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Killing spinor-valued forms and their integrability conditions

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Abstract
We study invariant systems of PDEs defining Killing vector-valued forms, and then we specialize to Killing spinor-valued forms. We give a detailed treatment of their prolongation and integrability conditions by relating the pointwise values of solutions to the curvature of the underlying manifold. As an example, we completely solve the equations on model spaces of constant curvature producing brand-new solutions which do not come from the tensor product of Killing spinors and Killing–Yano forms.

Keywords Killing-type equations · Prolongation of differential systems · Projective invariance · Spinor-valued differential forms · Cone construction · Constant curvature space

Mathematics Subject Classification 35N10 · 53A20 · 53A55 · 53B20 · 53C21 · 58J70

1 Introduction

Killing equations are a class of invariant overdetermined systems of partial differential equations, appearing naturally in many problems related to (pseudo-)Riemannian geometry. One of the most prominent examples are the Killing vectors, corresponding to infinitesimal isometries of Riemannian manifolds. In the present article, we focus on another specific example in the hierarchy of Killing equations, termed *Killing spinor-valued forms*. We introduce relevant Killing equations and deduce their properties mostly implied by integrability of the differential system in question. We shall start our analysis in a rather general context and then gradually specialize to the cases of most authors' interest. As an application of general results, we shall completely resolve the Killing equations on model spaces of constant curvature.

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The main motivation for the study of Killing spinor-valued forms is that they are a natural generalization of both Killing spinors and Killing–Yano forms. The Killing spinors and Killing–Yano forms play a dominant role in the geometrical analysis on Riemannian manifolds, e.g., the study of Dirac and Laplace operators and the associated eigenvalue problems. Subsequently, the two examples of Killing-type equations gained their own interest in theoretical physics, too.

A central question in the subject asks for (pseudo-)Riemannian manifolds admitting nontrivial solutions of Killing-type equations, and their relation to the underlying geometric structure for which they occur. To some extent, this question is answered by the *integrability conditions* which relate the solutions with the curvature properties of manifolds. Moreover, the Killing spinors and Killing–Yano forms are closely related to special Riemannian structures, e.g., Sasakian and G_2-manifolds.

The general interest in Killing spinors was stimulated and accelerated by Friedrich's inequality [10] for the eigenvalues of Dirac operator. He also proved that a Riemannian manifold admitting Killing spinors is Einstein, which is a direct consequence of the first integrability condition. The so-called *cone construction*, cf. [2], relates Killing spinors with parallel spinors on the metric cone and thus allows a classification via holonomy of the cone.

Killing–Yano forms were introduced by Yano in [19] within the framework of his study of Killing vectors and harmonic tensors. Integrability conditions and cone construction for (special) Killing–Yano forms were deduced in [15]. The relevant Killing equations are also examples of the so-called *first BGG operator* in projective geometry. Namely, they can be efficiently described in the context of parabolic geometries using tractor calculus, cf. [13, 14].

Killing spinor-valued forms already appeared in theoretical physics in the construction of Kaluza–Klein supergravity, cf. [7, 8]. A systematic treatment of Killing spinor-valued forms can be found in [17], the main result being the cone construction for special Killing spinor-valued forms. The present paper is a continuation of this effort.

Here is a brief summary of the content of our article. In Sect. 2, we start by deducing a *prolongation* of the defining Killing equation. In general, the prolongation procedure transforms the original differential system into a closed one by introducing new indeterminate variables for undetermined components of first derivatives. This corresponds to certain extension of the initial bundle to a larger one equipped with suitable connection, such that the original system of equations is equivalent to the equation for parallel sections with respect to the newly constructed connection. The prolongation allows to write down the integrability conditions in an explicit way. Our approach is based on direct computations that are guided by representation-theoretical considerations. We shall also analyze intrinsic projective invariance of Killing–Yano forms and compare our results with those obtained by more abstract methods based on the tractor calculus.

We shall consider vector-valued differential forms that take values in an arbitrary vector bundle equipped with linear connection. The presence of vector values, when compared to the scalar-valued case, yields additional terms induced by the curvature acting just on the values. Later on, we shall specialize to (pseudo-)Riemannian manifolds and spinor-valued differential forms. Note that all our results are valid in arbitrary signature of the metric. On the other hand, we will not attempt to cover other generalizations such as affine connections with torsion or conformal Killing equations that would complicate our computations.

In Sect. 3, we shall generalize *special* Killing–Yano forms that are accommodated to the cone construction equivalence mentioned above. An example of this kind is the contact form on a Sasakian manifold.

Section 4 is devoted to considerations of spinor-valued differential forms and specializing former achievements into the language of spinor calculus.

In Sect. 5, we briefly review the cone construction and the already-known equivalences for Killing equations. Namely, we present explicit formulas for solutions which we later use in the example of spaces of constant curvature.

In Sect. 6, we employ the cone construction and the integrability conditions with the aim to describe all Killing spinor-valued forms on model spaces of constant curvature. We discover additional solutions in degree 1 and describe them explicitly by means of a new variant of the cone construction. These solutions are the first example of Killing spinor-valued forms that are not spanned by tensor product of Killing spinors and Killing–Yano forms.

2 Killing vector-valued differential forms

Let $(M, \nabla^{\mathsf{aff}})$ be a smooth manifold of dimension n, ∇^{aff} a torsion-free affine connection and $(\mathrm{V}, \nabla^{\mathrm{V}})$ a real or complex vector bundle over M equipped with a linear connection. We denote by ∇ the linear connection combined from ∇^{aff} and ∇^{V}, acting on mixed tensors built out of the tangent bundle $\mathrm{T}M$ and V by means of duals and tensor products. The situation of most interest for us is the vector-valued, or more specifically, V -valued differential forms which we will call just V-valued forms for short. We use the notation based on superscripts indicating the origin of all objects involved, e.g., the curvature operators $\mathcal{R}^{\mathsf{aff}}$, $\mathcal{R}^{\mathrm{V}}$ and $\mathcal{R}$ associated with ∇^{aff}, ∇^{V} and ∇, respectively.

Definition 1 Let $\boldsymbol{\Phi}$ be a V-valued form of degree $p \in \{0, \dots, n\}$. Then, $\boldsymbol{\Phi}$ is a *Killing* V *-valued form* provided there exists a V-valued form $\boldsymbol{\Xi}$ of degree $p+1$ such that

$$\nabla_X \boldsymbol{\Phi} = X \lrcorner \boldsymbol{\Xi}, \quad \text{for all } X \in \mathcal{X}(M), \tag{1}$$

where $X \lrcorner \boldsymbol{\Xi}$ denotes the contraction of the form $\boldsymbol{\Xi}$ by vector field X.

In other words, $\boldsymbol{\Phi}$ is Killing if and only if its covariant derivative is totally skew-symmetric. The form $\boldsymbol{\Xi}$ is hence uniquely determined as the normalized skew-symmetrization of the covariant derivative,

$$\boldsymbol{\Xi} = \tfrac{1}{p+1} \,\text{skew-symm.}(\nabla \boldsymbol{\Phi}) = \tfrac{1}{p+1}\, \mathrm{d}^{\mathrm{V}} \boldsymbol{\Phi}, \tag{2}$$

which equals the exterior covariant derivative of $\boldsymbol{\Phi}$. Because Eq. (1) does not imply for $p = 0$ any restriction on $\boldsymbol{\Phi}$, we will assume $p \geq 1$ for the rest of this section. On the other hand, for $p = n$, we set $\bigwedge^{n+1} \mathrm{T}^* M$ to be the zero vector bundle and (1) is thus equivalent to $\boldsymbol{\Phi}$ being parallel.

In the scalar-valued case, the solutions are often termed *Killing–Yano forms*, and we stick to this terminology in order to clearly distinguish between the scalar-valued case and the general vector-valued one. It is well known that (1) is a projectively invariant system of partial differential equations, and hence an invariant of the projective class of affine connections $[\nabla^{\mathsf{aff}}]$. A more detailed discussion around this observation is given in Sect. 2.2.

2.1 Killing connection

In order to deduce a prolongation and integrability conditions for the differential system (1), we decompose the action of curvature on $\boldsymbol{\Phi}$ into components according to their tensor symmetry types. This approach yields the prolongation in an invariant form, see Sect. 2.2 for further discussion. In fact, the value in the vector bundle V does not play a serious role and the whole procedure is parallel to the one for the scalar-valued Killing–Yano forms, cf. [15]. We have

$$\mathcal{R}_{X,Y}\boldsymbol{\Phi} = \nabla^2_{X,Y}\boldsymbol{\Phi} - \nabla^2_{Y,X}\boldsymbol{\Phi}, \tag{3}$$

which is a V-valued covariant $(p+2)$-tensor skew-symmetric separately in the first two and the remaining p indices. By the Littlewood–Richardson rule, the corresponding decomposition is

$$(2) \otimes (p) \simeq (p, 2) \oplus (p+1, 1) \oplus (p+2). \tag{4}$$

This result corresponds to the tensor product decomposition of irreducible representations for the general linear group. Here, we denote by $(c_1, c_2, \dots)$ the space of tensors with symmetries corresponding to the *dual* partition, i.e., the Young diagram with c_1 boxes in the first column, c_2 boxes in the second column, etc. For example, $(p+2)$ is the totally skew-symmetric component, and on the other hand, $(p, 2)$ is the component such that the skew-symmetrization over any subset of $p+1$ indices vanishes. In the case $p=1$, the decomposition degenerates and the term $(p, 2)$ disappears.

Now suppose that $\boldsymbol{\Phi}$ is a Killing V-valued form. From (1) and (3), we get

$$\mathcal{R}_{X,Y}\boldsymbol{\Phi} = Y\lrcorner(\nabla_X \Xi) - X\lrcorner(\nabla_Y \Xi). \tag{5}$$

The right-hand side depends linearly just on the first covariant derivative of Ξ, which in general decomposes according to

$$(1) \otimes (p+1) \simeq (p+1, 1) \oplus (p+2). \tag{6}$$

Comparing the decompositions (4) and (6), we conclude that the $(p, 2)$-type component of $\mathcal{R}\boldsymbol{\Phi}$ vanishes, and $\mathcal{R}\boldsymbol{\Phi}$ may be sufficient for computing the covariant derivative of Ξ. In what follows, we confirm these ideas and deduce explicit formulas.

Let us denote the partially and totally skew-symmetrized action of the curvature on skew-symmetric forms

$$\mathcal{R}_X \wedge \boldsymbol{\Phi} = \sum_{j=1}^{n} e^j \wedge (\mathcal{R}_{X,e_j}\boldsymbol{\Phi}), \tag{7}$$

$$\mathcal{R} \wedge \boldsymbol{\Phi} = \tfrac{1}{2}\sum_{i=1}^{n} e^i \wedge (\mathcal{R}_{e_i} \wedge \boldsymbol{\Phi}) = \tfrac{1}{2}\sum_{i,j=1}^{n} e^i \wedge e^j \wedge (\mathcal{R}_{e_i,e_j}\boldsymbol{\Phi}), \tag{8}$$

where $\{e_1, \dots, e_n\}$ is a tangent frame and $\{e^1, \dots, e^n\}$ its dual coframe. The three components with different tensor symmetries are given by

$$(\mathcal{R}\boldsymbol{\Phi})^{(p+2)}_{X,Y} = \tfrac{2}{(p+1)(p+2)}\, Y\lrcorner(X\lrcorner(\mathcal{R} \wedge \boldsymbol{\Phi})), \tag{9}$$

$$
\begin{aligned}
(\mathcal{R}\boldsymbol{\Phi})^{(p+1,1)}_{X,Y} = \tfrac{1}{p}\Big(& Y\lrcorner(\mathcal{R}_X \wedge \boldsymbol{\Phi}) - X\lrcorner(\mathcal{R}_Y \wedge \boldsymbol{\Phi}) \\
& - \tfrac{4}{p+2}\, Y\lrcorner(X\lrcorner(\mathcal{R} \wedge \boldsymbol{\Phi}))\Big),
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
(\mathcal{R}\boldsymbol{\Phi})^{(p,2)}_{X,Y} &= \mathcal{R}_{X,Y}\boldsymbol{\Phi} - (\mathcal{R}\boldsymbol{\Phi})^{(p+1,1)}_{X,Y} - (\mathcal{R}\boldsymbol{\Phi})^{(p+2)}_{X,Y} \\
&= \mathcal{R}_{X,Y}\boldsymbol{\Phi} - \tfrac{1}{p}\Big(Y\lrcorner(\mathcal{R}_X \wedge \boldsymbol{\Phi}) - X\lrcorner(\mathcal{R}_Y \wedge \boldsymbol{\Phi}) \\
&\quad - \tfrac{2}{p+1}\, Y\lrcorner(X\lrcorner(\mathcal{R} \wedge \boldsymbol{\Phi}))\Big).
\end{aligned} \tag{11}
$$

A straightforward computation verifies that the components indeed have the appropriate symmetries. It also easily follows that the $(p, 2)$-type component vanishes automatically for $p = 1$.

Proposition 2 *Let $\boldsymbol{\Phi}$ be a Killing* V*-valued form of degree $p \geq 1$ and $\boldsymbol{\Xi}$ the corresponding* V*-valued form of degree $p + 1$. Then, it holds*

$$
\nabla_X \boldsymbol{\Xi} = \tfrac{1}{p}\Big(\mathcal{R}_X \wedge \boldsymbol{\Phi} - \tfrac{1}{p+1}\, X\lrcorner(\mathcal{R} \wedge \boldsymbol{\Phi})\Big), \quad \text{for all } X \in \mathcal{X}(M), \tag{12}
$$

as well as

$$
(\mathcal{R}\boldsymbol{\Phi})^{(p,2)}_{X,Y} = 0, \quad \text{for all } X, Y \in \mathcal{X}(M). \tag{13}
$$

Equivalently, $\mathcal{R}_{X,Y}\boldsymbol{\Phi}$ is completely determined by $\mathcal{R}_X \wedge \boldsymbol{\Phi}$.

Proof By (5), (7), (8), we have

$$
\begin{aligned}
\mathcal{R}_X \wedge \boldsymbol{\Phi} &= \sum_{j=1}^{n} e^j \wedge (e_j\lrcorner(\nabla_X \boldsymbol{\Xi}) - X\lrcorner(\nabla_{e_j} \boldsymbol{\Xi})) \\
&= (p+1)\nabla_X \boldsymbol{\Xi} - \sum_{j=1}^{n} (\langle e^j, X\rangle \nabla_{e_j} \boldsymbol{\Xi} - X\lrcorner(e^j \wedge (\nabla_{e_j} \boldsymbol{\Xi}))) \\
&= p\nabla_X \boldsymbol{\Xi} + X\lrcorner\left(\sum_{j=1}^{n} e^j \wedge (\nabla_{e_j} \boldsymbol{\Xi})\right), \\
\mathcal{R} \wedge \boldsymbol{\Phi} &= \frac{1}{2}\sum_{i=1}^{n} e^i \wedge \left(p\nabla_{e_i} \boldsymbol{\Xi} + e_i\lrcorner\left(\sum_{j=1}^{n} e^j \wedge (\nabla_{e_j} \boldsymbol{\Xi})\right)\right) \\
&= (p+1)\sum_{j=1}^{n} e^j \wedge (\nabla_{e_j} \boldsymbol{\Xi}),
\end{aligned}
$$

proving (12). Then, we substitute (12) into (5),

$$\begin{aligned}\mathcal{R}_{X,Y}\boldsymbol{\Phi} =&Y\lrcorner\Big(\tfrac{1}{p}\Big(\mathcal{R}_X \wedge \boldsymbol{\Phi} - \tfrac{1}{p+1}\, X\lrcorner(\mathcal{R}\wedge\boldsymbol{\Phi})\Big)\Big)\\ &- X\lrcorner\Big(\tfrac{1}{p}\Big(\mathcal{R}_Y \wedge \boldsymbol{\Phi} - \tfrac{1}{p+1}\, Y\lrcorner(\mathcal{R}\wedge\boldsymbol{\Phi})\Big)\Big)\\ =&\tfrac{1}{p}\Big(Y\lrcorner(\mathcal{R}_X\wedge\boldsymbol{\Phi}) - X\lrcorner(\mathcal{R}_Y\wedge\boldsymbol{\Phi}) - \tfrac{2}{p+1}\, Y\lrcorner(X\lrcorner(\mathcal{R}\wedge\boldsymbol{\Phi}))\Big),\end{aligned}$$

proving (13). The last statement is a consequence of (13) and the fact that $\mathcal{R}\wedge\boldsymbol{\Phi}$ is just skew-symmetrization of $\mathcal{R}_X\wedge\boldsymbol{\Phi}$. ◻

The prolongation of vector-valued Killing forms then easily follows from (12). The appropriate prolongation vector bundle is the direct sum of V-valued p-form and $(p+1)$-form bundles,

$$\mathrm{K}^p = \Big(\bigwedge^p \mathrm{T}^*M \oplus \bigwedge^{p+1}\mathrm{T}^*M\Big)\otimes \mathrm{V}, \tag{14}$$

and the prolongation connection $\widetilde{\nabla}$ on K^p, called the *Killing connection*, is given by

$$\widetilde{\nabla}_X \begin{pmatrix}\boldsymbol{\Phi}\\ \boldsymbol{\Xi}\end{pmatrix} = \begin{pmatrix}\nabla_X\boldsymbol{\Phi} - X\lrcorner\boldsymbol{\Xi}\\ \nabla_X\boldsymbol{\Xi} - \frac{1}{p}\Big(\mathcal{R}_X\wedge\boldsymbol{\Phi} - \frac{1}{p+1}\,X\lrcorner(\mathcal{R}\wedge\boldsymbol{\Phi})\Big)\end{pmatrix} \tag{15}$$

for $\boldsymbol{\Phi}\in\Omega^p(M,\mathrm{V})$, $\boldsymbol{\Xi}\in\Omega^{p+1}(M,\mathrm{V})$.

Corollary 3 *The* V*-valued Killing forms of degree* $p\geq 1$ *are in one-to-one correspondence with sections of* K^p,

$$\boldsymbol{\Phi}\in\Omega^p(M,\mathrm{V}) \;\leftrightarrow\; \boldsymbol{\Theta} = \begin{pmatrix}\boldsymbol{\Phi}\\ \boldsymbol{\Xi}\end{pmatrix}\in\Gamma(\mathrm{K}^p), \quad \text{where } \boldsymbol{\Xi} \text{ is given by (2),}$$

which are parallel with respect to the Killing connection $\widetilde{\nabla}$. *In particular, the maximal possible dimension of the solution space on a connected manifold is* $\operatorname{rank}\mathrm{K}^p = \binom{n+1}{p+1}\operatorname{rank}\mathrm{V}$.

2.2 Projective invariance

As we have already noted, Eq. (1) is projectively invariant. To be precise, it is invariant under a projective change of the affine connection ∇^{aff} when considered acting on appropriately weighted differential forms. This is well known in the case of the scalar-valued Killing–Yano forms. In any case, note that the linear connection ∇^{V} on the value bundle V must remain fixed. The aim of this part is to compare the Killing connection in (15) with the standard projective tractor connection.

Now we shall briefly recall the projective tractor calculus. For more detail, see references [3, 9, 13, 14], and for a more systematic approach to Cartan and parabolic geometries, we refer to the monograph [6], Sections 4.1.5 and 5.2.6 devoted to projective structures. Two torsion-free affine connections ∇^{aff} and $\widehat{\nabla}^{\mathsf{aff}}$ are projectively equivalent if there exists a 1-form $\Upsilon\in\Omega^1(M)$ such that

$$\widehat{\nabla}^{\mathsf{aff}}_X Y = \nabla^{\mathsf{aff}}_X Y + \Upsilon(X)Y + \Upsilon(Y)X, \quad \text{for all } X, Y\in\mathcal{X}(M). \tag{16}$$

A *projective structure* on M is an equivalence class $[\nabla^{\mathsf{aff}}]$ of torsion-free affine connections. The curvature of ∇^{aff} can be decomposed with respect to the general linear group as

$$\mathcal{R}^{\mathsf{aff}}_{X,Y}Z = \mathcal{W}_{X,Y}Z + \mathcal{P}(Y,Z)X - \mathcal{P}(X,Z)Y + \beta(X,Y)Z, \tag{17}$$

where $\mathcal{W}$ is the totally trace-free *projective Weyl tensor*, $\mathcal{P}$ is the *projective Schouten tensor* and β is a skew-symmetric 2-form. Note that the projective Weyl tensor is independent of a representative connection in the class $[\nabla^{\mathsf{aff}}]$ and hence an invariant of the projective structure.

We define the *projective w-density bundles*, $w \in \mathbb{R}$, as oriented line bundles

$$\mathcal{E}(w) = \Big(\big(\textstyle\bigwedge^n \mathrm{T}^*M\big)^{\otimes 2} \Big)^{-\frac{w}{2(n+1)}}, \tag{18}$$

where $\bigwedge^n \mathrm{T}^*M$ is the canonical line bundle of M. If W is a vector bundle over M, we denote the corresponding *weighted bundles* $\mathrm{W}(w) = \mathrm{W} \otimes \mathcal{E}(w)$. The affine connection ∇^{aff} canonically extends to the density bundles as well, and a positive section σ of $\mathcal{E}(1)$ parallel with respect to ∇^{aff} is called *projective scale*. A choice of scale σ trivializes the density bundles, or more generally, induces bundle isomorphisms

$$\mathrm{W} \overset{\sigma}{\simeq} \mathrm{W}(w), \qquad v \mapsto v \otimes \sigma^w, \quad \text{for all } v \in \Gamma(W). \tag{19}$$

The only curvature component acting nontrivially on $\mathcal{E}(w)$ is the last one in (17) given by β,

$$\mathcal{R}^{\mathsf{aff}}_{X,Y}\sigma = w\beta(X,Y)\sigma, \quad \text{for all } \sigma \in \Gamma(\mathcal{E}(w)). \tag{20}$$

It is convenient for our purposes to assume that ∇^{aff} is such that $\beta = 0$, which means by (20) that ∇^{aff} admits locally a scale. Note that this assumption is always satisfied if $\nabla^{\mathsf{aff}} = \nabla^g$ is the Levi-Civita connection of a (pseudo-)Riemannian metric g. In this case, we have canonical global scale

$$\sigma^g = \Big(|\det(g)| \Big)^{-\frac{1}{2(n+1)}} \tag{21}$$

induced by the metric. The existence of a scale or vanishing of β also implies that the Schouten tensor $\mathcal{P}$ is symmetric.

The standard *projective tractor bundle* can be defined as a direct sum

$$\mathbb{T}M = \mathrm{T}M(-1) \oplus \mathcal{E}(-1) \tag{22}$$

equipped with the linear *tractor connection* $\nabla^{\mathbb{T}}$ given by

$$\nabla^{\mathbb{T}}_X \begin{pmatrix} Y \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla^{\mathsf{aff}}_X Y + \rho X \\ \nabla^{\mathsf{aff}}_X \rho - \mathcal{P}(X,Y) \end{pmatrix}, \quad \text{for all} \quad \begin{matrix} Y \in \Gamma(\mathrm{T}M(-1)), \\ \rho \in \Gamma(\mathcal{E}(-1)). \end{matrix} \tag{23}$$

While splitting (22) depends on a representative connection in the projective class, the tractor bundle itself and the tractor connection are projective invariants. The tractor connection naturally extends to other tractor bundles which are constructed as tensors generated by $\mathbb{T}M$ and its dual $\mathbb{T}^*M$. In particular, the skew-symmetric tractor $(p+1)$-forms split as

$$\textstyle\bigwedge^{p+1} \mathbb{T}^*M = \big(\bigwedge^p \mathrm{T}^*M\big)(p+1) \oplus \big(\bigwedge^{p+1} \mathrm{T}^*M\big)(p+1), \tag{24}$$

and the tractor connection is given by

$$\nabla^{\mathbb{T}}_X \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \nabla^{\mathsf{aff}}_X \alpha - X \lrcorner \beta \\ \nabla^{\mathsf{aff}}_X \beta + (X \lrcorner \mathcal{P}) \wedge \alpha \end{pmatrix} \tag{25}$$

for all $\alpha \in \Gamma\big(\big(\bigwedge^p \mathrm{T}^* M\big)(p+1)\big)$ and $\beta \in \Gamma\big(\big(\bigwedge^{p+1} \mathrm{T}^* M\big)(p+1)\big)$. Now we observe that in the presence of a scale we can identify the prolongation vector bundle K^p in (14) with $\bigwedge^{p+1} \mathbb{T}^* M \otimes \mathrm{V}$ via isomorphisms (19). Moreover, we immediately see that the top slot of (25) coincides with the top slot of (15) and corresponds to Eq. (1) defining Killing–Yano forms in the case of scalar-valued forms.

We proceed with the general case of vector-valued Killing forms and hence couple the tractor connection $\nabla^{\mathbb{T}}$ with the connection ∇^{V} on the vector bundle V yielding a connection $\nabla^{\mathbb{T}\mathrm{V}}$ acting on tractor-vector tensors built out of $\mathbb{T}M$ and V, e.g., V-valued skew-symmetric tractor forms. Formula (25) for the tractor connection now becomes simply

$$\nabla^{\mathbb{T}\mathrm{V}}_X \begin{pmatrix} \boldsymbol{\Phi} \\ \Xi \end{pmatrix} = \begin{pmatrix} \nabla_X \boldsymbol{\Phi} - X \lrcorner \Xi \\ \nabla_X \Xi + (X \lrcorner \mathcal{P}) \wedge \boldsymbol{\Phi} \end{pmatrix}, \tag{26}$$

for all $\boldsymbol{\Phi} \in \Gamma\big(\big(\bigwedge^p \mathrm{T}^* M\big)(p+1) \otimes \mathrm{V}\big)$ and $\Xi \in \Gamma\big(\big(\bigwedge^{p+1} \mathrm{T}^* M\big)(p+1) \otimes \mathrm{V}\big)$.

Since the covariant derivative ∇ is constructed from ∇^{aff} and ∇^{V}, we can split the curvature,

$$\mathcal{R} = \mathcal{R}^{\nabla} = \mathcal{R}^{\mathsf{aff}} + \mathcal{R}^{\mathrm{V}}, \tag{27}$$

into the parts $\mathcal{R}^{\mathsf{aff}}$ and $\mathcal{R}^{\mathrm{V}}$ which act separately on the form part and the value part, respectively. The curvature on the form part depends just on the affine connection ∇^{aff} and can be computed for any form $\boldsymbol{\Phi}$ as

$$\mathcal{R}^{\mathsf{aff}}_{X,Y} \boldsymbol{\Phi} = \sum_{k=1}^{n} (\mathcal{R}^{\mathsf{aff}}_{X,Y} e^k) \wedge (e_k \lrcorner \boldsymbol{\Phi}) = -\sum_{k=1}^{n} e^k \wedge ((\mathcal{R}^{\mathsf{aff}}_{X,Y} e_k) \lrcorner \boldsymbol{\Phi}). \tag{28}$$

In order to compare the curvature terms in the bottom slot of (26) and (15), we compute the skew-symmetrizations as defined in (7) and (8),

$$\begin{aligned} \mathcal{R}^{\mathsf{aff}}_X \wedge \boldsymbol{\Phi} &= -\sum_{j,k=1}^{n} e^j \wedge e^k \wedge ((\mathcal{R}^{\mathsf{aff}}_{X,e_j} e_k) \lrcorner \boldsymbol{\Phi}) \\ &= -\tfrac{1}{2} \sum_{j,k=1}^{n} e^j \wedge e^k \wedge ((\mathcal{R}^{\mathsf{aff}}_{X,e_j} e_k - \mathcal{R}^{\mathsf{aff}}_{X,e_k} e_j) \lrcorner \boldsymbol{\Phi}) \\ &= \tfrac{1}{2} \sum_{j,k=1}^{n} e^j \wedge e^k \wedge ((\mathcal{R}^{\mathsf{aff}}_{e_j,e_k} X) \lrcorner \boldsymbol{\Phi}), \end{aligned} \tag{29}$$

$$\begin{aligned} \mathcal{R}^{\mathsf{aff}} \wedge \boldsymbol{\Phi} &= -\tfrac{1}{2} \sum_{i,j,k=1}^{n} e^i \wedge e^j \wedge e^k \wedge ((\mathcal{R}^{\mathsf{aff}}_{e_i,e_j} e_k) \lrcorner \boldsymbol{\Phi}) \\ &= -\tfrac{1}{6} \sum_{i,j,k=1}^{n} e^i \wedge e^j \wedge e^k \wedge (\mathrm{cycl.}_{(i,j,k)} (\mathcal{R}^{\mathsf{aff}}_{e_i,e_j} e_k) \lrcorner \boldsymbol{\Phi}) \\ &= 0. \end{aligned} \tag{30}$$

We have used repeatedly the first Bianchi identity for $\mathcal{R}^{\mathsf{aff}}$ in the previous computations.

Remark In the case of scalar-valued forms, we have $\mathcal{R} = \mathcal{R}^{\mathsf{aff}}$, and it is well known that (30) is equivalent to the first Bianchi identity. The completely skew-symmetrized curvature terms are the newly arising components in the vector-valued case when compared to the case of Killing–Yano forms, cf. [15].

Proposition 4 *Assume that* ∇^{aff} *admits a scale, i.e., there is a* ∇^{aff}*-parallel section* σ *of* $\mathcal{E}(1)$. *Then, under the identification of weighted forms with their unweighted counterparts via the isomorphisms* (19), *the Killing connection defined in* (15) *is given by*

$$\widetilde{\nabla}_X \begin{pmatrix} \boldsymbol{\Phi} \\ \boldsymbol{\Xi} \end{pmatrix} = \nabla^{\mathbb{T}\mathrm{V}}_X \begin{pmatrix} \boldsymbol{\Phi} \\ \boldsymbol{\Xi} \end{pmatrix} - \frac{1}{p} \begin{pmatrix} 0 \\ \mathcal{W}_X \wedge \boldsymbol{\Phi} + \mathcal{R}^{\mathrm{V}}_X \wedge \boldsymbol{\Phi} - \frac{1}{p+1} X \lrcorner (\mathcal{R}^{\mathrm{V}} \wedge \boldsymbol{\Phi}) \end{pmatrix}, \tag{31}$$

for all $\boldsymbol{\Phi} \in \Omega^p(M, \mathrm{V})$ *and* $\boldsymbol{\Xi} \in \Omega^{p+1}(M, \mathrm{V})$. *Here, the skew-symmetrized actions of* $\mathcal{W}$ *and* $\mathcal{R}^{\mathrm{V}}$ *on* $\boldsymbol{\Phi}$ *are defined as in* (7) *and* (8).

Proof Firstly, since the scale σ is required to be ∇^{aff}-parallel, isomorphisms (19) preserve the covariant derivatives and the curvature actions. As already noted, the presence of a scale also implies $\beta = 0$ and that the Schouten tensor $\mathcal{P}$ is symmetric. It remains to show that the curvature terms in the bottom slot of (26) and (15) are equal. Indeed, the $\mathcal{R}^{\mathrm{V}}$-part is left unchanged and for the $\mathcal{R}^{\mathsf{aff}}$-part we compute using (29), (17) with $\beta = 0$, and the symmetry of $\mathcal{P}$,

$$\begin{aligned} \mathcal{R}^{\mathsf{aff}}_X \wedge \boldsymbol{\Phi} &= \mathcal{W}_X \wedge \boldsymbol{\Phi} + \frac{1}{2} \sum_{j,k=1}^{n} e^j \wedge e^k \wedge ((\mathcal{P}(e_k, X)\, e_j - \mathcal{P}(e_j, X)\, e_k) \lrcorner \boldsymbol{\Phi}) \\ &= \mathcal{W}_X \wedge \boldsymbol{\Phi} - \sum_{j,k=1}^{n} \mathcal{P}(X, e_j)\, e^j \wedge e^k \wedge (e_k \lrcorner \boldsymbol{\Phi}) \\ &= \mathcal{W}_X \wedge \boldsymbol{\Phi} - p\, (X \lrcorner \mathcal{P}) \wedge \boldsymbol{\Phi}, \end{aligned}$$

which together with (30) proves (31). □

Since the Weyl tensor $\mathcal{W}$ is an invariant of the projective structure and ∇^{V} and hence also $\mathcal{R}^{\mathrm{V}}$ are fixed independently of ∇^{aff}, Eq. (31) implies that $\widetilde{\nabla}$ is also an invariant of the projective structure when considered on V-valued tractor forms. The scale becomes redundant when appropriately weighted forms are considered; hence, by Corollary 2, we get:

Corollary 5 *The Killing Eq.* (1) *is projectively invariant when considered on weighted* V *-valued p-forms of weight* $w = p + 1$.

The corollary can be also proved directly by considering two representative connections ∇^{aff} and $\hat{\nabla}^{\mathsf{aff}}$ in the projective class and employing appropriate transformation of $\boldsymbol{\Xi}$ induced by the change of splitting (24), cf. also [16]. After some curvature rearrangements based on the first Bianchi identity, a special case of formula (31) can also be found in [9], p. 50.

Remark A method of constructing an invariant prolongation connection via tractors in the broad context of parabolic geometries was developed in [13, 14]. In particular, in

Section 3.2 of [14], an explicit formula was derived for the case of skew-symmetric contravariant projective tractors dual to our case of tractor forms. There is a sign mistake in the relevant formula in [14], the corrected prolongation connection $\widetilde{\nabla}$ is

$$\widetilde{\nabla}_a \begin{pmatrix} \sigma^{\mathbf{c}} \\ \rho^{\dot{\mathbf{c}}} \end{pmatrix} = \nabla^{\mathbb{T}}_a \begin{pmatrix} \sigma^{\mathbf{c}} \\ \rho^{\dot{\mathbf{c}}} \end{pmatrix} - \frac{\ell(\ell-1)}{2(n-\ell)} \begin{pmatrix} 0 \\ \mathcal{W}_{pr}{}^{c^2}{}_a \sigma^{pr\ddot{\mathbf{c}}} \end{pmatrix} \tag{32}$$

and it agrees with our formula (31) in the special case of scalar-valued forms ($\mathcal{R}^{\mathrm{V}} = 0$), and consequently also with [15]. This can be verified by a straightforward application of the bundle isomorphism

$$\bigwedge^{\ell} \mathbb{T}M \simeq \bigwedge^{n+1-\ell} \mathbb{T}^*M \tag{33}$$

induced by a constant nonzero tractor volume form $\Omega \in \Gamma\big(\bigwedge^{n+1} \mathbb{T}^*M\big)$. We remark that it is sufficient that Ω exists at least locally; hence, the last statement holds even in the nonorientable case.

The contravariant form corresponds to the prolongation of the equation

$$\nabla_X \sigma = -\tfrac{1}{\ell} X \wedge \rho, \quad \text{for all } X \in \mathcal{X}(M), \tag{34}$$

where $\sigma \in \Gamma\big(\bigwedge^{\ell} \mathrm{T}M\big)$ and $\rho \in \Gamma\big(\bigwedge^{\ell-1} \mathrm{T}M\big)$ for $\ell < n$. For $\ell = 1$, the solutions are called *concircular vector fields*, see [18] or [12]. Equation (34) is clearly just a dual form of our basic Eq. (1). Later, we introduce similar Eqs. (41) and (42) in the presence of a (pseudo-)Riemannian metric.

It is also worth noting that formula (15) for the Killing connection is not the only possibility to prolong Eq. (1). However, Proposition 4 implies that our particular form of the prolongation has the advantage of being projectively invariant.

2.3 Higher integrability conditions

We have already proved the first integrability condition (13) in the second part of Proposition 2. We are going to derive an explicit formula also for the second integrability condition. This is especially important in the case $p = 1$ when the first condition (13) becomes empty. As it turns out, the second integrability condition involves a modification of the total curvature on V-valued forms.

Proposition 6 *Let Φ be a Killing* V*-valued form of degree $p \geq 1$ and Ξ the corresponding* V*-valued form of degree $p+1$. Then, it holds*

$$(\widehat{\mathcal{R}}\Xi)^{(p+1,2)}_{X,Y} = ((\nabla\mathcal{R}) \barwedge \Phi)_{X,Y}, \quad \text{for all } X, Y \in \mathcal{X}(M), \tag{35}$$

where the modified curvature $\widehat{\mathcal{R}}$ acting on Ξ is given by

$$\widehat{\mathcal{R}} = (p+2)\mathcal{R} - \mathcal{R}^{\mathsf{aff}} = (p+1)\mathcal{R}^{\mathsf{aff}} + (p+2)\mathcal{R}^{\mathrm{V}}, \tag{36}$$

and the action of the derivative of the curvature on Φ is given by

$$((\nabla\mathcal{R}) \barwedge \boldsymbol{\Phi})_{X,Y} = (\nabla_X\mathcal{R})_Y \wedge \boldsymbol{\Phi} - (\nabla_Y\mathcal{R})_X \wedge \boldsymbol{\Phi} - \\ - \tfrac{1}{p+1}(Y\lrcorner((\nabla_X\mathcal{R}) \wedge \boldsymbol{\Phi}) - X\lrcorner((\nabla_Y\mathcal{R}) \wedge \boldsymbol{\Phi})). \tag{37}$$

The symmetry components of the action of curvature or its first derivative are defined as in (7)–(11).

Proof First, we compute the action of the curvature on Ξ using (12) and (1):

$$\begin{aligned} p\mathcal{R}_{X,Y}\Xi =&\mathcal{R}_Y \wedge (X\lrcorner\Xi) - \mathcal{R}_X \wedge (Y\lrcorner\Xi) \\ &- \tfrac{1}{p+1}(Y\lrcorner(\mathcal{R} \wedge (X\lrcorner\Xi)) - X\lrcorner(\mathcal{R} \wedge (Y\lrcorner\Xi))) \\ &+ (\nabla_X\mathcal{R})_Y \wedge \boldsymbol{\Phi} - (\nabla_Y\mathcal{R})_X \wedge \boldsymbol{\Phi} \\ &- \tfrac{1}{p+1}(Y\lrcorner((\nabla_X\mathcal{R}) \wedge \boldsymbol{\Phi}) - X\lrcorner((\nabla_Y\mathcal{R}) \wedge \boldsymbol{\Phi})) \end{aligned}$$

In the next, we compute the terms containing the contraction of Ξ using (7) and (8):

$$\begin{aligned} \mathcal{R}_Y \wedge (X\lrcorner\Xi) &= \sum_{j=1}^{n} e^j \wedge ((\mathcal{R}^{\mathsf{aff}}_{Y,e_j}X)\lrcorner\Xi + X\lrcorner(\mathcal{R}_{Y,e_j}\Xi)) \\ &= \sum_{j=1}^{n} e^j \wedge ((\mathcal{R}^{\mathsf{aff}}_{Y,e_j}X)\lrcorner\Xi) - \mathcal{R}_{X,Y}\Xi - X\lrcorner(\mathcal{R}_Y \wedge \Xi), \\ \mathcal{R} \wedge (X\lrcorner\Xi) &= \tfrac{1}{2}\sum_{i,j=1}^{n} e^i \wedge e^j \wedge ((\mathcal{R}^{\mathsf{aff}}_{e_i,e_j}X)\lrcorner\Xi) - \mathcal{R}_X \wedge \Xi + X\lrcorner(\mathcal{R} \wedge \Xi). \end{aligned}$$

Collecting all the terms containing Ξ and using (28), (29), (30) and the first Bianchi identity (recall that $\mathcal{R}^{\mathsf{aff}}$ acts just on the form part), we get

$$\begin{aligned} &p\mathcal{R}_{X,Y}\Xi - \mathcal{R}_Y \wedge (X\lrcorner\Xi) + \mathcal{R}_X \wedge (Y\lrcorner\Xi) \\ &\quad + \tfrac{1}{p+1}(Y\lrcorner(\mathcal{R} \wedge (X\lrcorner\Xi)) - X\lrcorner(\mathcal{R} \wedge (Y\lrcorner\Xi))) \\ &= (p+2)\Big(\mathcal{R}_{X,Y}\Xi - \tfrac{1}{p+1}\Big(Y\lrcorner(\mathcal{R}_X \wedge \Xi) - X\lrcorner(\mathcal{R}_Y \wedge \Xi) \\ &\quad - \tfrac{2}{p+2}\, Y\lrcorner(X\lrcorner(\mathcal{R} \wedge \Xi))\Big)\Big) \\ &\quad - \Big(\mathcal{R}^{\mathsf{aff}}_{X,Y}\Xi - \tfrac{1}{p+1}\Big(Y\lrcorner(\mathcal{R}^{\mathsf{aff}}_X \wedge \Xi) - X\lrcorner(\mathcal{R}^{\mathsf{aff}}_Y \wedge \Xi) \\ &\quad - \tfrac{2}{p+2}\, Y\lrcorner(X\lrcorner(\mathcal{R}^{\mathsf{aff}} \wedge \Xi))\Big)\Big). \end{aligned}$$

Finally, (35) follows by (11) for Ξ (recall that the degree of Ξ is $p+1$). □

In general, there are many other integrability conditions for Killing V-valued forms resulting from the prolongation procedure as formulated in Corollary 3. The complete set of integrability conditions for the equation $\widetilde{\nabla}\Theta = 0$ is given as the annihilator of the infinitesimal holonomy algebra $\mathfrak{hol}(\mathrm{K}^p, \widetilde{\nabla})$. The infinitesimal holonomy algebra is pointwise generated by curvature and all its derivatives; hence, the integrability conditions are

$$(\widetilde{\nabla}^k_{Z_1,\dots,Z_k}\widetilde{\mathcal{R}})_{X,Y}\Theta = 0, \qquad \begin{array}{l}\text{for all } X, Y, Z_1, \dots, Z_k \in \mathcal{X}(M) \\ \text{and } k = 0, 1, \dots .\end{array} \tag{38}$$

In other words, the integrability conditions reflect the obstructions to flatness of the prolongation tractor bundle with respect to the Killing connection.

However, we find it more convenient for further application to formulate the integrability conditions directly in terms of the components Φ, Ξ as in formulas (13) and (35). This corresponds to splitting (38) into individual slots. In fact, Eq. (13) appears in the top slot of (38) for $k = 0$, and Eq. (35) appears in the bottom slot of (38) for $k = 0$ as well as in an equivalent form in the top slot for $k = 1$.

3 Special Killing vector-valued differential forms

From now on, (M, g) is assumed to be a (pseudo-)Riemannian manifold and the affine connection ∇^{aff} will always be the Levi-Civita connection $\nabla^{\mathsf{aff}} = \nabla^{g}$. We will denote the corresponding (pseudo-)Riemannian curvature by $\mathcal{R}^{g}$.

The curvature operator at a point $x \in M$ takes values in the orthogonal Lie algebra $\mathfrak{so}(\mathrm{T}_xM, g)$, identified with $\bigwedge^2 \mathrm{T}_xM$ by an isomorphism ρ:

$$\rho(X \wedge Y)\, Z = g(Y, Z)\, X - g(X, Z)\, Y, \quad \text{for all } X, Y, Z \in \mathrm{T}_xM. \tag{39}$$

The action on skew-symmetric forms is induced by the tensor product action,

$$\begin{aligned} \rho(X \wedge Y)\, \Phi &= X \lrcorner (Y^\flat \wedge \Phi) - Y \lrcorner (X^\flat \wedge \Phi) \\ &= X^\flat \wedge (Y \lrcorner \Phi) - Y^\flat \wedge (X \lrcorner \Phi), \\ &\text{for all } X, Y \in \mathrm{T}_xM, \\ &\Phi \in \textstyle\bigwedge^p \mathrm{T}^*_xM, \end{aligned} \tag{40}$$

where $X^\flat$ denotes the metric dual of the vector X with respect to g. The presence of metric allows us to dualize the notion of a Killing form.

Definition 7 Let Ξ be a V-valued form of degree $p \in \{1, \dots, n\}$. We say that Ξ is a $\star$-*Killing* V-*valued form* if there exists a V-valued form Φ of degree $p - 1$ such that

$$\nabla_X \Xi = X^\flat \wedge \Phi, \quad \text{for all } X \in \mathcal{X}(M). \tag{41}$$

The $\star$-Killing forms are just the Hodge star duals of Killing forms. An interested reader can find more detailed treatment of the scalar-valued case in [15]. There is a special important class of Killing forms given by matching pairs of a Killing and a $\star$-Killing form. Note that in the following definition we allow the (nontrivial) case $p = 0$.

Definition 8 Let Φ be a Killing V-valued form of degree $p \in \{0, \dots, n\}$ and Ξ the corresponding V-valued form of degree $p + 1$. We say that Φ is a *special Killing* V-*valued form* if there exists a constant $c \in \mathbb{R}$ such that in addition to Eq. (1) it holds

$$\nabla_X \Xi = -cX^\flat \wedge \Phi, \quad \text{for all } X \in \mathcal{X}(M). \tag{42}$$

The main significance of special Killing forms stems from their close relationship with the *metric cone construction*, see Sect. 5 for more details. We mention an example of *Sasakian structures*, equivalent to special Killing–Yano 1-forms of constant length 1, cf. [15].

We note that the system of Eqs. (1) and (42) is already in a closed form, and there is no need to prolong it. In the following proposition, we give the first integrability condition which in fact characterizes special Killing forms among Killing forms.

Proposition 9 *Let Φ be a Killing* V*-valued form. Then, it is a special Killing* V*-valued form with the corresponding constant $c \in \mathbb{R}$ if and only if it holds*

$$\mathcal{R}_{X,Y}\Phi = c\rho^{\mathsf{aff}}(X \wedge Y)\,\Phi, \quad \text{for all } X, Y \in \mathcal{X}(M), \tag{43}$$

where the superscript '`aff`*' emphasizes that it acts only on the form part of Φ.*

Proof First, suppose that Φ is a special Killing V-valued form of degree p and Ξ the corresponding V-valued form of degree $p+1$. Using (1), (42) and (40), we compute the action of the curvature proving (43):

$$\begin{aligned}\mathcal{R}_{X,Y}\Phi &= \nabla^2_{X,Y}\Phi - \nabla^2_{Y,X}\Phi = Y\lrcorner(\nabla_X \Xi) - X\lrcorner(\nabla_Y \Xi)\\ &= -c(Y\lrcorner(X^\flat \wedge \Phi) - X\lrcorner(Y^\flat \wedge \Phi)) = c\rho^{\mathsf{aff}}(X \wedge Y)\,\Phi.\end{aligned}$$

On the other hand, suppose that Φ is Killing and (43) holds. We will compute in a tangent frame $\{e_1, \ldots, e_n\}$, with $g_{ij} = g(e_i, e_j)$ the corresponding metric components. The skew-symmetrized actions of the curvature as defined in (7) and (8) are

$$\begin{aligned}\mathcal{R}_X \wedge \Phi &= c\sum_{j=1}^{n} e^j \wedge (\rho^{\mathsf{aff}}(X \wedge e_j)\,\Phi)\\ &= c\sum_{j=1}^{n} e^j \wedge (X^\flat \wedge (e_j\lrcorner\Phi) - (e_j)^\flat \wedge (X\lrcorner\Phi))\\ &= -c\left(X^\flat \wedge \left(\sum_{j=1}^{n} e^j \wedge (e_j\lrcorner\Phi)\right) + \sum_{i,j=1}^{n} g_{ij}\, e^j \wedge e^i \wedge (X\lrcorner\Phi)\right)\\ &= -cpX^\flat \wedge \Phi,\\ \mathcal{R} \wedge \Phi &= -\tfrac{cp}{2}\sum_{i=1}^{n} e^i \wedge (e_i)^\flat \wedge \Phi = -\tfrac{cp}{2}\sum_{i,j=1}^{n} g_{ij}\, e^i \wedge e^j \wedge \Phi = 0.\end{aligned}$$

Now (42) follows by using Proposition 2 and substituting into (12). □

We conclude this section with a few higher integrability conditions.

Proposition 10 *Let Φ be a special Killing* V*-valued p-form, Ξ the corresponding* V*-valued $(p+1)$-form and $c \in \mathbb{R}$ the associated Killing constant. Then, the following equalities hold for all $X, Y, Z \in \mathcal{X}(M)$:*

$$\mathcal{R}_{X,Y}\Xi = c\rho^{\mathsf{aff}}(X \wedge Y)\,\Xi, \tag{44}$$

$$(\nabla_X \mathcal{R})_{Y,Z}\Phi = -((\mathcal{R}^g_{Y,Z} - c\rho(Y \wedge Z))\,X)\lrcorner\Xi, \tag{45}$$

$$(\nabla_X \mathcal{R})_{Y,Z} \Xi = c((\mathcal{R}^g_{Y,Z} - c\rho(Y \wedge Z))X)^\flat \wedge \boldsymbol{\Phi}. \tag{46}$$

Proof The proof of (44) is analogous to that of (43), and we just compute the action of the curvature using (1), (42) and (40):

$$\begin{aligned}\mathcal{R}_{X,Y}\Xi &= \nabla^2_{X,Y}\Xi - \nabla^2_{Y,X}\Xi = -c(Y^\flat \wedge (\nabla_X \boldsymbol{\Phi}) - X^\flat \wedge (\nabla_Y \boldsymbol{\Phi})) \\ &= -c(Y^\flat \wedge (X \lrcorner \Xi) - X^\flat \wedge (Y \lrcorner \Xi)) = c\rho^{\mathsf{aff}}(X \wedge Y)\,\Xi.\end{aligned}$$

Equations (45) and (46) follow from (43) and (44) by differentiating while noticing that $\rho(\cdot \wedge \cdot)$ is covariantly constant. ☐

Remark The integrability conditions can be elegantly formulated in terms of a curvature modified by the scalar curvature component in the Ricci decomposition,

$$\mathcal{R}^c_{X,Y} := \mathcal{R}_{X,Y} - c\rho^{\mathsf{aff}}(X \wedge Y). \tag{47}$$

The second term is covariantly constant and thus we have $\nabla\mathcal{R} = \nabla\mathcal{R}^c$, so the modified curvature can be conveniently incorporated into all the higher integrability conditions.

This is related to the *concircular curvature* consisting just of the trace-free Ricci and (conformal) Weyl components, cf. [18]. The modification has also nice interpretation from the point of view of Cartan geometries and the phenomenon of *model mutation*, cf. [6] Sections 1.1.2 and 1.5.1. It is the modification we would get if we had chosen the round sphere (for $c > 0$) or the hyperbolic space (for $c < 0$), respectively, as the homogeneous model space for the Riemannian geometry instead of the Euclidean space (and analogously in the pseudo-Riemannian case).

4 Killing spinor-valued differential forms

From now on, (M, g) is assumed to be oriented and spin with a fixed chosen spin structure. We denote the corresponding complex spinor bundle by ΣM and the operation of Clifford multiplication by vectors on spinors by '$\cdot$.' The Levi-Civita connection ∇^g lifts to a spin connection on ΣM, and we denote it by abuse of notation ∇^g too. We briefly recall the notion of Killing spinors and their basic properties relevant for our needs, see, for example, the monograph [4].

Definition 11 A spinor field Ψ is a *Killing spinor* if there exists a constant $a \in \mathbb{C}$ such that

$$\nabla^g_X \Psi = aX \cdot \Psi, \quad \text{for all } X \in \mathcal{X}(M). \tag{48}$$

The constant a is called the *Killing number* of Ψ.

Equation (48) is already in the closed form, and we can readily write down the prolongation connection, called again the *Killing connection* on spinors,

$$\nabla^a_X \Psi = \nabla^g_X \Psi - aX \cdot \Psi, \quad X \in \mathcal{X}(M),\ \Psi \in \Gamma(\Sigma M). \tag{49}$$

Corollary 12 *The Killing spinors with fixed Killing number $a \in \mathbb{C}$ are sections of ΣM parallel with respect to the Killing connection ∇^a. In particular, the maximal possible dimension of the solution space on a connected manifold is* rank $\Sigma M = 2^{\lfloor \frac{n}{2} \rfloor}$.

By this corollary or directly from (48), it follows easily the first integrability condition. We recall the action of the Lie algebra $\mathfrak{spin}(\mathrm{T}_x M, g) \simeq \mathfrak{so}(\mathrm{T}_x M, g)$ on spinors in terms of the isomorphism ρ in (39),

$$\rho(X \wedge Y)\Psi = -\tfrac{1}{4}[X\cdot, Y\cdot]\Psi = -\tfrac{1}{2}(X \cdot Y \cdot + g(X, Y))\Psi, \tag{50}$$

where $[\cdot, \cdot]$ denotes the commutator. The curvature of the Killing connection ∇^a defined in (49) is given by

$$\begin{aligned} \mathcal{R}^a_{X,Y}\Psi &= \nabla^a_X(\nabla^a_Y\Psi) - \nabla^a_Y(\nabla^a_X\Psi) - \nabla^a_{[X,Y]}\Psi \\ &= \mathcal{R}^g_{X,Y}\Psi + a^2[X\cdot, Y\cdot]\Psi = \mathcal{R}^g_{X,Y}\Psi - 4a^2\rho(X \wedge Y)\Psi. \end{aligned} \tag{51}$$

Proposition 13 *Let Ψ be a Killing spinor with Killing number $a \in \mathbb{C}$. Then,*

$$\mathcal{R}^g_{X,Y}\Psi = -a^2\,[X\cdot, Y\cdot]\Psi = 4a^2\rho(X \wedge Y)\Psi, \quad \text{for all } X, Y \in \mathcal{X}(M). \tag{52}$$

By (52), it follows the well-known fact that a manifold admitting Killing spinors has constant *scalar curvature*, related to the Killing number by

$$\mathrm{Scal}^g = 4n(n-1)a^2. \tag{53}$$

In the Riemannian case, it also follows that the manifold is Einstein.

We define the Killing spinor-valued forms as a special case of Killing vector-valued forms, where $\nabla^{\mathsf{aff}} = \nabla^g$ as before and the vector bundle of values is the spinor bundle $\mathrm{V} = \Sigma M$ equipped with the connection $\nabla^{\mathrm{V}} = \nabla^a$ given above for arbitrary $a \in \mathbb{C}$. We can now reformulate Definitions 1 and 8 in terms of the Levi-Civita spin connection ∇^g and the Killing number $a \in \mathbb{C}$. In particular, by abuse of notation, ∇^g denotes the usual tensor product connection acting on both the form and the spinor part of a spinor-valued form by ∇^g.

Definition 14 Let Φ be a spinor-valued form of degree $p \in \{0, \dots, n\}$. We say that Φ is a *Killing spinor-valued form* if there exist a constant $a \in \mathbb{C}$ and a spinor-valued form Ξ of degree $p+1$ such that

$$\nabla^g_X\Phi = aX \cdot \Phi + X \lrcorner \Xi, \quad \text{for all } X \in \mathcal{X}(M). \tag{54}$$

We say that Φ is a *special Killing spinor-valued form* if there exists another constant $c \in \mathbb{R}$ such that both (54) and the equation

$$\nabla^g_X\Xi = aX \cdot \Xi - cX^\flat \wedge \Phi, \quad \text{for all } X \in \mathcal{X}(M) \tag{55}$$

are satisfied.

In order to develop calculus for spinor-valued forms, it is convenient to express the Clifford multiplication $\cdot : \mathrm{T}M \otimes \Sigma M \to \Sigma M$ as a 1-form with values in the endomorphisms of the spinor bundle $\gamma\cdot \in \Omega^1(M, \mathrm{End}\,(\Sigma M))$,

$$\gamma\cdot = \sum\nolimits_{i=1}^{n} e^i \otimes (e_i\cdot). \tag{56}$$

In terms of $\gamma\cdot$, the defining equation of the Clifford algebra can be expressed as

$$\mathrm{symm.}(\gamma\cdot \otimes \gamma\cdot) = -2g, \tag{57}$$

where symm. denotes the symmetrization in form indices. Note that the Clifford-valued form $\gamma\cdot$ is parallel with respect to the Levi-Civita spin connection ∇^g because the Clifford multiplication is equivariant with respect to the spin group. We will also frequently use the following algebraic formulas,

$$X\cdot(\gamma\cdot\wedge\boldsymbol{\Phi}) + \gamma\cdot\wedge(X\cdot\boldsymbol{\Phi}) = -2X^\flat\wedge\boldsymbol{\Phi}, \tag{58}$$

$$X\lrcorner(\gamma\cdot\wedge\boldsymbol{\Phi}) + \gamma\cdot\wedge(X\lrcorner\boldsymbol{\Phi}) = X\cdot\boldsymbol{\Phi}, \tag{59}$$

$$X\cdot(\gamma^\sharp\cdot\lrcorner\boldsymbol{\Phi}) + \gamma^\sharp\cdot\lrcorner(X\cdot\boldsymbol{\Phi}) = -2X\lrcorner\boldsymbol{\Phi}, \tag{60}$$

$$X^\flat\wedge(\gamma^\sharp\cdot\lrcorner\boldsymbol{\Phi}) + \gamma^\sharp\cdot\lrcorner(X^\flat\wedge\boldsymbol{\Phi}) = X\cdot\boldsymbol{\Phi}, \tag{61}$$

as a consequence of (56) and (57). Here,

$$\gamma^\sharp\cdot = \sum\nolimits_{i,j=1}^{n} g^{ij}\, e_i \otimes (e_j\cdot) \tag{62}$$

is the metric dual of the 1-form $\gamma\cdot$ with respect to g and $g^{ij} = g(e^i, e^j) = (g_{ij})^{-1}$ are the entries of the inverse metric. By (56)–(62),

$$\gamma^\sharp\cdot\lrcorner(\gamma\cdot\wedge\boldsymbol{\Phi}) - \gamma\cdot\wedge(\gamma^\sharp\cdot\lrcorner\boldsymbol{\Phi}) = (2p-n)\,\boldsymbol{\Phi}, \tag{63}$$

for all spinor-valued forms $\boldsymbol{\Phi}$ of degree p.

In the present case, Eq. (2) which determines Ξ specializes to

$$\Xi = \tfrac{1}{p+1}\,\mathrm{d}^a\boldsymbol{\Phi} = \tfrac{1}{p+1}(\mathrm{d}^g\boldsymbol{\Phi} - a\gamma\cdot\wedge\boldsymbol{\Phi}), \tag{64}$$

where d^a and d^g denote the respective exterior covariant derivatives with ∇^a and ∇^g acting on the spinor values. We can now substitute Ξ into (54) to get another equivalent form of the defining equation,

$$\nabla^g_X\boldsymbol{\Phi} = a\Big(X\cdot\boldsymbol{\Phi} - \tfrac{1}{p+1}\,X\lrcorner(\gamma\cdot\wedge\boldsymbol{\Phi})\Big) + \tfrac{1}{p+1}\,X\lrcorner\mathrm{d}^g\boldsymbol{\Phi}. \tag{65}$$

Similarly, we can substitute Ξ into (55), and by (58), (59) get

$$\begin{aligned}
\nabla^g_X(\mathrm{d}^g\boldsymbol{\Phi}) &= \nabla^g_X((p+1)\,\Xi + a\gamma\cdot\wedge\boldsymbol{\Phi})\\
&=(p+1)(aX\cdot\Xi - cX^\flat\wedge\boldsymbol{\Phi}) + a\gamma\cdot\wedge(aX\cdot\boldsymbol{\Phi} + X\lrcorner\Xi)\\
&= -c(p+1)\,X^\flat\wedge\boldsymbol{\Phi} + a\Big(X\cdot\mathrm{d}^g\boldsymbol{\Phi} + \tfrac{1}{p+1}\,\gamma\cdot\wedge(X\lrcorner\mathrm{d}^g\boldsymbol{\Phi})\Big)\\
&\quad + a^2\Big(2X^\flat\wedge\boldsymbol{\Phi} + \tfrac{2p+1}{p+1}\,\gamma\cdot\wedge(X\cdot\boldsymbol{\Phi}) + \tfrac{1}{p+1}\,\gamma\cdot\wedge(\gamma\cdot\wedge(X\lrcorner\boldsymbol{\Phi}))\Big).
\end{aligned} \tag{66}$$

The resulting formulas (65) and (66) agree with the definitions of (special) Killing spinor-valued forms introduced in [17].

5 Cone construction

We shall briefly recall the cone construction following reference [17], which provides a useful description of special Killing spinor-valued forms. As it turns out in Sect. 6, most of the Killing spinor-valued forms on spaces of constant curvature are special and hence arise from this construction.

Let (M, g) be a spin (pseudo-)Riemannian manifold of signature (n_+, n_-) with a fixed spin structure, and for $\varepsilon = \pm 1$ we define

$$\begin{aligned} \overline{n}_+ &= n_+ + 1, & \overline{n}_- &= n_-, & \text{if } \varepsilon &= +1, \\ \overline{n}_+ &= n_+, & \overline{n}_- &= n_- + 1, & \text{if } \varepsilon &= -1. \end{aligned} \tag{67}$$

The *ε-metric cone* over (M, g) is the product manifold $\overline{M} = \mathbb{R}_+ \times M$ with the warped product metric $\overline{g}$ of signature $(\overline{n}_+, \overline{n}_-)$,

$$\overline{g} = \varepsilon \mathrm{d}r^2 + r^2 g, \tag{68}$$

where r is the coordinate function on $\mathbb{R}_+$. The original manifold (M, g) is isometrically embedded in $(\overline{M}, \overline{g})$ as the hypersurface defined by $r = 1$, and the outer unit normal is given by the radial vector field ∂_r. Note that $\overline{g}(\partial_r, \partial_r) = \varepsilon$.

5.1 Tangent bundle and forms on hypersurfaces

Let us consider the restricted vector bundle $\overline{\mathrm{T}}M = \mathrm{T}\overline{M}|_M$ over M. It splits orthogonally into the normal and tangent bundle of M,

$$\overline{\mathrm{T}}M = \mathrm{N}M \oplus \mathrm{T}M. \tag{69}$$

The normal bundle $\mathrm{N}M$ is in our case trivialized by the outer unit normal ∂_r. Hence, we can write the normal and tangent projections as

$$\begin{aligned} \pi^{\mathrm{N}}(\overline{v}) &= \overline{g}(\partial_r, \overline{v}), \\ \pi^{\mathrm{T}}(\overline{v}) &= \overline{v} - \varepsilon \overline{g}(\partial_r, \overline{v})\, \partial_r, \end{aligned} \qquad \text{for all } \overline{v} \in \overline{\mathrm{T}}M. \tag{70}$$

Accordingly, the decomposition of differential forms is

$$\bigwedge\nolimits^{p+1} \overline{\mathrm{T}}^* M = \left(\mathrm{N}^*M \otimes \bigwedge\nolimits^{p} \mathrm{T}^*M\right) \oplus \bigwedge\nolimits^{p+1} \mathrm{T}^*M, \tag{71}$$

and the corresponding projections are given by

$$\begin{aligned} \pi^{\mathrm{N}}(\overline{\alpha}) &= \partial_r \lrcorner \overline{\alpha}, \\ \pi^{\mathrm{T}}(\overline{\alpha}) &= \overline{\alpha} - \mathrm{d}r \wedge (\partial_r \lrcorner \overline{\alpha}), \end{aligned} \qquad \text{for all } \overline{\alpha} \in \bigwedge\nolimits^{p+1} \overline{\mathrm{T}}^* M. \tag{72}$$

The *shape operator* of M regarded as embedded in its ε-metric cone is $\mathrm{S}(X) = -X$ and hence the respective Levi-Civita connections ∇^g and $\overline{\nabla}^g$ are related by

$$\nabla^g_X Y = \overline{\nabla}^g_X Y + \varepsilon g(X, Y)\, \partial_r, \quad \text{for all } X, Y \in \mathcal{X}(M), \tag{73}$$

cf. the formulas for $\overline{\nabla^g}$ in [15, 17]. It is convenient to extend ∇^g to the whole vector bundle $\overline{\mathrm{T}}M$ so that the normal vector field ∂_r is covariantly constant, which yields the general formula

$$\nabla^g_X = \overline{\nabla^g_X} + \varepsilon\overline{\rho}(\partial_r \wedge X), \quad \text{for all } X \in \mathcal{X}(M), \tag{74}$$

where $\overline{\rho} : \bigwedge^2 \overline{\mathrm{T}}_x M \to \mathfrak{so}(\overline{\mathrm{T}}_x M, \overline{g})$ is defined analogously as in (39). In particular, using (40) we get for skew-symmetric forms

$$\nabla^g_X \overline{\alpha} = \overline{\nabla^g_X}\overline{\alpha} + \mathrm{d}r \wedge (X \lrcorner \overline{\alpha}) - \varepsilon X^\flat \wedge (\partial_r \lrcorner \overline{\alpha}), \tag{75}$$

for all $X \in \mathcal{X}(M)$, $\overline{\alpha} \in \Gamma(\bigwedge^p \overline{\mathrm{T}}^* M)$. By construction, the extended connection ∇^g commutes with the projections π^{N} and π^{T}.

5.2 Spinors on hypersurfaces

The description of spinor bundles on a hypersurface can be found in [2, 5]. Here, we discuss the case of our interest for signature (n_+, n_-) and space- or time-like normal depending on $\varepsilon = \pm 1$.

Firstly, we can naturally realize the Clifford algebra $\mathrm{Cl}\,(n, \mathbb{C})$ as a subalgebra of $\mathrm{Cl}\,(n+1, \mathbb{C})$, and the corresponding complex spinor space Σ_n as a $\mathrm{Cl}\,(n, \mathbb{C})$-submodule of Σ_{n+1}. We also recall that $\mathrm{Cl}\,(n, \mathbb{C})$ is isomorphic to the even part $\mathrm{Cl}\,_0(n+1, \mathbb{C})$ of $\mathrm{Cl}\,(n+1, \mathbb{C})$. In particular, there are two such isomorphisms $\varphi_\pm : \mathrm{Cl}\,(n, \mathbb{C}) \xrightarrow{\sim} \mathrm{Cl}\,_0(n+1, \mathbb{C})$ given on the generators by

$$\varphi_\pm(e_i) = \mp\sqrt{\varepsilon}\, e_0 \cdot e_i, \quad i = 1, \dots, n, \tag{76}$$

where $\{e_0, \dots, e_n\}$ is an orthonormal basis of $(\mathbb{R}^{n+1}, \overline{g})$ such that $\overline{g}(e_0, e_0) = \varepsilon$. All formulas are valid for both choices of the square root sign, and we fix $\sqrt{\varepsilon} = 1$ for $\varepsilon = 1$ and $\sqrt{\varepsilon} = \mathrm{i}$ for $\varepsilon = -1$, respectively. There are also corresponding endomorphisms $f_\pm : \Sigma_{n+1} \to \Sigma_{n+1}$,

$$f_\pm(\overline{\psi}) = (1 \mp \sqrt{\varepsilon}\, e_0) \cdot \overline{\psi}, \quad \text{for all } \overline{\psi} \in \Sigma_{n+1}, \tag{77}$$

which intertwine the restricted representations of $\mathrm{Cl}\,(n, \mathbb{C})$ and $\mathrm{Cl}\,_0(n+1, \mathbb{C})$ on Σ_{n+1} with respect to the isomorphisms $\varphi_\pm$. The mappings f_+ and f_- are up to a scalar multiple mutual inverses,

$$f_\pm \circ f_\mp = (1 - \varepsilon\, e_0^2) = 2, \tag{78}$$

and so are linear isomorphisms. Restricting $f_\pm$ to the subspace Σ_n, we get

$$\begin{aligned} \Sigma_{n+1} &= f_+(\Sigma_n) = f_-(\Sigma_n), && \text{if } n \text{ is even}, \\ \Sigma_{n+1} &= f_+(\Sigma_n) \oplus f_-(\Sigma_n), && \text{if } n \text{ is odd}, \end{aligned} \tag{79}$$

as $\mathrm{Cl}\,_0(n+1, \mathbb{C})$-modules. In particular, the odd case agrees with the decomposition of Σ_{n+1} to *half-spinors*, so adopting suitable sign convention we have

$$\Sigma^+_{n+1} = f_+(\Sigma_n), \quad \Sigma^-_{n+1} = f_-(\Sigma_n). \tag{80}$$

Recall that we assume a fixed spin structure on M and since the ε-metric cone $\overline{M} = \mathbb{R}_+ \times M$ is homotopy equivalent to M, there is a unique compatible spin structure on $\overline{M}$. Now we pass to the associated complex spinor bundles ΣM and $\Sigma\overline{M}$ and denote the restricted bundle $\overline{\Sigma M} = \Sigma\overline{M}|_M$. Compatibility of the spin structures implies that ΣM is naturally a subbundle of $\overline{\Sigma M}$. The linear automorphisms $f_\pm$ induce bundle automorphisms $F_\pm : \overline{\Sigma M} \to \overline{\Sigma M}$,

$$F_\pm(\overline{\Psi}) = (1 \mp \sqrt{\varepsilon}\,\partial_r)\cdot\overline{\Psi}, \quad \text{for all } \overline{\Psi} \in \overline{\Sigma M}. \tag{81}$$

Since $f_\pm$ are intertwining with respect to $\varphi_\pm$, we have the identity

$$F_\pm(X\cdot\overline{\Psi}) = \mp\sqrt{\varepsilon}\,\partial_r\cdot X\cdot F_\pm(\overline{\Psi}), \quad \text{for all } X \in \mathcal{X}(M),\ \overline{\Psi} \in \Gamma(\overline{\Sigma M}). \tag{82}$$

Formula (74) together with (50) yields the spin connection ∇^g on $\overline{\Sigma M}$,

$$\nabla^g_X\overline{\Psi} = \overline{\nabla}^g_X\overline{\Psi} - \tfrac{1}{2}\,\varepsilon\,\partial_r\cdot X\cdot\overline{\Psi}, \quad \text{for all } X \in \mathcal{X}(M),\ \overline{\Psi} \in \Gamma(\overline{\Sigma M}). \tag{83}$$

Finally, we can obtain the spin connection ∇^g on ΣM by pulling back along the bundle map F_+ or F_- and restricting to ΣM. Due to $\nabla^g\partial_r = 0$, we have $\nabla^g F_\pm = 0$ and so ∇^g commutes with $F_\pm$; hence, formula (83) remains unchanged after the pullback.

5.3 Equivalences for Killing equations

The decomposition (71) suggests that $(p+1)$-forms over the extended tangent bundle $\overline{\mathrm{T}}M$ are isomorphic to the prolongation vector bundle K^p defined in (14). Moreover, if we decompose the Levi-Civita connection $\overline{\nabla}^g$ on the cone into individual slots, cf. (75), we basically obtain the defining Eqs. (1) and (42) for *special* Killing–Yano forms. Hence, we arrive at the following equivalence, see [15] or [17] for a detailed proof.

Proposition 15 *Let α and β be differential forms on M of degrees p and $p+1$, respectively, and define a differential form $\overline{\theta}$ on the ε-metric cone $\overline{M}$ by*

$$\overline{\theta} = r^p \mathrm{d}r \wedge \pi_2^*(\alpha) + r^{p+1}\pi_2^*(\beta), \tag{84}$$

where π_2^* denotes the pullback along the canonical projection $\pi_2 : \overline{M} \to M$. Then, $\overline{\theta}$ is parallel with respect to $\overline{\nabla}^g$ if and only if α is special Killing–Yano form with the corresponding $(p+1)$-form β and the Killing constant $c = \varepsilon$.

Conversely, any parallel differential form $\overline{\theta}$ of degree $p+1$ on $\overline{M}$ arises this way with α and β given by the normal and tangent projections, respectively,

$$\alpha = \pi^{\mathrm{N}}(\overline{\theta}|_M), \quad \beta = \pi^{\mathrm{T}}(\overline{\theta}|_M). \tag{85}$$

The homogeneity factors r^p and r^{p+1} in (84) ensure that $\overline{\theta}$ is parallel in the direction ∂_r. This is equivalent to the projective weight $w = p+1$ which appears in (24). There is a related construction of the so-called *Thomas projective cone* equipped with an affine connection which is equivalent to the standard projective tractor connection, see [1]. In particular, for Einstein manifolds, the two cone constructions essentially coincide, and special Killing–Yano forms are thus equivalent to parallel tractor forms. This approach is further exploited in [11].

Regarding spinors, we combine (83), (82) and the fact that ∇^g commutes with $F_\pm$ producing formula

$$\overline{\nabla^g_X}(F_\pm(\overline{\Psi})) = F_\pm(\nabla^g_X\overline{\Psi} \mp \tfrac{1}{2}\sqrt{\varepsilon}\, X \cdot \overline{\Psi}). \tag{86}$$

In other words, the pullback of $\overline{\nabla^g}$ along $F_\pm$ is the Killing connection ∇^a from (49) with Killing number $a = \pm\frac{1}{2}\sqrt{\varepsilon}$. For a detailed proof of the following proposition, see [2] or [17].

Proposition 16 *Let Ψ be a spinor field on M and define a spinor field $\overline{\Psi}_\pm$ on the ε-metric cone $\overline{M}$ by*

$$\overline{\Psi}_\pm = \pi_2^*(F_\pm(\Psi)), \tag{87}$$

where we canonically identify the pullback bundle $\pi_2^*(\Sigma M)$ with $\Sigma\overline{M}$. Then, $\overline{\Psi}_\pm$ is parallel with respect to $\overline{\nabla^g}$ if and only if Ψ is Killing spinor with the Killing number $a = \pm\frac{1}{2}\sqrt{\varepsilon}$.

Conversely, in order to associate a Killing spinor Ψ with parallel spinor field $\overline{\Psi}$ we need to be a bit careful and take into account relations (79), (80). For n even, we have $\overline{\Sigma M} = \Sigma M$; hence, we may choose parallel spinor field $\overline{\Psi} \in \Gamma(\Sigma\overline{M})$ arbitrarily and it produces two Killing spinors $\Psi_\pm$, one for each sign of the Killing number. In the odd case, we have to restrict ourselves just to half-spinor fields $\overline{\Psi} \in \Gamma(\Sigma^\pm\overline{M})$, such that each produces just one Killing spinor Ψ_+ or Ψ_- depending on a half-spinor subbundle chosen. However, this does not produce any restriction since $\overline{\nabla^g}$ preserves the splitting (79) and so we can decompose any parallel spinor field into parallel half-spinor fields. In any case, based on (78), we get formula

$$\Psi_\pm = \tfrac{1}{2}\, F_\mp(\overline{\Psi}|_M). \tag{88}$$

Analogous equivalence for special Killing spinor-valued forms follows as a straightforward combination of the previous two cases, see [17] for details.

Proposition 17 *Let Φ and Ξ be spinor-valued differential forms on M of degrees p and $p+1$, respectively, and define a spinor-valued differential form $\overline{\Theta}_\pm$ on the ε-metric cone $\overline{M}$ by*

$$\overline{\Theta}_\pm = r^p \mathrm{d}r \wedge \pi_2^*(F_\pm(\Phi)) + r^{p+1}\pi_2^*(F_\pm(\Xi)). \tag{89}$$

Then, $\overline{\Theta}_\pm$ is parallel with respect to $\overline{\nabla^g}$ if and only if Φ is special Killing spinor-valued form with the corresponding $(p+1)$-form Ξ, the Killing number $a = \pm\frac{1}{2}\sqrt{\varepsilon}$ and the Killing constant $c = \varepsilon$.

For the converse, the same considerations as in the case of spinors apply. Hence, all special Killing spinor-valued p-forms with $a = \pm\frac{1}{2}\sqrt{\varepsilon}$ and $c = \varepsilon$ are given by the formula

$$\Phi_\pm = \tfrac{1}{2}\, F_\mp(\pi^{\mathrm{N}}(\overline{\Theta}|_M)), \quad \Xi_\pm = \tfrac{1}{2}\, F_\mp(\pi^{\mathrm{T}}(\overline{\Theta}|_M)), \tag{90}$$

for all parallel spinor-valued differential forms $\overline{\Theta} \in \Omega^{p+1}(\overline{M}, \Sigma\overline{M})$ if n is even, and $\overline{\Theta} \in \Omega^{p+1}(\overline{M}, \Sigma^\pm\overline{M})$ if n is odd.

6 Spaces of constant curvature

In this section, we describe the full sets of solutions of the Killing equations of our interest on (pseudo-)Riemannian spaces of nonzero constant curvature. Without loss of generality, we may assume that the sectional curvature is equal to $\varepsilon = \pm 1$. We will explicitly realize the space as a quadratic hypersurface in (pseudo-) Euclidean space.

Let (n_+, n_-) be arbitrary signature such that $n = n_+ + n_- \geq 2$. We take the (pseudo-) Euclidean space $\mathbb{R}^{n+1}$ with the inner product $\overline{g}$ of signature $(\overline{n}_+, \overline{n}_-)$ given by (67) according to the sign of ε,

$$\overline{g} = \varepsilon(\mathrm{d}x_0)^2 + \sum_{i=1}^{n_+}(\mathrm{d}x_i)^2 - \sum_{j=n_++1}^{n}(\mathrm{d}x_j)^2, \tag{91}$$

where $x_0, \dots, x_n$ are the standard coordinates on $\mathbb{R}^{n+1}$. We define manifold M_ε to be the connected component of the quadric

$$\overline{g}(x,x) = \varepsilon, \tag{92}$$

which contains the point $(1, 0, \dots, 0)$. The manifold M_ε obviously inherits a (pseudo-)Riemannian metric g of signature (n_+, n_-).

Conversely, the ε-metric cone $(\overline{M_\varepsilon}, \overline{g})$ over (M, g) is just a connected open submanifold of $(\mathbb{R}^{n+1}, \overline{g})$; in particular, the radial coordinate is given by

$$r(x) = \sqrt{\varepsilon\overline{g}(x,x)}, \quad \text{for all } x \in \overline{M_\varepsilon}. \tag{93}$$

We can also identify the outer unit normal of M_ε with the position vector,

$$\partial_r(x) = x, \quad \text{for all } x \in M_\varepsilon. \tag{94}$$

The Levi-Civita connection on the cone $(\overline{M}, \overline{g})$ is simply restriction of the ordinary partial derivative $\overline{\nabla^g} = \partial$ on $\mathbb{R}^{n+1}$. Using formula (73) for ∇^g, we can verify that M_ε has indeed constant sectional curvature equal to ε,

$$\mathcal{R}^g_{X,Y} = \varepsilon\rho(X \wedge Y), \quad \text{for all } X, Y \in \mathcal{X}(M_\varepsilon), \tag{95}$$

where ρ is the natural isomorphism from (39).

6.1 Killing differential forms and spinors

The cone correspondences discussed in Propositions 15, 16, 17 yield all special Killing–Yano forms, Killing spinors and special Killing spinor-valued forms on M_ε with $a = \pm\frac{1}{2}\sqrt{\varepsilon}$ and $c = \varepsilon$, by means of forms and spinors over $\mathbb{R}^{n+1}$ regarded as constant sections on the cone $\overline{M_\varepsilon}$. Substituting (72), (81) and (94) into (85), (88), (90), we get explicit formulas for the solutions,

$$\begin{aligned} \alpha(x) &= x \lrcorner \overline{\theta}, \\ \beta(x) &= \overline{\theta} - \mathrm{d}r \wedge (x \lrcorner \overline{\theta}), \qquad \text{where } \overline{\theta} \in \textstyle\bigwedge^{p+1}(\mathbb{R}^{n+1})^*, \end{aligned} \tag{96}$$

$$\Psi_\pm(x) = \tfrac{1}{2}(1 \pm \sqrt{\varepsilon}\, x) \cdot \overline{\Psi}, \qquad \text{where } \overline{\Psi} \in \begin{cases} \Sigma_{n+1}, & \text{for } n \text{ even}, \\ \Sigma^\pm_{n+1}, & \text{for } n \text{ odd}, \end{cases} \tag{97}$$

$$
\begin{aligned}
\Phi_{\pm}(x) &= \tfrac{1}{2}(1 \pm \sqrt{\varepsilon}\, x) \cdot (x \lrcorner \overline{\Theta}),\\
\Xi_{\pm}(x) &= \tfrac{1}{2}(1 \pm \sqrt{\varepsilon}\, x) \cdot (\overline{\Theta} - \mathrm{d}r \wedge (x \lrcorner \overline{\Theta})),\\
&\text{where } \overline{\Theta} \in \begin{cases} \bigwedge^{p+1}(\mathbb{R}^{n+1})^* \otimes \Sigma_{n+1}, & \text{for } n \text{ even},\\ \bigwedge^{p+1}(\mathbb{R}^{n+1})^* \otimes \Sigma^{\pm}_{n+1}, & \text{for } n \text{ odd}, \end{cases}
\end{aligned}
\tag{98}
$$

for all form degrees $p = 0, \dots, n$.

As for general (not necessarily special) Killing forms, recall that they are meaningful only in degrees $p \geq 1$. The dimension of the solution spaces attains its universal upper bound for Killing forms and Killing spinors by Corollaries 3 and 12, respectively. Hence, since M_ε is connected, the above formulas give all Killing–Yano forms, Killing spinors and Killing spinor-valued forms on M_ε with $a = \pm\frac{1}{2}\sqrt{\varepsilon}$. As for Killing spinors, the other Killing numbers cannot occur by (53). Hence, it remains to discuss Killing spinor-valued forms with $a \neq \pm\frac{1}{2}\sqrt{\varepsilon}$.

6.2 Other Killing numbers

To determine possible Killing numbers a admitting nontrivial Killing spinor-valued forms is more involved. Relying on the first integrability condition (13), we employ separately the curvature $\mathcal{R}^{\mathsf{aff}} = \mathcal{R}^g$ acting on the form part and $\mathcal{R}^{\mathrm{V}} = \mathcal{R}^a$ acting on the value (spinor) part. By (95) and (40), we have

$$
\mathcal{R}^{\mathsf{aff}}_{X,Y}\Phi = \varepsilon(X^\flat \wedge (Y \lrcorner \Phi) - Y^\flat \wedge (X \lrcorner \Phi)),
\tag{99}
$$

and using (7), (8) and (11), we compute

$$
\begin{aligned}
\mathcal{R}^{\mathsf{aff}}_X \wedge \Phi &= \varepsilon \sum_{j=1}^n e^j \wedge (X^\flat \wedge (e_j \lrcorner \Phi) - (e_j)^\flat \wedge (X \lrcorner \Phi))\\
&= \varepsilon\left(-X^\flat \wedge \sum_{j=1}^n e^j \wedge (e_j \lrcorner \Phi) - \sum_{j,k=1}^n g_{jk}\, e^j \wedge e^k \wedge (X \lrcorner \Phi)\right)\\
&= -\varepsilon p\, X^\flat \wedge \Phi,
\end{aligned}
\tag{100}
$$

$$
\mathcal{R}^{\mathsf{aff}} \wedge \Phi = -\varepsilon p \sum_{i=1}^n e^i \wedge (e_i)^\flat \wedge \Phi = -\varepsilon p \sum_{i,j=1}^n g_{ij}\, e^i \wedge e^j \wedge \Phi = 0,
\tag{101}
$$

$$
\begin{aligned}
(\mathcal{R}^{\mathsf{aff}}\Phi)^{(p,2)}_{X,Y} &= \varepsilon(X^\flat \wedge (Y \lrcorner \Phi) - Y^\flat \wedge (X \lrcorner \Phi)\\
&\quad + Y \lrcorner (X^\flat \wedge \Phi) - X \lrcorner (Y^\flat \wedge \Phi)) = 0.
\end{aligned}
\tag{102}
$$

For the value part, we have by (51), (95) and (50),

$$
\mathcal{R}^{\mathrm{V}}_{X,Y}\Phi = -\tfrac{1}{2}(\varepsilon - 4a^2)(X \cdot Y \cdot + g(X, Y))\, \Phi,
\tag{103}
$$

and using again (7), (8), (11) and also (56), we compute

$$\mathcal{R}^{\mathrm{V}}_X \wedge \boldsymbol{\Phi} = -\tfrac{1}{2}(\varepsilon - 4a^2) \sum_{j=1}^{n} e^j \wedge ((X \cdot e_j \cdot + g(X, e_j))\, \boldsymbol{\Phi}) \\ = -\tfrac{1}{2}(\varepsilon - 4a^2)(X \cdot (\gamma \cdot \wedge \boldsymbol{\Phi}) + X^\flat \wedge \boldsymbol{\Phi}), \tag{104}$$

$$\mathcal{R}^{\mathrm{V}} \wedge \boldsymbol{\Phi} = -\tfrac{1}{4}(\varepsilon - 4a^2) \sum_{i=1}^{n} e^i \wedge (e_i \cdot (\gamma \cdot \wedge \boldsymbol{\Phi}) + (e_i)^\flat \wedge \boldsymbol{\Phi}) \\ = -\tfrac{1}{4}(\varepsilon - 4a^2)\left(\gamma \cdot \wedge \gamma \cdot \wedge \boldsymbol{\Phi} + \sum_{i,j=1}^{n} g_{ij}\, e^i \wedge e^j \wedge \boldsymbol{\Phi} \right) \\ = -\tfrac{1}{4}(\varepsilon - 4a^2)\, \gamma \cdot \wedge \gamma \cdot \wedge \boldsymbol{\Phi}, \tag{105}$$

$$(\mathcal{R}^{\mathrm{V}} \boldsymbol{\Phi})^{(p,2)}_{X,Y} = -\tfrac{1}{2}(\varepsilon - 4a^2)\big((X \cdot Y \cdot + g(X, Y))\, \boldsymbol{\Phi} \\ - \tfrac{1}{p}\big(Y \lrcorner (X \cdot (\gamma \cdot \wedge \boldsymbol{\Phi}) + X^\flat \wedge \boldsymbol{\Phi}) \\ - X \lrcorner (Y \cdot (\gamma \cdot \wedge \boldsymbol{\Phi}) + Y^\flat \wedge \boldsymbol{\Phi}) \\ - \tfrac{1}{p+1}\, Y \lrcorner (X \lrcorner (\gamma \cdot \wedge \gamma \cdot \wedge \boldsymbol{\Phi}))\big)\big). \tag{106}$$

Altogether, on M_ε, we have $(\mathcal{R}\boldsymbol{\Phi})^{(p,2)} = (\mathcal{R}^{\mathrm{V}}\boldsymbol{\Phi})^{(p,2)}$ by the previous formulas.

Now we shall prove that there are no nontrivial solutions with Killing number $a \neq \pm\frac{1}{2}\sqrt{\varepsilon}$ for $p \geq 2$. Recall that for $p = 1$ the component $(\mathcal{R}\boldsymbol{\Phi})^{(p,2)}$ of the curvature action vanishes automatically, and as we shall observe later on, there exist additional Killing spinor-valued 1-forms on M_ε.

Lemma 18 *Let $\boldsymbol{\Phi}$ be a Killing spinor-valued p-form on M_ε with $a \neq \pm\frac{1}{2}\sqrt{\varepsilon}$ and $p \geq 2$. Then, it holds*

$$\gamma^\sharp \cdot \lrcorner (\gamma^\sharp \cdot \lrcorner \boldsymbol{\Phi}) = 0. \tag{107}$$

Proof We compute the following curvature operator built from $(\mathcal{R}\boldsymbol{\Phi})^{(p,2)}$,

$$r_1((\mathcal{R}\boldsymbol{\Phi})^{(p,2)}) = \sum_{i,j,k,l=1}^{n} g^{ik} g^{jl}\, e_i \lrcorner \big(e_j \lrcorner (\mathcal{R}\boldsymbol{\Phi})^{(p,2)}_{e_k, e_l}\big) \\ = -\tfrac{1}{2}(\varepsilon - 4a^2)\left(\sum_{i,j,k,l=1}^{n} g^{ik} g^{jl}\, e_i \lrcorner (e_k \cdot (e_j \lrcorner (e_l \cdot \boldsymbol{\Phi}))) \\ + \sum_{i,j=1}^{n} g^{ij}\, e_i \lrcorner (e_j \lrcorner \boldsymbol{\Phi}) \right) \\ = -\tfrac{1}{2}(\varepsilon - 4a^2)\, \gamma^\sharp \cdot \lrcorner (\gamma^\sharp \cdot \lrcorner \boldsymbol{\Phi}).$$

Now the claim follows from Proposition 2. □

Lemma 19 *Let $\boldsymbol{\Phi}$ be a Killing spinor-valued p-form on M_ε with $a \neq \pm\frac{1}{2}\sqrt{\varepsilon}$ and $p \geq 2$. Then, it holds*

$$\gamma^{\sharp}\cdot\lrcorner\boldsymbol{\Phi} = 0. \tag{108}$$

Proof Again we compute a curvature operator built from $(\mathcal{R}\boldsymbol{\Phi})^{(p,2)}$,

$$\begin{aligned}
r_2((\mathcal{R}\boldsymbol{\Phi})^{(p,2)}) &= \sum_{i,j,k,l=1}^{n} g^{ik} g^{jl}\, e_i \cdot \left(e_j \lrcorner (\mathcal{R}\boldsymbol{\Phi})^{(p,2)}_{e_k,e_l}\right) \\
&= -\tfrac{1}{2}(\varepsilon - 4a^2)\Big(\sum_{i,j,k,l=1}^{n} g^{ik} g^{jl}\, e_i \cdot e_k \cdot (e_j \lrcorner (e_l \cdot \boldsymbol{\Phi})) \\
&+ \sum_{i,j=1}^{n} g^{ij}\, e_i \cdot (e_j \lrcorner \boldsymbol{\Phi}) \\
&- \tfrac{1}{p}\Big(\sum_{i,j,k,l=1}^{n} g^{ik} g^{jl}\, e_k \lrcorner (e_i \cdot (e_j \lrcorner (e_l \cdot (\gamma\cdot \wedge \boldsymbol{\Phi})))) \\
&+ \sum_{i,j,k,l=1}^{n} g^{ik} g^{jl}\, e_k \lrcorner (e_i \cdot (e_j \lrcorner ((e_l)^{\flat} \wedge \boldsymbol{\Phi}))) \Big)\Big) \\
&= \tfrac{1}{2p}(\varepsilon - 4a^2)((np + n - 2p)\, \gamma^{\sharp}\cdot\lrcorner\boldsymbol{\Phi} + \gamma^{\sharp}\cdot\lrcorner(\gamma^{\sharp}\cdot\lrcorner(\gamma\cdot \wedge \boldsymbol{\Phi}))),
\end{aligned}$$

and rearrange the second term using (63),

$$= \tfrac{1}{2p}(\varepsilon - 4a^2)((n+2)(p-1)\, \gamma^{\sharp}\cdot\lrcorner\boldsymbol{\Phi} + \gamma\cdot \wedge (\gamma^{\sharp}\cdot\lrcorner(\gamma^{\sharp}\cdot\lrcorner\boldsymbol{\Phi}))).$$

Now the claim follows from Proposition 2 and Lemma 18. □

Proposition 20 *There are no nontrivial Killing spinor-valued p-forms on* M_ε *with* $a \neq \pm\frac{1}{2}\sqrt{\varepsilon}$ *and* $p \geq 2$.

Proof Again we compute a curvature operator built from $(\mathcal{R}\boldsymbol{\Phi})^{(p,2)}$,

$$
\begin{aligned}
r_3((\mathcal{R}\boldsymbol{\Phi})^{(p,2)}) &= \sum_{i,j,k,l=1}^{n} g^{ik} g^{jl}\, e_i \cdot e_j \cdot (\mathcal{R}\boldsymbol{\Phi})^{(p,2)}_{e_k,e_l} \\
&= -\tfrac{1}{2}(\varepsilon - 4a^2)\Big(- \sum_{i,j,k,l=1}^{n} g^{ik} g^{jl}\, e_i \cdot e_k \cdot e_j \cdot e_l \cdot \boldsymbol{\Phi} \\
&- 2 \sum_{i,l=1}^{n} g^{il}\, e_i \cdot e_l \cdot \boldsymbol{\Phi} \\
&+ \sum_{i,j=1}^{n} g^{ij}\, e_i \cdot e_j \cdot \boldsymbol{\Phi} \\
&- \tfrac{1}{p}\Big(- \sum_{i,j,k,l=1}^{n} g^{ik} g^{jl}\, e_i \cdot e_k \cdot (e_l \lrcorner (e_j \cdot (\gamma \cdot \wedge \boldsymbol{\Phi}))) \\
&- 2 \sum_{i,l=1}^{n} g^{il}\, e_l \lrcorner (e_i \cdot (\gamma \cdot \wedge \boldsymbol{\Phi})) \\
&- \sum_{i,j,k,l=1}^{n} g^{ik} g^{jl}\, (e_k)^\flat \wedge (e_i \cdot (e_l \lrcorner (e_j \cdot \boldsymbol{\Phi}))) \\
&+ \sum_{i,j=1}^{n} g^{ij}\, e_i \cdot e_j \cdot \boldsymbol{\Phi} \\
&- \sum_{i,j,k,l=1}^{n} g^{ik} g^{jl}\, e_k \lrcorner (e_i \cdot e_j \cdot e_l \cdot (\gamma \cdot \wedge \boldsymbol{\Phi})) \\
&- \sum_{i,j,k,l=1}^{n} g^{ik} g^{jl}\, e_k \lrcorner (e_i \cdot ((e_l)^\flat \wedge (e_j \cdot \boldsymbol{\Phi}))) \\
&+ \tfrac{1}{p+1} \sum_{i,j,k,l=1}^{n} g^{ik} g^{jl}\, e_k \lrcorner (e_i \cdot (e_l \lrcorner (e_j \cdot (\gamma \cdot \wedge \gamma \cdot \wedge \boldsymbol{\Phi})))) \Big)\Big) \\
&= \tfrac{1}{2p}(\varepsilon - 4a^2)\big(n(np - p - 1)\, \boldsymbol{\Phi} - \gamma \cdot \wedge (\gamma^\sharp \cdot \lrcorner \boldsymbol{\Phi}) \\
&+ (2n-3)\, \gamma^\sharp \cdot \lrcorner (\gamma \cdot \wedge \boldsymbol{\Phi}) \\
&+ \tfrac{1}{p+1}\, \gamma^\sharp \cdot \lrcorner (\gamma^\sharp \cdot \lrcorner (\gamma \cdot \wedge \gamma \cdot \wedge \boldsymbol{\Phi}))\big),
\end{aligned}
$$

and rearrange the last two terms using (63),

$$
\begin{aligned}
&= \tfrac{1}{2(p+1)}(\varepsilon - 4a^2)\big((n+1)(n+2)(p-1)\, \boldsymbol{\Phi} \\
&+ \tfrac{1}{p}(2(n+2)(p-1)\, \gamma \cdot \wedge (\gamma^\sharp \cdot \lrcorner \boldsymbol{\Phi}) \\
&+ \gamma \cdot \wedge \gamma \cdot \wedge (\gamma^\sharp \cdot \lrcorner (\gamma^\sharp \cdot \lrcorner \boldsymbol{\Phi})))\big).
\end{aligned}
$$

Now the claim follows from Proposition 2 and Lemma 19. □

Before we proceed to the degree 1 case, we conclude this part discussing all the possible special Killing spinor-valued forms in degree 0 on M_ε.

Proposition 21 *Let Φ be a nontrivial special Killing spinor-valued* 0*-form on* M_ε *and* Ξ *the corresponding spinor-valued* 1*-form. Then, the Killing number is necessarily* $a = \pm\frac{1}{2}\sqrt{\varepsilon}$, *and the constant c is either*

(a) $c = \varepsilon$, *in which case* Φ *and* Ξ *are given by formula* (98), *or*
(b) $c = 0$, *in which case* Φ *is a Killing spinor and* $\Xi = 0$.

Proof First, we employ the first integrability condition (43) in Proposition 9. Since Φ is of degree 0, the right-hand side vanishes, and by (103), Eq. (43) reads

$$(\varepsilon - 4a^2)\,\rho(X \wedge Y)\,\Phi = -\tfrac{1}{2}(\varepsilon - 4a^2)\,(X \cdot Y \cdot + g(X, Y))\,\Phi = 0.$$

Note that this is the same condition as (52) for Killing spinors, and we must again have $a = \pm\frac{1}{2}\sqrt{\varepsilon}$. In more detail, we can argue that the spin representation of the spin Lie algebra contains no trivial summands. Alternatively, we can just compute the operator r_3 as in Proposition 20,

$$0 = r_3(\mathcal{R}\Phi) = \tfrac{1}{2}(\varepsilon - 4a^2)n(n-1)\,\Phi.$$

As for the second part, we employ the second integrability condition (44) in Proposition 10. Because $a = \pm\frac{1}{2}\sqrt{\varepsilon}$, we have also $\mathcal{R}^{\mathrm{V}} = \mathcal{R}^{a} = 0$ and by (95) Eq. (44) reads

$$(\varepsilon - c)\,\rho(X \wedge Y)^{\mathsf{aff}}\,\Xi = (\varepsilon - c)(X \lrcorner (Y^\flat \wedge \Xi) - Y \lrcorner (X^\flat \wedge \Xi)) = 0.$$

Hence, we must have either $c = \varepsilon$ or $\Xi = 0$. Again, we can argue that the representation of the spin Lie algebra on spinor-valued 1-forms contains no trivial summands, or to compute the operator q (sometimes called the *curved Casimir operator*),

$$\begin{aligned} 0 &= q((\mathcal{R} - c\rho^{\mathsf{aff}})\,\Xi) = (\varepsilon - c)\,q(\rho^{\mathsf{aff}}\,\Xi) \\ &= (\varepsilon - c)\sum_{i,j,k,l=1}^{n} g^{ik} g^{il}\,(e_i)^\flat \wedge (e_j \lrcorner (\rho^{\mathsf{aff}}(e_k \wedge e_l)\,\Xi)) \\ &= -(\varepsilon - c)(n-1)\,\Xi. \end{aligned}$$

Finally, the case $\Xi = 0$ implies $c = 0$ by the second defining Eq. (42). □

6.3 Additional solutions in degree 1

To resolve the case of Killing spinor-valued 1-forms on the space M_ε of constant curvature, we need to employ also the second integrability condition (35). So let Φ' be a Killing spinor-valued 1-form with Killing number $a' \neq \pm\frac{1}{2}\sqrt{\varepsilon}$ and the corresponding spinor-valued 2-form Ξ'. The left-hand side of (35) is just a multiple of the (2, 2)-symmetry type component, so by (36), (102) and (106) we have

$$(\widehat{\mathcal{R}}\Xi')^{(2,2)}_{X,Y} = -\tfrac{3}{2}(\varepsilon - 4(a')^2)\big((X\cdot Y\cdot + g(X,Y))\,\boldsymbol{\Phi}' - \tfrac{1}{2}\big(Y\lrcorner(X\cdot(\gamma\cdot\wedge\boldsymbol{\Phi}') + X^\flat\wedge\boldsymbol{\Phi}') - X\lrcorner(Y\cdot(\gamma\cdot\wedge\boldsymbol{\Phi}') + Y^\flat\wedge\boldsymbol{\Phi}') - - \tfrac{1}{3}\,Y\lrcorner(X\lrcorner(\gamma\cdot\wedge\gamma\cdot\wedge\boldsymbol{\Phi}'))\big)\big), \tag{109}$$

noting that the degree of Ξ' is $p+1=2$. In order to compute the right-hand side, we need the covariant derivative of the curvature. Note that the isomorphism ρ from (39) and (50) is invariant with respect to the spin group and hence we have $\nabla^g\rho = 0$. By (95) and (103), we have

$$(\nabla_X\mathcal{R}^{\mathsf{aff}})_{Y,Z} = \varepsilon\,(\nabla^g_X\rho)(Y\wedge Z) = 0 \tag{110}$$

$$\begin{aligned}(\nabla_X\mathcal{R}^{\mathrm{V}})_{Y,Z} &= (\varepsilon - 4(a')^2)(\nabla^{a'}_X\rho)(Y\wedge Z)\\ &= (\varepsilon - 4(a')^2)\big((\nabla^g_X\rho)(Y\wedge Z) - a'[X\cdot,\rho(Y\wedge Z)]\big)\\ &= a'(\varepsilon - 4(a')^2)(g(X,Z)\,Y - g(X,Y)\,Z)\cdot,\end{aligned} \tag{111}$$

for all $X,Y,Z\in\mathcal{X}(M_\varepsilon)$. Next, we compute the particular action of $\nabla\mathcal{R}$ on $\boldsymbol{\Phi}'$ defined by Eq. (37),

$$\begin{aligned}(\nabla_X\mathcal{R})_Y\wedge\boldsymbol{\Phi}' &= a'(\varepsilon - 4(a')^2)\sum\nolimits_{j=1}^n e^j\wedge(g(X,e_j)\,Y - g(X,Y)\,e_j)\cdot\boldsymbol{\Phi}'\\ &= a'(\varepsilon - 4(a')^2)(X^\flat\wedge(Y\cdot\boldsymbol{\Phi}') - g(X,Y)\,\gamma\cdot\wedge\boldsymbol{\Phi}'),\end{aligned} \tag{112}$$

$$\begin{aligned}(\nabla_X\mathcal{R})\wedge\boldsymbol{\Phi}' &= \tfrac{1}{2}\,a'(\varepsilon - 4(a')^2)\sum\nolimits_{i=1}^n e^i\wedge(X^\flat\wedge(e_i\cdot\boldsymbol{\Phi}')\\ &\quad - g(X,e_i)\,\gamma\cdot\wedge\boldsymbol{\Phi}'),\\ &= -a'(\varepsilon - 4(a')^2)\,X^\flat\wedge(\gamma\cdot\wedge\boldsymbol{\Phi}'),\end{aligned} \tag{113}$$

$$\begin{aligned}((\nabla\mathcal{R})\barwedge\boldsymbol{\Phi}')_{X,Y} &= a'(\varepsilon - 4(a')^2)\big(X^\flat\wedge(Y\cdot\boldsymbol{\Phi}') - Y^\flat\wedge(X\cdot\boldsymbol{\Phi}')\\ &\quad + \tfrac{1}{2}(Y\lrcorner(X^\flat\wedge\gamma\cdot\wedge\boldsymbol{\Phi}')\\ &\quad - X\lrcorner(Y^\flat\wedge\gamma\cdot\wedge\boldsymbol{\Phi}'))\big).\end{aligned} \tag{114}$$

Now we proceed similarly to Lemmas 18, 19 and Proposition 20 and compare the operators $r1$, $r2$ and $r3$ applied to both sides of (35).

Lemma 22 *Let $\boldsymbol{\Phi}'$ be a Killing spinor-valued 1-form on M_ε with $a'\neq\pm\frac{1}{2}\sqrt{\varepsilon}$ and Ξ' the corresponding spinor-valued 2-form. Then, it holds*

$$\gamma^\sharp\cdot\lrcorner(\gamma^\sharp\cdot\lrcorner\Xi') = \tfrac{4}{3}\,a'(n-1)\,\gamma^\sharp\cdot\lrcorner\boldsymbol{\Phi}'. \tag{115}$$

Proof From (109) and the computation in Lemma 18, we have

$$r_1((\widehat{\mathcal{R}}\Xi')^{(2,2)}) = -\tfrac{3}{2}(\varepsilon - 4(a')^2)\,\gamma^\sharp\cdot\lrcorner(\gamma^\sharp\cdot\lrcorner\Xi').$$

For the right-hand side, we compute using (114),

$$
\begin{aligned}
&r_1((\nabla\mathcal{R})\barwedge\boldsymbol{\Phi}')\\
&\quad=a'(\varepsilon-4(a')^2)\Big(-\sum\nolimits_{i,j,k,l=1}^{n}g^{ik}g^{jl}\,e_i\lrcorner((e_k)^\flat\wedge(e_j\lrcorner(e_l\cdot\boldsymbol{\Phi}')))\quad+\sum\nolimits_{i,l=1}^{n}g^{il}\,e_i\lrcorner(e_l\cdot\boldsymbol{\Phi}')\\
&\qquad-\sum\nolimits_{i,j,k,l=1}^{n}g^{ik}g^{jl}\,e_i\lrcorner(e_k\cdot(e_j\lrcorner((e_l)^\flat\wedge\boldsymbol{\Phi}')))\Big)\\
&\qquad=-2a'(\varepsilon-4(a')^2)(n-1)\,\gamma^\sharp\cdot\lrcorner\boldsymbol{\Phi}'.
\end{aligned}
$$

Now the claim follows from Proposition 6. ◻

Lemma 23 *Let $\boldsymbol{\Phi}'$ be a Killing spinor-valued 1-form on M_ε with $a'\neq\pm\frac{1}{2}\sqrt{\varepsilon}$ and $\boldsymbol{\Xi}'$ the corresponding spinor-valued 2-form. Then, it holds*

$$
\gamma^\sharp\cdot\lrcorner\boldsymbol{\Xi}'=\tfrac{2}{3}\,a'((n-2)\,\boldsymbol{\Phi}'-\gamma\cdot\wedge(\gamma^\sharp\cdot\lrcorner\boldsymbol{\Phi}')). \tag{116}
$$

Proof From (109) and the computation in Lemma 19, we have

$$
r_2((\widehat{\mathcal{R}}\boldsymbol{\Xi}')^{(2,2)})=\tfrac{3}{4}(\varepsilon-4(a')^2)((n+2)\,\gamma^\sharp\cdot\lrcorner\boldsymbol{\Xi}'+\gamma\cdot\wedge(\gamma^\sharp\cdot\lrcorner(\gamma^\sharp\cdot\lrcorner\boldsymbol{\Xi}'))).
$$

For the right-hand side, we compute using (114),

$$
\begin{aligned}
&r_2((\nabla\mathcal{R})\barwedge\boldsymbol{\Phi}')\\
&\quad=a'(\varepsilon-4(a')^2)\Big(-\sum_{i,j,k,l=1}^{n}g^{ik}g^{jl}\,(e_k)^\flat\wedge(e_i\cdot(e_j\lrcorner(e_l\cdot\boldsymbol{\Phi}')))\\
&\quad+\sum_{i,l=1}^{n}g^{il}\,e_i\cdot e_l\cdot\boldsymbol{\Phi}'\\
&\quad-\sum_{i,j,k,l=1}^{n}g^{ik}g^{jl}\,e_i\cdot e_k\cdot(e_j\lrcorner((e_l)^\flat\wedge\boldsymbol{\Phi}'))\\
&\quad+\tfrac{1}{2}\sum_{i,j,k,l=1}^{n}g^{ik}g^{jl}\,e_k\lrcorner(e_i\cdot(e_j\lrcorner((e_l)^\flat\wedge\gamma\cdot\wedge\boldsymbol{\Phi}')))\Big)\\
&\quad=a'(\varepsilon-4(a')^2)\big(n(n-2)\,\boldsymbol{\Phi}'-\gamma\cdot\wedge(\gamma^\sharp\cdot\lrcorner\boldsymbol{\Phi}')+\tfrac{1}{2}(n-2)\,\gamma^\sharp\cdot\lrcorner(\gamma\cdot\wedge\boldsymbol{\Phi}')\big),
\end{aligned}
$$

and rearrange the last term using (63),

$$
=\tfrac{1}{2}\,a'(\varepsilon-4(a')^2)((n-2)(n+2)\,\boldsymbol{\Phi}'+(n-4)\,\gamma\cdot\wedge(\gamma^\sharp\cdot\lrcorner\boldsymbol{\Phi}')).
$$

Now the claim follows from Proposition 6 and Lemma 22. ◻

Proposition 24 *Let $\boldsymbol{\Phi}'$ be a Killing spinor-valued 1-form on M_ε with $a'\neq\pm\frac{1}{2}\sqrt{\varepsilon}$ and $\boldsymbol{\Xi}'$ the corresponding spinor-valued 2-form. Then, it holds*

$$
\boldsymbol{\Xi}'=-\tfrac{2}{3}\,a'\gamma\cdot\wedge\boldsymbol{\Phi}'. \tag{117}
$$

In other words, $\boldsymbol{\Phi}'$ satisfies differential equation

$$
\nabla^g_X\boldsymbol{\Phi}'=a'\big(X\cdot\boldsymbol{\Phi}'-\tfrac{2}{3}\,X\lrcorner(\gamma\cdot\wedge\boldsymbol{\Phi}')\big),\quad\text{for all } X\in\mathcal{X}(M_\varepsilon). \tag{118}
$$

Proof From (109) and the computation in Proposition 20, we have

$$r_3((\widehat{\mathcal{R}}\Xi')^{(2,2)}) = \tfrac{1}{2}(\varepsilon - 4(a')^2)\big((n+1)(n+2)\,\Xi' + (n+2)\,\gamma\cdot\wedge(\gamma^\sharp\cdot\lrcorner\Xi') + \tfrac{1}{2}\,\gamma\cdot\wedge\gamma\cdot\wedge(\gamma^\sharp\cdot\lrcorner(\gamma^\sharp\cdot\lrcorner\Xi'))\big).$$

For the right-hand side, we compute using (114),

$$\begin{aligned}
&r_3((\nabla\mathcal{R})\,\overline{\wedge}\,\boldsymbol{\Phi}') \\
&\quad = a'(\varepsilon - 4(a')^2)\Big(\sum_{i,j,k,l=1}^{n} g^{ik}g^{jl}\,(e_k)^\flat\wedge(e_i\cdot e_j\cdot e_l\cdot\boldsymbol{\Phi}') \\
&\quad + \sum_{i,j,k,l=1}^{n} g^{ik}g^{jl}\,e_i\cdot e_k\cdot((e_l)^\flat\wedge(e_j\cdot\boldsymbol{\Phi}')) \\
&\quad + 2\sum_{i,l=1}^{n} g^{il}\,(e_l)^\flat\wedge(e_i\cdot\boldsymbol{\Phi}') \\
&\quad + \tfrac{1}{2}\Big(-\sum_{i,j,k,l=1}^{n} g^{ik}g^{jl}\,(e_k)^\flat\wedge(e_i\cdot(e_l\lrcorner(e_j\cdot(\gamma\cdot\wedge\boldsymbol{\Phi}')))) \\
&\quad + \sum_{i,j=1}^{n} g^{ij}\,e_i\cdot e_j\cdot(\gamma\cdot\wedge\boldsymbol{\Phi}') \\
&\quad - \sum_{i,j,k,l=1}^{n} g^{ik}g^{jl}\,e_k\lrcorner(e_i\cdot((e_l)^\flat\wedge(e_j\cdot(\gamma\cdot\wedge\boldsymbol{\Phi}'))))\Big)\Big) \\
&\quad = -\tfrac{1}{2}\,a'(\varepsilon - 4(a')^2)((5n-4)\,\gamma\cdot\wedge\boldsymbol{\Phi}' \\
&\quad + \gamma\cdot\wedge(\gamma^\sharp\cdot\lrcorner(\gamma\cdot\wedge\boldsymbol{\Phi}')) + \gamma^\sharp\cdot\lrcorner(\gamma\cdot\wedge\gamma\cdot\wedge\boldsymbol{\Phi}')),
\end{aligned}$$

and rearrange the last two terms using (63),

$$= -a'(\varepsilon - 4(a')^2)((n+2)\,\gamma\cdot\wedge\boldsymbol{\Phi}' + \gamma\cdot\wedge\gamma\cdot\wedge(\gamma^\sharp\cdot\lrcorner\boldsymbol{\Phi}')).$$

Now the claim follows from Proposition 6 and Lemma 23. □

Consequently, it remains to describe solutions of the stronger Eq. (118). It turns out that for $a' \neq 0$ the solutions are just algebraic transformation of suitable special Killing spinor-valued forms in degree 0. The transformation works in general, so we point out that the following two propositions apply to any spin (pseudo-)Riemannian manifold and not just M_ε.

Proposition 25 *Let M be an arbitrary spin (pseudo-)Riemannian manifold. Then, spinor-valued differential 1-forms $\boldsymbol{\Phi}'$ on M solving (118) with Killing number $a' \neq 0$ bijectively correspond to special Killing spinor-valued 0-forms $\boldsymbol{\Phi}$ on M with Killing number $a = \frac{1}{3}\,a'$, constant $c = 4a^2$ and the corresponding spinor-valued 1-form Ξ such that*

$$\gamma^\sharp\cdot\lrcorner\Xi = -2a\boldsymbol{\Phi}. \tag{119}$$

The correspondence is given by formulas

$$\boldsymbol{\Phi} = -\tfrac{1}{2a(n+1)}\,\gamma^\sharp\cdot\lrcorner\boldsymbol{\Phi}', \quad \Xi = \boldsymbol{\Phi}' + \tfrac{1}{n+1}\,\gamma\cdot\wedge(\gamma^\sharp\cdot\lrcorner\boldsymbol{\Phi}'), \tag{120}$$

$$\boldsymbol{\Phi}' = \Xi + 2a\gamma\cdot \wedge \boldsymbol{\Phi} = \Xi - \gamma\cdot \wedge (\gamma^\sharp \cdot \lrcorner \Xi). \tag{121}$$

Proof Let us first note that since $\boldsymbol{\Phi}'$ has degree 1, we have $\gamma^\sharp\cdot\lrcorner(\gamma^\sharp\cdot\lrcorner\boldsymbol{\Phi}') = 0$ and $X\lrcorner(\gamma^\sharp\cdot\lrcorner\boldsymbol{\Phi}') = 0$, which we shall use repeatedly in the proof. This gives an additional insight why there is no analogous construction in higher degrees.

Now let $\boldsymbol{\Phi}'$ be a solution of (118) with $a' \neq 0$ and define $\boldsymbol{\Phi}$ and Ξ by formulas (120). Using (63), we immediately get Eq. (119),

$$\gamma^\sharp\cdot\lrcorner\Xi = \gamma^\sharp\cdot\lrcorner\boldsymbol{\Phi}' - \tfrac{n}{n+1}\gamma^\sharp\cdot\lrcorner\boldsymbol{\Phi}' = -2a\boldsymbol{\Phi}.$$

Then, we compute covariant derivatives using assumption (118), formulas for computing with spinor-valued forms (58)–(61), (63), and also $a = \frac{1}{3}a'$,

$$\begin{aligned}
\nabla^g_X(\gamma^\sharp\cdot\lrcorner\boldsymbol{\Phi}') &= \gamma^\sharp\cdot\lrcorner(\nabla^g_X\boldsymbol{\Phi}') \\
&= a'\big(- 2X\lrcorner\boldsymbol{\Phi}' - X\cdot(\gamma^\sharp\cdot\lrcorner\boldsymbol{\Phi}') + \tfrac{2}{3}X\lrcorner(\gamma^\sharp\cdot\lrcorner(\gamma\cdot\wedge\boldsymbol{\Phi}'))\big) \\
&= -a(X\cdot(\gamma^\sharp\cdot\lrcorner\boldsymbol{\Phi}') + 2(n+1)\,X\lrcorner\boldsymbol{\Phi}'), \\
\nabla^g_X(\gamma\cdot\wedge(\gamma^\sharp\cdot\lrcorner\boldsymbol{\Phi}')) &= \gamma\cdot\wedge(\nabla^g_X(\gamma^\sharp\cdot\lrcorner\boldsymbol{\Phi}')) \\
&= -a(-2X^\flat\wedge(\gamma^\sharp\cdot\lrcorner\boldsymbol{\Phi}') - X\cdot(\gamma\cdot\wedge(\gamma^\sharp\cdot\lrcorner\boldsymbol{\Phi}')) + \\
&\quad + 2(n+1)(X\cdot\boldsymbol{\Phi}' - X\lrcorner(\gamma\cdot\wedge\boldsymbol{\Phi}'))).
\end{aligned}$$

The defining Eqs. (54), (55) of special Killing spinor-valued forms then follow using also (120) and $c = 4a^2$,

$$\begin{aligned}
\nabla^g_X\boldsymbol{\Phi} &= - aX\cdot\boldsymbol{\Phi} + X\lrcorner\Xi - \tfrac{1}{n+1}X\lrcorner(\gamma\cdot\wedge(\gamma^\sharp\cdot\lrcorner\boldsymbol{\Phi}')) \\
&= aX\cdot\boldsymbol{\Phi} + X\lrcorner\Xi, \\
\nabla^g_X\Xi &= a\big(3X\cdot\boldsymbol{\Phi}' - 2X\lrcorner(\gamma\cdot\wedge\boldsymbol{\Phi}') \\
&\quad + \tfrac{2}{n+1}X^\flat\wedge(\gamma^\sharp\cdot\lrcorner\boldsymbol{\Phi}') + \tfrac{1}{n+1}X\cdot(\gamma\cdot\wedge(\gamma^\sharp\cdot\lrcorner\boldsymbol{\Phi}')) \\
&\quad - 2X\cdot\boldsymbol{\Phi}' + 2X\lrcorner(\gamma\cdot\wedge\boldsymbol{\Phi}')\big) \\
&= aX\cdot\Xi - cX^\flat\wedge\boldsymbol{\Phi}.
\end{aligned}$$

Conversely, suppose that $\boldsymbol{\Phi}$ and Ξ satisfy (54), (55) with $a \neq 0$ and $c = 4a^2$. We again compute covariant derivative using formulas (58)–(61) and (63),

$$\begin{aligned}
\nabla^g_X(\gamma\cdot\wedge\boldsymbol{\Phi}) &= \gamma\cdot\wedge(\nabla^g_X\boldsymbol{\Phi}) \\
&= a(-2X^\flat\wedge\boldsymbol{\Phi} - X\cdot(\gamma\cdot\wedge\boldsymbol{\Phi})) + X\cdot\Xi - X\lrcorner(\gamma\cdot\wedge\Xi).
\end{aligned}$$

Now we define $\boldsymbol{\Phi}'$ by formula (121) and further compute using also $a' = 3a$,

$$\begin{aligned}
\nabla^g_X\boldsymbol{\Phi}' &= aX\cdot\Xi - 4a^2X^\flat\wedge\boldsymbol{\Phi} \\
&\quad - 2a^2(2X^\flat\wedge\boldsymbol{\Phi} + X\cdot(\gamma\cdot\wedge\boldsymbol{\Phi})) + 2a(X\cdot\Xi - X\lrcorner(\gamma\cdot\wedge\Xi)) \\
&= 3aX\cdot\Xi - 2aX\lrcorner(\gamma\cdot\wedge\Xi) \\
&\quad + 6a^2X\cdot(\gamma\cdot\wedge\boldsymbol{\Phi}) - 4a^2X\lrcorner(\gamma\cdot\wedge\gamma\cdot\wedge\boldsymbol{\Phi}) \\
&= a'(X\cdot\boldsymbol{\Phi}' - \tfrac{2}{3}X\lrcorner(\gamma\cdot\wedge\boldsymbol{\Phi}')),
\end{aligned}$$

proving that Φ' solves (118). Finally, a straightforward computation using (119) and (63) verifies that the formulas (120) and (121) are inverse to each other. □

The last proposition allows to translate the cone correspondence from Proposition 17 to Killing spinor-valued 1-forms satisfying (118). As it turns out, condition (119) has a nice representation-theoretic formulation in terms of the corresponding parallel spinor-valued 1-form $\overline{\Theta}$ on the cone $\overline{M}$. In order to see it, we recall the well-known invariant decomposition of spinor-valued 1-forms corresponding on the level of vector spaces to

$$(\mathbb{R}^n)^* \otimes \Sigma_n \simeq \Sigma_n \oplus \Sigma_n^{\frac{3}{2}}. \tag{122}$$

The respective projections are given by

$$\begin{aligned} \pi^{\Sigma}(\Theta) &= \gamma^{\sharp}\cdot \lrcorner \Theta, \\ \pi^{\frac{3}{2}}(\Theta) &= \Theta + \tfrac{1}{n}\gamma\cdot \wedge (\gamma^{\sharp}\cdot\lrcorner\Theta), \end{aligned} \qquad \text{for all } \Theta \in (\mathbb{R}^n)^* \otimes \Sigma_n, \tag{123}$$

and the space $\Sigma^{\frac{3}{2}}$ of *primitive* spinor-valued 1-forms is simply the kernel of π^{Σ}. In even dimensions, we define also the spaces $\Sigma^{\frac{3}{2}\pm}$ of primitive half-spinor-valued 1-forms as the kernel of π^{Σ} restricted to half-spinor-valued forms.

Proposition 26 *Let M be a spin (pseudo-)Riemannian manifold and Φ' be a spinor-valued differential 1-form on M. Define a spinor-valued differential 1-form $\overline{\Theta}_{\pm}$ on the ε-metric cone $\overline{M}$ over M by*

$$\begin{aligned} \overline{\Theta}_{\pm} &= r\,\overline{\pi}^{\frac{3}{2}}(\pi_2^*(F_{\pm}(\Phi'))) \\ &= r\big(\pi_2^*(F_{\pm}(\Phi')) + \tfrac{1}{n+1}\,\overline{\gamma}\cdot \wedge (\overline{\gamma}^{\sharp}\cdot\lrcorner\pi_2^*(F_{\pm}(\Phi')))\big), \end{aligned} \tag{124}$$

where $\overline{\gamma}\cdot$ and $\overline{\gamma}^{\sharp}\cdot$ are the Clifford multiplication form and its metric dual on $\overline{M}$. Then, $\overline{\Theta}_{\pm}$ is primitive by construction, and it is parallel with respect to $\overline{\nabla^g}$ if and only if Φ' is a solution of (118) *with the Killing number $a' = \pm\frac{3}{2}\sqrt{\varepsilon}$.*

Conversely, any parallel primitive (half-)spinor-valued 1-form $\overline{\Theta}$ on $\overline{M}$, in particular, $\overline{\Theta} \in \Gamma(\Sigma^{\frac{3}{2}}\overline{M})$ for n even, and $\overline{\Theta} \in \Gamma(\Sigma^{\frac{3}{2}\pm}\overline{M})$ for n odd, arises this way with $\Phi'_{\pm}$ given by

$$\Phi'_{\pm} = \tfrac{1}{2}\big(F_{\mp}(\pi^{\mathrm{T}}(\overline{\Theta}|_M)) \pm \sqrt{\varepsilon}\,\gamma\cdot \wedge F_{\mp}(\pi^{\mathrm{N}}(\overline{\Theta}|_M))\big). \tag{125}$$

Proof The proof is based on repeated application of the relationship between the Clifford multiplication forms $\gamma\cdot$ and $\overline{\gamma}\cdot$ on M and $\overline{M}$, respectively. Taking into account (57) and (68), we easily deduce

$$\overline{\gamma}\cdot = \mathrm{d}r \otimes (\partial_r\cdot) + r\,\pi_2^*(\gamma\cdot), \quad \overline{\gamma}^{\sharp}\cdot = \varepsilon\,\partial_r \otimes (\partial_r\cdot) + \tfrac{1}{r}\,\pi_2^*(\gamma^{\sharp}\cdot).$$

We recall that π_2^* denotes the pullback along the projection $\pi_2 : \overline{M} \to M$, and we also canonically identify the pullback bundles $\pi_2^*(\mathrm{T}M)$ and $\pi_2^*(\Sigma M)$ with $\mathrm{T}\overline{M}$ and $\Sigma\overline{M}$, respectively.

First, we define the 1-form $\overline{\Theta}_{\pm}$ on $\overline{M}$ by (89) and show that it is primitive if and only if the forms Φ and Ξ on M are related by (119),

$$\begin{aligned}\overline{\gamma}^{\sharp}\cdot\lrcorner\overline{\Theta}_{\pm} &= \overline{\gamma}^{\sharp}\cdot\lrcorner(\mathrm{d}r\wedge\pi_2^*(F_{\pm}(\boldsymbol{\Phi}))+r\,\pi_2^*(F_{\pm}(\Xi)))\\ &= \partial_r\lrcorner(\mathrm{d}r\wedge\pi_2^*(\varepsilon\,\partial_r\cdot F_{\pm}(\boldsymbol{\Phi})))+\pi_2^*(\gamma^{\sharp}\cdot\lrcorner F_{\pm}(\boldsymbol{\Phi}))\\ &= \pi_2^*(F_{\mp}(\pm\sqrt{\varepsilon}\,\boldsymbol{\Phi}+\gamma^{\sharp}\cdot\lrcorner\Xi))\\ &= \pi_2^*(F_{\mp}(2a\boldsymbol{\Phi}+\gamma^{\sharp}\cdot\lrcorner\Xi)),\end{aligned}$$

where we used (81), (57) and $a=\frac{1}{3}a'=\pm\frac{1}{2}\sqrt{\varepsilon}$. The claimed correspondence between $\boldsymbol{\Phi}'$ and $\overline{\Theta}$ now follows from Propositions 17 and 25. Formula (125) follows immediately by substituting (90) for $\boldsymbol{\Phi}$ and Ξ into (121).

As for (124), we substitute (120) for $\boldsymbol{\Phi}$ and Ξ into (89) and compute

$$\begin{aligned}\overline{\Theta}_{\pm} &= \mp\frac{1}{\sqrt{\varepsilon(n+1)}}\,\mathrm{d}r\wedge\pi_2^*(F_{\pm}(\gamma^{\sharp}\cdot\lrcorner\boldsymbol{\Phi}'))\\ &\quad + r\,\pi_2^*\Big(F_{\pm}\Big(\boldsymbol{\Phi}'+\frac{1}{n+1}\gamma\cdot\wedge(\gamma^{\sharp}\cdot\lrcorner\boldsymbol{\Phi}')\Big)\Big)\\ &= \frac{1}{n+1}\,r\,\mathrm{d}r\wedge(\partial_r\cdot(\overline{\gamma}^{\sharp}\cdot\lrcorner\pi_2^*(F_{\pm}(\gamma^{\sharp}\cdot\lrcorner\boldsymbol{\Phi}'))))\\ &\quad + r\,\pi_2^*(F_{\pm}(\boldsymbol{\Phi}'))+\frac{1}{n+1}\,r^2\pi_2^*(\gamma\cdot)\wedge(\overline{\gamma}^{\sharp}\cdot\lrcorner\pi_2^*(F_{\pm}(\boldsymbol{\Phi}')))\\ &= r\Big(\pi_2^*(F_{\pm}(\boldsymbol{\Phi}'))+\frac{1}{n+1}\,\overline{\gamma}\cdot\wedge(\overline{\gamma}^{\sharp}\cdot\lrcorner\pi_2^*(F_{\pm}(\boldsymbol{\Phi}')))\Big),\end{aligned}$$

which completes the proof. □

Now we return to the example space M_ε of constant curvature. Substituting (72), (81) and (94) into (125), we get explicit formulas for Killing spinor-valued 1-forms $\boldsymbol{\Phi}'_{\pm}$ on M_ε which have the Killing number $a'=\pm\frac{3}{2}\sqrt{\varepsilon}$,

$$\begin{aligned}\boldsymbol{\Phi}'_{\pm}(x) &= \frac{1}{2}\Big((1\pm\sqrt{\varepsilon}\,x)\cdot(\overline{\Theta}-\mathrm{d}r\wedge(x\lrcorner\overline{\Theta}))\\ &\quad \pm\sqrt{\varepsilon}\,\gamma\cdot\wedge((1\pm\sqrt{\varepsilon}\,x)\cdot(x\lrcorner\overline{\Theta}))\Big),\end{aligned}\tag{126}$$

where $\overline{\Theta}\in\Sigma^{\frac{3}{2}}_{n+1}$ for n even, and $\overline{\Theta}\in\Sigma^{\frac{3}{2}\pm}_{n+1}$ for n odd, regarded as a constant section of $\Sigma^{\frac{3}{2}}\overline{M}$ or $\Sigma^{\frac{3}{2}\pm}\overline{M}$, respectively. Note that these additional solutions are *not* in the span of tensor products $\alpha\otimes\boldsymbol{\Psi}$ of a Killing–Yano form and a Killing spinor. This is simply due to the fact that there are no nontrivial Killing spinors on M_ε with Killing number $a'=\pm\frac{3}{2}\sqrt{\varepsilon}$.

By Propositions 24 and 25, there are no nontrivial Killing spinor-valued 1-forms on M_ε with Killing number

$$a'\neq 0,\ \pm\tfrac{1}{2}\sqrt{\varepsilon},\ \pm\tfrac{3}{2}\sqrt{\varepsilon}$$

because, by Proposition 21, there are no nontrivial special Killing 0-forms on M_ε with Killing number $a\neq\pm\frac{1}{2}\sqrt{\varepsilon}$. Finally, we resolve the remaining case $a'=0$ of Eq. (118) and hence complete our discussion of all Killing spinor-valued forms on M_ε. In this case, the equation simply requires ∇^g-parallel spinor-valued 1-forms.

Proposition 27 *There are no nontrivial ∇^g-parallel spinor-valued 1-forms on the space M_ε.*

Proof Suppose that $\boldsymbol{\Phi}$ is a ∇^g-parallel spinor-valued 1-form. Then, the first integrability condition requires that $\boldsymbol{\Phi}$ is annihilated by the curvature of ∇^g and thus we have by (95)

$$0 = \mathcal{R}^g_{X,Y}\boldsymbol{\Phi} = \varepsilon\rho(X \wedge Y)\,\boldsymbol{\Phi}.$$

Hence, $\boldsymbol{\Phi} = 0$ since the representation of the spin Lie algebra on spinor-valued 1-forms contains no trivial summands. Alternatively, we can compute the operator r_2 as in Lemma 19,

$$r_2(\mathcal{R}^g\boldsymbol{\Phi}) = -\tfrac{1}{2}\,\varepsilon(n-1)\,\gamma^{\sharp}\cdot\lrcorner\boldsymbol{\Phi},$$

and then the operator q as in Proposition 21,

$$q(\mathcal{R}^g\boldsymbol{\Phi}) = -\big(n - \tfrac{1}{2}\big)\,\boldsymbol{\Phi} - \tfrac{1}{2}\,\gamma\cdot\wedge(\gamma^{\sharp}\cdot\lrcorner\boldsymbol{\Phi}),$$

and the claim follows. □

7 Final remarks and comments

We have shown that application of the integrability conditions revealed unexpected Killing spinor-valued 1-forms on spaces of constant curvature. Our results can be regarded as the first example resulting from the investigation of Killing spinor-valued forms that is not implied by known results on Killing–Yano forms and Killing spinors. Apparently, the application toward explicit examples is computationally rather complicated to do by hand even in the simplest case of spaces of constant curvature.

However, as Eq. (38) suggests, all the integrability conditions can be applied algorithmically, and in many cases this approach is sufficient to completely determine the space of solutions. The second author has implemented an algorithm for solving the three types of Killing equations on homogeneous spaces using a computer algebra system, see [20]. In fact, the additional solutions on spaces of constant curvature were originally discovered this way. Computed examples include the *Berger spheres* in dimensions 3, 5, 7 which are Sasakian manifolds, the *Aloff–Wallach space* $N(1,1)$ which is a nontrivial 3-Sasakian manifold, and the seven-sphere equipped with G_2-structure. As a result, a new type of solutions appears in the 3-Sasakian case, and this case will be discussed in a separate article.

It is worth of notice that the relationship between the existence of Killing spinor-valued forms and the *Einstein manifolds* is not clear. Contrary to Killing spinors in the Riemannian case, the Einstein condition imposed on curvature is not a direct consequence of the integrability conditions. On the other hand, there are not known counterexamples. For example, computer-aided computations produced no solutions on Berger spheres with non-Einstein metrics.

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