Annals of Global Analysis and Geometry (2019) 56:361–401
https://doi.org/10.1007/s10455-019-09671-y

Invariant Ricci-flat metrics of cohomogeneity one with Wallach spaces as principal orbits

Hanci Chi[1]

Received: 12 March 2019 / Accepted: 19 June 2019 / Published online: 27 June 2019
© Springer Nature B.V. 2019

Abstract

We construct a continuous 1-parameter family of smooth complete Ricci-flat metrics of cohomogeneity one on vector bundles over $\mathbb{CP}^2$, $\mathbb{HP}^2$ and $\mathbb{OP}^2$ with respective principal orbits G/K the Wallach spaces $SU(3)/T^2$, $Sp(3)/(Sp(1)Sp(1)Sp(1))$ and $F_4/\mathrm{Spin}(8)$. Almost all the Ricci-flat metrics constructed have generic holonomy. The only exception is the complete G_2 metric discovered in Bryant and Salamon (Duke Math J 58(3):829–850, 1989) and Gibbons et al. (Commun Math Phys 127(3):529–553, 1990). It lies in the interior of the 1-parameter family on $\bigwedge^2_- \mathbb{CP}^2$. All the Ricci-flat metrics constructed have asymptotically conical limits given by the metric cone over a suitable multiple of the normal Einstein metric on G/K.

Keywords Noncompact Ricci-flat manifold · G_2 holonomy · Cohomogeneity one manifold

Contents

✉ Hanci Chi
chih4@mcmaster.ca

1 McMaster University, Hamilton, Canada

1 Introduction

1.1 Background and main result

A Riemannian manifold (M, g) is *Ricci-flat* if its Ricci curvature vanishes:

$$\mathrm{Ric}(g) = 0. \tag{1.1}$$

A Ricci-flat manifold is the Euclidean analogy of a vacuum solution of the Einstein field equations.

In this article, we study complete non-compact Ricci-flat manifolds of cohomogeneity one. A Riemannian manifold (M, g) is of cohomogeneity one if a Lie Group G acts isometrically on M such that the principal orbit G/K is of codimension one. The Ricci-flat condition (1.1) is then reduced to a system of ODEs.

Many examples of cohomogeneity one Ricci-flat metrics have special holonomy. These include the first example of an inhomogeneous Einstein metric, which is also a Kähler metric. It was constructed in [10] on a non-compact open set of $\mathbb{C}^n$. A complete Calabi–Yau metric was constructed on $T^*\mathbb{S}^2$ independently in [11,22]. The construction was generalized to $T^*\mathbb{CP}^n$ in [11] and those Ricci-flat metrics are hyper-Kähler. Cohomogeneity one Kähler–Einstein metrics were constructed on complex line bundles over a product of compact Kähler–Einstein manifolds in [2,18]. Complete metrics with G_2 or Spin(7) holonomy can be found in [7,16,17,24,25].

Ricci-flat metrics with generic holonomy, for example, were constructed on various vector bundles in [2,4,13,31]. It is further shown in [8,9] that for infinitely many dimensions, there exist examples which are homeomorphic but not diffeomorphic. The case where the isotropy representation of the principal orbit contains exactly two inequivalent irreducible summands was studied in [4,33]. In this article, we consider examples with three inequivalent summands. Specifically, let (G, H, K) be one of

$$\begin{aligned} &\text{I. } (SU(3), S(U(2)U(1)), S(U(1)U(1)U(1))),\\ &\text{II. } (Sp(3), Sp(2)Sp(1), Sp(1)Sp(1)Sp(1)),\\ &\text{III. } (F_4, \mathrm{Spin}(9), \mathrm{Spin}(8)). \end{aligned} \tag{1.2}$$

For these triples, we construct Ricci-flat metrics on the corresponding cohomogeneity one vector bundles M with unit sphere bundle $H/K \hookrightarrow G/K \to G/H$. The singular orbits G/H's are, respectively, $\mathbb{CP}^2$, $\mathbb{HP}^2$ and $\mathbb{OP}^2$. The principal orbits G/K's are *Wallach spaces*. They appeared explicitly in Wallach classification of even dimensional homogeneous manifolds with positive sectional curvature [30]. Throughout this paper, the letters j, k, l will denote three distinct numbers in $\{1, 2, 3\}$ whenever more than one of them appear in a formula together. Let $d = \dim(H/K)$ and $n = \dim(G/K)$. As will be shown in Sect. 2.2, each M is in fact an irreducible (sub)bundle of $\bigwedge^d_- T^*(G/H)$.

In all three cases, we can rescale the normal metric on G/K to a metric Q, whose restriction on H/K is the standard metric with constant sectional curvature 1. Take Q as the background metric for G/K. As will be shown in Sect. 2.1, the isotropy representation $\mathfrak{g}/\mathfrak{k}$ has $\mathbb{Z}_3$-symmetry among its three inequivalent irreducible summands. By Schur's lemma, any G-invariant metric on G/K has the form

$$g_{G/K} = f_1^2\, Q|_{\mathfrak{p}_1} \oplus f_2^2\, Q|_{\mathfrak{p}_2} \oplus f_3^2\, Q|_{\mathfrak{p}_3} \tag{1.3}$$

for some $f_j > 0$. Correspondingly, the Ricci endomorphism r of G/K, defined by $g_{G/K}(r(\cdot), \cdot) = \mathrm{Ric}(\cdot, \cdot)$, has the form

$$Q(r(\cdot), \cdot) = r_1 \; Q|_{\mathfrak{p}_1} \oplus r_2 \; Q|_{\mathfrak{p}_2} \oplus r_3 \; Q|_{\mathfrak{p}_3}, \tag{1.4}$$

where

$$r_j = \frac{a}{f_j^2} + b\left(\frac{f_j^2}{f_k^2 f_l^2} - \frac{f_k^2}{f_j^2 f_l^2} - \frac{f_l^2}{f_j^2 f_k^2}\right) \tag{1.5}$$

for some constants a and b. Their values were computed in [28], as shown in Table 1.

Remark 1.1 A basic observation on a and b is $a - 2b = d - 1$. This is not surprising since Q is the sectional curvature 1 metric on $\mathbb{S}^d$. Another observation is $a - 6b \geq 0$, where the equality is achieved in Case I. These observations are frequently used in this article, especially in Sects. 3.1 and 3.2.

Note that all three possible $\frac{f_j^2}{f_k^2 f_l^2}$'s appear in (1.5). An important motivation for our choices of principal orbits to consider is to study the complications that arise from the simultaneous presence of the terms $\frac{f_1^2}{f_2^2 f_3^2}, \frac{f_2^2}{f_1^2 f_3^2}$ and $\frac{f_3^2}{f_1^2 f_2^2}$. If two of f_j's are identical, say $f_2 \equiv f_3$, the Ricci endomorphism takes a simpler form, with $r_1 = \frac{a-2b}{f_1^2} + b\frac{f_1^2}{f_2^4}$ and $r_2 \equiv r_3 = \frac{a}{f_2^2} - b\frac{f_1^2}{f_2^4}$. The Ricci-flat ODE system for this special case then reduces to the one for $\mathfrak{g}/\mathfrak{k}$ with two inequivalent irreducible summands considered in [4,33]. It is noteworthy that the functional $\widehat{\mathcal{G}}$ introduced in [4] does not have any positive real root for Case I. Nevertheless, the two summands case can be viewed as the subsystem of the ODE system studied in this article. The invariant compact set constructed in Sect. 3.1 can be used to prove the existence of complete Ricci-flat metric for this special case. With the condition $f_2 \equiv f_3$ relaxed, we prove the following theorem.

Theorem 1.2 *There exists a continuous 1-parameter family of non-homothetic complete smooth invariant Ricci-flat metrics on each M.*

Remark 1.3 Ricci-flat metrics constructed in Case II and Case III all have generic holonomy. In Case I, the 1-parameter family of smooth Ricci-flat metrics contains in its interior the complete smooth G_2 metric that was first constructed in [7,25]. The other metrics in that family all have generic holonomy. Therefore, for M in Case I, the moduli space $\mathcal{M}_{G_2}$ of G_2 metric is *not* isolated in $\mathcal{M}_0$ the moduli space of Ricci-flat metric in the C^0 sense. Such a phenomenon cannot occur on a simply connected spin closed manifold, for example, by Theorem 3.1 in [32].

Table 1 Constants a and b for all cases

Case	d	n	a	b
I	2	6	$\frac{3}{2}$	$\frac{1}{4}$
II	4	12	4	$\frac{1}{2}$
III	8	24	9	1

Definition 1.4 Let (N, g_N) and (M, g_M) be Riemannian manifolds of respective dimension n and $n+1$. Let t be the geodesic distance from some point on M. Then M has one *asymptotically conical (AC) end* if there exists a compact subset $\check{M} \subset M$ such that $M \backslash \check{M}$ is diffeomorphic to $(1, \infty) \times N$ with $g_M = dt^2 + t^2 g_N + o(1)$ as $t \to \infty$.

With further analysis on the asymptotic behavior of Ricci-flat metrics in Theorem 1.2, we are able to prove the following:

Theorem 1.5 *Each Ricci-flat metric in Theorem 1.2 has an AC end with limit the metric cone over a suitable multiple of the normal Einstein metric on G/K.*

Remark 1.6 In Case I, the normal Einstein metric on the principal orbit $SU(3)/T^2$ admits a (strict) nearly Kähler structure. Hence the metric cone over G/K is the singular G_2 metric which was first constructed in [6]. The other two principal orbits, however, do not admit (strict) nearly Kähler structure [21].

1.2 Organization

This paper is structured as followings. In Sects. 2.1 and 2.2, we discuss some details of the geometry of the cohomogeneity one manifolds M. Based on the work in [23], we reduce (1.1) to a system of ODEs (2.9) with a conservation law (2.10). A G-invariant Ricci-flat metric around G/H is hence represented by an integral curve defined on $[0, \epsilon)$. We derive the condition for smooth extension to G/H using Lemma 1.1 in [23]. If in addition, the integral curve is defined on $[0, \infty)$, the corresponding Ricci-flat metric is complete.

In Sect. 2.3, we apply the coordinate change introduced in [19,20]. The ODE system is transformed to a polynomial one. Invariant Einstein metrics on G/H and G/K are transformed to critical points of the new system. We carry out linearizations at these critical points and prove the local existence of invariant Ricci-flat metrics around G/H. An integral curve defined on $[0, \epsilon)$ is transformed to a new one that is defined on $(-\infty, \epsilon')$ for some $\epsilon' \in \mathbb{R}$. Each integral curve represents a Ricci-flat metric on M up to homothety. It is determined by a parameter s_1 that controls the principal curvature of G/H at $t = 0$. To show the completeness of the metric is equivalent to proving that the new integral curve is defined on $\mathbb{R}$.

The proof of completeness of the metric is divided into two sections. In Sect. 3.1, we construct a compact invariant set whose boundary contains critical points that represent the invariant metric on G/H and the normal Einstein metric on G/K. The construction is almost the same for all three cases with a little difference in Case I. Section 3.2 proves that as long as s_1 is close enough to zero, integral curves of Ricci-flat metrics enter the compact invariant set constructed in Sect. 3.1 in finite time, hence proving the completeness.

In Sect. 4, we analyze the asymptotic behavior of all the Ricci-flat metrics constructed in Sect. 3.2. There also exist solutions to the polynomial system that represent singular Ricci-flat metrics. They are discussed in Sect. 5. Results in this article are summarized by a plot at the end.

With similar techniques introduced in Sect. 3, we can also show that there exists a 2-parameter family of Poincaré–Einstein metrics on each M. More details will appear in another upcoming article.

2 Local solution near singular orbit

2.1 Cohomogeneity one Ricci-flat equation

In this section, we derive the system of ODEs whose solutions give Ricci-flat metrics of cohomogeneity one on M.

Since M is of cohomogeneity one, there is a G-diffeomorphism between $M\backslash(G/H)$ and $(0,\infty)\times G/K$. We construct a Ricci-flat metric g on M by setting $(0,\infty)$ as a geodesic and assigning a G-invariant metric $g_{G/K}$ to each hypersurface $\{t\}\times G/K$, i.e., define

$$g = dt^2 + g_{G/K}(t) \tag{2.1}$$

on M. By [23], if $g_{G/K}(t)$ satisfies

$$\dot{g}_{G/K} = 2g_{G/K}(L\cdot,\cdot), \tag{2.2}$$
$$\dot{L} = -\mathrm{tr}(L)L + r, \tag{2.3}$$
$$\mathrm{tr}(\dot{L}) = -\mathrm{tr}(L^2), \tag{2.4}$$
$$d(\mathrm{tr}(L)) + \delta^{\nabla}L = 0, \tag{2.5}$$

on $(0,\epsilon)$, where $\delta^{\nabla}:\Omega^1(G/K,T(G/K))\to T^*(G/K)$ is the divergence operator composed with the musical isomorphism, then g is a Ricci-flat metric on $(0,\epsilon)\times G/K$.

Note that (2.2) provides a formula for computing $L(t)$ the shape operator of hypersurface $\{t\}\times G/K$ for each $t\in(0,\epsilon)$. By [1,23], Eq. (2.5) automatically holds for a C^3 metric satisfying (2.2) and (2.3) if there exists a singular orbit of dimension smaller than $\dim(G/K)$. Canceling the term $\mathrm{tr}(\dot{L})$ using (2.3) and (2.4) yields the conservation law

$$R - (\mathrm{tr}(L))^2 + \mathrm{tr}(L^2) = 0. \tag{2.6}$$

We shall focus on deriving specific formulas for (2.2), (2.3) and (2.6) on M. It requires a closer look at isotropy representations of G/K and G/H. We fix notations first. Each irreducible complex representation is characterized by inner products between the dominant weight and simple roots on nodes of the corresponding Dynkin diagram. We use $[a]$ for class $A_1=B_1=C_1$; $[a,b]$ for $C_2=B_2$ with the shorter root on the right end; $[a,b,c,d]$ for B_4 with the shorter root on the right end. Furthermore, let $\mathfrak{t}$ be the Lie algebra of $S(U(1)U(1)U(1))$. Choose Q-orthogonal decomposition $\mathfrak{t}=\mathfrak{t}_1\oplus\mathfrak{t}_2$, where

$$\mathfrak{t}_1 = \mathrm{span}_{\mathbb{R}}\left\{\begin{bmatrix} i & & \\ & -i & \\ & & 0\end{bmatrix}\right\},\quad \mathfrak{t}_2 = \mathrm{span}_{\mathbb{R}}\left\{\begin{bmatrix} i & & \\ & i & \\ & & -2i\end{bmatrix}\right\}.$$

Let θ_j^a denote the complexified irreducible representation of circle generated by $\mathfrak{t}_j$ with weight a. We use Λ_8 and $\Delta_8^{\pm}$ to respectively denote the complexified standard representation and spin representations of $\mathrm{Spin}(8)$. We use $\mathbb{I}$ to denote the trivial representation.

Proposition 2.1 *The formula of $g_{G/K}$ is given by* (1.3).

Proof With (G,H,K) listed in (1.2), we have the following Q-orthogonal decomposition for $\mathfrak{g}$:

$$\begin{aligned}\mathfrak{g} &= \mathfrak{h}\oplus\mathfrak{q} \quad \text{as a representation of } Ad(G)|_H\\ &= (\mathfrak{k}\oplus\mathfrak{p}_1)\oplus(\mathfrak{p}_2\oplus\mathfrak{p}_3) \quad \text{as a representation of } Ad(G)|_K\,.\end{aligned} \tag{2.7}$$

Irreducible K-modules $\mathfrak{p}_j$'s are all of dimension d, but they are inequivalent to each other. Specifically, we have Table 2.

By Schur's lemma, a G-invariant metric on G/K has the form of (1.3). □

Proposition 2.2 *The formula of Ricci endomorphism on $(G/K, g_{G/K})$ is given by (1.4) and (1.5) with constants a and b listed in Table 1.*

Proof Since the Ricci endomorphism is also G-invariant, it has the form of (1.4). To compute its formula, use (7.39) in [3] to derive the scalar curvature on G/K and then apply variation. For each case, since $[\mathfrak{p}_j, \mathfrak{p}_j] \subset \mathfrak{k}$ and $[\mathfrak{p}_j, \mathfrak{p}_k] \subset \mathfrak{p}_l$, each r_j in (1.4) has the form of (1.5). □

Take M as an associated vector bundle to principal H-bundle $G \to G/H$ of cohomogeneity one. As the orbit space is of dimension one, the action of H on the unit sphere of $\mathbb{R}^{d+1}$ must be transitive. Then the group K is taken as an isotropy group of a fixed nonzero element in $\mathbb{R}^{d+1}$, say $v_0 = (1, 0, \ldots, 0)$. It is clear that $H/K = \mathbb{S}^d$. Hence G/K is indeed a unit sphere bundle over G/H. In this setting, $g_{G/K}(t)$ is an $S^2(\mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3)^K$-valued function with each f_j in (1.3) as a positive function. Correspondingly, the Ricci endomorphism r in (1.4) is also an $S^2(\mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3)^K$-valued function.

Proposition 2.3 *For (G, H, K) listed in (1.2), Ricci-flat conditions (2.2) (2.3) and (2.6), respectively, become*

$$L = \frac{\dot{f}_1}{f_1} Q|_{\mathfrak{p}_1} \oplus \frac{\dot{f}_2}{f_2} Q|_{\mathfrak{p}_2} \oplus \frac{\dot{f}_3}{f_3} Q|_{\mathfrak{p}_3}, \tag{2.8}$$

$$\frac{\ddot{f}_j}{f_j} - \left(\frac{\dot{f}_j}{f_j}\right)^2 = -\left(d\frac{\dot{f}_1}{f_1} + d\frac{\dot{f}_2}{f_2} + d\frac{\dot{f}_3}{f_3}\right)\frac{\dot{f}_j}{f_j} + r_j, \quad j = 1, 2, 3 \tag{2.9}$$

and

$$-d\sum_{j=1}^{3}\left(\frac{\dot{f}_j}{f_j}\right)^2 = -\left(\sum_{j=1}^{3} d\frac{\dot{f}_j}{f_j}\right)^2 + R. \tag{2.10}$$

Proof The proof is complete by computation results in Propositions 2.1 and 2.2. □

In summary, constructing a smooth complete cohomogeneity one Ricci-flat metric on M is essentially equivalent to solving $g_{G/K}(t)$ that satisfies (2.8), (2.9) and (2.10). The fundamental theorem of ODE guarantees the existence of solution on neighborhood around $\{t_0\} \times G/K$ for any $t_0 \in (0, \infty)$. In order to have a smooth complete Ricci-flat metric on M, we need to show that

1. (Smooth extension) the solution exists on a tubular neighborhood around G/H and extends smoothly to the singular orbit;
2. (Completeness) the solution exists on $(0, \infty) \times G/K$.

We discuss the smooth extension in Sects. 2.2 and 2.3. The proof for completeness is in Sect. 3.

Table 2 Representations $\mathfrak{p}_j \otimes \mathbb{C}$

Case	$\mathfrak{p}_1 \otimes \mathbb{C}$	$\mathfrak{p}_2 \otimes \mathbb{C}$	$\mathfrak{p}_3 \otimes \mathbb{C}$
I	$\theta_1^2 \otimes \mathbb{I}$	$\theta_1^1 \otimes \theta_2^3$	$\theta_1^{-1} \otimes \theta_2^3$
II	$[1] \otimes [1] \otimes \mathbb{I}$	$[1] \otimes \mathbb{I} \otimes [1]$	$\mathbb{I} \otimes [1] \otimes [1]$
III	Λ_8	Δ_8^+	Δ_8^-

Table 3 Representations $\chi \otimes \mathbb{C}$ and $\mathfrak{q} \otimes \mathbb{C}$

Case	$\chi \otimes \mathbb{C}$ as an H-module	$\chi \otimes \mathbb{C}$ as a K-module	$\mathfrak{q} \otimes \mathbb{C}$ as an H-module
I	$[2] \otimes \mathbb{I}$	$\mathbb{R} \oplus (\theta_1^2 \otimes \mathbb{I})$	$([1] \otimes \theta_2^3) \oplus ([1] \otimes \theta_2^{-3})$
II	$[1, 0] \otimes \mathbb{I}$	$\mathbb{R} \oplus ([1] \otimes [1] \otimes \mathbb{I})$	$[0, 1] \otimes [1]$
III	$[1, 0, 0, 0]$	$\mathbb{R} \oplus \Lambda_8$	$[0, 0, 0, 1]$

2.2 Smoothness extension

It is not difficult to guarantee the smoothness of $g_{G/K}(t)$ at $t = 0$ as a $S^2(\mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3)$-valued function. However, the smooth function does not guarantee the smooth extension of $g = dt^2 + g_{G/K}(t)$ as a metric on G/H as $t \to 0$. By Lemma 1.1 in [23], the question boils down to studying the slice representation $\chi = \mathbb{R}^{d+1}$ of M and the isotropy representation $\mathfrak{q}$ of G/H. We rephrase the lemma below.

Lemma 2.4 [23] *Let $g(t) : [0, \infty) \to S^2(\chi \oplus \mathfrak{q})^K$ be a smooth curve with Taylor expansion at $t = 0$ as $\sum_{l=0}^{\infty} g_l t^l$. Let $W_l = \mathrm{Hom}(S^l(\chi), S^2(\chi \oplus \mathfrak{q}))^H$ be the space of H-equivariant homogeneous polynomials of degree l. Let $\iota \colon W_l \to S^2(\chi \oplus \mathfrak{q})$ denote the evaluation map at $v_0 = (1, 0, \ldots, 0)$. Then the map $g(t)$ has a smooth extension to G/H as a symmetric tensor if and only if $g_l \in \iota(W_l)$ for all l.*

To compute W_l, we need to identify χ and $\mathfrak{q}$ first. Since H acts transitively on H/K, the slice representation $\chi = \mathbb{R}^{d+1}$ of M is irreducible and hence can be identified. Recall that $\mathfrak{q}$ is an irreducible H-module in decomposition (2.7). Hence we have Table 3.

Remark 2.5 Recall the background metric Q on G/K is chosen that $Q|_{\mathfrak{p}_1}$ is the standard metric on $\mathbb{S}^d$. Therefore, the Euclidean inner product $\langle\cdot,\cdot\rangle$ on χ can be written in "polar coordinate" as $dt^2 + t^2\, Q|_{\mathfrak{p}_1}$. As shown in the first column of Table 3, the action of H is essentially the standard representation of $\mathrm{Spin}(d + 1)$ on χ and it preserves $\langle\cdot,\cdot\rangle$. In the following discussion, we take $\langle\cdot,\cdot\rangle \oplus Q|_{\mathfrak{p}_2} \oplus Q|_{\mathfrak{p}_3}$ as the background metric of $T_pM = \chi \oplus T_p(G/H)$ for $p = [H] \in G/H$.

Compare the second column of Table 3 to the first column of Table 2. It is clear that $\chi = \mathbb{R} \oplus \mathfrak{p}_1$ as a K-module. Since χ and $\mathfrak{q}$ are inequivalent H-modules, we have

$$S^2(\chi \oplus \mathfrak{q})^K = S^2(\chi)^K \oplus S^2(\mathfrak{q})^K. \tag{2.11}$$

Hence we have decomposition $W_l = W_l^+ \oplus W_l^-$ where W_l^+ and W_l^- are respectively valued in $S^2(\chi)$ and $S^2(\mathfrak{q})$. We are ready to compute each $W_l^{\pm}$.

Proposition 2.6 *For each M, we have*

$$W_l^+ \cong \begin{cases} \mathbb{R} & l = 0 \\ 0 & l \equiv 1 \mod 2 \\ \mathbb{R}^2 & l \equiv 0 \mod 2, \quad l \geq 2 \end{cases}, \quad W_l^- \cong \mathbb{R}$$

Proof From Table 3, we can derive the decomposition of complexified symmetric products $S^l(\chi) \otimes \mathbb{C}$ and $S^l(\mathfrak{q}) \otimes \mathbb{C}$ as H-modules, as shown in Table 4 below. The proof is complete. □

Table 4 $S^l(\chi)\otimes\mathbb{C}$ and $S^2(\mathfrak{q})\otimes\mathbb{C}$ as H-modules

Case	$S^{2m-1}(\chi)\otimes\mathbb{C}$	$S^{2m}(\chi)\otimes\mathbb{C}$	$S^2(\mathfrak{q})\otimes\mathbb{C}$
I	$\bigoplus_{i=1}^m([4i-2]\otimes\mathbb{I})$	$\bigoplus_{i=0}^m([4i]\otimes\mathbb{I})$	$([2]\otimes\theta_2^6)\oplus([2]\otimes\theta_2^{-6})\oplus([2]\otimes\mathbb{I})\oplus\mathbb{I}$
II	$\bigoplus_{i=1}^m([2i-1,0]\otimes\mathbb{I})$	$\bigoplus_{i=0}^m([2i,0]\otimes\mathbb{I})$	$([0,2]\otimes[2])\oplus([1,0]\otimes\mathbb{I})\oplus\mathbb{I}$
III	$\bigoplus_{i=1}^m[2i-1,0,0,0]$	$\bigoplus_{i=0}^m[2i,0,0,0]$	$[0,0,0,2]\oplus[1,0,0,0]\oplus\mathbb{I}$

Table 5 Real matrix representation of left multiplication of $\mathbf{x}\in\mathbb{F}$

Case	I	II	III
$\mathsf{L}_\mathbf{x}$	$\begin{bmatrix} x_1 & -x_2 \\ x_2 & x_1 \end{bmatrix}$	$\begin{bmatrix} x_1 & -x_2 & -x_3 & -x_4 \\ x_2 & x_1 & -x_4 & x_3 \\ x_3 & x_4 & x_1 & -x_2 \\ x_4 & -x_3 & x_2 & x_1 \end{bmatrix}$	$\begin{bmatrix} x_1 & -x_2 & -x_3 & -x_4 & -x_5 & -x_6 & -x_7 & -x_8 \\ x_2 & x_1 & -x_4 & x_3 & -x_6 & x_5 & x_8 & -x_7 \\ x_3 & x_4 & x_1 & -x_2 & -x_7 & -x_8 & x_5 & x_6 \\ x_4 & -x_3 & x_2 & x_1 & -x_8 & x_7 & -x_6 & x_5 \\ x_5 & x_6 & x_7 & x_8 & x_1 & -x_2 & -x_3 & -x_4 \\ x_6 & -x_5 & x_8 & -x_7 & x_2 & x_1 & x_4 & -x_3 \\ x_7 & -x_8 & -x_5 & x_6 & x_3 & -x_4 & x_1 & x_2 \\ x_8 & x_7 & -x_6 & -x_5 & x_4 & x_3 & -x_2 & x_1 \end{bmatrix}$

In order to apply Lemma 2.4, we need to find generators of each $W_l^\pm$ in Proposition 2.6. Note that $W_l^\pm$ can be viewed as subspaces of $W_{l+2}^\pm$ by multiplying each element with $\sum_{i=0}^d x_i^2$. Hence we only need to find generators of $W_0^\pm$, W_2^+ and W_1^-. It is clear that W_0^+ is spanned by $I_{d+1}\in S^2(\chi)$ and W_0^- is spanned by $I_{2d}\in S^2(\mathfrak{q})$. It is also clear that W_2^+ is generated by the identity map and $(\sum_{i=0}^d x_i^2)I_{d+1}$. Note that the identity map in the form of a homogeneous polynomial is a symmetric matrix Π with $\Pi_{ij}=x_ix_j$ for $i,j\in\{0,1,\dots,d\}$.

The computation for W_1^- is a bit more complicated. We follow Chapter 14 in [26] and consider $\chi=\mathbb{R}\oplus\mathbb{F}$ with $\mathbb{F}$ as one of $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ for cases I, II and III, respectively.

Proposition 2.7 *W_1^- is generated by the $\mathbb{R}$-linear map*

$$\Phi\colon \chi\to S^2(\mathfrak{q})$$
$$(x_0,\mathbf{x})\mapsto\begin{bmatrix} x_0I_d & \mathsf{L}_\mathbf{x} \\ \mathsf{L}_{\bar{\mathbf{x}}} & -x_0I_d, \end{bmatrix}$$

where $\mathsf{L}_\mathbf{x}$ is the real matrix representation of left multiplication of $\mathbf{x}\in\mathbb{F}$, as shown in Table 5 below.

Proof Consider $i\Phi(\chi)$ a subspace of $\mathbb{C}\otimes_\mathbb{R}S^2(\mathfrak{q})$. Since $(i\Phi(x_0,\mathbf{x}))^2=-(x_0^2+\|\mathbf{x}\|^2)I_{d+1}$, it is clear that the matrix multiplication of $i\Phi(\chi)$ generates a Clifford algebra and hence $\mathrm{Spin}(d+1)$. Specifically, the group is generated by elements $\Xi(y_0,\mathbf{y}):=\Phi(-1,\mathbf{0})\Phi(y_0,\mathbf{y})$ with $y_0^2+\|\mathbf{y}\|^2=1$. Since each $\mathbb{F}$ is an alternative algebra that satisfies Moufang identity, computations show

$$Ad(\Xi(y_0,\mathbf{y}))(\Phi(x_0,\mathbf{x}))=\Xi(y_0,\mathbf{y})(\Phi(x_0,\mathbf{x}))\Xi(y_0,\mathbf{y})^{-1}=\Phi(z_0,\mathbf{z}),\tag{2.12}$$

where $z_0=(y_0^2-\|\mathbf{y}\|)x_0+2y_0\langle\mathbf{y},\mathbf{x}\rangle$ and $\mathbf{z}=y_0^2\mathbf{x}-2x_0y_0\mathbf{y}-(\mathbf{y}\bar{\mathbf{x}})\mathbf{y}$. Hence $\Phi(\chi)$ is an $Ad_{\mathrm{Spin}(d+1)}$-invariant subspace in $S^2(\mathfrak{q})$. Moreover, since

$$(Ad(\Xi(y_0,\mathbf{y}))(\Phi(x_0,\mathbf{x})))^2=(\Phi(x_0,\mathbf{x}))^2=(x_0^2+\|\mathbf{x}\|^2)I_{d+1},$$

The adjoint action on $\Phi(\chi)$ induces the standard representation Λ_{d+1} on $\mathbb{R}^{d+1}$. Therefore,

$$\Phi\colon (\chi, \Lambda_{d+1}) \to \left(\Phi(\chi), Ad_{\mathrm{Spin}(d+1)}\right)$$

is H-equivariant and generates W_1^-. □

With the generators known, we are ready to prove the following proposition.

Proposition 2.8 *The necessary and sufficient conditions for a metric $g = dt^2 + g_{G/K}(t)$ on M to extend to a smooth metric in a tubular neighborhood of the singular orbit G/H are*

$$\lim_{t\to 0}(f_1, f_2, f_3, \dot f_1, \dot f_2, \dot f_3) = (0, h_0, h_0, 1, -h_1, h_1) \tag{2.13}$$

for some $h_0 > 0$ and $h_1 \in \mathbb{R}$.

Proof The metric g in LHS of (2.1) can be identified with a map

$$g(t)\colon [0, \epsilon) \to S^2(\chi)^K \oplus S^2(\mathfrak{q})^K \tag{2.14}$$

with Taylor expansion

$$g(t) = \sum_{l=0}^{\infty} g_l t^l. \tag{2.15}$$

Write $g(t) = D(t) \oplus J(t)$, where $D(t)\colon [0,\infty) \to S^2(\chi)$ and $J(t)\colon [0,\infty) \to S^2(\mathfrak{q})$. The Taylor expansion (2.15) can be rewritten as

$$\begin{aligned} D(t) &= D_0 + D_1 t + D_2 t^2 + \ldots \\ J(t) &= J_0 + J_1 t + J_2 t^2 + \ldots \end{aligned} \tag{2.16}$$

Since $W_2^+/W_0^+ \cong \mathbb{R}$, in principle there is a free variable for the second derivative of a smooth $D(t)$. However, with the geometric setting that t is a unit speed geodesic, the choice of D_2 is in fact determined by D_0. Hence we take $D_0 = I_{d+1}$ and D_2 must be a multiple of $\left((\sum_{i=0}^d x_i^2) I_{d+1} - \Pi\right)(v_0) = \begin{bmatrix} 0 & \\ & I_d \end{bmatrix}$ with the multiplier determined by the choice of D_0. Since H/K is and irreducible sphere, it is expected that there is no indeterminacy from $D(t)$. By Lemma 2.4, the smooth condition for $D(t)$ with respect to background metric $\langle\cdot,\cdot\rangle$ is $D(t) = I_{d+1} + O(t^2)$. This is consistent with Lemma 9.114 in [3].

As g degenerates to an invariant metric on G/H and the isotropy representation of G/H is irreducible, J_0 is a positive multiple of I_{2d}. The evaluation of Φ at v_0 in Proposition 2.7 is $\begin{bmatrix} I_d & \\ & -I_d \end{bmatrix}$. Hence by Lemma 2.4, the smoothness condition for $J(t)$ is

$$J(t) = \begin{bmatrix} f_2^2(t) I_d & \\ & f_3^2(t) I_d \end{bmatrix} = c_0 I_{2d} + c_1 \begin{bmatrix} I_d & \\ & -I_d \end{bmatrix} t + O(t^2)$$

for some $c_0 > 0$ and $c_1 \in \mathbb{R}$.

Recall 2.5, note that $\langle\cdot,\cdot\rangle = dt^2 + t^2\, Q|_{\mathfrak{p}_1}$. Switch the background metric to $dt^2 + Q$, we conclude that the smoothness condition for g is

$$\begin{aligned} f_1^2(t) &= t^2 + O(t^4) \\ f_2^2(t) &= c_0 + c_1 t + O(t^2) \\ f_3^2(t) &= c_0 - c_1 t + O(t^2) \end{aligned}$$

Then the proof is complete. □

Table 6 H-decomposition of $\bigwedge^d \mathfrak{q}\otimes\mathbb{C}$ and dimension of each summand

Case	H	H-decomposition of $\bigwedge^d \mathfrak{q}\otimes\mathbb{C}$ and Dimension of each Summand
I	$S(U(2)U(1))$	$\bigwedge^2\left(([1]\otimes\theta_2^3)\oplus([1]\otimes\theta_2^3)\right)=(\mathbb{I}\otimes\theta_2^6)\oplus(\mathbb{I}\otimes\theta_2^{-6})\oplus\mathbb{I}\oplus\{[2]\otimes\mathbb{I}\}$ $\mathbf{6=1+1+1+\{3\}}$
II	$Sp(2)Sp(1)$	$\bigwedge^4[01]\otimes[1]=[01]\otimes[2]\oplus[02]\otimes\mathbb{I}\oplus\mathbb{I}\otimes[4]\oplus\mathbb{I}\oplus\{[02]\otimes[2]\oplus[01]\otimes\mathbb{I}\}$ $\mathbf{70=15+14+5+1+\{30+5\}}$
III	Spin(9)	$\bigwedge^8[0001]=[2010]\oplus[0020]\oplus[1002]\oplus[0200]\oplus[4000]\oplus[0010]\oplus[2000]\oplus\mathbb{I}$ $\oplus\{[2002]\oplus[0110]\oplus[1010]\oplus[3000]\oplus[0002]\oplus[1000]\}$ $\mathbf{12870=2457+1980+924+495+450+84+44+1}$ $\mathbf{+\{3900+1650+594+156+126+9\}}$

Remark 2.9 The Ricci-flat ODE system (2.9) and (2.10) is invariant under the homothetic change $\kappa^2(dt^2+g_{G/K})$ with $ds=\kappa dt$. The smooth initial condition 2.13 is transformed to $(0,\kappa h_0,\kappa h_0,1,h_1,-h_1)$. Hence if we abuse the notation. Multiplying h_0 by $\kappa>0$ while having $\dot{f}_j(0)$ unchanged give the smooth initial condition for metrics in the same homothetic family. Therefore, in the original coordinate, h_1 is the free variable that gives non-homothetic metrics. As shown in (2.27), only h_1 matters in producing different curves in the polynomial system.

Combine the analysis in Proposition 2.8 with the main result in [23], we conclude that there exists a 1-parameter family of Ricci-flat metric on a neighborhood around G/H in M. We derive the same result in Sect. 2.3 using a new coordinate.

Remark 2.10 Note that we always have $\lim_{t\to 0}\frac{\dot{f}_3}{f_3}+\frac{\dot{f}_2}{f_2}=0$, i.e., the mean curvature of G/H vanishes at $t=0$. This is consistent with Corollary 1.1 in [27]. The last two components of (2.13) shows that the smooth extension does not require G/H to be totally geodesic. If h_1 in (2.13) vanishes, then we recover cases in [4,33] with $f_2\equiv f_3$.

Remark 2.11 It is worth pointing out that Eqs. (2.9) and (2.10) are symmetric among f_1, f_2 and f_3. Therefore, initial condition (2.13) has two other counterparts where f_2 or f_3 collapses initially depending how H is embedded in G. Without loss of generality, we will consider initial condition (2.13) in this article.

We end this section by identifying each vector bundle M as a (sub)bundle of ASD d-form of lowest rank. Table 6 lists out H-decomposition of $\bigwedge^d \mathfrak{q}\otimes\mathbb{C}$ and dimension of each irreducible summand. The subspace $\bigwedge^d_- \mathfrak{q}\otimes\mathbb{C}$ consists of summands in brace brackets. Decomposition below is mostly computed via software LiE, with reference in [5,12,29].

For Case I, it is known that the trivial representation generates the invariant Kähler form on $\mathbb{CP}^2$. The bundle that we study in this paper is the associated bundle with respect to representation $[2]\otimes\mathbb{I}$, which is the bundle of ASD 2-form $\bigwedge^2_- T^*\mathbb{CP}^2$ that admits a complete smooth G_2 metric [7,25].

For Case II, the trivial representation generates a canonical 4-form for Quaternionic Kähler manifolds, as described in [29]. Explicitly, given a Quaternionic Kähler manifold with a triple of complex structures (I, J, K) and corresponding symplectic forms $(\omega_I, \omega_J, \omega_K)$, the canonical 4-form is defined as $\Omega = \omega_I \wedge \omega_I + \omega_J \wedge \omega_J + \omega_K \wedge \omega_K$. By Table 3, M is an associate bundle with respect to representation $[0, 1] \otimes \mathbb{I}$ in $\bigwedge_-^4 \mathfrak{q}_2 \otimes \mathbb{C}$. Therefore, M is indeed an irreducible subbundle of $\bigwedge_-^4 T^*\mathbb{HP}^2$.

For Case III, the trivial representation generates the canonical 8-form, whose existence is proved in [5]. Explicit formula for the canonical 8-form can be found in [12]. The nine-dimensional representation $[1, 0, 0, 0]$ is the (twisted) adjoint representation of Spin(9) on $\mathbb{R}^9$. Similar to Case II, the bundle that we consider in this paper is an irreducible subbundle of $\bigwedge_-^8 T^*\mathbb{OP}^2$.

In conclusion, the name "(sub)bundle of ASD d-form of lowest rank" for M is justified.

2.3 Coordinate change and linearization

We apply the coordinate change introduced in [19,20] to the Ricci-flat system in this section. The original ODE system is transformed to a polynomial one. As described in Remark 2.16, some critical points of the new system carry geometric data. Linearizations at these critical points provide guidance on how integral curves potentially behave, which help us to construct a compact invariant set in Sect. 3 to prove the completeness.

As predicted by the result in the previous section (Remark 2.9), analysis on the new system shows that there exists a 1-parameter family of integral curves with each represents a homothetic class of Ricci-flat metrics on a neighborhood around G/H.

Consider

$$d\eta = \mathrm{tr}(L)dt. \tag{2.17}$$

Define

$$X_j := \frac{\frac{\dot{f}_j}{f_j}}{\mathrm{tr}(L)}, \quad Z_j := \frac{\frac{f_j}{f_k f_l}}{\mathrm{tr}(L)}. \tag{2.18}$$

And define

$$\mathcal{R}_j := \frac{r_j}{(\mathrm{tr}(L))^2} = aZ_kZ_l + b\left(Z_j^2 - Z_k^2 - Z_l^2\right), \quad \mathcal{G} := \sum_{j=1}^3 dX_j^2, \quad \mathcal{H} := \sum_{j=1}^3 dX_j.$$

Use $'$ to denote derivative with respect to η. In the new coordinates given by (2.17) and (2.18), the system (2.9) is transformed to

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ Z_1 \\ Z_2 \\ Z_3 \end{bmatrix}' = V(X_1, X_2, X_3, Z_1, Z_2, Z_3) := \begin{bmatrix} X_1(\mathcal{G} - 1) + \mathcal{R}_1 \\ X_2(\mathcal{G} - 1) + \mathcal{R}_2 \\ X_3(\mathcal{G} - 1) + \mathcal{R}_3 \\ Z_1\left(\mathcal{G} - \frac{\mathcal{H}}{d} + 2X_1\right) \\ Z_2\left(\mathcal{G} - \frac{\mathcal{H}}{d} + 2X_2\right) \\ Z_3\left(\mathcal{G} - \frac{\mathcal{H}}{d} + 2X_3\right) \end{bmatrix}, \tag{2.19}$$

and the conservation law (2.10) becomes

$$C: \mathcal{G} - 1 + d\sum_j \mathcal{R}_j = 0. \tag{2.20}$$

As $\left(\frac{1}{\mathrm{tr}(L)}\right)' = \frac{\mathcal{G}}{\mathrm{tr}(L)}$, the original variables can be recovered by

$$t = \int_{\eta_0}^{\eta} \exp\left(\int_{\tilde{\eta}_0}^{\tilde{\eta}} \mathcal{G} \mathrm{d}\tilde{\tilde{\eta}} + \tilde{t}_0\right) \mathrm{d}\tilde{\eta} + t_0, \quad f_j = \frac{\exp\left(\int_{\eta_0}^{\eta} \mathcal{G} \mathrm{d}\tilde{\eta} + t_0\right)}{\sqrt{Z_k Z_l}}. \tag{2.21}$$

Remark 2.12 The new variables X_j's record the relative size of each principal curvature of G/K. Variables Z_j's carry the data of relative size of each f_j's. Note that $\frac{Z_j}{Z_k} = \frac{f_j^2}{f_k^2}$.

In the original coordinates, a smooth solution to (2.9) is an integral curve with variable $t \in [0, \epsilon)$. Since by (2.17), $\lim_{t\to 0} \eta = \lim_{t\to 0} \ln\left(f_1^d f_2^d f_3^d\right) + \hat{\eta} = -\infty$, the original solution is transformed to an integral curve with variable $\eta \in (-\infty, \epsilon')$ for some $\epsilon' \in \mathbb{R}$. Note that the graph of the integral curve does not change when homothetic change is applied to the original variable. Hence each integral curve to the new system represent a solution in the original coordinate up to homothety.

Remark 2.13 It is clear that the symmetry mentioned in Remark 2.11 remains among pairs (X_j, Z_j)'s in the new system (2.19) with (2.20). In addition, by the observation on Z_j's derivative. It is clear that they do not change sign along the integral curve. Without loss of generality, we focus on the region where these three variables are positive. This observation provides basic estimates needed in our construction of compact invariant set [the set P introduced in (3.1)].

Remark 2.14 It is clear that $\mathcal{H} \equiv 1$ by the definition variable X_j. In fact, since $\mathcal{H}' = (\mathcal{H} - 1)(\mathcal{G} - 1)$ on C, the set $C \cap \{\mathcal{H} \equiv 1\}$ is flow-invariant. Furthermore, $C \cap \{\mathcal{H} \equiv 1\}$ is diffeomorphic to a level set

$$dX_1^2 + dX_2^2 + d\left(\frac{1}{d} - X_1 - X_2\right)^2 - 1 + d\sum_j \mathcal{R}_j = 0$$

in $\mathbb{R}^5$. Therefore, $C \cap \{\mathcal{H} \equiv 1\}$ is a four-dimensional smooth manifold by the inverse function theorem. System (2.19) can be restricted to a four-dimensional subsystem on $C \cap \{\mathcal{H} \equiv 1\}$.

Proposition 2.15 *The complete list of critical points of system* (2.19) *in* $C \cap \{\mathcal{H} \equiv 1\}$ *is the following:*

I. *the set* $\left\{(x_1, x_2, x_3, 0, 0, 0) \mid \sum_{j=1}^3 x_j^2 = \frac{1}{d}, \quad \sum_{j=1}^3 x_j = \frac{1}{d}\right\}$;
II. $\left(-\frac{1}{d}, \frac{1}{d}, \frac{1}{d}, \pm\frac{1}{d}\sqrt{\frac{3-d}{b}}, 0, 0\right)$ *and its counterparts with pairs* (X_j, Z_j)*'s permuted. This critical point occurs only for Case I;*
III. $\left(\frac{1}{d}, 0, 0, 0, \pm\frac{1}{d}, \pm\frac{1}{d}\right)$ *and its counterparts with pairs* (X_j, Z_j)*'s permuted;*
IV. $\left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \pm\frac{2b}{d-1}\frac{1}{n}\sqrt{\frac{(n-1)(d-1)}{b(a+2b)}}, \pm\frac{2b}{d-1}\frac{1}{n}\sqrt{\frac{(n-1)(d-1)}{b(a+2b)}}, \pm\frac{1}{n}\sqrt{\frac{(n-1)(d-1)}{b(a+2b)}}\right)$ *and its counterparts with pairs* (X_j, Z_j)*'s permuted;*
V. $\left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \pm\frac{1}{n}\sqrt{\frac{n-1}{a-b}}, \pm\frac{1}{n}\sqrt{\frac{n-1}{a-b}}, \pm\frac{1}{n}\sqrt{\frac{n-1}{a-b}}\right)$.

Proof The proof is processed by direct computations. □

By Remark 2.13, we focus on critical points with nonnegative Z_j's.

Remark 2.16 Some critical points in Proposition 2.15 have further geometric significance.

- $p_0 := \left(\frac{1}{d}, 0, 0, 0, \frac{1}{d}, \frac{1}{d}\right)$
This critical point is the initial condition (2.13) under the new coordinate (2.17) and (2.18), i.e., (2.13) becomes $\lim_{\eta\to-\infty}(X_1, X_2, X_3, Z_1, Z_2, Z_3) = p_0$. Hence we study integral curves emanating from p_0. In order to prove the completeness, we construct a compact invariant set in Sect. 3 that contains p_0 in its boundary and traps the integral curve initially.
By Remarks 2.11 and 2.13, its two other counterparts $p_0' = \left(0, \frac{1}{d}, 0, \frac{1}{d}, 0, \frac{1}{d}\right)$ and $p_0'' = \left(0, 0, \frac{1}{d}, \frac{1}{d}, \frac{1}{d}, 0\right)$ also have the similar geometric meaning depending on how H is embedded in G.
- $p_1 := \left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \frac{1}{n}\sqrt{\frac{n-1}{a-b}}, \frac{1}{n}\sqrt{\frac{n-1}{a-b}}, \frac{1}{n}\sqrt{\frac{n-1}{a-b}}\right)$
This critical point is symmetric among all (X_j, Z_j)'s. Note that $\frac{f_j^2}{f_k^2}(p_1) = \frac{Z_j}{Z_k}(p_1) = 1$, all f_j's are equal at this point. We prove in Sect. 4 that p_1 represents an AC end for the complete Ricci-flat metric represented by the integral curve emanating from p_0. The conical limit is a metric cone over a suitable multiple of the normal Einstein metric on G/K.
- $p_2 := \left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \frac{2b}{d-1}\frac{1}{n}\sqrt{\frac{(n-1)(d-1)}{b(a+2b)}}, \frac{2b}{d-1}\frac{1}{n}\sqrt{\frac{(n-1)(d-1)}{b(a+2b)}}, \frac{1}{n}\sqrt{\frac{(n-1)(d-1)}{b(a+2b)}}\right)$
Since $r_j(p_2)$ are all equal, this point represent an invariant Einstein metric on G/K other than the one represented by p_1. In the following text, we call the metric the "alternative Einstein metric." For Case I, it is a Kähler–Einstein metric. It has two other counterparts with permuted Z_j's.
Although we do not find any integral curve with its limit as p_2, we show in Sect. 5 that there exists an integral curve emanating from p_2 and tends to p_1, representing a singular Ricci-flat metric with a conical singularity and an AC end.

The linearization $\mathcal{L}$ of vector field V in (2.19) is

$$\begin{bmatrix} \mathcal{G}-1+2dX_1^2 & 2dX_1X_2 & 2dX_1X_3 & 2bZ_1 & aZ_3-2bZ_2 & aZ_2-2bZ_3 \\ 2dX_1X_2 & \mathcal{G}-1+2dX_2^2 & 2dX_2X_3 & aZ_3-2bZ_1 & 2bZ_2 & aZ_1-2bZ_3 \\ 2dX_1X_3 & 2dX_2X_3 & \mathcal{G}-1+2dX_3^2 & aZ_2-2bZ_1 & aZ_1-2bZ_2 & 2bZ_3 \\ (2dX_1+1)Z_1 & (2dX_2-1)Z_1 & (2dX_3-1)Z_1 & \mathcal{G}-\frac{\mathcal{H}}{d}+2X_1 & 0 & 0 \\ (2dX_1-1)Z_2 & (2dX_2+1)Z_2 & (2dX_3-1)Z_2 & 0 & \mathcal{G}-\frac{\mathcal{H}}{d}+2X_2 & 0 \\ (2dX_1-1)Z_3 & (2dX_2-1)Z_3 & (2dX_3+1)Z_3 & 0 & 0 & \mathcal{G}-\frac{\mathcal{H}}{d}+2X_3 \end{bmatrix} \tag{2.22}$$

With (2.22), we can compute the dimension of the unstable subspace at p_0. As we are considering system (2.19) on $C \cap \{\mathcal{H} \equiv 1\}$, we require each unstable eigenvector to be tangent to $C \cap \{\mathcal{H} \equiv 1\}$. The normal vector field to the hypersurfaces C and $\{\mathcal{H} \equiv 1\}$ are respectively

$$N_C = \begin{bmatrix} 2dX_1 \\ 2dX_2 \\ 2dX_3 \\ adZ_2+adZ_3-2bdZ_1 \\ adZ_1+adZ_3-2bdZ_2 \\ adZ_2+adZ_1-2bdZ_3 \end{bmatrix}, \quad N_{\{\mathcal{H}\equiv 1\}} = \begin{bmatrix} 1\\1\\1\\0\\0\\0 \end{bmatrix}. \tag{2.23}$$

Lemma 2.17 *The unstable subspace of system (2.19) at p_0, restricted on $C \cap \{\mathcal{H} \equiv 1\}$, is of dimension 2.*

Proof Hence the linearization at p_0 is

$$\mathcal{L}(p_0) = \begin{bmatrix} \frac{3}{d}-1 & 0 & 0 & 0 & \frac{a-2b}{d} & \frac{a-2b}{d} \\ 0 & \frac{1}{d}-1 & 0 & \frac{a}{d} & \frac{2b}{d} & -\frac{2b}{d} \\ 0 & 0 & \frac{1}{d}-1 & \frac{a}{d} & -\frac{2b}{d} & \frac{2b}{d} \\ 0 & 0 & 0 & \frac{2}{d} & 0 & 0 \\ \frac{1}{d} & \frac{1}{d} & -\frac{1}{d} & 0 & 0 & 0 \\ \frac{1}{d} & -\frac{1}{d} & \frac{1}{d} & 0 & 0 & 0 \end{bmatrix}. \tag{2.24}$$

Eigenvalues and corresponding eigenvectors of (2.24) are

$$\lambda_1 = \frac{1}{d}, \quad \lambda_2 = \lambda_3 = \frac{2}{d}, \quad \lambda_4 = \lambda_5 = \frac{1}{d} - 1, \quad \lambda_6 = -1.$$

$$v_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ -2 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ \frac{a}{d+1} \\ \frac{a}{d+1} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1-d \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad v_5 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$v_6 = \begin{bmatrix} 0 \\ 4b \\ -4b \\ 0 \\ -1 \\ 1 \end{bmatrix}. \tag{2.25}$$

With Remarks 1.1, 2.14 and (2.23), it is clear that

$$T_{p_0}(C \cap \{\mathcal{H} \equiv 1\}) = \operatorname{span}\{v_1, (d+1)v_3 - av_2, 2v_4 + (d-1)v_5, v_6\}.$$

By (2.25), an unstable subspace at p_0 is spanned by v_1 and $(d+1)v_3 - av_2$. □

Solutions of the linearized equations at p_0 have the form

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} = p_0 + s_0 e^{\frac{2\eta}{d}}((d+1)v_4 - av_3) + s_1 e^{\frac{\eta}{d}} v_1 = \begin{bmatrix} \frac{1}{d} \\ 0 \\ 0 \\ 0 \\ \frac{1}{d} \\ \frac{1}{d} \end{bmatrix} + s_0 e^{\frac{2\eta}{d}} \begin{bmatrix} -2a \\ a \\ a \\ d+1 \\ -a \\ -a \end{bmatrix} + s_1 e^{\frac{\eta}{d}} \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ -2 \\ 2 \end{bmatrix}, \tag{2.26}$$

for some $s_0 > 0$ and $s_1 \in \mathbb{R}$. Recall Remark 2.13. In order to let Z_1 be positive initially, the assumption $s_0 > 0$ is necessary.

It is clear that there is a 1 to 1 correspondence between the germ of linearized solution (2.26) around p_0 and $[s_0 : s_1^2]$ in $\mathbb{RP}^2$. We fix $s_0 > 0$ in the following text. By Hartman–Grobman theorem, there is a 1 to 1 correspondence between each (2.26) and local solution to

(2.19). Hence for a fixed $s_0 > 0$, there is no ambiguity to use γ_{s_1} to denote an integral curve to system (2.19) on (2.20) with

$$\gamma_{s_1} \sim p_0 + s_0 e^{\frac{2\eta}{d}}((d+1)v_4 - av_3) + s_1 e^{\frac{\eta}{d}} v_1$$

near p_0.

Analysis above shows that there exists a 1-parameters family of short-time existing integral curves of system (2.19) on (2.20). Since each curve corresponds to a homothetic class of Ricci-flat metrics defined on a neighborhood around singular orbit G/H, there exists a 1-parameters family of non-homothetic Ricci-flat metrics defined on a neighborhood around G/H. Recall Remark 2.9, the result is consistent with the main theorem in [23].

Remark 2.18 By the unstable version of Theorem 4.5 in [15], from (2.13) we know that

$$\frac{2h_1}{\sqrt{d}} = \lim_{t\to 0} \frac{\left(\frac{\dot{f}_3}{f_3} - \frac{\dot{f}_2}{f_2}\right)\sqrt{f_2 f_3}}{\sqrt{\operatorname{tr}(L) f_1}} = \lim_{\eta\to\infty} \frac{X_3 - X_2}{\sqrt{Z_1}} = \frac{2s_1}{\sqrt{(d+1)s_0}}. \tag{2.27}$$

Hence the parameter s_1 vanishes if and only if h_1 does. The solution with $s_1 = 0$ corresponds to the subsystem of (2.19) where $(X_2, Z_2) \equiv (X_3, Z_3)$ is imposed, which corresponds to the subsystem of the original system (2.9) where $f_2 \equiv f_3$ is imposed. The reduced system is essentially the same as the one for the case where the isotropy representation has two inequivalent irreducible summands. For Case I, γ_0 represents the smooth complete G_2 metric in [7,25]. For Case II and Case III, Ricci-flat metrics with $s_1 = 0$ are proved to be complete in [4,33].

Our construction does not assume the vanishing of s_1. By the symmetry of the ODE system, we mainly focus on the situation where $s_1 \geq 0$ without loss of generality.

Suppose an integral curve γ_{s_1} is defined on $\mathbb{R}$, then by Lemma 5.1 in [9], functions $f_j(t)$'s are defined on $[0, \infty)$. Therefore, Theorem 1.2 is proved once γ_{s_1} is shown to be defined on $\mathbb{R}$.

3 Completeness

With smooth extension of metrics represented by γ_{s_1} proved, the next step is to show that γ_{s_1} is defined on $\mathbb{R}$ so that the Ricci-flat metric it represents is complete. Our construction is divided into two parts. The first part is to find an appropriate compact invariant set $\hat{S}_3$ with p_0 sitting on its boundary. Although p_0 is in the boundary of $\hat{S}_3$, integral curves are not trapped in the set initially unless $s_1 = 0$. In the second step, we construct another compact set that serves as an *entrance zone*. It traps γ_{s_1} initially as long as s_1 is close enough to zero. Moreover, integral curves trapped in this set cannot escape through some part of its boundary and they are forced to enter $\hat{S}_3$. Hence such a γ_{s_1} must be defined on $\mathbb{R}$.

3.1 Compact invariant set

We describe the first step in this section. There is a subtle difference between the compact invariant set for Case I and ones for Cases II and III. We first construct the set for Cases II and III since it is simpler.

Let $\rho=\sqrt{\frac{a+2b}{2}}$. It is clear that $\rho\geq 1$ and equality holds exactly in Case I. Define

$$\begin{aligned} P&=\{Z_1,Z_2,Z_3\geq 0\}\\ \tilde{S_3}&=\bigcap_{j=1}^{2}\left\{Z_3-Z_j\geq 0,\quad X_3-X_j+\rho(Z_3-Z_j)\geq 0,\quad X_3\geq 0\right\}. \end{aligned} \tag{3.1}$$

And define

$$S_3=C\cap\{\mathcal{H}\equiv 1\}\cap P\cap\tilde{S_3}. \tag{3.2}$$

Before doing further analysis on S_3, we give some explanations as to why it is constructed in this way. Note that the positivity of Z_j's are immediate by Remark 2.13. The first inequality in $\tilde{S_3}$ is to require Z_3 to be the largest variable among Z_j's. Equivalently, it requires f_3 to be the largest among f_j's in the original coordinate. This condition is indicated by the subscript of $\tilde{S_3}$ and S_3. A direct consequence of this assumption is that we can assume $X_3\geq 0$ along γ_{s_1} as shown in (3.9).

It is easy to check that $p_0\in S_3$ hence the set is nonempty. Each inequality in (3.2) defines a closed subset in $\mathbb{R}^7$ whose boundary is defined by the equality. Therefore, a point $x\in\partial S_3$ if there exists at least one defining inequality in (3.1) reaches equality at x. For Case II and III, functions

$$X_3,Z_1,Z_3-Z_2,X_3-X_2+\rho(Z_3-Z_2) \tag{3.3}$$

among those in (3.1) vanish at p_0. The point is hence in ∂S_3. Substitute (2.26) to functions in (3.3). It is clear that γ_{s_1} is trapped in S_3 initially if $s_1\geq 0$. By Remark 2.18, we know that γ_0 is trapped in ∂S_3 with $(X_2,Z_2)\equiv(X_3,Z_3)$.

Proposition 3.1 *In the set $S_3\cap\{2bZ_3-a(Z_1+Z_2)\leq 0\}$, we have estimate*

$$Z_1+Z_2\leq 2\sqrt{\frac{n-1}{n^2(a-b)}}. \tag{3.4}$$

Proof By the conservation law (2.20), it follows that

$$0\geq\frac{1}{n}-1+da(Z_2Z_3+Z_1Z_3+Z_1Z_2)-db(Z_1^2+Z_2^2+Z_3^2). \tag{3.5}$$

Note that the RHS of (3.5) is symmetric between Z_1 and Z_2. It is convenient to find the maximum of Z_1+Z_2 on $S_3\cap\{Z_2\geq Z_1\}$ first. By the symmetry between Z_1 and Z_2 in (3.5), such a maximum is the maximum of Z_1+Z_2 in S_3. With the assumption $Z_2\geq Z_1$, we write $Z_1=\nu Z_2$ for some $\nu\in[0,1]$. Fix such a ν. Then (3.5) becomes

$$\begin{aligned} 0&\geq\frac{1}{n}-1+da(Z_2Z_3+\nu Z_2Z_3+\nu Z_2^2)-db(\nu^2Z_2^2+Z_2^2+Z_3^2)\\ &=\frac{1}{n}-1+d(-bZ_3^2+a(1+\nu)Z_2Z_3+(a\nu-b(1+\nu^2))Z_2^2). \end{aligned} \tag{3.6}$$

Define $\mathcal{F}(Z_3)=-bZ_3^2+a(1+\nu)Z_2Z_3+(a\nu-b(1+\nu^2))Z_2^2$. Consider the set $S_3\cap\{2bZ_3-a(Z_1+Z_2)\leq 0\}\cap\{Z_1=\nu Z_2\}$, we have

$$Z_2\leq Z_3\leq\frac{a}{2b}(1+\nu)Z_2.$$

Hence for any fixed ν and Z_2, the minimum of $\mathcal{F}$ in $S_3 \cap \{2bZ_3 - a(Z_1+Z_2) \leq 0\} \cap \{Z_1 = \nu Z_2\}$ is reached at $Z_3 = Z_2$. Therefore, computation (3.6) continues as

$$\begin{aligned} 0 &\geq \frac{1}{n} - 1 + d\left(-b + a(1+\nu) + (\nu a - b(1+\nu^2))\right) Z_2^2 \\ &= \frac{1}{n} - 1 + d\left(-b\nu^2 + 2a\nu + a - 2b\right) Z_2^2. \end{aligned} \tag{3.7}$$

The coefficient of Z_2^2 in (3.7) can be easily checked to be positive. It follows that

$$\begin{aligned} (Z_1+Z_2)^2 = (1+\nu)^2 Z_2^2 &\leq \left(1 - \frac{1}{n}\right) \frac{(1+\nu)^2}{d(-b\nu^2 + 2a\nu + a - 2b)} \\ &= \left(1 - \frac{1}{n}\right) \frac{1}{d\left(-b + 2(a+b)\frac{1}{1+\nu} - (a+3b)\frac{1}{(1+\nu)^2}\right)}. \end{aligned} \tag{3.8}$$

Consider function $h\left(\frac{1}{1+\nu}\right) = -(a+3b)\frac{1}{(1+\nu)^2} + 2(a+b)\frac{1}{1+\nu} - b$. Since by Remark 1.1, we have $\frac{1}{2} \leq \frac{a+b}{a+3b} \leq 1$, the minimum of h is either $h\left(\frac{1}{2}\right)$ or $h(1)$. Computation shows $h\left(\frac{1}{2}\right) < h(1)$. We conclude that $(Z_1+Z_2)^2 \leq \left(1 - \frac{1}{n}\right)\frac{1}{d}\frac{1}{h\left(\frac{1}{2}\right)} = 4\frac{n-1}{n^2(a-b)}$. Hence the proof is complete. Note that the equality in (3.4) is reached by p_1. □

Proposition 3.2 *For Cases II and III, integral curves γ_{s_1} to system (2.19) on $C_0 \cap \{\mathcal{H} \equiv 1\}$ emanating from p_0 with $s_1 \geq 0$ do not escape S_3.*

Proof Two perspectives can be taken in the following computations that frequently appear through out this article. First is to view algebraic expressions in (3.1) as functions along γ_{s_1} and they all vanish at p_0. Integral curves emanating from p_0 being trapped in S_3 initially is equivalent to these defining functions being positive near p_0. To show that γ_{s_1} does not escape S_3 is to show the nonnegativity of these functions along the integral curves. Suppose one of these functions vanishes at some point along the integral curves for the first time. We want to show that its derivative at that point is nonnegative.

The second perspective is to consider ∂S_3 as a union of subsets of a collection of linear and quadratic varieties. Require the restriction of the vector field V in (2.19) on each of these subsets to point inward S_3. If such a requirement is met, then it is impossible for the integral curves to escape if they are initially in S_3. Both perspectives lead to the same computation of inner product between V and the gradient of each defining function in (3.1). Then require the inner product to be nonnegative if the gradient points inward S_3. It might not be true that the inner product is nonnegative on each variety globally. But all we need is the nonnegativity on its subsets that ∂S_3 consists of.

By definition of S_3, we automatically have

$$\mathcal{R}_3 = aZ_1Z_2 + b(Z_3^2 - Z_1^2 - Z_2^2) = \begin{cases} Z_2(aZ_1 - bZ_2) + b(Z_3^2 - Z_1^2) \geq 0 \text{ if } Z_1 \geq Z_2 \\ Z_1(aZ_2 - bZ_1) + b(Z_3^2 - Z_2^2) \geq 0 \text{ if } Z_2 \geq Z_1 \end{cases}. \tag{3.9}$$

On $X_3 = 0$, we have $\langle \nabla(X_3), V \rangle|_{X_3=0} = \mathcal{R}_3 \geq 0$ by (3.9). Hence X_3 is nonnegative along every γ_{s_1} that is trapped in S_3 initially.

Next we need to show that the integral curves cannot escape from the part of $\partial\tilde{S}_3$ that is in ∂S_3. For distinct $j, k \in \{1, 2\}$, it follows that

$$\begin{aligned}\langle\nabla(Z_3 - Z_j), V\rangle\big|_{Z_3-Z_j=0} &= Z_3\left(\mathcal{G} - \frac{1}{d} + 2X_3\right) - Z_j\left(\mathcal{G} - \frac{1}{d} + 2X_j\right)\\ &= 2Z_3(X_3 - X_j) \quad \text{since } Z_3 - Z_j = 0\\ &\geq 2\rho Z_3(Z_j - Z_3) \quad \text{by definition of } S_3\\ &= 0 \quad \text{since } Z_3 - Z_j = 0.\end{aligned}$$

Although it is not clear if $X_3 - X_j \geq 0$ along γ_{s_1}, we impose a weaker condition, which is the second inequality in $\tilde{S}_3$. What it means is to allow $Z_3 - Z_j$ to decrease, yet the rate of its decreasing cannot be too steep so that $Z_3 - Z_j$ increases before it could decrease to zero. Fortunately, the weaker condition does hold along the integral curves.

$$\begin{aligned}&\langle\nabla(X_3 - X_j + \rho(Z_3 - Z_j)), V\rangle\big|_{X_3-X_j+\rho(Z_3-Z_j)=0}\\ &= (X_3 - X_j + \rho(Z_3 - Z_j))(\mathcal{G} - 1) + \mathcal{R}_3 - \mathcal{R}_j + \rho Z_3\left(1 - \frac{1}{d} + 2X_3\right)\\ &\quad - \rho Z_j\left(1 - \frac{1}{d} + 2X_j\right)\\ &= (Z_3 - Z_j)\left(2b(Z_3 + Z_j) - aZ_k + \rho\left(1 - \frac{1}{d}\right) + 2\rho X_3 - 2\rho^2 Z_j\right) \quad . \quad (3.10)\\ &\quad \text{since } X_j = X_3 + \rho(Z_3 - Z_j)\\ &\geq (Z_3 - Z_j)\left(2bZ_3 - a(Z_j + Z_k) + \rho\left(1 - \frac{1}{d}\right)\right) \quad \text{since } X_3 \geq 0 \text{ in } S_3\\ &= (Z_3 - Z_j)\left(2bZ_3 - a(Z_1 + Z_2) + \rho\left(1 - \frac{1}{d}\right)\right)\end{aligned}$$

If $2bZ_3 - a(Z_1 + Z_2) \geq 0$, then the last line of computation above is obviously nonnegative. If $2bZ_3 - a(Z_1 + Z_2) \leq 0$, then (3.10) continues as

$$\geq (Z_3 - Z_j)\left((b - a)(Z_1 + Z_2) + \rho\left(1 - \frac{1}{d}\right)\right) \tag{3.11}$$

since $Z_3 \geq \frac{Z_1+Z_2}{2}$ in S_3. Apply Proposition 3.1, we know that (3.11) is nonnegative if

$$\frac{\rho(d-1)}{d(a-b)} \geq 2\sqrt{\frac{n-1}{n^2(a-b)}}. \tag{3.12}$$

Straightforward computations show that

Case	ρ	$\frac{\rho(d-1)}{d(a-b)}$	$2\sqrt{\frac{n-1}{n^2(a-b)}}$
I	1	$\frac{2}{5}$	$\frac{2}{3}$
II	$\sqrt{\frac{5}{2}}$	$\frac{3\sqrt{10}}{28} \approx 0.339$	$\frac{\sqrt{154}}{42} \approx 0.295$
III	$\sqrt{\frac{11}{2}}$	$\frac{7\sqrt{22}}{128} \approx 0.257$	$\frac{\sqrt{46}}{48} \approx 0.141$

Inequality (3.12) holds only for Cases II and III. Hence for Cases II and III, integral curves γ_{s_1} emanating from p_0 does not escape S_3 if $s_1 \geq 0$.

Although estimate (3.4) is sharp in S_3, inequality (3.10) has room to be improved as we dropped a nonnegative term $2\rho X_3$ in the computation. It turns out (3.10) can be proved to be nonnegative for Case I with an additional inequality, as demonstrated in Proposition 3.4. □

We move on to Case I. Recall that the construction in Proposition 3.2 is not successful just because inequality (3.12) does not hold in this case. To fix this issue, an additional inequality is needed. Define

$$F_j := X_k + X_l - Z_j. \tag{3.13}$$

Computations show

$$\langle \nabla F_j, V \rangle = F_j (\mathcal{G} - 1) + \frac{3Z_j}{2}\left(\frac{1}{3}F_j - F_k - F_l\right).$$

Remark 3.3 The condition $F_1 \equiv F_2 \equiv F_3 \equiv 0$ is in fact the G_2 condition on cohomogeneity one manifold with principal orbit $SU(3)/T^2$. Hence $\cap_{j=1}^{3}\{F_j \equiv 0\}$ is flow-invariant and it contains the integral curve γ_0 that represents the complete smooth G_2 metric on M, which is firstly discovered in [7,25].

In the following text, we still use $\tilde{S_3}$ and S_3 to denote invariant sets constructed. If necessary, we use the phrase such as "S_3 for Case I" to refer to the case in particular. Define

$$\tilde{S_3} = \bigcap_{j=1}^{2}\left\{Z_3 - Z_j \geq 0, \quad F_j - F_3 \geq 0, \quad X_3 \geq 0\right\} \cap \{3F_1 + 3F_2 - F_3 \geq 0\}. \tag{3.14}$$

And define

$$S_3 = C \cap \{\mathcal{H} \equiv 1\} \cap P \cap \tilde{S_3}. \tag{3.15}$$

Note that $F_j - F_3 \geq 0$ is simply the second defining inequality in the $\tilde{S_3}$ in (3.1) with $\rho = 1$.

It is easy to check that $p_0 \in S_3$ hence S_3 is nonempty. Since functions $X_3, Z_1, Z_3 - Z_2, F_j - F_3$ and $3F_1 + 3F_2 - F_3$ vanish at p_0 among those in (3.14), the point is in ∂S_3. With the same argument as the one for Case II and III, we know that γ_{s_1} is trapped in S_3 initially if $s_1 \geq 0$.

Proposition 3.4 *Integral curves γ_{s_1} to system (2.19) on $C \cap \{\mathcal{H} \equiv 1\}$ emanating from p_0 with $s_1 \geq 0$ do not escape S_3.*

Proof The idea of proving Proposition 3.4 is the same as the one of Proposition 3.2. Besides, almost all computations for Proposition 3.2 still hold except the one for $F_j - F_3 \geq 0$ since (3.12) is not true for Case I. With the additional inequality, it follows that

$$
\begin{aligned}
&\langle \nabla(F_j - F_3), V\rangle\big|_{F_j-F_3=0}\\
&= (F_j - F_3)(\mathcal{G}-1) + \frac{3Z_j}{2}\left(\frac{1}{3}F_j - F_k - F_3\right) - \frac{3Z_3}{2}\left(\frac{1}{3}F_3 - F_j - F_k\right)\\
&= \frac{3Z_j}{2}\left(\frac{1}{3}F_j - F_k - F_3\right) - \frac{3Z_3}{2}\left(\frac{1}{3}F_3 - F_j - F_k\right) \quad \text{since } F_j = F_3\\
&= \frac{3Z_j}{2}\left(\frac{1}{3}F_j - F_k - F_j\right) - \frac{3Z_3}{2}\left(\frac{1}{3}F_j - F_j - F_k\right) \quad \text{since } F_j = F_3\\
&= F_k\frac{3}{2}(Z_3 - Z_j) + F_j(Z_3 - Z_j)\\
&= \frac{1}{2}(Z_3 - Z_j)(3F_j + 3F_k - F_3) \quad \text{since } F_j = F_3\\
&\geq 0.
\end{aligned}
$$

Notice that we do not drop any nonnegative term in the computation above like we do in (3.10). The estimate for $\langle \nabla(F_j - F_3), V\rangle\big|_{F_j-F_3=0}$ hence becomes sharper. Finally, we need to show that the additional inequality holds along the integral curves. Indeed, since

$$
\begin{aligned}
&\langle \nabla(3F_1 + 3F_2 - F_3), V\rangle|_{3F_1+3F_2-F_3=0}\\
&= (3F_1 + 3F_2 - F_3)(\mathcal{G}-1)\\
&\quad + \frac{3Z_1}{2}(F_1 - 3F_2 - 3F_3) + \frac{3Z_2}{2}(F_2 - 3F_1 - 3F_3) - \frac{3Z_3}{2}\left(\frac{1}{3}F_3 - F_1 - F_2\right)\\
&= \frac{3Z_1}{2}(F_1 - 3F_2 - 3F_3) + \frac{3Z_2}{2}(F_2 - 3F_1 - 3F_3) \quad \text{since } 3F_1 + 3F_2 - F_3 = 0,\\
&= \frac{3Z_1}{2}(4F_1 - 4F_3) + \frac{3Z_2}{2}(4F_2 - 4F_3) \quad \text{since } 3F_1 + 3F_2 - F_3 = 0\\
&\geq 0 \quad \text{definition of } S_3 \text{ for } i = 1
\end{aligned}
$$

$3F_1 + 3F_2 - F_3$ remains nonnegative along the integral curves. Therefore, integral curves γ_{s_1} do not escape S_3 in Case I if $s_1 \geq 0$. □

Remark 3.5 One may want to integrate the additional inequality in S_3 for Case I to the other two cases so that all cases can be discussed by a single construction. Specifically, one can define

$$F_j := X_k + X_l - \rho Z_j.$$

Then the additional inequality analogous to $3F_1 + 3F_2 - F_3 \geq 0$ for Cases II and III is $aF_1 + aF_2 - 2bF_3 \geq 0$. But

$$
\begin{aligned}
&\langle \nabla(aF_1 + aF_2 - 2bF_3), V\rangle|_{aF_1+aF_2-2bF_3=0}\\
&= \frac{aZ_1}{k}(a+2b)(F_1 - F_3) + \frac{aZ_2}{k}(a+2b)(F_2 - F_3) + \frac{\zeta}{k}(aZ_1 + aZ_2 - 2bZ_3),
\end{aligned}
$$

where $\zeta = \frac{(3-d)a-(2+2d)b}{2d} \leq 0$. It only vanishes in Case I. Hence whether $aF_1 + aF_2 - 2bF_3$ is nonnegative along the integral curves in S_3 is not clear. The analogous F_j defined for Case II and Case III may not have too much meaning after all because there is no special holonomy for odd dimension other than 7.

We are ready to construct the compact invariant set mentioned at the beginning of this section. Define

$$\hat{S}_3 = S_3 \cap \{Z_1 + Z_2 - Z_3 \geq 0\} \cap \{Z_1(X_1 - X_3) + Z_2(X_2 - X_3) \geq 0\}$$

for all three cases. We have the following lemma.

Lemma 3.6 *$\hat{S}_3$ is a compact invariant set.*

Proof Because $Z_1 + Z_2 - Z_3 \geq 0$ in $\hat{S}_3$, we can apply Proposition 3.1 so that $Z_1 + Z_2$ is bounded above. Then all Z_j's are bounded in $\hat{S}_3$. By conservation law (2.20), we immediately conclude that all variables are bounded. The compactness of $\hat{S}_3$ is hence proved.

To check that $\hat{S}_3$ is flow-invariant, consider the hyperplane $Z_1 + Z_2 - Z_3 = 0$. It follows that

$$\begin{aligned}\langle \nabla(Z_1 + Z_2 - Z_3), V\rangle|_{Z_1+Z_2-Z_3=0} &= (Z_1 + Z_2 - Z_3)\left(\mathcal{G} - \frac{1}{d}\right)\\ &\quad + 2Z_1X_1 + 2Z_2X_2 - 2Z_3X_3\\ &= 2Z_1(X_1 - X_3) + 2Z_2(X_2 - X_3) \quad \text{since } Z_1 + Z_2 - Z_3 = 0\\ &\geq 0 \quad \text{definition of } \hat{S}_3\end{aligned}.$$

On hypersurface $Z_1(X_1 - X_3) + Z_2(X_2 - X_3) = 0$, we have

$$\begin{aligned}&\langle \nabla(Z_1(X_1 - X_3) + Z_2(X_2 - X_3)), V\rangle|_{Z_1(X_1-X_3)+Z_2(X_2-X_3)=0}\\ &= \left\langle \nabla\left(Z_3\left(\frac{Z_1}{Z_3}(X_1 - X_3) + \frac{Z_2}{Z_3}(X_2 - X_3)\right)\right), V\right\rangle\Bigg|_{Z_1(X_1-X_3)+Z_2(X_2-X_3)=0}\\ &= Z_3\left(\mathcal{G} - \frac{1}{d} + 2X_3\right)\left(\frac{Z_1}{Z_3}(X_1 - X_3) + \frac{Z_2}{Z_3}(X_2 - X_3)\right)\\ &\quad + Z_3\left(2\frac{Z_1}{Z_3}(X_1 - X_3)^2 + 2\frac{Z_2}{Z_3}(X_2 - X_3)^2\right)\\ &\quad + Z_1\left((X_1 - X_3)(\mathcal{G} - 1) + \mathcal{R}_1 - \mathcal{R}_3\right) + Z_2\left((X_2 - X_3)(\mathcal{G} - 1) + \mathcal{R}_2 - \mathcal{R}_3\right)\\ &= 2Z_1(X_1 - X_3)^2 + 2Z_2(X_2 - X_3)^2 + Z_1(\mathcal{R}_1 - \mathcal{R}_3) + Z_2(\mathcal{R}_2 - \mathcal{R}_3)\\ &\text{since } Z_1(X_1 - X_3) + Z_2(X_2 - X_3) = 0\\ &\geq Z_1(\mathcal{R}_1 - \mathcal{R}_3) + Z_2(\mathcal{R}_2 - \mathcal{R}_3)\\ &= Z_1(Z_3 - Z_1)(aZ_2 - 2b(Z_3 + Z_1)) + Z_2(Z_3 - Z_2)(aZ_1 - 2b(Z_3 + Z_2))\end{aligned} \tag{3.16}$$

For distinct $j, k \in \{1, 2\}$, take $A_j = Z_j(Z_3 - Z_j)$ and $B_j = aZ_k - 2b(Z_3 + Z_j)$. Apply identity

$$A_1B_1 + A_2B_2 = \frac{1}{2}((A_1 + A_2)(B_1 + B_2) + (A_1 - A_2)(B_1 - B_2)).$$

Then the computation (3.16) continues as

$$\begin{aligned}&= \frac{1}{2}(Z_1(Z_3 - Z_1) + Z_2(Z_3 - Z_2))((a - 2b)(Z_1 + Z_2) - 4bZ_3)\\ &\quad + \frac{1}{2}(Z_1 - Z_2)^2(Z_1 + Z_2 - Z_3)(a + 2b)\\ &\geq \frac{1}{2}(Z_1(Z_3 - Z_1) + Z_2(Z_3 - Z_2))(a - 6b)Z_3 \quad \text{since } Z_1 + Z_2 \geq Z_3\\ &\geq 0 \quad \text{Remark 1.1}\end{aligned}. \tag{3.17}$$

Therefore, $\hat{S}_3$ is flow-invariant. □

Remark 3.7 By the symmetry between (X_2, Z_2) and (X_3, Z_3), constructions of S_3 and $\hat{S}_3$ above can be carried over to defining S_2 and $\hat{S}_2$. With the same arguments, it can be shown that γ_{s_1} does not escape S_2 whenever $s_1 \leq 0$ and $\hat{S}_2$ is a compact invariant set.

Remark 3.8 It is clear that $p_0 \in \partial \hat{S}_3$. One can check that γ_0 is trapped in $\hat{S}_3$ initially. Hence the long time existence for γ_0 is proved. By Remark 2.18, it is trapped in $\hat{S}_3 \cap \{X_2 \equiv X_3, Z_2 \equiv Z_3\}$. Hence $\hat{S}_3$ can be used to prove the long time existence for the special case where $(X_2, Z_2) \equiv (X_3, Z_3)$ is imposed. In fact, the compact invariant set for cohomogeneity one manifolds of two summands can be constructed by a little modification on $\hat{S}_3 \cap \{X_2 \equiv X_3, Z_2 \equiv Z_3\}$, reproducing the same result in [4,33]. For Case I in particular, γ_0 represents the complete G_2 metric discovered in [7,25].

Remark 3.9 Not only $\mathcal{R}_3$ is nonnegative in $\hat{S}_3$. This is in fact the case for all $\mathcal{R}_j$'s. For distinct $j, k \in \{1, 2\}$, we have

$$\begin{aligned}\mathcal{R}_j = aZ_3Z_k + b(Z_j^2 - Z_k^2 - Z_3^2) &\geq aZ_3Z_k + b(Z_j^2 + Z_k^2 - (Z_j + Z_k)^2) \quad \text{by definition of } \hat{S}_3\\ &= aZ_3Z_k - 2bZ_jZ_k\\ &\geq (a-2b)Z_jZ_k \quad \text{by definition of } \hat{S}_3\\ &\geq 0\end{aligned}.$$

Therefore, one geometric feature of complete Ricci-flat metrics represented by γ_0 is that hypersurface has positive Ricci tensor for all $t \in (0, \infty)$. As discussed in Remark 3.23, Ricci-flat metrics represented by γ_{s_1} with $s_1 \neq 0$ does not hold such a property.

Although γ_{s_1} is trapped in S_3 if $s_1 \geq 0$, functions $Z_1 + Z_2 - Z_3$ and $Z_1(X_1 - X_3) + Z_2(X_2 - X_3)$ are negative initially if $s_1 > 0$. Hence γ_{s_1} is not trapped in $\hat{S}_3$ initially if $s_1 > 0$. To include the case where $s_1 > 0$, we need to enlarge $\hat{S}_3$ a little bit so that it initially traps all γ_{s_1} with s_1 close enough to zero. That leads us to the second step of our construction.

3.2 Entrance zone

In this section, we assume $s_1 > 0$ and work with the set S_3. We construct an entrance zone that forces γ_{s_1} to enter $\hat{S}_3$ eventually. Our goal is to show that for all small enough $s_1 > 0$, γ_{s_1} will enter $\hat{S}_3$ in a compact set. As shown in computation (3.16), it is more convenient to compute with variables $\omega_1 = \frac{Z_1}{Z_3}$ and $\omega_2 = \frac{Z_2}{Z_3}$, whose respective derivatives are $\omega_1' = 2\omega_1(X_1 - X_3)$ and $\omega_2' = 2\omega_2(X_2 - X_3)$. By the definition of S_3, we have $Z_3 \geq Z_1, Z_2$. Therefore $\omega_1, \omega_2 \in [0, 1]$. For another point of view, we can also consider the problem on $\omega_1\omega_2$-plane as shown Fig. 1. Whatever γ_{s_1} looks like, we can always project its Z_1 and Z_2 coordinate to $\omega_1\omega_2$-plane. And we want to prove the projection is bounded away from $(0, 0)$ and hopefully going through the line

$$l_0 : \omega_1 + \omega_2 - 1 = 0,$$

which is the projection of hyperplane $Z_1 + Z_2 - Z_3 = 0$. Note that any homogeneous variety in Z_j's of degree D can be projected to an algebraic curve on $\omega_1\omega_2$-plane by dividing by Z_3^D. Before the construction, we establish the following basic fact.

Proposition 3.10 *$\frac{Z_1}{Z_2}$ is strictly increasing along γ_{s_1} as long as $Z_2 > Z_1$.*

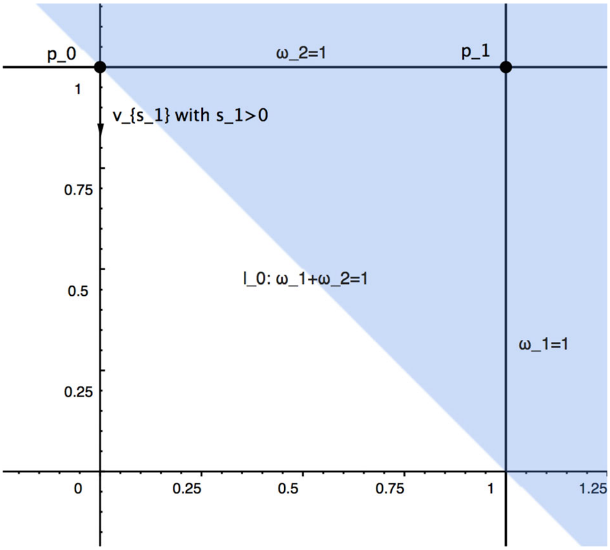


Fig. 1 Projection to $\omega_1\omega_2$-plane

Proof Initially we have $(X_1 - X_2)(p_0) = \frac{1}{d}$. If $Z_2 > Z_1$, we have

$$
\begin{aligned}
(X_1 - X_2)'\big|_{X_1 - X_2 = 0} &= (X_1 - X_2)(\mathcal{G} - 1) + \mathcal{R}_1 - \mathcal{R}_2 \\
&= (Z_2 - Z_1)(aZ_3 - 2bZ_1 - 2bZ_2) \\
&\geq (Z_2 - Z_1)(a - 4b)Z_2 \quad \text{since } Z_3 \geq Z_2 > Z_1 \\
&> 0 \quad \text{Remark 1.1}
\end{aligned}. \tag{3.18}
$$

Hence $X_1 - X_2 > 0$ along γ_{s_1} when $Z_2 > Z_1$. But then

$$\left(\frac{Z_1}{Z_2}\right)' = 2\frac{Z_1}{Z_2}(X_1 - X_2) > 0 \tag{3.19}$$

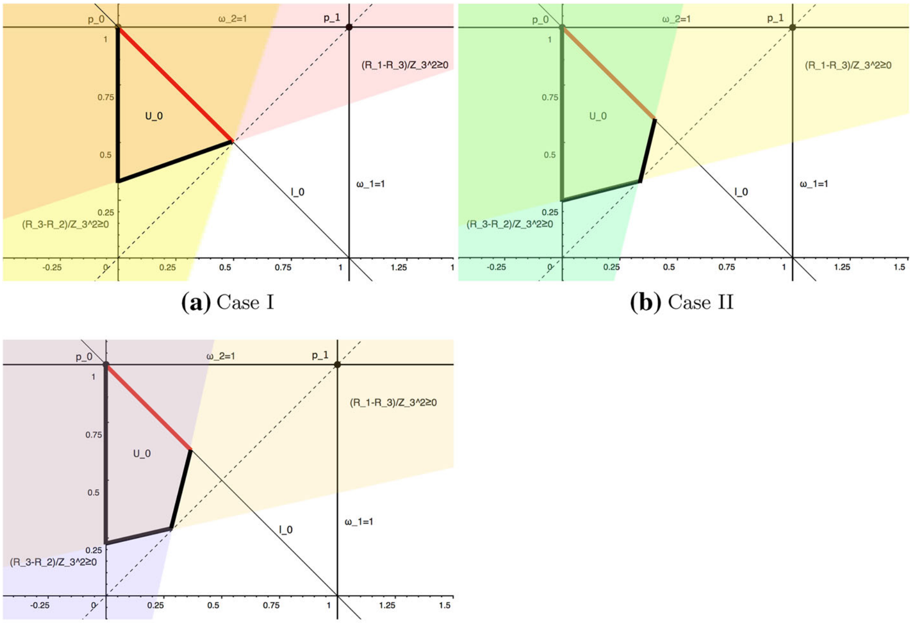
when $Z_2 > Z_1$. Therefore $\frac{Z_1}{Z_2}$ is strictly increasing along γ_{s_1} as long as $Z_2 > Z_1$. □

Substitute solution (2.26) of linearized equation to $\mathcal{R}_1 - \mathcal{R}_3$ and $\mathcal{R}_3 - \mathcal{R}_2$. It is clear that they are positive initially. Hence at the beginning, the integral curve is trapped in

$$U_0 = S_3 \cap \{Z_1 + Z_2 - Z_3 \leq 0, \mathcal{R}_1 - \mathcal{R}_3 \geq 0, \mathcal{R}_3 - \mathcal{R}_2 \geq 0\}, \tag{3.20}$$

whose projection on $\omega_1\omega_2$-plane for all three cases is illustrated in Fig. 2.

By Proposition 3.10, we know that in principal, the projection of γ_{s_1} on $\omega_1\omega_2$-plane can get arbitrarily closed to $\omega_1 - \omega_2 = 0$, represented the dashed lines in Fig. 2. Therefore, an integral curve that is initially trapped in U_0 has to escape. The question is whether it will escape U_0 through $Z_1 + Z_2 - Z_3 = 0$, represented by the red line segment. It turns out that a subset of U_0 can be constructed in a way that it contains a part of $Z_1 + Z_2 - Z_3 = 0$ and γ_{s_1} has to escape that subset through $Z_1 + Z_2 - Z_3 = 0$. Specifically, the construction is based on the following three ideas.



(a) Case I **(b)** Case II **(c)** Case III

Fig. 2 Projection of U_0 (enclosed by bold line segments) on $\omega_1\omega_2$-plane for all three cases

1. Since $Z_1 + Z_2 - Z_3 \leq 0$ initially along γ_{s_1}, the main task is to bound Z_3 from above. For computation conveniences, we prefer to bound Z_3 from above by some homogeneous algebraic varieties in Z_j's. In other words, defining inequalities of the entrance zone should include $Z_1 + Z_2 - Z_3 \leq 0$ and $B(Z_1, Z_2, Z_3) \geq 0$ for some homogeneous polynomial B in Z_j's.

2. In order to show that γ_{s_1} does not escape through $B = 0$, we need to show that $\langle \nabla(B), V\rangle|_{B=0}$ is nonnegative along γ_{s_1} in the entrance zone. This idea is discussed in the proof of Proposition 3.2. It might be difficult to determine the sign of $\langle \nabla(B), V\rangle|_{B=0}$ even we are allowed to mod out $B = 0$ in the computation result. But notice that $B' := \langle \nabla(P), V\rangle|_{B=0} = 0$ vanishes at p_0, and inequality $B' \geq 0$ can potentially be added to the definition of the entrance zone.

3. If we want to impose $B' \geq 0$, the trade-off is to show that $\langle \nabla(B'), V\rangle\big|_{B'=0} \geq 0$ along γ_{s_1} in the entrance zone. The homogeneous polynomial B that we find consists of two parameters. They allow us to tune the entrance zone to satisfy some technical inequalities. Once these inequalities are satisfied, we can show that $\langle \nabla(B'), V\rangle\big|_{B'=0} \geq 0$ in the entrance zone and γ_{s_1} is forced to escape through $Z_1 + Z_2 - Z_3 = 0$.

We first proceed the construction by having those technical inequalities in part 3 ready. In this process, the first parameter for B is introduced and how they interact with these technical inequalities are explained. Then we reveal the definition for B and its last parameter.

Proposition 3.11 *In S_3, $X_2 + X_3 > 0$ along γ_{s_1} always.*

Proof It is clear that $X_2 + X_3$ is positive initially along the curves. Since

$$\begin{aligned}\langle \nabla (X_2 + X_3), V\rangle|_{X_2+X_3=0} &= (X_2 + X_3)(\mathcal{G} - 1) + \mathcal{R}_2 + \mathcal{R}_3\\ &= \mathcal{R}_2 + \mathcal{R}_3 \quad \text{since } X_2 + X_3 = 0\\ &\geq Z_1(a(Z_2 + Z_3) - 2bZ_1)\\ &> 0 \quad \text{since } Z_3 \geq Z_1 \text{ and } a - 2b = d - 1 > 0\end{aligned}, \tag{3.21}$$

$X_2 + X_3$ stays positive along γ_{s_1}. □

Proposition 3.12 *For any fixed $\delta \geq 0$, $X_3 - (1 + \delta)X_2 > 0$ initially along γ_{s_1} and stay positive in the region where $\mathcal{R}_3 - (1 + \delta)\mathcal{R}_2 \geq 0$.*

Proof Substitute solution (2.26) of linearized equation to $X_3 - (1 + \delta)X_2$. We have

$$(2 + \delta)s_1 e^{\frac{\eta}{d}} - a\delta s_0 e^{\frac{2\eta}{d}} \sim (2 + \delta)s_1 e^{\frac{\eta}{d}} > 0$$

near p_0. Since

$$\begin{aligned}&\langle \nabla (X_3 - (1 + \delta X_2)), V\rangle|_{X_3-(1+\delta)X_2=0}\\ &= (X_3 - (1 + \delta)X_2)\,(\mathcal{G} - 1) + \mathcal{R}_3 - (1 + \delta)\mathcal{R}_2\\ &= \mathcal{R}_3 - (1 + \delta)\mathcal{R}_2 \quad \text{since } X_3 - (1 + \delta)X_2 = 0,\end{aligned} \tag{3.22}$$

the proof is complete. □

Define

$$U_\delta = U_0 \cap \{\mathcal{R}_3 - (1 + \delta)\mathcal{R}_2 \geq 0\}. \tag{3.23}$$

It is easy to check that U_δ is a subset of U_0 and γ_{s_1} is initially trapped in U_δ if $s_1 > 0$. Therefore, $X_3 - (1 + \delta)X_2 > 0$ when γ_{s_1} is in U_δ by Proposition 3.12.

The fixed value of δ needs to be picked in a certain range for the following two technical reasons. Firstly, we want inequality $X_3 - (1 + \delta)X_2 > 0$ to hold at least until γ_{s_1} enters $\hat{S}_3$. Hence by Proposition 3.12, we need to pick δ that make $\mathcal{R}_3 - (1 + \delta)\mathcal{R}_2 \geq 0$ contains a subset of $Z_1 + Z_2 - Z_3 = 0$. Secondly, because $U_0 \subset S_3 \cap \{Z_2 - Z_1 > 0\}$ and the behavior of γ_{s_1} is better known in U_0, we want γ_{s_1} passes though the part of $Z_1 + Z_2 - Z_3 = 0$ that $Z_2 - Z_1 \geq 0$ is satisfied. In summary, we have the following proposition.

Proposition 3.13 *If $\delta \in \left(\frac{6b-a}{2(d-1)}, \frac{4b}{d-1}\right)$, then $\{\mathcal{R}_3 - (1 + \delta)\mathcal{R}_2 \geq 0\}$ contains a subset of $\{Z_1 + Z_2 - Z_3 = 0\} \cap \{Z_2 - Z_1 > 0\}$ in U_0.*

Proof If $\delta \in \left(\frac{6b-a}{2(d-1)}, \frac{4b}{d-1}\right)$, then we have $\frac{4b-(d-1)\delta}{(d-1)(1+\delta)} \in (0, 1)$. Suppose $\frac{4b-(d-1)\delta}{(d-1)(1+\delta)} \geq \frac{Z_1}{Z_2}$, then

$$\begin{aligned}&(\mathcal{R}_3 - (1 + \delta)\mathcal{R}_2)|_{Z_1+Z_2-Z_3=0}\\ &= (Z_3 - Z_2)(2b(Z_3 + Z_2) - aZ_1) - \delta(aZ_1Z_3 + b(Z_2^2 - Z_1^2 - Z_3^2))\\ &\quad \text{since } Z_1 + Z_2 - Z_3 = 0\\ &= Z_1(2b(2Z_2 + Z_1) - aZ_1) - \delta(aZ_1Z_2 + aZ_1^2 - 2bZ_1^2 - 2bZ_1Z_2)\cdot\\ &\quad \text{since } Z_1 + Z_2 - Z_3 = 0\\ &= Z_1(-(d - 1)(1 + \delta)Z_1 + (4b - (d - 1)\delta)Z_2) \quad \text{Remark 1.1}\\ &\geq 0\end{aligned} \tag{3.24}$$

The proof is complete. □

Remark 3.14 Perhaps a better way to illustrate Proposition 3.13 is to consider the projection on the $\omega_1\omega_2$-plane. For $\mathcal{R}_3 - (1+\delta)\mathcal{R}_2 = 0$, we obtain an algebraic curve

$$l_1 : (1-\omega_2)(2b(1+\omega_2) - a\omega_1) - \delta(a\omega_1 + b(\omega_2^2 - \omega_1^2 - 1)) = 0.$$

Straightforward computation shows that l_1 intersect with $\omega_1 + \omega_2 = 1$ at points $(0, 1)$ and $\left(\frac{4b-\delta(d-1)}{a+2b}, \frac{(d-1)(1+\delta)}{a+2b}\right)$. If $\delta \in \left(\frac{6b-a}{d-1}, \frac{4b}{d-1}\right)$, then the second intersection point $\left(\frac{4b-\delta(d-1)}{a+2b}, \frac{(d-1)(1+\delta)}{a+2b}\right)$ is in the region where $\omega_2 - \omega_1 > 0$. Hence U_δ, denoted by the darker area in Fig. 3, can include a segment of l_0 in U_0, represented by the bold segment, that is away from $\omega_1 - \omega_2 = 0$.

Remark 3.15 Note that Case I is the only case where the admissible δ must be positive.

The entrance zone we construct is a subset of U_δ. We impose $\delta \in \left(0, \frac{4b}{d-1}\right)$. As shown in the following technical proposition, $\delta > 0$ is needed for the sake of conveniences. The first parameter in the definition of B is also introduced.

Proposition 3.16 *In U_δ, we can find a p large enough such that*

$$((X_1 - X_2) + (p-1)(X_3 - X_2))(X_1 - X_2 + (p+1)(X_3 - X_2)) \geq \frac{1-\mathcal{G}}{d(d-1)} \tag{3.25}$$

along γ_{s_1} in U_δ.

Proof Since $X_1 = \frac{1}{d} - X_2 - X_3$, we can write inequality (3.25) with respect to $\tilde{X} = X_3 + X_2$ and $\tilde{Y} = X_3 - X_2$. Straightforward computation shows that inequality (3.25) is equivalent to

$$\begin{aligned}&\left(\left(p-\frac{1}{2}\right)\left(p+\frac{3}{2}\right) + \frac{1}{2(d-1)}\right)\tilde{Y}^2 - \left(3p + \frac{3}{2}\right)\tilde{X}\tilde{Y}\\ &+\left(\frac{9}{4} + \frac{3}{2(d-1)}\right)\tilde{X}^2 + \frac{2p+1}{d}\tilde{Y} - \frac{3d-1}{d(d-1)}\tilde{X} \geq 0.\end{aligned} \tag{3.26}$$

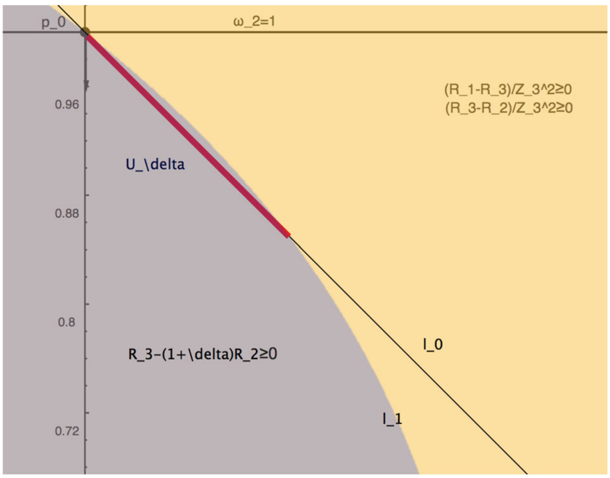


Fig. 3 $\delta = 0.7$ for Case I

Note that $\tilde{X}$ and $\tilde{Y}$ are positive along γ_{s_1} in U_δ by Propositions 3.11 and 3.12. Moreover, in U_δ, we have $X_3 - (1+\delta)X_2 > 0$ along γ_{s_1} by Proposition 3.12. Rewrite this condition in terms of $\tilde{X}$ and $\tilde{Y}$ so we have $(2+\delta)\tilde{Y} - \delta\tilde{X} > 0$ along γ_{s_1} in U_δ. Hence the LHS of (3.26) is larger than

$$\begin{aligned}&\left(\left(\left(p-\frac{1}{2}\right)\left(p+\frac{3}{2}\right)+\frac{1}{2(d-1)}\right)\frac{\delta}{2+\delta}-\left(3p+\frac{3}{2}\right)\right)\tilde{X}\tilde{Y}\\ &+\left(\frac{9}{4}+\frac{3}{2(d-1)}\right)\tilde{X}^2+\left(\frac{2p+1}{d}\frac{\delta}{2+\delta}-\frac{3d-1}{d(d-1)}\right)\tilde{X}\end{aligned}. \tag{3.27}$$

Since $\delta \in \left(0, \frac{4b}{d-1}\right)$ is fixed, we can choose p large enough so that

$$\begin{aligned}&\left(\left(p-\frac{1}{2}\right)\left(p+\frac{3}{2}\right)+\frac{1}{2(d-1)}\right)\frac{\delta}{2+\delta}\geq 3p+\frac{3}{2}\\ &\frac{2p+1}{d}\frac{\delta}{2+\delta}\geq\frac{3d-1}{d(d-1)}\end{aligned} \tag{3.28}$$

are satisfied. Then inequality (3.26) is satisfied. □

Now we are ready to reveal the definition for B and its last parameter. Define

$$B_{p,k}(Z_1, Z_2, Z_3) := kZ_1Z_3^{p+1} - Z_2^p(Z_3 - Z_2)^2.$$

For a fixed $\delta \in \left(0, \frac{4b}{d-1}\right)$, choose a p that satisfies inequalities (3.28). Then define

$$\begin{aligned}U_{(\delta,p,k)} &= S_3 \cap \{Z_1 + Z_2 - Z_3 \leq 0\} \cap \{B_{p,k} \geq 0\}\\ &\cap \{(Z_3 - Z_2)(X_1 - X_3) + (p(Z_3 - Z_2) - 2Z_2)(X_3 - X_2) \geq 0\},\end{aligned} \tag{3.29}$$

More requirements on the choice of p and k are added later. Before that, we prove the following.

Proposition 3.17 *For any fixed $k > 0$, γ_{s_1} is initially trapped in $U_{(\delta,p,k)}$ as long as $s_1 \in \left(0, \sqrt{\frac{ks_0(d+1)}{16d}}\right)$.*

Proof With discussion in Sect. 3.1, we know that γ_{s_1} is initially in S_3 if $s_1 > 0$. Since all inequalities presented in (3.29) reach equality at p_0, we need to substitute solution (2.26) of linearized equation in each one of them. For $Z_1 + Z_2 - Z_3$, we have

$$(d+1)s_0e^{\frac{2\eta}{d}} - 4s_1e^{\frac{\eta}{d}} \sim -4s_1e^{\frac{\eta}{d}} < 0 \tag{3.30}$$

if $s_1 < 0$.

Substitute solution (2.26) of linearized equation to $kZ_1Z_3^{p+1} - Z_2^p(Z_3 - Z_2)^2$, we have

$$\begin{aligned}&k(d+1)s_0e^{\frac{2\eta}{d}}\left(\frac{1}{d}-as_0e^{\frac{2\eta}{d}}+2s_1e^{\frac{\eta}{d}}\right)^{p+1}-\left(\frac{1}{d}-as_0e^{\frac{2\eta}{d}}-2s_1e^{\frac{\eta}{d}}\right)^p 16s_1^2e^{\frac{2\eta}{d}}\\ &\sim\left(\frac{1}{d}\right)^p\left(\frac{ks_0(d+1)}{d}-16s_1^2\right)e^{\frac{2\eta}{d}}.\end{aligned} \tag{3.31}$$

Hence $kZ_1Z_3^{p+1} - Z_2^p(Z_3 - Z_2)^2 > 0$ initially along the projection of γ_{s_1} when $s_1^2 < \frac{ks_0(d+1)}{16d}$.

Finally, for $(Z_3 - Z_2)(X_1 - X_3) + (p(Z_3 - Z_2) - 2Z_2)(X_3 - X_2)$, we have

$$4s_1 e^{\frac{\eta}{d}} \left(\frac{1}{d} - 3as_0 e^{\frac{2\eta}{d}} - s_1 e^{\frac{\eta}{d}}\right) + \left(4ps_1 e^{\frac{\eta}{d}} - 2\left(\frac{1}{d} - as_0 e^{\frac{2\eta}{d}} - 2s_1 e^{\frac{\eta}{d}}\right)\right)$$
$$2s_1 e^{\frac{\eta}{d}} \sim (4 + 8p)s_1^2 e^{\frac{2\eta}{d}} > 0. \tag{3.32}$$

Hence γ_{s_1} is indeed trapped in $U_{(\delta,p,k)}$ initially when $s_1 \in \left(0, \sqrt{\frac{ks_0(d+1)}{16d}}\right)$. □

We now specify our choice for p and k. Projected to the $\omega_1\omega_2$-plane, the first two inequalities in (3.29) is equivalent to

$$\frac{\omega_2^p(1 - \omega_2)^2}{k} \leq \omega_1 \leq 1 - \omega_2.$$

Write l_0 as a function $\mathcal{C}_0(\omega_2) = 1 - \omega_2$. Define $l_2 \colon \mathcal{C}_2(\omega_2) = \frac{\omega_2^p(1-\omega_2)^2}{k}$. It is clear that $\mathcal{C}_0 - \mathcal{C}_2 = 0$ at $\omega_2 = 1$. Our goal is to choose p and k so that $\mathcal{C}_0 - \mathcal{C}_2$ vanishes again at some $\omega_* < 1$. Then we define $\hat{U}_{(\delta,p,k)}$ to be the compact subset of $U_{(\delta,p,k)}$ where $\omega_2 \in [\omega_*, 1]$ and $\mathcal{C}_0 > \mathcal{C}_2$ for $\omega_2 \in (\omega_*, 1)$. Moreover, because we want to utilize Proposition 3.16, parameters p and k are chosen to guarantee that ω_* is not too small so that $\hat{U}_{(\delta,p,k)} \subset U_\delta$. Specifically, we have the following proposition.

Proposition 3.18 *Let $p \geq 2$ be a fixed number large enough that it satisfies inequalities (3.28) and*

$$\frac{p}{p+1} \geq \frac{(d-1)(1+\delta)}{a+2b}. \tag{3.33}$$

Let $k > 0$ be a number small enough so that

$$k < \left(\frac{p}{p+1}\right)^p \frac{1}{p+1}. \tag{3.34}$$

Then there exists some $\omega_* \in \left(\frac{p}{p+1}, 1\right)$ such that

$$\hat{U}_{(\delta,p,k)} := U_{(\delta,p,k)} \cap \{Z_2 - \omega_* Z_3 \geq 0\} \tag{3.35}$$

is a compact subset of U_δ.

Proof Although the proposition is true as long as $p > 0$, the technical condition $p \geq 2$ is imposed for computations in Lemma 4.2 and (3.44). We first claim that p exists. Because δ is a fixed number in $(0, \frac{4b}{d-1})$, we have $\frac{(d-1)(1+\delta)}{a+2b} < \frac{d-1+4b}{a+2b} = 1$. Hence we can choose p large enough on top of inequalities (3.28) to satisfies inequalities (3.33).

Consider the function

$$\mathcal{C} = \mathcal{C}_0 - \mathcal{C}_2 = 1 - \omega_2 - \frac{\omega_2^p(1 - \omega_2)^2}{k} = \frac{1 - \omega_2}{k}\left(k - \omega_2^p(1 - \omega_2)\right). \tag{3.36}$$

It is clear that $\mathcal{C}$ vanishes at $\omega_2 = 1$ and $\mathcal{C} > 0$ near that point. Let $\tilde{\mathcal{C}} = k - \omega_2^p(1 - \omega_2)$. Since $\frac{d\tilde{\mathcal{C}}}{d\omega_2} = \omega_2^{p-1}(\omega_2 - p(1 - \omega_2))$, we have

Therefore, for an arbitrary p, inequality (3.34) is satisfied if and only if $\tilde{\mathcal{C}}\left(\frac{p}{p+1}\right) < 0$. Then there exists some $\omega_* \in \left(\frac{p}{p+1}, 1\right)$ such that $\tilde{\mathcal{C}}(\omega_*) = 0$ and $\tilde{\mathcal{C}}(\omega_2) > 0$ in $(\omega_*, 1)$. Since $\omega_2 \leq 1$, that means for such an ω_*, we must have $\mathcal{C}(\omega_*) = 0$ and $\mathcal{C}(\omega_2) > 0$ in $(\omega_*, 1)$.

ω_2	0	$\left(0, \frac{p}{p+1}\right)$	$\frac{p}{p+1}$	$\left(\frac{p}{p+1}, 1\right)$	1
$\frac{d\tilde{C}}{d\omega_2}$	0	<0	0	>0	1
$\tilde{C}$	k	Decrease	Local Minimum	Increase	k

The ω_2-coordinate of the intersection point between l_0 and l_1 is $\frac{(d-1)(1+\delta)}{a+2b}$. Since $\delta > 0$, by (3.33) and Remark 1.1, the root ω_* discussed above satisfies

$$\omega_* > \frac{p}{p+1} \geq \frac{(d-1)(1+\delta)}{a+2b} > \frac{a-2b}{a+2b} \tag{3.37}$$

We are ready to prove that $\hat{U}_{(\delta,p,k)} \subset U_\delta$. In other words, with our choice of p and k above, inequalities in the definition (3.29) of $U_{(\delta,p,k)}$ and (3.35) of $\hat{U}_{(\delta,p,k)}$ imply all inequalities in definition (3.20) of U_0 and (3.23) of U_δ.

Firstly, we need to show $\hat{U}_{(\delta,p,k)} \subset U_0$. With $-Z_1 \geq Z_2 - Z_3$ and $Z_2 \geq \omega_* Z_3$ satisfied in S_3, we have

$$\begin{aligned}\mathcal{R}_1 - \mathcal{R}_3 &= (Z_3 - Z_1)(aZ_2 - 2bZ_1 - 2bZ_3)\\ &\geq (Z_3 - Z_1)((a+2b)Z_3\omega_* - 4bZ_3)\\ &\geq (Z_3 - Z_1)((a+2b)Z_3\omega_* - (a-2b)Z_3) \quad \text{Remark 1.1}\\ &\geq 0 \quad \text{by (3.37) and definition of } S_3\end{aligned} \tag{3.38}$$

and

$$\begin{aligned}\mathcal{R}_3 - \mathcal{R}_2 &= (Z_3 - Z_2)(2bZ_3 + 2bZ_2 - aZ_1)\\ &\geq (Z_3 - Z_2)((a+2b)\omega_* Z_3 - (a-2b)Z_3).\\ &\geq 0 \quad \text{by (3.37) and definition of } S_3\end{aligned} \tag{3.39}$$

Hence $\hat{U}_{(\delta,p,k)} \subset U_0$.

In $\hat{U}_{(\delta,p,k)}$, we have

$$\begin{aligned}\mathcal{R}_3 - (1+\delta)\mathcal{R}_2 &= (Z_3 - Z_2)(2b(Z_3 + Z_2) - aZ_1) - \delta(aZ_1Z_3 + b(Z_2^2 - Z_1^2 - Z_3^2))\\ &= 2b(Z_3^2 - Z_2^2) - aZ_1(Z_3 - Z_2) - \delta aZ_1Z_3 - \delta bZ_2^2 + \delta bZ_1^2 + \delta bZ_3^2\\ &= (2+\delta)bZ_3^2 - (1+\delta)aZ_1Z_3 + \delta bZ_1^2 - (\delta b + 2b)Z_2^2 + aZ_1Z_2.\end{aligned} \tag{3.40}$$

Treat the result of the computation above as a function of Z_3. It is a parabola centered at $\frac{(1+\delta)a}{(2+\delta)2b}Z_1$. By (3.37), it is clear that $\frac{1}{1-\omega_*} > \frac{a+2b}{4b}$. Since $\delta \in \left(0, \frac{4b}{a-2b}\right)$, it is straightforward to deduce that $\frac{a+2b}{4b} > \frac{(1+\delta)a}{(2+\delta)2b}$. From $Z_2 \geq \omega_* Z_3 \geq \omega_*(Z_1 + Z_2)$ we also deduce

$$Z_2 \geq \frac{\omega_*}{1-\omega*} Z_1. \tag{3.41}$$

Therefore, we know that $Z_1 + Z_2 \geq \frac{1}{1-\omega_*}Z_1 \geq \frac{a+2b}{4b}Z_1 \geq \frac{(1+\delta)a}{(2+\delta)2b}Z_1$ in $\hat{U}_{(\delta,p,k)}$. Hence

$$\begin{aligned}\mathcal{R}_3 - (1+\delta)\mathcal{R}_2 &\geq (\mathcal{R}_3 - (1+\delta)\mathcal{R}_2)|_{Z_3 = Z_1 + Z_2}.\\ &\geq 0 \quad \text{by (3.24)}\end{aligned} \tag{3.42}$$

Finally, we need to show that $\hat{U}_{(\delta,p,k)}$ is compact. Since $Z_2 - \omega_* Z_3 \geq 0$, we automatically have $Z_1 + Z_2 \geq \omega_* Z_3$ in $\hat{U}_{(\delta,p,k)}$. By (3.37), we can deduce $\omega_* > \frac{a-2b}{a+2b} > \frac{2b}{a}$ in $\hat{U}_{(\delta,p,k)}$, where the last inequality is from Remark 1.1. Hence $a(Z_1 + Z_2) - 2bZ_3 \geq 0$ in $\hat{U}_{(\delta,p,k)}$. Proposition 3.1 can be applied and all Z_j's are bounded above. By the conservation law (2.20), we know that all variables are bounded. Hence $\hat{U}_{(\delta,p,k)}$ is compact. The proof is complete. □

We are ready to show that $\hat{U}_{(\delta,p,k)}$ is the entrance zone. An example of $\hat{U}_{(\delta,p,k)}$ is shown in Fig. 4.

Lemma 3.19 *For $s_1 \in \left(0, \sqrt{\frac{k(d+1)s_0}{16d}}\right)$ and suitable choice of δ, p and k as described above, the integral curve γ_{s_1} escapes $\hat{U}_{(\delta,p,k)}$ through $Z_1 + Z_2 - Z_3 = 0$.*

Proof Suppose γ_{s_1} does not escape through $Z_1 + Z_2 - Z_3 = 0$, then it can only escape through either $kZ_1 Z_3^{p+1} - Z_2^p (Z_3 - Z_2)^2 = 0$ or $(Z_3 - Z_2)(X_1 - X_3) + (p(Z_3 - Z_2) - 2Z_2)(X_3 - X_2) = 0$. We prove that these situations are impossible.

Since

$$\begin{aligned}
&\langle \nabla (kZ_1 Z_3^{p+1} - Z_2^p (Z_3 - Z_2)^2), V\rangle\Big|_{kZ_1 Z_3^{p+1} - Z_2^p (Z_3-Z_2)^2=0}\\
&= \langle \nabla (Z_3^{p+2}(k\omega_1 - \omega_2^p (1-\omega_2)^2)), V\rangle\Big|_{kZ_1 Z_3^{p+1} - Z_2^p (Z_3-Z_2)^2=0}\\
&= (p+2) Z_3^{p+2}\left(\mathcal{G} - \frac{1}{d} + 2X_3\right)(k\omega_1 - \omega_2^p (1-\omega_2)^2)\\
&\quad + Z_3^{p+2}(2k\omega_1 (X_1 - X_3) - 2p\omega_2^p (X_2 - X_3)(1-\omega_2)^2\\
&\quad + 4\omega_2^p (1-\omega_2)\omega_2 (X_2 - X_3))\\
&= Z_3^{p+2}(2k\omega_1 (X_1 - X_3) - 2p\omega_2^p (X_2 - X_3)(1-\omega_2)^2\\
&\quad + 4\omega_2^p (1-\omega_2)\omega_2 (X_2 - X_3))\\
&\quad \text{since } kZ_1 Z_3^{p+1} - Z_2^p (Z_3 - Z_2)^2 = 0
\end{aligned}$$

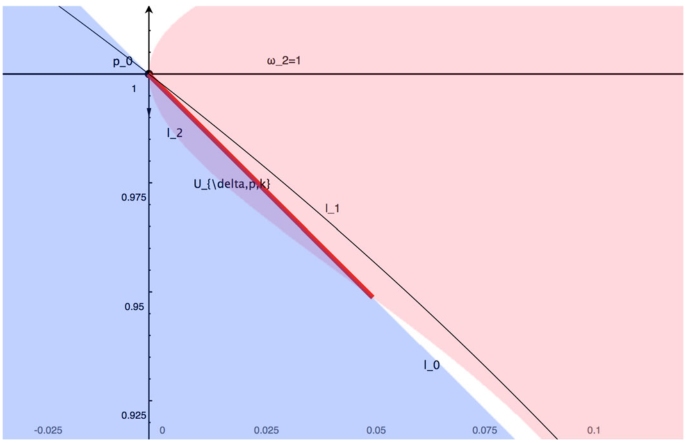


Fig. 4 $\delta = 0.7$, $p = 12$, $k = \frac{1}{13+1}\left(\frac{12}{13}\right)^{12}$ for Case I

$$
\begin{aligned}
&= Z_3^{p+2}(2\omega_2^p(1-\omega_2)^2(X_1-X_3) - 2p\omega_2^p(X_2-X_3)(1-\omega_2)^2 \\
&\quad +4\omega_2^p(1-\omega_2)\omega_2(X_2-X_3)) \\
&\quad \text{since } kZ_1Z_3^{p+1} - Z_2^p(Z_3-Z_2)^2 = 0 \\
&= 2Z_2^p(Z_3-Z_2)((Z_3-Z_2)(X_1-X_3) + (p(Z_3-Z_2)-2Z_2)(X_3-X_2)) \\
&\geq 0 \quad \text{definition of } \hat{U}_{(\delta,p,k)},
\end{aligned}
\tag{3.43}
$$

it is impossible for γ_{s_1} to escape $\hat{U}_{(\delta,p,k)}$ through $kZ_1Z_3^{p+1} - Z_2^p(Z_3-Z_2)^2 = 0$.

For the other defining inequality, we have

$$
\begin{aligned}
&\langle \nabla((Z_3-Z_2)(X_1-X_3) \\
&\quad +(p(Z_3-Z_2)-2Z_2)(X_3-X_2)), V\rangle|_{(Z_3-Z_2)(X_1-X_3)+(p(Z_3-Z_2)-2Z_2)(X_3-X_2)=0} \\
&= \langle \nabla(Z_3((1-\omega_2)(X_1-X_3) \\
&\quad +(p(1-\omega_2)-2\omega_2)(X_3-X_2))), V\rangle|_{(Z_3-Z_2)(X_1-X_3)+(p(Z_3-Z_2)-2Z_2)(X_3-X_2)=0} \\
&= Z_3\left(\mathcal{G} - \frac{1}{d} + 2X_3\right)((1-\omega_2)(X_1-X_3) + (p(1-\omega_2)-2\omega_2)(X_3-X_2))) \\
&\quad +Z_3((1-\omega_2)(X_1-X_3) + (p(1-\omega_2)-2\omega_2)(X_3-X_2)))(\mathcal{G}-1) \\
&\quad +Z_3((1-\omega_2)(\mathcal{R}_1-\mathcal{R}_3) + (p(1-\omega_2)-2\omega_2)(\mathcal{R}_3-\mathcal{R}_2)) \\
&\quad +Z_3(2\omega_2(X_3-X_2)(X_1-X_3) + 2(p+2)\omega_2(X_3-X_2)^2) \\
&= Z_3((1-\omega_2)(\mathcal{R}_1-\mathcal{R}_3) + (p(1-\omega_2)-2\omega_2)(\mathcal{R}_3-\mathcal{R}_2)) \\
&\quad +Z_3(2\omega_2(X_3-X_2)(X_1-X_3) + 2(p+2)\omega_2(X_3-X_2)^2) \\
&\quad \text{since } (Z_3-Z_2)(X_1-X_3) + (p(Z_3-Z_2)-2Z_2)(X_3-X_2)) = 0 \\
&= (Z_3-Z_2)(\mathcal{R}_1-\mathcal{R}_3) + (p(Z_3-Z_2)-2Z_2)(\mathcal{R}_3-\mathcal{R}_2) \\
&\quad +2Z_2(X_3-X_2)(X_1-X_3) + 2(p+2)Z_2(X_3-X_2)^2 \\
&= (Z_3-Z_2)(\mathcal{R}_1-\mathcal{R}_3+p(\mathcal{R}_3-\mathcal{R}_2)) - 2Z_2(Z_3-Z_2)(2bZ_3+2bZ_2-aZ_1) \\
&\quad +2Z_2(X_3-X_2)((X_1-X_3)+(p+2)(X_3-X_2)) \\
&= (Z_3-Z_2)(\mathcal{R}_1-\mathcal{R}_3+p(\mathcal{R}_3-\mathcal{R}_2)+2Z_2(aZ_1-2bZ_2-2bZ_3)) \\
&\quad +(Z_3-Z_2)((X_1-X_2)+(p-1)(X_3-X_2))(X_1-X_2+(p+1)(X_3-X_2)) \\
&\quad \text{since } 2Z_2(X_3-X_2) = (Z_3-Z_2)(X_1-X_3) + p(Z_3-Z_2)(X_3-X_2).
\end{aligned}
\tag{3.44}
$$

Because $\hat{U}_{(\delta,p,k)} \subset U_\delta$, we can apply Proposition 3.16 to the last line of (3.44) and continue the computation as

$$
\begin{aligned}
&\geq (Z_3-Z_2)\left(\mathcal{R}_1-\mathcal{R}_3+p(\mathcal{R}_3-\mathcal{R}_2)+2Z_2(aZ_1-2bZ_2-2bZ_3)+\frac{1-\mathcal{G}}{d(d-1)}\right) \\
&= (Z_3-Z_2)\left(\mathcal{R}_1-\mathcal{R}_3+p(\mathcal{R}_3-\mathcal{R}_2)+2Z_2(aZ_1-2bZ_2-2bZ_3)\right. \\
&\quad \left.+\frac{1}{d-1}(\mathcal{R}_1+\mathcal{R}_2+\mathcal{R}_3)\right) \quad \text{by (2.20)} \\
&= \left(2-\frac{1}{d-1}\right)bZ_1^2 + \left(aZ_3((p+1)\omega_2-p)+\frac{a}{d}(Z_2+Z_3)\right)Z_1 \\
&\quad +Z_3^2\left(-\left(2bp+4b+\frac{b}{d-1}\right)\omega_2^2+\left(\frac{a}{d-1}+a-4b\right)\omega_2+\left(2pb-2b-\frac{b}{d-1}\right)\right).
\end{aligned}
\tag{3.45}
$$

The first term of the computation result above is obviously positive. The second term is positive because $\omega_2 \geq \omega_* > \frac{p}{p+1}$ in $\hat{U}_{(\delta,p,k)}$. The positivity of the last term depends on the one of parabola

$$\pi(\omega_2) = -\left(2bp + 4b + \frac{b}{d-1}\right)\omega_2^2 + \left(\frac{a}{d-1} + a - 4b\right)\omega_2 + \left(2pb - 2b - \frac{b}{d-1}\right).$$

Since we impose $p \geq 2$, it is clear that $\pi(0)$ is positive. As the coefficient of the first term is negative, we know that π has two roots with different signs. It is easy to verify that $\pi(1) = 0$. Then we conclude that π is nonnegative for all $\omega_2 \in [0, 1]$. Therefore, the computation of (3.44) is nonnegative and only vanishes when $Z_1 = 0$ and $Z_2 = Z_3$.

Notice that there is no need to check the possibility that γ_{s_1} may escape through $Z_2 - \omega_* Z_3 = 0$. Because when the equality of $Z_2 - \omega_* Z_3 \geq 0$ is reached at some point $\gamma_{s_1}(\eta_*)$, it implies that the function $\mathcal{C}$ in (3.36) vanishes at that point. Specifically, we have $1 - \omega_2 = \frac{\omega_2^p(1-\omega_2)^2}{k}$ at that point. But then

$$Z_1 Z_3^{p+1} \leq Z_3^{p+1}(Z_3 - Z_2) = \frac{Z_2^p(Z_3 - Z_2)^2}{k} \leq Z_1 Z_3^{p+1},$$

which implies $kZ_1 Z_3^{p+1} - Z_2^p(Z_3 - Z_2)^2 = 0$ at that point and this case is included in the computation at the beginning of the proof. □

Proposition 3.20 *The only critical points in $\hat{U}_{(\delta,p,k)}$ are p_0 and those of Type I.*

Proof By Proposition 2.15, it is clear that p_0 and critical points of Type I are in $\hat{U}_{(\delta,p,k)}$. We first eliminate critical points with negative Z_j entry. Since $\hat{U}_{(\delta,p,k)} \subset S_3$, we can eliminate critical points with Z_3 smaller than the other two Z_j's. Because $X_3 \geq 0$ in S_3, there is no critical points of Type II. Since $Z_2 \geq pZ_1 \geq Z_1$ in $\hat{U}_{(\delta,p,k)}$ by (3.37) and (3.41), there is no critical points other than p_0 and those of Type I in $\hat{U}_{(\delta,p,k)}$. □

Proposition 3.21 *The function $Z_1Z_2Z_3$ stays positive and increases along γ_{s_1}.*

Proof Since $\mathcal{H} \equiv 1$, it is clear that $\mathcal{G} \geq \frac{1}{n}$. Hence

$$(Z_1Z_2Z_3)' = Z_1Z_2Z_3\left(3\mathcal{G} - \frac{1}{d}\right) \geq 0. \tag{3.46}$$

Since $Z_1Z_2Z_3$ is initially positive along γ_{s_1}, the proof is complete. □

We are ready to prove the completeness of Ricci-flat metrics represented by γ_{s_1} with s_1 close enough to zero.

Lemma 3.22 *There exists a $k > 0$ such that an unstable integral curve γ_{s_1} to (2.19) on $C \cap \{\mathcal{H} \equiv 1\}$ emanating from p_0 is defined on $\mathbb{R}$ if $s_1 \in \left(-\sqrt{\frac{k(d+1)s_0}{16d}}, \sqrt{\frac{k(d+1)s_0}{16d}}\right)$.*

Proof If $s_1 > 0$, the curve γ_{s_1} is initially trapped in $\hat{U}_{(\delta,p,k)}$ as long as $s_1 \in \left(0, \sqrt{\frac{k(d+1)s_0}{16d}}\right)$.

The function $Z_1 + Z_2 - Z_3$ vanishes at p_0 and it is negative along γ_{s_1} in $\hat{U}_{(\delta,p,k)}$. By Lemma 3.19, the function $Z_1 + Z_2 - Z_3$ must vanish at $\gamma_{s_1}(\eta_*)$ for some $\eta_* \in \mathbb{R}$. Then we must have $(Z_1 + Z_2 - Z_3)'(\gamma_{s_1}(\eta_*)) \geq 0$. But

$$\begin{aligned}(Z_1 + Z_2 - Z_3)'(\gamma_{s_1}(\eta_*)) &= \langle \nabla(Z_1 + Z_2 - Z_3), V\rangle(\gamma_{s_1}(\eta_*))\\ &= \left(\langle \nabla(Z_1 + Z_2 - Z_3), V\rangle|_{Z_1+Z_2-Z_3=0}\right)(\gamma_{s_1}(\eta_*)) \quad (3.47)\\ &= (Z_1(X_1 - X_3) + Z_2(X_2 - X_3))(\gamma_{s_1}(\eta_*)).\end{aligned}$$

Hence $\gamma_{s_1}(\eta_*)$ is in $\partial\hat{S}_3$.

By Proposition 3.18, we know that $\hat{U}_{(\delta,p,k)}$ is in U_0, where $\mathcal{R}_1-\mathcal{R}_3\geq 0$ and $\mathcal{R}_3-\mathcal{R}_2\geq 0$ hold. Then with the similar argument in Proposition 3.12, we know that $X_1 > X_3 > X_2$ along γ_{s_1} in $\hat{U}_{(\delta,p,k)}$. Hence the intersection point $\gamma_{s_1}(\eta_*)$ is not p_0. By Proposition 3.21, we know that $\gamma_{s_1}(\eta_*)$ cannot be a critical point of Type I. By Proposition 3.20, we know that $\gamma_{s_1}(\eta_*)$ is not a critical point. Then by Lemma 3.6, γ_{s_1} continue to flows inward $\hat{S}_3$ from $\gamma_{s_1}(\eta_*)$ and never escape. Therefore, such a γ_{s_1} is defined on $\mathbb{R}$.

By symmetry, similar result can be obtained for $s_1\in\left(-\sqrt{\frac{k(d+1)s_0}{16d}},0\right)$. If $s_1=0$, then we are back to the special case by Remark 3.8. □

By the discussion at the end of Sect. 2.3, Lemma 3.22 proves the first half of Theorem 1.2.

Remark 3.23 For γ_{s_1} with $s_1\in\left(0,\sqrt{\frac{k(d+1)s_0}{16d}}\right)$, it can be shown that $\mathcal{R}_2$ is negative initially by substituting (2.26). Hence the Ricci-flat metrics represented does not have the property introduced in Remark 3.9. By straightforward computation, however, it processes a weaker condition that the scalar curvature of each hypersurface remain positive.

4 Asymptotic limit

In this section, we study the asymptotic behavior of complete Ricci-flat metrics constructed above. Each integral curve γ_{s_1} mentioned below satisfies the condition in Lemma 3.22, i.e., each γ_{s_1} is trapped in $\hat{U}_{(\delta,p,k)}$ initially and then enter $\hat{S}_3$ in finite time.

Lemma 4.1 *Let γ_{s_1} be a long time existing integral curve that intersects with $\hat{S}_3$ at a non-critical point $\gamma_{s_1}(\eta_*)$. Then function $\omega_1+\omega_2>1$ along $\gamma_{s_1}(\eta)$ for $\eta\in(\eta_*,\infty)$.*

Proof Note that $(\omega_1+\omega_2)(\gamma_{s_1}(\eta_*))=1$. By Lemma 3.6, we know that $\gamma_{s_1}(\eta)\in\hat{S}_3$ for $\eta\geq\eta_*$. We have

$$\begin{aligned}(\omega_1+\omega_2)'(\gamma_{s_1}(\eta_*))&=(2\omega_1(X_1-X_3)+2\omega_2(X_2-X_3))(\gamma_{s_1}(\eta_*))\\&\geq 0\quad\text{by definition of }\hat{S}_3.\end{aligned}\tag{4.1}$$

Suppose $(\omega_1+\omega_2)'(\gamma_{s_1}(\eta_*))=0$. Recall in the proofs of Lemma 3.22, we know that $X_1>X_3>X_2$ at $\gamma_{s_1}(\eta_*)$. By (3.16) and (3.17), we have

$$(\omega_1+\omega_2)''(\gamma_{s_1}(\eta_*))\geq\left(4\omega_1(X_1-X_3)^2+4\omega_2(X_2-X_3)^2\right)(\gamma_{s_1}(\eta_*))>0\quad.\tag{4.2}$$

Suppose there exists $\eta_1\in(\eta_*,\infty)$ that $(\omega_1+\omega_1)(\gamma_{s_1}(\eta_1))=1$. We know from the computation above that there exists $\eta_2\in(\eta_*,\eta_1)$ such that $(\omega_1+\omega_2)(\gamma_{s_1}(\eta_2))>1$. By mean value theorem, there exists $\eta_3\in[\eta_2,\eta_1]$ such that $(\omega_1+\omega_2)'(\gamma_{s_1}(\eta_3))=(2\omega_1(X_1-X_3)+2\omega_2(X_2-X_3))(\gamma_{s_1}(\eta_3))<0$, a contradiction to the definition of $\hat{S}_3$. □

Lemma 4.2 *The variable X_3 is smaller than $\frac{1}{n}$ along integral curves γ_{s_1}.*

Proof Since $\mathcal{H} \equiv 1$, $X_3 \leq \frac{1}{n}$ is equivalent to $X_1+X_2-2X_3 \geq 0$. The function $X_1+X_2-2X_3$ is positive at p_0. Suppose the function vanishes along γ_{s_1} at some point in $\hat{U}_{(\delta,p,k)}$, then we have

$$\begin{aligned}(X_1+X_2-2X_3)'\big|_{X_1+X_2-2X_3=0} &= (X_1+X_2-2X_3)(\mathcal{G}-1)+\mathcal{R}_1+\mathcal{R}_2-2\mathcal{R}_3\\ &= \mathcal{R}_1+\mathcal{R}_2-2\mathcal{R}_3 \quad \text{since } X_1+X_2-2X_3=0 \\ &= a(Z_2Z_3+Z_1Z_3-2Z_1Z_2)-2b(2Z_3^2-Z_1^2-Z_2^2).\end{aligned} \tag{4.3}$$

Consider the computation result above as a function

$$\mathcal{J}(Z_3) = -4bZ_3^2 + a(Z_1+Z_2)Z_3 + 2bZ_1^2 + 2bZ_2^2 - 2aZ_1Z_2.$$

Since $Z_1+Z_2 \leq Z_3 \leq \frac{Z_2}{\omega_*}$ in $\hat{U}_{(\delta,p,k)}$, the positivity of $\mathcal{J}$ is implied by those of $\mathcal{J}(Z_1+Z_2)$ and $\mathcal{J}\left(\frac{Z_2}{\omega_*}\right)$. With the choice $p \geq 2$, inequality (3.37) implies $\omega_* > \frac{p}{p+1} \geq \frac{2}{3} \geq \frac{4b}{a}$. Hence it is sufficient to prove a stronger condition: the positivity of $\mathcal{J}(Z_1+Z_2)$ and $\mathcal{J}\left(\frac{a}{4b}Z_2\right)$. We have

$$\begin{aligned}\mathcal{J}(Z_1+Z_2) &= (a-2b)(Z_1^2+Z_2^2)-8bZ_1Z_2\\ &\geq 4b(Z_1-Z_2)^2 \quad \text{Remark 1.1} \\ &\geq 0\end{aligned} \quad . \tag{4.4}$$

And we have

$$\begin{aligned}\mathcal{J}\left(\frac{a}{4b}Z_2\right) &= \left(\frac{a^2}{4b}-2a\right)Z_1Z_2+2b(Z_1^2+Z_2^2)\\ &\geq \left(\frac{a^2}{4b}-2a\right)Z_1Z_2+4bZ_1Z_2 \\ &\geq 0\end{aligned} \quad . \tag{4.5}$$

All Z_j's are positive along γ_{s_1}. Hence by (4.4) and (4.5), computation (4.3) can vanish only if $Z_1 = Z_2 = \frac{Z_3}{2}$. But with $p \geq 2$ imposed, $Z_2 \geq \omega_* Z_3 \geq \frac{2}{3}Z_3 \geq \frac{Z_3}{2}$ in $\hat{U}_{(\delta,p,k)}$. Hence $\mathcal{J}$ can only vanish at the origin of Z-space, which is impossible for γ_{s_1} to reach by (3.46). Therefore, $X_1+X_2-2X_3$ never vanishes along γ_{s_1} at least till γ_{s_1} intersect with $\partial\hat{S}_3$ at some $\gamma_{s_1}(\eta_*)$.

γ_{s_1} is in $\hat{S}_3$ for $\eta \in [\eta_*, \infty)$. The function $X_1+X_2-2X_3$ is positive at $\gamma_{s_1}(\eta_*)$. Suppose the function vanishes at $\gamma_{s_1}(\eta_{**})$ with $\eta_{**} \in (\eta_*, \infty)$ in $\hat{S}_3$, then at that point

$$\begin{aligned}&(X_1+X_2-2X_3)'\big|_{X_1+X_2-2X_3=0} = \mathcal{R}_1+\mathcal{R}_2-2\mathcal{R}_3\\ &= a(Z_2Z_3+Z_1Z_3-2Z_1Z_2)-2b(2Z_3^2-Z_1^2-Z_2^2)\\ &\geq a(Z_2Z_3+Z_1Z_3-2Z_1Z_2)-\frac{a}{3}(2Z_3^2-Z_1^2-Z_2^2) \quad \text{Remark 1.1}\\ &\geq \frac{a}{3}(Z_1+Z_2-Z_3)(2Z_3-Z_1-Z_2)\\ &\geq 0 \quad \text{definition of } \hat{S}_3\end{aligned} . \tag{4.6}$$

Suppose computation above vanishes at $\gamma_{s_1}(\eta_{**})$. By Proposition 3.21, there is no need to consider the case where each Z_j vanishes. One possibility is that $Z_1 = Z_2 = Z_3$ at that point. But then $X_3 - X_j + \rho(Z_3 - Z_j) = X_3 - X_j = \frac{1}{n} - X_j \geq 0$ at that point by the definition of $\hat{S}_3$. Then we must have $X_j = \frac{1}{n}$ for each j. Hence the point must be the critical point p_1, a contradiction. The other possibility, where $Z_1 = Z_2 = \frac{Z_3}{2}$ at that point, is ruled out by Lemma 4.1. Hence $X_3 < \frac{1}{n}$ along γ_{s_1} all the way. □

We can now describe the asymptotic limit of γ_{s_1}.

Lemma 4.3 *The integral curve γ_{s_1} converges to p_1.*

Proof Since γ_{s_1} does not hit any critical point in $\hat{U}_{(\delta,p,k)}$ by Lemma 3.22, we can focus on the behavior of the integral curve in the set $\hat{S}_3$. By Proposition 3.21, we know that $Z_1Z_2Z_3$ converges to some $m_1 > 0$ along γ_{s_1}. In addition, since

$$(Z_1Z_2)' = 2Z_1Z_2(\mathcal{G} - X_3) \geq 2Z_1Z_2\left(\frac{1}{n} - X_3\right) \geq 0$$

by Lemma 4.2, we know that Z_1Z_2 converges to some $m_2 > 0$ along γ_{s_1}. Therefore, the ω-limit set is a subset of

$$\left\{\left(x_1, x_2, x_3, z, \frac{m_2}{z}, \frac{m_1}{m_2}\right)\right\} \cap C \cap \{\mathcal{H} \equiv 1\}.$$

Since the ω-limit set is invariant and $Z_1Z_2Z_3 \equiv m_1$ in the set, the ω-limit set must be a subset of

$$\left\{\left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, z, \frac{m_2}{z}, \frac{m_1}{m_2}\right)\right\} \cap C \cap \{\mathcal{H} \equiv 1\},$$

a finite set. Hence the ω-limit set is a set of critical points with all X-coordinates be $\frac{1}{n}$. Hence the set must be a subset of $\{p_1, p_2\}$. For Case II and Case III, p_2 is not in $\hat{S}_3$. For Case I, we know that p_2 is not in the ω-limit set by Lemma 4.1. Hence the ω-limit set is the singleton $\{p_1\}$.

Consider $p_1 = \left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \alpha, \alpha, \alpha\right)$, where $\alpha = \frac{1}{n}\sqrt{\frac{n-1}{a-b}}$. By (2.22), the linearization at p_1 is

$$\mathcal{L}(p_1) = \begin{bmatrix} \frac{5}{3n} - 1 & \frac{2}{3n} & \frac{2}{3n} & 2b\alpha & (a-2b)\alpha & (a-2b)\alpha \\ \frac{2}{3n} & \frac{5}{3n} - 1 & \frac{2}{3n} & (a-2b)\alpha & 2b\alpha & (a-2b)\alpha \\ \frac{2}{3n} & \frac{2}{3n} & \frac{5}{3n} - 1 & (a-2b)\alpha & (a-2b)\alpha & 2b\alpha \\ \frac{5}{3}\alpha & -\frac{1}{3}\alpha & -\frac{1}{3}\alpha & 0 & 0 & 0 \\ -\frac{1}{3}\alpha & \frac{5}{3}\alpha & -\frac{1}{3}\alpha & 0 & 0 & 0 \\ -\frac{1}{3}\alpha & -\frac{1}{3}\alpha & \frac{5}{3}\alpha & 0 & 0 & 0 \end{bmatrix}. \tag{4.7}$$

Its eigenvalues and corresponding eigenvectors are

$$\lambda_1 = \frac{1}{n} - 1, \quad \lambda_2 = \lambda_3 = \beta_1, \quad \lambda_4 = \lambda_5 = \beta_2, \quad \lambda_6 = \frac{2}{n}.$$

$$v_1 = \begin{bmatrix} n-1 \\ n-1 \\ n-1 \\ -n\alpha \\ -n\alpha \\ -n\alpha \end{bmatrix}, v_2 = \begin{bmatrix} -\frac{\beta_1}{2\alpha} \\ \frac{\beta_1}{2\alpha} \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} -\frac{\beta_1}{2\alpha} \\ 0 \\ \frac{\beta_1}{2\alpha} \\ -1 \\ 0 \\ 1 \end{bmatrix}, v_4 = \begin{bmatrix} -\frac{\beta_2}{2\alpha} \\ \frac{\beta_2}{2\alpha} \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, v_5 = \begin{bmatrix} -\frac{\beta_2}{2\alpha} \\ 0 \\ \frac{\beta_2}{2\alpha} \\ -1 \\ 0 \\ 1 \end{bmatrix}, v_6 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ n\alpha \\ n\alpha \\ n\alpha \end{bmatrix},$$

where

$$\beta_1 = -\frac{n-1+\sqrt{(n-1)^2-8n^2\alpha^2(a-4b)}}{2n} < 0,$$
$$\beta_2 = -\frac{n-1-\sqrt{(n-1)^2-8n^2\alpha^2(a-4b)}}{2n} < 0.$$

Evaluate (2.23) at p_1, it is clear that $T_{p_1}(C \cap \{\mathcal{H} \equiv 1\}) = \text{span}\{v_2, v_3, v_4, v_5\}$. Critical point p_1 is a sink. Hence $\lim_{\eta\to\infty} \gamma_{s_1} = p_1$. □

Lemma 4.4 *Ricci-flat metrics represented by γ_{s_1} are AC.*

Proof For each j, we have

$$\lim_{t\to\infty} \dot{f}_j = \lim_{\eta\to\infty} \frac{X_j}{\sqrt{Z_k Z_l}} = \sqrt{\frac{a-b}{n-1}}. \tag{4.8}$$

Therefore by Definition 1.4, the Ricci-flat metric represented by γ_{s_1} has conical asymptotic limit $dt^2 + t^2\frac{a-b}{n-1}Q$. □

Lemmas 4.3 and 4.4 imply Theorem 1.5.

5 Singular Ricci-flat metrics

This section is dedicated to singular Ricci-flat metrics. Note that critical points p_1 and p_2 can be viewed as integral curves defined on $\mathbb{R}$. They correspond to singular Ricci-flat metrics $g = dt + t^2\frac{a-b}{n-1}Q$. This is consistent with the fact that the Euclidean metric cone over a proper scaled homogeneous Einstein manifold is Ricci-flat. For Case I in particular, the normal Einstein metric on G/K is strict nearly Kähler. Hence the metric cone represented by p_1 is the singular G_2 metric discovered in [6]. Note that functions F_j's in (3.13) do note vanish at p_2. Therefore, the Euclidean metric cone over the Kähler–Einstein metric has generic holonomy.

There are also singular Ricci-flat metrics represented by nontrivial integral curves. Recall Remark 3.3, The cohomogeneity one G_2 condition is given by $F_j \equiv 0$ for each j. Eliminate X_j's in the conservation law C shows that

$$\begin{aligned}\blacktriangle &= C \cap \{\mathcal{H} \equiv 1\} \cap P \cap \{F_1 \equiv F_2 \equiv F_3 \equiv 0\}\\ &= \{Z_1 + Z_2 + Z_3 - 1 \equiv 0\} \cap P \cap \{F_1 \equiv F_2 \equiv F_3 \equiv 0\}\end{aligned}$$

is an invariant two-dimensional plane with boundary. The projection of $\blacktriangle$ in Z-space is plotted in Fig. 5. Black squares are critical points of Type II. Linearization at these points shows that they are sources. Furthermore, for any $\xi \in \mathbb{R}$, $\blacktriangle \cap \{Z_3(Z_1 - Z_2) - \xi Z_2(Z_1 - Z_3) \equiv 0\}$ is a pair of integral curves that connects three critical points. If $\xi \neq 0, 1$, then these two integral curves connect p_1 with two distinct critical points of Type II. These integral curves represent singular cohomogeneity one G_2 metrics on $(0, \infty) \times G/K$ that do not have smooth extension to G/H [14,16]. They all share the same AC limit as the metric cone over G/K equipped with the normal Einstein metric.

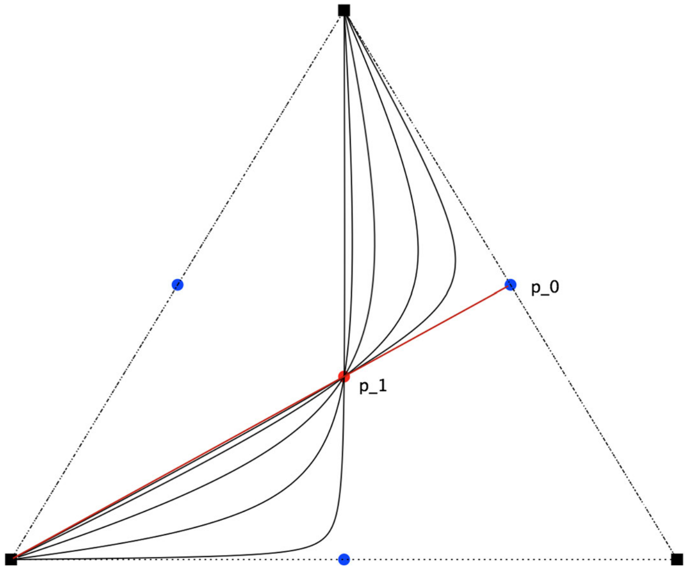


Fig. 5 Integral curves on ▲ with $0 < \xi \leq 1$

When $\xi = 0, 1$, then one of the integral curve connects a critical point of Type II with p_1 and the other one connects a critical point of Type III with p_1. In particular, if $\xi = 1$, then we recover γ_0 that represents the G_2 metric, connecting p_0 and p_1.

There are singular metrics with generic holonomy. We construct a new compact invariant set whose boundary includes p_1 and p_2. Consider

$$\check{S}_3 = S_3 \cap \{X_1 \equiv X_2, Z_1 \equiv Z_2\} \cap \{X_1 + X_2 - 2X_3 \geq 0\} \cap \left\{(d-1)^2 Z_1 Z_2 - 4b^2 Z_3^2 \geq 0\right\}.$$

Proposition 5.1 *$\check{S}_3$ is a compact invariant set.*

Proof It is easy to show that $\{X_1 \equiv X_2, Z_1 \equiv Z_2\}$ is flow-invariant. In fact, even if we define $\check{S}_3$ without $\{X_1 \equiv X_2, Z_1 \equiv Z_2\}$, the set is still compact and invariant. However, considering the subsystem does make the computation easier.

In $\check{S}_3$, we have

$$4b^2 Z_3^2 \leq (d-1)^2 Z_1 Z_2 < a^2 Z_1 Z_2 \leq a^2 (Z_1 + Z_2)^2. \tag{5.1}$$

Hence we can apply Proposition 3.1 and conclude that inequality (3.4) holds in $\check{S}_3$. As Z_3 is bounded above by $\frac{d-1}{2b}\sqrt{Z_1 Z_2}$ in $\check{S}_3$, the compactness follows by (2.20).

To show that $\check{S}_3$ is invariant, consider

$$\begin{aligned}
&\langle \nabla(X_1+X_2-2X_3), V\rangle|_{X_1+X_2-2X_3=0} = (X_1+X_2-2X_3)(\mathcal{G}-1)+\mathcal{R}_1+\mathcal{R}_2-2\mathcal{R}_3\\
&\quad = 2\mathcal{R}_2-2\mathcal{R}_3 \quad \text{since } Z_1\equiv Z_2 \text{ in } \check{S}_3 \text{ and } X_1+X_2-2X_3=0\\
&\quad = 2(Z_3-Z_2)((d-1)\sqrt{Z_1Z_2}-2bZ_3) \quad \text{since } Z_1\equiv Z_2 \text{ in } \check{S}_3\\
&\quad \geq 0 \quad \text{by definition of } \check{S}_3
\end{aligned}.$$

Moreover, we have

$$\begin{aligned}
&\langle \nabla((d-1)^2Z_1Z_2-4b^2Z_3^2), V\rangle\big|_{(d-1)^2Z_1Z_2-4b^2Z_3^2=0}\\
&\quad = \nabla\left(Z_3^2\left((d-1)^2\frac{Z_1Z_2}{Z_3^2}-4b^2\right)\right)\Bigg|_{(d-1)^2Z_1Z_2-4b^2Z_3^2=0}\\
&\quad = \left((d-1)^2Z_1Z_2-4b^2Z_3\right)\left(\mathcal{G}-\frac{1}{d}+2X_3\right)+2(d-1)^2Z_1Z_2\,(X_1+X_2-2X_3)\\
&\quad = 2(d-1)^2Z_1Z_2\,(X_1+X_2-2X_3) \quad \text{since } (d-1)^2Z_1Z_2-4b^2Z_3^2=0\\
&\quad \geq 0 \quad \text{by definition of } \check{S}_3
\end{aligned}.$$

Hence $\check{S}_3$ is a compact invariant set. □

Lemma 5.2 *There exists an integral curve Γ defined on $\mathbb{R}$ emanating from p_2 in $\check{S}_3$.*

Proof Consider $p_2=\left(\frac{1}{n},\frac{1}{n},\frac{1}{n},\frac{2}{n}\sqrt{\frac{(n-1)b}{(d-1)(a+2b)}},\frac{2}{n}\sqrt{\frac{(n-1)b}{(d-1)(a+2b)}},\frac{1}{n}\sqrt{\frac{(n-1)(d-1)}{b(a+2b)}}\right)$. For simplicity, denote $Z_*=\frac{2}{n}\sqrt{\frac{(n-1)b}{(d-1)(a+2b)}}$. The linearization at p_2 is

$$\mathcal{L}(p_2)=\begin{bmatrix}
\frac{5}{9d}-1 & \frac{2}{9d} & \frac{2}{9d} & 2bZ_* & \left(\frac{a(d-1)}{2b}-2b\right)Z_* & 2bZ_*\\
\frac{2}{9d} & \frac{5}{9d}-1 & \frac{2}{9d} & \left(\frac{a(d-1)}{2b}-2b\right)Z_* & 2bZ_* & 2bZ_*\\
\frac{2}{9d} & \frac{2}{9d} & \frac{5}{9d}-1 & (d-1)Z_* & (d-1)Z_* & (d-1)Z_*\\
\frac{5}{3}Z_* & -\frac{1}{3}Z_* & -\frac{1}{3}Z_* & 0 & 0 & 0\\
-\frac{1}{3}Z_* & \frac{5}{3}Z_* & -\frac{1}{3}Z_* & 0 & 0 & 0\\
-\frac{d-1}{6b}Z_* & -\frac{d-1}{6b}Z_* & -\frac{5(d-1)}{6b}Z_* & 0 & 0 & 0
\end{bmatrix} \tag{5.2}$$

Straightforward computation shows that for all cases, $L(p_2)$ is a hyperbolic critical point that has only one unstable eigenvalues with the corresponding eigenvector as

$$\check{\lambda}=\frac{1}{2n}\left(\sqrt{(n-1)^2+96n(d-1)(a-4b)Z_*^2}-(n-1)\right),\quad \check{v}=\begin{bmatrix} b\check{\lambda}\\ b\check{\lambda}\\ -2b\check{\lambda}\\ 2bZ_*\\ 2bZ_*\\ -2(d-1)Z_*\end{bmatrix}$$

Evaluate (2.23) at p_2, it is clear that $\check{v}$ are tangent to $C \cap \{\mathcal{H} \equiv 1\}$. Fix $\check{s}_0 > 0$, there exists a unique trajectory Γ emanating from p_2 with $\Gamma \sim p_2 + \check{s}_0 e^{\check{\lambda}\eta}\check{v}$.

It is easy to check that $p_2 \in \partial \check{S}_3$ with only $X_1 + X_2 - 2X_3$ and $(d-1)^2 Z_1 Z_2 - 4b^2 Z_3^2$ vanished at p_2. By straightforward computation, we know that Γ is trapped in $\check{S}_3$ initially. The integral curve is hence defined on $\mathbb{R}$. Functions f_j's that correspond to solutions Γ are defined on $[0, \infty)$. □

Lemma 5.3 *The integral curve Γ converges to p_1.*

Proof Since $\check{S}_3$ is a compact invariant set with $X_3 \leq \frac{1}{n}$. Arguments in Lemmas 4.3 and 4.4 carry over. Hence for Γ, we have $\lim_{\eta\to\infty} \Gamma = p_1$. □

For each j, we have

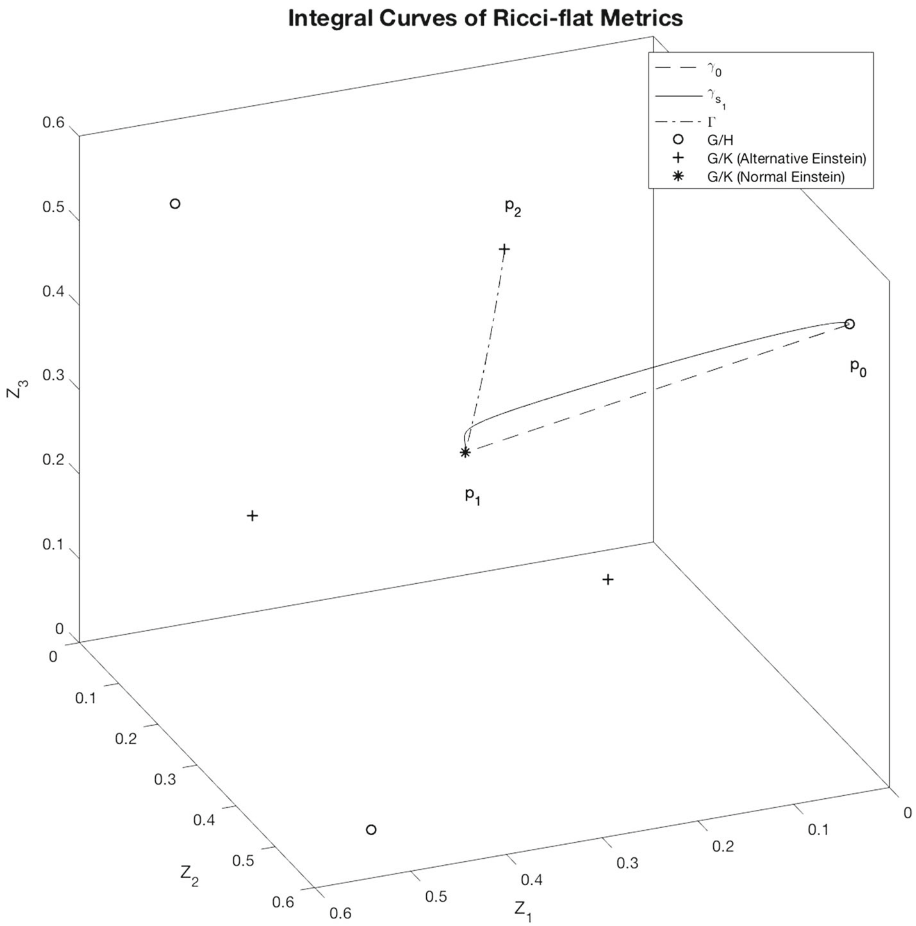
$$\begin{aligned} \lim_{t\to 0} \dot{f}_j &= \lim_{\eta\to -\infty} \frac{X_j}{\sqrt{Z_k Z_3}} = \frac{\check{\lambda}}{2Z_*} \quad j, k \in \{1, 2\} \\ \lim_{t\to 0} \dot{f}_3 &= \lim_{\eta\to -\infty} \frac{X_3}{\sqrt{Z_1 Z_2}} = \frac{b\check{\lambda}}{(d-1)Z_*} \end{aligned} . \tag{5.3}$$

Hence $f_1 = f_2 \sim \frac{\check{\lambda}}{2Z_*}t$ and $f_3 \sim \frac{b\check{\lambda}}{(d-1)Z_*}t$ as $t \to 0$. Since $\lim_{t\to\infty} \dot{f}_1 = \lim_{t\to\infty} \dot{f}_2 = \frac{d-1}{2b}\lim_{t\to\infty} \dot{f}_3$, Γ represents a singular metric whose end at $t \to 0$ is a conical singularity as a metric cone over the alternative Einstein metrics. Lemmas 5.2 and 5.3 then prove the following theorem.

Theorem 5.4 *Up to homothety, there exists a unique singular Ricci-flat metric on $(0, \infty) \times G/K$ that at the end with $t \to 0$, it admits conical singularity as the metric cone over G/K with alternative Einstein metric. It has an AC limit at the end with $t \to \infty$ as the metric cone over G/K with normal Einstein metric.*

Results of this article can be summarized by the plot in the following page. It shows the projection of integral curves to (2.19) on the Z-space for Case I. It is computed by MATLAB using the 4th order Runge–Kutta method.

Integral curves	Metric type
γ_0	Smooth metric with vanished principal curvatures on G/H
$\gamma_{s_1}, s_1 \neq 0$	Smooth metrics with nonzero principal curvatures on G/H
Γ	Conical Singularity as alternative Einstein metric on G/K



Acknowledgements The author is grateful to his PhD supervisor, Prof. McKenzie Wang for his guidance and encouragement.

References

1. Back, A.: Local theory of equivariant Einstein metrics and Ricci realizability on Kervaire spheres. Preprint (1986)
2. Bérard-Bergery, L.: Sur de nouvelles variétés riemanniennes d'Einstein. Inst. Élie. Cartan **6**, 1–60 (1982)
3. Besse, A.L.: Einstein Manifolds. Classics in Mathematics. Springer, Berlin (2008). **(Reprint of the 1987 edition)**
4. Böhm, C.: Inhomogeneous Einstein metrics on low-dimensional spheres and other low-dimensional spaces. Invent. Math. **134**(1), 145–176 (1998)
5. Brown, R.B., Gray, A.: Riemannian manifolds with holonomy group Spin(9). In: Differential geometry (in honor of Kentaro Yano), pp. 41–59. Kinokuniya, Tokyo (1972)
6. Bryant, R.L.: Metrics with exceptional holonomy. Ann. Math. **126**(3) 525–576 (1987)
7. Bryant, R.L., Salamon, S.M.: On the construction of some complete metrics with exceptional holonomy. Duke Math. J. **58**(3), 829–850 (1989)
8. Buzano, M., Dancer, A.S., Gallaugher, M., Wang, M.Y.: Non-Kähler expanding Ricci solitons, Einstein metrics, and exotic cone structures. Pacific J. Math. **273**(2), 369–394 (2015)

9. Buzano, M., Dancer, A.S., Wang, M.: A family of steady Ricci solitons and Ricci flat metrics. Comm. Anal. Geom. **23**(3), 611–638 (2015)
10. Calabi, E.: A construction of nonhomogeneous Einstein metrics. In Differential geometry (Proc. Sympos. Pure Math., vol. XXVII, Stanford Univ., Stanford, Calif., 1973), Part 2, pp. 17–24. American Mathematical Society, Providence, RI (1975)
11. Calabi, E.: Métriques Kählériennes et fibrés holomorphes. Ann. Sci. École Norm. Sup. **12**(2), 269–294 (1979)
12. Castrillón López, M., Gadea, P.M., Mykytyuk, I.V.: The canonical eight-form on manifolds with holonomy group Spin(9). Int. J. Geom. Methods Mod. Phys. **7**(7), 1159–1183 (2010)
13. Chen, D.: Examples of einstein manifolds in odd dimensions. Ann. Global Anal. Geom. **40**(3), 339 (2011)
14. Cleyton, R., Swann, A.: Cohomogeneity one G_2-structures. J. Geom. Phys. **44**(2–3), 202–220 (2002)
15. Coddington, E.A., Levinson, N.: Theory of Ordinary Differential Equations. McGraw-Hill Book Company Inc, New York (1955)
16. Cvetič, M., Gibbons, G.W., Lü, H., Pope, C.N.: Cohomogeneity one manifolds of Spin(7) and G_2 holonomy. Phys. Rev. D **65**(10), 106004 (2002)
17. Cvetič, M., Gibbons, G.W., Lü, H., Pope, C.N.: New cohomogeneity one metrics with Spin(7) holonomy. J. Geom. Phys. **49**(3–4), 350–365 (2004)
18. Dancer, A.S., Wang, M.Y.: Kähler–Einstein metrics of cohomogeneity one. Math. Ann. **312**(3), 503–526 (1998)
19. Dancer, A.S., Wang, M.Y.: Non-Kähler expanding Ricci solitons. Int. Math. Res. Not. IMRN **6**, 1107–1133 (2009)
20. Dancer, A.S., Wang, M.Y.: Some new examples of non-Kähler Ricci solitons. Math. Res. Lett. **16**(2), 349–363 (2009)
21. Dávila, J.C.G., Martín Cabrera, F.: Homogeneous nearly Kähler manifolds. Ann. Global Anal. Geom. **42**(2), 147–170 (2012)
22. Eguchi, T., Hanson, A.J.: Self-dual solutions to Euclidean gravity. Ann. Physics **120**(1), 82–106 (1979)
23. Eschenburg, J.-H., Wang, M.Y.: The initial value problem for cohomogeneity one Einstein metrics. J. Geom. Anal. **10**(1), 109–137 (2000)
24. Foscolo, L., Haskins, M., Nordström, J.: Infinitely many new families of complete cohomogeneity one G_2-manifolds: G_2 analogues of the Taub-NUT and Eguchi–Hanson spaces (2018). arXiv:1805.02612 [hep-th]
25. Gibbons, G.W., Page, D.N., Pope, C.N.: Einstein metrics on S^3, R^3 and R^4 bundles. Comm. Math. Phys. **127**(3), 529–553 (1990)
26. Harvey, F.R.: Spinors and Calibrations. Perspectives in Mathematics. Elsevier Science, Amsterdam (1990)
27. Hsiang, W., Lawson Jr., H.B.: Minimal submanifolds of low cohomogeneity. J. Differential Geom. **5**, 1–38 (1971)
28. Nikonorov, Y.G.: Classification of generalized Wallach spaces. Geom. Dedicata **181**, 193–212 (2016)
29. Salamon, S.M.: Riemannian geometry and holonomy groups, volume 201 of Pitman Research Notes in Mathematics Series. Longman Scientific & Technical, Harlow; copublished in the United States with Wiley, New York (1989)
30. Wallach, N.R.: Compact homogeneous Riemannian manifolds with strictly positive curvature. Ann. of Math. **2**(96), 277–295 (1972)
31. Wang, J., Wang, M.Y.: Einstein metrics on S^2-bundles. Math. Ann. **310**(3), 497–526 (1998)
32. Wang, M.Y.: Preserving parallel spinors under metric deformations. Indiana Univ. Math. J. **40**(3), 815–844 (1991)
33. Wink, M.: Cohomogeneity one Ricci solitons from Hopf fibrations (2017). arXiv preprint arXiv:1706.09712

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.