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Lower bound of null injectivity radius without curvature assumptions in a family of null cones

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Abstract
We establish a new lower bound for the null injectivity radius of a null cone. The idea is to use a function closely related to the null second fundamental form which codifies the directional expansion of the null cone along any null geodesic in it. This approach uses first derivatives of the metric instead of curvature bounds. The technique is applied to a family of null cones with the null conjugate radius less or equal to the null crossing radius, or folded due to the curvature in a precise way that we call geometric fold. This condition is necessary because there are trivial examples with bounded null injectivity radius due to global identifications. We show two examples where we compute the real null injectivity radius and the lower bound provided in this paper, in order to compare both quantities. We also give an analogous result for Riemannian manifolds.

Keywords Null injectivity radius · Lorentzian manifolds · Null conjugate point · Null crossing point · Null cone

Mathematics Subject Classification 53C50 · 53C22

1 Introduction

The notion of null injectivity radius for null cones in Lorentzian geometry has geometrical interest because it is formally analogous to the Riemannian case and it reflects a combination of curvature and topological properties of both the null cone and the ambient space. For applications, a lower bound of the null injectivity radius allows to improve

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the study of properties of solutions of wave equations in Lorentzian manifolds, and it is important in the study of regularity properties of space-times satisfying the Einstein field equations.

The study of null injectivity radius was initiated recently in [2,5,10,12]. The technique used in all of them is to define a spacelike foliation that covers the null cone. They impose bounds on several quantities related to the leaves of the foliation and codify the lower bound of the null injectivity radius with the parameter of the integral curves of the normal unit vector field of the foliation. So it is necessary to control the geometry of the foliation and the way it influences the estimation of the bound.

The existence of conjugate points in Riemannian and Lorentzian geometry depends on the curvature, so on second derivatives of the metric. The same is true if we restrict to a null cone in Lorentzian geometry. In all cases, they are localized using curvature bounds as in a Schoenberg–Morse-like theorem in Riemannian geometry, [11]. The bounds on the sectional curvature in the Riemannian case are replaced by bounds on the null sectional curvature in the Lorentzian case when we deal with null conjugate points, see [5,6,8,9]. Explicitly,

Theorem 1 [8] *Let M be a Lorentzian manifold and $\gamma : [0, a] \to M$ a null geodesic such that $\gamma(a)$ is the first conjugate point to $\gamma(0)$ along γ. Let $c > 0$ be a constant.*

1. *If $c^2 \leq \mathcal{K}_{\gamma'}(\Pi)$ for all null plane containing γ', then $a \leq \frac{\pi}{c}$.*
2. *If $\mathcal{K}_{\gamma'}(\Pi) \leq c^2$ for all null plane containing γ', then $\frac{\pi}{c} \leq a$.*

On the other hand, null crossing points are produced by the existence of two or more null geodesics from the vertex to another point of the null cone and it can be due to different possibilities such as curvature behavior or the global topology of the manifold. The localization of crossing points is more subtle than conjugate ones. Recently, it has been shown how to introduce an auxiliary Riemannian metric, called rigged metric, in a null cone to determine its conjugate points, including its multiplicities, [8]. The difficulties in handling this metric come from the fact that it is not geodesically complete and the vertex and the crossing points themselves do not belong to the region of the null cone where it is a submanifold. Even more, we do not know this region. We succeed with the control of conjugate points, but the case of crossing points is still open.

The main results in this paper are Theorems 2 and 3. They show that for null cones folded by some effect of the curvature, there exists a lower bound that depends on first derivatives of the metric. The idea comes from the fact that the null second fundamental form of a null cone is negative definite near the vertex for spacelike vectors and it codifies the directional expansion of the flow of the null position vector field. Basically, while the flow is expanding, there is not null conjugate (neither null crossing points, perhaps except for global topological effects). An example of this global topological behavior is the flat Lorentzian manifold $\mathbb{S}^1 \times \mathbb{S}^1$. When the null second fundamental form changes its sign, its flow ends expansion and conjugate points can appear. The parameter used in our approach is the affine parameter of null geodesics normalized by a fixed timelike unit vector at the vertex of the null cone. On the opposite, the bound is not optimal, see Sect. 4. The global topological issue mentioned above is present in the simply connected case too, see Example 1. This is the main reason to restrict ourselves to a family of null cones suitably folded. These ideas have a direct application to the Riemannian case which we discuss in the last section.

2 Preliminaries

Let (M, g) be a time-oriented and connected Lorentzian manifold with dimension $n \geq 3$. Let $p \in M$ be a point being fixed. We call $\widehat{\theta} \subset T_pM$ the maximal domain of $\exp_p$ and $\widehat{C}_p \subset T_pM$ the set of future null vectors. The future null cone at p is defined as

$$C_p = \exp_p(\widehat{\theta} \cap \widehat{C}_p),$$

i.e., the set of points that can be reached by a future null geodesic from p. Observe that it is possible that $p \in C_p$. This occurs when there exists a null geodesic loop with base at p.

Fixing a timelike and past vector $e \in T_pM$, we define

$$\widehat{S}_t = \{v \in \widehat{C}_p \;/\; g(e, v) = t\},$$
$$\widehat{S}_{(0,t)} = \{v \in \widehat{C}_p \;/\; g(e, v) < t\}$$

and $S_t = \exp_p\big(\widehat{\theta} \cap \widehat{S}_t\big)$, $S_{(0,t)} = \exp_p\big(\widehat{\theta} \cap \widehat{S}_{(0,t)}\big)$. Obviously, it holds $\partial \widehat{S}_{(0,t)} = \widehat{S}_t \cup \{0\}$.

The future null radius of definition ρ_p, the future null crossing radius ℓ_p and the future null conjugate radius c_p are

$$\rho_p = \sup\{t \geq 0 \;/\; \widehat{S}_{(0,t)} \subset \widehat{\theta}\},$$
$$\ell_p = \sup\{t < \rho_p \;/\; \exp_p : \widehat{S}_{(0,t)} \to S_{(0,t)} \text{ is bijective}\}$$
$$c_p = \sup\{t < \rho_p \;/\; \exp_p : \widehat{S}_{(0,t)} \to S_{(0,t)} \text{ is a local diffeomorphism}\}.$$

Finally, the future null injectivity radius at p is $i_p = \min\{\ell_p, c_p\}$, that is, the supremum of $t \in \mathbb{R}$ such that $\exp_p : \widehat{S}_{(0,t)} \to S_{(0,t)}$ is a diffeomorphism. These parameters depend on the chosen vector e, and some of them could be infinite.

Remark 1 Observe that a Jacobi field along a null geodesic which vanishes at two points is orthogonal to the null geodesic at every point; thus, a null vector $u \in \widehat{S}_{(0,t)}$ is a singular point of $\exp_p : \widehat{\theta} \to M$ if and only if it is a singular point of the restriction $\exp_p : \widehat{S}_{(0,t)} \to M$. Therefore, there is not null conjugate point to p along a null geodesic $\exp_p(sv)$, $0 \leq s < c_p$, where $v \in \widehat{S}_1$.

Definition 1 If $x = \exp_p(v_1) = \exp_p(v_2) \in C_p$, where $v_1, v_2 \in \widehat{\theta} \cap \widehat{C}_p$ with $v_1 \neq v_2$, then x is called a null crossing point. If moreover $x \in S_{\ell_p}$, then it is called a first null crossing point.

If $c_p < \rho_p$, there is a null conjugate point to p in S_{c_p}.

Given a future null geodesic γ starting at p and normalized by $g(e, \gamma'(0)) = 1$, we can consider

$$s_0 = \inf\{s \geq 0 \;/\; \gamma(s) \in I^+(p)\}.$$

If s_0 exists and it is positive, then $s_0 = \sup\{s \geq 0 : \gamma(s) \notin I^+(p)\}$ and $\gamma(s_0)$ is called a cut point, [1]. Observe that in this case, $\gamma(s_0) \in \partial I^+(p)$.

There are several reasonable definitions for a future null cone. In [4], it is defined as we did above, but in [2,5,10] it is defined as the border of the chronological future of p, $\partial I^+(p)$. Observe that in general, $C_p \not\subset \partial I^+(p)$ as a flat vertical cylinder shows.

To show the relationship with the usual definition in the literature, observe that if U is a normal convex neighborhood of p and $S_{(0,t)} \subset U$, then it is easy to check that $S_{(0,t)} \subset \partial I^+(p, U)$. If M is globally hyperbolic, then we have the following.

Lemma 1 *If M is globally hyperbolic, then a point in S_{i_p} is a null crossing or null conjugate point if and only if it is a cut point. Moreover, $S_{(0,i_p)} \cup S_{i_p} \subset \partial I^+(p) \subset C_p$.*

Proof It follows from [1, Theorem 9.15] and [14, p. 298, Theorem 51]. □

Fixing a (locally defined) null section ξ on a null hypersurface L, the null second fundamental form with respect to ξ is defined as the symmetric tensor $B(v, w) = -g(\nabla_v \xi, w)$ for all $v, w \in TL$. A screen distribution $\mathcal{S}$ is a complementary distribution to $span(\xi)$ on TL. If $V, W \in \mathfrak{X}(L)$, then

$$\nabla_V W = \nabla_V^L W + B(V, W)N, \tag{1}$$

where $\nabla_V^L W$ is tangent to L and N is the unique (locally defined) null transverse vector field determined by ξ and $\mathcal{S}$ and normalized so that $g(N, \xi) = 1$. The values of the null second fundamental form depend on the chosen null section ξ, but its sign only depends on the time orientation of ξ. In this paper, we chose ξ to be future.

Definition 2 Let $\gamma : I \to M$ be a future null geodesic starting at p and normalized by $g(e, \gamma'(0)) = 1$. Call

$$A_\gamma = \{X \in \mathfrak{X}(M) \ / \ X \circ \gamma \in \mathfrak{X}(\gamma)^\perp\}.$$

Let $B_\gamma : I \times A_\gamma \times A_\gamma \to \mathbb{R}$ be the map defined by

$$B_\gamma(s, X, Y) = g(\gamma', \nabla_X Y)_s.$$

Definition 3 We say that $S_{(0,t)}$ is full expanding (resp. semiexpanding) if for any future null geodesic $\gamma : (0, t) \to M$ starting at p and normalized by $g(e, \gamma'(0)) = 1$, $B_\gamma(s, X, Y) < 0$ (resp. $B_\gamma(s, X, Y) \le 0$) for all $s \in (0, t)$ and $X, Y \in A_\gamma$ nonproportional to $\gamma'(s)$ at $\gamma(s)$.

It is clear that if L is a null hypersurface contained in C_p, both $B_\gamma(s, X, Y)$ and $B(X, Y)_{\gamma(s)} = -g(\nabla_X P, Y)_{\gamma(s)}$ share their signs for any $X, Y \in A_\gamma$, where P is the position vector field based at p. Since it holds $B(v, P) = 0$ for all $v \in TL$, the above definition is equivalent to say that the restriction of B to any screen distribution is a negative definite (semidefinite) symmetric bilinear form. Observe that the map B_γ is not a tensor.

The name in Definition 3 is justified because the null second fundamental form B is interpretable as part of a kind of divergence of P.

Obviously, null cones in Minkowski space are full expanding, so the same can be expected, at least locally, for an arbitrary Lorentzian manifold.

Lemma 2 *There exists $0 < t < \rho_p$ such that $S_{(0,t)}$ is fully expanding.*

Proof Take $(x_0, \ldots, x_n)$ a normal coordinate system at p. We can suppose that there is a neighborhood U of p such that ∂_{x_0} and ∇x_0 are timelike vector fields, ∂_{x_i} are spacelike for $i > 0$ and $\max\{|x_i|, |g_{ij}|, |g^{ij}|\} \le C$ for some constant C. Take t such that $S_{(0,t)} \subset U$. Observe that the position vector field is $P = \sum_i x_i \partial_{x_i}$. If we take v an arbitrary unit vector tangent to the null cone such that $v \in (\nabla x_0)^\perp$, then

$$|g(\nabla_v P, v) - g(v, v)| = \left| \sum_{i,j,k,l} v_i v_j x_k \Gamma_{ik}^l g_{lj} \right|.$$

Since $g(\partial_{x_j}, \nabla x_0) = 0$, for $j > 0$, $span\{v, \partial_{x_j}\}$ is spacelike, the Cauchy inequality gives $\left|\sum_k v_k g_{kj}\right| \le |g_{jj}|$ and

$$|v_i| = \left| \sum_{k,j} g^{ij} v_k g_{kj} \right| \le nC^2.$$

Therefore, since $\Gamma^k_{ij}(p) = 0$, shrinking U if necessary, there is an arbitrary small $\delta > 0$ such that $|g(\nabla_v P, v) - g(v, v)| \leq \delta$ for all $v \in (\nabla x_0)^\perp$, which implies that $B(v, v) = -g(\nabla_v P, v)$ restricted to $(\nabla x_0)^\perp$ is negative definite. Varying the coordinate x_0, we have that $S_{(0,t)}$ is fully expanding. □

3 Main result

Obviously, $S_{(0,i_p)}$ is a null hypersurface, but C_p maybe not due to the presence of null conjugate or null crossing points.

We tackle the problem of finding a lower bound of the null injectivity radius encoding both null conjugate and null crossing radius using the preferred affine parametrization of null geodesics starting at any point $p \in M$, determined by a timelike past vector $e \in T_p M$. That is, for any null geodesic γ starting at p, we use the affine parameter normalized by $g(\gamma'(0), e) = 1$. We say that the geodesic (or $\gamma'(0)$) is e-normalized. We emphasize that the value of the bound depends on the choice of this vector.

We use the fully expanding property of $S_{(0,t)}$ (see Definition 3), which imposes a bound on first derivatives of the metric, to determine the lower bound. This is an improvement with respect to the usual techniques used in the literature which use second derivatives instead. On the opposite, we mention that the bound is not optimal and the technique used is not applicable to any null cone.

Definition 4 Take a point $x \in C_p$ and suppose that there are open sets $\widehat{U} \subset T_p M$ and $x \in U \subset M$ such that $\widehat{U} \cap \widehat{C}_p \neq \emptyset$ and $\exp_p : \widehat{U} \to U$ is a diffeomorphism. We call a regular part of the null cone C_p through x to the null hypersurface $L = \exp_p(\widehat{U} \cap \widehat{C}_p)$.

Of course, if x is not a null conjugate point along a null geodesic, then there exists a regular part of the null cone through x. On the other hand, if x is a null crossing point, which is not a null conjugate point, then there are different regular parts of C_p through x.

We always take the position vector field P (which is future directed) as the null section in regular parts L of the null cone C_p.

Definition 5 Let L be a regular part of C_p and $x \in L$. A shield of L at x is a hypersurface given by $\pi = \exp_x(T_x L \cap \widehat{\Theta})$, where $\widehat{\Theta} \subset T_x M$ is a connected open neighborhood of the origin.

A shield π at x is a (nonnecessarily null) hypersurface formed by radial geodesics which are tangent to L at x. In particular, π and L share a null geodesic through x. Observe that, shrinking it if necessary, L is locally an achronal set, that is, every point in L has a neighborhood U such that $U \cap L$ is achronal in $(U, g_{|U})$, see [7, Remark 3.8]. So there is a well-defined notion of future and past of L at least locally.

If $\gamma : (-\varepsilon, \varepsilon) \to \pi$ is an arbitrary nonnull geodesic in the shield at x, using Eq. (1) it satisfies

$$0 = \nabla_{\gamma'(0)}\gamma' = \nabla^L_{\gamma'(0)}\gamma' + B(\gamma'(0), \gamma'(0))N_x.$$

This means that the shield is a kind of zero local height of L near x, being the positive height the side of positive B. So if B is negative semidefinite along spacelike directions, it means that any curve $\alpha : (-\varepsilon, \varepsilon) \to L$ with $\alpha'(0) = \gamma'(0) \in T_x L$ initially enters in the nonpositive side of the shield. Equivalently, the shield is in the local causal past of L near x. If

B is negative definite along spacelike directions, then the shield is in the local chronological past of L near x, except for a null geodesic through x. We give details in the following lemma.

Lemma 3 *Let L be a regular part of C_p and π a shield of L at $x \in L$. The null second fundamental form B is negative semidefinite in spacelike directions at T_xL if and only if there exists a neighborhood U of x in M such that $\pi \cap U \subset J^-(L \cap U, U)$. Moreover, if B is negative definite in spacelike directions at T_xL, then $(\pi - L) \cap U \subset I^-(L \cap U, U)$.*

Proof Since L and π are tangent at x, shrinking π if necessary there exist a timelike, past and transverse (to π) vector field E over π and a diffeomorphism

$$\Phi : (-\varepsilon, \varepsilon) \times \pi \to U \subset M$$

given by $\Phi(t, z) = \exp_z(tE_z)$. Obviously, $\Phi(0, z) = z$ and $\Phi_{*(0,z)}(\partial_t) = E_z$. Take a vector field F defined over π such that it is orthogonal to it, $g(F, E) = 1$ and F_x is a future null vector. Since π is formed by radial geodesic from x tangent to L, $g(\nabla_v F, v) = 0$ for all $v \in T_xL$. Take $\alpha : (-\varepsilon, \varepsilon) \to L \cap U$ a spacelike curve such that $\alpha(0) = x$. If we write $\alpha(s) = \Phi(t(s), x(s))$, then

$$\alpha'(s) = t'(s)\Phi_{*(t,x)}(\partial_t) + V(s),$$

where $V(s) = \Phi_{*(t,x)}(x'(s))$. Taking into account that $\alpha'(0) \in T_x\pi$, it follows that $t(0) = t'(0) = 0$ and $\alpha'(0) = V(0) = x'(0)$. If we derive once again and evaluate in $s = 0$, we get

$$\alpha''(0) = t''(0)E_x + V'(0);$$

thus,

$$g(\alpha''(0), F) = t''(0) + g(V'(0), F). \tag{2}$$

Now, we want to show that $g(V'(0), F) = 0$. For this, call $X(t, s) = \Phi(t, x(s))$. Since $V(s) = X_s(t(s), s)$, we have

$$V'(0) = X_{st}(0, 0)t'(0) + X_{ss}(0, 0) = x''(0),$$

and therefore

$$g(V'(0), F) = -g(x'(0), \nabla_{x'(0)} F) = 0.$$

On the other hand, since $\nabla_{\alpha'}\alpha' = \tan \nabla_{\alpha'}\alpha' + B(\alpha'(0), \alpha'(0))N$, being N the null, past and transverse vector field to L, it holds

$$B(\alpha'(0), \alpha'(0))g(N_x, F_x) = t''(0).$$

Since $g(N_x, F_x) > 0$, then $B(\alpha'(0), \alpha'(0))$ and $t''(0)$ have the same sign and the conclusion easily follows. □

Suppose that $q_1, q_2 \in S_{(0,i_p)}$ are not on the same null geodesic from p. If we have a geodesic joining them and M is globally hyperbolic, then it must be spacelike, since it holds $S_{(0,i_p)} \subset \partial I^+(p)$, Lemma 1. This is also true if we restrict to a convex neighborhood of p.

The following lemma justifies, at least locally, Definition 3.

Lemma 4 *Take $0 < t < \rho_p$ such that $S_{(0,t)}$ is fully expanding and it is contained in a normal convex neighborhood U of p. If $\alpha : [0, 1] \to U$ is a (necessarily spacelike) geodesic with $\alpha(0), \alpha(1) \in S_{(0,t)}$ not on the same null geodesic from p, then $\alpha(t) \in I^+(p)$ for all $t \in (0, 1)$.*

Proof Suppose that $\alpha : [0, 1] \to M$ is a spacelike geodesic with $q = \alpha(0), \alpha(1) \in S_{(0,t)}$ but there exists $s_0 \in (0, 1)$ such that $\alpha(s_0) \notin I^+(p)$. Write $\alpha(s) = \exp_q(sw)$, $0 \le s \le 1$, and consider a spacelike plane Π in $T_q M$ with $w \in \Pi$. Now, $\mathcal{C} = \exp_q^{-1}(S_{(0,t)}) \cap \Pi$ is a closed curve through the origin in $T_q M$, and the image by $\exp_q$ of the interior of $\mathcal{C}$ is inside $I^+(p, U)$. The straight line sw intersects $\mathcal{C}$ for $s = 0$ and $s = 1$, but $s_0 v$ is in the exterior of $\mathcal{C}$. Thus, there is $\nu \in T_q M$ such that the straight line $s\nu$ is tangent to $\mathcal{C}$ at some point $s_1 \nu$ and it is in its interior for $s \in (s_1 - \varepsilon, s_1 + \varepsilon) - \{s_1\}$. In other words, the spacelike geodesic $\sigma(s) = \exp_q(s\nu)$ is tangent to $\exp_q(\Pi) \cap S_{(0,t)}$ at $x = \exp_q(s_1 \nu)$ and $\sigma(s) \in I^+(p)$ for $s \in (s_1 - \varepsilon, s_1 + \varepsilon) - \{s_1\}$. But σ is in the shield of $S_{(0,t)}$ at x, so from Lemma 3 it is in $I^-(S_{(0,t)}, U)$ near the point x. In this way, there is a point in $I^-(S_{(0,t)}, U) \cap I^+(p, U)$ which is a contradiction because U is a convex neighborhood. □

We want to avoid the case where a null geodesic starting at p intersects itself to produce a first null crossing point, because this case does not fit the proof of our main theorem. By this reason, from now on we suppose that the Lorentzian manifold is causal.

Null cones can be divided into two types depending on the presence of a particular effect of the curvature in the fold phenomenon. Our main theorem can be applied only to one of these types, which we call null cone geometrically folded.

Definition 6 Let M be causal and $p \in M$. Given $0 < t < s \le i_p$, we say that $S_{(t,s)}$ is locally convex folding if for any $t_0 \in (t, s)$ and $z \in S_{t_0}$ there exists a neighborhood U of z such that any geodesic $\alpha : [0, 1] \to U$ with $z = \alpha(0), \alpha(1) \in S_{t_0}$ holds $\alpha(r) \in I^+(S_{(0,s)} \cap U, U)$ for every $r \in (0, 1)$. The null cone C_p has a geometric fold if $\ell_p < c_p$ and there exists $\varepsilon > 0$ such that for any δ with $0 < \delta < \varepsilon$, $S_{(\ell_p - \delta, \ell_p)}$ is not locally convex folding.

It is fairly intuitive that geometric folds depend mainly on the curvature. We show an example of a null cone which has not geometric fold.

Example 1 Let S^2 be a hemisphere glued to a flat cylinder $[0, \infty) \times \mathbb{S}^1$ along the boundary to obtain a cigar-like surface Σ. Call g_0 the metric on Σ. Let $(M, g) = (\mathbb{R} \times \Sigma, -dt^2 + g_0)$ be the direct product which is a simply connected Lorentzian manifold. Take a point $x = (t_0, y_0) \in M$ where y_0 is in the cylindrical part of Σ, far away from the hemisphere. Take $e = -\partial_t \in T_x M$ as a timelike vector for normalization of geodesics through x. A null cone with vertex at x has as a $t = cte$ section the set $(t, B_x(t))$ where $B_x(t)$ is a geodesic ball with center x and radius t.

Now we can state the main theorem.

Theorem 2 *Let M be a causal Lorentzian manifold and $p \in M$. Suppose $S_{(0,t)}$ $(t \le \rho_p)$ is fully expanding, that is, for any future null geodesic $\gamma : (0, t) \to M$ starting at p and normalized by $g(e, \gamma'(0)) = 1$, the map B_γ in Definition 2 verifies $B_\gamma(s, X, Y) < 0$ for all $s \in (0, t)$ and for all $X, Y \in A_\gamma$ which are nonproportional to $\gamma'(s)$ at $\gamma(s)$. If one of the following is true:*

1. $c_p \le \ell_p$.
2. *C_p has a geometric fold, that is, $\ell_p < c_p$ and there exists $\varepsilon > 0$ such that for any δ with $0 < \delta < \varepsilon$, $S_{(\ell_p - \delta, \ell_p)}$ is not locally convex folding,*

then $i_p \ge t$.

Proof Take $u \in \widehat{S}_{(0,t)}$ and $\exp_p(u) \in L \subset C_p$, being L a regular part of the null cone. If we consider the geodesic variation of an e-normalized null geodesic $X(s, r) = \exp_p(s(u + rv))$, for $s \in (0,1)$, then $J(s) = X_r(s,0)$ is a Jacobi vector field with $J(0) = 0$ and the position vector field is given by $P_{X(s,r)} = sX_s(s,r)$. Thus, $X_{sr}(s,0) = \frac{1}{s}\nabla_{X_r} P$ and for the values of the parameter s such that $\exp_p(su) \in L$ we have

$$g(J', J)|_s = g(\nabla_J P, J) = -B(J(s), J(s)),$$

where B is the null second fundamental form of L. Therefore, $t < c_p$. If $c_p \leq \ell_p$, the theorem follows. To show point 2, we suppose $i_p = \ell_p < c_p$. If $t > i_p$, by hypothesis, for any small $\delta > 0$, there exist $t_0 \in (i_p - \delta, i_p)$ and $z \in S_{t_0}$ such that for any neighborhood U of z, there is a geodesic $\alpha : [0,1] \to U$ from $\alpha(0) = z$ to $\alpha(1) \in S_{t_0}$, which holds $\alpha(r) \in J^-(S_{(0,i_p)} \cap U, U)$ for some $r \in (0,1)$. We can take a sequence of spacelike geodesics α_n such that $y_n = \alpha_n(1) \to z$ in S_{t_0}, and a corresponding sequence of vectors $v_n \to v \in T_z S_{t_0}$ where $v_n = k_n \alpha_n'(0)$ is normalized to $|v_n| = 1$ and v is spacelike because $T_z S_{t_0}$ is spacelike. The geodesic γ_v with initial condition v enters initially in $J^+(S_{(0,i_p)} \cap U, U)$ by construction. Take the shield π at z. The geodesic γ_v above is spacelike and belongs to the shield π. But observe that $t > i_p$ implies $S_{(0,i_p)}$ is fully expanding, so we can apply Lemma 3 to get

$$(\pi - S_{(0,i_p)}) \cap U \subset I^-(S_{(0,i_p)} \cap U, U),$$

contradiction. □

Example 1 shows that the theorem is not true for null cones without geometric fold, even if M is simply connected.

Observe that the lower bound provided by this result is not optimal because the family of maps B_γ could become positive far away of the presence of a null conjugate point of p along γ, or null crossing points in the null cone, see the next section.

4 Examples

The aim of this section is to compare the true null injectivity radius i_p of a null cone with the lower bound given by the Theorem 2 in two particular examples.

4.1 The closed Friedmann cosmological model

Let $\tau : (0, 2\pi) \to (0, \pi)$ be the diffeomorphism given by $\tau(\theta) = \frac{\theta - \sin(\theta)}{2}$ and take

$$(M, g) = \big((0,\pi) \times \mathbb{S}^3, -dt^2 + f(t)^2 g_0\big),$$

where $f(t) = \frac{1-\cos(\tau^{-1}(t))}{2}$. In this case, we take the rigging vector field $\zeta = -f\partial_t$ according to the rigging construction given in [8]. Fix a point $p_0 = (t_0, x_0) \in M$ with $0 < t_0 < \frac{\pi}{2}$ and let $\gamma : [0,a) \to M$ be a maximal e-normalized null geodesic with $\gamma(0) = p_0$ and $e = \zeta_{p_0}$. In fact, the rigged vector field ξ on C_{p_0} is g-geodesic, so γ is an integral curve of it. If we call

$$\begin{aligned} a(t) &= \int_{t_0}^{t} f(r)\mathrm{d}r, \\ \alpha(s) &= a^{-1}(s), \\ b(s) &= \int_0^s \frac{1}{f(\alpha(r))^2}\mathrm{d}r, \end{aligned}$$

then we know that $\gamma(s) = (\alpha(s), x(s))$, where $x(s) = \exp^{\mathbb{S}^3}_{x_0}(b(s)u)$ and $u \in T_{x_0}\mathbb{S}^3$ is a unit vector, [7, Proposition 3.1].

If $\gamma(s_1)$ is the first null conjugate point of p_0 along γ, then $x(s_1)$ is the first conjugate point in $\mathbb{S}^3$ along the geodesic $\exp^{\mathbb{S}^3}_{x_0}(su)$, [3]. Therefore, $x(s_1) = -x_0$ and $b(s_1) = \pi$.

If $\tau(\theta_0) = t_0$ for $0 < \theta_0 < \pi$, then

$$a(\tau(\theta)) = \frac{3\theta}{8} - \frac{\sin\theta}{2} + \frac{\sin 2\theta}{16} - \frac{3\theta_0}{8} + \frac{\sin\theta_0}{2} - \frac{\sin 2\theta_0}{16}$$

and $b(a(\tau(\theta))) = \theta - \theta_0$ for $\theta_0 \leq \theta < 2\pi$. Therefore, the first conjugate point occurs at $s_1 = \frac{3\pi}{8} + \sin\theta_0$, so $c_{p_0} = \frac{3\pi}{8} + \sin\theta_0$ with respect to $-f\partial_t$. It is easy to check that null crossing points coincide with conjugate points; thus, $i_{p_0} = c_{p_0}$.

On the other hand, from [7, Proposition 3.3], if a point (t, x) lies in a regular part of the future null cone with vertex at p_0, then it is totally umbilic and the null mean curvature is given by

$$\begin{aligned} 2H &= \frac{-1}{f(t)^2}\left(f'(t) + \frac{1}{\tan\left(\int_{t_0}^{t} \frac{1}{f(r)}\mathrm{d}r\right)}\right) \\ &= \frac{-1}{f(t)^2}\left(f'(t) + \frac{1}{\tan(b(a(t)))}\right) \\ &= \frac{-1}{y(\theta)^2}\left(\frac{y'(\theta)}{y(\theta)} + \frac{1}{\tan(\theta - \theta_0)}\right), \end{aligned}$$

where $y(\theta) = \frac{1-\cos\theta}{2}$. Observe that the minus sign here is because we have considered the null section ξ on the null cone with opposite sign that in [7]. It vanishes for $\theta = \frac{2}{3}(\theta_0+\pi)$, and thus the lower bound estimated by Theorem 2 is $a\left(\tau(\frac{2}{3}(\theta_0 + \pi))\right)$. If we consider the quotient $c(\theta_0) = \frac{a\left(\tau(\frac{2}{3}(\theta_0+\pi))\right)}{\frac{3\pi}{8}+\sin\theta_0}$ for $0 < \theta < \pi$, then it is an increasing function with $c(0) \approx 0.25$ and $c(\pi) \approx 0.75$. Thus, the given estimate varies from $\frac{1}{4}$ to $\frac{3}{4}$ of the real null injectivity radius.

4.2 Odd-dimensional Lorentzian spheres

If we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, then the Euclidean product is given by

$$g_E(z, w) = Re(z \cdot \overline{w}) = Re\left(\sum_{k=1}^{n} z_k\overline{w_k}\right),$$

for all $z, w \in \mathbb{C}^n$. Consider the Hopf vector field $\zeta_z = iz$ defined on $\mathbb{S}^{2n-1}$, which is a Killing unit vector field. It holds $\nabla_X\zeta = iX$, where ∇ is the Euclidean Levi-Civita connection.

If we take the Lorentzian metric

$$g_L(X, Y) = g_E(X, Y) - 2g_E(\zeta, X)g_E(\zeta, Y), \tag{3}$$

for all $X, Y \in \mathfrak{X}(\mathbb{S}^{2n-1})$, then its Levi-Civita connection ∇^L is given by

$$\nabla^L_X Y = \nabla^S_X Y - 2g_E(\zeta, X)\nabla^S_Y\zeta - 2g_E(\zeta, Y)\nabla^S_X\zeta, \tag{4}$$

where ∇^S is the Levi-Civita connection of the Euclidean sphere, [13]. Now, $\left(\mathbb{S}^{2n-1}, g_L\right)$ is a geodesically complete Lorentzian manifold and ζ is a Killing, unit and timelike vector field on it. We chose ζ as a rigging vector field, see again the construction in [8].

Fix the point $p_0 = (1, \ldots, 0) \in \mathbb{S}^{2n-1}$ and call $a = 2 + \sqrt{2}$ and $b = 2 - \sqrt{2}$. The (future) null cone C_{p_0} of the Lorentzian sphere $\mathbb{S}^{2n-1}$ at the point p_0 can be parametrized as $\Phi : (0, \infty) \times \mathbb{S}^{2n-3} \to \mathbb{S}^{2n-1}$ given by $\Phi(t, v) = (A(t), B(t)v)$, where as before, we are considering $\mathbb{S}^{2n-3} \subset \mathbb{C}^{n-1}$ and

$$A(t) = \frac{be^{-ait} + ae^{-bit}}{4},$$
$$D(t) = \frac{a-b}{8} i\left(e^{-ait} - ae^{-bit}\right).$$

In other words, $\gamma_v(t) = \Phi(t, v)$ is a null geodesic with $\gamma_v(0) = p_0$. Note that $\gamma_v(t) \neq p_0$ for all $t > 0$ and all $v \in \mathbb{S}^{2n-3}$, and $g_L(e, \gamma_v'(0)) = 1$, where we chose $e = \zeta_{p_0}$. Moreover, the first conjugate point of p_0 along $\gamma_v(t)$ is $\gamma_v\left(\frac{\pi}{2\sqrt{2}}\right)$, [6, Teorema 5.4.8.]. It is easy to show that $\Phi : \left(0, \frac{\pi}{\sqrt{2}}\right) \times \mathbb{S}^{2n-3} \to \mathbb{S}^{2n-1}$ is injective, so the null injectivity radius is

$$i_{p_0} = \frac{\pi}{2\sqrt{2}}. \tag{5}$$

Now, we compute the bound given in Theorem 2. Define $\xi_{\Phi(t,v)} = \gamma_v'(t) = \big(A'(t), D'(t)v\big)$, which is a well-defined null vector field on any regular part of the null cone C_{p_0} and is just the rigged vector field induced by ζ. Consider the screen distribution given by

$$\mathcal{T}_{\Phi(t,v)} = \{(0, D(t)w) : w \in \mathbb{C}^{n-1}, w \in T_v\mathbb{S}^{2n-3}\},$$

which is a well-defined screen except for $t = \frac{\pi k}{\sqrt{2}}$, $k \in \mathbb{Z}$. If $W = (0, D(t)w) \in \mathcal{T}_{\Phi(t,v)}$, then

$$g_L(\zeta, W) = -g_E(\zeta, W) = |D(t)|^2 Im(v\overline{w}),$$

so, the screen distribution $\mathcal{T}$ does not coincide with the induced by ζ.

Now, we compute $B(W_1, W_2)$ where $W_1, W_2 \in \mathcal{T}_{\Phi(t,v)}$ are of the form $W_1 = (0, D(t)w_1)$ and $W_2 = (0, D(t)w_2)$ and $\Phi(t, v)$ lies in a regular part of the null cone. Using Eqs. (3) and (4), we have

$$\begin{aligned} g_L\left(\nabla^L_{W_1}\xi, W_2\right) &= g_E\big(\nabla_{W_1}\xi, W_2\big) + 2g_E(W_2, \zeta)g_E\big(\nabla_{W_1}\zeta, \xi\big) \\ &\quad + 2g_E(W_1, \zeta)g_E\big(\nabla_{W_2}\zeta, \xi\big) + 2g_E\big(\nabla_{W_1}\zeta, W_2\big). \end{aligned}$$

Since $\nabla_{W_1}\xi$ is identified with $\big(0, D'(t)w_1\big)$ and $\nabla_{W_1}\zeta$ with $(0, iD(t)w_1)$, we have that

$$\begin{aligned} g_E\big(\nabla_{W_1}\xi, W_2\big) &= Re\Big(D'(t)\overline{D(t)}\Big)Re(w_1\overline{w}_2) + 2|D(t)|^2 Im(w_1\overline{w}_2), \\ g_E\big(\nabla_{W_1}\zeta, \xi\big) &= Re\Big(D'(t)\overline{D(t)}\Big)Im(v\overline{w}_1) \\ g_E\big(\nabla_{W_1}\zeta, W_2\big) &= -|D(t)|^2 Im(w_1\overline{w}_2) \end{aligned}$$

where we have used that $2|D(t)|^2 = -Im\Big(D'(t)\overline{D(t)}\Big)$ which follows from $g_E(\xi, W) = -2g_E(\zeta, W)$ for $W \in \mathcal{T}_{\Phi(t,v)}$. Therefore, the null second fundamental form at a point $\Phi(t, v)$ is given by

$$B(W_1, W_2) = -Re\Big(D'(t)\overline{D(t)}\Big)\Big(Re(w_1\overline{w}_2) - 4|D(t)|^2 Im(v\overline{w}_1)Im(v\overline{w}_2)\Big).$$

Taking into account that $Im(v\overline{w}_1) = -Re(iv\overline{w}_1) = -g_E(\Upsilon_v, w_1)$, where Υ is the Hopf vector field on $\mathbb{S}^{2n-3}$, we can write the null second fundamental form at a point $\Phi(t, v)$ as $B : T_v\mathbb{S}^{2n-3} \times T_v\mathbb{S}^{2n-3} \to \mathbb{R}$ given by

$$B(W_1, W_2) = -Re\Big(D'(t)\overline{D(t)}\Big)\big(g_E(w_1, w_2) - 4|D(t)|^2 g_E(\Upsilon_v, w_1)g_E(\Upsilon_v, w_2)\big).$$

Using an orthonormal basis of $T_v\mathbb{S}^{2n-3}$ which contains Υ_v, it is easy to check that B is negative defined in any screen distribution until $t = \frac{\pi}{4\sqrt{2}}$, so the estimate given in Theorem 2 is half of the true null injectivity radius (5).

5 Application to Riemannian manifolds

In this section, we apply the above ideas to Riemannian manifolds. For this, we adapt the notations and definitions to be close to those used above for the Lorentzian geometry. Fixing a point x in a Riemannian manifold (N, g_0), suppose that $\widehat{\theta}$ is the maximal definition domain of $\exp_x$ and take

$$\begin{aligned}
\widehat{S}_t &= \{v \in T_xN : |v| = t\},\\
\widehat{B}_t &= \{v \in T_xN : |v| < t\},\\
S_t &= \exp_x\big(\widehat{S}_t \cap \widehat{\theta}\big),\\
B_t &= \exp_x\big(\widehat{B}_t \cap \widehat{\theta}\big),
\end{aligned}$$

which are called geodesic sphere and geodesic ball of radius t, respectively.

The radius of definition ρ_x^R, the crossing radius ℓ_x^R and the conjugate radius c_x^R and the injectivity radius i_x^R are

$$\begin{aligned}
\rho_x^R &= \sup\{t \geq 0 \;/\; \widehat{B}_t \subset \widehat{\theta}\},\\
\ell_x^R &= \sup\{t < \rho_x^R \;/\; \exp_x : \widehat{B}_t \to B_t \text{ is bijective}\},\\
c_x^R &= \sup\{t < \rho_x^R \;/\; \exp_x : \widehat{B}_t \to B_t \text{ is a local diffeomorphism}\},\\
i_x^R &= \min\{\ell_x^R, c_x^R\}.
\end{aligned}$$

Take $z \in S_t$ and suppose that there are open sets $\widehat{U} \subset T_xN$ and $z \in U \subset N$ such that $\exp_x : \widehat{U} \to U$ is a diffeomorphism. A regular part of the geodesic sphere S_t through z is $\exp_x(\widehat{U} \cap \widehat{S}_t)$. We say that S_t is fully expanding if for every unit geodesic starting at p, the map $B_\gamma(s, X, Y) = g(\gamma'(s), \nabla_X Y)$ is negative for $X, Y \in \mathfrak{X}(M)$ with $X \circ \gamma \in \mathfrak{X}(\gamma)^\perp$.

Let $0 < t < s \leq i_p^R$. The set $B_s - B_t$ is locally convex folding if for any $t_0 \in (t, s)$ and $z \in S_{t_0}$ there exists a neighborhood U of z such that any geodesic $\alpha : [0, 1] \to U$ with $z = \alpha(0)$, $\alpha(1) \in S_{t_0}$ holds $\alpha(r) \in B_{t_0} \cap U$ for every $r \in (0, 1)$. The ball $B_{\rho_p^R}$ is geometrically folded if $\ell_p^R < c_p^R$ and there exists $\varepsilon > 0$ such that for any $0 < \delta < \varepsilon$, $B_{i_p^R} - B_{i_p^R - \delta}$ is not locally convex folding.

Now, we consider the Lorentzian product $(M, g) = (\mathbb{R} \times N, -ds^2 + g_0)$ which is clearly causal, the point $p = (0, x)$ and the vector $e = -\partial s|_p$. It is straightforward to check that $c_x^R = c_p$ and $\ell_x^R \leq \ell_p$. Since radial geodesics in a normal ball globally minimize the distance, it follows that given two vectors $v_1, v_2 \in \widehat{B}_{\ell_x^R} \cup \widehat{S}_{\ell_x^R}$ such that $\exp_x(v_1) = \exp_x(v_2)$, then necessarily $v_1, v_2 \in \widehat{S}_{\ell_x^R}$. Therefore, it holds that $i_x^R = i_p$ and a direct application of Theorem 2 to the Lorentzian manifold (M, g) gives us the following.

Theorem 3 *Let (N, g_0) be a Riemannian manifold and $x \in N$. If there exists $t_0 \leq \rho_x^R$ such that the geodesic spheres S_t are fully expanding for all $t \in (0, t_0)$, that is, for every unit geodesic starting at p, the map $B_\gamma(s, X, Y) = g(\gamma'(s), \nabla_X Y)$ is negative for $X, Y \in \mathfrak{X}(M)$ with $X \circ \gamma \in \mathfrak{X}(\gamma)^\perp$, and one of the following is true:*

1. $c_p \leq \ell_p$.
2. *The ball $B_{\rho_p^R}$ has a geometric fold, that is, $\ell_p^R < c_p^R$ and there exists $\varepsilon > 0$ such that for any $0 < \delta < \varepsilon$, $B_{i_p^R} - B_{i_p^R - \delta}$ is not locally convex folding.*

Then, $i_p^R \geq t_0$.

Instead of considering the Lorentzian product $\mathbb{R} \times N$, we can also adapt the proof of Theorem 2 to the Riemannian case. This can be done in a quite straightforward way with the obvious modifications.

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