \geq 1$ and that $A_p = 0$ for $a = 0$. This means that the modules appearing in theorem 4 are examples of what has been described in [24] as 'infinite-dimensional discrete series' (with eigenvalues $K_m + a$ for $a \in \mathbb{Z}^+$).

Because we have realised the Higgs algebra H_3 in terms of differential operators which remain invariant under a suitable action of the orthogonal group SO(m), it comes as no surprise that also the Casimir operators are related. To illustrate this, hence obtaining an explicit formula for the central element C_h for the algebra H_3 in terms of the Casimir elements C_4 and C_2 for the Lie algebra $\mathfrak{so}(m)$. First of all, recall that

$$c_q := \sum_{1 \leq i_1, \ldots, i_q \leq m} e_{i_1 i_2} e_{i_2 i_3} \ldots e_{i_q i_1} \in \mathcal{U}\big(\mathfrak{so}(m)\big)$$

denotes a central element in the universal enveloping algebra for $\mathfrak{so}(m)$. Note that this notation is slightly different from the one used earlier in this paper, but when considering the regular representation one has that

$$C_2(H) = \sum_{i<j} e_{ij}^2 = -\frac{1}{2} \sum_{i,j} e_{ij}e_{ji} = -\frac{1}{2}c_2.$$

According to [15], the eigenvalue of the Casimir operator c_q on an irreducible representation $\mathbb{V}_\rho$ is given by the number

$$\rho(c_q) = \sum_{\lambda} \frac{\dim \mathbb{V}_\lambda}{\dim \mathbb{V}_\rho} w(\lambda;\rho)^q, \tag{20}$$

where $\lambda \in \mathfrak{h}^*$ runs over all the highest weights of irreducible components $\mathbb{V}_\lambda$ appearing in the tensor product $\mathbb{V}_\rho \otimes \mathbb{C}^m$. These λ are all of the form $\rho \pm e_j$, provided that the dominant weight condition is satisfied. The number $w(\lambda;\rho)$ is given by

$$w(\lambda;\rho) := \frac{1}{2}\big(\|\lambda + \delta\| - \|\rho + \delta\| - (m-1)\big),$$

with δ half the sum of the positive roots. In our case, for $\rho = (k, \ell)$, we may have up to five components in the tensor product $\mathbb{V}_\rho \otimes \mathbb{C}^m$ (three when $k = \ell$): for $j \in \{1, 2, 3\}$, we have highest weights $\lambda_j^+ := \rho + e_j$, and for $j \in \{1, 2\}$ we have $\lambda_j^- := \rho - e_j$. The numbers $w(\lambda;\rho)$ which correspond to these weights are given by

$$\begin{aligned} w(\lambda_1^+;\rho) &= k \quad w(\lambda_2^+;\rho) = \ell - 1 \quad w(\lambda_3^+;\rho) = -2 \\ w(\lambda_1^-;\rho) &= 2-k-m \quad w(\lambda_2^-;\rho) = 3-\ell-m. \end{aligned}$$

We also need the so-called relative dimensions, given by

$$\frac{d(\lambda)}{d(\rho)} = \frac{\dim \mathbb{V}_\lambda}{\dim \mathbb{V}_\rho} = \big(2w(\lambda;\rho) + m\big) \prod_{\lambda' \neq \lambda} \frac{w(\lambda;\rho) + w(\lambda';\rho) + m - 1}{w(\lambda;\rho) - w(\lambda';\rho)}.$$

Straightforward but cumbersome calculations give the following numbers in our case:

$$\begin{aligned}
\frac{d(\lambda_1^+)}{d(\rho)} &= \frac{(k-\ell+2)(m+k-3)(m+2k)(m+\ell+k-2)}{(k+2)(k-\ell+1)(m+2k-2)(m+\ell+k-3)} \\
\frac{d(\lambda_2^+)}{d(\rho)} &= \frac{(k-\ell)(m+\ell-4)(m+\ell+k-2)(m+2\ell-2)}{(\ell+1)(k-\ell+1)(m+\ell+k-3)(m+2\ell-4)} \\
\frac{d(\lambda_3^+)}{d(\rho)} &= \frac{\ell(k+1)(m-4)(m+k-3)(m+\ell-4)}{(k+2)(\ell+1)(m+k-4)(m+\ell-5)} \\
\frac{d(\lambda_1^-)}{d(\rho)} &= \frac{(k+1)(k-\ell)(m+2k-4)(m+\ell+k-4)}{(k-\ell+1)(m+k-4)(m+2k-2)(m+\ell+k-3)} \\
\frac{d(\lambda_2^-)}{d(\rho)} &= \frac{\ell(k-\ell+2)(m+\ell+k-4)(m+2\ell-6)}{(k-\ell+1)(m+\ell-5)(m+\ell+k-3)(m+2\ell-4)}.
\end{aligned}$$

Using these numbers, we can then finally invoke formula (20) to calculate the eigenvalues for the Casimir operators c_2 and c_4 for the Lie algebra $\mathfrak{so}(m)$:

$$\rho(c_2) = \sum_\lambda \frac{d(\lambda)w(\lambda,\rho)^2}{d(\rho)} = 2\big(k^2 + (m-2)k + \ell^2 + (m-4)\ell\big)$$
$$\rho(c_4) = \sum_\lambda \frac{d(\lambda)w(\lambda,\rho)^4}{d(\rho)} = \mu_3 m^3 + \mu_2 m^2 + \mu_1 m + \mu_0,$$

where the integer coefficients μ_j (in terms of m, k and ℓ) are given by

$$\begin{aligned}
\mu_3 &= k + \ell\\
\mu_2 &= 3(k^2+\ell^2) - (7k + 13\ell)\\
\mu_1 &= 4(k^3+\ell^3) - (13k^2 + 25\ell^2) + (16k + 54\ell)\\
\mu_0 &= 2(k^4+\ell^4) - (8k^3 + 16\ell^3) + (14k^2 + 50\ell^2) - (12k + 72\ell).
\end{aligned}$$

Note that we will need to replace $\ell = k - 2j$ in what follows, as these are the components appearing in the decomposition of the kernel $\ker \Delta_G$. We will now compare the Casimir operators c_2 and c_4 with the Casimir operator C_h for the Higgs algebra, given by formula (18). In [13], it was shown that the algebra of invariant differential operators on the Grassmannian $\mathrm{Gr}_o(m,2)$ is generated by c_2 and c_4, so we can expect $|\underline{x}\wedge\underline{u}|^2\Delta_G$ and C_h to be realised in terms of these Casimir operators for $\mathfrak{so}(m)$. In [23], it was shown that the operator $|\underline{x}\wedge\underline{u}|^2\Delta_G$ is algebraically independent from c_2, which means that c_4 will indeed be needed. We then arrive at the following conclusion, which can be seen as the generalisation of the classical operator identity

$$-\frac{1}{2}c_2 = C_2(H) = |\underline{x}|^2\Delta_x - \mathbb{E}_x(\mathbb{E}_x + m - 2)$$

on the sphere $S^{m-1} \subset \mathbb{R}^m$.

Theorem 7 *One has the following operator identity on* $\mathcal{C}^\infty\big(\mathrm{Gr}_o(m,2)\big)$:

$$16C_h = -\frac{1}{4}c_4 + \frac{1}{8}c_2^2 + \frac{(m-2)(2m-9)}{8}c_2 + \frac{m(m-2)(m-4)(m-6)}{16}.$$

Proof Plugging in the operators $N_\pm$ and N_0 which define our model (see above), we find that

$$\begin{aligned}
16C_h = {} & |\underline{x}\wedge\underline{u}|^2\Delta_G + \frac{1}{2}\big(\rho(c_2) + (m^2 - 6m + 6)\big)\Big(\mathbb{E} + \frac{m}{2}\Big)\Big(\mathbb{E} + \frac{m}{2} - 2\Big)\\
& - \Big(\mathbb{E} + \frac{m}{2}\Big)^2\Big(\mathbb{E} + \frac{m}{2} - 2\Big)^2.
\end{aligned}$$

In view of the invariance, the only thing left to do now is to compare the action of this operator C_h and the invariant operator

$$C_{\mathrm{comb}} := \alpha c_4 + \beta c_2^2 + \gamma c_2 + \delta$$

on the element $v_{k,j} \in \ker \Delta_G$ (in order to determine the unknown coefficients α, β, γ and δ). An easy calculation shows that

$$16C[v_{k,j}] = \Big(k + \frac{m}{2}\Big)\Big(k + \frac{m}{2} - 2\Big)\Big(k + \frac{m}{2} - 2j - 1\Big)\Big(k + \frac{m}{2} - 2j - 3\Big)v_{k,j},$$

whereas previous calculations tell us that the eigenvalues for c_4 and c_2 on $v_{k,j}$ are given by

$$\rho(c_4) = 32j^4 + 4k^4 + \text{lower order terms}$$
$$\rho(c_2) = 64j^4 + 16k^4 + \text{lower order terms.}$$

Comparing the coefficients in (j^4, k^4), we immediately see that $\alpha + 2\beta = 0$ and also $4\alpha + 16\beta = 1$. Hence, we get $(\alpha, \beta) = (-\frac{1}{4}, \frac{1}{8})$. In a completely similar fashion, one can then also determine the coefficients γ and δ, which leads to the desired result. □

3 Pizzetti formula on the Grassmannian

In this section, we will derive a Pizzetti formula for the integral of a suitable function f on the Grassmannian $\mathrm{Gr}_o(m, 2)$. This formula first appeared in [21], but has since then be used in a variety of contexts. We mention [6] and [4] for applications in super analysis and random matrix theory. The idea behind Pizzetti formulae is that one can express the integral of a suitable function over a manifold in terms of a formal series expressed in terms of an invariant differential operator. Pizzetti's original formula, for instance, expresses the integral of a function $f(\underline{x})$ on $\mathbb{R}^m$ whose Taylor expansion at the origin converges in a neighbourhood of the unit sphere S^{m-1} as

$$\int_{S^{m-1}} f \mathrm{d}\sigma := \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{m}{2}\right)}{4^n n! \Gamma\left(n + \frac{m}{2}\right)} (\Delta_x^n f)(\underline{0}),$$

where Δ_x denotes the Laplace operator on $\mathbb{R}^m$. In order to find a similar formula for the manifold $\mathrm{Gr}_o(m, 2)$, we first introduce the following:

Definition 2 The space $\mathcal{A}_G(\mathbb{R}^{m\times 2})$ contains functions $f(\underline{x}, \underline{u})$ which belong to $C^\infty(\Omega)$ with $\langle \underline{u}, \underline{\partial}_x \rangle f = \langle \underline{x}, \underline{\partial}_u \rangle f = 0$, where $\Omega \subset \mathbb{R}^{m\times 2}$ denotes an open subset such that $\{\lambda A g_2 : A \in \mathrm{St}(m, 2), \lambda \in [0, 1], g_2 \in \mathrm{SO}(2)\} \subset \Omega$, and such that the Taylor series of f at the origin converges uniformly in Ω.

The guiding principle behind our generalisation of the Pizzetti formula is the following: we are looking for a functional I_G on $\mathcal{A}_G(\mathbb{R}^{m\times 2})$ which satisfies two conditions. One must have that

(i) $I_G\left(|\underline{x} \wedge \underline{u}|^2 f\right) = I_G(f)$, a condition which essentially says that one can restrict to $|\underline{x} \wedge \underline{u}|^2 = 1$.
(ii) $I_G(Mf) = I_G(f)$ for all $M \in \mathrm{SO}(m)$. This condition then expresses the invariance of the measure $\mathrm{d}\mu$ appearing in the integral (see below, in theorem 8).

Condition (*ii*) allows us to prove the following:

Lemma 1 *The action of I_G restricted to $\mathcal{H}_k^G(\mathbb{R}^{m\times 2}, \mathbb{C})$ is trivial for all strictly positive integers $k > 0$.*

Proof This easily follows from Schur's Lemma, since I_G commutes with the action of SO(m) on functions $f(\underline{x}, \underline{u})$. This means that ker($I_G$) defines an invariant subspace of $\mathcal{H}_k^G(\mathbb{R}^{m\times 2}, \mathbb{C})$. As this space decomposes into a direct sum of subspaces isomorphic with $\mathcal{H}_{k,k-2j}(\mathbb{R}^{m\times 2}, \mathbb{C})$, and as the image of the functional I_G has at most dimension $d = 1$, the statement follows from the fact that the dimension of $\mathcal{H}_{k,k-2j}(\mathbb{R}^{m\times 2}, \mathbb{C})$ is always (strictly) bigger than 1 if $k > 0$. □

If we now start from an arbitrary function

$$f(\underline{x}, \underline{u}) = \sum_{k=0}^{\infty} f_k(\underline{x}, \underline{u}) \in \mathcal{A}_G(\mathbb{R}^{m\times 2}),$$

we can decompose each homogeneous (polynomial) component $f_k(\underline{x}, \underline{u})$ into its SO(m)-irreducible building blocks by means of

$$f_k(\underline{x}, \underline{u}) = \sum_{a=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{b=0}^{\lfloor \frac{k}{2} \rfloor - a} |\underline{x} \wedge \underline{u}|^{2a} \pi_0 \left(|\underline{u}|^{2b} H_{k-2a,k-2(a+b)}(\underline{x}, \underline{u}) \right).$$

It is then clear that the integral of f_k reduces to the projection of f_k onto the trivial component $H_{0,0}$, which clearly only exists for $k \in \mathbb{Z}^+$ even, with $2a = k$ and $b = 0$. The integral of the full function $f(\underline{x}, \underline{u})$ can then be seen as the sum of the projection on each of the trivial components appearing in the building blocks $f_{2k}(\underline{x}, \underline{u})$ appearing in the Taylor expansion.

Theorem 8 *The integral $I_G(f)$ of a function $f(\underline{x}, \underline{u}) \in \mathcal{A}_G(\mathbb{R}^{m\times 2})$ over the oriented Grassmann manifold* $\mathrm{Gr}_o(m, 2)$ *can be defined as*

$$\int_{\mathrm{Gr}_o(m,2)} f \mathrm{d}\mu := V_{m,2} \sum_{k=0}^{\infty} \frac{\Gamma(m-1)}{(2k+1)!\Gamma(2k+m-1)} \left(\Delta_G^k f \right)(\underline{0}, \underline{0}). \tag{21}$$

The constant $V_{m,2}$ hereby represents the volume of $\mathrm{Gr}_o(m, 2)$.

Proof We first of all recall from the proof for theorem 5 that the action of Δ_G on a function of the form $|\underline{x} \wedge \underline{u}|^{2a}$ times a polynomial in $ker(\Delta_G)$ is to eat away a factor $|\underline{x} \wedge \underline{u}|^2$ (up to a non-trivial constant). This means that putting $\underline{x} = \underline{u} = \underline{0}$ at the end of the calculations will ensure that non-trivial summands $\mathcal{H}_{k,k-2j}$ (with $k > 0$) will never contribute to the final expression. The constant appearing inside the summation is the unique constant $c_k \in \mathbb{R}$ for which $c_k \Delta_G^k |\underline{x} \wedge \underline{u}|^{2k} = 1$. This follows from relation (17), hereby taking into account that the action of T on the trivial representation for $\mathfrak{so}(m)$ is equal to zero:

$$\begin{aligned} \Delta_G |\underline{x} \wedge \underline{u}|^{2k} &= [\Delta_G, |\underline{x} \wedge \underline{u}|^{2k}](1) \\ &= 2k|\underline{x} \wedge \underline{u}|^{2k-2}(D + 2k - 2)\big((2k+1)(D + 2k - 3) + T\big)(1) \\ &= 2k(2k+1)(2k+m-2)(2k+m-3)|\underline{x} \wedge \underline{u}|^{2k-2}. \end{aligned}$$

A repeated application of this then gives that

$$c_k^{-1} = (2k+1)! \frac{\Gamma(2k+m-1)}{\Gamma(m-1)}.$$

The constant appearing outside the summation in (21) ensures that the integral of the constant function 1 reproduces the volume $V_{m,2}$ of the real Grassmannian $\mathrm{Gr}_o(m,2)$, see, e.g. [1]:

$$V_{m,2} = \frac{\mathrm{Vol}\big(\mathrm{O}(m)\big)}{\mathrm{Vol}\big(\mathrm{O}(m-2)\big)\mathrm{Vol}\big(\mathrm{O}(2)\big)}$$

Together, this proves the theorem. □

Using this result, one can show that the decomposition of $L^2\big(\mathrm{Gr}_o(m,2)\big)$ into irreducible summands for the regular action of $\mathrm{SO}(m)$ is orthogonal with respect to the following inner product:

Definition 3 For two functions f and g on $\mathrm{Gr}_o(m,2)$, the inner product is given by

$$\langle f, g\rangle := \int_{\mathrm{Gr}_o(m,2)} f^c g \mathrm{d}\mu.$$

In the classical case (harmonic analysis on $\mathbb{R}^m$), one has the trivial but very useful identity which says that

$$\begin{aligned} \Delta_x : \mathcal{H}_k(\mathbb{R}^m,\mathbb{C})\mathcal{H}_\ell(\mathbb{R}^m,\mathbb{C}) &\to \mathcal{H}_{k-1}(\mathbb{R}^m,\mathbb{C})\mathcal{H}_{\ell-1}(\mathbb{R}^m,\mathbb{C}) \\ H_k(\underline{x})H_\ell(\underline{x}) &\mapsto 2\sum_{j=1}^m (\partial_{x_j}H_k(\underline{x}))(\partial_{x_j}H_\ell(\underline{x})). \end{aligned} \tag{22}$$

Note that this has to be seen as an identity on $\mathcal{P}(\mathbb{R}^m,\mathbb{C})$, in the sense that both harmonic polynomials take $\underline{x} \in \mathbb{R}^m$ as their argument. An easy consequence of this is the fact that for $k > \ell$ in $\mathbb{Z}^+$, one gets that

$$\Delta_x^{\ell+1}\big(H_k(\underline{x})H_\ell(\underline{x})\big) = 0. \tag{23}$$

Property (22) does not hold for the operator Δ_G, but one does have the following:

Lemma 2 *For arbitrary integers $k > \ell \in \mathbb{Z}^+$, one has that*

$$\Delta_G^{\ell+1}\big(H_k^G(\underline{x},\underline{u})H_\ell^G(\underline{x},\underline{u})\big) = 0,$$

where $H_k^G \in \mathcal{H}_k^G(\mathbb{R}^{m\times 2},\mathbb{C})$ and $H_\ell^G \in \mathcal{H}_\ell^G(\mathbb{R}^{m\times 2},\mathbb{C})$.

Proof If $H_k^G(\underline{x},\underline{u})$ and $H_\ell^G(\underline{x},\underline{u})$ belong to $\mathcal{H}_k^G$ and $\mathcal{H}_\ell^G$, respectively, it is clear that their product has a total degree $(k+\ell, k+\ell)$ in the variables $(\underline{x},\underline{u})$. The operator Δ_G is of degree $(-2,-2)$ and is given by

$$\Delta_G = \sum_{a<b}\partial_{ab}^2 = \sum_{a<b}\big(\partial_{x_a}^2\partial_{u_b}^2 - 2\partial_{x_a}\partial_{x_b}\partial_{u_a}\partial_{u_b} + \partial_{x_b}^2\partial_{u_a}^2\big).$$

This means that its action on a product of two polynomials reduces to a linear combination containing terms of the form $(D_{p_1,p_2}H_k^G)(D_{q_1,q_2}H_\ell^G)$, where each $D_{i,j}$ denotes a constant coefficient differential operator of degree $(-i,-j)$ in the variables $(\underline{x},\underline{u})$. These integers satisfy the requirement that $p_1 + q_1 = p_2 + q_2 = 2$. Moreover, in view of the fact that we are considering the action of Δ_G on elements in $\mathcal{H}_k^G$ and $\mathcal{H}_\ell^G$, it is clear that the terms with $p_1 = p_2 = 2$ and $q_1 = q_2 = 2$ will not appear in the summation above. Indeed, for these choices one has that $D_{2,2}$ (summed over $a < b$) reduces to the operator Δ_G itself. But this means that q_1 and q_2 cannot both be zero at the same time, or in other words: the linear combination from above only contains polynomial factors $D_{q_1,q_2} H_\ell^G$ with at most degree $(\ell - 1)$ in either the variable $\underline{x}$ or $\underline{u} \in \mathbb{R}^m$. Hence, acting ℓ times with Δ_G produces terms which no longer depend on $\underline{x}$ or $\underline{u}$ so that one further application of the operator Δ_G kills everything. □

Proposition 2 *For $k \neq \ell$, one has that $\mathcal{H}_k^G(\mathbb{R}^{m\times 2}, \mathbb{C}) \perp \mathcal{H}_\ell^G(\mathbb{R}^{m\times 2}, \mathbb{C})$.*

Proof We may assume that $k > \ell$ without loss of generality, and so the previous lemma allows us to conclude that

$$H_k^G(\underline{x}, \underline{u}) H_\ell^G(\underline{x}, \underline{u}) = \sum_{j=0}^{\ell} |\underline{x} \wedge \underline{u}|^{2j} H_{k+\ell-2j}^G(\underline{x}, \underline{u}).$$

This clearly shows that the trivial representation $\mathbb{C}$ does not appear as a subspace, which means that the Pizzetti integral will indeed give zero. □

For $k = \ell$, we still have an orthogonality relation for the different summands inside the homogeneous kernel space for the operator Δ_G:

Proposition 3 *For all $0 \leq i \neq j \leq \lfloor \frac{k}{2} \rfloor$, one has that*

$$\pi_0\left(|\underline{u}|^{2i}\mathcal{H}_{k,k-2i}(\mathbb{R}^{m\times 2}, \mathbb{C})\right) \perp \pi_0\left(|\underline{u}|^{2j}\mathcal{H}_{k,k-2j}(\mathbb{R}^{m\times 2}, \mathbb{C})\right).$$

Proof First of all, we note that we can safely ignore the factors $|\underline{u}|^2$ and the projection operator π_0 for our intents and purposes, since these are all expressed in terms of $\mathfrak{so}(m)$ -invariant operators. We will therefore show that the trivial summand $\mathbb{C}$ is not contained in $\mathcal{H}_{k,k-2i} \otimes \mathcal{H}_{k,k-2j}$ (with $i \neq j$). A simple way to see this goes as follows: every irreducible subspace $\mathbb{V}_\alpha \subset \mathbb{V}_\lambda \otimes \mathbb{V}_\mu$ is characterised by a highest weight of the form $\alpha = \lambda + \nu$, where ν is a weight which appears in the representation $\mathbb{V}_\mu$ (see [18]). As $\nu = -\mu$ is the lowest weight appearing in the module $\mathbb{V}_\mu$, this implies that $\mathbb{V}_{\lambda-\mu}$ is the smallest possible summand (up to a lexicographic ordering). In our case, for $i < j$, this means that the smallest weight α in the tensor product $\mathcal{H}_{k,k-2i} \otimes \mathcal{H}_{k,k-2j}$ would be given by $\alpha = (2j - 2i, 0, \ldots, 0)$. This proves the claim. □

Remark 5 In order to see that the module $\mathbb{V}_\alpha$ with $\alpha = (2j - 2i, 0, \ldots, 0)$ indeed appears inside the tensor product, one can argue as follows: choose $i < j$ and identify $\mathcal{H}_{k,k-2j}$ with its dual space (replacing each variable x_a or u_b with the associated partial derivative). One can then find a copy of the space $\mathcal{H}_{2(j-i)}(\mathbb{R}^m, \mathbb{C})$ inside the tensor product $\mathcal{H}_{k,k-2i} \otimes \mathcal{H}_{k,k-2j}$ as the result of a 'maximal contraction' of the indices. This corresponds to the action of the differential operator $H_{k,k-2j}(\underline{\partial}_x, \underline{\partial}_u)$ on $H_{k,k-2i}(\underline{x}, \underline{u})$, which may give a polynomial $P_{2(j-i)}(\underline{u})$ in the variable $\underline{u} \in \mathbb{R}^m$. Because Δ_u commutes with the (constant coefficient) differential operator $H_{k,k-2j}(\underline{\partial}_x, \underline{\partial}_u)$, it is clear that $P_{2(j-i)}(\underline{u}) \in \mathcal{H}_{2(j-i)}(\mathbb{R}^m)$ is indeed harmonic.

Note that in the classical case (on the sphere), the Pizzetti formula can be expressed in terms of the normalised Bessel function (at least formally). Indeed, as mentioned in, for instance, [4] one has that

$$\int_{S^{m-1}} f \mathrm{d}\sigma = \left(\Psi_{\frac{m}{2}-1}\left(\sqrt{-\Delta_x}\right) f \right)(\underline{0}),$$

where the function $\Psi_a(x)$ is defined in terms of the Bessel function of the first kind by means of

$$\Psi_a(x) := \frac{\Gamma(a) J_a(x)}{\left(\frac{x}{2}\right)^a}.$$

In the case of the Grassmann manifold $\mathrm{Gr}_o(m, 2)$, a similar formula exists:

Theorem 9 *The integral over* $\mathrm{Gr}_o(m, 2)$ *can be formally expressed as*

$$\int_{\mathrm{Gr}_o(m,2)} f \mathrm{d}\mu = V_{m,2} \frac{(m-2)!}{2^m \left(\sqrt{\Delta_G}\right)^{\frac{m-1}{2}}} \left(I_{m-3}\left(2\sqrt[4]{\Delta_G}\right) - J_{m-3}\left(2\sqrt[4]{\Delta_G}\right) \right) f \Big|_{(\underline{0},\underline{0})},$$

with $I_a(x)$ *and* $J_a(x)$ *the modified and standard Bessel function of the first kind, respectively.*

Proof Invoking Legendre's duplication formula for the Gamma function, we have that

$$(2k+1)! = \frac{2^{1+2k} k! \Gamma\left(k + \frac{3}{2}\right)}{\sqrt{\pi}}$$

$$(m+2k-2)! = \frac{2^{m+2k-2} \Gamma\left(k + \frac{m-1}{2}\right) \Gamma\left(k + \frac{m}{2}\right)}{\sqrt{\pi}}.$$

Using the Pochammer symbol $(a)_k = a(a+1)\dots(a+k-1)$, this means that

$$\sum_{k=0}^{\infty} \frac{\Gamma(m-1)\Delta_G^k}{(2k+1)!\Gamma(2k+m-1)} = \sum_{k=0}^{\infty} \frac{1}{k! \left(\frac{3}{2}\right)_k \left(\frac{m-1}{2}\right)_k \left(\frac{m}{2}\right)_k} \left(\frac{\Delta_G}{16}\right)^k.$$

This is the standard form for a hypergeometric function of the form

$${}_0F_3\left(\frac{3}{2}, \frac{m-1}{2}, \frac{m}{2}; \frac{x}{16}\right),$$

where the argument x is to be formally replaced by the operator Δ_G. In [2], the author showed that

$${}_0F_3\left(\frac{3}{2}, c, c+\frac{1}{2}; x\right) = \frac{\Gamma(2c)}{2^{1+2c}(\sqrt{x})^c} \left(I_{2c-2}\left(4\sqrt[4]{x}\right) - J_{2c-2}\left(4\sqrt[4]{x}\right) \right).$$

This then leads to the desired result. □

4 Refinement of the Stiefel Pizzetti integral

In this section, we will show how the formula obtained in (21) can also be extracted from the formula for the Pizzetti integral over the Stiefel manifold St(m, 2). This formula was obtained in [4], and looks as follows:

Theorem 10 *The integral of a function* $f(\underline{x}, \underline{x})$ *on the Stiefel manifold* $\mathrm{St}(m, 2)$ *is given by*

$$\int_{\mathrm{St}(m,2)} f \mathrm{d}\mu = \sum_{j=0}^{\infty} \frac{\Gamma\left(\frac{m}{2}\right)}{4^j \Gamma\left(j + \frac{m}{2}\right)} \sum_{\ell=0}^{\lfloor \frac{j}{2} \rfloor} \frac{\Gamma\left(\frac{m-1}{2}\right)}{\Gamma\left(\ell + \frac{m-1}{2}\right)} \frac{I_1^{j-2\ell} I_2^{\ell} f}{(j-2l)!\ell!}(\underline{0}, \underline{0}),$$

where $I_1 = \Delta_x + \Delta_u$ *and* $I_2 = \Delta_x \Delta_u - \langle \underline{\partial}_x, \underline{\partial}_u \rangle^2$. (*As a matter of fact, this is the operator* Δ_G .)

In the recent paper [12], the authors studied this Pizzetti formula from a distributional approach. In this section, we will show how the formula for the integral over $\mathrm{Gr}_o(m, 2)$ can be derived from the formula above, by means of a symmetry argument. For this purpose, we have to extract the $\mathfrak{sl}(2)$-invariant part of the formal operator defined above as a series in terms of I_1 and I_2. As $I_2 = \Delta_G$, this amounts to projecting I_1^k (with $k \in \mathbb{Z}^+$) to its invariant piece. To do so, we will switch to the Fourier image (working with $|\underline{x}|^2$, $|\underline{u}|^2$ and $\langle \underline{x}, \underline{u} \rangle$), as we can then exploit our knowledge about polynomials and how these decompose under the action of $\mathfrak{sl}(2)$. This means that we have to investigate how the polynomial $(|\underline{x}|^2 + |\underline{u}|^2)^k$ decomposes under the action of $\mathfrak{sl}(2)$. In view of the fact that $\mathbb{V}_2 \cong \mathrm{span}(|\underline{x}|^2, \langle \underline{x}, \underline{u} \rangle, |\underline{u}|^2)$, this is equivalent with decomposing $\mathrm{Sym}(\mathbb{V}_2)$ into irreducible summands. For instance, if $k = 2$ we find that

$$(|\underline{x}|^2 + |\underline{u}|^2)^2 = \left(|\underline{x}|^4 + \frac{2}{3}(|\underline{x}|^2|\underline{u}|^2 + 2\langle \underline{x}, \underline{u} \rangle^2) + |\underline{u}|^4 \right) + \frac{4}{3}|\underline{x} \wedge \underline{u}|^2,$$

where the first term between brackets belongs to $\mathbb{V}_4$ (generated by $|\underline{x}|^4$). In this case, one would find that

$$I_1^2 = (\Delta_x + \Delta_u)^2 = \frac{4}{3}\Delta_G + D_4,$$

where D_4 is an operator belonging to the $\mathfrak{sl}(2)$-module $\mathbb{V}_4$ as generated by the highest weight Δ_u^2, which can thus be ignored (as we are interested in the invariant piece). For the general case, we first prove a few results:

Lemma 3 *The vector space of* $\mathfrak{so}(m)$*-invariant polynomials Sym*$(\mathbb{V}_2)$ *in two vector variables* $(\underline{x}, \underline{u}) \in \mathbb{R}^{m\times 2}$ *which have a fixed total degree of homogeneity is invariant under the action of* $\mathfrak{sl}(2)$.

Proof If we put $2k = \deg(\underline{x}) + \deg(\underline{u})$, it is easy to see that the operators $X = \langle \underline{x}, \underline{\partial}_u \rangle$ and $Y = \langle \underline{u}, \underline{\partial}_x \rangle$ indeed preserve the total degree. Note hereby that this degree is necessarily even, as the basic generators in $\mathbb{V}_2$ all have degree 2. □

Let us then introduce the notation $\mathcal{I}_{2k}$ for this space of fixed total degree $2k$. An easy counting argument shows that

$$\dim(\mathcal{I}_{2k}) = \dim \operatorname{span}\left(|\underline{x}|^{2a}\langle \underline{x}, \underline{u}\rangle^b |\underline{u}|^{2c} \,:\, \deg(\underline{x}) + \deg(\underline{u}) = 2k \right)$$
$$= \frac{1}{2}(k+1)(k+2)$$

Next, we recall that the invariant polynomial $|\underline{x}|^{2k} \in \mathcal{I}_{2k}$ generates a copy of the irreducible module $\mathbb{V}_{2k}$ for $\mathfrak{sl}(2)$ with dimension $(2k+1)$. These spaces then appear in the following result:

Theorem 11 *The spaces $\mathcal{I}_{2k}$ (with $k \in \mathbb{Z}^+$ an arbitrary integer) decompose as follows under the action of $\mathfrak{sl}(2)$:*

$$\mathcal{I}_{4k} \cong \bigoplus_{j=0}^{k} |\underline{x} \wedge \underline{u}|^{2j} \mathbb{V}_{4(k-j)}$$
$$\mathcal{I}_{4k+2} \cong \bigoplus_{j=0}^{k} |\underline{x} \wedge \underline{u}|^{2j} \mathbb{V}_{4(k-j)+2}.$$

Proof It suffices to note that the spaces appearing at the right-hand side are clearly subspaces of the left-hand side, after which a simple counting argument proves that their dimensions sum up to the total dimension of the space $\mathcal{I}_{4k}$ and $\mathcal{I}_{4k+2}$, respectively. □

We will now project $(|\underline{x}|^2 + |\underline{u}|^2)^{2k}$ onto the trivial component. To do so, we will focus on the component with weight zero, for the action of the operator $H = \mathbb{E}_x - \mathbb{E}_u \in \mathfrak{sl}(2)$. Using the result above, we thus get an equality of the form

$$\binom{2k}{k} |\underline{x}|^{2k} |\underline{u}|^{2k} = \sum_{j=0}^{k} \gamma_j^{(k)} |\underline{x} \wedge \underline{u}|^{2j} w_0(\mathbb{V}_{4(k-j)}), \tag{24}$$

whereby $w_0(\mathbb{V}_{2a})$ stands for the zero-weight component in the representation. Recalling that this representation is generated by the highest weight $|\underline{x}|^{2a}$, it is clear that we may define $w_0(\mathbb{V}_{2a}) := \langle \underline{u}, \underline{\partial}_x \rangle^a |\underline{x}|^{2a}$. Note that this invariant polynomial (for the regular action of the orthogonal Lie algebra) belongs to $\ker(\Delta_G)$, which easily follows from the observation that

$$[\Delta_G, \langle \underline{u}, \underline{\partial}_x \rangle] = 0 \;\Rightarrow\; \Delta_G \langle \underline{u}, \underline{\partial}_x \rangle^a |\underline{x}|^{2a} = \langle \underline{u}, \underline{\partial}_x \rangle^a \Delta_G |\underline{x}|^{2a} = 0.$$

At first sight, it may seem strange that $w_0(\mathbb{V}_{2a}) \in \ker \Delta_G$, because we have characterised the $\mathfrak{so}(m)$-invariant subspaces of $\ker(\Delta_G)$ in theorem 1, and the trivial summand can only appear inside the polynomials of degree zero. However, we would like to stress that $w_0(\mathbb{V}_{2a})$ is not in the kernel of the skew Euler operators X and $Y \in \mathfrak{sl}(2)$, which means that this theorem does not apply to the polynomial $\langle \underline{u}, \underline{\partial}_x \rangle^a |\underline{x}|^{2a}$. This is important, because it means that we cannot employ the power of the Higgs algebra H_3 obtained in theorem 4: this realisation only holds when Δ_G and $|\underline{x} \wedge \underline{u}|^2$ are seen as operators acting on polynomials in the kernel of the skew Euler operators. However, we do have the following:

Lemma 4 *For all integers $k > 0$, one has that*

$$\ker \Delta_G^k\Big|_{\mathcal{I}_{4k}} = \bigoplus_{j=0}^{k-1} |\underline{x}\wedge\underline{u}|^{2j}\mathbb{V}_{4(k-j)}.$$

Proof As the operator Δ_G is homogeneous of total degree -4, one has that $\Delta_G^k \in \mathrm{Hom}(\mathcal{I}_{4k},\mathcal{I}_0)$. As Δ_G is $\mathfrak{sl}(2)$-invariant, this then means that all the spaces $|\underline{x}\wedge\underline{u}|^{2j}\mathbb{V}_{4(k-j)}$ with $0\le j<k$ sit inside the kernel of the operator Δ_G^k. On the other hand, it follows from theorem 5 that Δ_G^k acts surjectively on $\mathcal{I}_0\cong\mathbb{V}_0$. Indeed, $|\underline{x}\wedge\underline{u}|^{2k}$ can be seen as the weight vector $v_{0,0}(k)$ in the module $\mathbb{V}_{0,0}^\infty$ for the algebra H_3. □

Getting back to expression (24), we want to solve the equation

$$\binom{2k}{k}|\underline{x}|^{2k}|\underline{u}|^{2k} = \sum_{j=0}^{k}\gamma_j^{(k)}|\underline{x}\wedge\underline{u}|^{2j}\langle\underline{u},\underline{\partial}_x\rangle^{2(k-j)}|\underline{x}|^{4(k-j)} \tag{25}$$

for the constant $\gamma_k^{(k)}$. For instance, if $2k=2$ we have

$$2|\underline{x}|^2|\underline{u}|^2 = \gamma_0^{(1)}\langle\underline{u},\underline{\partial}_x\rangle^2|\underline{x}|^4+\gamma_1^{(1)}|\underline{x}\wedge\underline{u}|^2,$$

and acting with Δ_G on both sides of this equation tells us that

$$2\Delta_G\big(|\underline{x}|^2|\underline{u}|^2\big) = 8m(m-1) = \gamma_1^{(1)}\Delta_G|\underline{x}\wedge\underline{u}|^2 = 6m(m-1)\gamma_1^{(1)},$$

after which we once again find that $\gamma_1^{(1)}=\frac{4}{3}$. The general situation is the topic of the following technical lemma:

Lemma 5 *The value for the constant* $\gamma_k^{(k)}$ *in expression* (25) *is given by*

$$\gamma_k^{(k)} = \frac{2^{2k}}{2k+1}.$$

Proof In view of the previous lemma, it suffices to let the operator Δ_G^k act on both sides of the equation above. On the right-hand side, this leads to

$$\begin{aligned}\Delta_G^k\sum_{j=0}^{k}\gamma_j^{(k)}|\underline{x}\wedge\underline{u}|^{2j}\langle\underline{u},\underline{\partial}_x\rangle^{2(k-j)}|\underline{x}|^{4(k-j)} &= \gamma_k^{(k)}\Delta_G^k|\underline{x}\wedge\underline{u}|^{2k}\\ &= \gamma_k^{(k)}(2k+1)!\,\frac{\Gamma(2k+m-1)}{\Gamma(m-1)},\end{aligned}$$

see the proof of Theorem 8. On the left-hand side, we will use the fact that

$$\Delta_G^k|\underline{x}|^{2k}|\underline{u}|^{2k} = \sum_{j=0}^{k}\binom{k}{j}(-1)^j\langle\underline{\partial}_x,\underline{\partial}_u\rangle^{2j}\Delta_x^{k-j}\Delta_u^{k-j}|\underline{x}|^{2k}|\underline{u}|^{2k}.$$

For a fixed index j, we first of all note that

$$\Delta_x^{k-j}|\underline{x}|^{2k}\Delta_u^{k-j}|\underline{u}|^{2k} = \left(2^{2(k-j)}\frac{\Gamma(k+1)\Gamma\left(k+\frac{m}{2}\right)}{\Gamma(j+1)\Gamma\left(j+\frac{m}{2}\right)}\right)^2|\underline{x}|^{2j}|\underline{u}|^{2j}.$$

Next, ignoring the constants for now, we note that

$$\begin{aligned}\langle \underline{\partial}_x, \underline{\partial}_u \rangle^{2j} |\underline{x}|^{2j} |\underline{u}|^{2j} &= 4j^2 \langle \underline{\partial}_x, \underline{\partial}_u \rangle^{2j-1} \langle \underline{x}, \underline{u} \rangle |\underline{x}|^{2j-2} |\underline{u}|^{2j-2} \\ &= 4j^2 \left[\langle \underline{\partial}_x, \underline{\partial}_u \rangle^{2j-1}, \langle \underline{x}, \underline{u} \rangle \right] |\underline{x}|^{2j-2} |\underline{u}|^{2j-2},\end{aligned}$$

where we have introduced a commutator in the last line. This is due to the fact that the action of $\langle \underline{\partial}_x, \underline{\partial}_u \rangle^{2j-1}$ on the remaining power of $|\underline{x}|^2 |\underline{u}|^2$ is zero. The advantage is that we can now exploit the fact that

$$\mathfrak{sl}(2) = \mathrm{Alg}(X, Y, H) \cong \mathrm{Alg}\big(\langle \underline{x}, \underline{u} \rangle, -\langle \underline{\partial}_x, \underline{\partial}_u \rangle, \mathbb{E}_x + \mathbb{E}_u + m\big),$$

because in the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}(2))$, one has that

$$[Y^a, X] = -a(H + a - 1)Y^{a-1} \qquad (\forall a \in \mathbb{Z}^+).$$

This relation (proved by induction in the parameter a) leads to

$$\langle \underline{\partial}_x, \underline{\partial}_u \rangle^{2j} |\underline{x}|^{2j} |\underline{u}|^{2j} = 4j^2(2j-1)(m+2j-2) \langle \underline{\partial}_x, \underline{\partial}_u \rangle^{2j-2} |\underline{x}|^{2j-2} |\underline{u}|^{2j-2},$$

which means that the result follows by induction on the parameter j. Using the fact that $\langle \underline{\partial}_x, \underline{\partial}_u \rangle^2 |\underline{x}|^2 |\underline{u}|^2 = 4m$, we thus find that

$$\langle \underline{\partial}_x, \underline{\partial}_u \rangle^{2j} |\underline{x}|^{2j} |\underline{u}|^{2j} = 4^{2j} \big(j!\big)^2 \frac{\Gamma\Big(j+\frac{1}{2}\Big)\Gamma\Big(j+\frac{m}{2}\Big)}{\Gamma\Big(\frac{1}{2}\Big)\Gamma\Big(\frac{m}{2}\Big)}.$$

This then means that

$$\Delta_G^k |\underline{x}|^{2k} |\underline{u}|^{2k} = \left(\sum_{j=0}^{k} \binom{k}{j} (-1)^j \frac{\Gamma\Big(j+\frac{1}{2}\Big)\Gamma\Big(\frac{m}{2}\Big)}{\Gamma\Big(\frac{1}{2}\Big)\Gamma\Big(j+\frac{m}{2}\Big)} \right) \frac{2^{4k}(k!)^2 \Gamma^2\Big(k+\frac{m}{2}\Big)}{\Gamma^2\Big(\frac{m}{2}\Big)}.$$

Introducing the Pochammer symbol $(a)_j = a(a+1)\ldots(a+j-1)$, the sum between brackets can now be rewritten in terms of a hypergeometric function:

$$\sum_{j=0}^{k} \frac{1}{j!} \frac{(-k)_j \Big(\frac{1}{2}\Big)_j}{\Big(\frac{m}{2}\Big)_j} = {}_2F_1\Big(-k, \frac{1}{2}; \frac{m}{2}; 1\Big) = \frac{\Gamma\Big(\frac{m}{2}\Big)\Gamma\Big(k+\frac{m-1}{2}\Big)}{\Gamma\Big(k+\frac{m}{2}\Big)\Gamma\Big(\frac{m-1}{2}\Big)}.$$

Bringing everything together, we thus find that

$$\gamma_k^{(k)} = \frac{2^{2k}\Gamma(m-1)\Gamma\Big(k+\frac{m-1}{2}\Big)\Gamma\Big(k+\frac{m}{2}\Big)}{(2k+1)\Gamma(2k+m-1)\Gamma\Big(\frac{m-1}{2}\Big)\Gamma\Big(\frac{m}{2}\Big)}.$$

Using Legendre's duplication formula for the Gamma function, this then leads to the desired result. □

If we then get back to theorem 10, we first of all note that the terms with odd indices $j \in 2\mathbb{Z}^+ + 1$ in the formula for the Stiefel manifold $\mathrm{St}(m, 2)$ do not contribute to the Pizzetti integral over $\mathrm{Gr}_o(m,2)$ in view of theorem 11. However, for even indices $j \in 2\mathbb{Z}^+$, one has a contribution of the form

$$I_1^{2j-2\ell} = (\Delta_x + \Delta_u)^{2(j-\ell)} \xrightarrow{\pi_0} \frac{2^{2(j-\ell)}}{1+2(j-\ell)} I_2^{j-\ell}.$$

This thus means that the 'trivial' part of the Pizzetti integral over the Stiefel manifold is found to be

$$\sum_{j=0}^{\infty} \frac{\Gamma\left(\frac{m}{2}\right)}{4^{2j}\Gamma\left(2j+\frac{m}{2}\right)} \sum_{\ell=0}^{j} \frac{\Gamma\left(\frac{m-1}{2}\right)}{\Gamma\left(\ell+\frac{m-1}{2}\right)} \frac{2^{2(j-\ell)}\Delta_G^j}{(1+2j-2\ell)!\ell!}$$

We now claim that for fixed j, the coefficient of Δ_G^j reduces to the numerical constant appearing in the formula for the Pizzetti integral, see (21).

Lemma 6 *For a fixed index $j \in \mathbb{Z}^+$, one has that*

$$\sum_{\ell=0}^{j} \frac{\Gamma\left(\frac{m-1}{2}\right)}{2^{2\ell}\Gamma\left(\ell+\frac{m-1}{2}\right)\ell!(1+2j-2\ell)!} = \frac{1}{(2j+1)!} \frac{\Gamma\left(\frac{m-1}{2}\right)\Gamma\left(2j+\frac{m}{2}\right)}{\Gamma\left(j+\frac{m-1}{2}\right)\Gamma\left(j+\frac{m}{2}\right)}.$$

Proof We will try to rewrite the left-hand side as a hypergeometric function evaluated at $x = 1$, as this leads to a closed formula. First of all, the sum on the left can be written as

$$\frac{1}{(2j+1)!} \sum_{\ell=0}^{j} \frac{1}{\ell!} \frac{(2j+1)2j(2j-1)\ldots(2j+2-2\ell)}{2^{2\ell}\left(\frac{m-1}{2}\right)_\ell}.$$

For a fixed index ℓ, we can now divide each of the 2ℓ factors in the nominator by 2 and use a factor $(-1)^{2\ell}$ to obtain

$$\frac{1}{(2j+1)!} \sum_{\ell=0}^{j} \frac{1}{\ell!} \frac{\left(-j-\frac{1}{2}\right)_\ell (-j)_\ell}{\left(\frac{m-1}{2}\right)_\ell} = \frac{{}_2F_1\left(-j-\frac{1}{2}, -j; \frac{m-1}{2}; 1\right)}{(2j+1)!},$$

which is exactly what needed to be found. □

Putting everything together, we have thus obtained the following:

Theorem 12 *Up to a normalisation factor, the Pizzetti formula for the Grassmannian* $\mathrm{Gr}_o(m,2)$ *can be obtained as the* $\mathfrak{sl}(2)$*-invariant part of the formal power series defining the Pizzetti formula for the Stiefel manifold* $\mathrm{St}(m,2)$.

5 Conclusion and outlook

In this paper, we have generalised the classical Pizzetti formula, which was first developed for the integral on the sphere $S^{m-1} \subset \mathbb{R}^m$ and later extended to the Stiefel manifolds $\mathrm{St}(m, k)$, to the oriented Grassmannian $\mathrm{Gr}_o(m, 2)$. The novelty of our approach lies in the fact that we were able to do this in terms of a dual symmetry algebra (the Higgs algebra), which allows to see this formula as an algebraic projection on an irreducible representation for this algebra (characterised by a certain lowest weight) realised inside the full space of smooth functions on the Grassmannian. In an upcoming paper, we will see how this approach can be extended to more complicated Grassmann manifolds.

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