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On the mean curvature of submanifolds with nullity

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Abstract
In this paper, we investigate geometric conditions for isometric immersions with positive index of relative nullity to be cylinders. There is an abundance of noncylindrical n-dimensional minimal submanifolds with index of relative nullity $n-2$, fully described by Dajczer and Florit (Ill J Math 45:735–755, 2001) in terms of a certain class of elliptic surfaces. Opposed to this, we prove that nonminimal n-dimensional submanifolds in space forms of any codimension are locally cylinders provided that they carry a totally geodesic distribution of rank $n-2 \geq 2$, which is contained in the relative nullity distribution, such that the length of the mean curvature vector field is constant along each leaf. The case of dimension $n=3$ turns out to be special. We show that there exist elliptic three-dimensional submanifolds in spheres satisfying the above properties. In fact, we provide a parametrization of three-dimensional submanifolds as unit tangent bundles of minimal surfaces in the Euclidean space whose first curvature ellipse is nowhere a circle and its second one is everywhere a circle. Moreover, we provide several applications to submanifolds whose mean curvature vector field has constant length, a much weaker condition than being parallel.

Keywords Index of relative nullity · Relative nullity distribution · Mean curvature · Cylinder · Elliptic submanifolds · Minimal surfaces · Curvature ellipse

Mathematics Subject Classification Primary 53C42; Secondary 53C40 · 53B25

1 Introduction

A fundamental concept in the theory of submanifolds is the index of relative nullity introduced by Chern and Kuiper [4]. At a point $x \in M^n$ the *index of relative nullity* $\nu(x)$ of an isometric immersion $f : M^n \to \mathbb{Q}_c^m$ is the dimension of the *relative nullity* tangent subspace $\Delta_f(x)$ of f at x, that is, the kernel of the second fundamental form α^f at that point. Here, $\mathbb{Q}_c^m$ is the simply connected space form with curvature c, that is, the Euclidean space $\mathbb{R}^m$, the sphere $\mathbb{S}^m$ or the hyperbolic space $\mathbb{H}^m$, according to whether

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$c = 0, c = 1$ or $c = -1$, respectively. The kernels form an integrable distribution along any open subset where the index is constant and the images under f of the leaves of the foliation are totally geodesic submanifolds in the ambient space.

Cylinders are the simplest examples of submanifolds with positive index of relative nullity. An isometric immersion $f : M^n \to \mathbb{R}^m$ is said to be a *k-cylinder* if the manifold M^n splits as a Riemannian product $M^n = M^{n-k} \times \mathbb{R}^k$ and there is an isometric immersion $g : M^{n-k} \to \mathbb{R}^{m-k}$ such that $f = g \times \mathrm{id}_{\mathbb{R}^k}$. A natural problem in submanifold theory is to find geometric conditions for an isometric immersion with index of relative nullity $\nu \geq k > 0$ at any point to be a k-cylinder.

A fundamental result asserting that an isometric immersion $f : M^n \to \mathbb{R}^m$ of a complete Riemannian manifold with positive index of relative nullity must be a cylinder is Hartman's theorem [21] that requires the Ricci curvature of M^n to be nonnegative. Even for hypersurfaces, the same conclusion does not hold if instead we assume that the Ricci curvature is nonpositive. Notice that the latter is always the case if f is a minimal immersion. Counterexamples easy to construct are the complete irreducible ruled hypersurfaces of any dimension discussed in [7, p. 409].

The cylindricity of minimal submanifolds was studied in [8, 23] under global assumptions. These results are truly global in nature since there are plenty of (noncomplete) examples of minimal submanifolds of any dimension n with constant index $\nu = n - 2$ that are not part of a cylinder on any open subset. They can be all locally parametrically described in terms of a certain class of elliptic surfaces (see [5, Th. 22]). Some of the many papers containing characterizations of submanifolds as cylinders without the requirement of minimality are [6, 20, 21, 26].

In this paper, we deal with nonminimal n-dimensional submanifolds of arbitrary codimension and index of relative nullity $\nu \geq n - 2$ at any point. Our aim is to provide geometric conditions, in terms of the mean curvature, for an isometric immersion to be a cylinder. The choice of the geometric condition is inspired by the observation that cylinders are endowed with a totally geodesic distribution contained in the relative nullity distribution, such that the mean curvature is constant along each leaf. Throughout the paper, the *mean curvature* of an isometric immersion f is defined as the length $H = \|\mathcal{H}\|$ of the *mean curvature vector field* given by $\mathcal{H} = \mathrm{trace}(\alpha^f)/n$.

The following result provides a characterization of cylinders of dimension $n \geq 4$.

Theorem 1 *Let $f : M^n \to \mathbb{Q}_c^{n+p}$, $n \geq 4$, be an isometric immersion such that M^n carries a totally geodesic distribution D of rank $n - 2$ satisfying $D(x) \subseteq \Delta_f(x)$ for any $x \in M^n$. If the mean curvature of f is constant along each leaf of D, then either f is minimal or $c = 0$ and f is locally a $(n-2)$-cylinder over a surface on the open subset where the mean curvature is positive. Moreover, the submanifold is globally a cylinder if the leaves of D are complete.*

It is interesting that the above theorem fails for substantial three-dimensional submanifolds of codimension $p \geq 2$. Being substantial means that the codimension cannot be reduced. We show that besides cylinders, there exist elliptic three-dimensional submanifolds in spheres satisfying the properties assumed in Theorem 1. Thus, the submanifolds being three-dimensional are special. The notion of elliptic submanifolds was introduced in [5]. In fact, the following result allows a parametrization of them in terms of minimal surfaces in the Euclidean space, the so-called *bipolar parametrization*, using the following construction.

Let $g : L^2 \to \mathbb{R}^{n+1}, n \geq 5$, be a minimal surface. The map $\Phi_g : T^1L \to \mathbb{S}^n$ defined on the unit tangent bundle of L^2 and given by

$$\Phi_g(x, w) = g_{*_x} w \tag{1}$$

parametrizes (outside singular points) an immersion with index of relative nullity at least one at any point.

Theorem 2 *Let $f : M^3 \to \mathbb{Q}_c^{3+p}$ be an isometric immersion such that M^3 carries a totally geodesic distribution D of rank one satisfying $D(x) \subseteq \Delta_f(x)$ for any $x \in M^3$. If the mean curvature of f is constant along each integral curve of D, then one of the following holds*:

- (i) *The immersion f is minimal.*
- (ii) $c = 0$ *and f is locally a cylinder over a surface.*
- (iii) $c = 1$ *and the immersion f is elliptic and locally parametrized by* (1), *where* $g : L^2 \to \mathbb{R}^{n+1}, n \geq 5$, *is a minimal surface whose first curvature ellipse is nowhere a circle and the second curvature ellipse is everywhere a circle.*

Minimal surfaces satisfying the conditions in part (iii) of the above theorem can be constructed using the Weierstrass representation by choosing appropriately the holomorphic data. It is worth noticing that minimal surfaces in the Euclidean space that satisfy the Ricci condition, or equivalently are locally isometric to a minimal surface in $\mathbb{R}^3$, fulfill these conditions (see Sect. 6 for details). These surfaces were classified by Lawson [25].

The above results allow us to provide applications to submanifolds with constant mean curvature and not necessarily constant positive index of relative nullity.

Having constant mean curvature is a much weaker restriction on the mean curvature vector field than being parallel in the normal bundle. One can check that three-dimensional elliptic submanifolds described in Theorem 2 do not have parallel mean curvature vector field along the totally geodesic distribution. Combining this with Theorem 1, it follows that a submanifold is locally a cylinder provided that it carries a totally geodesic distribution of rank $n - 2 \geq 1$ that is contained in the relative nullity distribution, along which the mean curvature vector field is parallel in the normal connection.

Opposed to the fact that there is an abundance of noncylindrical n-dimensional minimal submanifolds with index of relative nullity $n - 2$ (see [5]), we prove the following result for submanifolds with constant positive mean curvature.

Theorem 3 *Let $f : M^n \to \mathbb{Q}_c^{n+p}, n \geq 3$, be a nonminimal isometric immersion with index of relative nullity $\nu \geq n - 2$ at any point. If the mean curvature of f is constant and either $n \geq 4$ or $n = 3$ and $p = 1$, then $c = 0$. Moreover, there exists an open dense subset $V \subseteq M^n$ such that every point has a neighborhood $U \subseteq V$ so that $f(U)$ is an open subset of the image of a cylinder either over a surface in $\mathbb{R}^{p+2}$, or over a curve in $\mathbb{R}^{p+1}$ with constant first Frenet curvature.*

The following is an immediate consequence of the above result due to real analyticity of hypersurfaces with constant mean curvature.

Corollary 4 *Let $f : M^n \to \mathbb{Q}_c^{n+1}, n \geq 3$, be a nonminimal isometric immersion with index of relative nullity $\nu \geq n - 2$. If the mean curvature of f is constant, then $c = 0$ and $f(M)$ is an open subset of the image of a cylinder over a surface in $\mathbb{R}^3$ of constant mean curvature.*

The next result extends Corollary 1 in [3] for hypersurfaces in every space form without any global assumption.

Corollary 5 *Let $f : M^n \to \mathbb{Q}_c^{n+1}, n \geq 3$, be an isometric immersion with constant mean curvature. If M^n has sectional curvature $K \leq c$, then either f is minimal or $c = 0$ and f(M) is an open subset of the image of a cylinder over a surface in $\mathbb{R}^3$ of constant mean curvature. In the latter case, f is a cylinder over a circle provided that M^n is complete.*

The following rigidity result that was proved in [6] for $c = 0$ is another consequence of our main results.

Corollary 6 *Any nonminimal isometric immersion $f : M^n \to \mathbb{Q}_c^{n+1}, n \geq 3$, with constant mean curvature is rigid, unless $c = 0$ and f(M) is an open subset of the image of a cylinder over a surface in $\mathbb{R}^3$ of constant mean curvature.*

Our next result extends to any dimension a well-known theorem for constant mean curvature surfaces due to Klotz and Osserman [24] (see [2] for another extension).

Theorem 7 *Let $f : M^n \to \mathbb{Q}_c^{n+1}, n \geq 3$, be an isometric immersion with constant mean curvature, where $c = 0$ or $c = 1$. If M^n is complete and its extrinsic curvature does not change sign, then either f is minimal or totally umbilical or a cylinder over a sphere of dimension $1 \leq k < n$.*

For submanifolds with constant mean curvature of codimension two, we prove the following.

Theorem 8 *Let $f : M^n \to \mathbb{R}^{n+2}, n \geq 3$, be a nonminimal isometric immersion with constant mean curvature. If the sectional curvature of M^n is nonpositive, then there exists an open dense subset $V \subseteq M^n$ such that every point has a neighborhood $U \subseteq V$ where one of the following holds:*

(i) *The neighborhood U splits as a Riemannian product $U = M^2 \times W^{n-2}$ such that $f|_U = g \times j$ is a product, where $g : M^2 \to \mathbb{R}^4$ is a surface with constant mean curvature and $j : W^{n-2} \to \mathbb{R}^{n-2}$ is the inclusion.*

(ii) *The immersion on U is a composition $f|_U = h \circ F$, where $h = \gamma \times id_{\mathbb{R}^{n-1}} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n+2}$ is cylinder over a unit speed plane curve $\gamma(s)$ with curvature $k(s)$ and $F : M^n \to \mathbb{R}^{n+1}$ is a hypersurface. Moreover, the mean curvature H_F of F is given by*

$$H_F^2 = H_f^2 - \frac{1}{n^2}k^2 \circ F_a \big(1 - \langle \xi, a \rangle^2\big)^2,$$

where $F_a = \langle F, a \rangle$ and $\langle \xi, a \rangle$ are the height functions of F and its Gauss map ξ relative to the unit vector $a = \partial/\partial s$, respectively.

(iii) *The neighborhood U splits as a Riemannian product $U = M_1^2 \times M_2^2 \times W^{n-4}$ such that $f|_U = g_1 \times g_2 \times j$ is a product, where $g_i : M_i^2 \to \mathbb{R}^3, i = 1, 2$, are surfaces with constant mean curvature and $j : W^{n-4} \to \mathbb{R}^{n-4}$ is the inclusion.*

For constant sectional curvature submanifolds with constant mean curvature of codimension two, we prove the following theorem that extends results in [11, 14].

Theorem 9 *Let* $f : M^n_{\tilde{c}} \to \mathbb{Q}^{n+2}_c, n \geq 3,$ *be an isometric immersion of a Riemannian manifold of constant sectional curvature* $\tilde{c}$. *If the mean curvature of f is constant and either* $n \geq 4$ *or* $n = 3$ *and* $c = \tilde{c}$, *then one of the following holds*:

(i) *f is totally geodesic or totally umbilical.*
(ii) $\tilde{c} = c = 0$ *and* $f = g \times j$, *where* $g : M^2 \to \mathbb{R}^4$ *is a flat surface with constant mean curvature and* $j : W \to \mathbb{R}^{n-2}$ *is an inclusion.*
(iii) $\tilde{c} = 0, c = -1$ *and f is a composition* $f = i \circ F$, *where* $i : \mathbb{R}^{n+1} \to \mathbb{H}^{n+2}$ *is the inclusion as a horosphere and* $F : M^n_{\tilde{c}} \to \mathbb{R}^{n+1}$ *is cylinder over a circle.*

Cylinder theorems for complete minimal Kähler submanifolds were proved in [9, 19]. For Kähler submanifolds with constant mean curvature, we prove the following results.

Theorem 10 *Let* $f : M^n \to \mathbb{R}^{n+1}, n \geq 4,$ *be an isometric immersion with constant mean curvature. If* M^n *is Kähler, then either f is minimal or f(M) is an open subset of the image of a cylinder over a surface in* $\mathbb{R}^3$ *with constant mean curvature.*

Theorem 11 *Let* $f : M^n \to \mathbb{R}^{n+2}, n \geq 4,$ *be a nonminimal isometric immersion of a Kähler manifold* M^n *with constant mean curvature. If the Ricci curvature or the holomorphic curvature of* M^n *is nonnegative, then there exists an open dense subset* $V \subseteq M^n$ *such that every point has a neighborhood* $U \subseteq V$ *where* $f|_U$ *is as in Theorem 8.*

The paper is organized as follows: In Sect. 2, we recall well-known results about the relative nullity distribution, totally geodesic distributions that are contained in the relative nullity distribution, as well as results about their splitting tensor. In Sect. 3, we fix the notation, give some preliminaries and prove auxiliary results that will be used in the proofs of our main theorems. Section 4 is devoted to the proof of Theorem 1. In Sect. 5, we recall the notion of elliptic submanifolds, as well as the associated notion of higher curvature ellipses. We also discuss the polar and bipolar surfaces of elliptic submanifolds. In Sect. 6, we study the case of three-dimensional submanifolds. We provide a parametrization for these submanifolds in terms of certain elliptic surfaces, the so-called *polar parametrization* (see Theorem 21). Based on this, we give the proof of Theorem 2. We conclude this section by showing that minimal surfaces in the Euclidean space that are locally isometric to a minimal surface in $\mathbb{R}^3$ satisfy the conditions in part (iii) of Theorem 2. In Sect. 7, we prove Theorem 3 and the applications of our main results on submanifolds with constant mean curvature. In addition, we provide examples of submanifolds as in part (ii) of Theorems 8 and 9.

2 The relative nullity distribution

In this section, we recall some basic facts from the theory of isometric immersions that will be used throughout the paper.

Let $M^n, n \geq 3$, be a Riemannian manifold and let $f : M^n \to \mathbb{Q}^m_c$ be an isometric immersion into a space form $\mathbb{Q}^m_c$. The *relative nullity* subspace $\Delta_f(x)$ of *f* at $x \in M^n$ is the kernel of its second fundamental form $\alpha^f : TM \times TM \to N_fM$ with values in the normal bundle, that is,

$$\Delta_f(x) = \{X \in T_xM : \alpha^f(X,Y) = 0 \text{ for all } Y \in T_xM\}.$$

The dimension $\nu(x)$ of $\Delta_f(x)$ is called the *index of relative nullity* of f at $x \in M^n$.

A smooth distribution $D \subset TM$ on M^n is *totally geodesic* if $\nabla_T S \in \Gamma(D)$ whenever $T, S \in \Gamma(D)$. Let D be a smooth distribution on M^n and $D^\perp$ denote the distribution on M^n that assigns to each $x \in M^n$ the orthogonal complement of $D(x)$ in T_xM. We write $X = X^v + X^h$ according to the orthogonal splitting $TM = D \oplus D^\perp$ and denote $\nabla^h_X Y = (\nabla_X Y)^h$ for all $X, Y \in TM$, where ∇ is the Levi-Civitá connection on M^n. The *splitting tensor* $C : D \times D^\perp \to D^\perp$ is given by

$$C(T,X) = -\nabla^h_X T$$

for any $T \in D$ and $X \in D^\perp$.

When D is a totally geodesic distribution such that $D(x) \subseteq \Delta_f(x)$ for all $x \in M^n$, the following differential equation for the tensor $\mathcal{C}_T = C(T,\cdot)$ is well-known to hold (cf. [7] or [12]):

$$\nabla^h_S \mathcal{C}_T = \mathcal{C}_T \circ \mathcal{C}_S + \mathcal{C}_{\nabla_S T} + c\langle S,T\rangle I, \tag{2}$$

where I is the identity endomorphism of $D^\perp$. Here $\nabla^h_S \mathcal{C}_T \in \Gamma(\mathrm{End}(D^\perp))$ is defined by

$$(\nabla^h_S \mathcal{C}_T)X = \nabla^h_S \mathcal{C}_T X - \mathcal{C}_T \nabla^h_S X$$

for all $T, S \in D$ and $X \in D^\perp$. The Codazzi equation gives

$$\nabla_T A_\xi = A_\xi \circ \mathcal{C}_T + A_{\nabla^\perp_T \xi} \tag{3}$$

for any $T \in D$, where the shape operator A_ξ with respect to the normal direction ξ is restricted to $D^\perp$ and $\nabla^\perp$ stands for the normal connection of f. In particular, the endomorphism $A_\xi \circ \mathcal{C}_T$ of $D^\perp$ is symmetric, that is,

$$A_\xi \circ \mathcal{C}_T = \mathcal{C}^t_T \circ A_\xi. \tag{4}$$

For later use, we recall the following known results.

Proposition 12 [12, *Prop*. 7.4] *Let $f : M^n \to \mathbb{Q}^m_c$ be an isometric immersion such that M^n carries a smooth totally geodesic distribution D of rank $0 < k < n$ satisfying $D(x) \subseteq \Delta_f(x)$ for all $x \in M^n$. If the splitting tensor C vanishes, then $c = 0$ and f is locally a k-cylinder.*

Proposition 13 [12, *Prop*. 1.18] *For an isometric immersion $f : M^n \to \mathbb{Q}^m_c$, the following assertions hold*:

(i) *The index of relative nullity ν is upper semicontinuous. In particular, the subset*

$$M_0 = \{x \in M^n : \nu(x) = \nu_0\},$$

where ν attains its minimum value ν_0 is open.

(ii) *The relative nullity distribution $x \mapsto \Delta_f(x)$ is smooth on any subset of M^n where ν is constant.*

(iii) *If $U \subseteq M^n$ is an open subset where ν is constant, then Δ_f is a totally geodesic (hence integrable) distribution on U and the restriction of f to each leaf is totally geodesic.*

3 Auxiliary results

The aim of this section is to prove several lemmas that will be used in the proofs of our main results.

Throughout this section, we assume that $f : M^n \to \mathbb{Q}_c^{n+p}, n \geq 3$, is a nonminimal isometric immersion such that M^n carries a smooth totally geodesic distribution D of rank $n-2$ satisfying $D(x) \subseteq \Delta_f(x)$ for any $x \in M^n$. We also assume that the mean curvature of f is constant along each leaf of D.

Hereafter, we work on the open subset where the mean curvature is positive and choose a local orthonormal frame $\xi_{n+1}, \dots, \xi_{n+p}$ in the normal bundle $N_f M$, such that ξ_{n+1} is collinear to the mean curvature vector field. We also choose a local orthonormal frame $e_1, \dots, e_n$ in the tangent bundle TM such that e_1, e_2 span $D^\perp$ and diagonalize $A_{\xi_{n+1}}|_{D^\perp}$, where $A_{\xi_{n+1}}$ denotes the shape operator of f with respect to ξ_{n+1}. Then, we have $A_{\xi_{n+1}} e_i = k_i e_i,\ i = 1, 2$, and consequently the mean curvature is given by $nH = k_1 + k_2$, where k_1, k_2 are the principal curvatures.

Since the mean curvature is positive, at least one of the principal curvatures k_1 and k_2 has to be different from zero. In the sequel, we assume without loss of generality, that $k_1 \neq 0$ and define the function

$$\rho = -\frac{k_2}{k_1}.$$

On the open subset where the mean curvature is positive we have

$$k_1 = -\frac{nH}{\rho - 1} \quad \text{and} \quad k_2 = \frac{n\rho H}{\rho - 1}. \tag{5}$$

The above-mentioned notation is used throughout the paper.

The following lemma gives the form of the splitting tensor.

Lemma 14 *On the open subset where the mean curvature is positive, the splitting tensor is given by*

$$\mathcal{C}_T = \psi_1(T) L_1 + \psi_2(T) L_2$$

for any $T \in \Gamma(D)$, where ψ_1, ψ_2 are 1*-forms dual to the vector fields $\nabla_{e_2} e_2, \nabla_{e_1} e_2$, respectively, and $L_1, L_2 \in \Gamma(\mathrm{End}(D^\perp))$ are defined by $L_1 e_1 = \rho e_1 = -L_2 e_2$ and $L_1 e_2 = e_2 = L_2 e_1$. Moreover, the following holds*:

$$T(k_1) = \rho k_1 \psi_1(T) + \sum_{\alpha=n+2}^{n+p} \langle \nabla_T^\perp \xi_{n+1}, \xi_\alpha \rangle \langle A_{\xi_\alpha} e_1, e_1 \rangle, \tag{6}$$

$$T(k_2) = k_2\psi_1(T) - \sum_{\alpha=n+2}^{n+p} \langle \nabla_T^{\perp}\xi_{n+1}, \xi_\alpha\rangle\langle A_{\xi_\alpha}e_1, e_1\rangle, \tag{7}$$

$$(k_1 - k_2)\omega(T) = k_2\psi_2(T) + \sum_{\alpha=n+2}^{n+p} \langle \nabla_T^{\perp}\xi_{n+1}, \xi_\alpha\rangle\langle A_{\xi_\alpha}e_1, e_2\rangle, \tag{8}$$

$$(k_1 - k_2)\omega(T) = -\rho k_1\psi_2(T) + \sum_{\alpha=n+2}^{n+p} \langle \nabla_T^{\perp}\xi_{n+1}, \xi_\alpha\rangle\langle A_{\xi_\alpha}e_1, e_2\rangle \tag{9}$$

for any $T \in \Gamma(D)$, *where* ω *denotes the connection form given* by $\omega = \langle \nabla e_1, e_2\rangle$.

Proof From the Codazzi equation, we have

$$\left(\nabla_T A_{\xi_{n+1}}\right)e_i - \left(\nabla_{e_i} A_{\xi_{n+1}}\right)T = A_{\nabla_T^{\perp}\xi_{n+1}}e_i - A_{\nabla_{e_i}^{\perp}\xi_{n+1}}T$$

for any $T \in \Gamma(D)$ and $i = 1, 2$. The above is equivalent to the following:

$$T(k_1) = k_1\langle \nabla_{e_1}e_1, T\rangle + \sum_{\alpha=n+2}^{n+p} \langle \nabla_T^{\perp}\xi_{n+1}, \xi_\alpha\rangle\langle A_{\xi_\alpha}e_1, e_1\rangle,$$
$$T(k_2) = k_2\langle \nabla_{e_2}e_2, T\rangle - \sum_{\alpha=n+2}^{n+p} \langle \nabla_T^{\perp}\xi_{n+1}, \xi_\alpha\rangle\langle A_{\xi_\alpha}e_1, e_1\rangle,$$
$$(k_1 - k_2)\omega(T) = k_2\langle \nabla_{e_1}e_2, T\rangle + \sum_{\alpha=n+2}^{n+p} \langle \nabla_T^{\perp}\xi_{n+1}, \xi_\alpha\rangle\langle A_{\xi_\alpha}e_1, e_2\rangle,$$
$$(k_1 - k_2)\omega(T) = k_1\langle \nabla_{e_2}e_1, T\rangle + \sum_{\alpha=n+2}^{n+p} \langle \nabla_T^{\perp}\xi_{n+1}, \xi_\alpha\rangle\langle A_{\xi_\alpha}e_1, e_2\rangle.$$

Using the assumption that the mean curvature is constant along each leaf of the distribution D, the first two equations imply

$$\langle \nabla_{e_1}e_1, T\rangle = \rho\langle \nabla_{e_2}e_2, T\rangle$$

for any $T \in \Gamma(D)$. Additionally, the last two equations yield

$$\langle \nabla_{e_2}e_1, T\rangle = -\rho\langle \nabla_{e_1}e_2, T\rangle.$$

Now the structure of the splitting tensor and (6)-(9) follow easily from the above. □

Lemma 15 *Let* $e_r, r \geq 3$, *be an orthonormal frame of the distribution* D. *Then the functions* $u_r := \psi_1(e_r)$ *and* $v_r := \psi_2(e_r)$ *satisfy*

$$2\rho(u_r u_s + v_r v_s) - c\delta_{rs} = \frac{\rho - 1}{nH} \sum_{\alpha=n+2}^{n+p} \langle \nabla_{e_r}^{\perp}\xi_{n+1}, \xi_\alpha\rangle\left(u_s\langle A_{\xi_\alpha}e_1, e_1\rangle - v_s\langle A_{\xi_\alpha}e_1, e_2\rangle\right) \tag{10}$$

for all $r, s \geq 3$, *where* δ_{rs} *is the Kronecker delta.*

Proof Using Lemma 14, we have

$$(\nabla^h_{e_r}\mathcal{C}_{e_s}) = e_r(u_s)L_1 + e_r(v_s)L_2 + u_s\nabla^h_{e_r}L_1 + v_s\nabla^h_{e_r}L_2 \tag{11}$$

for any $r, s \geq 3$. A direct computation yields

$$(\nabla^h_{e_r}L_1)e_1 = -(\nabla^h_{e_r}L_2)e_2 = e_r(\rho)e_1 + (\rho-1)\omega(e_r)e_2, \tag{12}$$

$$(\nabla^h_{e_r}L_1)e_2 = (\nabla^h_{e_r}L_2)e_1 = (\rho-1)\omega(e_r)e_1. \tag{13}$$

Then (6) and (7) imply that

$$e_r(\rho) = -\rho(\rho-1)u_r + \frac{(\rho-1)^2}{nH}\sum_{\alpha=n+2}^{n+p}\langle\nabla^\perp_{e_r}\xi_{n+1},\xi_\alpha\rangle\langle A_{\xi_\alpha}e_1,e_1\rangle. \tag{14}$$

From (2), we know that the splitting tensor satisfies

$$(\nabla^h_{e_r}\mathcal{C}_{e_s})e_i = \mathcal{C}_{e_s}\circ\mathcal{C}_{e_r}e_i + \mathcal{C}_{\nabla_{e_r}e_s}e_i + c\delta_{rs}e_i \tag{15}$$

for any $r, s \geq 3$ and $i = 1, 2$.

Let ω_{rs} be the connection form given by $\omega_{rs} = \langle\nabla e_r, e_s\rangle$ for all $r, s \geq 3$. Using (11)-(14), we find that (15) for $i = 1$ is equivalent to

$$\begin{aligned}\rho e_r(u_s) =& \rho(2\rho-1)u_ru_s - \rho v_rv_s - (\rho-1)v_s\omega(e_r)\\ &- u_s\frac{(\rho-1)^2}{nH}\sum_{\alpha=n+2}^{n+p}\langle\nabla^\perp_{e_r}\xi_{n+1},\xi_\alpha\rangle\langle A_{\xi_\alpha}e_1,e_1\rangle + \rho\sum_{t\geq 3}^{n}\omega_{st}(e_r)u_t + c\delta_{rs}\end{aligned} \tag{16}$$

and

$$e_r(v_s) = \rho u_rv_s + u_sv_r - (\rho-1)u_s\omega(e_r) + \sum_{t\geq 3}^{n}\omega_{st}(e_r)v_t \tag{17}$$

for all $r, s \geq 3$. Moreover, (15) for $i = 2$ implies that

$$e_r(u_s) = u_ru_s - \rho v_rv_s + (\rho-1)v_s\omega(e_r) + \sum_{t\geq 3}^{n}\omega_{st}(e_r)u_t + c\delta_{rs} \tag{18}$$

for all $r, s \geq 3$.

Combining (16) and (17), we obtain

$$2\rho u_ru_s + \rho v_rv_s - c\delta_{rs} - v_s(\rho+1)\omega(e_r) = u_s\frac{\rho-1}{nH}\sum_{\alpha=n+2}^{n+p}\langle\nabla^\perp_{e_r}\xi_{n+1},\xi_\alpha\rangle\langle A_{\xi_\alpha}e_1,e_1\rangle.$$

Using (5), it is easily seen that (9) is written as

$$(\rho+1)\omega(e_r) = -\rho v_r - \frac{\rho-1}{nH}\sum_{\alpha=n+2}^{n+p}\langle\nabla^\perp_{e_r}\xi_{n+1},\xi_\alpha\rangle\langle A_{\xi_\alpha}e_1,e_2\rangle, \tag{19}$$

and now (10) follows directly from the above two equations. □

We recall that the *first normal space* $N_1^f(x)$ of the immersion f at a point $x \in M^n$ is the subspace of its normal space $N_f M(x)$ spanned by the image of its second fundamental form α^f at x, that is,

$$N_1^f(x) = \operatorname{span}\left\{\alpha^f(X, Y) : X, Y \in T_x M\right\}.$$

The rank condition and the symmetry of the second fundamental form imply that $\dim N_1^f(x) \leq 3$ for all $x \in M^n$.

Consider the open subset

$$M_3 = \left\{x \in M^n : \dim N_1^f(x) = 3\right\}.$$

Lemma 16 *The splitting tensor vanishes on the open subset* $M_3^* := M_3 \setminus \{x \in M^n : H(x) = 0\}$.

Proof On the subset M_3^*, we consider the orthogonal splitting $N_1^f = \hat{N}_1^f \oplus \operatorname{span}\{\mathcal{H}\}$. Choose the local frame such that ξ_{n+1} is collinear to the mean curvature vector field $\mathcal{H}$, and ξ_{n+2}, ξ_{n+3} span the plane bundle $\hat{N}_1^f$. Then, we have

$$\operatorname{trace} A_{\xi_{n+2}}|_{D^\perp} = 0 = \operatorname{trace} A_{\xi_{n+3}}|_{D^\perp}.$$

Hence, we obtain

$$A_{\xi_{n+2}}|_{D^\perp} \circ J = J^t \circ A_{\xi_{n+2}}|_{D^\perp} \text{ and } A_{\xi_{n+3}}|_{D^\perp} \circ J = J^t \circ A_{\xi_{n+3}}|_{D^\perp},$$

where J denotes the unique, up to a sign, almost complex structure acting on the plane bundle $D^\perp$.

It follows using (4) that

$$A_{\xi_{n+2}}|_{D^\perp} \circ \mathcal{C}_T = \mathcal{C}_T^t \circ A_{\xi_{n+2}}|_{D^\perp} \text{ and } A_{\xi_{n+3}}|_{D^\perp} \circ \mathcal{C}_T = \mathcal{C}_T^t \circ A_{\xi_{n+3}}|_{D^\perp}$$

for any $T \in \Gamma(D)$. Since $\hat{N}_1^f$ is a plane bundle, the above imply that $\mathcal{C}_T \in \operatorname{span}\{I, J\} \subseteq \operatorname{End}(D^\perp)$. This, combined with Lemma 14, yields

$$(\rho - 1)\psi_1(T) = 0 \text{ and } (\rho - 1)\psi_2(T) = 0$$

for any $T \in \Gamma(D)$. Thus, the splitting tensor vanishes identically on M_3^*. □

Hereafter, we assume that M_3 is not dense on M^n and consider the open subset

$$M_2 = \left\{x \in M^n \setminus \overline{M}_3 : \dim N_1^f(x) = 2\right\}.$$

In the sequel, we assume that the open subset $M_2^* := M_2 \setminus \{x \in M^n : H(x) = 0\}$ is non-empty. Choose a local orthonormal frame such that ξ_{n+1} and ξ_{n+2} span the plane bundle N_1^f on this subset and ξ_{n+1} is collinear to the mean curvature vector field. Thus, there exist smooth functions λ, μ such that

$$A_{\xi_{n+2}} e_1 = \lambda e_1 + \mu e_2, \; A_{\xi_{n+2}} e_2 = \mu e_1 - \lambda e_2 \text{ and } \lambda^2 + \mu^2 > 0.$$

We proceed with some auxiliary lemmas.

Lemma 17 *The plane bundle N_1^f is parallel in the normal connection along the distribution D on the subset M_2^*. Moreover, the following holds*:

$$\mu\psi_1(T) = -\lambda\psi_2(T), \tag{20}$$

$$\mu\phi(T) = -(\lambda^2 + \mu^2)\frac{\rho - 1}{nH}\psi_2(T), \tag{21}$$

$$T(\mu) + 2\lambda\omega(T) + (\rho + 1)\lambda\psi_2(T) = 0, \tag{22}$$

$$T(\lambda) - 2\mu\omega(T) - \mu\rho\psi_2(T) - \lambda\psi_1(T) = \frac{n\rho H}{\rho - 1}\phi(T), \tag{23}$$

$$T(\lambda) - 2\mu\omega(T) - \mu\psi_2(T) - \lambda\rho\psi_1(T) = \frac{nH}{\rho - 1}\phi(T) \tag{24}$$

for any $T \in \Gamma(D)$, where ϕ is the normal connection form given by $\phi = \langle \nabla^\perp \xi_{n+1}, \xi_{n+2} \rangle$.

Proof It follows from (3) that

$$\langle \nabla_T^\perp \xi_\alpha, \xi \rangle = 0 \ \text{ if } \ \alpha = n+1, n+2$$

for any $T \in \Gamma(D)$ and any $\xi \in \Gamma(N_1^{f\perp})$. Thus, the subbundle N_1^f is parallel in the normal connection along the distribution D.

Moreover, from (3) we have

$$(\nabla_T A_{\xi_{n+2}})e_i = A_{\xi_{n+2}} \circ \mathcal{C}_T e_i + A_{\nabla_T^\perp \xi_{n+2}} e_i, \quad i = 1, 2,$$

for any $T \in \Gamma(D)$. Bearing in mind the form of the splitting tensor given in Lemma 14, the above equations yield directly (23), (24) and the following

$$T(\mu) + 2\lambda\omega(T) + \lambda\rho\psi_2(T) - \mu\psi_1(T) = 0,$$
$$T(\mu) + 2\lambda\omega(T) - \mu\rho\psi_1(T) + \lambda\psi_2(T) = 0$$

for any $T \in \Gamma(D)$. Subtracting the above equations, we obtain (20). Similarly, (21) follows by subtracting (23), (24) and using (20). Finally, plugging (20) into the first of the above equations, we obtain (22). □

Now suppose that the subset $M_3 \cup M_2$ is not dense on M^n and consider the open subset

$$M_1 = \left\{ x \in M^n \setminus \overline{M_3 \cup M_2} \, : \, \dim N_1^f(x) = 1 \right\}.$$

Lemma 18 *If the subset $M_1^* := M_1 \setminus \{x \in M^n : H(x) = 0\}$ is nonempty, then $c = 0$ and $f|_{M_1^*}$ is locally a cylinder either over a surface in $\mathbb{R}^{p+2}$ or over a curve in $\mathbb{R}^{p+1}$.*

Proof On the subset M_1^* we choose a local orthonormal frame $\xi_{n+1}, \ldots, \xi_{n+p}$ in the normal bundle such that ξ_{n+1} is collinear to the mean curvature vector field. Then we have $A_{\xi_\alpha} = 0$ for all $\alpha \geq n+2$. The Codazzi equation yields

$$A_{\nabla^\perp_{e_i}\xi_\alpha} e_r = A_{\nabla^\perp_{e_r}\xi_\alpha} e_i$$

for all $\alpha \geq n+2, i = 1, 2$, and $r \geq 3$. Thus, we obtain $\nabla^\perp_{e_r}\xi_{n+1} = 0$ and Lemma 15 gives

$$2\rho(u_r u_s + v_r v_s) = c\delta_{rs} \tag{25}$$

for all $r \geq 3$. Moreover, (14) becomes

$$e_r(\rho) = -\rho(\rho - 1)u_r.$$

Differentiating (25) with respect to e_r and using the above along with (17) and (18), we obtain

$$\rho u_r(\rho - 3)(u_r^2 + v_r^2) - 2c\rho u_r + 2\rho \sum_{s\geq 3}^{n} \omega_{rs}(e_r)\big(u_s u_r + v_s v_r\big) = 0$$

for all $r \geq 3$. In view of (25), the above equation simplifies to the following

$$c(\rho + 1)u_r = 0.$$

Now we prove that $c = 0$. Arguing indirectly, we suppose that $c \neq 0$. Assume that the open set of points where $\rho \neq -1$ is nonempty. On this subset, we have $u_r = 0$ for all $r \geq 3$. Thus, (25) becomes $2\rho v_r^2 = c$ for all $r \geq 3$. Using (19), (18) yields $2\rho^2 v_r^2 = c(\rho + 1)$, which is a contradiction. Assume now that the set of points where $\rho = -1$ has nonempty interior. On this subset, (6) yields $u_r = 0$ and (8) implies that $v_r = 0$, which contradicts the assumption that $c \neq 0$.

Hence, $c = 0$ and (25) becomes

$$\rho(u_r^2 + v_r^2) = 0$$

for all $r \geq 3$. If $\rho \neq 0$, then the splitting tensor vanishes and Proposition 12 implies that f is locally a cylinder over a surface. If the subset of points where $\rho = 0$ has nonempty interior, then the Codazzi equation implies that the tangent bundle splits as an orthogonal sum of two parallel distributions one of which has rank $n - 1$. Thus, the manifold splits locally as a Riemannian product by the De Rham decomposition theorem. Since the second fundamental form is adapted to this splitting, the result follows from [12, Th. 8.4]. □

4 Submanifolds of dimension $n \geq 4$

We are now ready to give the proof of our first main result.

Proof of Theorem 1 If the open subset M_3^* is nonempty, then Lemma 16 implies that the splitting tensor vanishes identically on it. Then, by Proposition 12 the immersion f is locally a cylinder over a surface on M_3^*.

Now assume that the subset M_3 is not dense on M^n and suppose that M_2^* is nonempty. Hereafter, we work on M_2^*. Due to the choice of the local orthonormal frame ξ_{n+1}, ξ_{n+2} in the normal subbundle N_1^f, and using (20) and (21), (10) of Lemma 15 takes the following form

$$v_r v_s \left(\lambda^2 + \mu^2\right)\left(2\rho - \left(\lambda^2 + \mu^2\right)\frac{(\rho - 1)^2}{n^2H^2}\right) = c\mu^2\delta_{rs} \tag{26}$$

for any $r, s \geq 3$.

We claim that $v_r = 0$ for any $r \geq 3$. In fact, at points where

$$2\rho - \left(\lambda^2 + \mu^2\right)\frac{(\rho - 1)^2}{n^2H^2} \neq 0,$$

it follows from (26) that

$$v_r^2 = \frac{c\mu^2}{\left(\lambda^2 + \mu^2\right)\left(2\rho - \left(\lambda^2 + \mu^2\right)\frac{(\rho-1)^2}{n^2H^2}\right)}$$

for any $r \geq 3$ and $v_r v_s = 0$ for $r \neq s \geq 3$. Thus, $v_r = 0$ for any $r \geq 3$ at those points.

It remains to prove that the same holds on the subset $U \subseteq M_2^*$ of points where

$$2\rho - \left(\lambda^2 + \mu^2\right)\frac{(\rho - 1)^2}{n^2H^2} = 0.$$

Notice that because of (5), the subset U is the set of points where

$$\lambda^2 + \mu^2 = -2k_1k_2. \tag{27}$$

In order to prove that $v_r = 0$ for any $r \geq 3$ on U, we suppose that the interior of U is non-empty. Suppose to the contrary that there exists $r_0 \geq 3$ such that $v_{r_0} \neq 0$ on an open subset of U. Differentiating (27) with respect to e_{r_0} and using (5), (6), (7), (22) and (23), we obtain

$$\lambda^2 u_{r_0} - \lambda\mu v_{r_0} + (\rho + 1)k_1k_2u_{r_0} = \lambda(k_1 - 2k_2)\phi(e_{r_0}).$$

Multiplying by μ the above and using (21), we find that

$$\mu u_{r_0}\left(\lambda^2 + (\rho + 1)k_1k_2\right) = \lambda v_{r_0}\left(\mu^2 - (\lambda^2 + \mu^2)(k_1 - 2k_2)\frac{\rho - 1}{nH}\right).$$

Taking into account (5), (20) and (27), the above yields

$$\lambda v_{r_0}(\rho + 1)(\lambda^2 + \mu^2) = 0.$$

Due to (27), we conclude that $\lambda = 0$ and consequently $\mu \neq 0$. Then, it follows from (20) that $u_s = 0$ for any $s \geq 3$. It is easily seen from (16) and (18) for $s = r_0$ that

$$\rho v_{r_0}^2 + (\rho - 1)v_{r_0}\omega(e_{r_0}) - c = 0 \quad \text{and} \quad \rho v_{r_0}^2 - (\rho - 1)v_{r_0}\omega(e_{r_0}) - c = 0.$$

Hence, $\omega(e_{r_0}) = 0$, and consequently (19) yields

$$\rho v_{r_0} + \frac{\rho - 1}{nH}\mu\phi(e_{r_0}) = 0.$$

Using (5), (21) and (27), we find that $\rho = 0$, which contradicts (27). Thus, we have proved the claim that $v_r = 0$ for any $r \geq 3$.

Now, we claim that $u_r = 0$ for any $r \geq 3$. It follows using (20) that $\mu u_r = 0$ for any $r \geq 3$. Obviously, the function u_r vanishes at points where $\mu \neq 0$.

Assume that the set of points where $\mu = 0$ has nonempty interior and argue on this subset. Since $\lambda \neq 0$ on this subset, it follows from (22) that $\omega(e_r) = 0$ for any $r \geq 3$, and consequently (23) and (24) yield

$$\phi(e_r) = \frac{\rho - 1}{nH}\lambda u_r \;\text{ and }\; e_r(\lambda) = (\rho + 1)\lambda u_r \tag{28}$$

for all $r \geq 3$. Using the first of the above equations, (10) is written equivalently as

$$u_r^2\left(2\rho - \frac{\lambda^2(\rho - 1)^2}{n^2H^2}\right) = c \tag{29}$$

for all $r \geq 3$.

Since we already proved that $v_r = 0$ for all $r \geq 3$, Lemma 14 implies that the image of the splitting tensor $C : D \to \mathrm{End}(D^\perp)$ satisfies $\dim \mathrm{Im}\, C \leq 1$. Thus, $\dim \ker\, C \geq n - 3$.

Now suppose that $\dim \ker\, C = n - 3$. Then, there exists a unique $r_0 \geq 3$ such that $u_{r_0} \neq 0$ and $u_s = 0$ for any $s \neq r_0$. Thus, (29) implies that $c = 0$ and

$$2\rho = \frac{\lambda^2(\rho - 1)^2}{n^2H^2}.$$

On account of (5), the above equation becomes $\lambda^2 = -2k_1k_2 > 0$. Differentiating this equation with respect to e_{r_0} and using (5), (6), (7) and the second of (28), we obtain $2\lambda^2 + k_1k_2 = 0$, which contradicts the previous equation. Thus, the splitting tensor vanishes identically on the subset M_2^* and consequently, by Proposition 12, the immersion f is locally a cylinder over a surface.

If the open subset M_1^* is nonempty, then Lemma 18 implies that f is locally a cylinder over a surface or over a curve. □

5 Elliptic submanifolds

In this section, we recall from [5] the notion of elliptic submanifolds of a space form as well as several of their basic properties.

Let $f : M^n \to \mathbb{Q}_c^m$ be an isometric immersion. The *ℓ^{th}-normal space* $N_\ell^f(x)$ of f at $x \in M^n$ for $\ell \geq 1$ is defined as

$$N_\ell^f(x) = \mathrm{span}\left\{\alpha_{\ell+1}^f\left(X_1, \ldots, X_{\ell+1}\right) : X_1, \ldots, X_{\ell+1} \in T_xM\right\}.$$

Here $\alpha_2^f = \alpha^f$ and for $s \geq 3$ the so-called *s^{th}-fundamental form* is the symmetric tensor $\alpha_s^f : TM \times \cdots \times TM \to N_fM$ defined inductively by

$$\alpha_s^f(X_1, \ldots, X_s) = \pi^{s-1}\left(\nabla_{X_s}^\perp \cdots \nabla_{X_3}^\perp \alpha^f(X_2, X_1)\right),$$

where π^k stands for the projection onto $(N_1^f \oplus \cdots \oplus N_{k-1}^f)^\perp$.

An isometric immersion $f : M^n \to \mathbb{Q}_c^m$ is called *elliptic* if M^n carries a totally geodesic distribution D of rank $n-2$ satisfying $D(x) \subseteq \Delta_f(x)$ for any $x \in M^n$ and there exists an (necessary unique up to a sign) almost complex structure $J : D^\perp \to D^\perp$ such that the second fundamental form satisfies

$$\alpha^f(X,X) + \alpha^f(JX,JX) = 0$$

for all $X \in D^\perp$. Notice that J is orthogonal if and only f is minimal.

Assume that $f : M^n \to \mathbb{Q}_c^m$ is substantial and elliptic. Assume also that f is *nicely curved* which means that for any $\ell \geq 1$ all subspaces $N_\ell^f(x)$ have constant dimension and thus form subbundles of the normal bundle. Notice that any f is nicely curved along connected components of an open dense subset of M^n. Then, along that subset the normal bundle splits orthogonally and smoothly as

$$N_f M = N_1^f \oplus \cdots \oplus N_{\tau_f}^f, \tag{30}$$

where all N_ℓ^f's have rank two, except possibly the last one that has rank one in case the codimension is odd. Thus, the induced bundle $f^*T\mathbb{Q}_c^m$ splits as

$$f^*T\mathbb{Q}_c^m = f_*D \oplus N_0^f \oplus N_1^f \oplus \cdots \oplus N_{\tau_f}^f,$$

where $N_0^f = f_*D^\perp$. Setting

$$\tau_f^o = \begin{cases} \tau_f & \text{if } m-n \text{ is even} \\ \tau_f - 1 & \text{if } m-n \text{ is odd} \end{cases}$$

it turns out that the almost complex structure J on $D^\perp$ induces an almost complex structure J_ℓ on each N_ℓ^f, $0 \leq \ell \leq \tau_f^o$, defined by

$$J_\ell \alpha_{\ell+1}^f\big(X_1, \ldots, X_\ell, X_{\ell+1}\big) = \alpha_{\ell+1}^f\big(X_1, \ldots, X_\ell, JX_{\ell+1}\big),$$

where $\alpha_1^f = f_*$.

The ℓ^{th}*-order curvature ellipse* $\mathcal{E}_\ell^f(x) \subset N_\ell^f(x)$ of f at $x \in M^n$ for $0 \leq \ell \leq \tau_f^o$ is

$$\mathcal{E}_\ell^f(x) = \big\{\alpha_{\ell+1}^f(Z_\theta, \ldots, Z_\theta) \,:\, Z_\theta = \cos\theta Z + \sin\theta JZ \ \text{ and } \ \theta \in [0,\pi)\big\},$$

where $Z \in D^\perp(x)$ has unit length and satisfies $\langle Z, JZ\rangle = 0$. From ellipticity, such a Z always exists and $\mathcal{E}_\ell^f(x)$ is indeed an ellipse.

We say that the curvature ellipse $\mathcal{E}_\ell^f$ of an elliptic submanifold f is a *circle* for some $0 \leq \ell \leq \tau_f^o$ if all ellipses $\mathcal{E}_\ell^f(x)$ are circles. That the curvature ellipse $\mathcal{E}_\ell^f$ in a circle is equivalent to the almost complex structure J_ℓ being orthogonal. Notice that $\mathcal{E}_0^f$ is a circle if and only if f is minimal.

Let $f : M^n \to \mathbb{Q}_c^{m-c}, c \in \{0,1\}$, be a substantial nicely curved elliptic submanifold. Assume that M^n is the saturation of a fixed cross section $L^2 \subset M^n$ to the foliation of the distribution D. The subbundles in the orthogonal splitting (30) are parallel in the normal connection (and thus in $\mathbb{Q}_c^{m-c}$) along D. Hence, each N_ℓ^f can be seen as a vector bundle along the surface L^2.

A *polar surface* to f is an immersion h of L^2 defined as follows:

(a) If $m-n-c$ is odd, then the polar surface $h : L^2 \to \mathbb{S}^{m-1}$ is the spherical image of the unit normal field spanning $N_{\tau_f}^f$.

(b) If $m-n-c$ is even, then the polar surface $h : L^2 \to \mathbb{R}^m$ is any surface such that $h_*T_xL = N_{\tau_f}^f(x)$ up to parallel identification in $\mathbb{R}^m$.

Polar surfaces always exist since in case (b) any elliptic submanifold admits locally many polar surfaces.

The almost complex structure J on $D^\perp$ induces an almost complex structure $\tilde{J}$ on TL defined by $P \circ \tilde{J} = J \circ P$, where $P : TL \to D^\perp$ is the orthogonal projection. It turns out that a polar surface to an elliptic submanifold is necessarily elliptic. Moreover, if the elliptic submanifold has a circular curvature ellipse then its polar surface has the same property at the "corresponding" normal bundle. As a matter of fact, up to parallel identification it holds that

$$N_s^h = N^f_{\tau_f^o - s} \ \text{ and } \ J_s^h = \left(J^f_{\tau_f^o - s}\right)^t, \ \ 0 \le s \le \tau_f^o. \tag{31}$$

In particular, the polar surface is nicely curved.

A *bipolar surface* to f is any polar surface to a polar surface to f. In particular, if we are in case $f : M^3 \to \mathbb{S}^{m-1}$, then a bipolar surface to f is a nicely curved elliptic surface $g : L^2 \to \mathbb{R}^m$.

6 Three-dimensional submanifolds

In this section, we study the case of three-dimensional submanifolds and we provide the proof of Theorem 2. To this purpose, we need the following results.

Proposition 19 *Let $f : M^3 \to \mathbb{Q}_c^{3+p}$ be an isometric immersion such that M^3 carries a totally geodesic distribution D of rank one satisfying $D(x) \subseteq \Delta_f(x)$ for any $x \in M^3$. If the mean curvature of f is constant along each integral curve of D and the normal bundle of f is flat, then f is minimal or $c = 0$ and f is locally a cylinder.*

Proof Assume that f is nonminimal. If the open subset M_3^* is nonempty, then Lemma 16 and Proposition 12 imply that the immersion f is locally a cylinder over a surface.

Now suppose that the open subset M_2^* is nonempty and argue on it. Having flat normal bundle implies that $\mu = 0$ and according to (20), we obtain $v_3 = 0$. Consequently, (18) is written as

$$e_3(u_3) = u_3^2 + c. \tag{32}$$

Comparing (23) and (24), we obtain

$$\phi(e_3) = \frac{\rho - 1}{nH} \lambda u_3.$$

Thus,

$$e_3(\lambda) = (\rho + 1)\lambda u_3 \tag{33}$$

and consequently (14) becomes

$$e_3(\rho) = u_3(\rho - 1)(\tau - \rho), \tag{34}$$

where τ is the function given by

$$\tau = \frac{\lambda^2(\rho-1)^2}{n^2H^2}.$$

Moreover, (10) is written as $u_3^2(2\rho - \tau) = c$. Differentiating with respect to e_3 and using (32)-(34), we derive that

$$u_3^2(\rho+1)(\rho-\tau) = 0.$$

Now we claim that $u_3 = 0$. Arguing indirectly, we suppose that $u_3 \neq 0$ on an open subset. Observe that $\rho \neq -1$ due to our assumption and (6). Hence, $\rho = \tau$, or equivalently $\rho n^2H^2 = \lambda^2(\rho-1)^2$ and $e_3(\rho) = 0$ by (34). Thus, $e_3(\lambda) = 0$, which contradicts (33) since $\lambda \neq 0$. This proves the claim that $u_3 = 0$ and consequently the splitting tensor vanishes. That the immersion *f* is locally a cylinder on M_2^* follows from Proposition 12.

If the open subset M_1^* is nonempty, then Lemma 18 implies that the immersion f is locally a cylinder over a surface or over a curve. □

Proposition 20 *Let $f : M^3 \to \mathbb{Q}_c^{3+p}$ be a nonminimal isometric immersion such that M^3 carries a totally geodesic distribution D of rank one satisfying $D(x) \subseteq \Delta_f(x)$ for any $x \in M^3$. If the mean curvature of f is constant along each integral curve of D and f is not locally a cylinder, then the splitting tensor of f is an almost complex structure on $D^\perp$. Moreover, f is a spherical elliptic submanifold with respect to this almost complex structure and its first curvature ellipse is a circle.*

Proof Since by assumption the immersion *f* is not a cylinder on any open subset, it follows from Proposition 12, Lemmas 16 and 18 that the open subsets M_3^* and M_1^* are both empty.

Proposition 19 implies that the immersion *f* has nonflat normal bundle on M_2^*. Thus, we have $\mu \neq 0$ and $\rho \neq -1$. Using (20) and (21), it is easily seen that (10), (14), (17), (19), (22) and (23) are written as

$$\begin{aligned} \omega(e_3) &= -\frac{\rho-\tau}{\rho+1}v_3, \\ e_3(\rho) &= \frac{\lambda}{\mu}(\rho-1)(\rho-\tau)v_3, \end{aligned} \tag{35}$$

$$\begin{aligned} e_3(\mu) &= -\frac{\lambda}{\rho+1}\big(2\tau+\rho^2+1\big)v_3, \\ e_3(\lambda) &= \Big(\frac{2\mu}{\rho+1}\tau - \frac{2\mu\rho}{\rho+1} - \frac{\lambda^2}{\mu}(\rho+1)\Big)v_3, \end{aligned} \tag{36}$$

$$\begin{aligned} e_3(v_3) &= \frac{\lambda}{\mu(\rho+1)}\big((\rho-1)\tau - (2\rho^2+\rho+1)\big)v_3^2, \\ (\lambda^2+\mu^2)&(2\rho-\tau)v_3^2 = c\mu^2, \end{aligned} \tag{37}$$

where τ is the function given by

$$\tau = (\lambda^2+\mu^2)\frac{(\rho-1)^2}{n^2H^2}.$$

By differentiating (37) and using all the above equations, we obtain

$$\lambda(\lambda^2+\mu^2)\Big(\rho(5\rho^2+6\rho+5)-(4\rho^2+2\rho+4)\tau-2\tau^2\Big)v_3^3=c\lambda\mu^2v_3.$$

We claim that $\lambda v_3=0$. Arguing indirectly, we assume that the open subset where $\lambda v_3\neq 0$ is nonempty. Thus, comparing the above equation with (37), we derive that $\tau=\rho$. This along with (35) implies that $e_3(\tau)=e_3(\rho)=0$. By the definition of τ, it follows that $e_3(\lambda^2+\mu^2)=0$. Using the above equations, it is easy to see that

$$e_3(\lambda^2+\mu^2)=-2\frac{\lambda}{\mu}(\lambda^2+\mu^2)(\rho+1)v_3,$$

which is a contradiction and this proves our claim.

Now we claim that v_3 cannot vanish on any open subset. Arguing indirectly, we suppose that $v_3=0$ on an open subset. Then (20) implies that $u_3=0$. By Lemma 14, the splitting tensor vanishes and consequently the immersion f would be a cylinder by Proposition 12. This contradicts our assumption.

Since we already proved that $\lambda v_3=0$, we obtain $\lambda=0$ and (20) implies that $u_3=0$. It follows from (36) that

$$\mu^2=\frac{\rho n^2H^2}{(\rho-1)^2}. \tag{38}$$

In particular, we have $\rho>0$. This, along with (37) yields

$$\rho v_3^2=c. \tag{39}$$

Hence, $c=1$. Now observe that the splitting tensor satisfies $\mathcal{C}_3^2=-I$, where I is the identity endomorphism of $D^\perp$, that is, $\mathcal{C}_3$ is an almost complex structure $J:D^\perp\to D^\perp$. Using (39) and the fact that the shape operator A_{ξ_5} satisfies $A_{\xi_5}e_i=\mu e_j$ for $i\neq j=1,2$, we easily verify that the second fundamental form of f satisfies $\alpha^f(Je_1,e_2)=\alpha^f(e_1,Je_2)$. This is equivalent to the ellipticity of the immersion f.

In order to prove that the first curvature ellipse of f is a circle, it is equivalent to prove that the vector fields $\alpha^f(e_1,e_1)$ and $\alpha^f(e_1,Je_1)$ are of the same length and perpendicular. Obviously, they are perpendicular since

$$\alpha^f(e_1,e_1)=k_1\xi_4 \ \text{ and } \ \alpha^f(e_1,Je_1)=\mu v_3\xi_5.$$

Using (5) and (38), we obtain

$$\frac{\|\alpha^f(e_1,Je_1)\|^2}{\|\alpha^f(e_1,e_1)\|^2}=\rho v_3^2.$$

Bearing in mind (39), we conclude that the first curvature ellipse is a circle. □

The following result parametrizes all three-dimensional submanifolds in spheres that carry a totally geodesic distribution of rank one, contained in the relative nullity distribution, such that the mean curvature is constant along each integral curve. This parametrization, given in terms of their polar surfaces, was introduced in [5] as the *polar parametrization*.

Theorem 21 *Let* $h:L^2\to\mathbb{Q}_c^{N+1}, c\in\{0,1\}, N\geq 5$, *be a nicely curved elliptic surface of substantial even codimension, such that the curvature ellipses* $\mathcal{E}^h_{\tau_h-2},\mathcal{E}^h_{\tau_h}$ *are circles and*

$\mathcal{E}^h_{\tau_h-1}$ is nowhere a circle. Then, the map $\Psi_h : M^3 \to \mathbb{S}^{N+c}$ defined on the circle bundle $M^3 = UN^h_{\tau_h} = \{(x,w) \in N^h_{\tau_h} : \|w\| = 1\}$ by $\Psi_h(x,w) = w$ is a nonminimal elliptic isometric immersion with polar surface h. Moreover, M^3 carries a totally geodesic distribution D of rank one satisfying $D(p) \subseteq \Delta_{\Psi_h}(p)$ for any $p \in M^3$ such that the mean curvature of Ψ_h is constant along each integral curve of D.

Conversely, let $f : M^3 \to \mathbb{S}^{3+p}, p \geq 2$, be a substantial nonminimal isometric immersion such that M^3 carries a totally geodesic distribution D of rank one satisfying $D(x) \subseteq \Delta_f(x)$ for any $x \in M^3$. If the mean curvature of f is constant along each integral curve of D, then f is elliptic and there exists an open dense subset of M^3 such that for each point there exist a neighborhood U, and a local isometry $F : U \to UN^h_{\tau_h}$ such that $f = \Psi_h \circ F$, where h is a polar surface to f with curvature ellipses as above.

Proof Let $h : L^2 \to \mathbb{Q}^{N+1}_c, c \in \{0,1\}$, be a substantial elliptic surface, where $N = 2m+3, m \geq 1$. Choose a local orthonormal frame e_1, e_2 in the tangent bundle of L^2 such that the almost complex structure J of the elliptic surface is given by

$$Je_1 = be_2 \ \text{ and } \ Je_2 = -\frac{1}{b}e_1,$$

where b is a positive smooth function.

We argue for the case where $m \geq 2$. The case where $m = 1$ can be handled in a similar manner. We know from (30) that the normal bundle splits orthogonally as

$$N_hL = N^h_1 \oplus \cdots \oplus N^h_{m-1} \oplus N^h_m \oplus N^h_{m+1}.$$

Let $\zeta_3, \ldots, \zeta_{2m+4}$ be an orthonormal frame in the normal bundle, defined on an open subset $V \subseteq L^2$, such that $\zeta_{2s+1}, \zeta_{2s+2}$ span the plane subbundle N^h_s for any $1 \leq s \leq m+1$. The corresponding normal connection forms $\omega_{\alpha\beta}$ are given by $\omega_{\alpha\beta} = \langle \nabla^\perp \zeta_\alpha, \zeta_\beta \rangle, \alpha, \beta = 3, \ldots, 2m+4$.

Due to our hypothesis, we may choose the frame such that

$$\alpha^h_m(e_1, \ldots, e_1) = \kappa_{m-1}\zeta_{2m-1}, \quad \alpha^h_m(e_1, \ldots, e_1, e_2) = \frac{\kappa_{m-1}}{b}\zeta_{2m}$$

and

$$\alpha^h_{m+2}(e_1, \ldots, e_1) = \kappa_{m+1}\zeta_{2m+3}, \quad \alpha^h_{m+2}(e_1, \ldots, e_1, e_2) = \frac{\kappa_{m+1}}{b}\zeta_{2m+4},$$

where $\kappa_{m-1}, \kappa_{m+1}$ denote the radii of the circular curvature ellipses $\mathcal{E}^h_{m-1}, \mathcal{E}^h_{m+1}$, respectively. Since the curvature ellipse $\mathcal{E}^h_m$ is nowhere a circle, we may choose $\zeta_{2m+1}, \zeta_{2m+2}$ to be collinear to the major and minor axes of this ellipse, respectively. Thus, we may write

$$\alpha^h_{m+1}(e_1, \ldots, e_1) = v_{11}\zeta_{2m+1} + v_{12}\zeta_{2m+2} \quad \text{and} \quad \alpha^h_{m+1}(e_1, \ldots, e_1, e_2) = v_{21}\zeta_{2m+1} + v_{22}\zeta_{2m+2},$$

where v_{ij} are smooth functions such that

$$b^2 v_{21}v_{22} + v_{11}v_{12} = 0, \quad \kappa_m = \left(v^2_{11} + b^2 v^2_{21}\right)^{1/2}, \quad \mu_m = \left(v^2_{12} + b^2 v^2_{22}\right)^{1/2} \tag{40}$$

and κ_m, μ_m denote the lengths of the semi-axes of the curvature ellipse $\mathcal{E}^h_m$.

Bearing in mind the definition of the higher fundamental forms, their symmetry and the ellipticity of the surface h, we have

$$\alpha^h_{s+1}\big(e_1,\dots,e_1,e_2\big)=\Big(\nabla^\perp_{e_2}\alpha^h_s\big(e_1,\dots,e_1\big)\Big)^{N^h_s}=\Big(\nabla^\perp_{e_1}\alpha^h_s\big(e_1,\dots,e_1,e_2\big)\Big)^{N^h_s},$$
$$\alpha^h_{s+1}\big(e_1,\dots,e_1\big)=-b^2\Big(\nabla^\perp_{e_2}\alpha^h_s\big(e_1,\dots,e_1,e_2\big)\Big)^{N^h_s}=\Big(\nabla^\perp_{e_1}\alpha^h_s\big(e_1,\dots,e_1\big)\Big)^{N^h_s}$$

for $s=m,m+1$, where $(\cdot)^{N^h_s}$ denotes taking the projection onto the normal subbundle N^h_s. From these, we obtain

$$\omega_{2m-1,2m+1}(e_1)=\frac{v_{11}}{\kappa_{m-1}},\quad \omega_{2m-1,2m+2}(e_1)=\frac{v_{12}}{\kappa_{m-1}},\tag{41}$$

$$\omega_{2m-1,2m+1}(e_2)=\frac{v_{21}}{\kappa_{m-1}},\quad \omega_{2m-1,2m+2}(e_2)=\frac{v_{22}}{\kappa_{m-1}},\tag{42}$$

$$\omega_{2m,2m+1}(e_1)=\frac{bv_{21}}{\kappa_{m-1}},\quad \omega_{2m,2m+2}(e_1)=\frac{bv_{22}}{\kappa_{m-1}},\tag{43}$$

$$\omega_{2m,2m+1}(e_2)=-\frac{v_{11}}{b\kappa_{m-1}},\quad \omega_{2m,2m+2}(e_2)=-\frac{v_{12}}{b\kappa_{m-1}},\tag{44}$$

$$\omega_{2m+1,2m+3}(e_1)=\frac{b\kappa_{m+1}}{\kappa_m\mu_m}v_{22},\quad \omega_{2m+1,2m+3}(e_2)=\frac{\kappa_{m+1}}{b\kappa_m\mu_m}v_{12},\tag{45}$$

$$\omega_{2m+1,2m+4}(e_1)=-\frac{\kappa_{m+1}}{\kappa_m\mu_m}v_{12},\quad \omega_{2m+1,2m+4}(e_2)=\frac{\kappa_{m+1}}{\kappa_m\mu_m}v_{22},\tag{46}$$

$$\omega_{2m+2,2m+3}(e_1)=-\frac{b\kappa_{m+1}}{\kappa_m\mu_m}v_{21},\quad \omega_{2m+2,2m+3}(e_2)=-\frac{\kappa_{m+1}}{b\kappa_m\mu_m}v_{11},\tag{47}$$

$$\omega_{2m+2,2m+4}(e_1)=\frac{\kappa_{m+1}}{\kappa_m\mu_m}v_{11},\quad \omega_{2m+2,2m+4}(e_2)=-\frac{\kappa_{m+1}}{\kappa_m\mu_m}v_{21}.\tag{48}$$

Let $\Pi : M^3\to L^2$ the natural projection of the circle bundle

$$M^3=UN^h_{\tau_h}=\big\{(x,\delta)\in N^h_{m+1} : \|\delta\|=1, x\in L^2\big\}.$$

We parametrize $\Pi^{-1}(V)$ by $V\times\mathbb{R}$ via the map

$$(x,\theta)\mapsto\big(x,\cos\theta\zeta_{2m+3}(x)+\sin\theta\zeta_{2m+4}(x)\big)$$

and consequently, we have

$$\Psi_h(x,\theta)=\cos\theta\zeta_{2m+3}+\sin\theta\zeta_{2m+4}.$$

Notice that $\nabla^\perp N^h_{m+1}\subseteq N^h_m\oplus N^h_{m+1}$. It is easily seen that

$$\Psi_{h_*}E_i = \big(\cos\theta\omega_{2m+3,2m+1}(e_i) + \sin\theta\omega_{2m+4,2m+1}(e_i)\big)\zeta_{2m+1} + \big(\cos\theta\omega_{2m+3,2m+2}(e_i) + \sin\theta\omega_{2m+4,2m+2}(e_i)\big)\zeta_{2m+2},$$

where the vector fields $E_i \in TM, i = 1, 2$, are given by

$$E_i = e_i - \omega_{2m+3,2m+4}(e_i)\frac{\partial}{\partial\theta}.$$

Using (45)-(48), we obtain

$$\Psi_{h_*}E_1 = \frac{\kappa_{m+1}}{\kappa_m\mu_m}\Big(\big(-bv_{22}\cos\theta + v_{12}\sin\theta\big)\zeta_{2m+1} + \big(bv_{21}\cos\theta - v_{11}\sin\theta\big)\zeta_{2m+2}\Big) \tag{49}$$

and

$$\Psi_{h_*}E_2 = \frac{\kappa_{m+1}}{\kappa_m\mu_m}\Big(-\big(\frac{v_{12}}{b}\cos\theta + v_{22}\sin\theta\big)\zeta_{2m+1} + \big(\frac{v_{11}}{b}\cos\theta + v_{21}\sin\theta\big)\zeta_{2m+2}\Big). \tag{50}$$

Additionally, we have

$$\Psi_{h_*}(\partial/\partial\theta) = -\sin\theta\zeta_{2m+3} + \cos\theta\zeta_{2m+4}. \tag{51}$$

It follows that the normal bundle of the isometric immersion Ψ_h is given by

$$N_{\Psi_h}M = c\ \mathrm{span}\{h\} \oplus N_1^h \oplus \cdots \oplus N_{m-2}^h \oplus N_{m-1}^h.$$

It is easy to see that the first normal bundle of Ψ_h is $N_1^{\Psi_h} = N_{m-1}^h$. Moreover, it follows easily that the distribution $D = \mathrm{span}\{\partial/\partial\theta\}$ is contained in the nullity distribution Δ_{Ψ_h} of Ψ_h. In particular, from (51) and the Gauss formula we derive that $\nabla_{\partial/\partial\theta}\partial/\partial\theta = 0$. This implies that the distribution D is totally geodesic.

It remains to show that the mean curvature of the immersion Ψ_h is constant along each integral curve of D. The shape operator $A_{\zeta_{2m-j}}$ of Ψ_h with respect to the normal direction $\zeta_{2m-j}, j = 0, 1$, is given by the Weingarten formula as

$$-\Psi_{h_*}\Big(A_{\zeta_{2m-j}}E_i\Big) = \nabla^{\perp}_{e_i}\zeta_{2m-j} - \Big(\tilde\nabla_{e_i}\zeta_{2m-j}\Big)^{N_{m-2}^h\oplus N_{m-1}^h} = \Big(\nabla^{\perp}_{e_i}\zeta_{2m-j}\Big)^{N_m^h}, \quad i = 1, 2, \tag{52}$$

since $\zeta_{2m-1}, \zeta_{2m} \in N_{m-1}^h$. Here, $\tilde\nabla$ stands for the induced connection of the induced bundle $h^*T\mathbb{Q}_c^{N+1}$. It follows from (52) using (41)-(44) that

$$\Psi_{h_*}\big(A_{\zeta_{2m-1}}E_1\big) = -\frac{1}{\kappa_{m-1}}\big(v_{11}\zeta_{2m+1} + v_{12}\zeta_{2m+2}\big), \tag{53}$$

$$\Psi_{h_*}\big(A_{\zeta_{2m-1}}E_2\big) = -\frac{1}{\kappa_{m-1}}\big(v_{21}\zeta_{2m+1} + v_{22}\zeta_{2m+2}\big), \tag{54}$$

$$\Psi_{h_*}\big(A_{\zeta_{2m}}E_1\big) = -\frac{b}{\kappa_{m-1}}\big(v_{21}\zeta_{2m+1} + v_{22}\zeta_{2m+2}\big), \tag{55}$$

$$\Psi_{h_*}\left(A_{\zeta_{2m}}E_2\right) = \frac{1}{b\kappa_{m-1}}\left(v_{11}\zeta_{2m+1} + v_{12}\zeta_{2m+2}\right). \tag{56}$$

We may set

$$A_{\zeta_{2m-1}}E_i = \lambda_{i1}E_1 + \lambda_{i2}E_2 \quad \text{and} \quad A_{\zeta_{2m}}E_i = \gamma_{i1}E_1 + \gamma_{i2}E_2, \;\; i = 1, 2, \tag{57}$$

where λ_{ij} and γ_{ij} are smooth functions on the manifold M^3. From (49), (53), (54) and the first one of (57), we obtain

$$\lambda_{11} = \frac{1}{\kappa_{m-1}\kappa_{m+1}}\left(\left(v_{11}^2 + v_{12}^2\right)\cos\theta + b\left(v_{11}v_{21} + v_{12}v_{22}\right)\sin\theta\right)$$

and

$$\lambda_{22} = \frac{1}{\kappa_{m-1}\kappa_{m+1}}\left(-b^2\left(v_{21}^2 + v_{22}^2\right)\cos\theta + b\left(v_{11}v_{21} + v_{12}v_{22}\right)\sin\theta\right).$$

Hence

$$\mathrm{trace}A_{\zeta_{2m-1}} = \frac{1}{\kappa_{m-1}\kappa_{m+1}}\left(\left(v_{11}^2 + v_{12}^2 - b^2v_{21}^2 - b^2v_{22}^2\right)\cos\theta + 2b\left(v_{11}v_{21} + v_{12}v_{22}\right)\sin\theta\right).$$

Similarly, from (50), (55), (56) and the second of (57), we find that

$$\gamma_{11} = \frac{1}{\kappa_{m-1}\kappa_{m+1}}\left(b\left(v_{11}v_{21} + v_{12}v_{22}\right)\cos\theta + b^2\left(v_{21}^2 + v_{22}^2\right)\sin\theta\right)$$

and

$$\gamma_{22} = \frac{1}{\kappa_{m-1}\kappa_{m+1}}\left(b\left(v_{11}v_{21} + v_{12}v_{22}\right)\cos\theta - \left(v_{11}^2 + v_{12}^2\right)\sin\theta\right).$$

Then, it follows that

$$\mathrm{trace}A_{\zeta_{2m}} = \frac{1}{\kappa_{m-1}\kappa_{m+1}}\left(2b\left(v_{11}v_{21} + v_{12}v_{22}\right)\cos\theta - \left(v_{11}^2 + v_{12}^2 - b^2v_{21}^2 - b^2v_{22}^2\right)\sin\theta\right).$$

Thus, the mean curvature of the isometric immersion Ψ_h is given by

$$\|\mathcal{H}_{\Psi_h}\|^2 = \frac{1}{(3\kappa_{m-1}\kappa_{m+1})^2}\left(\left(v_{11}^2 + v_{12}^2 + b^2v_{21}^2 + b^2v_{22}^2\right)^2 - 4\left(v_{11}^2 + b^2v_{21}^2\right)^2\left(v_{12}^2 + b^2v_{22}^2\right)^2\right).$$

Using (40), the above equation becomes

$$\|\mathcal{H}_{\Psi_h}\| = \frac{|\kappa_m^2 - \mu_m^2|}{3\kappa_{m-1}\kappa_{m+1}}.$$

It is clear that the mean curvature of the isometric immersion Ψ_h is constant along each integral curve of the distribution D. This completes the proof of the direct statement of the theorem for $m \geq 2$. The case $m = 1$ can be treated in a similar manner. In this case, the mean curvature of Ψ_h is given by

$$\|\mathcal{H}_{\Psi_h}\| = \frac{|\kappa_1^2 - \mu_1^2|}{3\kappa_2^2}.$$

Conversely, let $f : M^3 \to \mathbb{S}^{3+p}$ be a nonminimal isometric immersion. Suppose that M^3 carries a totally geodesic distribution D of rank one satisfying $D(x) \subseteq \Delta_f(x)$ for any $x \in M^3$ such that the mean curvature is constant along each integral curve of D. From Proposition 20, we know that f is an elliptic submanifold and its first curvature ellipse is a circle. Hereafter, we work on a connected component of an open dense subset where f is nicely curved.

Consider a polar surface $h : L^2 \to \mathbb{Q}_c^{p-c+4}$ to the immersion f, where $c = 0$ if p is even and $c = 1$ if p is odd. Notice that $\tau_f^0 = \tau_h - 1$. Using (31), we conclude that the curvature ellipse $\mathcal{E}^h_{\tau_h-2}$ of the surface h is a circle and the curvature ellipse $\mathcal{E}^h_{\tau_h-1}$ is nowhere a circle.

We claim that the last curvature ellipse $\mathcal{E}^h_{\tau_h}$ is a circle. Observe that $N^h_{\tau_h} = \mathrm{span}\{\xi, \eta\}$, where the sections ξ, η of the normal bundle N_hL are given by $\xi = f \circ \pi$ and $\eta = f_* e_3 \circ \pi$. Here π denotes the natural projection $\pi : M^3 \to L^2$ onto the fixed cross section $L^2 \subset M^3$ to the foliation generated by the distribution D.

Let $X_1, \dots, X_{\tau_h} \in TL$ be arbitrary vector fields. By (31), we have $N^h_{\tau_h-1} = N^f_0 = f_* D^\perp$. Thus, there exists $X \in \Gamma(D^\perp)$ such that

$$\alpha^h_{\tau_h}\left(X_1, \dots, X_{\tau_h}\right) = f_* X.$$

For every vector field $Y \in TL$ there exists a vector field $Z \in \Gamma(D^\perp)$ such that $Y = \pi_* Z$. Then we have

$$\begin{aligned}\alpha^h_{\tau_h+1}(X_1, \dots, X_{\tau_h}, Y) &= \left(\nabla^\perp_Y \alpha^h_{\tau_h}(X_1, \dots, X_{\tau_h})\right)^{N^h_{\tau_h}} \\ &= -\langle f_* X, f_* Z\rangle \xi - \langle f_* X, \tilde{\nabla}_Z f_* e_3\rangle \eta.\end{aligned}$$

Using the Gauss formula and the definition of the splitting tensor, the above equation becomes

$$\alpha^h_{\tau_h+1}(X_1, \dots, X_{\tau_h}, Y) = -\langle X, Z\rangle \xi + \langle X, \mathcal{C}_3 Z\rangle \eta.$$

From Proposition 20, we know that the splitting tensor in the direction of e_3 is the almost complex structure $J^f_0 : D^\perp \to D^\perp$ of f. Hence, we obtain

$$\alpha^h_{\tau_h+1}(X_1, \dots, X_{\tau_h}, Y) = -\langle X, Z\rangle \xi + \langle X, J^f_0 Z\rangle \eta.$$

On account of $\pi_* \circ J^f_0 = J^h_0 \circ \pi_*$, we have $J^h_0 Y = \pi_* J^f_0 Z$. Thus, it follows that

$$\alpha^h_{\tau_h+1}(X_1, \dots, X_{\tau_h}, J^h_0 Y) = -\langle X, J^h_0 Z\rangle \xi - \langle X, Z\rangle \eta.$$

Since ξ, η is an orthonormal frame of the subbundle $N^h_{\tau_h}$, it is now obvious that the normal vector fields $\alpha^h_{\tau_h+1}(X_1, \dots, X_{\tau_h+1}, Y)$ and $\alpha^h_{\tau_h+1}(X_1, \dots, X_{\tau_h}, J^h_0 Y)$ are of the same length and perpendicular. Hence, the last curvature ellipse of the polar surface h is a circle.

Finally, observe that the isometric immersion f is written as the composition $f = \Psi_h \circ F$, where $F : U \to UN^h_{\tau_h}$ is the local isometry given by $F(x) = (\pi(x), f(x))$, $x \in U$, and U is the saturation of the cross section $L^2 \subset M^3$. □

Remark 22 It follows from the computation of the mean curvature of the submanifold Ψ_h in the proof of Theorem 21, that the mean curvature is constant by properly choosing the elliptic surface h. Ejiri [13] proved that tubes in the direction of the second normal bundle of a pseudoholomorphic curve in the nearly Kähler sphere $\mathbb{S}^6$ have constant mean curvature. Opposed to our case, the index of relative nullity of these tubes is zero.

Proof of Theorem 2 Assume that the isometric immersion f is neither minimal nor locally a cylinder. Proposition 20 implies that f is spherical. Thus, from Theorem 21 we know that for each point on an open dense subset there exist an elliptic surface $h : L^2 \to \mathbb{Q}_c^{p-c+4}$, where $c = 0$ if p is even and $c = 1$ if p is odd, a neighborhood U and a local isometry $F : U \to UN^h_{\tau_h}$ such that $f = \Psi_h \circ F$. In fact, the elliptic surface h is a polar to f. Moreover, we know that the curvature ellipses $\mathcal{E}^h_{\tau_h-2}$ and $\mathcal{E}^h_{\tau_h}$ are circles, while the curvature ellipse $\mathcal{E}^h_{\tau_h-1}$ is nowhere a circle.

Now consider a bipolar surface g to f, that is, a polar surface to the elliptic surface h. Then it follows from (31) that the curvature ellipse $\mathcal{E}^g_0$ of g is a circle. This means that the bipolar surface is minimal. Furthermore, its first curvature ellipse is nowhere a circle and the second one is a circle. That the isometric immersion f is locally parametrized by (1) follows from the fact that $f = \Psi_h \circ F$ and $N^g_0 = N^h_{\tau_h}$. □

6.1 Minimal surfaces

The following proposition provides a way of constructing minimal surfaces in $\mathbb{R}^6$ that satisfy the properties that are required in part (iii) of Theorem 2.

Proposition 23 *Let $\hat{g} : M^2 \to \mathbb{R}^6$ be the minimal surface defined by*

$$\hat{g} = \cos\varphi g_\theta \oplus \sin\varphi g_{\theta+\pi/2},$$

where $g_\theta, \theta \in [0, \pi)$, is the associated family of a simply connected minimal surface $g : M^2 \to \mathbb{R}^3$ with negative Gaussian curvature, and $\oplus$ denotes the orthogonal sum with respect to an orthogonal decomposition of $\mathbb{R}^6$. If $\varphi \neq \pi/4$, then its first curvature ellipse is nowhere a circle and its second curvature ellipse is a circle.

Let $g : M \to \mathbb{R}^n$ be an oriented minimal surface. The complexified tangent bundle $TM \otimes \mathbb{C}$ is decomposed into the eigenspaces $T'M$ and $T''M$ of the complex structure J, corresponding to the eigenvalues i and $-i$. The r-th fundamental form α^g_r, which takes values in the normal subbundle N^g_{r-1}, can be complex linearly extended to $TM \otimes \mathbb{C}$ with values in the complexified vector bundle $N^g_{r-1} \otimes \mathbb{C}$ and then decomposed into its (p, q)-components, $p + q = r$, which are tensor products of p differential 1-forms vanishing on $T''M$ and q differential 1-forms vanishing on $T'M$. The minimality of g is equivalent to the vanishing of the $(1, 1)$-component of the second fundamental form. Hence, the (p, q)-components of α^g_r vanish unless $p = r$ or $p = 0$.

It is known (see [30, Lem. 3.1]) that the curvature ellipse of order $r - 1$ is a circle if and only if the $(r, 0)$-component of α^g_r is isotropic, that is

$$\langle \alpha^g_r(X, \ldots, X), \alpha^g_r(X, \ldots, X)\rangle = 0$$

for any $X \in T'M$, where $\langle \cdot, \cdot \rangle$ denotes the bilinear extension over the complex numbers of the Euclidean metric.

Proof of Proposition 23 Choose a local tangent orthonormal frame e_1, e_2 such that the shape operator A of g satisfies $AE = k\bar{E}$, where $E = e_1 + ie_2$ and k is a positive smooth function. The associated family satisfies $g_{\theta_*} = g_* \circ J_\theta$, where $J_\theta = \cos\theta I + \sin\theta J$ and I is the identity endomorphism of the tangent bundle. Then we have

$$\hat{g}_* E = e^{-i\theta}\big(\cos\varphi g_* E, -i\sin\varphi g_* E\big). \tag{58}$$

Using the Gauss formula and the fact that the shape operator A_θ of g_θ is given by $A_\theta = A \circ J_\theta$, we find that the second fundamental form $\hat{\alpha}$ of $\hat{g}$ satisfies

$$\hat{\alpha}(E,E) = 2ke^{-i\theta}(\cos\varphi N, -i\sin\varphi N), \tag{59}$$

where N is the Gauss map of g. It is obvious that $\hat{\alpha}(E,E)$ is not isotropic if $\varphi \neq \pi/4$, which implies that the first curvature ellipse of $\hat{g}$ is nowhere a circle.

Differentiating (59) with respect to E and using the Weingarten formula, we obtain

$$\tilde{\nabla}_E \hat{\alpha}(E,E) = 2e^{-i\theta}E(k)(\cos\varphi N, -i\sin\varphi N) - 2k^2 e^{-i\theta}\big(\cos\varphi g_* \bar{E}, -i\sin\varphi g_* \bar{E}\big),$$

where $\tilde{\nabla}$ is the connection of the induced bundle of $\hat{g}$. Since $\hat{g}_* E$ and $\hat{g}_* \bar{E}$ span $N_0^{\hat{g}} \otimes \mathbb{C}$, the above equation along with (58) yield

$$\big(\tilde{\nabla}_E \hat{\alpha}(E,E)\big)^{N_0^{\hat{g}} \otimes \mathbb{C}} = -2k^2 e^{-2i\theta}\cos 2\varphi \hat{g}_* \bar{E}.$$

It follows using (59) that $N_1^{\hat{g}} \otimes \mathbb{C} = \mathrm{span}_{\mathbb{C}}\{\xi, \eta\}$, where $\xi = (N, 0)$ and $\eta = (0, iN)$. Then, we find that

$$\big(\tilde{\nabla}_E \hat{\alpha}(E,E)\big)^{N_1^{\hat{g}} \otimes \mathbb{C}} = 2e^{-i\theta}E(k)(\cos\varphi N, -i\sin\varphi N).$$

Using the above and since the (3, 0)-component of the third fundamental form of $\hat{g}$ is given by

$$\hat{\alpha}_3(E,E,E) = \big(\tilde{\nabla}_E \hat{\alpha}(E,E)\big)^{\left(N_0^{\hat{g}} \otimes \mathbb{C} \oplus N_1^{\hat{g}} \otimes \mathbb{C}\right)^\perp},$$

we obtain

$$\hat{\alpha}_3(E,E,E) = k^2 e^{-i\theta}\sin 2\varphi\big(-\sin\varphi g_* \bar{E}, i\cos\varphi g_* \bar{E}\big).$$

Thus, the (3,0)-component of the third fundamental form of $\hat{g}$ is isotropic, and consequently the second curvature ellipse is a circle. □

7 Submanifolds with constant mean curvature

In this section, we provide the proofs of the applications of our main results to submanifolds with constant mean curvature.

Proof of Theorem 3 The manifold M^n is the disjoint union of the subsets

$$M_{n-i} = \{x \in M^n : \nu(x) = n - i\}, \quad i = 1, 2.$$

Assume that the subset M_{n-2} is nonempty. Then, using Proposition 13 it follows from Theorem 1 for $n \geq 4$, or Theorem 2 for $n = 3$ and $p = 1$, that the isometric immersion f is locally a cylinder over a surface on M_{n-2}.

Suppose that the interior $\text{int}(M_{n-1})$ of the subset M_{n-1} is nonempty. It follows from the Codazzi equation that the relative nullity distribution is parallel in the tangent bundle along $\text{int}(M_{n-1})$. Thus, the tangent bundle splits as an orthogonal sum of two parallel orthogonal distributions of rank one and $n-1$ on $\text{int}(M_{n-1})$. By the De Rham decomposition theorem, $\text{int}(M_{n-1})$ splits locally as a Riemannian product of two manifolds of dimension one and $n-1$. Then, the Gauss equation yields $c = 0$. Since the second fundamental form is adapted to the orthogonal decomposition of the tangent bundle, it follows that f is a cylinder over a curve in $\mathbb{R}^{p+1}$ with constant first Frenet curvature (see [12, Th. 8.4]).

Finally, observe that the open subset $V = \text{int}(M_{n-1}) \cup M_{n-2}$ is dense on M^n. □

In order to proceed to the proofs of the applications of our main results, we need to recall Florit's estimate of the index of relative nullity for isometric immersions with nonpositive extrinsic curvature. The *extrinsic curvature* of an isometric immersion $f : M^n \to \tilde{M}^{n+p}$ for any point $x \in M^n$ and any plane $\sigma \subset T_xM$ is given by

$$K_f(\sigma) = K_M(\sigma) - K_{\tilde{M}}(f_*\sigma),$$

where K_M and $K_{\tilde{M}}$ are the sectional curvatures of M^n and $\tilde{M}^{n+p}$, respectively. Florit [15] proved that the index of relative nullity satisfies $\nu \geq n - 2p$ at points where the extrinsic curvature of f is nonpositive.

Proof of Corollary 5 We have that the index of relative nullity of f satisfies $\nu \geq n-2$. Theorem 3 implies that $c = 0$ and, on an open dense subset, f splits locally as a cylinder over a surface in $\mathbb{R}^3$ of constant mean curvature. By real analyticity, the splitting is global. If M^n is complete, then the surface is also complete with nonnegative Gaussian curvature. That the surface is a cylinder over a circle follows from [24]. □

Proof of Corollary 6 Assume that the hypersurface is nonrigid. Then, the well-known Beez-Killing Theorem (see [12]) implies that the index of relative nullity satisfies $\nu \geq n-2$. The result follows from Corollary 4. □

Proof of Theorem 7 Suppose that the hypersurface is nonminimal.

At first assume that the extrinsic curvature is nonnegative. If $c = 0$, a result of Hartman [22] asserts that $f(M^n) = \mathbb{S}^k_R \times \mathbb{R}^{n-k}$, where $1 \leq k \leq n$. If $c = 1$, then M^n is compact by the Bonnet-Myers theorem. According to [28, Th. 2], f is totally umbilical.

In the case of nonpositive extrinsic curvature, the result follows from Corollary 5. □

Proof of Theorem 8 According to the aforementioned result due to Florit [15], we have $\nu \geq n-4$. Clearly the manifold M^n is the disjoint union of the subsets

$$M_{n-i} = \{x \in M^n : \nu(x) = n - i\}, \quad i = 1, 2, 3.$$

We distinguish the following cases.

Case I: Suppose that the subset M_{n-4} is nonempty. According to Proposition 13, this subset is open. Using [16, Th. 1], we have that on an open dense subset of M_{n-4} the immersion f is locally a product $f = f_1 \times f_2$ of two hypersurfaces $f_i : M^{n_i} \to \mathbb{R}^{n_i+1}, i = 1, 2,$ of nonpositive sectional curvature. The assumption that f has constant mean curvature implies that both hypersurfaces have constant mean curvature as well. Each hypersurface $f_i, i = 1, 2,$ has index of relative nullity $n_i - 2$. Then, it follows from Corollary 4 that the submanifold is locally as in part (iii) of the theorem.

Case II: Suppose that the interior of the subset M_{n-3} is nonempty. Due to [17, Th. 1], on an open dense subset of $\text{int}(M_{n-3})$, f is written locally as a composition $f = h \circ F$, where $h = \gamma \times id_{\mathbb{R}^{n-1}} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n+2}$ is cylinder over a unit speed plane curve $\gamma(s)$ with nonvanishing curvature $k(s)$ and $F : M^n \to \mathbb{R}^{n+1}$ is a hypersurface. The second fundamental form of f is given by

$$\alpha^f(X, Y) = h_* \alpha^F(X, Y) + \alpha^h\big(F_* X, F_* Y\big), \quad X, Y \in TM.$$

From this, we obtain $k\langle F_* T, \partial/\partial s\rangle^2 = 0$ for any $T \in \Delta_f$. This implies that the height function $F_a = \langle F, a\rangle$ relative to $a = \partial/\partial s$ is constant along the leaves of Δ_f. Then, the mean curvature vector field of f is given by

$$n\mathcal{H}_f = nH_F h_* \xi + k \circ F_a \|\text{grad} F_a\|^2 \eta,$$

where ξ, η stand for the Gauss maps of F and h, respectively. Using that

$$\|\text{grad} F_a\|^2 = 1 - \langle \xi, a\rangle^2,$$

it follows that the mean curvature of F is given as in part (ii) of the theorem.

Case III: Suppose that the subset $M_{n-2} \cup M_{n-1}$ has nonempty interior. Then Theorem 3 implies that the submanifold is locally as in part (i) of the theorem. □

Proof of Theorem 9 It follows from [12, Th. 5.1] that $\tilde{c} \geq c$ if $n \geq 4$. We distinguish the following cases.

Case I: Suppose that $\tilde{c} > c$. From [10, Prop. 9] or [29, Lem. 8], we have that the second fundamental form splits orthogonally and smoothly as

$$\alpha^f(\cdot, \cdot) = \beta(\cdot, \cdot) + \sqrt{\tilde{c} - c}\, \langle \cdot, \cdot \rangle \eta,$$

where η is a unit normal vector field and β is a flat bilinear form. Thus, the shape operator A_ξ, associated to a unit normal vector field ξ perpendicular to η, has $\text{rank} A_\xi \leq 1$. The mean curvature H of f is given by

$$H^2 = \frac{k^2}{n^2} + \frac{\tilde{c} - c}{n},$$

where $k = \text{trace} A_\xi$. Obviously, the function k is constant. If $k = 0$, then f is totally umbilical.

Assume now that $k \neq 0$. Let X be a unit vector field such that $A_\xi X = kX$. The Codazzi equation

$$(\nabla_X A_\eta)T - (\nabla_T A_\eta)X = A_{\nabla^\perp_X \eta} T - A_{\nabla^\perp_T \eta} X$$

implies that

$$\nabla^\perp_T \xi = \nabla^\perp_T \eta = 0$$

for any $T \in \ker A_\xi$. Moreover, from the Codazzi equation

$$(\nabla_X A_\xi)T - (\nabla_T A_\xi)X = A_{\nabla^\perp_X \xi} T - A_{\nabla^\perp_T \xi} X$$

it follows that

$$\nabla_X X = 0 \ \text{ and } \ \langle \nabla_S T, X\rangle = \frac{1}{k}\sqrt{\tilde{c} - c}\langle \nabla^\perp_T \xi, \eta\rangle\langle T, S\rangle$$

for any $T, S \in \ker A_\xi$. Hence, the distributions $D^1 = \operatorname{span}\{X\}$ is totally geodesic and $D^{n-1} = \ker A_\xi$ is umbilical. The flatness of the normal bundle implies that D^{n-1} is spherical. Thus the manifold splits locally as a warped product $M^n_{\tilde{c}} = M^1 \times_\rho M^{n-1}$ and f is a warped product of a curve and an umbilical submanifold (see [12, Th. 10.4 and Th. 10.21]). This implies that ρ is constant and the manifold splits locally as a Riemannian product $M^n_{\tilde{c}} = M^1 \times M^{n-1}$. Consequently, we have $\tilde{c} = 0$ and $c = -1$. Clearly M^{n-1} is flat and the second fundamental form is adapted to this decomposition. Then it follows that f is a composition $f = i \circ F$, where $i : \mathbb{R}^{n+1} \to \mathbb{H}^{n+2}$ is the inclusion as a horosphere and $F : M^n_{\tilde{c}} \to \mathbb{R}^{n+1}$ is the cylinder over a circle (see [12, Th. 8.4]).

Case II: Suppose that $c = \tilde{c}$. It is known that $\nu \geq n-2$ (see Example 1 and Corollary 1 in [27]). Then, the result follows from Theorem 3.

If $n = 3$, then Theorem 2 implies that either $c = 0$ and $f(M)$ is an open subset of a cylinder over a flat surface $g : M^2 \to \mathbb{R}^4$ of constant mean curvature, or $c = 1$ and f is parametrized by (1). In the latter case, it follows from Proposition 20 that f is either totally geodesic or elliptic. However, the ellipticity of f implies that the sectional curvature cannot be equal to one. □

Proof of Theorem 10 Assume that f is nonminimal. According to Abe [1], the index of relative nullity satisfies $\nu \geq n-2$. Corollary 4 implies that the hypersurface is a cylinder over a surface with constant mean curvature. □

Proof Using [18, Cor. 2], it follows that $\nu \geq n-4$. The rest of the proof is omitted since it is similar to the proof of Theorem 8. □

The following example produces submanifolds satisfying the conditions in part (ii) of Theorem 8 or 11.

Example 24 Let $F = g \times id_{\mathbb{R}^{n-2}} : U \times \mathbb{R}^{n-2} \to \mathbb{R}^{n+1}$ be a cylinder over a rotational surface $g(x,\theta) = (x\cos\theta, x\cos\theta, \varphi(x)), (x,\theta) \in U$, where $\varphi(x)$ is a smooth function. Consider a cylinder $h = \gamma \times id_{\mathbb{R}^n}$ in $\mathbb{R}^{n+2}$ over a unit speed plane curve γ with curvature k. Then the isometric immersion $f = h \circ F$ satisfies the conditions in part (ii) of Theorems 8 and 11, with constant curvature H and $a = (1, 0, \dots, 0)$, if the function $\varphi(x)$ solves the ordinary differential equation

$$\varphi\varphi'' - 1 - \varphi'^2 = \pm\varphi\sqrt{(1+\varphi'^2)\big(n^2H^2(1+\varphi'^2)^2 - k^2\big)}.$$

In particular, g can be chosen as a Delaunay surface and γ as the curve with curvature $k = c_0(1+\varphi'^2)$ for a constant c_0 such that $0 < |c_0| < n|H|$.

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